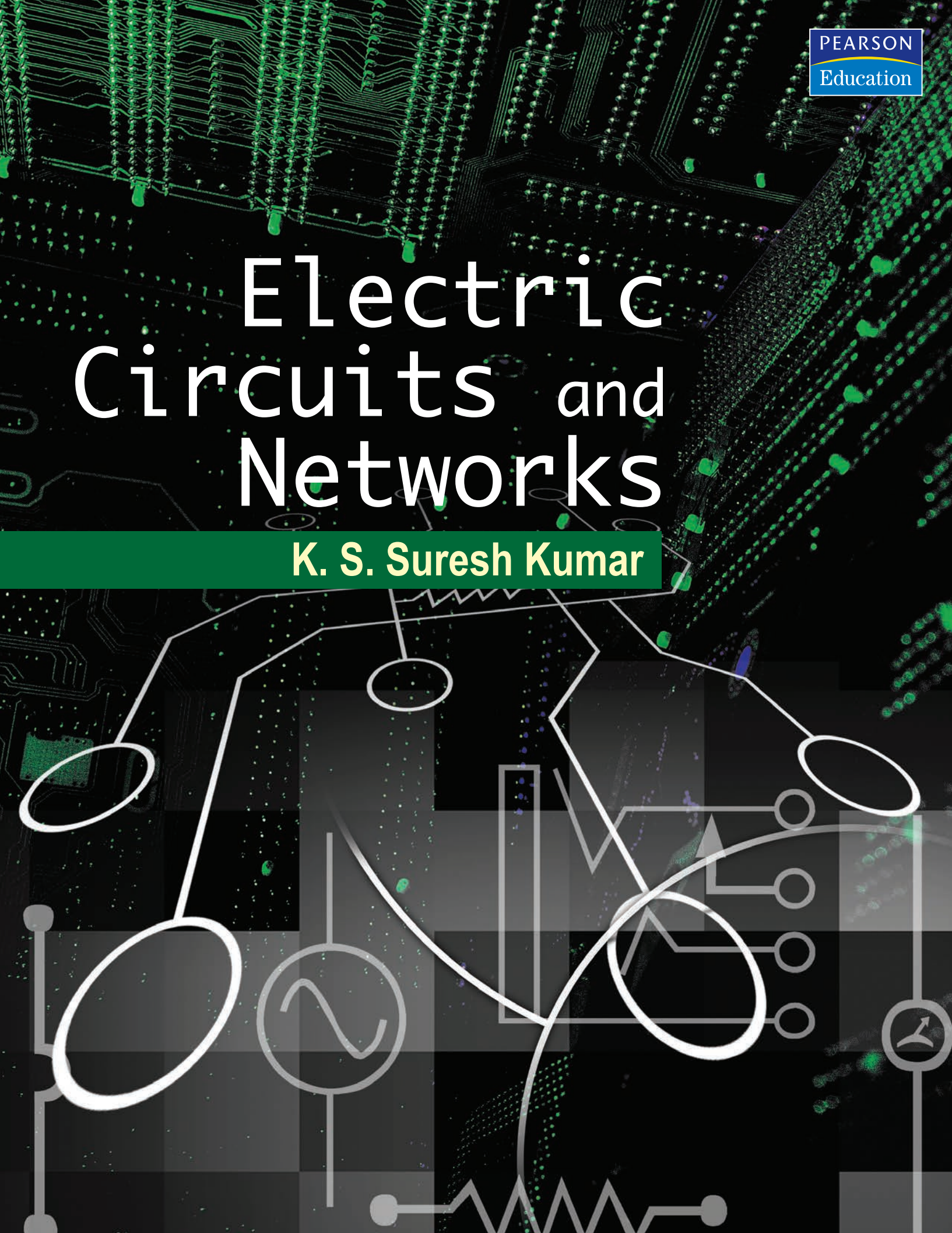


Electric Circuits and Networks

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This book is dedicated to the memory of

Karunakaran Sir

who was the class teacher for class X – C division during the academic year 1973–74 at
Government Boys' High School, Attingal, Thiruvananthapuram District, Kerala, India.

Layout at a Glance

Chapter Objectives: Chapter objectives provides a brief overview of the concepts to be discussed in the chapter.

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Circuit Theorems

CHAPTER OBJECTIVES

- Derive Superposition Theorem from the property of linearity of elements.
- Explain the two key theorems – Superposition Theorem and Substitution Theorem in detail.
- Derive other theorems like Compensation Theorem, Thevenin's Theorem, Norton's Theorem, Reciprocity Theorem and Maximum Power Transfer Theorem from these two key principles.
- Provide illustrations for applications of circuit theorems in circuit analysis through solved examples.
- Emphasise the use of Compensation Theorem, Thevenin's Theorem and Norton's Theorem in circuits containing dependent sources as a pointer to their applications in the study of Electronic Circuits.

This Chapter identifies the Substitution Theorem and Superposition Theorem as the two key theorems and shows how the other theorems may be extracted from them.

INTRODUCTION

The previous chapter showed that:

- (1) All the element voltages and element currents in a circuit can be obtained from its node voltages. The node voltages are governed by a matrix equation $YV = CU$, where V is the node voltage column vector, Y is the nodal conductance matrix of the circuit, U is the input column vector containing source functions of all independent voltage sources and current sources in the circuit and C is the input matrix. The values of conductances in the circuit and values of coefficients of linear dependent sources in the circuit decide the elements of Y -matrix. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources can make Y -matrix asymmetric. The C -matrix, in general, contain 0, 1, -1 and conductance values as well as dependent source coefficients.
- (2) An alternative formulation is given by a matrix equation $ZI = DU$, where I is the mesh current column vector, Z is the mesh resistance matrix of the circuit,

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The Operational Amplifier as a Circuit Element

CHAPTER OBJECTIVES

- To introduce Operational Amplifier (Opamp) as a circuit element and explain its features.
- To develop and illustrate the analysis of Opamp circuits using the Ideal Operational Amplifier (IOA) model.
- To explain the principles of operation of commonly employed linear Opamp circuits.
- To extend the IOA Model to include offset and input bias current effects.
- To illustrate the effect of voltage, current and slope limits at the Opamp output on circuit performance.

The introductory part of the chapter uses a MOSFET amplifier as an example to develop various concepts like bias point, small-signal and large-signal operation, linear and non-linear distortion, role of DC power supply, output limits, etc., in the context of amplifiers.

INTRODUCTION

We have developed certain powerful procedures of analysis and a set of powerful tools in the form of circuit theorems for memoryless circuits in the last two chapters. Memoryless circuits contain linear resistors and linear dependent sources and are driven by independent voltage sources and independent current sources. We continue our discussion on such circuits by introducing a very popular circuit element called Operational Amplifier (Opamp). It is an electronic amplifier that can be modelled by a voltage-controlled voltage source (VCVS).

We are familiar with four kinds of dependent sources – VCVS, CCVS, VCCS and CCCS. All these dependent sources are employed in modelling various kinds of electronic amplifiers. In fact, any interaction between two circuit variables that do not pertain to the same electrical element can be modelled by dependent sources. Coupled coils are modelled using dependent sources occasionally. However, the most frequent application of dependent source models occurs in modelling electronic devices and systems. Electronic amplifiers

Introduction: The Introduction gives a glimpse of how the content of this chapter evolve from the preceding chapter.

interesting source functions – unit impulse function $\delta(t)$ and unit step function $u(t)$. These functions are extremely important in Circuit Analysis.

3.1 THE RESISTOR

The physical basis for the two-terminal element, called resistor, was dealt in detail in Chap. 1. We revise briefly.

The source of e.m.f. in a circuit sets up charge distributions at the terminals of all the two-terminal elements connected in the circuit. This charge distribution at the terminals of a resistor sets up an electric field inside the conducting material in the resistor. The mobile electrons get accelerated by this electric field and move. But, their motion is impeded by frequent collisions with non-mobile atoms in the conducting substance. A steady situation in which the mobile electrons attain a constant average speed as a result of the aggregate effect of large number of collisions occur in the conducting material within a short time (called *relaxation time* of the conductor material in *Electromagnetic Field Theory*) of appearance of electric field. Once this steady situation occurs, the current through a linear resistor is proportional to the voltage appearing across it. The constant of proportionality is called 'resistance' of the resistor and has 'Ohm' (represented by ' Ω ') as its unit. Reciprocal of resistance is called 'conductance' of the resistor and its unit is 'Siemens' (represented by 'S'). The unit 'mho' is also used sometimes for conductance. The unit 'mho' is represented by inverted ' Ω ' – i.e., by Ω^{-1} .

Ohm's Law, an experimental law describing the relationship between voltage across a resistor and current through it, states that the voltage across a linear resistor at any instant t is proportional to the current passing through it at that instant provided the temperature of the resistor is kept constant. A resistor is called *linear* if it obeys Ohm's law. This is a kind of circular definition. We settle the matter by stating that we consider only those resistors that have a proportionality relationship between voltage and current in our study of circuits in this book.

The graphic symbol of a linear resistor and its element relationship is given below.

$$\begin{aligned} v(t) &= Ri(t) \text{ or } i(t) = Gv(t) \text{ for all } t \\ p(t) &= v(t)i(t) = Ri(t)^2 = \frac{[v(t)]^2}{R} = \frac{[i(t)]^2}{G} = G[v(t)]^2 \end{aligned}$$

where $p(t)$ is the power delivered to the resistor in Watts.

The resistor does not remember what was done to it previously. Its current response at a particular instant depends only on the voltage applied across at that instant. Therefore, a resistor is a *memoryless* element. Such an element needs to have same kind of wave-shape in both voltage and current. It is not capable of changing the wave-shape of a signal applied to it. It can only dissipate energy. Therefore, the power delivered to a positive resistor is always positive or zero.

3.1.1 Series Connection of Resistors

Consider the series connection of n resistors $R_1, R_2, \dots, R_n$ as in Fig. 3.1-1.

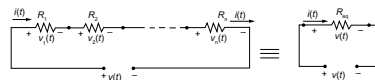


Fig. 3.1-1 Series Connection of Resistors and Its Equivalent

Voltage-current relation and power relations for a linear resistor obeying Ohm's Law.

Main headings and sub-headings: Well-organised main headings and sub-headings to guide the reader through and provide a lucid flow of the topic.

Circuits: Topics presented with clear circuits supported by analytical and conceptual ideas.

Thus, the only correct way to model a circuit that involves parallel connections of voltage sources (more generally, loops comprising only voltage sources) is to take into account the parasitic elements that are invariably associated with any practical voltage source. A somewhat detailed model for the two-source system is shown in Fig. 2.3-2.

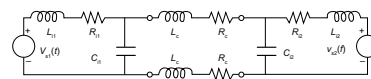


Fig. 2.3-2 A Detailed Model for a Circuit with Two Voltage Sources in Parallel

L_{c1} and L_{c2} represent the internal inductance of the sources, C_{t1} and C_{t2} represent the terminal capacitance of the sources and R_{i1} and R_{i2} represent the internal resistance of the sources. L_c and R_c represent the inductance and resistance of the connecting wires. Obviously, two practical voltage sources can be connected in parallel even if their open-circuit electromotive forces (e.m.f.s) are not equal at all t , only that they cannot be modelled by *ideal independent voltage source model*.

Two ideal independent current sources in series raise a similar issue (see Fig. 2.3-3). KCL requires that $i_{s1}(t) = i_{s2}(t)$ for all t . Even if this condition is satisfied, there is no way to obtain the voltages appearing across the current sources. Therefore, the correct model to be employed for practical current sources that appear in series in a circuit is a detailed model that takes into account the parasitic elements associated with any practical device. More generally, if there is a node in a circuit where only current sources are connected, then, those current sources cannot be modelled by *ideal independent current source model*.

Similar situations may arise in modelling practical dependent sources by *ideal dependent source models*. In all such cases we have to make the model more detailed in order to resolve the conflict that arises between Kirchhoff's laws and *ideal* nature of the model.

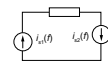


Fig. 2.3-3 Two Ideal Independent Current Sources in Series with Another Element

2.4 ANALYSIS OF A SINGLE-LOOP CIRCUIT

The circuit analysis problem involves finding the voltage variable and current variable of every element as functions of time, given the source functions. *Source functions* are the time-functions describing the e.m.f. of independent voltage sources and source currents of independent current sources. They are also called the *excitation functions*. If the circuit contains b -elements, there will be $2b$ variables to be solved for. Some of them will be known in the form of source functions, while others have to be solved for.

Element relation of each element gives us one equation per element. Thus, there are b equations arising out of element relations. The remaining b equations are provided by the interconnection constraints. These equations are obtained by applying KCL at all nodes except one and KVL in all meshes (in the case of a planar circuit).

Theoretically speaking, that is all there is to circuit analysis. However, systematic procedures for applying element relations, KVL equations and KCL equations would be highly desirable when it comes to analysis of complex circuits. Moreover, the fact that there are $2b - 2$ KCL equations for an n -node circuit and only $(n - 1)$ of them are independent, calls for a systematic procedure for writing KCL equations. Similarly, there will be b KVL equations for a circuit with b -loops and only $(b - n + 1)$ of them will be independent. This, again, calls for some systematic procedures for extracting a set of $(b - n + 1)$ independent KVL equations.

But, what is the use of a theorem that wants us to solve a circuit first and then replace part of the circuit by a source that has a value depending on the solution of the circuit? Obviously, such a theorem will not help us directly in solving circuits. The significance of this theorem lies in the fact that it can be used to construct theoretical arguments that lead to other powerful circuit theorems that indeed help us to solve circuit analysis problems in an elegant and efficient manner. Moreover, it does find application in circuit analysis in a slightly disguised form. We take up that disguised form of Substitution Theorem in Sect. 5.4.

part of the circuit that is being substituted and the remaining circuit except through the pair of terminals at which they are interconnected.

Subject to the constraints on unique solution and interaction only through the connecting terminals, we state the Substitution theorem as below (Fig. 5.3-8).

Let a circuit with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let $v(t)$ be the voltage that appears at the terminals between N_1 and N_2 and let $i(t)$ be the current flowing into N_2 from N_1 . Then, the network N_2 may be replaced by an independent current source of value $i(t)$ connected across the output of N_1 or an independent voltage source of value $v(t)$ connected across the output of N_1 without affecting any voltage or current variable within N_1 provided the resulting network has unique solution.

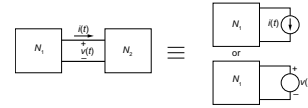


Fig. 5.3-8 The Substitution Theorem

5.4 COMPENSATION THEOREM

The circuit in Fig. 5.4-1(a) has a resistor marked as R . It has a nominal value of $2\ \Omega$. Mesh analysis was carried out to find the current in this resistor and the current was found to be 1 A as marked in the circuit as in Fig. 5.4-1(a).

Now, let us assume that the resistor value changes by ΔR to $R + \Delta R$. Correspondingly all circuit variables change by small quantities as shown in Fig. 5.4-1(b). The current through that resistor will also change to $i + \Delta i$. We can conduct a mesh analysis once again and get a new solution. However, we can do better than that. We can work out changes in variables everywhere by solving a single-source circuit and then construct the circuit solution by adding change to the initial solution value.

We apply Substitution theorem on the first circuit with R as the element that is being substituted and on the second circuit with $R + \Delta R$ as the part that is being substituted by an independent voltage source. The voltage source in the first circuit must be Ri V and the voltage source in the second circuit must be $(R + \Delta R)(i + \Delta i)$ V.

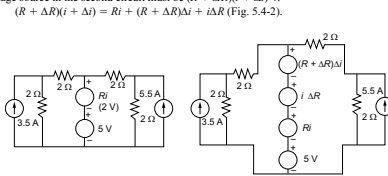


Fig. 5.4-2 Circuits After Applying Substitution Theorem

Stubs: Stubs in marginalia stress on important concepts. Additional information is also provided, wherever relevant.

and a_i is its 'coefficient of contribution'. The coefficient of contribution has the physical significance of contribution per unit input.

The coefficient of contribution, a_i , which is a constant for a time-invariant circuit, can be obtained by solving for $x(t)$ in a single-source circuit in which all independent sources other than the i^{th} one are deactivated by replacing independent voltage sources with short-circuits and independent current sources with open-circuits.

But, why should a linear combination $x = a_1 i_1 + a_2 i_2 + \dots + b_1 v_1 + b_2 v_2 + \dots$ be found term by term always? Is it possible to get it in subsets that contain more than one term? The third form of Superposition Theorem states that it can be done.

Superposition Theorem Form-3

'The response of any circuit variable in a multi-source linear memoryless circuit containing "n" independent sources can be obtained by adding response of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between such active source subsets among them'.

5.1.1 Linearity of a Circuit

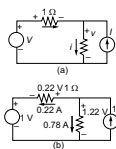
Why did the memoryless circuits we have been dealing with till now obey superposition principle? The elements of memoryless circuits were constrained to be linear time-invariant elements. We used only linear resistors and linear dependent sources. The v - i relations of all those elements obey superposition principle. As a result, all KCL and KVL equations in nodal analysis and mesh analysis had the form of linear combinations. Such KVL and KCL equations lead to nodal conductance matrix (and mesh resistance matrix) that contain only constants in the case of a time-invariant circuit (i.e., resistances are constants and coefficients of dependent sources are also constants). Similarly, the input matrix (C in nodal analysis and D in mesh analysis) will contain only constants in the case of circuits constructed using linear time-invariant elements. Thus, the solution for node voltage variables and mesh current variables will come out in the form of linear combination of independent source functions. And, after all Superposition Theorem is only a restatement of this fact. Therefore, Superposition Theorem holds in the circuit since we used only linear elements in constructing it except for independent sources which are non-linear. Hence, we conclude that a memoryless circuit constructed from a set of linear resistors, linear dependent sources and independent sources (they are non-linear elements) results in a circuit which obeys Superposition Theorem and hence, by definition, is a linear circuit.

Linearity of a circuit element and linearity of a circuit are two different concepts. An element is linear if its v - i relationship obeys principle of homogeneity and principle of additivity. A circuit is linear, if all circuit variables in it, without any exception, obey principle of homogeneity and principle of additivity, i.e., the principle of superposition. It may appear intuitively obvious that a circuit containing only linear elements will turn out to be a linear circuit. But, note that we used non-linear elements – independent sources are non-linear elements – and hence, it is not so apparent. The preceding discussion offers a plausibility reasoning to convince us that a circuit containing linear elements and independent sources will indeed be a linear circuit. But the mathematical proof for this apparently straightforward conclusion is somewhat formidable.

Linearity and Superposition appear so natural to us. But the fact is that most of the practical electrical and electronic circuits are non-linear in nature. Linearity, at best, is only an approximation that circuit analysts employ to make the analysis problem more tractable. We illustrate why Superposition Theorem does not hold for a circuit containing a non-linear element by an example. The circuit is shown in Fig. 5.1-3(a). The resistor R is a non-linear one with a v - i relation given by $v = 2i^2$ for $i \geq 0$ and $-2i^2$ for $i < 0$.

Superposition Theorem – Third form.

Linearity of a Circuit
Linearity of a circuit element and linearity of a circuit are two different concepts. A circuit is called linear if its solution obeys superposition principle. This is why we stated the Superposition Theorem with the adjective linear behind 'circuit'. Whether we view the statements on Superposition Theorem as a definition of linearity of a circuit or as an assertion of an important property of linear circuits is matter of viewpoint. There is indeed a bit of circularity in Linearity and Superposition Principle.

Fig. 5.1-3 (a) A Circuit Containing a Non-linear Resistor (b) Circuit Solution for $V = 1\text{ V}$ and $I = 1\text{ A}$

Pointer entries: Pointer entries located in the margin 'point' to significant discussions in the text to reiterate them.

We may recast the expressions that involve sum of the two sinusoidal functions in Eqn. 11.6-8 and 11.6-9 as single sinusoidal functions by employing trigonometric identities in the following manner.

$$v_c(t) = \sqrt{V_0^2 + \frac{L I_0^2}{C}} \cos(\omega_d t - \phi) \text{ V for } t \geq 0^+$$

$$i(t) = -\sqrt{I_0^2 + \frac{C V_0^2}{L}} \sin(\omega_d t - \phi) \text{ A for } t \geq 0^+, \quad (11.6-10)$$

where $\phi = \tan^{-1} \left(\frac{I_0 \sqrt{L/C}}{V_0} \right)$

These waveforms are shown in Fig. 11.6-5 for $L = 1 \text{ H}$, $C = 1 \text{ F}$, $V_0 = 2 \text{ V}$ and $I_0 = 1 \text{ A}$.

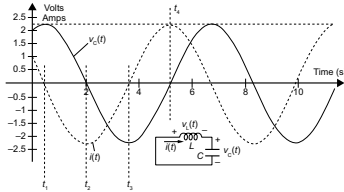


Fig. 11.6-5 Zero-Input Response of a LC Circuit ($L = 1 \text{ H}$, $C = 1 \text{ F}$, $V_0 = 2 \text{ V}$ and $I_0 = 1 \text{ A}$)

The initial voltage of 2 V across the capacitor appears across the inductor at $t = 0^+$ with a polarity such that the inductor current starts decreasing at the rate of $2 \text{ V}/1 \text{ H} = 2 \text{ A/s}$ from its initial value of 1 A. However, the circuit current is in a direction suitable for increasing the capacitor voltage. Hence, the capacitor voltage increases while the inductor current decreases. Under the action of increasing reverse voltage, the inductor current decreases more rapidly to reach zero at the instant t_1 . At that instant, the current and hence the energy storage in inductor are zero. The inductor had an initial energy of 0.5 J and the capacitor had an initial energy of 2 J. There was no dissipation in the circuit. Therefore, when the circuit current reaches zero, the capacitor must hold the total initial energy of 2.5 J in it. It will require 15 V across it (since $C = 1 \text{ F}$ and energy $= 0.5 C V^2$). Equation 11.6-10 predicts exactly this value as the amplitude of $v_c(t)$. When circuit current goes through zero, capacitor voltage must go through a positive or negative peak due to two reasons – firstly, the current through a capacitor is proportional to the rate of change of voltage across it and secondly that is the instant at which it will contain the maximum possible energy equal to the total initial energy. Therefore, $v_c(t)$ reaches a positive peak at t_1 .

With such a large reverse voltage across it, the inductor has to continue its current build up in the negative direction. But, with the current changing its direction, the capacitor

The source-free response (equivalently, the zero-input response) of a pure LC circuit will contain undying sinusoids with steady amplitudes. The amplitude of sinusoidal waveforms is decided by the total initial energy storage in the circuit and the circuit parameters. Circuit parameters, i.e., L and C decide the angular frequency of oscillations too – it is $(LC)^{-1/2}$ rad/s.

A Pure LC Circuit?

Strictly speaking, a pure LC circuit cannot exist in practice. The wire used to construct the inductor, the metal foil used in the capacitor and the connecting wires have non-zero resistance. The dielectric used in the capacitor will have non-zero conductivity. Thus, there will be some non-zero resistance left in any LC circuit.

continued

Graphical representations:
Graphical representations for figurative analysis of circuit behaviour.

Worked examples: Worked examples illustrate the theory explained in the text.

We solve the problem by finding the node voltage v_1 first. We express v_2 as $v_2 = v_1$ and write the node equation at the node where v_1 is assigned.

$$\frac{1}{R_1}(v_1 - v_i) + \frac{1}{R_2}v_1 + \frac{1}{R_3 + R_4}(v_1 - A(v_1 - v_i)) = 0$$

$$\text{Solving for } v_1, v_1 = \frac{\frac{1}{R_1} + \frac{A}{R_3 + R_4}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3 + R_4} + \frac{A}{R_3 + R_4}} v_i$$

Substituting the numerical values, we get, $v_1 = 0.9999 v_i$. Therefore, $v_2 = 0.0001 v_i$. The current in R_2 is $100000 v_i - 0.9999 v_i$ divided by 900075Ω . Therefore, it is equal to $9.9993 \times 10^{-6} v_i \text{ A}$.

Therefore, the voltage drop in $R_3 = 75 \times 9.9993 \times 10^{-6} v_i = 7.5 \times 10^{-4} v_i \text{ V}$.

$v_2 = 10 v_i - 7.5 \times 10^{-4} v_i \approx 10 v_i \text{ V}$. This is the same as the output predicted by the IOA model.

Let us repeat the calculations by assuming $A = 1000$, $R_1 = 200 \text{ k}\Omega$ and $R_2 = 1 \text{ k}\Omega$. Now, the node voltage $v_1 = 0.9901 v_i$, the differential input voltage $v_d = 0.0099 v_i$ and $v_2 = 9.86 v_i$. Thus, the gain will deviate by 1.4% away from its expected value of 10.

In general, the results predicted by the IOA model will be sufficiently accurate if the gain realised in the circuit is below 1% of the Opamp gain and the resistors used in the circuit are much higher than the Opamp output resistance and much lower than the Opamp input resistance.

A thumb rule for choosing the resistor values in a circuit containing Opamps and resistors may be arrived at as a result of these calculations on commonly used Opamp circuits.

The design rule for choosing the values for resistors in an Opamp circuit is that all resistors must be chosen to lie between $R_i/25$ and $25R_o$, where R_i and R_o are the input and output resistance of Opamps used in the circuit.

Voltage saturation at the output of an Opamp and the consequent clipping of output waveform are easy to understand. However, clipping at a level lower than the voltage saturation limit may take place under current-limited operation of Opamps. The next example illustrates this issue.

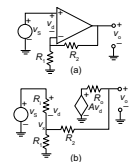


Fig. 6.8-1 (a) Non-Inverting Amplifier (b) Equivalent Circuit of Non-Inverting Amplifier

EXAMPLE: 6.8-2

The Opamp used in an inverting amplifier (Fig. 6.8-2) employs $\pm 12 \text{ V}$ supply. The output saturation limit of the Opamp at this power supply level is $\pm 10 \text{ V}$. The output current of Opamp is limited to $\pm 20 \text{ mA}$ with a supply voltage of $\pm 12 \text{ V}$. The feedback resistance draws negligible current from the output and the gain of the amplifier is -10 . Obtain the output of the amplifier if the input is a sine wave of 1 V amplitude and 10 Hz frequency and the load connected at the output is (i) $10 \text{ k}\Omega$ and (ii) 250Ω .

SOLUTION

(i) The gain of the amplifier is -10 . The input is $v_i(t) = 1 \sin 20\pi t \text{ V}$. Therefore, the output will be $v_o(t) = -10 \sin 20\pi t \text{ V}$ if the Opamp does not enter the non-linear range of operation at any instant. The peak voltage of the expected output is 10 V and this is just about equal to the voltage saturation limits. Therefore, clipping will not take place on this count. The maximum current that will be drawn by the $10 \text{ k}\Omega$ load will be $10 \text{ V}/10 \text{ k}\Omega = 1 \text{ mA}$ and that is well below the output current limit of Opamp. Therefore, the output in this case will be a pure sine wave given by $v_o(t) = -10 \sin 20\pi t \text{ V}$.

(ii) Clipping cannot take place in this case too due to the output voltage trying to exceed the saturation limits. However, if the output is really $-10 \sin 20\pi t \text{ V}$, then the load resistor will draw a current of $10/250 = 40 \text{ mA}$ at the peak of sine wave, but the Opamp output current is limited at $\pm 20 \text{ mA}$. The load resistor of 250Ω will draw 20 mA when the voltage across it is 5 V . This will happen at the 30° position on the sine wave. Thus, the output voltage will follow a sinusoid of 10 V amplitude until the 30° position, then, remain clipped at 5 V for the entire 30° to 150° range and again follow a sinusoidal variation for 150° to 180° in a half-cycle. Thus, output shows a clipping level of $\pm 5 \text{ V}$ for two-thirds of cycle period.

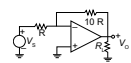


Fig. 6.8-2 The Inverting Amplifier in Example 6.8-2

Thus, n ideal independent voltage sources of voltage values $V_1, V_2, \dots, V_n$ each in series with a resistance, delivering power to a common load in parallel, can be replaced by a single ideal independent voltage source in series with a resistance. The value of voltage source is given by:

$$V_{eq} = \frac{\sum_{i=1}^n G_i V_i}{\sum_{i=1}^n G_i}; R_{eq} = \frac{1}{\sum_{i=1}^n G_i},$$

where $G_i = \frac{1}{R_i}$ for $i = 1$ to n .

This is known as Millman's Theorem. Millman's theorem is only a restatement of Source Transformation Theorem that is valid under a special context.

5.10 SUMMARY

- This chapter dealt with some circuit theorems that form an indispensable tool set in circuit analysis. Many of them were stated for linear time-invariant memoryless circuits. However, they are of wider applicability and will be extended to circuits containing inductors, capacitors and mutually coupled inductors in later chapters.
- Superposition theorem** is applicable only to linear circuits. It states that 'the response of any circuit variable in a multi-source linear memoryless circuit containing n independent sources can be obtained by adding the responses of the same circuit variable in n single-source circuits with i^{th} single-source circuit formed by keeping only i^{th} independent source active and all the remaining independent sources deactivated'.
- A more general form of Superposition Theorem states that 'the response of any circuit variable in a multi-source linear memoryless circuit containing n independent sources can be obtained by adding responses of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between such active source-subsets among them'.
- Substitution theorem** is applicable to any circuit satisfying certain stated constraints. Let a circuit with unique solution be represented as interconnection of the two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let $v(t)$ be the voltage that appears at the terminals between N_1 and N_2 and let $i(t)$ be the current flowing into N_1 from N_2 . Then, the network N_1 may be replaced by an independent current source of value $i_{sc}(t)$ in parallel with a resistance R_0 without affecting any voltage or current variable within N_1 , provided the resulting network has unique solution. $i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_0 is the equivalent resistance of the deactivated circuit ('dead circuit') seen from the terminals. This equivalent circuit for N_1 is called its *Norton's equivalent*.
- Reciprocity theorem** is applicable to linear time-invariant circuits with no dependent sources.
- Compensation theorem** is applicable to linear circuits and states that 'in a linear memoryless circuit, the change in circuit variables due to change in one resistor value from R to $R + \Delta R$ in the circuit can be obtained by solving a single-source circuit analysis problem with an independent voltage source of value ΔR in series with $R + \Delta R$, where i is the current flowing through the resistor before its value changed'.
- Thevenin's and Norton's Theorems are applicable to linear circuits. Let a network with unique solution be represented as interconnection of the two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent voltage source of value $v_{oc}(t)$ in series with a resistance R_0 without affecting any voltage or current variable within N_1 , provided the resulting network has unique solution. $v_{oc}(t)$ is the voltage that will appear across the terminals when they are kept open and R_0 is the equivalent resistance of the deactivated circuit ('dead circuit') seen from the terminals. This equivalent circuit for N_1 is called its *Thevenin's equivalent*.

Bulleted Summary: Bulleted Summary gives the essence of each chapter and enables quick recapitulation.

3.9 QUESTIONS

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- A large capacitor can absorb alternating currents in a circuit without contributing significant amount of alternating voltages in the circuit.
- The total energy delivered to a capacitor carrying a voltage V across it is $(\frac{1}{2})CV^2$ J and this energy is stored in its electric field. Stored energy in a capacitor is also given by $(\frac{1}{2})CQ^2$ J and $QV/2$ J. The capacitor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.
- A single capacitor C_{eq} can replace a set of n capacitors connected in series as far as changes in charge, changes in voltage and changes in total stored energy are concerned.
- $\frac{1}{C_{eq}} = \left[\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n} \right]$
- A single capacitor $C_{eq} = C_1 + C_2 + \dots + C_n$ can replace a set of n capacitors connected in parallel. The total charge, total current and total stored energy are shared by the various capacitors in direct proportion to capacitance value in a parallel connection of capacitors.

Review Questions: Review exercise questions help the reader to check the understanding of the topic.

3.9 QUESTIONS

[Passive sign convention is assumed throughout]

- What is meant by *linearity* of an electrical element? Show that a resistor satisfying Ohm's law is a linear element.
- What are series equivalent and parallel equivalent of n equal resistors?
- Show that a resistor in parallel with a short-circuit is a short-circuit.
- Show that a resistor in series with an open-circuit is an open-circuit.
- Show that the parallel equivalent of a set of resistors will be less than the resistor with the least value among them.
- How many different values of resistance can be obtained by using five resistors of equal value in series-parallel combinations? Enumerate them.
- Explain why an inductor needs an initial condition specification whereas a resistor does not.
- The voltage across a 0.1 H inductor is seen to be 7.5 V at $t = 7$ ms. What is the current in the inductor at that instant?
- The voltage across a 0.1 H inductor is seen to be constant at 10 V between 10 ms and 15 ms. The current through the inductor was 0.3 A at 12 ms. What is the current at 13.5 ms?
- The area under voltage waveform applied to a 10 mH inductor is 5 mV-s between 7 ms and 9 ms. If the current at 7 ms was 1 A how much is it at 9 ms?
- An inductor of 0.2 H has current of 2 A at $t = 0$ in it. The voltage applied across it is $38(t - 2)$. Find the current in it (a) at 1 s (b) at 3 s.
- An inductor of 2 H undergoes a flux linkage change of 7 Wb-T between 15 s and 17 s. What is the average voltage applied to the inductor during that interval?
- Two identical inductors L_1 and L_2 undergo a flux linkage change by 10 Wb-T. L_1 takes 2 s for this change and L_2 takes 20 s. What is the ratio of average voltage applied to the inductors during the relevant intervals?
- A 10 H has an initial energy equivalent to the energy consumed by a 40 W lamp in 1 h. Find the initial current in the inductor.
- A DC voltage source of 24 V is switched on to an initially relaxed inductor of 4 H through a 48 A fuse. Assume that the fuse acts instantaneously when current through it touches 48 A. How much time do we have to open the switch before the fuse blows?
- A DC source of 12 V is switched on to an inductor of 0.5 H at $t = 0$. The current in it is found to be 0 A at 5 s. Was there any initial stored energy in the inductor? If yes, how much?
- A symmetric triangular voltage waveform with a peak-to-peak value of 20 V and frequency 1 kHz is applied to an inductor from 0 s onwards. The inductor was carrying an initial current of 10 A. The inductor current is found to vary within $\pm 3\%$ of its initial current subsequently. What is the value of inductance?
- Two inductors of 1 H and 1.8 H with initial currents of 5 A and 2 A, respectively are connected in parallel. How much energy can be taken out from this parallel combination?
- Three inductors are connected in series and the current in the circuit is found to vary at the rate of 7 A/s at an instant when the applied voltage was at 14 V. The value of voltage measured across the third inductor at the same instant was 4 V. What is the value of the third inductor?
- Two inductors with zero initial energy were paralleled at $t = 0$ and a voltage source was applied across them. The rate of change of source current at 2 s is 5 A/s and the source voltage at that time was 2.5 V. It was also found that the first inductor had a stored energy that is twice that of the second inductor. Find the inductance values.
- How much time is required to charge a 10 mF capacitor with an initial voltage of -100 V to +100 V using a DC current source of value 10 mA?
- The voltage rating of a 10 μ F capacitor is 100 V. It is being charged by a 100 μ A pulse current source. Its initial voltage was -75 V. What is the maximum pulse width that the current source can have if we do not want to end up with a blown capacitor?
- The DC power supply in a PC uses 470 μ F capacitor across its DC output. The DC output value is normally 320 V. The PC can function without rebooting till the DC voltage across falls

Answers to Selected Problems

Chapter 1

1. (a) 432,000 Coulombs (b) 11.66 V (c) 88.18 AH (d) 3.76×10^3 J, 1.045 kWh
2. (a) 28 AH (b) 9 A (c) 2.94×10^3 J, 0.817 kWh 8.9 A, 71.2 AH
3. (a) 10 AH (b) 60 AH (c) 10 A (d) 2.133×10^3 J, 0.593 kWh
4. (a) a 100 Ω resistor (b) 0.36 mC (c) 1 mC (d) 5 mJ (e) 23 ms
5. (a) a 0.1 μ F capacitor (b) 1 μ C (c) 5 μ J (d) 3.75 μ J
6. (a) a 0.6 H inductor
- (b) $q(t) = \begin{cases} 0.1(1 - \cos 100t) \text{ coul} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$
- (c) $p(t) = \begin{cases} 3000 \sin 200t \text{ W} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$
- (d) $E(t) = \begin{cases} 30 \sin^2 100t \text{ J} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$
7. (a) -240 W (b) 120 J, increases
8. It is a resistor of 5 Ω and the current through it is $(2 - e^{-100t})$ A for $t \geq 0$.
9. (d) 0.85 A
10. (a) $v(t) = \begin{cases} 0 \text{ V} & \text{for } t \leq 0 \\ 20t \text{ V} & \text{for } 0 < t \leq 4 \text{ s} \\ (160 - 20t) \text{ V} & \text{for } 4 \text{ s} < t \leq 8 \text{ s} \\ 0 \text{ V} & \text{for } t > 8 \text{ s} \end{cases}$
- (b) $v(t) = \begin{cases} 0 \text{ V} & \text{for } t \leq 0 \\ 1 \text{ V} & \text{for } 0 < t \leq 4 \text{ s} \\ -1 \text{ V} & \text{for } 4 \text{ s} < t \leq 8 \text{ s} \\ 0 \text{ V} & \text{for } t > 8 \text{ s} \end{cases}$
- (c) $v(t) = \begin{cases} 0 \text{ V} & \text{for } t \leq 0 \\ 100t^2 \text{ V} & \text{for } 0 < t \leq 4 \text{ s} \\ (1600t - 100t^2 - 3200) \text{ V} & \text{for } 4 \text{ s} < t \leq 8 \text{ s} \\ 3200 \text{ V} & \text{for } t > 8 \text{ s} \end{cases}$
11. $v(t) = \begin{cases} 10 \text{ V} & \text{for } t < 0 \\ 10 - \frac{2}{3} \sin 1000\pi t \text{ V} & \text{for } t \geq 0 \end{cases}$
12. 9.97 A, 1.994 Wb-T, 9.94 J
- (b) $v_s = -10$ V (c) Yes, it is a DC source.
13. (a) (15 V, -3 A) and (-15 V, 3 A) (b) It is an active element and is a DC source. (c) No
14. 4 H, 1.98 H
15. 1 H, 0.7 H
17. 1 H, 0.5

18. (a) $\psi_1 = 0.139 \sin(100\pi t + 30^\circ)$ Wb-T
 $\psi_2 = 0.224 \sin(100\pi t + 42^\circ)$ Wb-T
- (b) $v_1 = 43.53 \cos(100\pi t + 30^\circ)$ Wb-T
 $v_2 = 70.35 \cos(100\pi t + 42^\circ)$ Wb-T
- (c) $\psi_1 = 0.139 \sin(100\pi t - 30^\circ)$ Wb-T
 $\psi_2 = 0.224 \sin(100\pi t + 138^\circ)$ Wb-T
 $v_1 = 43.53 \cos(100\pi t - 30^\circ)$ Wb-T
 $v_2 = 70.35 \cos(100\pi t + 138^\circ)$ Wb-T

Chapter 2

1. $v_s = -10$ V, $v_1 = 15$ V, $v_2 = -15$ V, $i_1 = -3$ A, $i_2 = 2$ A
2. (i) $v_1 = -15$ V, $v_2 = 15$ V, $v_3 = 10$ V, $i_1 = 3$ A, $i_2 = -5$ A, $i_3 = -8$ A, $i_4 = -5$ A
- (ii) Elements a , e and g (iii) Elements b , c , d and f .
- (iv) Elements b , c , d and f . Total power = 140 W (v) Elements a , e and g . Power absorbed = -140 W
3. (i) $i_1 = -i_2 - i_3 + i_4$, $i_1 = i_2 + i_3 - i_4$, $i_2 = -i_1 - i_3$, $i_3 = -i_1 - i_2$
- (ii) $v_1 = v_2 - v_3 - v_4$, $v_2 = -v_1 + v_3$, $v_3 = v_1 + v_2 - v_4$, $v_4 = v_1 - v_2$
4. (i) The elements are designated as $[a // (b+c)] + d + [(e+f)/g]$ where $//$ is parallel connection and $+$ is series connection. Then, $i_1 = -2$ A, $i_2 = 1$ A, $i_3 = -2$ A, $i_4 = 3$ A, $v_1 = 5$ V, $v_2 = -5$ V, $v_3 = 5$ V.
- (ii) 3
- (iii) $[5 \Omega // (2.5 \Omega + 10 \text{ V})] + [-1.5 \text{ V}] + [3.3333 \Omega // (2.5 \Omega + 15 \text{ V})]$ where $//$ stands for parallel connection and $+$ stands for series connection.
- (iv) $[5 \Omega // (2.5 \Omega + 2.5 \text{ A})] + [1 \text{ A}] + [3.3333 \Omega // (2.5 \Omega + 2 \text{ A})]$ where $//$ stands for parallel connection and $+$ stands for series connection.
5. (i) The elements are designated as $[a // (b+c)] + d + [(e+f)/g]$ where $//$ is parallel connection and $+$ is series connection. Then, $i_1 = -2$ A, $i_2 = 1$ A, $i_3 = -2$ A, $i_4 = 3$ A, $v_1 = 5$ V, $v_2 = -5$ V, $v_3 = 5$ V.
- (ii) Power delivered by a (1 A CS) = -5 W, Power delivered by d (1 A CS) = 5 W, Power delivered by g (3 A CS) = -30 W, Power delivered by b (5 V VS) = -10 W, Power delivered by c (10 V VS) = 20 W, Power delivered by e (5 V VS) = -10 W, Power delivered by f (15 V VS) = 30 W.
6. (i) The elements are designated as $[a // b] + c + d + [e // f]$ where $//$ is parallel connection and $+$ is series connection. Then, $i_1 = 1$ A, $i_2 = 1$ A, $i_3 = -3$ A, $v_1 = -10$ V, $v_2 = 5$ V, $v_3 = 10$ V.

Answers to Selected Problems:

Answers to Selected Problems given at the end of the book facilitate effortless verification of the solutions to chapter-end exercises.

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Preface

The field of electrical and electronic engineering is vast and diverse. However, two topics hold the key to the entire field. They are 'Circuit Theory' and 'Signals and Systems'. Both these topics provide a solid foundation for later learning, as well as for future professional activities.

This undergraduate textbook deals with one of these two pivotal subjects in detail. In addition, it connects 'Circuit Theory' and 'Signals and Systems', thereby preparing the student-reader for a more detailed study of this important subject either concurrently or subsequently.

The theory of electric circuits and networks, a subject derived from a more basic subject of electromagnetic fields, is the cornerstone of electrical and electronics engineering. Students need to master this subject well, and assimilate its basic concepts in order to become competent engineers.

Objectives

Primary Objective:- To serve as a textbook that will meet students' and instructors' need for a two- or three-semester course on electrical circuits and networks for undergraduate students of electrical and electronics engineering (EE), electronics and communications engineering (EC), and allied streams. This textbook introduces, explains and reinforces all the basic concepts of analysis of dynamic circuits in time-domain and frequency-domain.

Secondary Objective:- To use circuit theory as a carrier of the fundamentals of linear system and continuous signal analysis so that the students of EE and EC streams are well-prepared to take up a detailed study of higher level subjects like analog and digital electronics, pulse electronics, analog and digital communication systems, digital signal processing, control systems, and power electronics at a later stage.

Electric Circuits in EE and EC Curricula

The subject of electric circuits and networks is currently covered in two courses in Indian technical universities. The introductory portion is covered as a part of a course offered in the first year of undergraduate program. It is usually called basic electrical engineering. About half of the course time is devoted to Introductory Circuit Theory covering the basic principles, DC circuit analysis, circuit theorems and single frequency sinusoidal steady-state analysis using phasor theory. This course is usually a core course for *all disciplines*. Therefore, it is limited very much in its content and depth as far as topics in circuit theory are concerned. The course is aimed at giving an overview of electrical engineering to undergraduate students of all engineering disciplines.

Students of disciplines other than EE and EC need to be given a brief exposure to electrical machines, industrial electronics, power systems etc., in the third semester. Many universities include this content in the form of a course called electrical technology in the third semester for students of other engineering disciplines. This approach makes it necessary to teach them AC steady-state analysis of RLC circuits even before they can be told about transient response in such circuits. The EE students, in fact, need AC phasor analysis only from the fourth or fifth semester since they start on electric machines and power systems

only then. But the first year course on basic electrical engineering has to be a common course and hence even EE and EC students learn AC steady-state analysis before transient response.

The second course on circuits is usually taught in the third semester and is termed electric circuit theory for EE students and circuits and networks or network analysis for EC students. Few comments on these different course titles and course content are in order.

Traditionally, undergraduate circuit theory courses for the EE stream slant towards a ‘steady-state’ approach to teaching circuit theory. The syllabi of many universities in India contain extensive coverage on single-phase and three-phase circuits with the transients in RC and RL circuits postponed to the last module in the syllabus. The course instructor usually finds himself with insufficient contact hours towards the end of the semester to do full justice to this topic. The EE stream often orients circuits courses to serve as prerequisites for courses on electrical machines and power systems. This led to the EC stream preparing a different syllabus for their third semester circuit theory course—one that was expected to orient the student towards the dynamic behaviour of circuits in time-domain and analysis of dynamic behaviour in the frequency domain. But, in practice the syllabus for this subject is an attempt to crowd too many topics from network analysis and synthesise into what should have been a basic course on circuits.

Such a difference in orientation between the EE-stream syllabus and EC-stream syllabus for circuit theory is neither needed nor desirable. The demarcation line between EE and EC has blurred considerably over the last few years. In fact, students of both disciplines need good coverage of linear systems analysis or signals and systems in the third or fourth semester. Unfortunately linear systems analysis has gone out of the curriculum even in those universities which had introduced it earlier, and signals and systems has started making its appearance in the EC curriculum in many universities. But the EE stream is yet to lose its penchant for AC steady-state in many Indian technical universities.

The subject of electrical circuit theory is as *electronic* as it is *electric*. Inductors and capacitors do not get scared and behave differently when they see a transistor. Neither do they reach sinusoidal steady-state without going through a transient state just because they happen to be part of a power system or electrical machine.

Against this background, I state the pedagogical viewpoint I have adopted in writing this textbook.

Pedagogical Viewpoint

- With a few minor changes in emphasis here and there, both EE and EC students need the same Circuit Theory course.
- ‘Lumped Linear Electrical Circuits’ is an ideally suited subject to introduce and reinforce ‘Linear System’ concepts and ‘Signals and Systems’ concepts in the EE and EC undergraduate courses. This is especially important in view of shortage of course time which makes it difficult to introduce full-fledged courses in these two subjects. This textbook is organised along the flow of Linear Systems Analysis concepts.
- Circuit Theory is a very important *foundation course* for EE, EC and allied disciplines. The quality of teaching and intellectual capability of students varies widely in different sectors of technical educational institutions in India. Therefore, a good textbook on circuit theory has to be written explaining the basic concepts thoroughly and repeatedly, with the average students in mind—not the brilliant ones who manage to get into ivy-league institutions. Such a textbook will supplement good teaching in the case of students of premier institutions and, more importantly, save the average students from life-long confusion.
- The pages of a textbook on Circuit Theory are precious due to the reasons described above. Therefore, all extraneous matter should be dispensed with. The first in this category is the so-called historical vignettes aimed at motivating the students. I have avoided them and instead, used the valuable space to explain basic concepts from different points of view.

- The pre-engineering school curriculum in India prepares the students well in mathematics and physics. Engineering students have not yet become impatient enough to demand examples of practical applications of each and every basic concept introduced in subjects like Circuit Theory or Newtonian Mechanics. There is no need to keep on motivating the student by citing synthetic-looking examples of complex electrical and electronic systems when one is writing on basic topics in Circuit Theory. The pages can be used for providing more detailed explanation on basic concepts. The first year or second year undergraduate student is far away from a practical engineering application! I believe that a typical engineering student is willing to cover the distance patiently.
 - Circuit Theory is a foundation course. It is difficult to quote a practical application for each and every concept without spending considerable number of pages to describe the application and set the background. And the pedagogical impact of this wasteful exercise is doubtful. However, those applications that are within the general information level of an undergraduate student should be included. Thus, applications that require long explanations to fit them into the context must be avoided in the interest of saving pages for explanations on Circuit Theory concepts.
 - Circuit Theory is a basic subject. Therefore, all other topics that the students are going to learn in future semesters will be anchored on it. Hence, it should be possible to set pointers to applications in higher topics in a textbook on Circuit Theory. Such pointers can come in the form of worked examples or end-of-chapter problems that take up an idealised version of some practical application. An example would be to use an idealised form of fly-back switched mode converter and to show how the essential working of this converter can be understood from the inductance v - i relationship. In fact, all well-known switched mode power converter circuits can be employed in the chapter which deals with the v - i relation of an inductor. Similarly, switched-capacitor circuits can be introduced in the section dealing with the v - i relation of a capacitor.
 - Circuit Theory can be learnt well without simulation software. Circuit simulation packages are only tools. I am of the opinion that using simulation software becomes a source of distraction in a foundation course. A foundation course is aimed at flexing the student's intellect in order to encourage the growth of analytical capability in him/her.
 - An argument usually put forth in support of simulation software as an educational aid is that it helps one to study the response of circuits for various parameter sets and visualise the effect of such variations. That is precisely why I oppose it in a foundation course. Ability to visualise such things using his/her head and his/her ability for mental imagery is very much essential in an engineer. Let the student develop that first. He/she can seek the help of simulation software later when he/she is dealing with a complex circuit that goes beyond the limits of mental imagery.
- After all, we do not include a long chapter on waveform generators and another one on oscilloscopes in every Circuit Theory textbook. In fact, some of the modern-day waveform generators and oscilloscopes have so many features, that a chapter on each of them will not really be out of place. Yet, we do not spend pages of a Circuit Theory textbook for that. The same rule governs simulation software too.

Pedagogy

- Each chapter begins with a list of chapter objectives outlining the relative emphasis of topics covered in that chapter.
- Detailed summary covering all the important points made in the chapter is provided at the end of each chapter.
- Boxed entries and pointer entries located on the wide side margins highlight important concepts and reinforce them. Additional information is provided within these boxed entries wherever relevant.

- Large number of solved examples illustrating the concepts explained in the text is included. Simple formula-substitution kind of examples are avoided. There are about 250 such worked examples in the book.
- Numerous questions designed to provoke analytical thinking and to reinforce major concepts are included at the end of chapters. These questions may be short numerical problems or qualitative ones. There are about 270 such questions in the book.
- Ample number of problems included at the end of every chapter test the student's understanding. Section-wise organisation of these problems is *avoided intentionally*. I expect the student to assimilate the entire chapter and use all the concepts covered in that chapter (and from earlier chapters) to solve a problem if necessary. After all, no one tells him which concepts are relevant in solving a particular problem in the examination hall or in practical engineering. There will be about 450 such problems in the book. Answers to most of the problems are provided at the end of the book. A detailed solution manual is available at www.pearsoned.co.in/kssureshkumar for the course instructors.

Outline and Organisation

The book contains 17 chapters divided into 6 parts. The first three parts are intended to be used for basic electrical engineering course in the first year of undergraduate program. The remaining three parts are to be used for electric circuit theory for EE students and circuits and networks or network analysis for EC and allied disciplines. It may not be possible to cover these three parts entirely in one semester. A selection of sections to suit the course requirements is recommended.

Part I 'Basic Concepts', contains three chapters. The first chapter delves into the physics of two-terminal circuit elements briefly and deals with element relations, circuit variables, and sign convention. It also addresses the concepts of linearity, time-invariance and bilaterality properties of two-terminal elements. This chapter assumes that the reader has been introduced to the basic physics of electromagnetic fields in pre-engineering high-school physics. The chapter attempts to explain the important assumptions underlying circuit theory from the point of view of electromagnetic fields. The treatment is qualitative and not at all intended to be rigorous.

The second chapter covers the two basic laws – Kirchhoff's voltage law and Kirchhoff's current laws – in detail. Emphasis is placed on the applicability of these two laws under various conditions.

The third chapter looks into the v - i relationship of the resistor, the inductor and the capacitor. Series-parallel equivalents are also covered in this chapter. This chapter analyses the v - i relations of inductor and capacitor in great detail. The concept of 'memory' in circuit elements is introduced in this chapter and the electrical circuits are divided into two classes – *memoryless circuits* and *circuits with memory*. Circuits with memory are termed as *Dynamic Circuits* from that point onwards.

Part II 'Analysis of Memoryless Circuits', contains three chapters. Chapter 4 takes up the analysis of memoryless circuits containing independent voltage and current sources, linear resistors and linear memoryless dependent sources using node analysis and mesh analysis methods. An argument based on nodal admittance matrix (or mesh impedance matrix) and its cofactors is used to show that a memoryless circuit comprising memoryless linear two-terminal elements will be a *linear system* and that it will obey superposition principle.

The discussion then moves on to Chapter 5. This chapter systematically develops all important circuit theorems from the properties of a linear system.

The abstraction called a *linear dependent source* is given a concrete shape in Chapter 6 by introducing the Operational Amplifier (Opamp) as a memoryless circuit element. However, the reader will be given an introduction to feedback and stability *i.e.*, dynamics of Opamps at this stage itself. This chapter is an optional chapter in the syllabus for 'Basic Electrical Engineering'. It is a self contained chapter that can be suitably be shifted to some other course in a higher semester.

After the analysis of memoryless circuits, the book moves on to **Part III**, 'Sinusoidal Steady-State in Dynamic Circuits'. This part of the book starts with a detailed look at power and energy in periodic waveforms in Chapter 7. The periodic sinusoid is introduced and the principles governing its amplitude, frequency, and phase are made clear. The concept of cycle-average power in the context of periodic waveforms is covered in detail.

Chapter 8 begins with a qualitative description of the transient response and the forced response taking an RL circuit as an example, and illustrates how the sinusoidal steady-state can be solved by using the complex exponential function. It goes on to expound on phasor theory, transformation of the circuit into phasor domain, solving the circuit in phasor domain, and moving back to time-domain. It also introduces active power, reactive power and power factor and presents the basic ideas of frequency response.

Chapter 9 takes up three-phase balanced and unbalanced circuits and includes symmetrical components as well. Unbalanced three-phase circuits and symmetrical components may be optional in 'Basic Electrical Engineering' course.

Part IV, 'Time-Domain Analysis of Dynamic Circuits' contains three chapters. Chapter 10 is one of the key chapters in the book. It takes up a simple RL circuit and uses it as an example system to develop many important linear systems concepts. The complete response of an RL circuit to various kinds of inputs such as unit impulse, unit step, unit complex exponential, and unit sinusoid is fully delineated from various points of view in this chapter. Further, the need for, and sufficiency of initial current specification is thoroughly dealt with, and the concepts of time constant, rise and fall times, and bandwidth are clearly explained.

The response of a circuit is viewed as the sum of *transient response* and *forced response* on the one hand and as the sum of *zero-input response* and *zero-state response* on the other. The role of various response components is clearly spelt out. The application of superposition principle to zero-state and zero-input components is examined in detail.

Impulse response is shown to be an all-important response of a circuit. The equivalence between impulse excitation and non-zero initial conditions is established in this chapter. The chapter also shows how to derive the zero-state response to other inputs like unit step and unit ramp from impulse response in detail. The tendency of inductance to keep a circuit current smooth is pointed out and illustrated.

The notions of DC steady state, AC steady state and periodic steady state are explained in detail and illustrated through several worked examples. The chapter ends with a general method of solution to single time constant RL circuits in 'transient response + forced response' format as well as in 'zero-input response + zero-state response' format. This chapter places emphasis on impulse response as the key circuit response, keeping in mind the discussion on convolution integral in Chapter 12.

Chapter 11 takes up a similar analysis of RC and RLC circuits. Further, this chapter gradually introduces the concept of sinusoidal steady-state frequency response curves through RC and RLC circuits and sets the background for Fourier series in a later chapter. Specific examples where the excitation is in the form of a sum of harmonically related sinusoids containing three to five terms are used to illustrate the use of frequency response curves and to illustrate linear distortion. The conditions for distortion-free transmission of signals are briefly hinted at in this chapter and taken up for detailed study in Chapter 14.

Inconvenient circuit problems like shorting a charged capacitor, opening a current-carrying inductor, connecting two charged capacitors together, and connecting an uncharged capacitor across a DC supply require the inclusion of parasitic elements for correct explanation. Parasitic elements are emphasised at various places in chapters dealing with time-domain analysis.

Chapter 12 extends the differential equation based time-domain analysis to multi-node and multi-mesh circuits containing dependent sources. The issue of stability is brought out through illustrative examples containing dependent sources. The criterion for stability in linear circuits is hinted at and developed fully in later sections.

This chapter generalizes the time-domain approach and introduces the concept of 'signal space'. Every point in the complex signal space is viewed as a possible transient response term of

some linear circuit in complex exponential format or as a possible excitation function. The idea that a linear circuit can be represented as a set of points in the signal space is introduced to the reader in this manner. This will be a precursor to pole-zero representation in Chapter 15.

Impulse response is generalised for an n^{th} order system and circuit stability criterion is translated into absolute summability of impulse response in this chapter. The reader is reminded of the relation between step and ramp responses to impulse response and is prompted to ask the question: Can the zero-state response to any arbitrary input be determined from impulse response? The question is answered through the development of expansion of any input signal into a sum of delayed and scaled impulse functions, and convolution integral follows.

Two important results that follow from convolution integral are explained in detail. The first one is the relation between area of impulse response and steady-state value of step response. The second is the frequency response function in terms of impulse response. Once the sinusoidal steady-state frequency response is seen to be completely decided by impulse response, the question that arises is: Can the zero-state response to any arbitrary input be found out using frequency response function? The answer to this question defines what is meant by frequency-domain analysis and makes up Part V of the book.

Part V, 'Frequency-Domain Analysis of Dynamic Circuits', starts with Chapter 13 which answers the question posed in the previous paragraph, for a specific class of inputs – periodic inputs. This chapter expands a periodic waveform along the imaginary axis in signal space at discrete points. Fourier series in trigonometric and exponential forms are covered in detail in this chapter.

Chapter 14 extends the expansion of input functions along the imaginary axis in signal space for aperiodic waveforms through Fourier transforms. It also explains clearly how even periodic waveforms can be brought under the Fourier transform theory. The properties of Fourier transforms are explained and illustrated in detail. Significant insight into time-limiting and band-limiting of signals is provided in this chapter. This chapter introduces the notion of the system function and clearly shows that it is the same as the frequency response function. It thus answers the question raised earlier in the affirmative. This chapter also introduces the reader to continuous-time signal analysis.

Chapter 15 expands an arbitrary input signal along a line parallel to the vertical axis in a signal plane, that is, in terms of damped sinusoids of different frequencies rather than in terms of undamped sinusoids of different frequencies. This expansion is illustrated graphically in the case of a simple waveshape to convince the reader that an aperiodic signal can indeed be obtained by a large number of exponentially growing sinusoids and that there is nothing special about the expansion of a waveshape in terms of undamped sinusoids. This expansion of signals leads to the Laplace transform of the signal. Properties of the Laplace transform, use of the Laplace transform in solving differential equations and circuits, transfer functions, impedance functions, poles, and zeros follow. This chapter also includes a graphical interpretation of frequency response function in the s -plane. Stability criterion is re-visited and circuit theorems are generalised. This chapter winds up the section on frequency-domain analysis.

Part VI, 'An Introduction to Network Analysis' comprises two chapters. Chapter 16 deals with two-port networks and develops various two-port parameter sets. It also deals with passive constant- k and m -derived filter sections for four basic filtering functions. A study of active filters cannot be treated as part of circuit theory and is better covered in an analog electronics course. Hence it is not included in the text. However, standard active filter circuits are included in worked examples and problems in earlier chapters dealing with frequency response studies.

Chapter 17 provides an introduction to the study of topological properties of electrical networks. The reader is taken through an introduction to linear graphs, incidence matrix, circuit matrix and cut-set matrix and KCL/KVL equations in terms of topological matrices followed by nodal analysis, loop analysis and node-pair analysis of networks. This chapter and the book end with a brief exposure to Tellegen's theorem.

Prerequisites for Students

The student-reader is expected to have gone through basic level courses in electromagnetism, complex algebra, differential calculus and integral calculus. These are covered in the pre-engineering school curricula of all boards of senior/higher secondary school education in India.

Material for Further Study

The following books may be used as reference material for gaining further insight into the subject:

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To the Engineering Teacher

This is my first book, and I have tried to minimise errors as far as possible. However, there may be a few that escaped my attention. I request you to point out them to me so that I can incorporate suitable corrections in the future impressions of this book.

I would be grateful to you for any suggestion to improve the content or presentation of this book. Please send your suggestions directly to me at sureshks@nitc.ac.in or to the publisher.

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I learnt electric circuits and networks as an undergraduate at Indian Institute of Technology Madras, Chennai during 1976 -'83. The credit for the good things the reader finds in this book goes to my esteemed professors – Dr. Venkataseshiaiah, Dr. V. Bappeswara Rao, Dr. P. Sankaran,

Dr. M. Anthony Reddy, Dr. S. S. Yegnanarayanan, Dr. K. Ramar, Dr. G. Sreedhara Rao, Dr. B. Venugopal and Dr. R. Parthasarathy who taught me well. The faults, if any, in this book are mine.

I am indeed fortunate that my wife, Asha D, and my three children – Gayathri S, Gautham Suresh and Archana Suresh allow me considerable personal space that is very much essential for a venture like writing a textbook. I couldn't have written this book if they had not allowed me to be myself (with all my imperfections) over the past years.

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Part One

Basic Concepts

Circuit Variables and Circuit Elements

CHAPTER OBJECTIVES

- To introduce the concepts of electrostatic potential difference, voltage and electromotive force.
- To introduce the concepts of 'current density' and 'current intensity', and to explain the conduction process.
- To explain the 'quasi-static' approximation involved in Circuit Theory.
- To explain the need for and the role of quasi-static charge distributions over electrical devices.
- To define and explain various idealized two-terminal element models in use in Circuit Theory.
- To explain passive sign convention for two-terminal elements.
- To relate power and energy in a two-terminal element to its terminal voltage and current variables.
- To explain what is meant by 'lumped, linear, bilateral, passive and time-invariant circuit element'.
- To define and explain the mutual inductance element.
- To define and explain four types of linear dependent sources.

This chapter assumes that the reader has been introduced to the basic physics of electromagnetic fields in pre-engineering school physics. An attempt to explain the important assumptions underlying Circuit Theory from the point of view of electromagnetic fields has been made in this chapter. The treatment is qualitative and not at all intended to be rigorous.

INTRODUCTION

The profession of Electrical and Electronics Engineering deals with the generation, transmission and measurement of electric signals with signal power level varying from few nanowatts (10^{-9} W) to hundreds of megawatt (10^6 W) in various applications.

This textbook starts with a quite justifiable assumption that the reader is either a student of electrical engineering or of an allied engineering discipline or is an individual interested in the subject of Electric Circuits for any reason (possibly a teacher or a practicing engineer). Such a person is only too aware of the fact that modern life is inalienably dependent on electrical/electronic devices and systems.

This chapter introduces the language of circuit theory to the reader.

This chapter begins with a qualitative

continued

discussion on the assumptions inherent in modelling the electromagnetic phenomenon in an electrical system by a circuit model.

Physics of Electromagnetic Fields is taught presently at school level in many countries. Differential Calculus, Integral Calculus and Complex Algebra are covered in mathematics courses at school level in most of the countries in the world.

This textbook makes free use of Differential Calculus, Integral Calculus and Complex Algebra.

This chapter assumes that the reader possesses at least a cursory familiarity with the physics of electromagnetic fields.

Electric circuit is a mathematical model of an actual physical electrical system.

Circuit theory is an approximation of electromagnetic field theory.

The following questions arise naturally in the context of professional education in the field of electrical and electronics engineering:

- The field of electrical and electronics engineering is vast and diverse. Is there anything common between its various branches?
- Which topics hold the key to the entire field of electrical and electronics engineering?
- If a course-package containing minimum number of subjects is to be designed to impart professional undergraduate education in this field, then, which is/are the subject/s that has to be inevitably and prominently included in the package?
- Which subjects in the field of electrical and electronics engineering, if learnt thoroughly, will enable the student to learn all other sub-disciplines with relative ease and almost with no instruction at all?

The answers to the first three questions will be the same – ‘Circuit Theory’ and ‘Signals and Systems’. And these two subjects will appear prominently in the answer to the fourth question. Both these subjects provide a solid foundation for learning – both in the later courses as well as in the professional activities as a practicing engineer.

This textbook deals with one of these two ‘kingpin’ subjects in the entire field of electrical and electronics engineering in detail. In addition, it links ‘Circuit Theory’ to ‘Signals and Systems’ and thereby prepares the student-reader for a more detailed study of that important subject either concurrently or subsequently.

An *Electric Circuit* is a mathematical model of a real physical electrical system. Physical electrical systems consist of physical electrical devices connected together. *Electric Circuit* idealises the physical devices and converts the real physical system into a mathematical model. The mathematical model of a real physical electrical system is governed by a set of physical laws. Applying those physical laws on the mathematical circuit model and employing suitable mathematical technique result in the *circuit solution* that approximates the actual behaviour of the physical system to a remarkable degree of accuracy in practice.

Though the term *electric circuit* refers to the idealized mathematical model of a physical electrical system, it is also used to refer to the actual electrical system in common practice. However, when we refer to *electric circuit* in this text, we always mean a mathematical model.

Laws of electromagnetic fields govern the electrical behaviour of an actual electrical system. These laws, encoded in the form of four Maxwell’s equations, along with the constituent relations of electrical materials and boundary constraints, contain all the information concerning the electrical behaviour of a system under a set of specified conditions. However, extraction of the required information from these governing equations will turn out to be a formidable mathematical task even for simple electrical systems. The task will involve solution of partial differential equations involving functions of time and space variables in three dimensions subject to certain boundary conditions.

Circuit theory is a special kind of approximation of electromagnetic field theory. ‘Lumped parameter circuit theory’ converts the partial differential equations involving time and three space variables arising out of application of laws of electromagnetic fields into ordinary differential equations involving time alone. Circuit theory approximates electromagnetic field theory satisfactorily only if the physical electrical system satisfies certain assumptions. In this chapter, these assumptions are discussed first.

1.1 ELECTROMOTIVE FORCE, POTENTIAL AND VOLTAGE

Charge is the attribute of matter that is responsible for a force of interaction between two pieces of matter under certain conditions. Such an attribute was seen to be necessary as a result of experiments in the past which revealed the existence of a certain kind of interaction force between particles that could not be explained by other known sources of interaction forces.

Charge is bipolar. Two positively charged particles or two negatively charged particles repel each other. Two particles with charges of opposite polarity attract each other. Further, charge comes in integral multiples of a basic unit – the basic unit of charge is the charge of an electron. The value of electronic charge is -1.602×10^{-19} Coulombs. The SI unit of charge – i.e., Coulomb – represents the magnitude of charge possessed by 6.242×10^{18} electrons.

1.1.1 Force Between Two Moving Point Charges and Retardation Effect

The force experienced by a point charge of value q_2 moving with a velocity of $\vec{v}_2$ due to another point charge of value q_1 moving with a velocity of $\vec{v}_1$ at a distance r from it contains three components in general (Fig. 1.1-1). If the charges are moving slowly, the force components are given by approximate expressions as below.

The first component is directly proportional to the product $q_1 q_2$ and is inversely proportional to the square of distance between them. This component of force is directed along the line connecting them and is oriented away from q_1 . This component is governed by Coulomb's law; is termed '*the electrostatic force*' and is given by

$$\vec{F}_{12es} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \vec{u}_{12} \text{ N}$$

where, ϵ_0 is the dielectric permittivity of free space ($= 8.854 \times 10^{-12}$ F/m) and $\vec{u}_{12}$ is the unit vector directed from q_1 to q_2 .

The second component of force is the *magnetic force* and arises out of motion of charges. This component is given by

$$\vec{F}_{12m} = \frac{\mu_0 q_1 q_2}{4\pi r^2} \vec{v}_2 \times (\vec{v}_1 \times \vec{u}_{12}) \text{ N}$$

where, μ_0 is the magnetic permeability of free space ($= 4\pi \times 10^{-7}$ H/m) and $\vec{v}_1$ and $\vec{v}_2$ are the velocities of q_1 and q_2 , respectively.

The third component of force is the *induced electric force* and depends on relative acceleration of q_1 with respect to q_2 . It is given by

$$\vec{F}_{12ei} = -\frac{\mu_0 q_1 q_2}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}_1}{r} \right) \bigg|_{\vec{v}_2=0} \text{ N.}$$

Electromagnetic disturbances travel with a finite velocity – the velocity of light in the corresponding medium. Therefore, in general, the force experienced by q_2 at a time instant t depends on the position and velocity of q_1 at an earlier instant. Or, equivalently, the force that will be experienced by q_2 at $t + r/c$ depends on r and $\vec{v}_1$ at t , where c is the velocity of light. This effect is called the *retardation effect*. *The expressions described above ignored this retardation effect and assumed that the changes in relative position and velocity of q_1 are felt instantaneously at q_2 .*

Retardation effect can be ignored if (i) the speed of charges is such that no significant change can take place in the distance between charges during the time interval needed for electromagnetic disturbance to cover the distance and (ii) the acceleration of charges is such that no significant changes in the velocity of charges take place during the time interval needed for electromagnetic disturbance to cover the distance between charges. The first condition implies that the speed of charges must be small compared to that of velocity of light. This condition is met by almost any circuit since the drift speed associated with current flow in circuits is usually very small compared to that of velocity of light. However, though the speed of charge motion is small, it is quite possible that the charges accelerate and decelerate rapidly in circuits such that the second condition is not met.

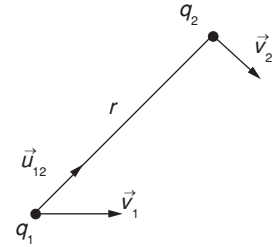


Fig. 1.1-1 Force Between Two Point Charges in Motion

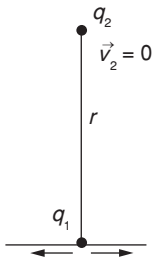


Fig. 1.1-2 Pertaining to Retardation Effect in a Two-charge System

Condition for ignoring retardation effect of electromagnetic phenomenon in electrical systems.

Consider the two point charges in Fig. 1.1-2. q_1 is oscillating with amplitude d and angular frequency ω rad/s. However, let us assume that $d \ll r$. Then, neither the distance between the charges nor the unit vector between them change much with time. Therefore, the electrostatic force experienced by q_2 is more or less constant in time. q_2 is at rest and hence it does not experience any magnetic force. However, it will experience a time-varying induced electric force. Let the horizontal position of q_1 be given by $x = d \sin(\omega t)$ m. Then, velocity of q_1 is $\vec{v}_1 = \omega d \cos \omega t$ m/s. The time taken by any electromagnetic disturbance to travel r meters is r/c , where c is the velocity of light in m/s. The quantity $\frac{\partial}{\partial t} \left(\frac{\vec{v}_1}{r} \right)$ that decides the induced electric force experienced by q_2 will change by $\frac{\omega^2 d}{r} [\sin \omega(t + r/c) - \sin \omega t]$ in that time interval. Retardation effect can be ignored only if this change is negligible – i.e., only if $\frac{\omega r}{c} \ll 1$.

In the context of circuit theory, ω is the highest angular frequency of time-varying signals present in the circuit and r is the largest physical dimension measured in the physical circuit. Then, retardation effect due to finite propagation speed of electromagnetic waves can be ignored in the analysis of an electrical system if the highest frequency present in the circuit and largest dimension of the system satisfy the inequality $\frac{\omega r}{c} \ll 1$.

An electrical system in which the retardation effect can be ignored is called a *quasi-static electrical system*. Electric circuit theory assumes that the electrical system that is modelled by an electrical circuit is a quasi-static system.

1.1.2 Electric Potential and Voltage

The force experienced by a charge q in arbitrary motion under quasi-static conditions due to another moving charge has three components as explained in Sect. 1.1.1. We club the second component, i.e., velocity dependent and the third component, i.e., acceleration dependent together into a composite component and term it as *the non-electrostatic component*. Thus, the force experienced by q due to another charge has an *electrostatic component* and a *non-electrostatic component*.

The force experienced by the same charge q in the presence of many charges can be obtained by adding the individual forces by vector addition. Thus, the electrostatic force experienced by the charge q due to a system of charges is a superposition of electrostatic force components from the individual charges. *Electrostatic field intensity vector*, $\vec{E}_s$, at a point in space is defined as the total electrostatic force vector acting on a unit test charge (i.e., q is taken as 1 C) located at that point. Then, the electrostatic force experienced by a charge q located at a point $P(x, y, z)$ is given in magnitude and direction by $q\vec{E}_s$ N. The SI unit of *electrostatic field intensity* is Newton per Coulomb.

Electrostatic field intensity due to a point charge falls in proportion to square of distance between location of charge and location at which the field is measured. Hence, the field intensity at infinitely large distances from a system of *finite amount* of charge tends to reach zero level. Therefore, a test charge of 1 C located at infinite distance away from the charge collection responsible for electrostatic field will experience zero electrostatic force.

Now assume that the test charge of 1 C that was at infinity initially is brought to point $P(x, y, z)$ by moving it quasi-statically through the electrostatic field. The agent who moves the charge has to apply a force that is numerically equal to $|\vec{E}_s|$ and opposite in direction. The total work *to be done* in moving the unit test charge from infinity to $P(x, y, z)$ is obtained by integrating the quantity $-\vec{E}_s(x, y, z) \cdot d\vec{l}$ over the path of travel, where $d\vec{l}$ is a small length element in the path of travel. This work is, by definition, the electric potential (*electrostatic potential to be precise*) at the point $P(x, y, z)$. It is usually designated by $V(x, y, z)$. Then,

$$V(x, y, z) = - \int_{\infty}^P \vec{E}_s(x, y, z) \cdot d\vec{l} \quad (1.1-1)$$

Definition of Electrostatic Potential at a point in space.

The unit of electric potential is Newton-meter per Coulomb or Joule per Coulomb. This unit is given the name ‘Volt’. This leads to another unit for electrostatic field intensity – namely, Volt/metre (V/m).

Electrostatic force field is a conservative force field. Hence, the value of this work integral will depend only on the end-points and not on the particular path that was traversed. Therefore, this work integral (and hence the potential at point $P(x, y, z)$) has a unique value that depends only on $P(x, y, z)$. Moreover, the conservative nature of electrostatic force field implies that the work done in taking a test charge around a *closed path* in that field is zero.

Equation 1.1-1 defines the potential at a point with respect to a point at infinity. The difference in potential at two points can likewise be interpreted as the work to be done to carry a unit test charge from one point to another. Specifically, let V_1 and V_2 be the electric potentials at point-1 and point-2 in space. Then, the potential of point-1 with respect to the point-2 is $V_1 - V_2$ volts and is equal to the work to be done in carrying a unit positive test charge from point-2 to point-1. This value is designated by V_{12} and is read as ‘potential of 1 with respect to 2’ or as ‘potential difference between 1 and 2’. In Circuit Analysis, this electrostatic potential difference between two points is called the ‘voltage between point-1 and point-2’. The same symbol that was used to designate the potential difference is used to designate ‘voltage’ too. Thus,

$$\begin{aligned} V_{ab} &= \text{‘Voltage between points } a \text{ and } b\text{’} \\ &= \text{‘Electrostatic potential at } a - \text{Electrostatic potential at } b\text{’} \\ &= \text{‘Work to be done in moving } a + 1 \text{ coulomb test charge from } b \text{ to } a\text{’} \\ &= -\int_b^a \vec{E}_s(x, y, z) \cdot d\vec{l} = \int_a^b \vec{E}_s(x, y, z) \cdot d\vec{l} \text{ over any path between } a \text{ and } b. \\ \text{Obviously, } V_{ba} &= -V_{ab} \end{aligned}$$

If V_{ab} is a positive quantity, we state that there is a ‘*potential rise*’ or ‘*voltage rise*’ from b to a . Equivalently, we can state that, there is a ‘*potential drop*’ or ‘*voltage drop*’ from a to b . If V_{ab} is a negative quantity, we state that, there is a ‘*potential rise*’ or ‘*voltage rise*’ from a to b . Equivalently, we can state that, there is a ‘*potential drop*’ or ‘*voltage drop*’ from b to a .

The work to be done in moving a unit positive test charge is positive when the charge is moved in a direction opposite to the direction of electrostatic field. Hence, the direction in which maximum voltage rise takes place and the direction of electrostatic field will be opposite at any point in space.

When a charge q is moved through a *voltage rise*, some non-electrostatic force has to work against the electrostatic force to effect the movement. The non-electrostatic force will do work on the charge in the process of moving it. This work done on the charge gets stored in the charge as increase in its potential energy. Hence, a charge q receives qV_{ab} J of potential energy when it is carried from b to a . If V_{ab} is positive – i.e., if there is a voltage rise from b to a – then, the potential energy level of charge q increases by qV_{ab} J in moving from b to a .

If V_{ab} is negative – i.e., if there is a voltage drop from b to a – then, the potential energy level of charge q decreases by $q|V_{ab}|$ J. The non-electrostatic force that maintains quasi-static condition during the movement of charge from b to a receives this energy.

Sustained and organised movement of charges through voltage rises and voltage drops in an electrostatic potential system is possible only if there are sources of non-electrostatic forces present in the electrical system. These non-electrostatic forces deliver and absorb the required amounts of energy to make movement of charges through voltage rises and voltage drops possible.

Obviously, a non-electrostatic force can fulfil its required role described above only if it has a component along the direction of motion of charge. Thus, a force that is always perpendicular to the velocity of a charge will not do. Such a force cannot deliver energy to the charge or absorb energy from it. One of the three force components that exist between

Definition of
Voltage between
two points a and b .

Positive charges that climb through a voltage rise gain potential energy and positive charges that fall through a voltage drop lose potential energy.

Equivalently, some non-electrostatic force delivers energy in the first case and absorbs energy in the second case.

Thus, some non-electrostatic force has to be present in an electrical system to make charges move continuously through voltage rises and voltage drops in that system.

Role of non-electrostatic forces in an electrical system.

two charges in quasi-static motion has this nature. This is the *magnetic force*. Magnetic force on a charge is always perpendicular to the velocity of the charge. Hence, magnetic force cannot be the non-electrostatic force that we require.

The *induced electric force*, a component of force of interaction between two charges in quasi-static motion, can deliver energy to moving charges and absorb energy from them. Thus, the induced electric force can be a source of the required kind of non-electrostatic force in an electrical system.

Electrical sources are the sources of non-electrostatic forces in an electrical system. Some of the sources make use of the *induced electric force* to generate the non-electrostatic force, e.g., DC generator, AC generator, etc. However, there are other sources of non-electrostatic forces. A dry cell, for instance, converts the internal chemical potential energy into a non-electrostatic force that acts on any charge carrier that transits through the conducting material inside the dry cell. We do not concern ourselves anymore with the exact nature of the non-electrostatic force available within an electrical source. It suffices for our purpose to understand that some non-electrostatic force is available within the electrical source.

1.1.3 Electromotive Force and Terminal Voltage of a Steady Source

Consider an electrical source on open circuit as in Fig. 1.1-3. A free charge located at a point inside the source will experience a non-electrostatic force as shown in the figure. This non-electrostatic force is expressed as a force field and the quantity $\vec{E}_e$ represents this force field. Thus, the non-electrostatic force experienced by a charge q located inside the source will be $q\vec{E}_e$ N, where $\vec{E}_e$ is the non-electrostatic field intensity vector. $\vec{E}_e$ may not be constant in magnitude and direction everywhere inside the source. However, $\vec{E}_e$ will not vary with time in the case of a *steady source*.

The source contains conducting material in which the conductors contain free electrons. The direction of $\vec{E}_e$ is also marked. The free electrons in the conducting substance will tend to move from top to bottom (electrons have negative charge) under the influence of non-electrostatic force. The first few electrons that move reach the bottom electrode (at B) and accumulate at that terminal. Electrons moving to B will cause an equal number of positive charges to appear at the terminal marked A. But then, such a segregation of charges will result in generation of electrostatic field inside (as well as outside) the source. Thus, the remaining free electrons inside the source will experience two forces – a non-electrostatic force that tends to move them towards the lower electrode B and an electrostatic force that tends to move them towards the upper electrode A. The source reaches a steady-state quickly. Under steady-state condition, the magnitude and spatial distribution of charges over the metallic electrodes at A and B are such that all the free electrons that are still within the source will experience zero net force and remain stationary (except for random thermal motion). Thus, the electrostatic field at a point inside will cancel the non-electrostatic field at that point under steady-state. The charge distribution at the terminals will arrange itself suitably such that this cancellation takes place at all points in the active region of source.

However, there is no non-electrostatic field outside the source. Therefore, a test charge kept at a point outside the source will experience an electrostatic force.

With reference to Fig. 1.1-4, let a unit test charge be carried from B to A through the path BOA – i.e., over a path that is outside the source. Some work has to be done for this. The required work will be positive since we are carrying a positive test charge from a negatively charged terminal to a positively charged terminal. The work that is required to be done is the voltage of A with respect to B – i.e., V_{AB} .

However, the work required to carry a unit test charge in a closed path in electrostatic field is zero due to conservative nature of electrostatic forces. Therefore, the work to be done *against the electrostatic force field* to carry the test charge in the path B–O–A–I–B

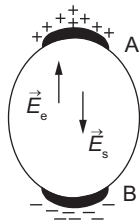


Fig. 1.1-3 A Steady Electrical Source Under Open-circuit Condition

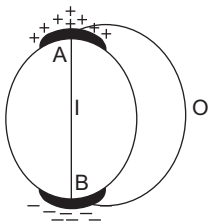


Fig. 1.1-4 Pertaining to Definition of e.m.f. and Open Circuit Voltage of a Source

is zero. That is, work to be done against electrostatic force to move a unit test charge from B to A over a path outside the source + work to be done against electrostatic force to move a unit test charge from A to B over a path inside the source is zero. Therefore, work to be done in path B–O–A = work to be done in path B–I–A. But the electrostatic field vector inside the source is equal to $-\vec{E}_e$. Therefore,

$$V_{AB} = - \int_{B-O-A} \vec{E}_s \cdot d\vec{l} = - \int_{B-I-A} \vec{E}_s \cdot d\vec{l} = \int_{B-I-A} \vec{E}_e \cdot d\vec{l}$$

But $\int_{B-I-A} \vec{E}_e \cdot d\vec{l}$ is the work done by the non-electrostatic force generated by the source on a unit positive charge when it moves through the source from negative terminal to positive terminal. This quantity is defined as the *Electromotive force* (e.m.f.) of the source and is usually represented by the symbol E . Therefore, the electrostatic potential difference between positive terminal and negative terminal of a source under open-circuited condition (that is, the open-circuit terminal voltage V_{AB}) is equal to the e.m.f. of the source.

E of a steady source is a constant. Such a source is called a DC voltage source. DC stands for ‘direct current’. The *terminal voltage* (V_{AB} in Fig. 1.1-3 and Fig. 1.1-4) of a practical steady source will be equal to the e.m.f. E only under open-circuit condition. Terminal voltage becomes less than E when the source is delivering some current due to the inevitably present voltage drop in the internal resistance of the source.

1.2 A VOLTAGE SOURCE WITH A RESISTANCE CONNECTED AT ITS TERMINALS

A piece of conducting substance is now connected to a steady voltage source with a steady-state static charge distribution at its terminals as shown in Fig. 1.1-3. The resulting system is shown in Fig. 1.2-1.

1.2.1 Steady-State Charge Distribution in the System

There was an electrostatic field due to the terminal charge distribution of the source in the space that is now occupied by the conducting substance. This field can change only if the terminal charge distributions change. Charges have to move and reach terminals in order to change the distribution there, which takes time. Therefore, the charge distribution at the terminals remain unaffected for a brief interval even after the conducting substance is connected across the source.

But this will result in large electrostatic forces on the free electrons in the metallic substance. These free electrons start migrating towards positive terminal A of the source and cancel the positive charge distribution there partially. Simultaneously, electrons are pulled from terminal B into conducting substance, thereby cancelling the charge distribution in the negative terminal partially. But this change in the charge distribution will affect the balance between electrostatic force and non-electrostatic force inside the source. Now, the non-electrostatic force will not be cancelled completely by the electrostatic force. The remnant non-electrostatic force will propel the free electrons present within the source towards the negative terminal (and an equal amount of positive charge gets propelled towards positive terminal). This will lead to a restoration of charge distribution at the terminals as well as creation of charge distribution on the surface of connecting wire and the conducting substance.

Soon a steady charge distribution is established as shown in Fig. 1.2-1. The non-electrostatic force and electrostatic force at any point inside the source will cancel each other under this condition. (The conducting substance inside the source is assumed to have

Electromotive Force and Terminal Voltage

Electromotive force of a source is the work done on a unit positive charge by the non-electrostatic force present within the source when the charge moves through the source from negative terminal to positive terminal.

Terminal voltage of a source is the electrostatic potential difference that exists between positive terminal and negative terminal.

Open-circuit terminal voltage = Electromotive force.

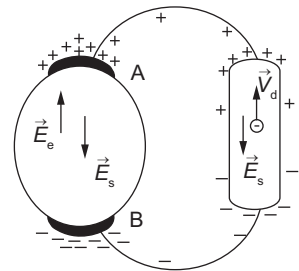


Fig. 1.2-1 Steady-state with a Resistance Connected Across a DC Voltage Source

The need for charge distribution everywhere under steady-state.

zero resistivity.) Therefore, the terminal voltage V_{AB} will be equal to the e.m.f. E of the source. The geometry of the system will decide the amount of positive and negative charges distributed on the terminal surface, wire surface and surface of the conductor. Once established, this charge distribution remains steady (in the case of a steady source) unless spatial arrangement is disturbed or the resistance is disconnected.

1.2.2 Drift Velocity and Current Density

This steady charge distribution will produce electrostatic field everywhere inside the conducting substance. Free electrons inside the conducting substance get accelerated by this electrostatic force. The velocity and kinetic energy of free electrons would have reached high levels in the absence of any impeding force. However, there is an impeding force. This force arises out of collisions suffered by the accelerating electrons. The electrons collide with ionised atoms in the conducting substance inelastically and lose some of the kinetic energy they acquire under the action of electrostatic force. The average effect of multitude of collisions suffered by an electron accelerating under electrostatic force is similar to that of friction. Thus, the inelastic collisions that take place between accelerating free electrons and ionised metal atoms result in a non-electrostatic force that is similar to friction. The accelerating electrons pick up kinetic energy first since they are losing potential energy by falling through a voltage drop. The kinetic energy is subsequently delivered to the lattice through inelastic collisions (or equivalently, to the non-electrostatic force that is manifested within the conducting substance as an average effect of multitude of inelastic collisions).

The free electrons inside the conducting substance reach a steady velocity in the direction opposite to that of electrostatic force under the action of electrostatic force and the non-electrostatic force arising out of collisions. This is somewhat similar to an object reaching a terminal velocity when it falls through a viscous medium. The terminal velocity attained by a free electron inside a conducting substance is called *drift velocity* and is denoted by $\bar{v}_d$. Experiments have revealed that this velocity is proportional to the electrostatic field intensity through a proportionality constant called mobility of the substance. Note that this is an average velocity that is super-imposed on the random thermal velocity of electrons.

Thus, the drift velocity of electrons at a point where the electrostatic field intensity is $\bar{E}_s$ is given by $\bar{v}_d = -\mu \bar{E}_s$, where μ is the mobility of the material and has m^2 per Volt-sec unit. Mobility of a material (μ) is a temperature dependent quantity and decreases with temperature in the case of metals. The negative sign is needed to account for the fact that negatively charged electrons move in a direction opposite to that of electrostatic field.

Consider a small area element $\Delta A \text{ m}^2$ at a point inside the conductor where the drift velocity is $\bar{v}_d$ and electrostatic field intensity is $\bar{E}_s$. Let the area element be taken perpendicular to the direction of $\bar{v}_d$ (or $\bar{E}_s$). Let N be the number of free electrons available in unit volume of the conducting material. Then, there will be $N(\Delta A) v_d$ free electrons present in a volume element constituted by $\Delta A \text{ m}^2$ and a length $v_d \text{ m}$ along the direction of drift velocity. All these electrons will cross the area element in the next second.

Negative charges crossing a surface in one direction is equivalent to positive charges crossing the same surface in the opposite direction. Hence, the total *positive* charge that crosses ΔA in one second in the direction of $\bar{E}_s$ is $N q_e (\Delta A) v_d$ coulombs where q_e is the *magnitude* of charge of an electron and v_d is the *drift speed*.

A vector quantity called ‘current density vector’ (denoted by $\bar{J}$) is defined at a point inside a conductor as the total *positive* charge that crosses *unit area* in *unit time* in the direction of electrostatic field at that point with the area kept perpendicular to the direction of electrostatic field at that point. (This definition is correct only for homogeneous and isotropic materials. Metallic conductors may be assumed to satisfy this requirement for all

A non-electrostatic force appears within a current carrying conductor due to collisions.

Steady-state charge distribution leads to free electrons acquiring drift velocity.

Current density and Conductivity.

practical purposes). It must be obvious that $\vec{J}$ at a point is proportional to $\vec{E}_s$. The dimension of current density is Coulomb per m² per second. Its unit is Ampere per m² since Coulomb/sec is called ‘Ampere’.

$\vec{J} = (N\mu q_e) \cdot \vec{E}_s = \sigma \vec{E}_s$, where σ is defined as the *conductivity* of the material and has ampere per volt per meter as its unit. Ampere per Volt is given a special name – ‘Siemens’. Therefore, the unit of conductivity will be Siemens/m.

Reciprocal of conductivity is called the *resistivity* of the material and is denoted by ρ . Its unit is Ohm-m. (Volt/Amp is given the name ‘Ohm’).

The value of conductivity (and resistivity) of a *linear, homogeneous, isotropic* material is a constant at a particular temperature. Conductivity and resistivity vary with temperature. In general, conductivity decreases with increasing temperature in the case of metallic conductors.

1.2.3 Current Intensity

Electric current intensity or, simply, *current intensity* through a surface is defined as the amount of charge that crosses the surface in unit time. ‘Current intensity’ is usually referred to as ‘current’ itself.

It is a scalar quantity. The definition begs a question – ‘charge that crosses the surface’ in which direction?

The direction of crossing implied in the definition is unambiguous in the case of a closed surface. It is in the direction of surface normal. Surface normal is drawn outwards in the case of a closed surface. Thus, the current intensity through a closed surface is the amount of charge that crosses the surface in one second *from inside to outside*. A positive current implies net positive charge flow out of the volume enclosed by the surface or net negative charge flow into the volume. A negative current implies net positive charge flow into the volume or net negative charge flow out of the volume.

However, there is ambiguity in interpreting current in the case of open surfaces if current is considered to be a pure scalar quantity. The surface normal for an open surface is not unique. The value of current through a surface in a given context can be positive or negative depending on the choice of surface normal. Therefore, the value of current intensity can be uniquely interpreted only if some direction is also associated with it. Therefore, it is customary in practice, especially in the case wire conductors with uniform cross-section, to refer to the *direction of current*. The current density in such conductors will be approximately uniform and hence current density will have same direction at all points in the cross-section of the conductor. *The direction of current intensity in such conductors (wire conductors of uniform cross-section) is taken to be same as the direction of current density itself.*

The unit of current is Coulomb/second and this unit is given a special name of ‘Ampere’. Current is denoted usually by I if it is a steady current (i.e., DC current) and by $i(t)$ (or by i) if it is a time-varying quantity.

Let $q(t)$ be the *total net charge* that crossed a given cross-section in a specified direction from $t = -\infty$ to $t = t$ and let $i(t)$ be the current flowing through that cross-section in the same direction at $t = t$. Then, the relation between these two quantities is given by the following equations.

$$\begin{aligned} i(t) &= \frac{dq(t)}{dt} \text{ A} \\ q(t) &= \int_{-\infty}^t i(t) dt = q(0) + \int_0^t i(t) dt \text{ C} \end{aligned} \quad (1.2-1)$$

Ohm's Law in point form

The relation $\vec{J} = \sigma \vec{E}_s$ is called the *point form of Ohm's Law*. It is an experimental law.

The conductivity σ of a material may vary from point to point in general. Further, even at a point its value may be different in different directions. And its value may depend on electrostatic field strength.

A conducting substance is called *linear homogeneous isotropic conductor* if σ at a particular temperature is constant everywhere in the material and is independent of direction and strength of electrostatic field.

Relation between current crossing a surface at t and net charge that crossed the surface till t .

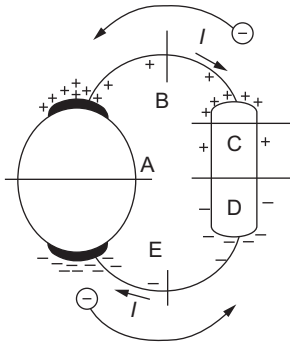


Fig. 1.2-2 A steady-current System – Current Through A, B, C, D and E have Same Value

The energy transfer mechanism in a conductor.

1.2.4 Conduction and Energy Transfer Process

Consider the steady voltage source with resistive load across it as shown in Fig. 1.2-2. Five cross-sections (A, B, C, D and E) are marked in the figure. Also, the directions of positive current flow and electron flow are marked. Electrons flow in the counter-clockwise direction in the circuit and positive current flows in the clockwise direction.

Consider the volume between the two cross-sections marked as B and C in Fig. 1.2-2. There is a surface charge distribution on this volume. (It is possible to show by employing equations of electromagnetic fields that there will be no charge distribution inside the volume in a homogeneous conducting substance under steady current conditions as well as under quasi-static current conditions. Charges will reside only on the surface.) This charge distribution remains stationary in time since the non-electrostatic force within the source of e.m.f. is assumed to be steady. Therefore, the amount of charge that crosses into the volume through the cross-section B in unit time must be same as the amount of charge that crosses out of the volume through the cross-section C in unit time – else the surface charge storage within this volume will change. Hence, the current through B must be the same as the current through C. Similar argument for other cross-sections will lead us to the conclusion that current through all cross-sections will have same value in this circuit.

The surface charge distribution present throughout the system is stationary. But that does not mean that the individual electrons that make this distribution stay put. For instance, a particular electron that is part of the current flow, may cancel a positive surface charge after crossing C. But that will result in an unbalance in the system and another electron will move out from surface and join the current stream leaving a positive charge on the surface. Thus, though the identity of individual charges that form the surface charge may not be preserved, the surface charge will appear stationary at a macroscopic level.

Consider an electron that is part of the current flow. The electrostatic field is oriented from positive terminal to negative terminal inside the source. The non-electrostatic field is oriented from negative terminal to positive terminal. When the conduction electron travels from positive terminal to negative terminal through the source it gains electrostatic potential energy. The non-electrostatic field does positive work to impart this extra potential energy to the electron. The conduction electron then flows through connecting wire to the negatively charged terminal of resistor. The electrostatic field inside the conductor tries to accelerate it and convert its potential energy into kinetic energy. The electron soon transfers its kinetic energy to the lattice through inelastic collisions with atoms. By the time it emerges at the positively charged terminal of the resistance, it would have lost all the extra potential energy it gained earlier to the lattice. The lattice energy appears as heat in the conductor.

Thus, electrons act as a medium for transferring energy from source to conductor. The electrostatic field present everywhere in the system is a facilitator of this energy transfer process. The non-electrostatic field in the source transfers the source energy into charge carriers flowing through them in the form of potential energy of the charged particles in an electrostatic field. The charged particles carry this potential energy with them into the conductor. The non-electrostatic force (i.e., the average effect of inelastic collisions) absorbs the potential energy of charges and transfers it to the lattice. The electrostatic field that is present within the conductor facilitates this process by converting the potential energy of charged particles into kinetic energy before they can deliver it to atoms through inelastic collision process.

Thus, electrostatic field permeating throughout the system is a necessary requirement for conduction and energy transfer process to take place in an electrical system. The required electrostatic field is created by surface charge distributions on conducting surfaces everywhere in the system.

1.2.5 Two-terminal Resistance Element

Consider the steady voltage source with resistive load across it shown in Fig. 1.2-3. Let us work out the electrostatic potential difference between e and f .

The work to be done *against electrostatic force* to carry a unit positive test charge around a closed path is zero. Therefore, the work to be done to take $+1$ C charge from f to e must be the same whether it is moved through a path that lies inside the conducting substance or outside. But the electrostatic field is given by $\vec{E}_s = \frac{\vec{J}}{\sigma}$ inside the conductor.

Therefore, $V_{ef} = -\frac{1}{\sigma} \int_f^e \vec{J} \cdot d\vec{l}$ with $d\vec{l}$ oriented from f to e . The value of this integral will be

same for any path through the conducting substance. However, evaluation of the integral to yield a closed-form result will be possible only in simple cases where the geometry of conductor has some kind of symmetry or other.

We consider a simple case of a conductor with uniform cross-section. The total current may be assumed to distribute itself uniformly throughout the cross-section in such a conductor. This results in a current density vector that has a constant magnitude of I/A (A is the area of cross-section) and direction parallel to the axis of conductor. This is a satisfactory assumption everywhere except at the connection ends. With this assumption, with l as the length of conductor and A as its uniform cross-sectional area, we get,

$$V_{ef} = -\frac{1}{\sigma} \int_f^e \vec{J} \cdot d\vec{l} = I \frac{l}{\sigma A} \left(\text{or } I \frac{\rho l}{A} \right) \text{ V} \quad (1.2-2)$$

Equation 1.2-2 relates the electrostatic potential difference across the connection points of a piece of conductor with uniform cross-section to the current flow through it. The proportionality constant is dependent on material property (conductivity or resistivity) and geometry of the conductor. This proportionality constant is called the *resistance parameter* R .

$$R = \frac{l}{\sigma A} = \frac{\rho l}{A} \text{ Ohm} \quad (1.2-3)$$

However, actual connection point between the resistive material and external circuit may not be accessible for observation of voltage. We measure the voltage across a resistance by connecting a voltmeter to the connecting wire on either side of the element. Assume that the voltmeter is connected across a - c . Then, the voltmeter will read the electrostatic potential difference V_{ac} . But,

$$V_{ac} = V_{ef} + \underbrace{\left(-\frac{1}{\sigma} \int_d^c \vec{J} \cdot d\vec{l} - \frac{1}{\sigma} \int_a^e \vec{J} \cdot d\vec{l} \right)}_{\text{evaluated over paths through connecting wire}}$$

Therefore, a unique voltage difference can be assigned to the conducting body only if the conductivity of connecting wires is infinitely large. This amounts to assuming that the connecting wires have zero resistance. However, it is to be noted that this does not imply thick connecting wires. In fact, Circuit Theory assumes that connecting wires have zero resistance *and* negligible thickness. The reason behind the assumption of negligible cross-section for connecting wires will be explained in a later section.

With this assumption, the electrostatic field inside connecting wires will be zero (since conductivity is infinite). Then, the electrostatic potential difference between the ends of conducting body has a unique value irrespective of which pair of points (a and b) on the connecting wire are chosen to measure it.

Now, a unique voltage and current variable pair can be assigned to the conducting body and its electrical behaviour can be described entirely in terms of these two variables.

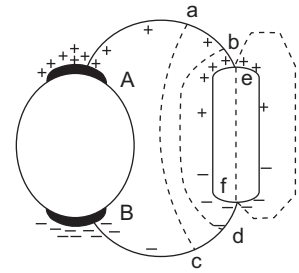


Fig. 1.2-3 Pertaining to Voltage Across a Two-terminal Resistance

Circuit Theory assumes that connecting wires are of material with infinite conductivity and that they are thin to the point of near-zero cross-section.



Fig. 1.2-4 Two-terminal Resistance

Ohm's Law for a linear resistor.

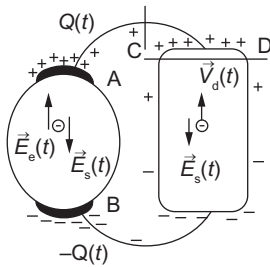


Fig. 1.2-5 A Time-varying Source e.m.f. with Conductor Across it

A metallic conductor requires surface charge distribution for conduction to take place inside. The surface charge will be time-varying when the sources in the system containing this conductor are time-varying.

Time-varying surface charge distribution implies that only a portion of total current entering the conductor through connecting wire is available for energy transfer to the lattice. The remaining portion will get spent in varying the surface charge distribution. Further, the value of current crossing various cross-sections of the conductor at the same instant will be different.

This model of a conducting body is called the *two-terminal resistance element* model. The symbol and element relation is shown in Fig. 1.2-4. The connecting leads leading to the resistive element on either side are of infinite conductivity and near-zero cross-section.

Ohm's Law, which is an experimental law, states that the voltage drop across a two-terminal resistance made of a linear conducting material is proportional to the current entering the element at the higher potential terminal at a constant temperature.

Resistivity and conductivity are functions of temperature. If the temperature range considered is small, resistivity may be approximated as $\rho(T) = \rho(T_0)[1 - \alpha(T - T_0)]$, where $\rho(T_0)$ is the known resistivity at temperature T_0 and α is the *temperature coefficient of resistivity*.

1.2.6 A Time-varying Voltage Source with Resistance Across it

Now, we consider a time-varying source of non-electrostatic force with a resistance load (Fig. 1.2-5).

We assume that the conducting matter inside the source has infinite conductivity. Hence, the non-electrostatic force at every point inside the source has to get cancelled by the electrostatic force at that point at all instants of time. We implicitly assumed that electromagnetic disturbances propagate with infinite speed in making this statement.

The non-electrostatic field inside the source is a function of time here. Thus $\vec{E}_e$ is a function of space and time. Therefore, $\vec{E}_s$ also has to be function of time and space in order to match $\vec{E}_e$. A time-varying $\vec{E}_s$ inside the source calls for a surface charge distribution that is time-varying. Thus, the charge distributed in the system $Q(t)$ is a time-varying function.

But, now the current flowing in the circuit has two functions to perform – (i) supply the time-varying surface charge requirement at each and every point in the system and (ii) transfer the source energy to the conducting substance. Therefore, the net charge crossing different cross-sections in unit time will not be the same. For instance, consider the two cross-sections marked C and D in Fig. 1.2-5. The volume between these two cross-sections has a certain quantity of charge distributed on its surface at t . The quantity of charge that has to get distributed in the same surface is different at $t + \Delta t$. Therefore, the current crossing D cannot be same as the current crossing C since a portion of the current crossing C will get used up in supplying the required change in surface charge distribution.

Thus, the current crossing various cross-sections will be different and there is no unique single value of current in the circuit at any instant. We restore uniqueness to the circuit current by resorting to certain assumptions.

First, we assume that the connecting wires are very thin. In this case, it is possible to show that the surface charge distribution on the surface of connecting wires will be extremely small in value compared to the charge distribution elsewhere. Thus, Circuit Theory assumes that, (i) connecting wires are made of material with infinite conductivity so that the electrostatic field inside connecting wires is zero and voltage drop in them is zero and (ii) connecting wires are so thin that there is virtually no surface charge distributed on their surface. With only negligible surface charge on their surface, the connecting wires will not divert any portion of current flowing through them in order to supply the changes in surface charge distribution. Then, the current through a section of connecting wire will be the same everywhere.

The next assumption used in Circuit Theory is that the current component that is needed to supply the changes in surface charge distribution at any point in the system is a negligible portion of the current at that point. Note that this does not amount to ignoring the electrostatic charge distribution altogether. That cannot be done. Charge distribution is essential for conduction and energy transfer to take place at all. It is only that we chose to ignore the diversion of current for creating a time-varying charge distribution at various points in the system. Obviously, this assumption will be satisfactory only if the source e.m.f. is a slowly varying one.

With these assumptions, the currents everywhere in the circuit in Fig. 1.2-5 become the same function of time. Then there is essentially no difference between an electrical system with a steady-source and an electrical system with time-varying source. The conductor in Fig. 1.2-5 can be modelled by a two-terminal resistance element satisfying Ohm's law on an instant to instant basis. If $v(t)$ is the electrostatic potential difference across the resistance and $i(t)$ is the current entering the higher potential terminal, then $v(t) = i(t)R$.

1.3 TWO-TERMINAL CAPACITANCE

Consider the electrical system shown in Fig. 1.3-1. A source of time-varying e.m.f. is connected to a pair of metallic electrodes A and B. We assume that the connecting wires are of infinite conductivity and near-zero cross-section. Further, we assume that the metallic electrodes are made of material with infinite conductivity. Therefore, the electrostatic field inside the two electrodes will be zero at all instants even when there is current flow in the electrode material.

The total surface charge distributed on the conducting surfaces in the system has two components – the charge distributed on the source terminal and the charge distributed on the electrode. The charge distributed on the connecting wire is negligible since the wire is assumed to be very thin.

The surface charges on the source terminals and electrodes will assume suitable magnitudes and suitable distributions such that (i) the non-electrostatic field in the source is cancelled by the electrostatic field created by the charge distributions on an instant to instant basis everywhere within and (ii) the electrostatic field everywhere inside the connecting wires and electrodes is zero at all time. Thus, $Q(t)$, the total charge distributed in the electrode and the manner in which it is distributed will depend on $\vec{E}_e(x, y, z, t)$ of the source, the spatial geometry of the entire system and material/medium dielectric properties. Therefore, $Q(t)$ will change if the source is moved without affecting the relative position of electrodes. The voltage between the electrodes A and B – i.e., $V_{AB}(t)$ – will be equal to the e.m.f. always; but the charge stored in the electrode system will vary with the spatial position of the source. Thus, a unique ratio between $Q(t)$ and $V_{AB}(t)$ will exist only for a particular spatial arrangement of source and electrodes. The ratio will change with the position of source and cannot be called a property of electrode arrangement alone.

All components in an electrical system will have static charge distributions at their terminals and on their surfaces. The electrostatic field at a point is the superposition of fields created by all these charge distributions. Thus, the voltage across terminals of one component will be decided by the work done in carrying a unit positive charge across the terminal pair against an electrostatic force that is decided by the static charge distributions in the entire electrical system. Thus, a simple ratio of the voltage across terminals of one component to the value of charge distributed at its terminals and surface cannot be defined in general.

Now, we introduce certain assumptions so that we can ascribe the ratio $Q(t)/V_{AB}(t)$ to the electrode pair A and B without any reference to the position of other elements in the system. We assume that the distance between electrodes and the physical dimensions of the two-electrode system are very small compared to the distance between the two-electrode system and other components in the electrical system. (The reader may think of a parallel plate capacitor of large capacitance value and wonder how such a capacitor can satisfy this requirement. That is, precisely why a parallel plate capacitor is found only in the pages of textbooks. A practical 'parallel plate capacitor' has two aluminium foils of large length rolled into a tight cylinder shape with a pair of dielectric films between them. Such an assembly of a pair of electrodes will satisfy the assumption stated above.)

Positive and negative charge distributions of equal magnitude kept close to each other will produce only negligible electrostatic field at distant points. Therefore, the charge distribution on a pair of electrodes that satisfy the assumption stated above will not affect the electrostatic field at the locations where other circuit components are located. And,

Circuit Theory assumes that the variation of surface charge requires only negligible current and that the value of current is the same in all cross-sections of the conductor.

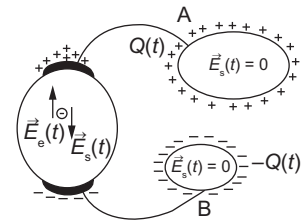


Fig. 1.3-1 A Time-varying e.m.f. Source with Two Electrodes

The ratio between charge and voltage of a pair of electrodes depends on relative position of electrode system with respect to other components.

Assumption required for defining a two-terminal capacitor.

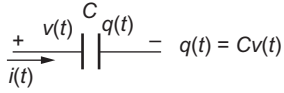


Fig. 1.3-2 A Two-terminal Capacitor

The assumption of 'locally confined electrostatic fields' in Circuit Theory.

charge distributions on other circuit components will not affect the electrostatic field at the location where this electrode pair is located. Therefore, the ratio of charge stored in the electrodes to voltage between the electrodes will depend only on the geometry of the electrode system and dielectric properties of the medium involved.

This unique and constant ratio associated with an electrode pair is defined as its capacitance value and the electrode system that satisfies the assumptions explained above is termed as a *two-terminal capacitor*. The magnitude of charge stored in one of the electrodes in a *linear capacitor* is proportional to the voltage across them. The symbol and variable assignment of a two-terminal capacitance is shown in Fig. 1.3-2.

In fact, Circuit Theory extends the assumption of '*locally confined stationary electrostatic field*' to all components in the circuit. It assumes that the electrostatic field created by the charge distribution residing on a particular component (remember that there is no charge distribution on wires; they are of near-zero cross-section. Therefore, charge distributions can be ascribed to components uniquely) is significant only in the vicinity of that component and is negligible at the locations of other components. This makes the electrostatic field around a component a function of its own charge distribution alone. Therefore, the potential difference across terminals of one component will be proportional to the charge distributed on them. Thus, assumption of '*locally confined stationary electrostatic field*' amounts to neglecting electrostatic coupling between various components. With this assumption, the voltage across a component becomes proportional to the total charge distributed on its terminals and conducting surfaces. The proportionality constant depends on the geometry of the component as well as on material dielectric properties. The fact that there has to be a certain amount of charge distributed on the surface of a component for a voltage difference to exist between its terminals is equivalently described as *the capacitive effect* present in the component. *Thus, every electrical component has a capacitive effect inherent in it.*

Therefore, a piece of conductor also has a capacitive effect associated with it. We ignored the current component that is required to support a time-varying charge distribution across a resistance in Sect. 1.2 in order to define a two-terminal resistance. This is equivalent to neglecting the capacitive effect that is invariably present in the resistance. There is no pure resistance element in practice. All resistors come with a capacitive effect. However, if the capacitance that is present across a resistor draws only negligible current in a given circumstance, then, it may be modelled by a two-terminal resistance.

The capacitance that is present across a two-terminal resistance is called the *parasitic capacitance* associated with it. The adjective '*parasitic*' gives us an impression that it is some second-order effect that has only nuisance value. That is not true – it arises out of the charge distribution that is required to make conduction possible in the resistance. Without this parasitic capacitance the resistor will not carry any current at all.

The relation between the charge stored in a capacitor and voltage across it is given by $q(t) = Cv(t)$. C , the capacitance value has Coulomb per Volt as its unit. This unit is given by a special name – 'Farad'. One Farad is too large a value for capacitance in practice. Practical capacitors have capacitance value ranging from few pFs ($1 \text{ pF} = 10^{-12} \text{ F}$) to few thousand μFs ($1 \mu\text{F} = 10^{-6} \text{ F}$). The value of C is a constant if the geometry of capacitor does not change with time and the material that is used as the dielectric between the metallic electrodes is linear, homogeneous and isotropic. If the value of C is a constant, then it is called as *linear capacitor*.

The current that has to flow into the positively charged electrode of the capacitor is given by the rate of change of the charge residing in that electrode. Therefore, the voltage across a linear capacitor is related to the current flowing into the positive electrode as given below.

Voltage–Current relations of a linear capacitor. $i(t)$ flows into the positive polarity of $v(t)$.

$$q(t) = Cv(t)$$

$$i(t) = C \frac{dv(t)}{dt}$$

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(t) dt = \frac{1}{C} v(0) + \frac{1}{C} \int_0^t i(t) dt$$

(1.3-1)

The current through a capacitor depends on the first derivative of voltage appearing across it. Therefore, the current flow through the parasitic capacitance that is inevitably present across any electrical element can be neglected in the circuit model for that element only if the rate of change of electrical quantities involved in the circuit is small. Thus, a two-terminal resistance will model a piece of conducting substance with sufficient accuracy only if the frequency of voltage and current variables in the circuit is sufficiently small.

We have seen that there is no purely resistive two-terminal element in the physical world. A parasitic capacitance always goes along with a resistance. However, is there a pure two-terminal capacitor in the real world?

Consider a parallel-plate capacitor with a current $i(t)$ flowing into its positive plate as shown in Fig. 1.3-3. The current entering the positive plate from the left has to deposit charge throughout the plate. Therefore, the current has to flow through the cross-section of the plate from left to right. The magnitude of current comes down with length travelled towards right. Specifically, the current crossing the cross-section of the plate at mid-point will be about $0.5 i(t)$. Thus, there is a linearly varying current crossing the cross-section of metallic electrode at any instance. This current flow meets with the impeding resistance of the metallic plate. Thus, there will be a resistive voltage drop along the length of the plate and the plates will no longer be equipotential surfaces. This resistive effect will produce power loss and heating in the capacitor.

There is yet another resistive effect present in a capacitor. A practical capacitor may use some dielectric material (like paper, polyester film, polypropylene film, etc.) between the electrodes in order to increase the capacitance value. The dielectric substance in between the electrodes will have a non-zero (though very small) conductivity; leading to a leakage current that flows from positive plate to negative plate. Thus, the current entering the capacitor gets partially diverted for supplying this leakage current and only the remaining portion is available to create the charge distributions on plates.

A two-terminal capacitor model can model a physical capacitor only if these two resistive effects – one in series and the other in parallel – can be neglected. The first resistive component can be made negligible by increasing the thickness of the electrodes and using a high conductivity metal to construct them. These two resistive effects are called the *parasitic resistive effects* in a capacitor.

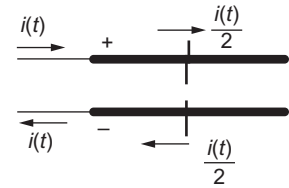


Fig. 1.3-3 Pertaining to the Discussion on Resistive Effect in a Capacitor

1.4 TWO-TERMINAL INDUCTANCE

A moving charge that is part of a steady-current flow (i.e., DC current) in a circuit is, in general, acted upon by non-electrostatic forces contributed by source regions, electrostatic forces contributed by charge distributions and frictional forces within conductors. A new force called *induced electric force* that acts on such a moving charge makes its appearance in circuits carrying unsteady (i.e., time-varying) currents. This new force component gives rise to a circuit element called *inductance*.

1.4.1 Induced Electromotive Force and its Location in a Circuit

Consider a source of e.m.f. with a short-circuit across as shown in Fig. 1.4-1. The conducting material inside the source is assumed to be of infinite conductivity. The shorting wire is assumed to be made of material of infinite conductivity and the cross-section of the wire is taken to be of near-zero dimension. These assumptions imply that there is no net force needed to make charged particles move inside the source as well as inside the shorting wire. Further, the static charge distribution on the surface of shorting wire is negligibly small since the wire is very thin.

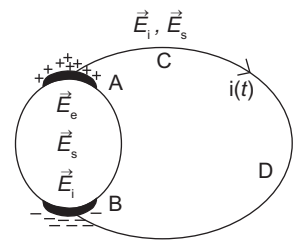


Fig. 1.4-1 A Shorted Source of Electromotive Force

Therefore, there can be no resistive voltage drops inside the source and the shorting wire. Hence, the current flow in the system will reach infinitely large level if the e.m.f. of the source is a steady value – i.e., if the source is a DC voltage source. Of course, the small resistances that are inevitably present within the source and in the shorting wire will limit the current in practice.

Now consider the situation with a time-varying e.m.f. in the source. The non-electrostatic field produced by the source in the source region – i.e., $\vec{E}_e$ – is a time-varying quantity. Therefore, the charge distribution on the source terminals will have to be time-varying in order to generate a time-varying electrostatic field inside the source for exact cancellation of time-varying $\vec{E}_e$. This time-varying charge distribution will, in turn, produce time-varying electrostatic field inside the shorting wire, resulting in a time-varying current flow in the wire. However, the conductivity of wire material is assumed to be infinity. Hence, we should expect a time-varying current of infinite magnitude in the wire. But, the current is observed to have finite amplitude in practice. If there is no resistive effect in the wire (conductivity is taken to be infinity), then what is the mechanism responsible for preventing the current from reaching infinitely high value?

The required mechanism arises out of the third component of force of interaction between two charges in arbitrary motion. We observed in Sect. 1.1 that this component is dependent on relative acceleration of interacting charges and is given by

$$\overline{F_{12ei}} = -\frac{\mu_0 q_1 q_2}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}_1}{r} \right) \Bigg|_{v_2=0} \quad \text{N where } \overline{F_{12ei}} \text{ is the component of force experienced by charge } q_2$$

due to charge q_1 and $\vec{v}_1$ is the velocity of q_1 . We termed this component of force as the *induced electric force*.

Thus, a moving charge can experience four kinds of force in general – (i) the force due to the non-electrostatic field inside a source acting on it, (ii) the force due to electrostatic field, (iii) the magnetic force due to other moving charges and (iv) the induced electric force from other charges which are accelerating with respect to the location of this charge. The first kind of force will be present only if the charge is inside a source region.

Magnetic force on a moving charge is in a direction perpendicular to the velocity of charge. Hence, magnetic force cannot change the energy possessed by a charged particle. Therefore, the magnetic force cannot affect the current flow in a circuit though it may produce mechanical forces in current carrying systems. Hence, we need to consider only the remaining three forces on a moving charge in circuit analysis.

The net induced electric force experienced by a charge located at a certain point in a circuit carrying *steady current* (i.e., a DC current) due to all the other moving charges in the circuit will be zero. We accept this statement without proof.

Hence, moving charges in a DC circuit do not experience any induced electric force provided there are no other circuits carrying time-varying currents in its vicinity.

However, the net induced electric force experienced by a charge located at a certain point in a circuit carrying *time-varying current* due to all the other moving charges in the circuit *will not be zero*. It will experience an induced electric force that will be proportional to its value. Thus, we can define induced electric field $\vec{E}_i$ at a point as the net induced electric force experienced by +1 C charge kept at that point. This field exists everywhere in space (including the source region) unlike the non-electrostatic field that is present only within the source.

Now, the force balance on charges inside the source requires that $\vec{E}_e + \vec{E}_s + \vec{E}_i = 0$ since the material inside the source has infinite conductivity. Similarly, the force balance condition inside the shorting wire requires that $\vec{E}_s + \vec{E}_i = 0$ since the shorting wire is of infinite conductivity.

Electrostatic field is a conservative field. Hence the work to be done against the electrostatic force in carrying a unit test charge around a closed loop is zero. Induced electric field is non-conservative and hence, the work to be done against the induced electric force in carrying a unit test charge around a closed loop is non-zero.

The electrostatic field inside the source and the shorting wire can be expressed in terms of induced electric field and the non-electrostatic field generated by the source as follows.

$$\text{Inside the source } \vec{E}_s = -(\vec{E}_e + \vec{E}_i)$$

$$\text{Inside the shorting wire } \vec{E}_s = -\vec{E}_i$$

It must be noted that all the field quantities appearing in these equations are functions of space as well as time, and that these equations are valid at all points inside the source and wire. Strictly speaking, they should have been expressed as below.

$$\text{Inside the source } \vec{E}_s(x, y, z, t) = -[\vec{E}_e(x, y, z, t) + \vec{E}_i(x, y, z, t)]$$

$$\text{Inside the shorting wire } \vec{E}_s(x, y, z, t) = -\vec{E}_i(x, y, z, t)$$

Let a +1C charge be taken around the circuit from B to A through the source and from A to B through the shorting wire. Then,

$$\oint -\vec{E}_s(x, y, z, t) \cdot d\vec{l} = 0 \text{ since electrostatic field is conservative. Therefore,}$$

$$\int_{\substack{\text{B to A} \\ \text{through the source}}} -\vec{E}_s(x, y, z, t) \cdot d\vec{l} + \int_{\substack{\text{A to B} \\ \text{through the wire}}} -\vec{E}_s(x, y, z, t) \cdot d\vec{l} = 0$$

$$\text{But } -\vec{E}_s(x, y, z, t) = [\vec{E}_e(x, y, z, t) + \vec{E}_i(x, y, z, t)] \text{ inside the source and}$$

$$-\vec{E}_s(x, y, z, t) = \vec{E}_i(x, y, z, t) \text{ inside the shorting wire. Therefore,}$$

$$\int_{\substack{\text{B to A} \\ \text{through the source}}} [\vec{E}_e(x, y, z, t) + \vec{E}_i(x, y, z, t)] \cdot d\vec{l} + \int_{\substack{\text{A to B} \\ \text{through the wire}}} \vec{E}_i(x, y, z, t) \cdot d\vec{l} = 0$$

$$\int_{\substack{\text{B to A}}} \vec{E}_e(x, y, z, t) \cdot d\vec{l} = \oint \vec{E}_i(x, y, z, t) \cdot d\vec{l}$$

The quantity on the left-hand side is the e.m.f. of the source – that is, it is the work done by the non-electrostatic force provided by the source when +1 C is taken through it from its negative terminal to the positive terminal. Similarly, the quantity on the right-hand side is the *negative* of work done by the non-conservative induced electric force when +1 C is taken through the loop in clockwise direction. That is, it is the *negative* e.m.f. due to the induced electric field in clockwise direction in the loop, which we term as the *induced e.m.f.* Obviously, the current in this circuit attains suitable magnitude at all instants such that the source e.m.f. and induced e.m.f. meet each other without leaving any net e.m.f. in the circuit loop.

Induced e.m.f. is present everywhere in the circuit whereas source e.m.f. is present only within the source.

The total induced e.m.f. acting in a circuit is distributed throughout the circuit.

1.4.2 Relation Between Induced Electromotive Force and Current

The induced electric field at a point in a circuit is the superposition of terms of the form

$$-\frac{\mu_0 q}{4\pi r^2} \frac{\partial}{\partial t} \left(\frac{\vec{v}}{r} \right), \text{ where } q \text{ is the charge per carrier and } \vec{v} \text{ is the carrier velocity and } r \text{ is the}$$

distance between the carrier and the point. All moving carriers in the circuit are to be considered in the vector summation. Since the current in the circuit is related to carrier velocity, we expect the summation to turn out to be proportional to $\frac{di(t)}{dt}$ in the circuit. Thus,

induced electric field at all points in space will be proportional to $\frac{di(t)}{dt}$. Electromotive force

of a force field is defined as the *work done by the field* when a unit test charge is taken through a closed path lying in that force field. Hence, the induced e.m.f. in any closed path

will be proportional to $\frac{di(t)}{dt}$. The proportionality constant will depend on the spatial geometry of the circuit and the magnetic properties of the medium involved. This proportionality constant is termed as the *inductance* of that closed path. Inductance is designated by the symbol L . If the geometry of the circuit does not vary with time, the value of L will be a constant.

Induced electric field at all points in space will be proportional to $\frac{di(t)}{dt}$.

1.4.3 Faraday's Law and Induced Electromotive Force

Faraday's law of electromagnetic induction states that the induced e.m.f. in a closed path in a circuit is equal to the time rate of change of flux linkage through that closed path. If the closed path is traversed in counter-clockwise direction and positive flux linkage is defined according to right-hand screw rule, then, this law states that induced electromotive force = $\frac{d\psi(t)}{dt}$, where $\psi(t)$ is the flux linkage through the closed path in 'Weber-turns' unit.

Faraday's law gives the total induced e.m.f. in a closed path. *However, Faraday's law cannot tell us where exactly this e.m.f. is located.* The discussion in Sect.1.4.1 has shown that the induced e.m.f. is distributed all around the closed path.

Determining the polarity of induced e.m.f. by using Faraday's law can be confusing at times for a beginner in circuit analysis. Alternatively, Lenz's law is a better option. Lenz's law, in effect, states that the induced electric field will be in such a direction that it opposes the change in current that is the cause for appearance of the induced electric field. With reference to Fig. 1.4-1, assume that the current $i(t)$ is increasing at some time instant. This means that all the charge carriers are accelerating in the direction of current flow at that instant. This acceleration is the cause of induced electric field in the wire and elsewhere. The direction of induced electric field inside the wire will be such that the induced electric force on a positive charge will tend to *decelerate* it. Thus, the induced e.m.f. will work against the source e.m.f.

Thus, the induced e.m.f. in a closed loop in a circuit with time-varying current is given by $L \frac{di(t)}{dt}$ (for static circuits; see Sect. 1.4.2) and by $\frac{d\psi(t)}{dt}$ (by Faraday's law) with the direction of e.m.f. as per Lenz's law. Therefore,

$$\begin{aligned} L \frac{di}{dt} &= \frac{d\psi(t)}{dt} \\ \frac{d(Li(t))}{dt} &= \frac{d\psi(t)}{dt} \\ \therefore \psi(t) &= Li(t) \end{aligned}$$

Inductance of a closed path is the flux linkage in that closed path for unit current.

Thus, inductance of a closed path is the flux linkage in that closed path for unit current. The unit of inductance is Weber-turns per ampere. This unit is given by a special name – 'Henry' and is represented by 'H'. Since $L \frac{di(t)}{dt}$ yields an e.m.f., inductance gets another unit – Volt-sec per ampere. It follows that the Volt-sec and Weber-turns refer to the same physical quantity.

1.4.4 The Issue of a Unique Voltage Across a Two-terminal Element

With reference to Fig. 1.4-2, a time-varying source of e.m.f. is connected to a conductor by thin connecting wires of infinite conductivity. The charge distributions at the source terminals and load terminals produce electrostatic field everywhere in space. The electrostatic field inside the source cancels the non-electrostatic field available inside the source and the induced electric field inside the source exactly. (The conductivity inside the source is assumed to be infinity.) Electrostatic field inside the connecting wires cancels the induced electric field inside them. Electrostatic field inside the conductor meets the frictional force arising out of collisions of charge carriers with atoms in the lattice and the induced electric force manifesting inside the conductor.

Three issues arise in this context.

- (i) The voltage across two points is the electrostatic potential difference between the two points. The voltage across the resistance is given by the potential difference between e and f. This voltage can be obtained by calculating the

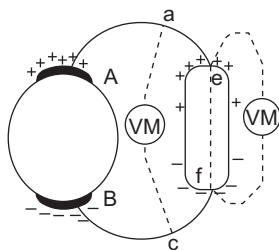


Fig. 1.4-2 Pertaining to Uniqueness of Terminal Voltage of a Two-terminal Element

work *to be* done in carrying a $+1\text{ C}$ charge from f to e through the inside of the conductor. But, the electrostatic field inside the conductor is equal to $-$ (frictional force field + induced electric field). Therefore, the terminal voltage of resistance will contain a resistive voltage drop plus a term that depends on $\frac{di(t)}{dt}$ (i.e., an inductive voltage drop). Therefore, the conductor can no longer be modelled as a pure two-terminal resistance.

- (ii) The voltmeter connected on the right of the conductor attempts to measure the terminal voltage of the resistance right across its terminals. However, the voltmeter connection creates a closed path comprising the resistance element, connecting leads and the meter. This closed path will have induced electric field everywhere inside the connecting leads as well as within the meter. Thus, the meter ends up reading the terminal voltage plus the induced e.m.f. in the voltmeter leads and in the meter internal circuit. Thus, the reading is in error. The amount of error will keep changing with geometry of voltmeter connection – that is, the reading will be different when the leads are disturbed into a new spatial configuration. The amount of error is dependent on the time-rate of change of flux linkage of the voltmeter loop.
- (iii) The voltmeter connected on the left of the conductor reads the actual terminal voltage of the resistance element plus the induced e.m.f. in the path f – c – VM – a – e . Thus, the reading includes the induced e.m.f. in the voltmeter leads and portions of connecting wire in the circuit.

Thus, no unique voltage can be assigned to the resistance by measurement. Therefore, we bring in certain assumptions. The first assumption is that the induced electric field (and also, the induced e.m.f.) inside the connecting wires everywhere in the circuit is negligible. The second assumption is that the induced electric field inside the conductor (or inside a capacitor) is negligible. Obviously, this is equivalent to ignoring the inductive effect present everywhere in the circuit.

No circuit can satisfy these assumptions exactly (except in DC circuits). Induced electric field is proportional to the rate of change of current. If the rate of change of current is low, then, the strength of induced electric field due to circuit current inside the sources, resistances, capacitors and connecting wires will be low compared to electrostatic field. Thus, the assumptions stated above can be employed if the rate of change of current in the circuit is sufficiently small.

With these assumptions, it becomes possible to model a physical resistor by an ideal two-terminal resistance model and a physical capacitor by an ideal two-terminal capacitance model. Further, voltages across a source, resistor and capacitor become unique even with time-varying currents in the circuit.

But, this does not mean that we will not make use of induced e.m.f. in a circuit at all.

A unique terminal voltage variable can be assigned for a circuit component only if the induced electric field in the connecting wires and in the space around the components of a circuit is negligible compared to electrostatic field that exists in the space around the components.

A unique terminal current variable can be assigned for a component in a circuit only if the surface charge distributed on the surface of connecting wires is negligibly small at all time.

1.4.5 The Two-terminal Inductance

An electrical device, in general, can have four kinds of force fields that can affect current flow at every point inside the device. They are:

- (i) Some non-electrostatic field arising out of some kind of potential energy stored within the device – for instance, the non-electrostatic field generated by chemical potential energy in a dry cell.
- (ii) Electrostatic field created by charge distributions on this device as well as other devices nearby.
- (iii) Induced electric field created by time-varying currents flowing in the circuit containing this device as well as in neighbouring circuits.
- (iv) Non-electrostatic force field arising out of frequent collision between moving charged particles and lattice atoms during conduction.

The model used in circuit theory for a device will depend on which of these are strong and which are negligible.

An ideal two-terminal capacitance has no induced electric field inside.

Electrostatic field will be present in all devices in an electrical system and cannot be ignored in any device. Electrostatic field inside *any* device is a function of charge distributions on *all* devices in the system. However, if the physical dimensions of the devices are small compared to spatial distance between the devices, then, the electrostatic field inside a particular device is determined uniquely by the charge distribution on its surface alone. Then, there exists a unique ratio between the electrostatic potential difference across its terminals and the total charge stored on its surface. This is how a two-terminal capacitance can be defined.

Thus, a two-terminal capacitance is a model for an electrical device that has only electrostatic field inside and that depends only on its own charge distribution. The non-electrostatic field existing in the metallic electrodes when current flows in them is ignored in an ideal two-terminal capacitance. The induced electric field that exists inside the device due to time-varying currents everywhere is also ignored in an ideal two-terminal capacitance.

An ideal two-terminal resistance has no induced electric field inside.

A piece of conductor with finite conductivity carrying a current will have electrostatic field, non-electrostatic field arising out of frictional forces and induced electric field due to time-varying currents in the circuit as well as in other circuits. An ideal two-terminal resistance models this piece of conductor by ignoring (i) the current component that is needed to build a time-varying charge distribution on its surface and (ii) the induced electric field inside the conductor.

Circuit Theory models a piece of connecting wire by ignoring all fields that exist within the wire and taking all of them to be zero at all instants. Thus, Circuit Theory assumes that there is no resistive drop across connecting wire; there is no induced e.m.f. in connecting wire and there are no charges distributed on the connecting wire. Such an element is called the *ideal short-circuit element*.

An ideal connecting wire (or short-circuit element) has no field of any kind inside.

An electrical source will have all the four kinds of fields inside. However, the *ideal two-terminal source model* of Circuit Theory attempts to model such a source by (i) ignoring the non-electrostatic field arising out of friction within conductor, (ii) ignoring the induced electric field inside in comparison with electrostatic field and (iii) ignoring the component of current needed to build a time-varying charge distribution at its terminals.

An ideal two-terminal source has no induced electric field and no non-electrostatic field arising out of friction against carrier movement.

And, the *ideal two-terminal inductance model* of Circuit Theory is a model for an electrical device in which there are only two fields – the induced electric field and the electrostatic field. It is not a source and hence there is no source field. It uses conducting substance and hence there is a non-electrostatic field arising out of collisions of charge carriers with lattice atoms when a current flows through it. But this field is ignored in comparison with the other fields. Further, the component of current needed to build a time-varying charge distribution on its surface is assumed to be negligibly small.

Consider a long piece of round conductor carrying a time-varying current as shown in Fig. 1.4-3(a). This wire is not a connection wire. It has a non-zero cross-sectional area. But it is indicated by a line as shown in the figure. The current entering the conductor is $i(t)$ and the same current leaves the conductor at far end. The value of current crossing any cross-section at a particular instant will be the same everywhere since we neglect retardation effect as well as the current that is required to build the surface charge distribution.

An ideal two-terminal inductance has no non-electrostatic field arising out of friction against carrier movement.

There is induced electric field at all points within this conductor. The induced electric

field at a point inside is the sum of terms of the form $-\frac{\mu_0 q}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}(t)}{r} \right)$, where q is the charge

per carrier and $\vec{v}(t)$ is the carrier velocity and r is the distance between the carrier and the point – as many terms as there are moving carriers in the conductor. All the charge carriers will be moving with same instantaneous velocity that is proportional to $i(t)$. But the distance between the point, at which the induced electric field is calculated, and the location of carrier (i.e., r) will be large for all those carriers that are moving at a far away location at the instant under consideration. Therefore, only those carriers that are presently moving within the

immediate vicinity of the point at which field is being calculated will contribute to the induced electric field significantly. Thus, the induced electric field will be relatively low everywhere, and, correspondingly the total induced e.m.f. in the long conductor will be relatively low. The induced field as well as the total induced e.m.f. will be proportional to $\frac{di(t)}{dt}$ since $\frac{\partial}{\partial t} \left(\frac{v(t)}{r} \right)$ that appears in the equation for induced electric field due to a moving charge is directly related to $\frac{di(t)}{dt}$.

The conductor is assumed to be of large conductivity. Then, the net force experienced by a charge carrier inside must be zero. Therefore, the induced electric field at every point within the conductor will be cancelled exactly by the electrostatic field created by the surface charge distribution. This charge distribution is shown in Fig. 1.4-3 assuming that $\frac{di(t)}{dt}$ is positive at the instant under consideration.

A physical inductor is constructed so as to strengthen the induced electric field and the induced e.m.f. inside the conductor forming the inductor.

As in Fig. 1.4-3(b), the same conductor is wound into a coil of four turns. Now the relative distances between moving carriers in various sections of the wire are reduced considerably. Hence, the induced electric field at any point in the conductor will have a value greater than the value when the entire conductor was stretched out in a straight-line as in Fig. 1.4-3(a). Therefore, the total induced e.m.f. will also be higher. Obviously, the value of induced e.m.f. will go up further if the turns can be kept closer.

As in Fig. 1.4-3(c), the same conductor is wound into a coil of lower diameter and higher turns. And the turns are kept closer. This structure will have higher induced electric field everywhere. The total induced e.m.f. will also be higher. If the wire has an insulation cover, then the turns can touch each other.

Thus, winding a long length of wire into an optimally sized and layered coil with turns touching each other will result in large induced electric field everywhere in the wire and large induced e.m.f. over the length of the wire when the current through the coil is time-varying. The induced electric field everywhere inside will be cancelled by the electrostatic field created by the surface charge distribution all along the wire surface. (The conductivity of wire material is assumed to be very large). Therefore, the electrostatic potential difference between the ends of the coil – i.e., the voltage difference between coil terminals – will be equal to the total induced e.m.f. in the coil. The polarity of voltage will follow Lenz's law.

Here, we described an air-cored coil. Air-cored inductor is essentially a long piece of wire that is arranged to occupy a small region of space of dimensions that are very small compared to its length. Such a spatial confinement of a long wire results in strengthening the induced e.m.f. in it. Further strengthening of induced electric field inside the wire can be attained by winding it around a core made of magnetic material (usually iron). If the core made of magnetic material is a closed structure, the induced electric field will be enhanced further. Moreover, a closed core structure confines the time-varying magnetic field to the core itself and reduces the magnetic flux linking rest of the circuit to negligible levels.

A physical inductor that is designed to strengthen the induced electric field within itself, while confining the time-varying flux-linkage to predominantly within itself, can be modelled by an ideal *two-terminal inductance model* provided the resistive voltage drop in the coil can be neglected and the capacitive effect due to surface charge distribution over the coil surface can be neglected.

The value of inductance depends on the geometry of the coil and core assembly and the magnetic properties of the core. *Inductance of a coil is proportional to the square of number of turns of the coil, area of a turn and magnetic permeability of the core material.*

The symbol and variable assignment for an ideal two-terminal inductance is shown in Fig. 1.4-4.

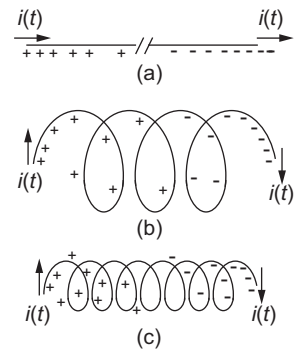


Fig. 1.4-3 Towards a Two-terminal Inductance

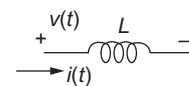


Fig. 1.4-4 A Two-terminal Inductance

Voltage–Current
relation of a two-
terminal inductance.

The governing equations of a linear two-terminal inductance are:

$$\begin{aligned}\psi(t) &= Li(t) \\ v(t) &= L \frac{di(t)}{dt}\end{aligned}\tag{1.4-1}$$

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt = \frac{1}{L} \int_{-\infty}^0 v(t) dt + \frac{1}{L} \int_0^t v(t) dt = i(0) + \frac{1}{L} \int_0^t v(t) dt$$

where $\psi(t)$ is the flux-linkage at t in Weber-turns, $v(t)$ is the voltage across the inductance and $i(t)$ is the current entering the higher potential terminal. $i(0)$ is the current in the inductor at $t = 0$.

Self-induction
versus Mutual
Induction.

A coil can have induced electric field and induced e.m.f. present in it due to accelerated motion of charges (i.e., time-varying current) in the circuit in which it is connected and/or due to accelerated motion of charges taking place in another physically separated circuit. The e.m.f. induced in the coil due to its own time-varying current is termed as *self-induced e.m.f.* and the e.m.f. induced in it due to current in another circuit is termed as *mutually induced e.m.f.* Self-induced e.m.f. is associated with an inductance value called *self-inductance*. Equation 1.4-1 describes the governing equations of self-inductance.

There is no region without induced electric field and induced e.m.f. in any circuit carrying time-varying currents. All devices and components of such a circuit are affected by electromagnetic induction. Thus, all devices have a little inductive effect associated with them. The associated inductance will be called as the *parasitic inductance* of the two-terminal element (unless it is a two-terminal inductance). Ideal two-terminal element models ignore the parasitic inductance in a resistor, capacitor, source and connecting wire.

1.5 IDEAL INDEPENDENT TWO-TERMINAL ELECTRICAL SOURCES

Electrical sources are devices that are capable of applying a non-electrostatic force on a charge that moves through the source region. They can deliver energy to the charged particle or absorb energy from it.

1.5.1 Ideal Independent Voltage Source

A two-terminal voltage source will have a non-electrostatic field at every point inside the source region. The charge distribution on the terminal surfaces of the source will create an electrostatic field at all points inside the source. The two fields cancel each other at all points at all instants under all conditions if the material inside the source is of infinite conductivity. The terminal voltage (which is an electrostatic potential difference) will always be equal to the internal e.m.f. in that case.

The conducting material inside the source will have finite conductivity in practice. Charge carriers moving inside such material require net non-zero force to work against collisions with lattice atoms. This will call for a difference between the internal non-electrostatic field and the electrostatic field to exist. Then, the terminal voltage will be different from the internal e.m.f. It will be less than the internal e.m.f. if the source is delivering positive current out of its positive terminal and it will be more than internal e.m.f. if it is absorbing positive current at its positive terminal. The difference between terminal voltage and internal e.m.f. is termed as the voltage developed across the *internal resistance* of the source.

A practical voltage source with time-varying internal e.m.f. will require a time-varying current flow component to support the time-varying surface charge distribution on its terminals. That is, a practical voltage source has a parasitic capacitance right across its terminals.

A practical voltage source will have induced electric field inside due to its own time-varying current as well as due to time-varying currents elsewhere in the circuit and in neighbouring circuits. This will affect the voltage appearing at its terminals. That is, a practical voltage source also has internal parasitic inductance. Thus, a detailed circuit model for a practical voltage source will be as shown in Fig. 1.5-1. L_i is a lumped parameter approximation for the inductive effect distributed within the source. R_i is a lumped resistance parameter that approximates the distributed resistive effect within the source. C_i is a lumped capacitance parameter that approximates the distributed capacitive effect within the source and at its terminals. $E(t)$ is the internal e.m.f. of the source. The + and – signs do not signify the polarity of charges at the terminals. Rather, the + sign indicates the point at which the potential difference is specified and – sign indicates the reference point for specifying the potential difference. Thus $V(t)$ is the voltage of the terminal marked with + with respect to the point marked with – sign at the time instant t in volts unit.

An *ideal voltage source* is the one in which all the three elements R_i , L_i and C_i are assumed to be negligible. Thus, the terminal voltage of an *ideal voltage source* is always equal to its internal e.m.f. quite independent of magnitude or wave-shape of current delivered or absorbed by it. Such an ideal voltage source is called as *an ideal independent voltage source* if the e.m.f. is a function of time only and does not depend on any other electrical or non-electrical variable. An ideal independent voltage source is specified by the following terminal equations.

$v(t) = E(t)$, a specified function of time,

$i(t)$ = Arbitrary, decided by the rest of the circuit in which this source is connected.

The symbol for a constant ideal independent voltage source (that is, a DC source) is shown in Fig. 1.5-2(a) and that of a time-varying ideal independent voltage source is shown in Fig. 1.5-2(b).

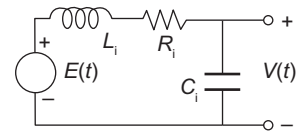


Fig. 1.5-1 Approximate Equivalent Circuit of a Practical Voltage Source

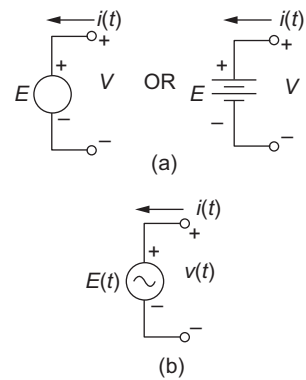


Fig. 1.5-2 Ideal Independent Voltage Source

1.5.2 Ideal Independent Current Source

An ideal independent current source delivers or absorbs a current at its terminals that is a specified function of time. Rest of the circuit in which it is connected decides its terminal voltage. The current delivered or absorbed by it does not depend on the voltage that appears across its terminals.

Practical current sources will have a parallel resistance and capacitance at its terminals representing the effect of finite conductivity within the source and charge distribution on its surface and terminals. These parasitic components are neglected in the ‘ideal independent current source’ model.

The symbol of a constant ideal independent current source (i.e., a DC source) is shown in Fig. 1.5-3(a) and that of a time-varying ideal independent current source is shown in Fig. 1.5-3(b).

An ideal independent current source is specified by the following equations:

$i(t) = I_s(t)$, a specified function of time.

$v(t)$ = Arbitrary, decided by the rest of the circuit in which this source is connected.

There are no ideal independent voltage sources and ideal independent current sources in practice. These are only models of practical sources that give reasonably accurate results provided they are not applied under extreme loading conditions. The ideal model will undoubtedly fail for a practical voltage source that is shorted or for a practical current source that is open-circuited. The short-circuit current in a DC voltage source is limited by its internal resistance while that of a time-varying voltage source is limited by internal resistance and internal inductance. Similarly, the open-circuit voltage that appears across a practical current source is limited by its internal resistance in the case of a DC source. It is limited by internal capacitance and resistance in the case of a time-varying current source.

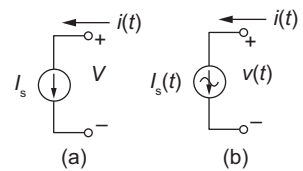


Fig. 1.5-3 Ideal Independent Current Source

Limitations of Ideal Source Model

Ideal Independent Voltage Source model can be used for a practical voltage source only when the current flowing in the source is a small fraction of its short-circuit current.

Ideal Independent Current Source model can be used for a practical current source only when the voltage appearing across the source is a small fraction of its open-circuit voltage.

'Ideal short-circuit element' is a special case of 'ideal independent voltage source' with zero voltage.

'Ideal open-circuit element' is a special case of 'ideal independent current source' with zero current.

In fact, ideal model for a voltage source models a practical voltage source accurately only when the current delivered/absorbed by it is a small fraction of its short-circuit current. Similarly, ideal model for a current source models a practical current source accurately only when the voltage appearing across its terminals is a small fraction of its open-circuit voltage.

1.5.3 Ideal Short-circuit Element and Ideal Open-circuit Element

Ideal two-terminal short-circuit element is the element that is used to model a piece of connecting wire in Circuit Theory. It is also used to model an ideal switch in closed condition. It has no resistance, no inductance and no charges distributed on it. The voltage across its terminals is constrained to remain at zero. It can carry an arbitrary current that is decided by rest of the circuit. Thus, definition of the ideal short-circuit element parallels that of an ideal independent voltage source. *Hence, an ideal short-circuit element may be thought of as a special case of an ideal independent voltage source with $E(t) = 0$ for all t .* It is described by the following equations.

$$v(t) = 0 \text{ V,}$$

$$i(t) = \text{Arbitrary, decided by the rest of the circuit in which this source is connected.}$$

Similarly, *an ideal open-circuit element is equivalent to an ideal independent current source with $I_s(t) = 0$.*

It is described by the following equations.

$$i(t) = 0 \text{ A,}$$

$$v(t) = \text{Arbitrary, decided by the rest of the circuit in which this source is connected.}$$

In practice, a short-circuit element has a little resistance and inductance in series. A practical open-circuit has a small capacitance shunting its terminals.

1.6 POWER AND ENERGY RELATIONS FOR TWO-TERMINAL ELEMENTS

An ideal two-terminal circuit element has a unique voltage variable assigned at its terminals and a unique current variable assigned to its terminals. The electrical behaviour of such an element can be described in terms of these two variables at all instants. Electromagnetic disturbances are assumed to travel instantaneously to all parts of such an element. This results in an electrical description that is independent of space variables for the element. Such an electrical description for an element is termed as *lumped parameter description*.

Further, ideal two-terminal elements have only one kind of electrical phenomena taking place inside them. The capacitive and inductive effects in a practical resistance are neglected in order to arrive at an ideal two-terminal resistance model. The resistive and inductive effects in a physical capacitor are neglected to model it by an ideal two-terminal capacitor. The capacitive and resistive effects in a physical inductor are neglected to arrive at the ideal two-terminal inductance model.

Moreover, lumped two-terminal elements confine the electromagnetic fields associated with them to the space inside them and in the immediate vicinity.

Such a two-terminal element can be represented in general by the symbol as given in Fig. 1.6-1. The variable assignment for the element is also shown in the figure.

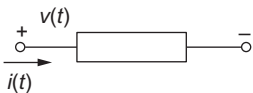


Fig. 1.6-1 An Ideal Two-terminal Element

1.6.1 Passive Sign Convention

Current in a wire has a direction associated with it. The actual direction associated with a current – i.e., the *current direction* – is the direction in which positive charges move.

Consider a wire section A–B with a cross-section identified at C as in Fig. 1.6-2. Four possible kinds of charge motion are depicted in (a) to (d) in this figure. The *direction*

of current in (a) and (d) is from left to right (from A to B). This is so because negative charge crossing a cross-section in one direction is equivalent to positive charge crossing the same cross-section in opposite direction. The direction of current in (b) and (c) is from right to left (from B to A) for the same reason.

However, it is not possible to decide the direction in which current will flow in an element that is a part of a circuit before we actually solve the circuit analysis problem. The actual direction in which positive charges flow through the element can be ascertained only after the circuit solution is obtained. But, despite this we need to assume some direction for current flow in each and every element in a circuit so that we can prepare the circuit equations needed for solving the circuit.

This is where the ‘reference direction for current’ comes in. We assign a particular direction along the element as the reference direction for current. That is, we assume that positive charge moves through the element in the direction chosen as the reference direction even before we arrive at the circuit solution. The circuit solution will either confirm our assumption or reveal that actual current direction is opposite to the direction we assumed. If the circuit solution returns a positive value for the element current, then positive charge flows through the element in the assumed reference direction (or negative charge flows in opposite direction). If the circuit solution results in a negative value for the element current, then positive charge flows in a direction opposite to reference direction (or negative charge flows in the reference direction).

A similar issue arises in the case of voltage across a two-terminal element. We cannot determine which terminal is the higher potential terminal before we actually solve the circuit. Hence, we choose one of the two terminals to be the higher potential terminal prior to solving the circuit. The circuit solution will either confirm it by returning a positive value for element voltage or correct us by returning a negative value for that element voltage.

But, the circuit solution can reveal the correct state of affairs as far as element current and element voltage are concerned only if the chosen reference directions for voltage across the element and current in the element are consistent with the element voltage–current relationship that was employed in solving the circuit. Consider the four different ways of selecting the reference directions for current and voltage of a two-terminal resistance as shown in Fig. 1.6-3. The correct statement for Ohm’s Law is shown by the side of each choice in Fig. 1.6-3. We can use any one of these four reference direction choices provided we employ the correct statement for Ohm’s Law. Similar v – i relation statements can be prepared for other two-terminal elements also.

However, we would like to avoid the confusion that may result from multiple choices available for reference polarities. We settle the matter once and for all by choosing one set of current and voltage reference directions for all two-terminal elements. The chosen reference directions will be as per the scheme marked as (i) in Fig. 1.6-3. That is, we assign positive polarity of voltage variable to one of the two terminals and then assign positive current flowing into that terminal from outside. This choice of reference directions for current and voltage of a two-terminal element is called as the ‘Passive Sign Convention’ (Fig. 1.6-4).

The choice shown in (iv) of Fig. 1.6-3 is also per passive sign convention. The choice of the terminal to assign positive polarity of voltage variable is arbitrary in passive sign convention.

The v – i relation for a two-terminal resistance with passive sign convention is $v(t) = Ri(t)$. It is $v(t) = L \frac{di(t)}{dt}$ for a two-terminal inductance and $i(t) = C \frac{dv(t)}{dt}$ for a two-terminal capacitance with passive sign convention.

1.6.2 Power and Energy in Two-terminal Elements

Consider a general two-terminal element shown in Fig. 1.6-4. Assume that at the instant t both $v(t)$ and $i(t)$ are positive.

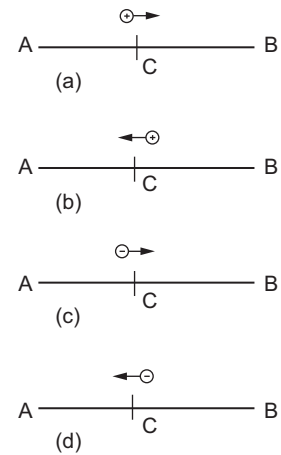


Fig. 1.6-2 Pertaining to the Discussion on Direction of Current

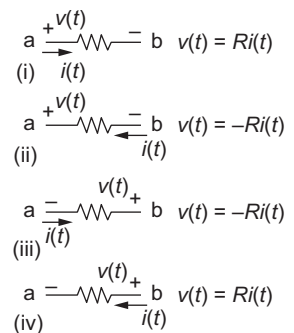


Fig. 1.6-3 Four Different Choices for Reference Directions and Corresponding Statements of Ohm’s Law

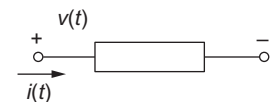


Fig. 1.6-4 Passive Sign Convention

Positive charges moving through a voltage drop lose potential energy.

This implies that positive charges are flowing into the element at that instant. We have seen earlier that positive charges moving through a voltage drop will lose their potential energy to the element. If the element is a resistance, then, the energy lost by positive charges moving from higher potential end to lower potential end will appear as heat in the resistive element. If the element is a capacitor, then, the energy lost by these charges will get stored in the capacitor as electrostatic energy storage. If the element is an inductance, the energy lost by charges will appear as increase in energy stored in the magnetic field in the element. If the element is a source, the energy lost by the charges will be absorbed by the source element and stored inside in some other form of energy.

Let us assume that Δq C of positive charge crossed the left terminal into the element in a time interval Δt centred around the time instant t . Then, the energy lost by these charges will be $\Delta E = v(t) \times \Delta q$ J since 1 C passing through a voltage drop of $v(t)$ V will lose $v(t)$ J of energy. *Energy lost by the charge is equal to energy delivered to the element.* Therefore, the energy delivered to the element over the time interval Δt is $v(t) \times \Delta q$ J. The average rate at which this energy is delivered to the element is given by $v(t) \frac{\Delta q}{\Delta t}$ J/s. The instantaneous rate at which energy is delivered to an element is defined as the *instantaneous power delivered to the element* and is denoted by $p(t)$. Therefore,

$$p(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta E}{\Delta t} = v(t) \times \lim_{\Delta t \rightarrow 0} \frac{\Delta q}{\Delta t} = v(t) \times i(t) \text{ J/s.}$$

The unit J/s is given the name ‘Watts’ and denoted as ‘W’.

Thus, the instantaneous power delivered to a two-terminal element is given by $p(t) = v(t)i(t)$ W, where $v(t)$ and $i(t)$ are the voltage across the element and current through the element as per passive sign convention.

Since instantaneous power $p(t)$ is the instantaneous rate at which energy is delivered to the element, total energy $E(t)$ that was delivered to the element from $t = -\infty$ to the current instant t is given by $E(t) = \int_{-\infty}^t p(t) dt$. (Note: We use the symbol $E(t)$ to denote the time-varying e.m.f. of a voltage source as well as the total energy delivered to a two-terminal element. The symbol will have to be interpreted contextually.) The relation between energy delivered to a two-terminal element and power delivered to it is summarised below.

Power and Energy delivered to a two-terminal element.

$$\begin{aligned} p(t) &= \frac{dE(t)}{dt} = v(t)i(t) \text{ W} \\ E(t) &= \int_{-\infty}^t p(t) dt = \int_{-\infty}^t v(t)i(t) dt \text{ J} \end{aligned} \quad (1.6-1)$$

Power delivered by a two-terminal element is obviously the negative of power delivered to it. Therefore,

Instantaneous power delivered by a two-terminal element = $-v(t)i(t)$ W, where $v(t)$ and $i(t)$ are instantaneous voltage and instantaneous current of the element as per passive sign convention.

Consider an *isolated circuit* that has no energy coupling of any kind with the surroundings. A circuit can get coupled to the surroundings by electrostatic/electromagnetic coupling with other physically separate circuits in the vicinity or by mechanical, thermal or optical interaction with the environment. We assume a circuit that has no such interaction when we refer to an *isolated circuit*. Obviously, the total energy in that circuit has to remain a constant in time. That is, the sum of energy delivered to all the elements in the circuit must remain constant. Let there be n two-terminal elements connected in such a circuit. Some of them may be electrical sources. Then,

$$E_1(t) + E_2(t) + \dots + E_n(t) = \text{Constant}$$

Differentiating both sides of this equation with respect to time, we get,

$$\frac{dE_1(t)}{dt} + \frac{dE_2(t)}{dt} + \dots + \frac{dE_n(t)}{dt} = 0$$

But each term in this equation is nothing but the instantaneous power delivered to the corresponding two-terminal element. Therefore,

$$\sum_{\substack{\text{Over all the} \\ \text{elements in an} \\ \text{isolated circuit}}} p_i(t) = 0 \quad (1.6-2)$$

Instantaneous power sums to zero over all two-terminal elements in an isolated circuit.

Thus, the sum of instantaneous power *delivered to* all elements in an *isolated* circuit is always zero. Or equivalently, the sum of instantaneous power *delivered by* all elements in an *isolated* circuit is always zero. This implies that total power *delivered by* the elements that deliver positive power at t must be equal to the total power *absorbed by* the elements that absorb positive power at that instant. This principle can be employed to check the solution of a circuit analysis problem.

Note that ‘power delivered to an element’ and ‘power absorbed by an element’ mean the same.

The instantaneous power delivered to a two-terminal element does not have to be positive at all instants of time. Neither does it have to be negative at all instants. It is always positive in the case of a resistance. But in all other cases, it can be positive or negative depending on the relative polarity of voltage and current in the element.

EXAMPLE: 1.6-1

The current through a two-terminal element is given by $i(t) = 10(1 - e^{-1000t})$ mA for all $t \geq 0$ and $= 0$ for $t < 0$. (i) Find the amount of charge that went through the element in $[0 \text{ ms}, 5 \text{ ms}]$. (ii) Find an expression for the charge that went through the element up to the time instant t . (iii) If the voltage across the element is a constant at 10 V and the current $i(t)$ flows out of positive terminal find the energy delivered by the element as a function of time.

SOLUTION

Refer to Fig. 1.6-5. $i(t)$ as per passive sign convention is $-10(1 - e^{-1000t})$ mA.

(i) Charge that went through the element in a time interval $[t_1, t_2]$ is given by

$\Delta q = \int_{t_1}^{t_2} i(t) dt$. Substituting the time-function for $i(t)$ and using limits $t_1 = 0$ and $t_2 = 5 \text{ ms}$, we get,

$$\begin{aligned} \Delta q &= \int_0^{0.005} -10(1 - e^{-1000t}) \times 10^{-3} dt \text{ C} \\ &= -10^{-2} (t + 10^{-3} e^{-1000t}) \Big|_0^{0.005} \text{ C} \\ &= -10^{-2} [0.005 + 10^{-3} e^{-5} - 10^{-3}] \text{ C} \\ &= -40 \mu\text{C}. \end{aligned}$$

(ii) The required expression is obtained by

$$\begin{aligned} q(t) &= \int_{-\infty}^t i(t) dt = \int_{-\infty}^0 i(t) dt + \int_0^t i(t) dt = \int_{-\infty}^0 0 dt + \int_0^t -10 \times 10^{-3} (1 - e^{-1000t}) dt \text{ C} \\ &= -0.01t - 10^{-5} e^{-1000t} \Big|_0^t = -0.01t - 10^{-5} e^{-1000t} + 10^{-5} \text{ C} \\ &= 10 [1 - t - e^{-t}] \mu\text{C with } t \text{ in ms.} \end{aligned}$$

(iii) The instantaneous power delivered to the element is $p(t) = v(t) i(t)$, where $v(t)$ and $i(t)$ are as per passive sign convention. Therefore, the power delivered by the

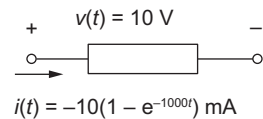


Fig. 1.6-5 The Two-terminal Element in Example 1.6-1

element is given by $-v(t)i(t)$. The energy delivered by the element is obtained by integrating this quantity as below.

$$\begin{aligned}\text{Energy delivered} &= \int_{-\infty}^t -v(t)i(t)dt = \int_{-\infty}^0 -10 \times 0 dt + \int_0^t -10 \times -10(1 - e^{-1000t})dt \text{ mJ} \\ &= (100t + 0.1e^{-1000t} - 0.1) \text{ mJ} \\ &= (0.1t + 0.1e^{-t} - 0.1) \text{ mJ with } t \text{ in ms.}\end{aligned}$$

EXAMPLE: 1.6-2

The voltage across a two-terminal element and current through it are given in Fig. 1.6-6. Passive sign convention may be assumed. Obtain the instantaneous power delivered to the element and the energy delivered to the element as functions of time.

SOLUTION

Instantaneous power delivered to the element is obtained by $p(t) = v(t)i(t)$. This waveform will contain straight-line segments since the voltage waveform contains straight-line segments and current waveform is a symmetric rectangular pulse waveform. The power waveform is shown in Fig. 1.6-7(a).

The energy delivered to the element is obtained by integrating the power delivered to the element from $t = -\infty$ to $t = t$. The equation of $p(t)$ in the interval $[0 \text{ ms}, 2 \text{ ms}]$ is that of a straight-line of slope 18 W/ms . Integrating this straight-line equation results in a parabolic curve for energy in that interval. The parabolic curve reaches 18 mJ value at 2 ms (since area of the triangle in $p(t)$ curve is $18 \text{ W} \times 2 \text{ ms} \times 0.5 = 18 \text{ mJ}$). Then $p(t)$ reverses polarity and remains negative and linear in the interval $[2 \text{ ms}, 4 \text{ ms}]$. This means

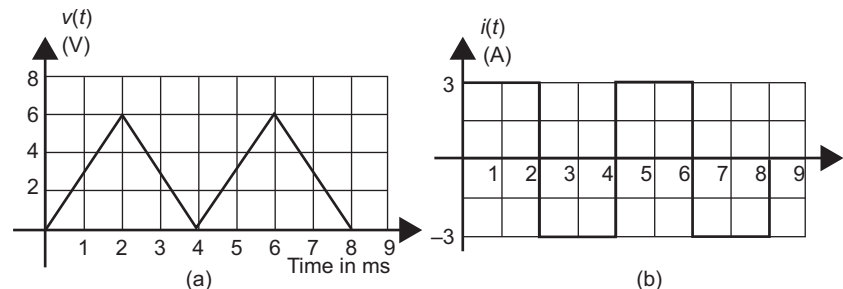


Fig. 1.6-6 Voltage and Current Waveform for Example 1.6-2

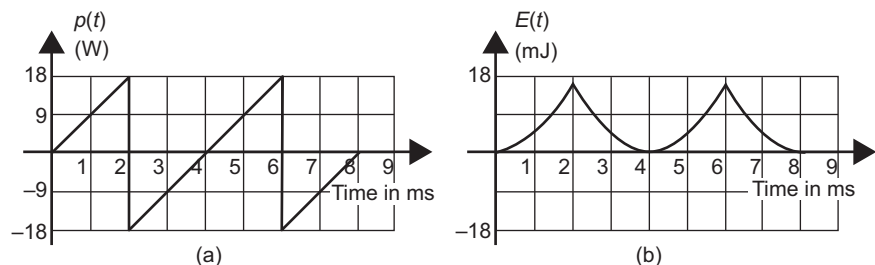


Fig. 1.6-7 (a) Waveform of Instantaneous Power and (b) Waveform of Energy in Example 1.6-2

that the element delivers power during this interval. The area of triangle in the power curve in the interval [2 ms, 4 ms] is again 18 mJ; but with a negative sign. Therefore, the total energy delivered to the element at the end of 4 ms period must be 18 mJ – 18 mJ = 0 mJ and the energy curve between 2 ms and 4 ms must be parabolic again. The variation of energy delivered to the element is shown in Fig. 1.6-7(b).

Note that the net energy delivered to the element at the end of 8 ms is zero. The element received a total of 36 mJ of energy during the intervals [0 ms, 2 ms] and [4 ms, 6 ms]. The element delivered a total of 32 mJ of energy during the two intervals [2 ms, 4 ms] and [6 ms, 8 ms].

EXAMPLE: 1.6-3

In charging a storage battery, it is found that energy of 2 watt-hour is expended in 30 min in sending 200 C through the battery. (i) What is the terminal voltage of the battery assuming that this voltage remains constant during the charging process? (ii) What is the magnitude of average charging current?

SOLUTION

(i) 200 C of charge went through the battery. Energy delivered to the battery is given by $\int_0^{30 \times 60} v(t)i(t)dt$.

The battery voltage is stated to be a constant during the charging process. Let this constant voltage be V volts. Then, the energy delivered over 1800 s is $\int_0^{1800} Vi(t)dt = V \int_0^{1800} i(t)dt = VQ$, where Q is the charge that went through the battery in the same time interval. Therefore, $VQ = 2 \text{ watt-hour} = 2 \times 3600 \text{ watt-sec} = 7200 \text{ J}$. Since Q is 200 C, $V = 7200/200 = 36 \text{ V}$.

(ii) The average charging current is the value of a constant current that will result in same charge flow over the same time interval. Therefore, the average charging current is $200 \text{ C}/1800 \text{ s} = 1/9 \text{ A}$.

EXAMPLE: 1.6-4

Find the current I in the direction marked in Fig. 1.6-8.

SOLUTION

The sum of power delivered by all elements in an isolated circuit must be zero at all instants. Power delivered by an element in a DC circuit = $-VI$, where V and I are its voltage and current variables as per passive sign convention.

The values of V and I for 10 V source = 10 V and -5 A

$\therefore$ Power delivered by 10 V source = 50 W

The values of V and I for 20 V source = 20 V and -20 A

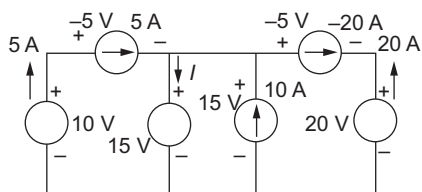


Fig. 1.6-8 Circuit for Example 1.6-4

$\therefore$ Power delivered by 20 V source = 400 W
 The values of V and I for 15 V source = 15 V and I A
 $\therefore$ Power delivered by 15 V source = $-15 I$ W
 The values of V and I for 5 A source = -5 V and 5 A
 $\therefore$ Power delivered by 5 A source = 25 W
 The values of V and I for 10 A source = 15 V and -10 A
 $\therefore$ Power delivered by 10 A source = 150 W
 The values of V and I for -20 A source = -5 V and -20 A
 $\therefore$ Power delivered by -20 A source = -100 W
 Sum of power delivered by all elements = $(50 + 400 - 15 I + 25 + 150 - 100)$ W
 $= (525 - 15 I)$ W
 This has to be equal to zero. Therefore, the value of I is 35 A.

1.7 CLASSIFICATION OF TWO-TERMINAL ELEMENTS

Circuit elements can be classified based on different criteria. Classifying elements based on the physical dimensions of the element results in two broad classes of circuits – *lumped parameter circuits* and *distributed parameter circuits*.

1.7.1 Lumped and Distributed Elements

Electromagnetic effects propagate within the circuit in the form of waves with a finite velocity. Hence, the time-variation of e.m.f. taking place within the electrical sources will be felt at different points in the circuit with different time delays. Therefore, the description of electrical phenomena in circuit elements, in general, will involve time and space variables. A circuit element cannot be described by a unique voltage and current variable pair in that case.

However, if the circuit dimensions and element dimensions are such that the time taken by electromagnetic waves to propagate over the largest dimension in the circuit is small compared to the *characteristic time of variation* of the e.m.f.s acting in the circuit, then, the retardation effect due to finite velocity of electromagnetic waves can be ignored and a simple circuit model for elements can be used.

Assume that the circuit contains many sources of sinusoidal nature and the maximum angular frequency of such source functions is ω_0 rad/s. That is, there is some voltage or current variable of the form $X \sin(\omega_0 t)$ present in one of the sources in the circuit. Then, this variable will complete one cycle of oscillation in $2\pi/\omega_0$ s. The *characteristic time of variation* in this circuit is then $2\pi/\omega_0$ s. That is, this is a measure of the minimum time-interval over which significant changes in circuit variables will take place. Now, let us assume that the largest dimension of any element in the circuit (including connecting wires) is d meters. Then, electromagnetic waves will take d/c s to cover this distance where c is the velocity of light in free space. If d/c is much less than $2\pi/\omega_0$, we may ignore the travel time of electromagnetic disturbances and model all the elements in the circuit by terminal voltage–current relationships. Note that this conclusion is valid only for operation at $\leq \omega_0$ rad/s.

The ‘characteristic time of variation’ of a circuit depends on the wave-shape of source functions present in the circuit. The source functions need not be sinusoidal always. However, it is possible to expand arbitrary time-functions in terms of sinusoidal functions under certain conditions. The highest frequency that appears in such expansions will have to be used to decide whether the circuit can be modelled by ignoring retardation effect. There is one kind of source function, which if present in a circuit, will not permit us to ignore retardation effect. That is, a source function that contains sudden, instantaneous

A **lumped element** has the following features.

(i) A unique terminal voltage and unique element current, which are functions of time only, can be defined for a lumped element. All the electrical phenomena that take place within can be described in terms of these two terminal variables.

(ii) The current that enters one of the terminals of a lumped element leaves through the other terminal without any loss of current within the element at any instant.

(iii) This means that there is no accumulation of net charge within the element at any instant. However, that does not mean that there is no charge distribution inside. But equal positive and negative charges are distributed over the surface of the element.

changes in values – a function that has step discontinuities. Obviously, the characteristic time of variation of such a function is zero.

An element is classified as a *lumped element* if the net effect of electrical phenomena taking place within that element can be described in terms of only its terminal voltage and current variables, irrespective of its internal details and geometry. This amounts to neglecting the retardation effect in the element. If the electrical description of an element calls for voltage and current variables that are functions of space variables over the element (in addition to time variable), the element is called as a *distributed element*.

An electrical device can be modelled by a *lumped model* only for a range of frequencies in the source functions in the circuit. The same electrical device may call for a *distributed model* if the source functions in the circuit vary rapidly enough to make retardation time within the device significant.

For instance, consider a solenoid coil of length 5 cm and diameter 1 cm with 100 turns of wire. One may be tempted to assume that the largest dimension of the coil is its length – i.e., 5 cm. It is not true. The largest dimension that we need here is the length of the wire and that is about 314 cm. The retardation time over this length $= 3.14/3 \times 10^8 = 10$ ns. If ω_0 is the highest frequency of sinusoidal components present in the sources within the circuit, then, the ‘characteristic time of variation’ is $2\pi/\omega_0$ s. If this time is 10 ns then ω_0 is 628 Mrad/s. The corresponding cyclic frequency will be 100 MHz. Thus, this coil can be modelled as a two-terminal lumped inductance with good accuracy if the circuit contains source sinusoidal components at 1 MHz or below. However, it will call for a distributed model if the sources contain >10 MHz sinusoidal components.

Consider a power transmission line of length 300 km. The retardation time over the length of the line is 1 ms. 50 Hz sinusoidal source functions have a waveform period of 20 ms. Hence, a lumped parameter model for this line amounts to ignoring 1 ms in comparison with 20 ms. But 20 ms is the time required for one full oscillation of source function. Significant change in function value takes place within a quarter cycle – i.e., in 5 ms. Obviously, this power line requires a distributed model even at 50 Hz.

A 1nF ceramic capacitor typically has two leads of 1.5 cm each. The retardation time over 3 cm is 100 ps ($1 \text{ ps} = 10^{-12} \text{ s}$). This corresponds to a frequency value of 10 GHz. Therefore, a lumped parameter model will be satisfactory for frequencies below 10 MHz. A distributed model will be necessary for frequencies >50 MHz.

All circuit elements of arbitrary dimensions can be modelled by lumped elements if all the sources are DC sources. But, no element, of any dimension whatsoever, can be modelled by lumped parameter model to obtain *detailed* circuit solution at and around the instants at which such DC sources are either switched into the circuit or switched out of the circuit. Such switching operations represent very rapid changes in circuit variables and retardation time cannot be ignored in comparison with infinitesimal intervals.

Lumped element
versus Distributed
element.

1.7.2 Linear and Non-linear Elements

Two-terminal elements are classified as *linear* or *non-linear* based on whether the voltage–current relationship of the element satisfies the *linearity property*.

Two variables of time – $x(t)$ and $y(t)$ – satisfy the *property of linearity* if the relation between them is *homogeneous* and *additive* at all t .

Let the relation between the variables be represented by $y(t) = f[x(t)]$.

$y(t) = f[x(t)]$ is *homogeneous* if $f[ax(t)] = af[x(t)]$ for any t , where a is any real number. That is, scaling the variable $x(t)$ by a real number a results in the scaling of the variable $y(t)$ by the same real number a .

$y(t) = f[x(t)]$ is *additive* if $f[x_1(t) + x_2(t)] = f[x_1(t)] + f[x_2(t)]$ for any t . That is, $y(t)$ corresponding to sum of two variables $x_1(t)$ and $x_2(t)$ is equal to the sum of $y(t)$ corresponding to $x_1(t)$ and $y(t)$ corresponding to $x_2(t)$ at any time instant.

Superposition
property and
linearity.

$y(t) = kx(t)$,
 $y(t) = k \frac{dx(t)}{dt}$ and
 $y(t) = k \int_{-\infty}^t x(t)dt$ are linear
relations for a real k .

We may combine the requirements of *homogeneity* and *additivity* into a composite requirement called *superposition property*.

$y(t) = f[x(t)]$ satisfies *superposition property* if

$f[a_1x_1(t) + a_2x_2(t)] = a_1f[x_1(t)] + a_2f[x_2(t)]$ for any combination of real numbers a_1 and a_2 and for any t .

Thus, a two-terminal element is linear if its v - i relationship satisfies the principle of *superposition*.

The simplest case of a *linear* relationship between two variables occurs when $y(t)$ is proportional to $x(t)$.

Let $y(t) = kx(t)$, where k is a real number.

Then, $f[a_1x_1(t) + a_2x_2(t)] = k \times [a_1x_1(t) + a_2x_2(t)] = ka_1x_1(t) + ka_2x_2(t) = a_1 \times kx_1(t) + a_2 \times kx_2(t) = a_1f[x_1(t)] + a_2f[x_2(t)]$.

Therefore, $y(t) = kx(t)$ is a *linear relation* for any real k .

But, a relation does not have to be algebraic for it to be a linear relation. Consider the relation, $y(t) = k \frac{dx(t)}{dt}$. Then,

$$\begin{aligned} f[a_1x_1(t) + a_2x_2(t)] &= k \times \left(\frac{d(a_1x_1(t) + a_2x_2(t))}{dt} \right) \\ &= a_1 \left(k \frac{dx_1(t)}{dt} \right) + a_2 \left(k \frac{dx_2(t)}{dt} \right) \\ &= a_1f[x_1(t)] + a_2f[x_2(t)] \end{aligned}$$

Therefore, $y(t) = k \frac{dx(t)}{dt}$ is a *linear relation*.

Similarly, it can be shown that $y(t) = k \int_{-\infty}^t x(t)dt$ too is a *linear relation*.

Beginners in Circuit Analysis often tend to equate the property of linearity to straight-line nature of functional relationship between the concerned variables. Consider the following relationship.

$y(t) = mx(t) + c$, where m and c are two real numbers. Obviously, the graph of this function will be a straight-line with c as its vertical-axis intercept. But this is not a *linear relation* in the sense of linearity as defined in Circuit Theory.

$$\begin{aligned} f[a_1x_1(t) + a_2x_2(t)] &= m[a_1x_1(t) + a_2x_2(t)] + c \\ &= a_1mx_1(t) + a_2mx_2(t) + c \\ \text{But, } a_1f[x_1(t)] + a_2f[x_2(t)] &= a_1mx_1(t) + a_2mx_2(t) + (a_1 + a_2)c \\ &\neq f[a_1x_1(t) + a_2x_2(t)] \end{aligned}$$

Therefore, $y(t) = mx(t) + c$ is not a *linear relation* in Circuit Theory. It does not satisfy the property of homogeneity and additivity.

Let us examine the linearity property of various two-terminal elements that we have discussed so far.

Consider a two-terminal resistance element. Its v - i relation is $v(t) = Ri(t)$. It is a linear element if the R parameter is a real constant or a function of time alone. The resistance of a piece of conductor is temperature dependent. It may depend on current level in certain cases. Thus, a two-terminal resistance is linear if the temperature is constant and the R parameter is either a constant or is an independent function of time alone.

A two-terminal inductance is described by $v(t) = \frac{d[Li(t)]}{dt}$ in general. If the inductance parameter L is a constant, then two-terminal inductance is a linear element. L can vary with time if the physical geometry of the device changes with time (but independent of electrical variables). The element is linear in that case too. But if L varies as a function of the current in it, then, the element is non-linear.

Two-terminal
resistance, inductance
and capacitance are
linear elements if R , L
and C are either
constants or
independent functions
of time.

Ideal independent
sources are non-linear
elements.

A two-terminal capacitance is described by $i(t) = \frac{d[Cv(t)]}{dt}$ in general. If the capacitance parameter C is a constant, then two-terminal capacitance is a linear element. C can vary with time if the physical geometry of the device changes with time (but independent of electrical variables). The element is linear in that case too. A tuning capacitor in a radio receiver is an example. But if C varies as a function of the charge in it, then, the element is a non-linear one.

A two-terminal ideal independent voltage source is described by the relations $v(t) = E(t)$ (an independently specified function of time) and $i(t) = \text{arbitrary}$. Obviously, this is a non-linear relationship. Thus, an ideal independent voltage source is a non-linear element. Similarly, an ideal independent current source is a non-linear element.

1.7.3 Bilateral and Non-bilateral Elements

Some elements have a v - i relation that depends on the direction of current flow in them. For example, a diode (Fig. 1.7-1). The v - i relation of this two-terminal element is not symmetrical about the vertical axis. This is a non-bilateral element. The current that will flow in the device when it is connected across a battery of V volts will depend on how it is connected. If the terminal marked a is connected to positive terminal of the battery the resulting current will be high. If the terminal marked b is connected to positive terminal of the battery, the resulting current flow will be low.

An element with a voltage-current relation that is odd-symmetric about the vertical axis in the v - i plane is called as a bilateral element. For example, a linear resistor, a linear inductor and a linear capacitor are bilateral elements.

A linear two-terminal element will always be bilateral. However, a multi-terminal element (with more than two terminals) can be non-bilateral even if it is a linear element.

A two-terminal element is non-linear if it is non-bilateral. However, the converse is not true. Consider the v - i curve of a non-linear resistor shown in Fig. 1.7-2. The element is non-linear; but bilateral.

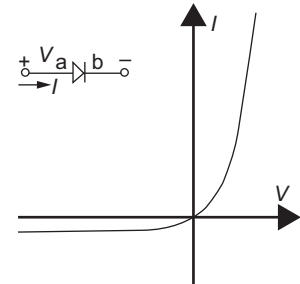


Fig. 1.7-1 Voltage-Current Relationship of a Diode, Non-bilateral

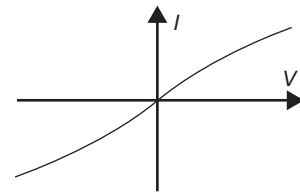


Fig. 1.7-2 V-I Curve for a Non-linear, Bilateral Resistance

1.7.4 Passive and Active Elements

The energy delivered to a two-terminal element from $t = -\infty$ to $t = t$ is given by $E(t) = \int_{-\infty}^t v(t)i(t)dt$. An element is called a *passive element* if the energy delivered to it is always non-negative for any t and for any possible terminal voltage-current conditions of the device. That is, an element is passive if $E(t) \geq 0$ for all t and for all permissible $[v(t), i(t)]$ combinations.

Consider a linear resistance with constant R . Then,

$$E(t) = \int_{-\infty}^t v(t)i(t)dt = R \int_{-\infty}^t [i(t)]^2 dt = \text{a positive number for any } t.$$

Therefore, a resistance is a passive element.

Consider a linear inductance element. Then,

$$E(t) = \int_{-\infty}^t v(t)i(t)dt = L \int_{-\infty}^t [i(t)] \frac{di(t)}{dt} dt = L \int_0^{i(t)} di(t) = \frac{L[i(t)]^2}{2} = \text{a positive number}$$

or zero. (We assume that L is positive). This energy is stored in the magnetic field inside the device. We have assumed that the energy storage inside the inductor at $t = -\infty$ is zero.

Thus, an inductor is a passive element. Similarly, it may be shown that $E(t) = \frac{C[v(t)]^2}{2}$ for a capacitor and that a capacitor is also a passive element.

Condition for passivity of a circuit element.

A positive-valued resistance is a dissipating element.

Inductors and capacitors can deliver energy temporarily; but not for indefinite time.

A positive-valued resistance is a dissipating element. It cannot deliver energy even for a short interval. This can be seen from the equation for power delivered to the resistor. $p(t) = v(t)i(t) = R [i(t)]^2$ is a positive number or zero. Therefore, the power delivered to a resistance cannot be negative at any t and hence resistance will always consume power.

A positive-valued inductance or capacitance is an energy-storing element. It can deliver the energy back to other elements. But it can deliver only as much energy as that was given to it earlier. It cannot generate energy and deliver it. Thus, the energy storage in an inductance or capacitance cannot be employed to deliver energy to other elements indefinitely. The instantaneous power delivered to such an element can be negative; but the area under $p(t)$ waveform from $-\infty$ to any t will be non-negative. In other words, the energy storage in an inductance or in a capacitance can only be zero or positive at any instant.

An independent source is an active element (since it is not a passive element). An independent voltage source can *deliver* any amount of current for any duration. Similarly, an independent current source is also able to deliver. Therefore, independent sources are *active elements*.

1.7.5 Time-invariant and Time-variant Elements

An element is 'time-invariant' if the values of parameters that characterise it are independent of time. Therefore, a two-terminal resistance is time-invariant if R is a constant; a two-terminal inductance is time-invariant if L is a constant and a two-terminal capacitance is time-invariant if C is a constant.

A synchronous generator driven at constant speed by some prime-mover is an example system that contains time-variant elements. The inductance value of various coils in the machine varies with time due to rotation.

In this book, we deal only with circuits comprising a *finite* number of 'lumped, linear, bilateral, time-invariant' two-terminal as well as multi-terminal elements interconnected.

1.8 MULTI-TERMINAL CIRCUIT ELEMENTS

Many electrical devices that are characterised by voltage and current variables at more than two terminals are in common use in electrical engineering. Transformers in power engineering, transistors in electronics engineering, etc., are some examples.

1.8.1 Mutual Inductance Element

In our discussion on inductance earlier, we noted that there is an induced electric field at every point in space around a circuit due to time-varying currents in that circuit as well as in other neighbouring circuits. The work that will be done by this induced electric field when a unit positive test charge is taken over a path is called the induced e.m.f. in that path. Thus, there can be an induced e.m.f. in any element due to time-varying current flowing in the circuit in which it is connected. This e.m.f. component is called *self-induced e.m.f.* Moreover, there can be induced e.m.f. in any element in a circuit due to time-varying currents in nearby, but physically separate, circuits. This e.m.f. is called *mutually induced e.m.f.*

Both self-induced and mutually induced e.m.f.s are neglected in modelling a physical electrical system by two-terminal elements except in the case of those devices that have been specifically designed to strengthen induced e.m.f. components. The self-induced e.m.f. in such a device was modelled by a two-terminal inductance element earlier.

Consider a two-coil system as shown in Fig. 1.8-1. Assume for a moment that $i_2(t)$ is zero. Then the induced e.m.f. in the first coil is entirely due to induced electric field created by its own current. Or, equivalently, the flux linkage in the first coil is entirely due to its own

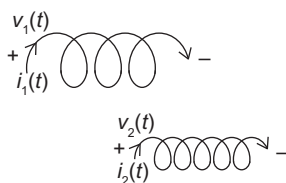


Fig. 1.8-1 A Two-coil Coupled System

current. The flux linkage per ampere in the first coil per unit current in it will give its self-inductance. Let it be L_1 . Then, the voltage that appears across first coil with the second coil kept open is $v_1(t) = L_1 \frac{di_1(t)}{dt}$ V and the flux linkage of first coil with second coil kept open is $\psi_1(t) = L_1 i_1(t)$ Weber-turns. Similarly, the voltage that appears across second coil with the first coil kept open is $v_2(t) = L_2 \frac{di_2(t)}{dt}$ V and the flux linkage of second coil with the first coil kept open is $\psi_2(t) = L_2 i_2(t)$ Weber-turns, where L_2 is the self-inductance of the second coil. Self-inductance of a coil in a multi-coil system is measured by measuring the inductance across the coil with all other coils kept open.

Time-varying current flow in the second coil will produce induced electric field at all points inside the first coil. Therefore, there will be an induced e.m.f. in the first coil due to $i_2(t)$ in the second coil even when the first coil is kept open. This e.m.f. will appear as a potential difference across its terminals. This e.m.f. is called the mutually induced e.m.f. in coil-1 due to coil-2. It will be proportional to $\frac{di_2(t)}{dt}$ and the proportionality constant is defined as M_{12} , the *mutual inductance between coil-1 and coil-2*. The induced electric field created at a point in coil-1 by an increasing current in coil-2 may add to the induced electric field created at the same point in coil-1 by an increasing current in coil-1 itself or may subtract from it. That depends on relative winding directions in the two coils.

The process of mutual induction may also be understood from flux linkage point of view. There is a flux linkage in the first coil due to the current in second coil even when first coil is kept open. This flux linkage is called the *mutual flux linkage in coil-1 due to coil-2*. The value of this mutual flux linkage will be proportional to $i_2(t)$ and the proportionality constant is the mutual inductance M_{12} . Thus M_{12} can be understood as the mutual flux linkage in coil-1 per unit current in coil-2. Rate of change of the mutual flux linkage gives the e.m.f. induced in coil-1 by current in coil-2. The flux linkage created in coil-1 by coil-2 may add to or subtract from the flux linkage created in coil-1 by its own current.

Hence, the voltage that appears across the coil-1 when both coils are carrying current will be $v_1(t) = L_1 \frac{di_1(t)}{dt} \pm M_{12} \frac{di_2(t)}{dt}$ V and total flux linkage in coil-1 is $\psi_1(t) = L_1 i_1(t) \pm M_{12} i_2(t)$.

Similarly, the voltage that appears across coil-2 is $v_2(t) = L_2 \frac{di_2(t)}{dt} \pm M_{21} \frac{di_1(t)}{dt}$ and total flux linkage in coil-2 is $\psi_2(t) = L_2 i_2(t) \pm M_{21} i_1(t)$, where M_{21} is the *mutual inductance between coil-2 and coil-1*.

The value of mutual inductance M_{12} can be measured by measuring the voltage that appears across the open-circuited coil-1 with a known time-varying voltage applied to coil-2. Similarly, the value of mutual inductance M_{21} can be measured by measuring the voltage that appears across the open-circuited coil-2 with a known time-varying voltage applied to coil-1.

Mutual induction arises out of magnetic coupling between two coils. Now, we make an important assumption. We assume that the two-coil system is constructed in such a way that there is magnetic coupling only between them and there is no magnetic coupling between these coils and any other element in the circuit or with the circuit loop itself. This is possible only if the two-coil system is designed to confine the magnetic field almost entirely within itself. A closed core structure employing a magnetic material with high magnetic permeability will be needed in practice to achieve this. Both the coils will be wound around the same core. With this assumption of *confinement of magnetic flux linkage entirely within the device itself*, we can model a two-coil system by an *ideal four-terminal element model*. The symbol of the model is shown in Fig. 1.8-2. L_1 and L_2 are the self-inductances of the coils. M is the mutual inductance between them. The two parallel lines between the coils indicate that they share a common core. The two 'dot' points marked by the side of coils help to decide the polarity of mutual e.m.f. in relation to the self-induced e.m.f. *Increasing current entering the dot point in one coil generates a mutual e.m.f. in other coil with positive polarity at its dot point.*

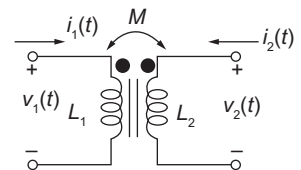


Fig. 1.8-2 A Four-terminal Element Model for a Two-coil System

Experimental measurements will reveal that $M_{12} = M_{21}$ within experimental error for any linear two-coil system.

Experiments will also reveal that $M \leq \sqrt{L_1 L_2}$, where $M = M_{12} = M_{21}$ for a two-coil system.

The number k defined as $k = \frac{M}{\sqrt{L_1 L_2}}$ is called the 'magnetic coupling coefficient'. It has a value between 0 and 1.

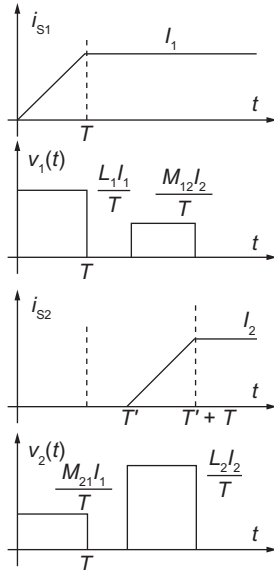


Fig. 1.8-3 Current Source Waveforms and Voltage Appearing Across them During the Current-build up in a Two-coil System

1.8.2 Why Should M_{12} be Equal to M_{21} ?

Consider an experiment with the two-coil system as shown in Fig. 1.8-2. We want to set up a DC current of I_1 in coil-1 and a DC current of I_2 in coil-2 starting from zero-current condition in both coils. We do this by applying an independent current source i_{s1} to coil-1 and the independent current source i_{s2} to coil-2 (Fig. 1.8-3).

The current in first coil ramps up linearly with a slope of I_1/T A/s. This results in $v_1(t) = L_1 I_1 / T$ V and $v_2(t) = M_{21} I_1 / T$ V during $[0, T]$ interval. But the current in second coil during that interval is zero and hence there is no power delivered to the coil-2 during this interval. There is power delivered to coil-1 during this interval. The total energy delivered to coil-1 during $[0, T]$ is $L_1 I_1^2 / 2$ J.

The current in coil-2 ramps up linearly during the interval $[T', T' + T]$ with a slope of I_2/T A/s. This results in $v_1(t) = M_{12} I_2 / T$ V and $v_2(t) = L_2 I_2 / T$ V during $[T', T' + T]$ interval. The total energy delivered to coil-2 is $L_2 I_2^2 / 2$ J. But, now the first coil current is at a constant level of I_1 and hence the mutually induced voltage across the first coil results in additional energy input into the first coil from the current source i_{s1} . This additional energy input is $\frac{M_{12} I_2}{T} \times I_1 \times T = M_{12} I_1 I_2$ J.

Therefore, the total energy delivered to the two-coil system in building up the currents is $\frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{12} I_1 I_2$ J.

Now, assume that we built up the currents in the coils to the same levels using a current source i_{s1} which had a waveform that is delayed by T' second for coil-1 and a current source i_{s2} which had a waveform that started at $t = 0$ for coil-2. Then the total energy required to build up the currents will be $\frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{12} I_1 I_2$ J.

The total energy required to be delivered to the system must be the same in these two situations. Otherwise, we can build up the currents using the procedure resulting in lower energy input and subsequently reduce the currents to zero using the other process (with suitably constructed current sources) to receive some net energy out of the system. That is not possible according to conservation law for energy. Therefore,

$$\frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{12} I_1 I_2 = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{21} I_1 I_2$$

$$\therefore M_{12} = M_{21}$$

Now, we know that the total energy storage in a two-coil system shown in Fig. 1.8-2 is given by $E(t) = \frac{L_1 [i_1(t)]^2}{2} + \frac{L_2 [i_2(t)]^2}{2} + M [i_1(t)] [i_2(t)]$ J. Two-coil system is a passive system. Therefore, the total energy storage in it cannot be negative for any permissible voltage-current condition. Therefore,

$$\frac{L_1 [i_1(t)]^2}{2} + \frac{L_2 [i_2(t)]^2}{2} + M i_1(t) i_2(t) \geq 0$$

$$\therefore L_1 [i_1(t)]^2 + L_2 [i_2(t)]^2 + 2M i_1(t) i_2(t) \geq 0$$

$$\text{i.e., } \left[\sqrt{L_1} i_1(t) + \sqrt{L_2} i_2(t) \right]^2 + 2(M - \sqrt{L_1 L_2}) i_1(t) i_2(t) \geq 0$$

This has to be true for any $[i_1(t), i_2(t)]$. Let us choose $\sqrt{L_1} i_1(t) + \sqrt{L_2} i_2(t) = 0$. This implies that $i_1(t)$ and $i_2(t)$ have opposite polarities for this choice. Therefore, the product of $i_1(t) \times i_2(t)$ will be negative. Therefore, $\sqrt{L_1 L_2}$ has to be $\geq M$ for the last inequality to be true.

$$\therefore M \leq \sqrt{L_1 L_2}$$

The equality sign applies only if the magnetic coupling is so tight that there is no leakage of magnetic flux from the common core and windings. The coupling coefficient

goes to unity under this situation. No physical coil system can have k equal to 1. However, coil systems in power transformers approach this value very closely.

1.8.3 Ideal Dependent Sources

Ideal dependent sources form the second category of multi-terminal elements that we employ in circuit analysis. These models are used extensively in analysis of electronic circuits to model devices like transistors, amplifiers, etc.

They have two terminal pairs. The first terminal pair *senses* either a voltage variable or a current variable at the location where this terminal pair is connected in the circuit. The second pair of terminals delivers either a voltage or a current to the location at which this terminal pair is connected in the circuit. *However, the source function delivered is a function of the variable sensed by the first terminal pair.* That is why they are called *dependent sources*. They are ideal in the sense that, (i) the first terminal pair does not affect the circuit variables in any way and (ii) the source function delivered by second terminal pair depends only on the variable sensed by the first terminal pair and on nothing else.

There are four dependent sources depending on the nature of circuit variable sensed by the first terminal pair and the nature of source function delivered by the second terminal pair.

A *Voltage-Controlled Voltage-Source (VCVS)* senses a voltage variable at some location in the circuit and delivers a source voltage that depends on the sensed voltage at some other location in the circuit. If the source voltage delivered is a linear function of the controlling voltage, the dependent source will be called as *linear VCVS*.

A *Voltage-Controlled Current-Source (VCCS)* senses a voltage variable at some location in the circuit and delivers a source current that depends on the sensed voltage at some other location in the circuit. If the source current delivered is a linear function of the controlling voltage, the dependent source will be called as *linear VCCS*.

A *Current-Controlled Current-Source (CCCS)* senses a current variable at some location in the circuit and delivers a source current that depends on the sensed current at some other location in the circuit. If the source current delivered is a linear function of the controlling current, the dependent source will be called as *linear CCCS*.

A *Current-Controlled Voltage-Source (CCVS)* senses a current variable at some location in the circuit and delivers a source voltage that depends on the sensed current at some other location in the circuit. If the source voltage delivered is a linear function of the controlling current, the dependent source will be called as *linear CCVS*.

The symbols used for the four *linear dependent sources* are shown in Fig. 1.8-4. k_v , k_p , k_z and k_y are real numbers. k_y is a trans-conductance, k_z is a trans-resistance, k_v and k_i are dimensionless.

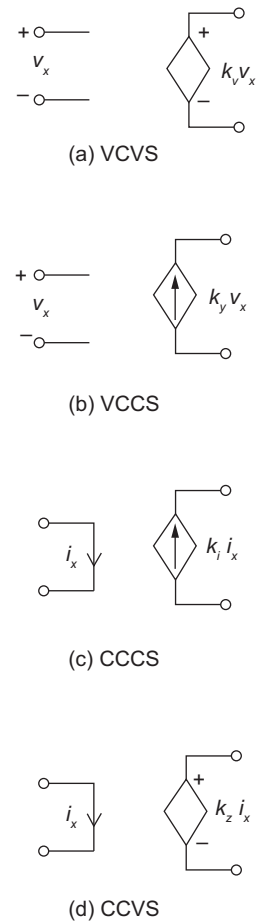


Fig. 1.8-4 Ideal Dependent Sources

1.9 SUMMARY

- *Electric Circuit* is a mathematical model of a real physical electrical system. It is an approximation of Electromagnetic Field Theory.
- Electromagnetic disturbances travel with a finite speed in electrical systems. Electric circuit theory assumes that the largest dimension in the circuit is so small that electromagnetic disturbances take negligible time to cover that distance compared to the time interval required for source quantities in the circuit to change significantly.
- In addition, *lumped parameter circuit theory* assumes the following:
 - (a) The electrostatic field created by charge distribution on an electrical device is confined to space within the device and in the immediate vicinity of the device predominantly. Thus, the terminal voltage across a device and the charge stored in that device can be related through a unique ratio.
 - (b) The induced electric field inside connecting wires and outside the devices is negligible. This makes it possible to assign a unique voltage variable to a device.

- (c) The connecting wires are of infinite conductivity and near-zero thickness. Thus, there are no charges distributed on their surface and no current component is needed to create charge distribution on them. This makes it possible to assign a unique current variable to an electrical device.
- (d) The component of current needed to create the time-varying charge distribution on the surface of the device is negligible except in those devices (capacitors) that are designed to make such current flow the dominant electrical phenomenon in them.
- (e) The induced electric field inside devices is negligible except in those devices (inductors) where induced electric field is the dominant electrical phenomenon by design. Such devices are designed to confine the time-varying flux linkage to space within them.
- (f) The conductivity of metallic conductors employed in capacitors and inductors is infinity. Hence, there is no resistive effect in them.
- A two-terminal element is a mathematical model of an electrical device with a terminal voltage variable $v(t)$ and current variable $i(t)$ assigned to it. These are functions of time only. The entire electrical behaviour of the device can be characterised by these two variables and a relation between them.
- Passive sign convention assigns reference polarity for $v(t)$ with + at one end and – at the other end of the element. Reference direction for $i(t)$ is such that it flows into the + polarity of $v(t)$ from outside the element. The power delivered to the element is given by $p(t) = v(t)i(t)$ with this sign convention.
- The symbols and v - i relations for the three passive two-terminal elements are shown in Fig. 1.9-1.

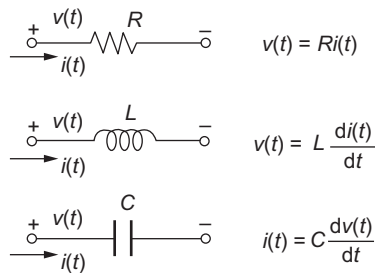


Fig. 1.9-1 Passive Two-terminal Elements

- Ideal independent voltage source is a two-terminal element with its voltage variable specified as a function of time and current variable as a free variable. Ideal independent current source is a two-terminal element with its current variable specified as a function of time and voltage variable as a free variable.
- Circuit elements are classified into *linear* and *non-linear* elements depending on whether their v - i relationships satisfy the *principle of superposition* or not. Independent sources are non-linear elements.
- An element is a *passive element* if the energy delivered to it from $-\infty$ to t is non-negative for all t and for all permissible $[v(t), i(t)]$ combinations. R , L , M and C are passive elements.
- A circuit element is a *time-invariant element* if the parameter of the element is a constant.
- Self-inductance of coil is the flux linkage in it when 1 A flows in it with all other coils in the vicinity kept open. Mutual inductance between two coils is the flux linkage in one coil when it is kept open with 1 A flowing in the other coil.
- For two coupled coils with self-inductance L_1 and L_2 , the mutual inductances M_{12} and M_{21} are equal. The maximum value mutual inductance can have is $\sqrt{L_1 L_2}$ Henry. The ratio between actual value of mutual inductance and its maximum possible value is defined as the *magnetic coupling coefficient*.
- Ideal dependent sources are four-terminal elements. The first terminal pair is connected at some location in the circuit to sense a voltage variable or a current variable there. The second terminal pair delivers either a voltage or a current at the location where it is connected. The value of voltage or current delivered is a function of *controlling variable* that is sensed by the first terminal pair.
- There are four kinds of dependent sources. They are Voltage-Controlled Voltage Source, Voltage-Controlled Current Source, Current-Controlled Voltage Source and Current-Controlled Current Source. A dependent source is a *linear* element if the source quantity is a linear function of the controlling-variable.

1.10 PROBLEMS

1. A fully charged lead-acid battery contains 120 Ampere-hours of charge in it. The terminal voltage of the battery is a function of charge remaining in it and is given by $V = 11(1 + 0.001C)$ V, where C represents the charge that remains in the battery in Ampere-hour units. The battery delivers a current of 10 A to a load. (a) Express the initial charge storage in the battery in

Coulombs. (b) What is the terminal voltage of the battery when it has been discharged to 50% level? (c) A voltage-sensing relay cuts out the battery from the load when the terminal voltage falls below 11.35 V. How long can the battery power the 10 A load? (d) What is the energy delivered by the battery to 10 A load in Joules and kW-h units if the load is kept powered till cut out takes place?

2. A fully charged battery contains 100 AH charge in it. The terminal voltage of battery is a function of the charge that remains in it and the current that is delivered by it. It is given as $V = (11 + 0.01C - 0.02I)$ V, where C is the AH (Ampere-hours) remaining in it and I is the current delivered by the battery in Amperes. This battery is delivering current to a pulsed load that draws current with a period of 5 s as shown in Fig. 1.10-1. The load is kept powered for 8 h. (a) What is the charge that remains in the battery after the load is switched off? (b) What is the value of constant load current that would have resulted in same charge consumption? (c) What is the energy consumed by the load (in Joules and kW-h) in 8 h? (d) What is the value of constant load current that would have consumed same energy in 8 h? What would have been the charge consumption with this value of load current?

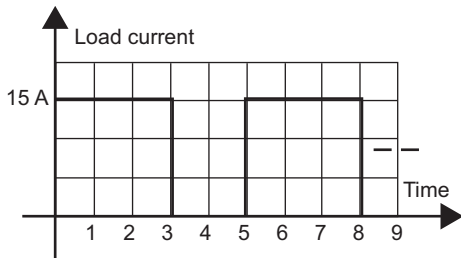


Fig. 1.10-1

3. The wave-shape of current that is delivered to a 12 V lead-acid battery charging is shown in Fig. 1.10-2. The initial charge in the battery is 36,000 C. The voltage across the battery is given by $V = 11.5 + 0.01C$ V, where C is the charge stored in the battery in AH units. The charging current is applied to the battery for 5 h. (a) Express the initial charge in the battery in AH. (b) What is the charge stored in the battery at the end of charging? (c) What is the constant charging current value that would have delivered the same amount of charge to the battery in the same time interval? (d) How much is the energy consumed in charging? [Hint: Assume that the battery voltage remains constant during a cycle of charging current.]

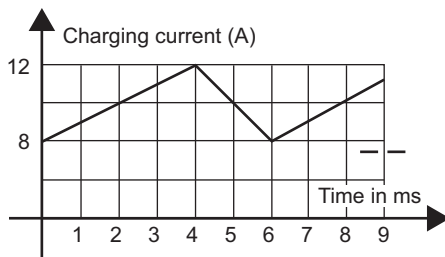


Fig. 1.10-2

4. The voltage across an ideal two-terminal passive element is $v(t) = 10e^{-100t}$ V for $t \geq 0$ and zero for $t < 0$. The current through the element is $i(t) = 0.1e^{-100t}$ A for $t \geq 0$ and zero for $t < 0$. (a) Identify the element and its parameter value. (b) What is the amount of charge that went through the element in the time interval $[0.01 \text{ s}, 0.05 \text{ s}]$? (c) What is the amount of

charge that went through the element in $[0, \infty)$ time-interval? (c) What is the ratio of instantaneous power delivered to it at 0.01 s to the corresponding value at 0 s? (d) What is the total energy delivered to the element? (e) What is the time at which the energy delivered to it reached 99% of total energy delivered to it? Assume that $v(t)$ and $i(t)$ given are as per passive sign convention.

5. The voltage across an ideal two-terminal passive element is $v(t) = 10(1 - e^{-1000t})$ V for $t \geq 0$ and zero for $t < 0$. The current through the element is $i(t) = 0.001e^{-1000t}$ for $t \geq 0$ and zero for $t < 0$. (a) Identify the element and its parameter value. (b) What is the amount of charge that went through the element in $[0, \infty)$ time-interval? (c) What is the total energy delivered to the element? (d) What is the time instant at which the power delivered to the element is a maximum? What is the value of this maximum power? What is the value of energy delivered to the element till that instant? Assume that $v(t)$ and $i(t)$ given are as per passive sign convention.
6. The voltage across an ideal two-terminal passive element is $v(t) = 600\cos(100t)$ V for $t \geq 0$ and zero for $t < 0$. The current through that element as per passive sign convention is $i(t) = 10\sin(100t)$ A for $t \geq 0$ and zero for $t < 0$. (a) Identify the element and its parameter value. (b) Find an expression for the charge that goes through the element as a function of time. (c) Find an expression for instantaneous power delivered to the element as a function of time. (d) Show that the energy delivered to the element till t is non-negative for all t .
7. The current that flows through an ideal independent voltage source with $v(t) = 12$ V is $i(t) = 10 + 10\cos 100\pi t$ A for $t \geq 0$ and 0 A for $t < 0$. Assume passive sign convention. (a) What is the power delivered by the source at $t = 1$ s? (b) What is the change in energy storage in the source between $t = 0$ and $t = 1$ s? Does the energy storage in source increase or decrease with time?

8. There are only three elements in an isolated circuit. Assume passive sign convention. The terminal voltage and current of first element are given by

$$v_1(t) = \begin{cases} 5 + 5(1 - e^{-100t}) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \quad \text{and}$$

$$i_1(t) = \begin{cases} e^{-100t} & \text{A for } t \geq 0 \\ 0 & \text{A for } t < 0 \end{cases}. \quad \text{Corresponding variables for}$$

the second element are $v_2(t) = v_1(t)$ and $i_2(t) = -2$ A. The voltage across the third element is $v_3(t) = v_1(t)$. Identify the third element assuming that it is a passive element; find its parameter value and the current through the third element as a function of time. [Hint: Sum of power delivered by all elements in a circuit is zero.]

9. The v - i characteristic of a passive two-terminal element as per passive sign convention is $v(t) = 100i(t) + 20i(t)|i(t)|$ V. (a) Show that this element is non-linear. (b) Show that this element is a passive element. (c) Show that it is a bilateral element. (d) Find the current flow through the element when the voltage across it is a constant at 100 V. There are

two possible values for the current. How do you choose the correct one?

10. The current $i(t)$ through a passive two-terminal element is a single pulse as shown in Fig. 1.10-3. Plot the voltage across the element if the device is (a) a resistance of $10\ \Omega$ (b) an inductance of $0.5\ \text{H}$ with zero initial energy storage and (c) a capacitance of $10,000\ \mu\text{F}$ with zero initial energy storage.

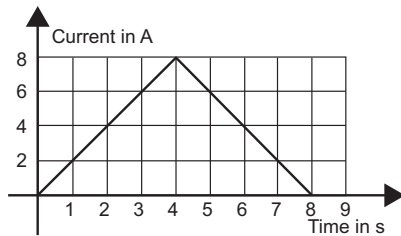


Fig. 1.10-3

11. A $1000\ \mu\text{F}$ two-terminal linear capacitor had a charge storage of $10\ \text{mC}$ across it at $t = 0$. The current delivered by the capacitor out of its positive terminal is given by $i(t) = 2\cos(1000\pi t)\ \text{A}$ for $t \geq 0$. Find and plot the voltage across the capacitor terminals as a function of time.
12. The voltage across a $0.2\ \text{H}$ two-terminal inductance is $v(t) = 10e^{-10t}\ \text{V}$ for $t \geq 0$. It was kept shorted for $t < 0$ with $i(t) = 0.5\ \text{A}$ circulating in it. Assume passive sign convention and find out the current in the inductance, flux linkage in it and energy storage in it at $t = 0.5\ \text{s}$.
13. (a) List the voltage and current values for all the elements in the circuit as in Fig. 1.10-4 as per passive sign convention. (b) Find the unknown voltage V_x . (c) The circuit is known to be a DC circuit. Can the nature of the two-terminal element across which V_x appears be identified?

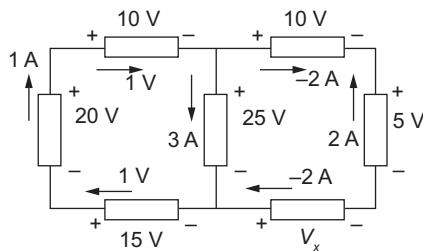


Fig. 1.10-4

14. An isolated circuit contains four elements. The $v-i$ values at a particular time instant for three of them as per passive sign convention are $(5\ \text{V}, 2\ \text{A})$, $(15\ \text{V}, 1\ \text{A})$ and $(10\ \text{V}, 2\ \text{A})$. The voltage across the fourth element at the same instant is seen to be $15\ \text{V}$. (a) Find all possible $v-i$ value combinations for the fourth element. (b) If the circuit is known to be a DC circuit that has been in the present state for a long time, identify whether the fourth element is a passive or active element. (c) Can the nature of fourth element be identified if the circuit is known to be a circuit with time-varying voltages and currents?
15. The self-inductance of one of the two coils wound on a common iron core is found to be $1\ \text{H}$. The second coil has double the number of turns compared to that of the first coil. The coupling coefficient is 0.99 . Find the self-inductance of the second coil and mutual inductance between them.
16. A current source with $i_s(t) = 10t\ \text{A}$ for $0 \leq t \leq 1\ \text{s}$ and zero for all other t is connected to one of the coils of a two-coil system with the second coil kept open. The voltage across current source is seen to be a rectangular pulse of amplitude $10\ \text{V}$ and duration $1\ \text{s}$. The voltage across the other coil is seen to be a rectangular pulse of amplitude $7\ \text{V}$ and duration $1\ \text{s}$. Find the self-inductance of first coil and mutual inductance between the coils.
17. A two-coil system with self-inductance values of $1\ \text{H}$ and $4\ \text{H}$ carries $2\ \text{A}$ in $1\ \text{H}$ coil and $1\ \text{A}$ in $4\ \text{H}$ coil. The total stored energy in the system is seen to be zero under this condition. Find the mutual inductance between the coils and coupling coefficient between them.
18. A two-coil system with $L_1 = 100\ \text{mH}$, $L_2 = 300\ \text{mH}$ and $k = 0.8$ has $i_1(t) = 1.2\sin(100\pi t)\ \text{A}$ and $i_2(t) = 0.5\cos(100\pi t)\ \text{A}$. Assuming the self-flux linkage and mutual flux linkage aid each other, (a) find time-domain expressions for flux linkage in coil-1 and coil-2, (b) find time-domain expressions for voltage across both coils. (c) Repeat (a) and (b) assuming that the self-induced e.m.f. and mutually induced e.m.f. oppose each other in the coils.

Basic Circuit Laws

CHAPTER OBJECTIVES

- To state and explain Kirchhoff's Current Law and Kirchhoff's Voltage Law.
- To explain the different ways of stating these laws.
- To illustrate, with examples, the application of these laws in Circuit Analysis.

This chapter is expected to make the reader proficient in preparing KCL and KVL equations for circuits of reasonable complexity. Moreover, the reader will gain experience in applying these laws along with the element relations, to arrive at a circuit solution for simple resistive circuits.

INTRODUCTION

The 'circuit model' of an electrical system under quasi-static conditions is obtained by modelling the electrical devices using ideal two-terminal elements or multi-terminal elements and interconnecting the elements by means of connecting wires that are assumed to be of infinite conductivity and have near-zero cross-sectional area. Interconnecting electrical elements into a 'circuit' will result in 'junctions', where the connecting wire-ends of two or more two-terminal elements or multi-terminal elements will join together. These junctions are called *nodes* in Circuit Analysis. Further, interconnection of elements will result in the formation of one or more closed paths involving two or more elements. Such closed conductive paths comprising elements and nodes are called *loops* in Circuit Analysis.

Each two-terminal element is completely described by two variables – one terminal voltage variable $v(t)$ and one element current variable $i(t)$. Passive sign convention is assumed in assigning reference directions for these variables. The *element relations* of two-terminal elements are known. The relation is in the form of an equation relating $v(t)$ and $i(t)$ in the case of passive elements. It is in the form of constraints on $v(t)$ or $i(t)$ in the case of ideal independent sources.

Nodes and Loops
in a Circuit.

Loops are closed paths traced through elements and nodes such that no node in the path is visited more than once in tracing the path.

In a lumped parameter circuit with b -elements, n -nodes and l -loops there will be $2b$ variables to be solved for. Assume that there is a unique solution for the circuit analysis problem.

Then *element relations* will provide b equations. The remaining b equations will come from constraints imposed on voltage and current variables between elements. Kirchhoff's laws govern these *interconnection equations* in a circuit.

Each four-terminal element will be described by two voltage variables and two current variables – one voltage variable and one current variable per terminal pair. Two relations tying up or constraining these variables will be available for each such four-terminal element. For instance, consider a voltage-controlled voltage source. There is an open-circuit across its first terminal pair and a voltage source across its second terminal pair.

We treat each four-terminal element as two two-terminal elements with some relation between their voltage/current variables, as far as element count in a circuit is concerned. Let there be b -elements, n -nodes and l -loops in a lumped parameter circuit. Then there are $2b$ variables – b terminal voltage variables and b element current variables – to be solved in the circuit. We call these variables the *element variables*.

Each two-terminal element (and a terminal pair of a four-terminal element) contributes either an equation relating its voltage variable to its current variable or a constraint equation which imposes a constraint on either its current variable or voltage variable. Thus, we get b equations in $2b$ variables from *element relations* alone. These equations are independent of the manner in which the circuit elements are interconnected. They depend only on the nature and parameter value of the individual elements. We call this set of b equations involving $2b$ element variables the *element equation set*.

We need another set of b independent equations on $2b$ element variables to solve for all the element variables. These equations will have to be independent of the *element equation set*. They come from the interconnection details of the circuit. *They depend only on how the elements are interconnected and will not depend on the nature or parameter value of elements*. That is, they depend only on the *topology* of the circuit. This set of b independent equations that summarises the constraints imposed on $2b$ element variables by the interconnection is called the *interconnection equation set*. ‘Element equation set’ and ‘interconnection equation set’ provide the complete set of equations needed to solve all the element variables in a circuit.

The interconnection equation set is obtained by applying two basic conservation laws of physics to the circuit. The laws of conservation of energy and charge have been restated in a form suitable to lumped parameter circuits. Gustav Robert Kirchhoff arrived at the required restatements of these conservation laws in 1857 and they are called *Kirchhoff's Voltage Law (KVL)* and *Kirchhoff's Current Law (KCL)*.

Kirchhoff's Voltage Law imposes a constraint on the voltage variables appearing in a loop in the circuit. Applying this law to a loop in the circuit results in a single constraint equation that involves an algebraic sum of all the voltage variables that appear in the loop. Kirchhoff's Current Law also imposes a constraint on the current variables appearing at a node in the circuit. Applying this law to a node in the circuit results in a single constraint equation involving an algebraic sum of all the current variables that appear at the node. These constraint equations have an algebraic form.

2.1 KIRCHHOFF'S VOLTAGE LAW (KVL)

Consider a DC circuit with many loops as shown in Fig. 2.1-1. Six two-terminal elements are interconnected in this four-node circuit. The interconnection results in seven loops in the circuit. The loops are 1–4–2, 2–5–3, 4–6–5, 1–6–3, 1–4–5–3, 2–4–6–3 and 1–6–5–2. The elements are numbered and the encircled numbers label the nodes.

The circuit is assumed to be in DC steady-state. That is, all the sources in the circuit are assumed to be constant values and all the circuit variables are assumed to be constants in time.

Thus, there is a steady charge distribution at the terminals and on the surface of each two-terminal element in the circuit. The charge distribution produces an electrostatic field everywhere in the circuit. The electrostatic field generated within an element and in the immediate vicinity of an element is proportional to the charge stored on that element (this is

Two nodes connected by a piece of connecting wire is equivalent to a single node.

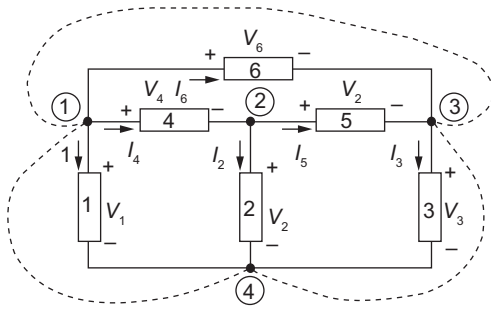


Fig. 2.1-1 A DC Circuit with 6-elements, 4-nodes, 7-loops

a standard assumption in lumped parameter circuit theory as pointed out in Chap. 1). The voltage variables marked in Fig. 2.1-1 are the electrostatic potential differences that exist between the terminals of elements. The connecting wires have zero resistance. In addition, there is no charge distribution on the surface of the connecting wire.

Imagine that we are carrying a unit positive test charge from node-4 back to the same node by moving it along the path shown by a dotted curve in Fig. 2.1-1 in the counter-clockwise direction. The path of travel touches node-1 and node-3. An electrostatic field is a conservative field. Only electrostatic field is present at points lying in the path of travel of the unit positive test charge. Therefore, the work to be done in moving the unit positive test charge around this closed path must be zero. v_1 J is the work to be done in moving a unit positive test charge from node-4 to node-1. v_6 J is the work to be done in moving a unit positive test charge from node-3 to node-1. And, v_3 J is the work to be done in moving a unit positive test charge from node-4 to node-3. Therefore, the work to be done in moving a unit positive test charge from node-4 to node-4 by moving along the dotted path in a counter-clockwise direction = (The work to be done in moving a unit positive test charge from node-4 to node-1) + (The work to be done in moving a unit positive test charge from node-1 to node-3) + (The work to be done in moving a unit positive test charge from node-3 to node-4) = $v_1 - v_6 - v_3$ J. This has to be zero. Therefore, the conservative nature of an electrostatic field leads to the following equation involving the three voltage variables appearing in the loop formed by element-1, element-6 and element-3.

$$v_1 - v_6 - v_3 = 0 \quad (2.1-1)$$

If we had taken a unit positive test charge around the same path in a clockwise direction, we would have obtained the following equation:

$$v_3 + v_6 - v_1 = 0 \quad (2.1-2)$$

Obviously, Eqn. 2.1-2 must be Eqn. 2.1-1 multiplied by -1 . Thus, the direction of traverse of the loop does not matter when we prepare the equation involving voltage variables appearing in that loop.

Now, we dispense with the dotted path altogether. Instead, we traverse the loop formed by element-1, element-6 and element-3 in the clockwise direction, starting from node-4. We collect the *voltage rise amounts across each element* as we go along and enter these quantities into a sum. Obviously, in this process we are calculating the total work to be done in carrying a unit positive test charge through a path outside the elements, but touching the nodes. Therefore, this sum must be zero. The *voltage rise* across element-1 in the direction of traverse (clockwise direction) is v_1 . The *voltage rise* across element-6 in the direction of traverse is $-v_6$. The *voltage rise* across element-3 in the direction of traverse is $-v_3$. Therefore, the sum of voltage rises encountered in traversing the loop formed by element-1, element-6 and element-3 in the clockwise direction is

$v_1 - v_6 - v_3$. We have already verified that this sum must be equal to zero due to the conservative nature of the electrostatic field.

If we collect the *voltage drop amounts across each element* as we traverse the loop in the clockwise direction and enter them as a sum, we get, $-v_1 + v_6 + v_3$. This sum is equal to zero since this is the work to be done in taking a unit positive test charge around the dotted path in a clockwise direction.

Similarly, we could have traversed the loop in a counter-clockwise direction and collected the *voltage rises*. The sum of *voltage rises* encountered will be $v_3 + v_6 - v_1$, which will be equal to zero. If *voltage drops* are collected instead, the sum of *voltage drops* will be $-v_3 - v_6 + v_1$, which will be equal to zero.

On the other hand, we could have entered the element voltages that we encounter when we traverse the loop in the *clockwise* direction in a sum, with the sign for a particular element voltage variable being the same as the *polarity of the variable that we meet first when we reach that element*. $-v_1 + v_6 + v_3$ is the result and that is equal to zero. The sum that is formed in this case is called the *algebraic sum of voltages*.

Alternatively, we could have formed the *algebraic sum of voltages* encountered when we traverse the loop in the *counter-clockwise* direction. $-v_3 - v_6 + v_1$ is the result and that is equal to zero.

Hence, the constraint appearing among voltage variables of elements in a loop can be obtained by one of the following methods:

- (i) Traversing the loop in a *clockwise* direction and equating the sum of *voltage rises* encountered to zero.
- (ii) Traversing the loop in a *clockwise* direction and equating the sum of *voltage drops* encountered to zero.
- (iii) Traversing the loop in a *counter-clockwise* direction and equating the sum of *voltage rises* encountered to zero.
- (iv) Traversing the loop in a *counter-clockwise* direction and equating the sum of *voltage drops* encountered to zero.
- (v) Traversing the loop in a *counter-clockwise* direction and equating the *algebraic sum of voltages* encountered to zero.
- (vi) Traversing the loop in a *clockwise* direction and equating the *algebraic sum of voltages* encountered to zero.

Different equivalent methods for obtaining the voltage constraint equation for a closed path in a circuit.

All the six methods will lead to the same constraint equation. However, in the interest of systematic formulation of circuit equations, it is imperative that we adhere to one method consistently. We will choose the last method in this textbook. Hence, in this textbook, voltage constraint equations are written by traversing the loop in a *clockwise* direction and equating the *algebraic sum of voltages* encountered to zero.

There was nothing special about the particular loop that was chosen to demonstrate the implications of the conservative nature of electrostatic field as far as voltage variables in a circuit are concerned. Hence, the same line of reasoning is applicable to all loops in the circuit. Therefore, at least for a DC circuit under steady-state, we can generalise the conclusions we observed above into the following law:

‘The algebraic sum of voltages in any closed path in a circuit is zero.’ This is called Kirchhoff’s Voltage Law.

Will this law hold good for circuits with time-varying voltage variables too? Now, consider the same circuit with time-varying voltage variables as in Fig. 2.1-2.

Lumped parameter circuit theory assumes that induced electric field component caused by time-varying currents in the circuit is negligible everywhere in the space surrounding the devices. Thus, the only force field that is present in the space outside the circuit elements is the field generated by the coulomb force (that is, the force that depends only on charges and the distance between charges as per inverse square law) arising out of charge distributions on the circuit elements. The quantity of charge stored in each element will change with time in a circuit containing time-varying sources. Therefore, the force field in

The method used in this book for writing voltage constraint equations.

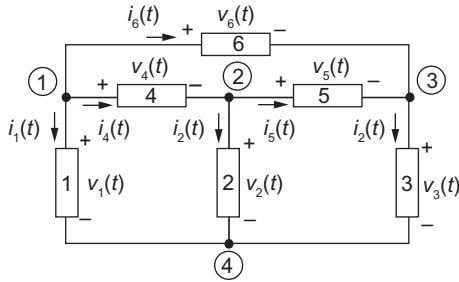


Fig. 2.1-2 A 4-node, 6-element, 7-loop Circuit with Time-Varying Voltage and Currents

the space outside the elements too will vary with time. However, at any instant t , the force field is dependent only on the charges stored in elements at that instant and the spatial distances involved. The term 'electrostatic field' does not imply that the value and direction of this field are constant with time. Rather, it means that the field arises out of 'coulomb force term'. Thus, we can use the term 'electrostatic field' to represent the force field arising out of 'coulomb force terms' even when the charge distributions on the elements vary with time.

The conservative nature of a force field arising out of 'coulomb force' is a direct result of the inverse square dependence on distances displayed by such forces. Therefore, if only coulomb force field is present in the space surrounding a circuit, the work integral – [i.e., $-\oint \vec{E}_s \cdot d\vec{l}$, where $\vec{E}_s$ is the 'coulomb force field' or electrostatic field], evaluated at any instant t over any closed path lying outside the circuit elements, will be zero irrespective of whether the 'coulomb force field' is time-varying or not. Therefore, the algebraic sum of instantaneous value of voltage variables in any loop in the circuit must be zero.

However, we have been accustomed to interpret the work integral $-\oint \vec{E}_s \cdot d\vec{l}$ as the work to be done in carrying a unit positive test charge around a closed path in the $\vec{E}_s$ field quasi-statically. We face a problem in carrying over this interpretation to a time-varying situation. We have to move this unit test charge slowly around the loop. But then, the work we calculate will be the work done against $\vec{E}_s$ at different instants at different locations since we cannot move that test charge with infinite speed in the closed path. That is not the same as the value of integral $-\oint \vec{E}_s \cdot d\vec{l}$ at a particular time-instant t .

However, the only field that is present in the space around the elements in a time-varying circuit is the electrostatic field. Therefore, the field outside the elements at any instant t will be the same as the *electrostatic field* that would exist in a *DC circuit* with the same elements and same geometry, but with all charge variables, voltage variables and current variables in the circuit frozen at the values they had at the time-instant t . Imagine that we are carrying a unit positive test charge around a closed loop in this *frozen circuit*. The work to be done in this process will be zero since the charge was taken around a closed path in a *steady conservative field*. Therefore, the algebraic sum of voltage variables in any loop in this *frozen circuit* must be zero. But, that will imply that, the algebraic sum of instantaneous value of voltage variables, at any instant t , in any loop in the circuit with time-varying voltages, must be equal to zero.

Therefore, KVL is valid under time-varying conditions too. The complete statement of KVL follows:

Kirchhoff's Voltage Law states that the algebraic sum of voltages in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.

It may alternatively be stated in terms of *voltage rises* or *voltage drops* as below.

Kirchhoff's Voltage Law states that the sum of 'voltage rises' in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.

Kirchhoff's Voltage Law states that the sum of 'voltage drops' in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.

The conservative nature of a force field arising out of 'coulomb force' is a direct result of the inverse square dependence on distances displayed by such forces. Such a force field remains a 'conservative field' even if it is varying with time.

Statement of Kirchhoff's Voltage Law for a lumped parameter circuit.

Alternative statements for Kirchhoff's Voltage Law.

The circuit in Fig. 2.1-2 has seven loops. The loops are 1-4-2, 2-5-3, 4-6-5, 1-6-3, 1-4-5-3, 2-4-6-3 and 1-6-5-2, where the numbers refer to the element labels.

The KVL equations for these loops are derived below:

$$\begin{aligned}\text{Loop 1-4-2} & : -v_1(t) + v_4(t) + v_2(t) = 0 \\ \text{Loop 2-5-3} & : -v_2(t) + v_5(t) + v_3(t) = 0 \\ \text{Loop 4-6-5} & : -v_4(t) + v_6(t) - v_5(t) = 0 \\ \text{Loop 1-6-3} & : -v_1(t) + v_6(t) + v_3(t) = 0 \\ \text{Loop 1-4-5-3} & : -v_1(t) + v_4(t) + v_5(t) + v_3(t) = 0 \\ \text{Loop 2-4-6-3} & : -v_2(t) - v_4(t) + v_6(t) + v_3(t) = 0 \\ \text{Loop 1-6-5-2} & : -v_1(t) + v_6(t) - v_5(t) + v_2(t) = 0\end{aligned}$$

We observe that the first three equations will form an independent set of three equations, but the fifth equation can be obtained by adding the first two equations together. When Loop 1-4-2 equation is added to Loop 2-5-3 equation, the term $v_2(t)$ appears twice with opposite signs. Thus, the resulting equation must be that of a loop formed by 1-4-5-3. Similarly, Loop 2-5-3 equation added to Loop 4-6-5 equation should result in Loop 2-4-6-3 equation. The sum of the first three equations must be the same as the fourth equation.

Thus, not all the seven equations are independent. In fact, in a non-degenerate circuit containing b -elements, n -nodes and l -loops, there will be exactly $(b - n + 1)$ loop equations that are independent. l will be greater than or equal to $(b - n + 1)$. We will prove these statements in the last chapter on Network Topology in this book. We accept these statements without proof at this point.

This does not mean that a random selection of $(b - n + 1)$ loop equations from the set of l -loop equations will be an independent set of loop equations. For instance, in the present example, there must be 3 (i.e., $6 - 4 + 1$) independent loop equations. However, the loop equations for Loop 1-4-2, Loop 2-5-3 and Loop 1-4-5-3 are not independent. The third can be obtained by adding the first two. We note that the first two loops are completely contained by the third loop. A *planar circuit* is one that can be drawn on paper without any crossing of connection wires. A *basic window* in a *planar circuit* is a loop that does not contain any other loop within it. It must be intuitively clear that the loop equations for the *basic windows* of the *planar circuit* will form an independent set of loop equations. These *basic windows* of a *planar circuit* are called its '*meshes*'.

Some KVL equations can be obtained by adding other KVL equations together.

The set of all loop equations in a circuit will not be an independent set of equations.

An arbitrary set of loop equations may not form an independent set of equations.

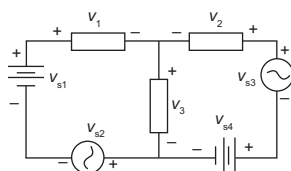


Fig. 2.1-3 Circuit for Example 2.1-1

EXAMPLE: 2.1-1

The source voltages of four independent voltage sources in the circuit in Fig. 2.1-3 are given as $v_{s1} = 10$ V, $v_{s2} = 10\sin 100t$ V, $v_{s3} = 10\cos 100t$ V and $v_{s4} = 10$ V. v_3 is observed to have a zero average value. The time-varying component of v_1 is seen to be $10\sin(100t - 30^\circ)$ V. Find v_1 , v_2 and v_3 as functions of time.

SOLUTION

v_3 is stated to have a zero average. This implies that v_3 has a zero DC content. Thus, v_3 can be written as $v_3 = A\sin(100t - \theta)$ V, where A and θ are to be found. v_1 is stated to have, a time-varying component of $10\sin(100t - 30^\circ)$ V. It may have a DC content too. Thus, $v_1 = B + 10\sin(100t - 30^\circ)$ V, where B is to be found. v_2 may contain both DC and time-varying components. Thus, $v_2 = C + D\sin(100t - \phi)$ V, where C , D and ϕ have to be found.

Apply KVL to the first loop. The KVL equation is:

$$\begin{aligned}-v_{s1} + v_1 + v_3 + v_{s2} &= 0 \\ \text{i.e., } -10 + (B + 10\sin(100t - 30^\circ)) + A\sin(100t - \theta) + 10\sin 100t &= 0\end{aligned}$$

This equation involves some constants and some sinusoidal functions. This equation is the result of applying KVL to a loop in a circuit. Therefore, this equation has to be true at all t . This equation can be split into two equations that have to be satisfied simultaneously. This is because a constant cannot be balanced by a sinusoidal function in an equation for all t .

$$\text{i.e., } -10 + B = 0 \text{ and } 10\sin(100t - 30^\circ) - A\sin(100t - \theta) + 10\sin 100t = 0$$

The first equation yields $B = 10$ V. Second equation is simplified by employing trigonometric identities as below:

$$5\sqrt{3}\sin 100t - 5\cos 100t + (A\cos\theta)\sin 100t - (A\sin\theta)\cos 100t + 10\sin 100t = 0$$

This equation can be true for all t only if the coefficient of $\sin 100t$ is zero and the coefficient of $\cos 100t$ is zero independently.

$$\therefore 5\sqrt{3} + A\cos\theta + 10 = 0 \text{ and } -5 - A\sin\theta = 0$$

$$\therefore A\cos\theta = -10 - 5\sqrt{3} = -18.66; A\sin\theta = -5$$

$$\therefore A = 19.32 \text{ and } \theta = \tan^{-1} \frac{-5}{-18.66} = 180 + \tan^{-1} \frac{5}{18.66} = 195^\circ$$

$$\therefore v_1 = 10 + 10\sin(100t - 30^\circ) \text{ V and } v_3 = 19.32 \sin(100t - 195^\circ) \text{ V.}$$

Now, apply KVL in the outer loop to get

$$-v_{s1} + v_1 - v_2 + v_{s3} + v_{s4} + v_{s2} = 0$$

$$\text{i.e., } -10 + (10 + 10\sin(100t - 30^\circ)) - C - D\sin(100t - \phi) + 10\cos(100t) + 10 + 10\sin 100t = 0$$

$$(10 + 10\sin(100t - 30^\circ)) - C - D\sin(100t - \phi) + 10\cos(100t) + 10\sin 100t = 0$$

$$\therefore C = 10 \text{ and}$$

$$10\sin(100t - 30^\circ) - D\sin(100t - \phi) + 10\cos 100t + 10\sin 100t = 0$$

$$8.66\sin 100t - 5\cos 100t - (D \cos\phi) \sin 100t + (D \sin\phi) \cos 100t + 10\cos 100t + 10\sin 100t = 0$$

$$\therefore D \cos\phi = 18.66; D \sin\phi = -5 \Rightarrow D = 19.32 \text{ and } \phi = \tan^{-1} \frac{-5}{18.66} = -15^\circ$$

$$\therefore v_2 = 10 + 19.32\sin(100t + 15^\circ) \text{ V.}$$

Therefore, $v_1 = 10 + 10\sin(100t - 30^\circ)$ V, $v_2 = 10 + 19.32\sin(100t + 15^\circ)$ V and $v_3 = 19.32\sin(100t - 195^\circ)$ V is the required answer.

The key to the solution of this problem is the point that KVL has to be satisfied at all time instants.

EXAMPLE: 2.1-2

Express the terminal voltages of elements 2, 3 and 5 in terms of terminal voltages of elements 1, 4 and 6 in the circuit in Fig. 2.1-4.

SOLUTION

Applying KVL in the Loop 1-4-2, we get, $-v_1(t) + v_4(t) + v_2(t) = 0$.

$$\therefore v_2(t) = v_1(t) - v_4(t)$$

Applying KVL in the Loop 4-5-6, we get, $v_4(t) + v_5(t) - v_6(t) = 0$.

$$\therefore v_5(t) = v_6(t) - v_4(t)$$

Applying KVL in the Loop 1-6-3, we get, $-v_1(t) + v_6(t) + v_3(t) = 0$.

$$\therefore v_3(t) = v_1(t) - v_6(t).$$

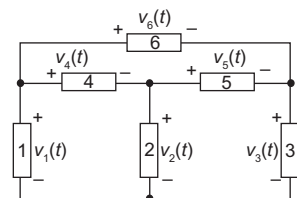


Fig. 2.1-4 Circuit for Example 2.1-2

Continuity
equation for currents.

2.2 KIRCHHOFF'S CURRENT LAW (KCL)

The law of conservation of charge states that charges can neither be created nor destroyed in a given volume. Hence, if the positive charge that flows into a volume at any instant t exceeds the positive charge that flows out of the volume at the same instant, then, the net charge stored inside the volume must be increasing at that instant. Similarly, if the positive charge that flows out of the volume exceeds the positive charge that flows into the volume at that instant, then, the net charge stored within must be decreasing at that instant. Therefore, the *net positive current* that flows into the volume at an instant t must be equal to the *rate of change of net charge* stored within that volume at that instant. A mathematical statement of this fact is called the *continuity equation for currents*.

If, for some reason or the other, the net charge inside the volume is either constrained to remain at zero at all instants or is constrained to remain at some constant value at all instants, then, the *net positive current* that flows into the volume must be zero at all instants.

Lumped parameter circuit theory assumes that the surface charge distribution on the surface of connecting wires is negligible at all instants of time. There is surface charge distribution on all circuit elements other than the connecting wires. In general, these surface charge distributions are time-varying too. However, the positive charge and the negative charge distributed on the surface of any two-terminal or four-terminal element are equal in magnitude at all time instants under quasi-static conditions. Therefore, if we consider a volume that contains some circuit elements completely within, those elements will contribute only a net charge of zero to the net charge storage within the volume. The situation would, however, be different if the volume intersects some element. For instance, consider a volume that encloses only one of the plates of a capacitor. Then, there will be net charge storage within the volume and that may change with time too.

Therefore, we restrict ourselves to a volume that intersects connecting wires at many places without enclosing or intersecting even a single circuit element or a volume that intersects connecting wires at many places and completely encloses one or more circuit elements. We do not permit the volume to intersect any circuit element.

Since an element completely enclosed within a volume does not contribute to net charge within the volume and since the connecting wires have only negligible surface charge distributions on them, it follows that, the net charge contained in a volume chosen the way suggested in the previous paragraph will be zero at all t . Therefore, the rate of change of net charge will also be zero at all t . Then, by continuity equation for currents, the *net positive current that flows into the volume through the wires must be zero at all time-instants*.

A *node* in a circuit is a part of the connecting wire. Therefore, there is no charge storage at a *node* in a circuit as per the assumptions employed by the lumped parameter circuit theory. Therefore, there is no rate of change of charge storage. Consider a special volume – a volume that encloses a node in a circuit and intersects all the wires connected at that node. Then, the reasoning outlined above leads us to the conclusion that the *net positive current* that flows into the volume through all the connecting wires that were intersected by the volume (i.e., all the wires connected together at that node) should be zero at all t . Equivalently, we may state that, *net positive current* that flows out of the volume must be zero at all t .

Consider a volume denoted by the *dotted* circle around node-2 in the circuit in Fig. 2.2-1. This volume intersects three wires and encloses the node-2. It does not enclose any circuit element nor does it intersect any circuit element other than the connecting wires. Therefore, the *net positive current flowing out of the volume* must be zero. This fact leads to the following equation:

$$-i_4(t) + i_2(t) + i_3(t) = 0 \quad (2.2-1)$$

$i_4(t)$ is a current that flows *into* the volume. We need a *minus* sign to make it a current that flows *out* of the volume. Hence, the minus sign in front of $i_4(t)$ in the Eqn. 2.2-1.

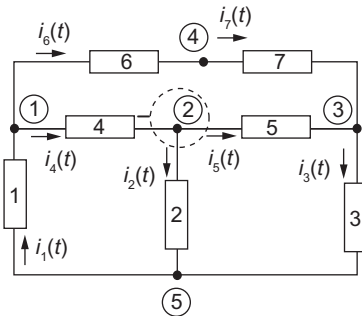


Fig. 2.2-1 Circuit for Illustrating Kirchhoff's Current Law

We could have arrived at an equation containing the same information as in Eqn. 2.2-1 by stipulating that *net positive current flowing into the volume* must be equal to zero. The equation that results will be:

$$i_4(t) - i_2(t) - i_5(t) = 0 \quad (2.2-2)$$

Equation 2.2-2 is, obviously, Eqn. 2.2-1 multiplied by -1 and contains the same information.

We could have arrived at Eqn. 2.2-1 by stipulating that the *algebraic sum* of currents *leaving* a node must be equal to zero. 'Algebraic sum' in this case implies that if a particular current variable has its *reference direction* pointing *towards* the node, then, it has to enter the equation with a *negative* sign. If a particular current variable has its *reference direction* pointing *away* from a node, it has to be entered in the equation with a *positive* sign.

Similarly, we could have arrived at Eqn. 2.2-2 by stipulating that the *algebraic sum* of currents *entering* a node must be equal to zero. 'Algebraic sum' in this case implies that if a particular current variable has its *reference direction* pointing *towards* the node, then, it has to enter the equation with a *positive* sign. If a particular current variable has its *reference direction* pointing *away* from a node, it has to be entered in the equation with a *negative* sign.

Obviously, all the four methods of arriving at the node equation are equivalent. However, in the interest of a systematic procedure, we use the stipulation that the *algebraic sum* of currents *leaving* a node must be equal to zero. We are ready to state the Kirchhoff's Current Law now.

Kirchhoff's Current Law (KCL) states that the algebraic sum of currents leaving a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.

KCL at a node can be stated in these alternative ways:

Kirchhoff's Current Law (KCL) states that the algebraic sum of currents entering a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.

Kirchhoff's Current Law (KCL) states that the sum of currents entering a node in a lumped parameter circuit through some wires must be equal to the sum of currents leaving the same node through the remaining wires on an instant-to-instant basis.

KCL equations at all nodes of the circuit shown in Fig. 2.2-1 are derived below:

$$\begin{aligned} \text{Node-1 : } & -i_1(t) + i_4(t) + i_6(t) = 0 \\ \text{Node-2 : } & -i_4(t) + i_2(t) + i_5(t) = 0 \\ \text{Node-3 : } & i_3(t) - i_5(t) - i_7(t) = 0 \\ \text{Node-4 : } & -i_6(t) + i_7(t) = 0 \\ \text{Node-5 : } & i_1(t) - i_2(t) - i_3(t) = 0 \end{aligned}$$

Statement of
Kirchhoff's Current
Law.

Alternative forms
of KCL.

There will be $(n-1)$ independent KCL equations at nodes in an n -node lumped parameter circuit.

'Supernode' defined.

We note that the sum of these equations will be of $0 = 0$ form. This indicates that these five equations do not form an independent set of equations. If we add all KCL equations derived for all the nodes of a circuit, a particular current variable that enters some equation with a positive sign will necessarily enter some other equation in the set with a negative sign. After all, an element has to get connected to two nodes. Therefore, all terms on the left-hand side of the sum will get cancelled. Assume that we discard the KCL equation at any one node. At least two elements must be connected to any node. Therefore, the sum of four of the five KCL equations will have at least two current variables present on the left-hand side. Therefore, any set of four KCL equations will be an independent set of equations. *Thus, in general, there will be $(n-1)$ independent KCL equations in an n -node circuit.*

We had earlier accepted the fact that there will be $(b-n+1)$ independent KVL equations for a b -element, n -node, l -loop lumped parameter circuit. $(b-n+1)$ independent KVL equations together with $(n-1)$ independent KCL equations make the required b interconnection equations needed to solve the circuit.

It is possible to arrive at a more general form of KCL applicable to lumped parameter circuits by considering a closed surface that encloses more than one node along with one or more elements. We have reasoned earlier in this section that the net charge contained inside such a closed surface must be equal to zero. Therefore, the algebraic sum of currents leaving such a closed surface must be equal to zero on an instant-to-instant basis. Such a closed surface will contain two or more nodes and all the elements that are connected between the nodes are within the closed surface. Such a closed surface is called a *supernode*.

How many nodes can a circuit with n nodes have? Taking two at a time, there are ${}^nC_2$ supernodes that contain two nodes each. Similarly, there are ${}^nC_3$ supernodes that contain three nodes each (see the circuit in Fig. 2.2-2).

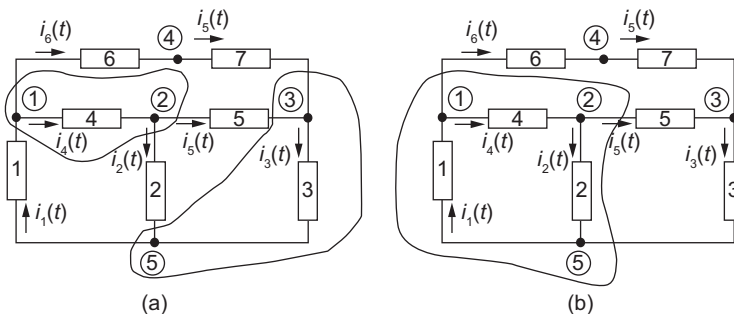


Fig. 2.2-2 (a) Circuit Showing Two Supernodes with Two Nodes Each
(b) Circuit Showing a Supernode that Contains Three Nodes

KCL for a supernode can be obtained by adding the KCL equations for all the nodes that are included within the supernode.

Two supernodes, each containing two nodes, are shown in the circuit in Fig. 2.2-2 (a). One supernode containing three nodes is shown in the circuit in Fig. 2.2-2 (b).

Kirchhoff's Current Law is applicable to supernodes too.

Therefore, the KCL equation for the supernode containing node-1 and node-2 is obtained as $-i_1(t) + i_6(t) + i_2(t) + i_5(t) = 0$. Obviously, this must be the sum of KCL equations written for node-1 and node-2. Similarly, the KCL equation for the supernode in the circuit Fig. 2.2-2 (b) must be the sum of KCL equations written for node-1, node-2 and node-5. This may be verified.

The total number of KCL equations that can be written for an n -node circuit is equal to ${}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$. This series has a sum equal to $2^n - 2$.

Only $(n-1)$ independent equations from these $2^n - 2$ KCL equations can be used for solving the circuit.

EXAMPLE: 2.2-1

Find the power delivered by all the sources in the circuit in Fig. 2.2-3.

SOLUTION

Currents through the voltage sources and voltage across the current sources have to be obtained first. The circuit with all the nodes and reference directions for variables identified is shown in Fig. 2.2-4.

Applying KCL at node-A, we get, $-5 - (-2) - i_1 = 0 \Rightarrow i_1 = -3$ A

Applying KCL at node-B, we get, $(-2) + (-2) - i_2 = 0 \Rightarrow i_2 = -4$ A

Applying KCL at node-C, we get, $5 - (-2) + i_3 = 0 \Rightarrow i_3 = -7$ A

Applying KVL in the loop $I_1 - V_1 - V_3$, we get, $v_1 - 10 - 5 = 0 \Rightarrow v_1 = 15$ V

Applying KVL in the loop $I_3 - V_1 - V_2$, we get, $v_3 - 10 + (-10) = 0 \Rightarrow v_3 = 20$ V

Applying KVL in the loop $I_2 - V_2 - V_3$, we get, $-v_2 - (-10) - 5 = 0 \Rightarrow v_2 = 5$ V

Power delivered by an element is given by $-vi$, where v and i are its voltage and current variables as per passive sign convention.

$\therefore$ Power delivered by I_1 source $= -v_1 \times 5$ A $= -75$ W

Power delivered by I_2 source $= -v_2 \times (-2)$ A $= 10$ W

Power delivered by I_3 source $= -v_3 \times (-2)$ A $= 40$ W

Power delivered by V_1 source $= -10$ V $\times i_1 = 30$ W

Power delivered by V_2 source $= -(-10$ V $\times i_2 = -40$ W

Power delivered by V_3 source $= -5$ V $\times i_3 = 35$ W

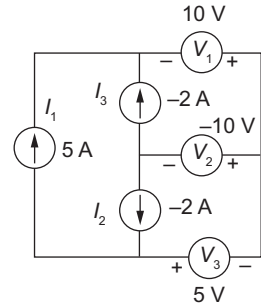


Fig. 2.2-3 Circuit for Example 2.2-1

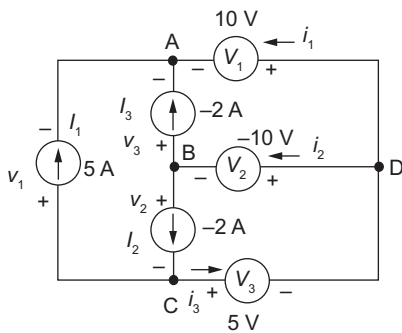


Fig. 2.2-4 Circuit with Nodes and Reference Directions Identified

EXAMPLE: 2.2-2

Express $i_2(t)$, $i_4(t)$ and $i_5(t)$ in terms of $i_1(t)$, $i_3(t)$ and $i_6(t)$ in the circuit shown in Fig. 2.2-5.

SOLUTION

Applying KCL at node-1, we get, $-i_1(t) + i_4(t) + i_6(t) = 0 \Rightarrow i_4(t) = i_1(t) - i_6(t)$

Applying KCL at node-3, we get, $-i_5(t) + i_3(t) - i_6(t) = 0 \Rightarrow i_5(t) = i_3(t) - i_6(t)$

Applying KCL at node-2, we get, $-i_4(t) + i_2(t) + i_5(t) = 0 \Rightarrow i_2(t) = i_4(t) - i_5(t) = i_1(t) - i_3(t)$

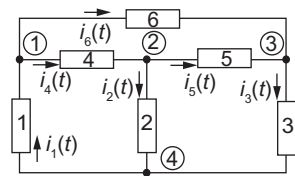


Fig. 2.2-5 Circuit for Example 2.2-2

EXAMPLE: 2.2-3

Fig. 2.2-6 shows the connecting wires in a part of a circuit. Some of the currents are specified for $t \geq 0$ in Fig. 2.2-6. The current i_2 is seen to be a constant in time. The current i_3 is seen to approach zero as $t \rightarrow \infty$. Find i_1 , i_2 and i_3 for $t \geq 0$.

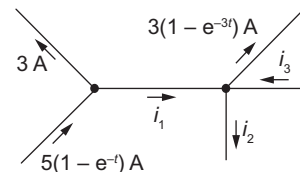


Fig. 2.2-6 Part of a Circuit Referred to in Example 2.2-3

SOLUTION

Applying KCL at node-1, we get, $3 + i_1 - 5(1 - e^{-t}) = 0 \Rightarrow i_1 = 2 - 5e^{-t}$ A
 i_2 is stated to be a constant in time. Let $i_2 = A$. i_3 is stated to approach zero as $t \rightarrow \infty$. This implies that there is no DC component in i_3 . It may contain both e^{-t} and e^{-3t} components. Let $i_3 = Ce^{-t} + De^{-3t}$ A. Then, applying KCL at the second node, we get,

$$\begin{aligned} -i_1 + 3(1 - e^{-3t}) - i_3 + i_2 &= 0 \\ \text{i.e., } -2 + 5e^{-t} + 3 - 3e^{-3t} - Ce^{-t} - De^{-3t} + A &= 0 \\ \text{i.e., } (A+1) + e^{-t}(5-C) + e^{-3t}(-3-D) &= 0 \end{aligned}$$

KCL remains true at all t . Hence, the last equation must be valid for all $t \geq 0$. No time-varying function can remain equal to a constant unless that function itself is a constant. Thus, $(A+1)$ term in the last equation cannot be balanced by e^{-t} and e^{-3t} terms for all $t \geq 0$. Therefore, $(A+1)$ has to be zero.

$$\therefore A = -1 \Rightarrow i_2 = -1 \text{ A.}$$

A term involving e^{-t} cannot get balanced by another term that involves e^{-3t} for all $t \geq 0$. Therefore, coefficient of e^{-t} must be zero and coefficient of e^{-3t} too must be zero. Therefore, $(5-C) = 0$ and $(3+D) = 0$.

$$\therefore C = 5 \text{ and } D = -3 \Rightarrow i_3 = (5e^{-t} - 3e^{-3t}) \text{ A.}$$

The solution is marked in Fig. 2.2-7.

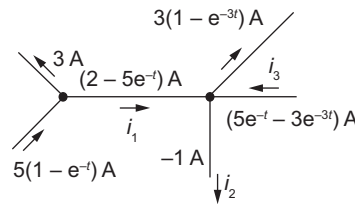


Fig. 2.2-7 Solution for Example 2.2-3

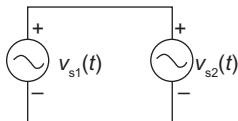


Fig. 2.3-1 Two Ideal Independent Voltage Sources in Parallel

2.3 INTERCONNECTIONS OF IDEAL SOURCES

Interconnecting two or more ideal voltage sources in a loop may lead to a degenerate circuit at times. Similarly, interconnecting two or more ideal current sources in series may lead to a degenerate circuit.

Consider the interconnection of two ideal independent voltage sources as shown in Fig. 2.3-1. KVL requires that $-v_{s1}(t) + v_{s2}(t) = 0$. Therefore, $v_{s1}(t)$ has to be equal to $v_{s2}(t)$ at all time instants. If they are not equal to each other, then, either KVL has to yield or the ideal sources have to yield. KVL cannot yield since it is only a disguised form of law of conservation of energy. Therefore, KVL has to be obeyed by a circuit at all instants.

There are two ways out of this impasse for a case where $v_{s1}(t) \neq v_{s2}(t)$. The first is to declare that *two ideal independent voltage sources cannot be connected in parallel unless their terminal voltages are equal to each other at all instants of time*. That is, we call such connections *illegal* if $v_{s1}(t) \neq v_{s2}(t)$, but that is an evasive method.

The correct way to resolve the issue is to recognise that *ideal independent voltage source* is a model for a practical electrical device, and, as in the case of any model, this model too has its range of applicability. Connecting a practical voltage source in parallel with another practical voltage source is a context in which *ideal independent voltage source model* cannot describe the circuit behaviour satisfactorily. In fact, the model is not satisfactory even when $v_{s1}(t) = v_{s2}(t)$. This is because there is no way to find out the amount of current that will flow in the circuit. Any amount of current can flow at any instant in such a circuit.

Thus, the only correct way to model a circuit that involves parallel connections of voltage sources (more generally, loops comprising only voltage sources) is to take into account the parasitic elements that are invariably associated with any practical voltage source. A somewhat detailed model for the two-source system is shown in Fig. 2.3-2.

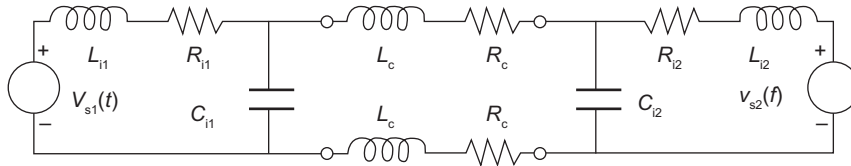


Fig. 2.3-2 A Detailed Model for a Circuit with Two Voltage Sources in Parallel

L_{i1} and L_{i2} represent the internal inductance of the sources, C_{i1} and C_{i2} represent the terminal capacitance of the sources and R_{i1} and R_{i2} represent the internal resistance of the sources. L_c and R_c represent the inductance and resistance of the connecting wires. Obviously, two practical voltage sources can be connected in parallel even if their open-circuit electromotive forces (e.m.f.s) are not equal at all t ; only that they cannot be modelled by *ideal independent voltage source model*.

Two ideal independent current sources in series raise a similar issue (see Fig. 2.3-3). KCL requires that $i_{s1}(t) = i_{s2}(t)$ for all t . Even if this condition is satisfied, there is no way to obtain the voltages appearing across the current sources. Therefore, the correct model to be employed for practical current sources that appear in series in a circuit is a detailed model that takes into account the parasitic elements associated with any practical device. More generally, if there is a node in a circuit where only current sources are connected, then, those current sources cannot be modelled by *ideal independent current source model*.

Similar situations may arise in modelling practical dependent sources by *ideal dependent source models*. In all such cases we have to make the model more detailed in order to resolve the conflict that arises between Kirchhoff's laws and *ideal* nature of the model.

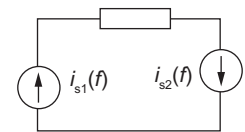


Fig. 2.3-3 Two Ideal Independent Current Sources in Series with Another Element

2.4 ANALYSIS OF A SINGLE-LOOP CIRCUIT

The circuit analysis problem involves finding the voltage variable and current variable of every element as functions of time, given the source functions. *Source functions* are the time-functions describing the e.m.f. of independent voltage sources and source currents of independent current sources. They are also called the *excitation functions*. If the circuit contains b -elements, there will be $2b$ variables to be solved for. Some of them will be known in the form of source functions, while others have to be solved for.

Element relation of each element gives us one equation per element. Thus, there are b equations arising out of element relations. The remaining b equations are provided by the interconnection constraints. These equations are obtained by applying KCL at all nodes except one and KVL in all meshes (in the case of a planar circuit).

Theoretically speaking, that is all there is to circuit analysis. However, systematic procedures for applying element relations, KVL equations and KCL equations would be highly desirable when it comes to analysis of complex circuits. Moreover, the fact that there are $2^n - 2$ KCL equations for an n -node circuit and only $(n - 1)$ of them are independent, calls for a systematic procedure for writing KCL equations. Similarly, there will be l KVL equations for a circuit with l -loops and only $(b - n + 1)$ of them will be independent. This, again, calls for some systematic procedures for extracting a set of $(b - n + 1)$ *independent* KVL equations.

Such systematic procedures are indeed available in Circuit Theory. We will look at such procedures in detail in later chapters. However, we try to gain some experience in applying Kirchhoff's laws and element relations to simple circuits in this section.

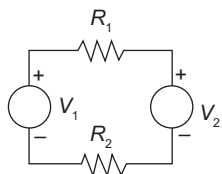


Fig. 2.4-1 Single-Loop in the Circuit in Example 2.4-1

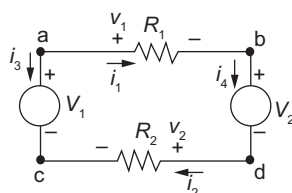


Fig. 2.4-2 Variable Assignment and Reference Directions in the Circuit in Example 2.4-1

EXAMPLE: 2.4-1

Refer to Fig. 2.4-1. $V_1 = 20\text{ V}$, $V_2 = 5\text{ V}$, $R_1 = 5\ \Omega$ and $R_2 = 2.5\ \Omega$. Find all element voltages and element currents. Also, find the power delivered to all elements.

SOLUTION

The first step in the analysis is to assign reference directions for variables. The passive sign convention is employed to decide the reference direction for current after reference polarity for voltage is decided arbitrarily. Or, reference polarity for voltage can be decided in compliance with passive sign convention after deciding reference direction for current arbitrarily. One has to start analysing from some point. Let us start at the first source terminal and move through the circuit in a clockwise direction.

The source function is already assigned a polarity. This does not prevent us from assigning another voltage variable to the first source with any polarity that we may decide. However, there is simply no reason to do so. Therefore, in the case of a voltage source we accept the polarity of source function itself as the reference polarity for element voltage. Obviously, a new voltage variable is also not required. The source function itself is the value of the voltage variable for an ideal independent voltage source.

Now, we have to assign a current variable with its reference direction entering the positive polarity from outside. This current is called i_3 (see Fig. 2.4-2.)

Next, we reach the resistor R_1 . We assign positive polarity of its voltage variable at the first terminal that we come across, that is the left terminal of the resistor. Then, its current variable must enter from left as per the passive sign convention.

Variable assignment and reference direction assignment for the remaining elements are completed in a similar manner and the final variable assignment is shown in Fig. 2.4-2. The nodes in the circuit are also identified in the circuit and labelled.

The next step in the analysis is to apply KCL at any three nodes. Let us apply KCL at node-a, node-b and node-c. The result will be that $-i_3 = i_1 = i_4 = i_2$. Next, we apply KVL in the loop to get $-V_1 + v_1 + V_2 + v_2 = 0$.

The next step in the analysis is to make use of the element relations. We have already made use of the element relations of the voltage sources to set the voltage variables for the sources at their source function values themselves. The remaining elements are two resistors. The element relation for a resistor is given by Ohm's Law. The relations are:

$$\begin{aligned} v_1 &= R_1 i_1 \\ v_2 &= R_2 i_2 \end{aligned}$$

Now, these relations as well as the fact that $-i_3 = i_1 = i_4 = i_2$ are applied in the KVL equation to simplify it as $-V_1 + R_1 i_1 + R_2 i_1 + V_2 = 0$. We note that we have eliminated i_2

by using $i_1 = i_2$. Solving this equation, we get, $i_1 = \frac{V_1 - V_2}{R_1 + R_2}\text{ A}$.

Now, the voltage across each resistor can be worked out by using its element relation once again.

$$v_1 = R_1 i_1 = \frac{R_1 (V_1 - V_2)}{R_1 + R_2}\text{ V and } v_2 = R_2 i_2 = R_2 i_1 = \frac{R_2 (V_1 - V_2)}{R_1 + R_2}\text{ V.}$$

The currents through the voltage sources can be noted as $i_3 = -i_1 = \frac{V_2 - V_1}{R_1 + R_2}$ A and $i_4 = i_1 = \frac{V_1 - V_2}{R_1 + R_2}$ A.

Substituting the numerical values in the example, we get,

$$i_1 = i_2 = -i_3 = i_4 = 2 \text{ A.}$$

$$v_1 = 10 \text{ V; } v_2 = 5 \text{ V}$$

This solution is marked in Fig. 2.4-3. We note that the -2 A flowing into the first source can be marked as $+2 \text{ A}$ if the reference direction is reversed. The value of -2 A implies that positive current flows out from the positive terminal of that source.

The power delivered to an element is given by vi , where v and i are its voltage variable and current variable, respectively, as per passive sign convention.

$$\text{Power delivered to } 20 \text{ V source} = (-2 \text{ A}) \times (+20 \text{ V}) = -40 \text{ W}$$

$$\text{Power delivered to } 5 \text{ V source} = (+2 \text{ A}) \times (+5 \text{ V}) = +10 \text{ W}$$

$$\text{Power delivered to } 5 \Omega \text{ resistance} = (+2 \text{ A}) \times (+10 \text{ V}) = +20 \text{ W}$$

$$\text{Power delivered to } 2.5 \Omega \text{ resistance} = (+2 \text{ A}) \times (+5 \text{ V}) = +10 \text{ W}$$

We note that the sum of power delivered to all elements is zero. The 20 V source delivers 40 W of power which is shared by the two resistors and the second source. The second source receives a power of 10 W from the first source and absorbs it.

Four elements were connected in this circuit with no two elements sharing the same pair of nodes. Two elements have only one common node. This kind of connection is called a *series connection of elements*. Obviously, in a *series connection of elements*, the same current will flow in the same direction in all elements.

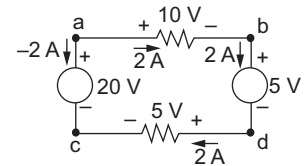


Fig. 2.4-3 Circuit Solution in Example 2.4-1

EXAMPLE: 2.4-2

Solve the circuit in Fig. 2.4-4 completely.

SOLUTION

This is a series connected, single-loop circuit. Therefore, the same current should flow through all the elements and it is constrained to be equal to 2 A in a counter-clockwise direction in the circuit by the independent current source that comes in series with other elements. The other elements have no role in deciding the circuit current.

However, the current source will have to pay a price for deciding the circuit current in an autocratic manner. The price it has to pay is the amount of voltage that it has to support across its terminals and the amount of power it has to deliver or absorb. The current source will have no control over these quantities. They will be decided by the rest of the elements in the circuit.

The circuit, after variable assignment and reference direction assignment, is shown in Fig. 2.4-5 (a).

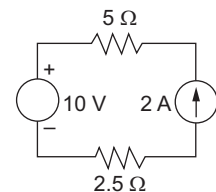


Fig. 2.4-4 Circuit for Example 2.4-2

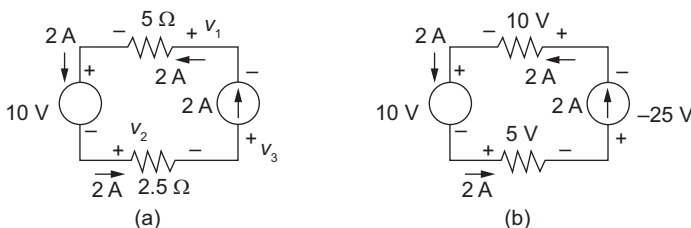


Fig. 2.4-5 (a) Variable and Reference Direction Assignment in the Circuit in Example 2.4-2 (b) Circuit Solution

Obviously, it is not necessary to assign a current variable to any element in the circuit if we choose the reference directions for all element currents to coincide with the actual direction of the current flow in the circuit. The polarity of element voltage variables is assigned as per passive sign convention.

Applying KVL in the loop, we get,

$$-10 \text{ V} - (5 \Omega \times 2 \text{ A}) - v_3 - (2.5 \Omega \times 2 \text{ A}) = 0$$

Therefore, $v_3 = -25 \text{ V}$. The circuit solution is marked in Fig. 2.4-5 (b).

We note that actually the upper terminal of current source is at a higher potential as compared to the lower terminal. This indicates that the current source is delivering power. However, 2 A flows into the positive polarity terminal of the 10 V voltage source. Therefore, the voltage source is absorbing power from the current source. Thus, the current source must be delivering all the power that is absorbed by the two resistors and the voltage source. Let us verify this.

Power delivered by current source	$= -(-25 \text{ V}) \times (2 \text{ A}) = 50 \text{ W}$
Power absorbed by 5 Ω resistor	$= 10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$
Power absorbed by 2.5 Ω resistor	$= 5 \text{ V} \times 2 \text{ A} = 10 \text{ W}$
Power absorbed by the voltage source	$= 10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$

Total power absorbed by power-absorbing elements is 50 W and that is equal to the power delivered by the current source.

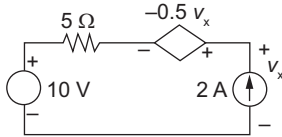


Fig. 2.4-6 Circuit for Example 2.4-3

EXAMPLE: 2.4-3

Solve the circuit in Fig. 2.4-6 completely.

SOLUTION

This single-loop circuit contains a voltage-controlled voltage source (VCVS) in series with other elements. The controlling variable of this VCVS is the voltage across the independent current source with positive polarity at current delivery point of the source.

A VCVS is a four-terminal element. The output terminal pair is connected in series with other elements in Fig. 2.4-6. But where is the input terminal pair?

Figure 2.4-7 shows the actual connections involved in the circuit. The input terminal pair of the VCVS is connected across the terminals of the independent current source and the output terminal pair is connected between the right-side terminal of the resistor and the left-side terminal of current source. The source function value of VCVS is $-0.5v_x$, where v_x is the voltage sensed by its input terminal pair.

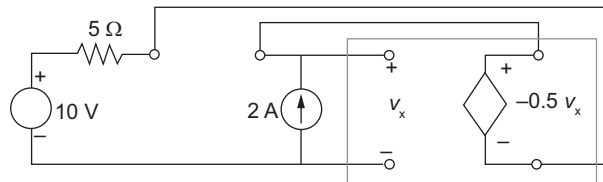


Fig. 2.4-7 Circuit in Example 2.4-3 Redrawn to Show the VCVS Terminal Connections

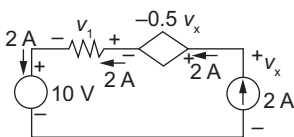


Fig. 2.4-8 Circuit Variable and Reference Direction Assignment in Example 2.4-3

The input terminal pair of an ideal VCVS is an open-circuit and does not affect the circuit behaviour in any manner. Therefore, it is sufficient to identify the controlling variable of a VCVS as in Fig. 2.4-6.

The variable and reference direction assignment is shown in Fig. 2.4-8. We note that we have not assigned any new voltage variable across the current source.

The voltage variable v_x itself is taken as its voltage variable. Therefore, the reference direction of current in current source and reference polarity for its voltage are not according to passive sign convention. This will not cause any problem in applying the element equation of current source in KVL. It is so since the voltage across a current source is independent of its current. However, the fact that we are not using the passive sign convention for this element has to be kept in mind when we calculate the power delivered by this source. Adhering to passive sign convention is important in the case of passive elements since the element relation of a passive element depends on the relative polarities of voltage and current in the element.

Applying KVL in the loop, we get,

$$\begin{aligned} -10 \text{ V} - v_1 - (-0.5v_x) + v_x &= 0 \\ \text{i.e., } -10 \text{ V} - (5 \Omega \times 2 \text{ A}) + 1.5v_x &= 0 \end{aligned}$$

$$\therefore v_x = \frac{40}{3} \text{ V} = 13.33 \text{ V}$$

The complete solution is marked in Fig. 2.4-9. Note the change in polarity markings across the dependent source.

Power absorbed by the 10 V voltage source	$= 10 \text{ V} \times 2 \text{ A}$	$= 20 \text{ W}$
Power absorbed by the 5Ω resistance	$= 10 \text{ V} \times 2 \text{ A}$	$= 20 \text{ W}$
Power delivered by the VCVS	$= (20/3) \text{ V} \times 2 \text{ A}$	$= 40/3 \text{ W}$
Power delivered by the 2 A current source	$= (40/3) \text{ V} \times 2 \text{ A}$	$= 80/3 \text{ W}$
Total power absorbed	$= 20 \text{ W} + 20 \text{ W}$	$= 40 \text{ W}$
Total power delivered	$= (40/3) \text{ W} + (80/3) \text{ W}$	$= 40 \text{ W}$

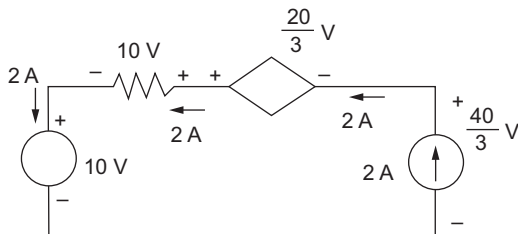
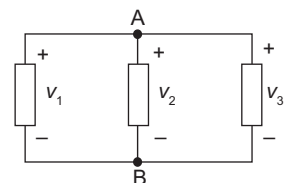


Fig. 2.4-9 Circuit Solution in Example 2.4-3

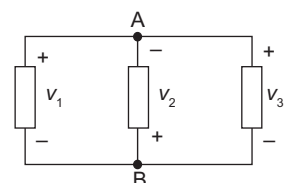
2.5 ANALYSIS OF A SINGLE-NODE-PAIR CIRCUIT

A set of circuit elements is said to be connected in *parallel* if they have two nodes in common. Fig. 2.5-1 shows three circuit elements connected in parallel. All the three elements have one of their terminals connected at node-A and the other terminal connected at node-B. Circuit in Fig. 2.5-1(a) shows reference polarity assignment for the element voltage variables for the three elements. The terminal connected to node-A is assigned the positive polarity in all the elements in this case. Circuit in Fig. 2.5-1(b) shows another possible polarity assignment. Here, the terminal that is connected to node-B is assigned the positive polarity of voltage variable in the case of the second element.

This circuit has two meshes. We can apply KVL in those meshes. KVL applied to meshes in circuit in Fig. 2.5-1(a) will show that $v_1 = v_2 = v_3$ in the circuit. However, KVL applied to meshes in circuit in Fig. 2.5-1(b) will show that $v_1 = -v_2 = v_3$. Thus, the terminal voltages of elements connected in parallel will have the same value at all time if the same reference polarity assignment is used for all of them. That is, parallel-connected elements have a common terminal voltage if reference polarity is the same for all of them. Therefore, it is a standard practice in circuit analysis to assign positive polarity to terminals connected to a common node in the case of a set of parallel-connected elements.



(a)



(b)

Fig. 2.5-1 Parallel Connection of Elements

A set of parallel-connected elements results in one node-pair. There will be only one independent KCL equation in a circuit containing just one node-pair. However, such a circuit may contain many meshes. Applying KVL in all those meshes will result in an already-noted conclusion that all the elements in parallel will have the same terminal voltage.

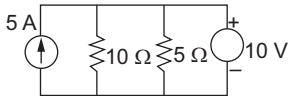


Fig. 2.5-2 Circuit for Example 2.5-1

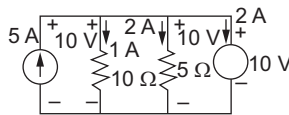


Fig. 2.5-3 Circuit Solution in Example 2.5-1

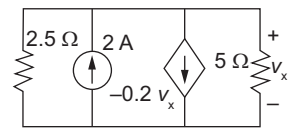


Fig. 2.5-4 Circuit for Example 2.5-2

Solve the circuit in Fig. 2.5-2 completely.

SOLUTION

The 10 V independent voltage source across the node-pair fixes the terminal voltage of all elements at 10 V. The current through 10 Ω will then be $10 \text{ V}/10 \Omega = 1 \text{ A}$ and the current through 5 Ω will be $10 \text{ V}/5 \Omega = 2 \text{ A}$. These currents flow from top node to bottom node.

Now, we apply KCL at the top node.

(Current flowing into the positive polarity of 10 V source) $-5 \text{ A} + 1 \text{ A} + 2 \text{ A} = 0$.

$\therefore$ Current flowing into the positive polarity of 10 V source = 2 A.

The circuit solution is marked in Fig. 2.5-3.

Power delivered by 5 A current source = $10 \text{ V} \times 5 \text{ A} = 50 \text{ W}$

Power absorbed by 10 Ω resistor = $10 \text{ V} \times 1 \text{ A} = 10 \text{ W}$

Power absorbed by 5 Ω resistor = $10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$

Power absorbed by 10 V current source = $10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$

Total power delivered = Total power absorbed

EXAMPLE: 2.5-1

Find the circuit solution for the circuit in Fig. 2.5-4.

SOLUTION

The variable assignment and reference polarity assignment are shown in Fig. 2.5-5.

The terminal voltage of all elements will be v_x with this polarity assignment. Now, $i_1 = v_x/2.5 \text{ A}$ and $i_2 = v_x/5 \text{ A}$. Applying KCL at the top node, we get,

$$i_1 - 2 + (-0.2v_x) - i_2 = 0$$

$$\text{i.e., } \frac{v_x}{2.5} - 2 + (-0.2v_x) + \frac{v_x}{5} = 0 \Rightarrow 0.4v_x = 2 \Rightarrow v_x = 5 \text{ V}$$

Now, $i_1 = 2 \text{ A}$ and $i_2 = 1 \text{ A}$. The circuit solution is marked in Fig. 2.5-6.

Power absorbed by 2.5 Ω resistor = $5 \text{ V} \times 2 \text{ A} = 10 \text{ W}$

Power absorbed by 5 Ω resistor = $5 \text{ V} \times 1 \text{ A} = 5 \text{ W}$

Power delivered by 2 A current source = $5 \text{ V} \times 2 \text{ A} = 10 \text{ W}$

Power delivered by the dependent current source = $5 \text{ V} \times 1 \text{ A} = 5 \text{ W}$

Total power delivered = Total power absorbed = 15 W

EXAMPLE: 2.5-2

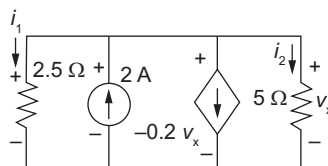


Fig. 2.5-5 Reference Polarity Assignment in the Circuit in Example 2.5-2

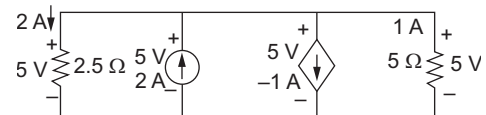


Fig. 2.5-6 Circuit Solution in Example 2.5-2

2.6 ANALYSIS OF MULTI-LOOP, MULTI-NODE CIRCUITS

The method of analysis of multi-loop, multi-node circuits, using element relations along with KCL and KVL equations, is illustrated through worked examples in this section.

EXAMPLE: 2. 6-1

Find the voltage across the dependent current source (v_y) in the circuit in Fig. 2.6-1.

SOLUTION

This circuit has six elements, three nodes and three meshes. We need to find out only v_y . Let us try to solve the circuit in terms of v_x and v_y without using any other new variables.

$$\text{We note that } v_{AD} = v_{CD} + v_{AC} = (10 - v_x) \text{ V.}$$

$$v_{AB} = v_{AD} - v_{BD} = (10 - v_x) - v_y \text{ V.}$$

$$v_{BC} = v_{BD} - v_{CD} = (v_y - 10) \text{ V.}$$

$$v_{CA} = v_x \text{ V.}$$

Note that we have essentially employed KVL in order to arrive at these relations. Now, we can express all the currents at node-A and node-B in terms of v_x and v_y . The currents going away from node-A are $(v_{AB}/3)$ A, $(v_{AD}/8)$ A and $(-v_{CA}/1)$ A. The sum of these three terms must be zero.

$$\therefore \frac{10 - v_x - v_y}{3} + \frac{10 - v_x}{8} - \frac{v_x}{1} = 0$$

The currents going away from node-B are $(-v_{AB}/3)$ A, $(v_{BC}/5)$ A and v_x . Sum of these terms must be zero.

$$\therefore \frac{-10 + v_x + v_y}{3} + \frac{v_y - 10}{5} + v_x = 0$$

Simplifying these two equations, we get,

$$35v_x + 8v_y = 110$$

$$20v_x + 8v_y = 80$$

Solving these two equations, we get, $v_x = 2 \text{ V}$; $v_y = 5 \text{ V}$

Therefore, the voltage across the dependent current source = 5 V.

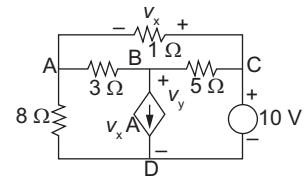


Fig. 2.6-1 Circuit for Example 2.6-1

EXAMPLE: 2.6-2

Find the power delivered by the voltage source and current source in the circuit shown in Fig. 2.6-2.

SOLUTION

We need to find out the current passing through the 20 V voltage source and the voltage across the 5 A current source. Refer to Fig. 2.6-3.

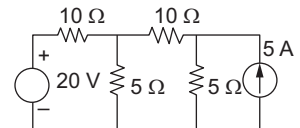


Fig. 2.6-2 Circuit for Example 2.6-2

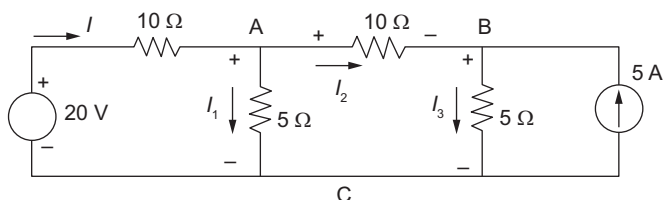


Fig. 2.6-3 Circuit in Example 2.6-2 with Variables Assigned

KVL in the first mesh gives,

$$-20 + 10i + V_{AC} = 0$$

$$\Rightarrow V_{AC} = (20 - 10i) \text{ V.}$$

$$\therefore i_1 = \frac{20 - 10i}{5} = 4 - 2i \text{ A.}$$

KCL has to be satisfied at node-A.

$$\therefore i_2 = i - i_1 = i - (4 - 2i) = 3i - 4 \text{ A.}$$

Applying KCL at node-B, we get,

$$-i_2 + i_3 - 5 = 0.$$

$$\Rightarrow i_3 = i_2 + 5 = (3i + 1) \text{ A.}$$

$$\text{Therefore, } V_{BC} = 5 \times (3i + 1) = 15i + 5 \text{ V.}$$

Now, we apply KVL on the outer loop of the circuit to get,

$$-20 + 10i + 10i_2 + V_{BC} = 0$$

$$\text{i.e., } -20 + 10i + (30i - 40) + (15i + 5) = 0$$

$$\text{i.e., } 55i = 55$$

$$\therefore i = 1 \text{ A}$$

$$\text{And, } V_{BC} = 15i + 5 = 20 \text{ V.}$$

Therefore, the current delivered by the voltage source is 1 A and the voltage appearing at the current source is 20 V.

Therefore, the power delivered by the voltage source = $20 \text{ V} \times 1 \text{ A} = 20 \text{ W}$

The power delivered by the current source = $20 \text{ V} \times 5 \text{ A} = 100 \text{ W}$

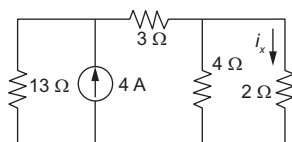


Fig. 2.6-4 Circuit for Example 2.6-3

Find i_x in the circuit shown in Fig. 2.6-4.

SOLUTION

The voltage across the 2Ω resistance is $2i_x$ V with positive polarity at the top terminal. Therefore, the current passing through 4Ω is $2i_x/4 = 0.5i_x$ A from the top terminal to the bottom terminal.

Therefore, the current through 3Ω must be $1.5i_x$ A from left to right by applying KCL to the node at which the three resistors are connected.

Therefore, the voltage across 13Ω must be $2i_x + 3 \times 1.5i_x = 6.5i_x$ V.

Therefore, the current through 13Ω must be $6.5i_x/13 = 0.5i_x$ A from top terminal to bottom terminal. Now, apply KCL at the current source node to get $1.5i_x + 0.5i_x = 4$.

$$\therefore i_x = 2 \text{ A.}$$

EXAMPLE: 2.6-3

EXAMPLE: 2.6-4

Find the ratio $\frac{V_0}{V_s}$ in the circuit shown in Fig. 2.6-5.

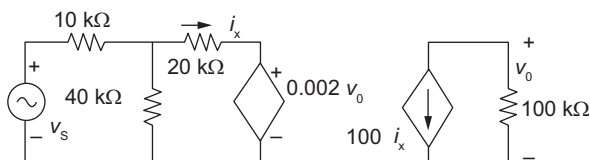


Fig. 2.6-5 Circuit for Example 2.6-4

SOLUTION

The current that flows through $100\text{ k}\Omega$ at the output side is $100i_x$ from the bottom terminal to the top terminal. Therefore, $v_o = -10^7 i_x$ V. Therefore, the voltage generated by the VCVS at the input side is $-0.002 \times 10^7 i_x = -2 \times 10^4 i_x$ V with polarity as shown in Fig. 2.6-5.

Therefore, voltage across $40\text{ k}\Omega$ resistance is $= 20 \times 10^3 i_x - 2 \times 10^4 i_x = 0$ V.

Therefore, current through $40\text{ k}\Omega$ resistance is $= 0$ A.

Applying KVL in the first mesh, we get, voltage drop across $10\text{ k}\Omega = v_s$ V.

Therefore, current through $10\text{ k}\Omega$ resistance $= 10^{-4} v_s$ A.

Current through $40\text{ k}\Omega$ is zero. Therefore, by KCL, $i_x = 10^{-4} v_s$ A.

We know that $v_o = -10^7 i_x$ V.

$v_o = -10^7 i_x = -10^7 \times 10^{-4} v_s = -1000 v_s$

$$\therefore \frac{v_o}{v_s} = -1000$$

2.7 SUMMARY

- Lumped parameter circuit theory assumes that a circuit element can be described by a terminal voltage variable and an element current variable. Unique terminal voltage for an element can be defined only if the induced electric field outside the devices can be neglected and the connecting wires have negligible resistance. Unique element current for an element can be defined only if the connecting wires are so thin that the surface charge distribution on their surface can be ignored.
- Let a circuit contain b -elements, l -loops and n -nodes, and let it have a unique solution. Then, the 'circuit analysis problem' involves solving for $2b$ variables – one voltage variable and one current variable per two-terminal element. b equations are obtained by using the element relation of b -elements. The remaining b equations come from constraints imposed on element voltage and current variables by the interconnection details. Kirchhoff's Current Law and Kirchhoff's Voltage Law give these equations. They depend only on the topology of the circuit and do not depend on the nature of elements.
- A *node* in a circuit is a junction point at which the connection leads of two or more elements join together. The *node* is a part of the connecting wire. Hence, according to the assumptions involved in lumped parameter circuit theory, there is negligible charge storage and negligible rate of change of charge storage at a node.
- Therefore, the net positive current that enters (or leaves) a node in a circuit through all the connecting wires connected at that node has to be zero. This is the Kirchhoff's Current Law.
- Kirchhoff's Current Law can be stated in three forms as below:
 - Kirchhoff's Current Law (KCL) states that the algebraic sum of currents leaving a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.
 - Kirchhoff's Current Law (KCL) states that the algebraic sum of currents entering a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.
 - Kirchhoff's Current Law (KCL) states that the sum of currents entering a node in a lumped parameter circuit through some wires must be equal to the sum of currents leaving the same node through the remaining wires on an instant-to-instant basis.
- Kirchhoff's Current Law is also valid for any closed surface that intersects connecting wires and encloses more than one node and one or more elements without intersecting any element. Such a closed surface is called a *supernode* of the circuit.
- KCL equations for any $(n - 1)$ nodes in an n -node circuit will form an independent set of equations.
- A loop in a circuit is a closed path traced through nodes, connecting wires and elements such that no node is visited more than once in one complete traversal of the closed path.
- Kirchhoff's Voltage Law for such a loop in a circuit can be stated in three equivalent forms as below:
 - Kirchhoff's Voltage Law states that the algebraic sum of voltages in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.

- (b) Kirchhoff's Voltage Law states that the sum of 'voltage rises' in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.
- (c) Kirchhoff's Voltage Law states that the sum of 'voltage drops' in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.
- A *planar* circuit is one that can be represented on a paper without any crossing of connecting wires. Those closed loops that do not contain other loops within them in a planar circuit are called its *meshes*.
- The KVL equations for meshes in a planar circuit will be an independent set of equations.
- A set of two-terminal elements is said to be *series-connected* if a common current can flow through them.
- A set of two-terminal elements is said to be *parallel-connected* if they share the same node-pair. In this case, it is possible to make the terminal voltage variables of all such elements a common variable by suitable reference polarity assignment for voltage variables.
- Kirchhoff's Current Law and Kirchhoff's Voltage Law are restatements of conservation law for charge and energy, respectively, in a form suitable for application in lumped parameter circuits. They have to be obeyed by all lumped parameter circuits under all conditions and at all time instants. If a circuit violates one of them apparently, it will imply that two-terminal models used to model some of the actual electrical devices are inappropriate for the analysis of that circuit.

2.8 PROBLEMS

- Some voltage and current variables are specified at an instant in the circuit in Fig. 2.8-1. Find the remaining variables at that instant as per the reference directions marked in Fig. 2.8-1.

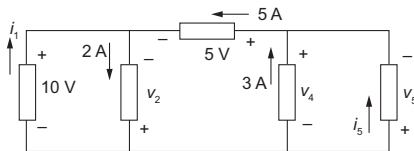


Fig. 2.8-1

- The circuit shown in Fig. 2.8-2 is a DC circuit containing only resistors and independent sources. Some voltage and current variables are specified in the circuit. (i) Find the remaining variables with reference directions as shown in Fig. 2.8-2. (ii) Identify those elements that *have to be* sources. (iii) Identify those elements that *can be* resistors. (iv) Identify the elements that receive positive power and calculate the total power absorbed by them. (v) Identify the elements that deliver positive power and calculate the total power absorbed by them.

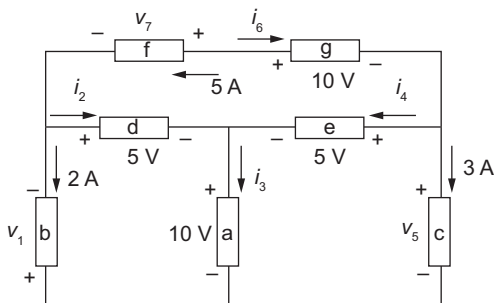


Fig. 2.8-2

- (i) Express all element currents in terms of i_3 , i_4 , i_7 and i_8 in the circuit in Fig. 2.8-3. (ii) Express all element voltage variables in terms of v_3 , v_4 , v_7 and v_8 .

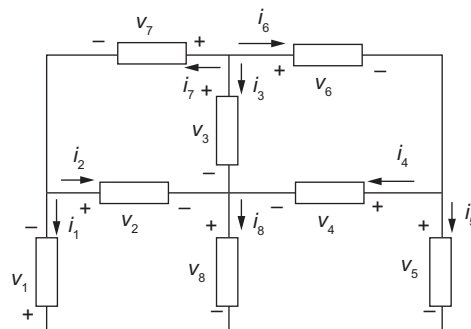


Fig. 2.8-3

- The DC circuit shown in Fig. 2.8-4 is known to contain only resistors and independent sources. (i) Find all the element

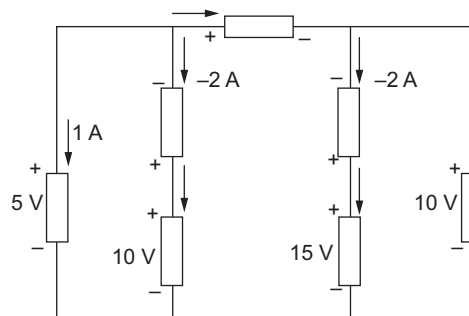


Fig. 2.8-4

voltage and current values in the circuit shown in Fig. 2.8-4 as per the reference directions marked in the circuit. (ii) What is the minimum number of independent sources that the circuit must contain? (iii) Draw the circuit configuration that contains the minimum number of independent sources, assuming that all sources are voltage sources. (iv) Draw the circuit configuration that contains the minimum number of independent sources, assuming that all sources are current sources.

5. (i) Show that the circuit solution for the circuit shown in Fig. 2.8-5 is the same as the solution for the circuit in Fig. 2.8-4. (ii) Calculate the power *delivered* by all current sources and voltage sources in the circuit and verify that the total power absorbed in the circuit is equal to the total power delivered in the circuit.

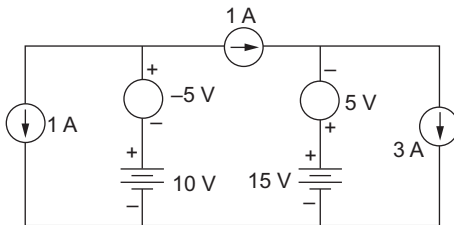


Fig. 2.8-5

6. (i) Complete the DC circuit solution for the circuit in Fig. 2.8-6. (ii) Construct *three* different circuits that have the same circuit solution as the circuit in Fig. 2.8-6 using three independent current sources and three independent voltage sources in all cases. Specify the source function values in each case.

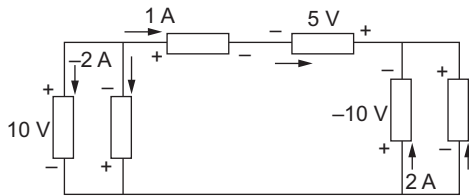


Fig. 2.8-6

7. Two identical DC practical voltage sources with internal e.m.f. of E V and internal resistance of $R \Omega$ each are paralleled to meet a steady load current demand of I A represented by an independent current source in the circuit shown in Fig. 2.8-7. (i) Derive expressions for the currents delivered by the sources (I_1 and I_2) and the terminal voltage V_t in terms of E , I and R . (ii) What is the ratio of I_1 to I_2 for an arbitrary value of R ? (iii) What is the value of this ratio when $R \rightarrow 0$ in an identical manner in both sources? (iv) What is the difference between I_1 and I_2 if $R = 0$ in both sources? (v) What is the ratio of I_1 to I_2 if $R = 0$? Can it be determined uniquely. [Hint: There is a difference between 'a quantity that tends to approach zero value' and 'a quantity that has a zero value'.]

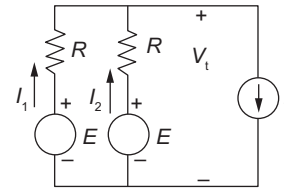


Fig. 2.8-7

8. Two identical DC practical current sources with a source current value of I A and internal resistance of $R \Omega$ each are connected in series to meet a steady voltage demand of V V represented by an independent voltage source in the circuit shown in Fig. 2.8-8. (i) Derive expressions for the voltages appearing across the sources (V_1 and V_2) and the terminal current I_t in terms of I , V and R . (ii) Show that the two current sources share the load voltage equally for any finite value of R , provided the resistance value remains equal for both sources. (iii) Show that the manner in which two current sources share the load voltage cannot be determined if the two current sources are *ideal*.

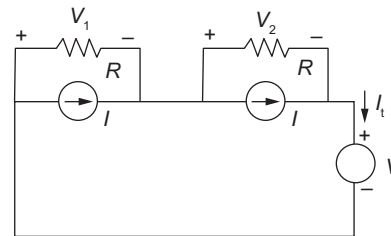


Fig. 2.8-8

9. Find the range of R such that the voltage V_o remains between 9 V and 9.5 V in the circuit in Fig. 2.8-9.

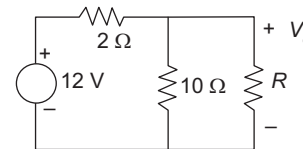


Fig. 2.8-9

10. Find the range of R such that the current I_o remains between 0.3 A and 0.35 A in the circuit shown in Fig. 2.8-10.

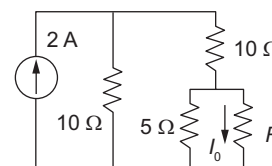


Fig. 2.8-10

11. Find the value of all resistors in the circuit in Fig. 2.8-11 and calculate the total power dissipated in them.

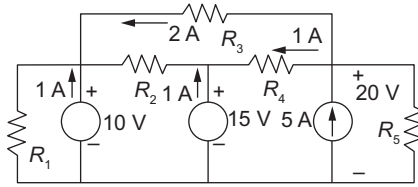


Fig. 2.8-11

12. Refer to Fig. 2.8-12. The voltage across 0.01 F capacitor is 30 V at $t = 0$. The voltage across 0.02 F capacitor is 60 V at $t = 0$. (i) Find the value of v_x at $t = 0$. (ii) Find the rate of change of capacitor voltage variables with respect to time at $t = 0$.

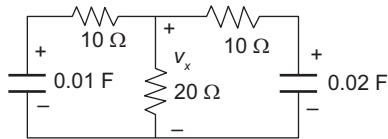


Fig. 2.8-12

13. Refer to Fig. 2.8-13. The currents flowing in the inductors at $t = 0$ are as marked in Fig. 2.8-13. (i) Find the value of i_x at $t = 0$. (ii) Find the time-rate of change of currents in the inductors at $t = 0$.

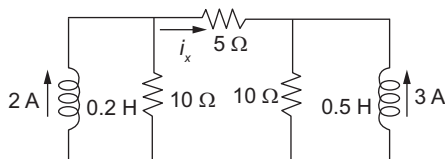


Fig. 2.8-13

14. Find the currents in all the resistors in the circuit shown in Fig. 2.8-14 by applying Kirchhoff's laws with $I_1 = 2$ A, $I_2 = 5$ A and $I_3 = 2$ A. [Hint: Write KCL equations at three nodes A, B and C in terms of voltage variables V_{AD} , V_{BD} and V_{CD} and resistance values.]

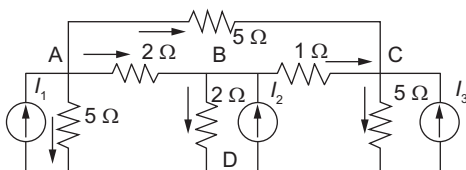


Fig. 2.8-14

15. Solve the circuit shown in Fig. 2.8-15 and find the power consumed by the resistors, power delivered by the independent source and power delivered by the dependent source.

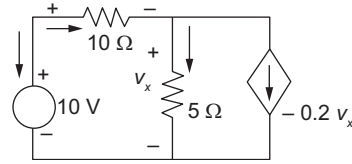


Fig. 2.8-15

16. Solve the circuit shown in Fig. 2.8-16 completely and find V_{AB} . Also, find the total power dissipated in the circuit and the power delivered by independent and dependent sources.

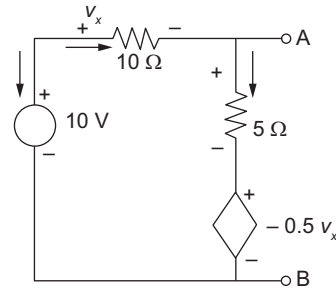


Fig. 2.8-16

17. Find the circuit current and power delivered by all the six elements in the circuit shown in Fig. 2.8-17.

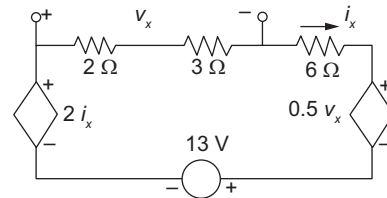


Fig. 2.8-17

18. Find the voltage across the parallel combination in the circuit shown in Fig. 2.8-18. Also, find the power absorbed by all the elements in the circuit.

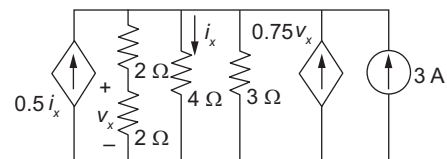


Fig. 2.8-18

19. Find the energy delivered to the 9 Ω resistor in the circuit shown in Fig. 2.8-19 during the time interval $[0, 2$ s] and

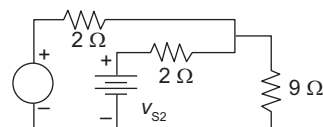


Fig. 2.8-19

the energy dissipated in the $2\ \Omega$ resistors if $v_{s1} = 10\sin 100\pi t$ V and $v_{s2} = 10$ V.

20. Find the charge delivered to the 6 V voltage source from $t = 0$ to $t = 2$ s in the circuit in Fig. 2.8-20. $i_{s1} = 2 + e^{-t}$ A for $t \geq 0$ and 0 A for $t < 0$. $i_{s2} = te^{-2t}$ A for $t \geq 0$ and 0 A for $t < 0$.

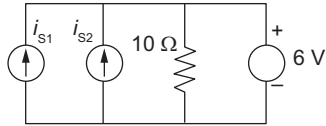


Fig. 2.8-20

21. Find the coefficients k_1 and k_2 for the dependent sources in the circuit in Fig. 2.8-21.

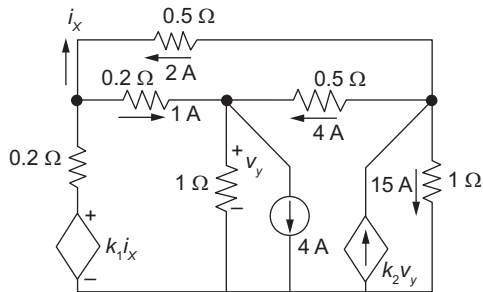


Fig. 2.8-21

22. Find the coefficients of the dependent sources (α and β) in the circuit shown in Fig. 2.8-22.

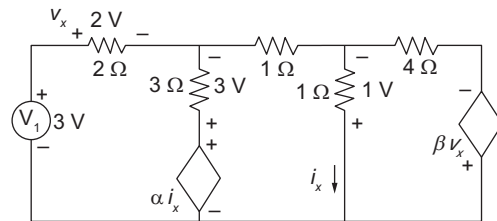


Fig. 2.8-22

Single Element Circuits

CHAPTER OBJECTIVES

- *Voltage–current relation of a resistor.*
- *Voltage, current and power division principle in series and parallel resistor combinations.*
- *Voltage–current relation and its various implications for an inductor.*
- *Initial current in an inductor and its significance.*
- *Series and parallel combinations of inductors.*
- *Need for better model for inductor in certain circuit situations.*
- *Voltage, current and power sharing in series and parallel connection of inductors.*
- *Voltage–current relation and its various implications for a capacitor.*
- *Initial voltage across a capacitor and its significance.*
- *Series and parallel combinations of capacitors.*
- *Need for better model for capacitor in certain circuit situations.*
- *Voltage, current and power sharing in series and parallel connection of capacitors.*
- *Simple single element circuits with independent voltage source and current source excitation.*
- *Unit impulse function and unit step function.*

The reader is expected to become proficient in visualising the voltage and current waveforms in inductor and capacitor when one function is known and the other is to be evaluated.

INTRODUCTION

In this chapter, we study simple circuits containing one type of element – resistor or inductor or capacitor – driven by one independent source, either an independent voltage source or independent current source. These circuits may contain more than one element, but all of them will be of same type except the source. The aim of this study is to understand the nature and behaviour of each element type thoroughly. We will also deal with ‘*series and parallel equivalents*’ that can be used to replace series or parallel connection of multiple elements of same type by one equivalent element of that type. Moreover, we deal with two

The current in a series circuit should be a common variable. This is evident if we apply KCL at the junction between one resistor and the next. Therefore, there is only one current variable and that is $i(t)$ as shown in Fig. 3.1-1. Applying KVL in the loop and employing the element relationship of a linear resistor, we get,

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + \dots + v_n(t) \text{ for all } t \\ \text{i.e., } v(t) &= R_1 i(t) + R_2 i(t) + \dots + R_n i(t) \text{ for all } t \\ \text{i.e., } v(t) &= [R_1 + R_2 + \dots + R_n] i(t) \text{ for all } t \\ \text{i.e., } v(t) &= R_{eq} i(t) \text{ for all } t, \text{ where } R_{eq} = \sum_{k=1}^n R_k. \end{aligned} \quad (3.1-1)$$

Thus, the entire series connection can be replaced by a single resistor of resistance value equal to the sum of resistance values of all the resistors in series. Further,

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + \dots + v_n(t) \text{ for all } t \\ \text{and } v_1(t) : v_2(t) : \dots : v_n(t) &= R_1 : R_2 : \dots : R_n \\ \text{i.e., } v_j(t) &= \frac{R_j}{\sum_{k=1}^n R_k} v(t) = \frac{R_j}{R_{eq}} v(t) \text{ for } j = 1 \text{ to } n \end{aligned} \quad (3.1-2)$$

The total voltage in a series combination divides in proportion to resistance value across various resistors. Since current is common and voltage is proportional to resistance value, the power delivered to individual resistor is also in proportion to its resistance value.

$$\begin{aligned} p(t) &= [v_1(t) + v_2(t) + \dots + v_n(t)] i(t) \text{ for all } t \\ &= [v_1(t) i(t) + v_2(t) i(t) + \dots + v_n(t) i(t)] \text{ for all } t \\ &= p_1(t) + p_2(t) + \dots + p_n(t) \\ \text{and } p_1(t) : p_2(t) : \dots : p_n(t) &= R_1 : R_2 : \dots : R_n \\ \text{i.e., } p_j(t) &= \frac{R_j}{\sum_{k=1}^n R_k} p(t) = \frac{R_j}{R_{eq}} p(t) \text{ for } j = 1 \text{ to } n \\ \text{and } p(t) &= \sum_{k=1}^n p_k(t) = [i(t)]^2 \sum_{k=1}^n R_k = R_{eq} [i(t)]^2 = \frac{[v(t)]^2}{R_{eq}}. \end{aligned} \quad (3.1-3)$$

Current is the common variable in a series connection.

Series Connection of Resistors

Series connection of many resistors can be replaced by a single resistor of resistance value equal to sum of resistance values of all the resistors in series.

Total voltage in a series combination divides in proportion to resistance value across various resistors.

Total power delivered to a series combination divides in proportion to resistance value across various resistors.

3.1.2 Parallel Connection of Resistors

Consider the parallel connection of n resistors $R_1, R_2, \dots, R_n$ as in Fig. 3.1-2.

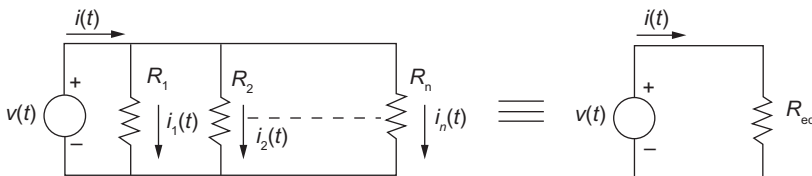


Fig. 3.1-2 Parallel Connection of Resistors and its Equivalent

The voltage across the parallel combination is the common variable. This is so because KVL will be violated in the loops formed by parallel connection otherwise. Applying KCL at the positive node of the voltage source and using the element equations of resistors, we get,

Voltage is the common variable in parallel connection.

$$\begin{aligned}
 i(t) &= i_1(t) + i_2(t) + \dots + i_n(t) \text{ for all } t \\
 \text{i.e., } i(t) &= G_1 v(t) + G_2 v(t) + \dots + G_n v(t) \text{ for all } t \\
 \text{i.e., } i(t) &= [G_1 + G_2 + \dots + G_n] i(t) \text{ for all } t \\
 \text{i.e., } v(t) &= G_{\text{eq}} i(t) \text{ for all } t, \text{ where } G_{\text{eq}} = \sum_{k=1}^n G_k \\
 \text{or equivalently } \frac{1}{R_{\text{eq}}} &= \sum_{k=1}^n \frac{1}{R_k} .
 \end{aligned} \tag{3.1-4}$$

Hence, the entire parallel combination can be replaced by a resistor with a conductance value equal to a sum of conductance values of all resistors in the parallel combination as far as the v - i relationship is concerned. Further,

$$\begin{aligned}
 i(t) &= i_1(t) + i_2(t) + \dots + i_n(t) \text{ for all } t \\
 \text{and } i_1(t) : i_2(t) : \dots : i_n(t) &= G_1 : G_2 : \dots : G_n \\
 \text{i.e., } i_j(t) &= \frac{G_j}{\sum_{k=1}^n G_k} i(t) = \frac{G_j}{G_{\text{eq}}} i(t) \text{ for } j = 1 \text{ to } n.
 \end{aligned} \tag{3.1-5}$$

Parallel Connection of Resistors

Parallel combination of many resistors can be replaced by a resistor with a conductance value equal to sum of conductance values of all resistors in the parallel combination.

Total current in a parallel combination divides among the various resistors in proportion to their conductance values.

Total power delivered to a parallel combination of resistors divides among the various resistors in proportion to their conductance values.

The total current in a parallel combination divides among the various resistors in proportion to their conductance values. Since voltage is common and current is proportional to conductance value, the power delivered to the individual resistor is also in proportion to its conductance value.

$$\begin{aligned}
 p(t) &= [i_1(t) + i_2(t) + \dots + i_n(t)] v(t) \text{ for all } t \\
 &= i_1(t) v(t) + i_2(t) v(t) + \dots + i_n(t) v(t) \text{ for all } t \\
 &= p_1(t) + p_2(t) + \dots + p_n(t) \\
 \text{and } p_1(t) : p_2(t) : \dots : p_n(t) &= G_1 : G_2 : \dots : G_n \\
 \text{i.e., } p_j(t) &= \frac{G_j}{G_{\text{eq}}} p(t) \text{ for } j = 1 \text{ to } n
 \end{aligned} \tag{3.1-6}$$

$$\text{and } p(t) = \sum_{k=1}^n p_k(t) = [v(t)]^2 \sum_{k=1}^n G_k = G_{\text{eq}} [v(t)]^2 = \frac{[v(t)]^2}{R_{\text{eq}}} = R_{\text{eq}} [i(t)]^2 .$$

A frequently occurring important special case in parallel combination is the one involving two resistors. The equivalent resistance and conductance as well as the current division relations are marked in the Fig. 3.1-3.

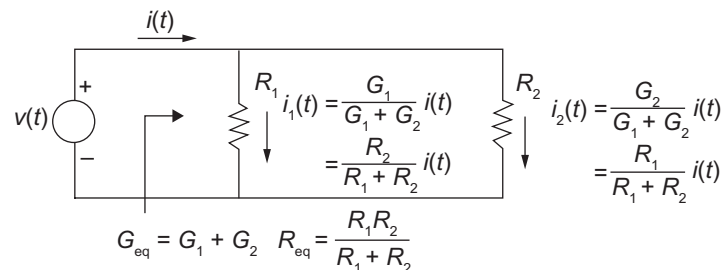


Fig. 3.1-3 Current Division in a Two-Resistor Parallel Connection

Current division principle applicable to parallel resistors.

The way total current divides between two parallel resistors is usually stated as *Current Division Principle*. This principle states that if the total current delivered to a parallel combination of two resistors R_1 and R_2 is I , then the current flowing in the resistor R_1 is $\frac{R_2}{R_1 + R_2} I$ and the current flowing in the resistor R_2 is $\frac{R_1}{R_1 + R_2} I$.

EXAMPLE: 3.1-1

Find R in the circuit in Fig. 3.1-4 such that v_o is zero.

SOLUTION

This circuit has two series combinations of resistors connected across the DC source. We take the negative of the DC source as the common reference point. The potential of node between $10\ \Omega$ and $25\ \Omega$ resistors will be given by voltage division principle in series combination.

The potential of the node marked $+$ $= 24 \times 25 / (25 + 10)$ V. But, we do not really need to calculate this.

We want the potential difference v_o to be zero. Therefore, the potential of the junction between $30\ \Omega$ and R with respect to the negative of the DC source must be same as the potential of the junction between $10\ \Omega$ and $25\ \Omega$. Therefore, $R / (30 + R) = 25 / 35 \Rightarrow R = 75\ \Omega$. Note that the value of R is independent of the DC source value.

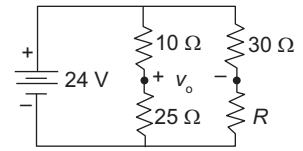


Fig. 3.1-4 Circuit for Example 3.1-1

EXAMPLE: 3.1-2

Two resistors in series combination can be used to create a voltage lower than the supply voltage by voltage division. Such an arrangement is called a *potential divider*. If one of the resistors is made a variable one, this arrangement can be used for making a variable DC voltage from a fixed DC supply. This arrangement is widely employed in electronic circuits. One such *fixed potential divider* is shown in Fig. 3.1-5. (i) What is the output voltage when there is no load connected across the output, i.e., $R = \infty$? (ii) What is the minimum value of R if the output voltage is not to fall below 95% of the value it has when there is no load connected?

SOLUTION

(i) The output when no-load is connected $= 25\text{ V} \times 1\text{ k}\Omega / 10\text{ k}\Omega = 2.5\text{ V}$

(ii) When R is connected, the parallel combination of R and $1\text{ k}\Omega$ will be less than $1\text{ k}\Omega$ and the voltage division ratio will be less than 0.1. This results in an output lower than 2.5 V. We want it to be more than $0.95 \times 2.5 = 2.375\text{ V}$.

$$\frac{25X}{9 + X} \geq 2.375 \Rightarrow 22.625X \geq 21.375 \Rightarrow X \geq 0.945\text{ k}\Omega$$

But X is the parallel combination of R and $1\text{ k}\Omega$

$$\therefore \frac{R}{R + 1} \geq 0.945 \Rightarrow R \geq 0.945R + 0.945 \Rightarrow R \geq 17.2\text{ k}\Omega$$

Therefore, the load resistor must be greater than $17.2\text{ k}\Omega$. This implies that we can draw $\approx 140\ \mu\text{A}$ before our potential divider output falls by more than 5% of its open-circuit output value.

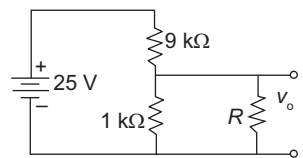


Fig. 3.1-5 Circuit for Example 3.1-2

EXAMPLE: 3.1-3

The power dissipated in the resistor marked with '*' in the circuit in Fig. 3.1-6 is found to be 12.5 W. (i) Find the value of R (ii) Find the total power delivered by the source (iii) Find the voltage, current and power dissipated for all the resistors.

SOLUTION

(i) To find an expression for power dissipated in the marked resistor in terms of R , we need an expression for current in it. We use series-parallel reductions successively as shown in Fig. 3.1-7.

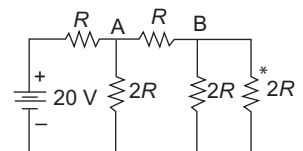


Fig. 3.1-6 Circuit for Example 3.1-3

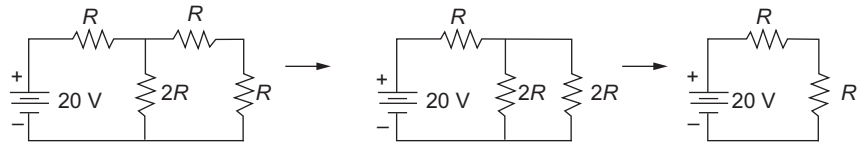


Fig. 3.1-7 Series-Parallel Reduction for Example 3.1-3

Therefore, the current delivered by the voltage source is $20/2R = 10/R$. The series-parallel reduction stages show that this current divides into two equal components at the node-A since equivalent resistance to the right of A is $2R$. Therefore, $5/R$ A reach node-B and split into $2.5/R$ A into the two equal resistors connected in parallel there. Therefore, the current in the marked resistor is $2.5/R$. Hence, the power dissipated in it is $2R \times 6.25/R^2 = 12.5/R$. Since this is stated to be 12.5 W, the value of $R = 1 \Omega$.

(ii) The equivalent resistance across the 20 V source is $2R = 2 \Omega$. Therefore, the current delivered by the source is 10 A and the power delivered by the source is $20 \times 10 = 200$ W.

(iii) The various currents, voltages and power values are marked in the Fig. 3.1-8.

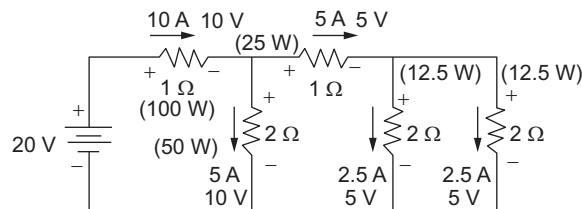


Fig. 3.1-8 Circuit Solution for Example 3.1-3

EXAMPLE: 3.1-4

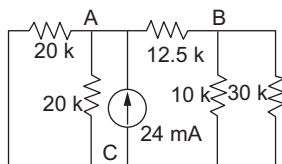


Fig. 3.1-9 Circuit for Example 3.1-4

All resistors in the circuit as shown in Fig. 3.1-9 are in k Ω . Solve the circuit for currents, voltages and power dissipated in all resistors. Also find the power delivered by the current source.

SOLUTION

The two 20 k resistors connected between A and C can be replaced by 10 k (n equal resistors of $R \Omega$ in parallel can be replaced by $R/n \Omega$). Similarly, the 10 k in parallel with 30 k between B and C can be replaced by $(10 \times 30)/(10 + 30) = 7.5$ k. The resulting circuit is shown in Fig. 3.1-10(b).

Now, there is 12.5 k + 7.5 k = 20 k on the right side of current source and 10 k on the left side of current source. The total current of 24 mA is divided in the ratio 0.05:0.1

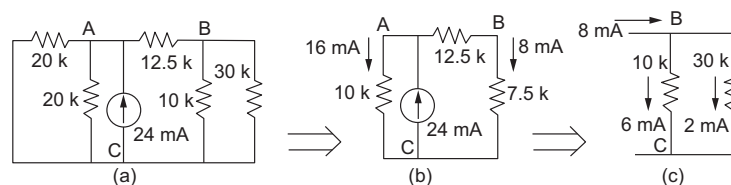


Fig. 3.1-10 Circuit Reduction for Example 3.1-4

between the right side and left side since the ratio of currents in a parallel combination is the conductance ratio. Therefore, current to the right is $1/3$ of $24 \text{ mA} = 8 \text{ mA}$ and current to the left is $2/3$ of $24 \text{ mA} = 16 \text{ mA}$ as shown in Fig. 3.1-10(b).

The 8 mA to the right side flows through 12.5 k and reaches the node-B where it gets divided into 10 k and 30 k . The conductance ratio here is $3:1$ and hence 75% of $8 \text{ mA} = 6 \text{ mA}$ goes into 10 k and 25% of $8 \text{ mA} = 2 \text{ mA}$ goes into 30 k as shown in Fig. 3.1-10(c).

Similarly, the 16 mA flowing to the left side from current source node will divide into 8 mA each in the 20 k resistors on the left as in Fig. 3.1-10(a).

Now, all resistor voltages may be obtained by multiplying resistor current by resistance value. Note that $\text{k}\Omega \times \text{mA}$ will give Volts.

Power delivered to resistor is given by product of voltage and current (since passive sign convention is followed).

The complete solution for the circuit is shown in Fig. 3.1-11. The power delivered to current source is negative valued since the current entering the positive polarity of voltage is -24 mA . This means that the current source is delivering a power of 3840 mW , which has to be equal to the sum of power dissipated in all resistors.

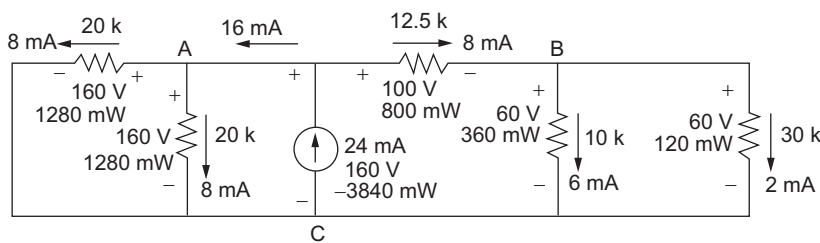


Fig. 3.1-11 Complete Circuit Solution for Example 3.1-4

EXAMPLE: 3.1-5

- (i) Find V_{DC} in the circuit in Fig. 3.1-12 if i_x is to be 2 mA . (ii) What is V_{DC} if i_x is to be 1.5 mA ? (iii) What is i_x if V_{DC} is 128 V ? (iv) Obtain the complete solution for the circuit with $V_{DC} = 96 \text{ V}$.

SOLUTION

- (i) We try to develop an expression for V_{DC} in terms of i_x . The 12.5 k and 17.5 k in series is replaced by 30 k as in Fig. 3.1-13(a). Further reduction is carried out by replacing 10 k and 30 k in parallel by its equivalent value of 7.5 k to arrive at circuit as in Fig. 3.1-13(b).

Now, i_x flows through 2.5 k and 7.5 k in series and develops a voltage drop of $10 i_x$ (i_x is in mA) across the 5 k resistor. This results in a current of $2 i_x$ in the 5 k resistor by Ohm's Law. Applying KCL at node-A as in Fig. 3.1-13(b) we get the current in 2 k resistor as $3 i_x$.

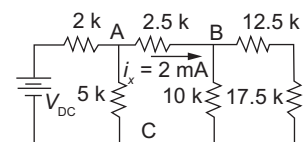


Fig. 3.1-12 Circuit for Example 3.1-5

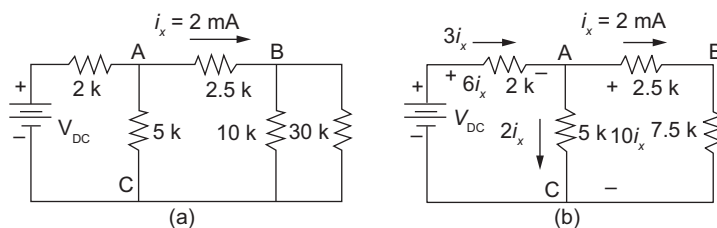


Fig. 3.1-13 Different Stages of Circuit Reduction in Example 3.1-5

Therefore, voltage drop across the 2 k resistor is $6i_x$. Applying KVL in the loop containing the DC voltage source, 2 k resistor and 5 k resistor, we get V_{DC} as $16i_x$.

Therefore, if i_x is 2 mA, then V_{DC} is 32 V.

(ii) If i_x is to be 1.5 mA, V_{DC} must be $16 \text{ k}\Omega \times 1.5 \text{ mA} = 24 \text{ V}$.

(iii) If V_{DC} is 128 V, then i_x must be $128/16 = 8 \text{ mA}$.

(iv) If V_{DC} is 96 V, i_x will be $96/16 = 6 \text{ mA}$. This 6 mA is shared by 10 k and 30 k in circuit as in Fig. 3.1-13(a) in the ratio of 1/10:1/30, i.e., 3:1. Therefore, the current in 10 k is 4.5 mA and current in 30 k is 1.5 mA. But the 30 k is actually series combination of 12.5 k and 17.5 k in the original circuit. Therefore, the current in 12.5 k and 17.5 k is 1.5 mA.

The voltage, v_{AC} , is $10i_x$ and hence it is 60 V. 60 V across 5 k produce 12 mA current in it. This results in $12 + 6 = 18 \text{ mA}$ in 2 k resistor in the original circuit in Fig. 3.1-12.

Now all the currents are known. Voltages across the resistors may be obtained by multiplying the current value by resistance value. Power delivered to the resistors may be obtained by multiplying the resistor voltage by current through it. The complete solution with all variable values marked is given in Fig. 3.1-14.

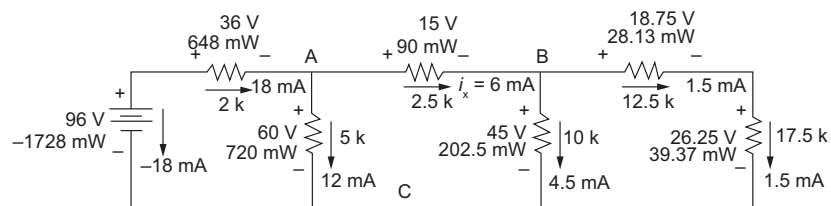


Fig. 3.1-14 Complete Solution with 96 V Source for Example 3.1-5

EXAMPLE: 3.1-6

- (i) Find the currents in all the resistors in the circuit in Fig. 3.1-15. (ii) Find the voltage appearing across the current source and the power delivered by it.

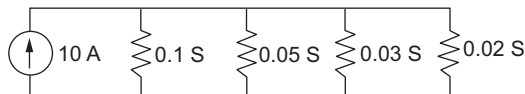


Fig. 3.1-15 Circuit for Example 3.1-6

SOLUTION

- (i) Total current in a parallel combination gets distributed in resistors as per the conductance ratio. The relevant ratio here is 0.1:0.05:0.03:0.02, i.e., 10:5:3:2. Therefore, the currents are 5 A, 2.5 A, 1.5 A and 1 A in 0.1 S, 0.05 S, 0.03 S and 0.02 S resistors, respectively.
- (ii) The equivalent conductance of a parallel combination is the sum of conductance values of the participating resistors. Hence, the equivalent conductance here is 0.2 S. Therefore, equivalent resistance is 5Ω . And hence, the voltage appearing across the current source is 50 V and the power delivered by it is 500 W.

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at $t = t_0$. This implies that, if v_0 is positive, the inductor current is on the rise, and, if v_0 is negative, it is on the fall. Notice that polarity of v_0 reveals the direction of current change and does not reveal anything about the polarity of the current unlike in the case of resistor. The current at $t = t_0$ can be positive or negative quite independent of whether it is increasing or decreasing.

What if v_0 is zero? What we can conclude about current in this case will depend on how this zero value was attained by $v(t)$. If $v(t)$ attained this value of zero at $t = t_0$ while it was moving from a negative value to positive value i.e., $v(t)$ was crossing zero in the upward direction, the inductor current will be at a local minimum. If $v(t)$ attained this value of zero at $t = t_0$ while it was moving from a positive value to negative value i.e., $v(t)$ was crossing zero in the downward direction, the inductor current will be at a local maximum. This will become clear if we keep in mind that the derivative of $v(t)$ will decide whether a critical point in $i(t)$ is a local maximum or local minimum. However, if $v(t)$ became zero at $t = t_0$ only because it is identically zero in some interval of time containing this time instant, it will imply that the inductor current is a constant at some value in that interval.

Instantaneous current in an inductor cannot be predicted from instantaneous value of voltage across it.

If instantaneous value of voltage is positive, the inductor current will be increasing at that instant, and, if it is negative the current will be decreasing at that instant.

When voltage across an inductor crosses zero in the downward direction, its current attains a local maximum and when it crosses zero in the upward direction, the inductor current attains a local minimum.

Voltage across an inductor carrying a constant current is zero.

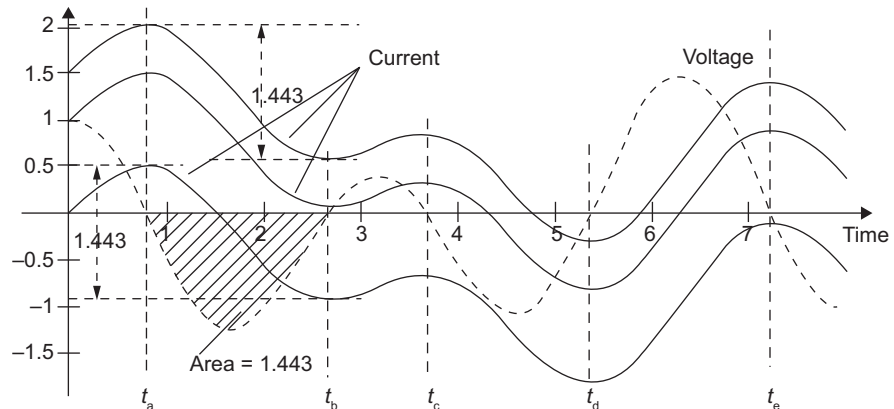


Fig. 3.2-1 Voltage–Current Relation in a 1 H Inductor

Consider the voltage–current relationship for a 1 H inductor as shown in Fig. 3.2-1. The *solid* curves represent three possible waveforms of current – all of them will have same first derivative waveform – and the *dotted* curve shows the applied voltage waveform. The three possible current waveforms are different only by constant values – notice that all three are parallel to each other. Derivative of a constant is zero and hence the voltage appearing across inductor in all the three cases will be represented by the same curve. Also, note that at the time instants marked as t_a , t_c and t_e , the $v(t)$ waveform crosses zero in the downward direction, and, $i(t)$ in all the cases attain local maxima at all the three instants. Similarly, at the time instants marked as t_b and t_d , the $v(t)$ waveform crosses zero in the upward direction, and, $i(t)$ in both cases reach local minima at those time instants. Further, the polarity of $v(t)$ is negative for all time instants between t_a and t_b , and, we observe that all the three current waveforms decrease in that interval. Similarly, polarity of $v(t)$ is positive for all time instants between t_b and t_c , and, we observe that all the three current waveforms increase in that interval.

Instantaneous current in an inductor cannot be predicted from instantaneous value of voltage across it.

If instantaneous value of voltage is positive, the inductor current will be increasing at that instant, and, if it is negative the current will be decreasing at that instant.

When the voltage across an inductor crosses zero in the downward direction, its current attains a local maximum and when it crosses zero in the upward direction, the inductor current attains a local minimum.

Voltage across an inductor carrying a constant current is zero.

3.2.2 Change in Inductor Current Function versus Area under Voltage Function

The relation between current and voltage of an inductor is reproduced below.

$$v(t) = L \frac{di(t)}{dt} \text{ and } i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt. \quad (3.2-1)$$

Consider two time instants t_1 and t_2 . Applying the integral form of v - i relationship, we get

$$i(t_2) - i(t_1) = \frac{1}{L} \int_{-\infty}^{t_2} v(t) dt - \frac{1}{L} \int_{-\infty}^{t_1} v(t) dt = \frac{1}{L} \int_{t_1}^{t_2} v(t) dt. \quad (3.2-2)$$

Thus, the *change* in inductor current is given by $(1/L) \times$ area under voltage function between the two instants under consideration. This is also expressed as $\Delta i = \frac{\text{volt-sec}}{L}$, where

Δi is the increase in inductor current $i(t)$ over $[t_1, t_2]$ and volt-sec is the area under $v(t)$ in the same interval.

Therefore, $i(t_2) = i(t_1) + \Delta i = i(t_1) + \frac{\text{volt-sec}}{L}$. We can also relate the volt-sec product to *change* in flux linkage in the inductor. *In fact, the volt-sec product itself is the change in flux linkage since $\Delta \psi = L \Delta i = \text{area under voltage function (volt-sec)}$.* Therefore, volt-sec and Wb-T are two units for the same quantity.

We can calculate only *change* in $i(t)$ given the $v(t)$ unless $v(t)$ is given for $(-\infty, t]$ interval. We can find the absolute instantaneous value of $i(t)$ if we know all the voltage applied to inductor from infinite past to the present instant. However, we need not insist on being given the $v(t)$ from $-\infty$ itself. It is enough that we know the area under $v(t)$ from $-\infty$ to some instant – say $t = t_0$ – and $v(t)$ itself from that instant onwards. This is so because we can split the integral in Eqn. 3.2-2 as shown in Eqn. 3.2-3.

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt = \frac{1}{L} \underbrace{\int_{-\infty}^{t_0} v(t) dt}_{I_0} + \frac{1}{L} \int_{t_0}^t v(t) dt. \quad (3.2-3)$$

Obviously, the first term on the right is the inductor current value at t_0 . Therefore, we can work out inductor current at an instant if we know its value at some reference instant and the voltage function applied to it from that reference instant onwards. This reference instant is usually set as $t = 0$ in analysis of circuits and the value of inductor current at $t = 0$ is termed as initial condition of inductor.

Change in inductor current over $[t_1, t_2]$, $\Delta i = (\text{Area under inductor voltage over } [t_1, t_2])/L$.

$i(t)$ at $t = t_2$ is $i(t)$ at $t = t_1$ plus Δi

$i(t) = I_0 + (\text{Area under inductor voltage over } [0, t])/L$,

where I_0 is the current in the inductor at $t = 0$ and is called initial condition of the inductor.

With reference to Fig. 3.2-1, the area under $v(t)$ between t_a and t_b is 1.443 volt-sec. The inductance value is 1 H. And, it is shown in the figure that all the three possible $i(t)$

Change in inductor current over a time interval is proportional to area under voltage waveform applied to it during that time interval.

Change in flux linkage in an inductor over a time interval is equal to area under voltage waveform applied to it during that time interval.

The area under voltage waveform applied to an inductor from $t = -\infty$ to $t = t_0$ can be summarised in the form of an initial value for inductor current at $t = t_0$.

waveforms undergo a change by -1.443 A in that interval, clearly demonstrating the relation between *change* in inductor current and volt-sec product dumped into the inductor during the relevant time interval. The voltage waveform as in the Fig. 3.2-1 is known only for $t \geq 0$. The three current curves shown in that figure represent three possible initial values for the inductor current at $t = 0$. The respective initial current values can be read off the curves at $t = 0$.

3.2.3 Average Applied Voltage for a Given Change in Inductor Current

Let us assume that we want to increase the current in an inductor L from I_1 to I_2 ($I_2 > I_1$) in a time interval of Δt . This change may be accomplished by applying any waveform for voltage provided the area under that waveform over Δt is $L(I_2 - I_1)$ volt-sec. This implies that irrespective of the exact waveform of voltage applied its average value over Δt has to be $L(I_2 - I_1)/\Delta t$ V.

Now, as Δt decreases – i.e., when we try to accomplish the required current change in shorter time interval – the average voltage to be applied increases. Thus, fast current changes in inductor require higher voltage to be applied across it.

3.2.4 Instantaneous Change in Inductor Current

It follows from Sect. 3.2.3, that the average voltage to be applied to cause a finite amount of change in inductor current increases to infinite value when we try to accomplish the change in current in zero time interval. We cannot bring about instantaneous change in inductor current unless we apply or support such an infinite voltage across the inductor.

Let us assume that we want to change the current in a 0.5 H inductor from 0 A to 2 A by applying a rectangular pulse voltage from $t = 0$. The voltage area content required is $0.5 \text{ H} \times 2 \text{ A} = 1$ volt-sec. The height of pulse will depend on the width of the pulse. Three cases are shown in Fig. 3.2-2.

When 2.5 V pulse lasting for 0.4 s is applied, the inductor current ramps up linearly from 0 A to 2 A in 0.4 s with a slope of 5 A/s. When 5 V pulse lasting for 0.2 s is applied, the inductor current ramps up linearly from 0 A to 2 A in 0.2 s with a slope of 10 A/s. When 10 V pulse lasting for 0.1 s is applied, the inductor current ramps up linearly from 0 A to 2 A in 0.1 s with a slope of 20 A/s. In all the three cases, we have kept the area under the voltage waveform at 1 volt-sec. Now, consider further shortening of pulse duration taking it to near-zero width. If we correspondingly increase the pulse height such that the area under the waveform remains at 1 volt-sec always, the change in inductor current will be 2 A always. However, the inductor current waveform will become steeper till it becomes a vertical waveform as pulse width $\rightarrow 0$ and pulse height $\rightarrow \infty$. Notice that though pulse height $\rightarrow \infty$ as width $\rightarrow 0$, its area is constrained to remain 1 volt-sec. Such an idealised waveform with zero width, undefined height and finite area-content of unity is called a *unit impulse function* and denoted by the symbol $\delta(t)$. Its mathematical definition is,

$$\delta(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined at } t = 0 & \text{and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \\ 0 & \text{for } 0^+ \leq t < \infty \end{cases}$$

where the time instant $t = 0^-$ is an instant which is arbitrarily close to $t = 0$ but, on its left side and time instant $t = 0^+$ is an instant which is arbitrarily close to $t = 0$ but, on its right side. Thus the interval $[0^-, 0^+]$ is of infinitesimal width; but 0 comes in the middle of this interval.

The graphical symbol used for $\delta(t)$ is shown in Fig. 3.2-2(b). The height of the arrow-terminated vertical line representing $\delta(t)$ is *not* the amplitude of the function (it is

The amount of current change required in an inductor decides the area-content under voltage waveform to be applied to it to bring about the change.

The time allowed to bring about the change in current decides the average voltage to be applied.

A unit impulse waveform can be considered as a limiting case of a rectangular pulse of unit area when its width is sent to infinitesimally small duration.

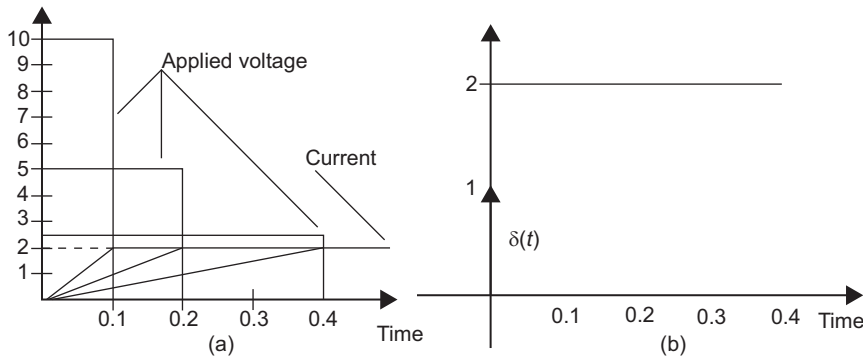


Fig. 3.2-2 Rectangular Pulse Application and Impulse Voltage

undefined at that point); rather it indicates the area-content of the waveform. The instantaneous change in inductor current from 0 A to 2 A is also shown in the same figure. Now look at the current waveform in the inductor. It is zero till 0^- and 2 A after 0^+ and a discontinuous jump at $t = 0$. This must be two times the integral of impulse function ($1/L = 2$ in this case). Let us verify this.

$$\int_{-\infty}^t \delta(t) dt = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined at } t = 0 \\ 1 & \text{for } t \geq 0^+ \end{cases} \quad (3.2-4)$$

This function is defined as a *unit step function* and is denoted by $u(t)$. Thus, when we apply a $\delta(t)$ voltage waveform to an inductor of inductance value L , the current in the inductor jumps up instantaneously by $1/L$ A. *Unit impulse voltage source* will dump 1 volt-sec of voltage area content into the inductor instantaneously. Equivalently, *unit impulse voltage source* will dump 1 Wb-T of flux linkage into the inductor instantaneously. The result will be a change in its current by $1/L$ A.

Strictly speaking, it is the flux linkage in an inductor that cannot be changed instantaneously. But in the case of an inductor that is not magnetically coupled to other inductors, this will amount to what we have stated above since flux linkage in such an inductor is proportional to its current. We will modify this statement suitably when we take up the study of coupled circuits later in the book.

3.2.5 Inductor with Alternating Voltage Across it

We consider the application of alternating voltage (AC voltage) across an inductor in this section. Alternating voltage is a voltage waveform that alternates between positive and negative voltages *periodically* and has a zero cyclic average. This means that the area under the voltage waveform during positive half-cycle and the area under the voltage waveform during the negative half-cycle are equal. The two half-cycles need not be equal in length. But the net area in a cycle has to be zero. This is equivalent to a *zero DC content* since the DC content of a cyclic waveform is its area-content over a cycle divided by the cycle period. It is possible to express a periodic waveform as a DC term plus a pure alternating term if there is a non-zero DC content in it.

Figure 3.2-3 shows the results of applying an alternating voltage waveform to an inductor with two values of inductance (1 H and 5 H) considered.

The *dotted* curve shows the applied voltage and *solid* curves show the current in the inductor. The integral of applied voltage is also shown in the figure. Both current curves show

Definition of unit step function and its relation to unit impulse function.

Instantaneous Change in Inductor Current

Current in an inductor cannot change instantaneously unless an impulse voltage is applied or supported in the circuit.

The current in an inductor L changes instantaneously by $1/L$ A when the circuit applies or supports unit impulse voltage across it.

Therefore, if a circuit does not apply or support impulse voltage, the currents in inductors in that circuit will be continuous functions of time.

local maxima and minima at voltage zero-crossing points. The area under one half-cycle of voltage is 1 volt-sec and the current in 1 H should change by 1 A over a half-cycle and current in 5 H should change by 0.2 A over a half-cycle. Figure 3.2-3 shows that the current in 1 H inductor varies between 1.4 A and 0.4 A with the initial condition of 0.4 A. The current varies between 0.6 A and 0.4 A in the case of a 5 H inductor with same initial condition.

We need to appreciate the following three points in this context.

With a specific area under a half-cycle of voltage waveform, the current in the inductor will change by an amount equal to that area value divided by L . In the next half-cycle it will vary by the same amount, but in opposite direction. *Thus, the peak-to-peak value of alternating component of inductor current will be equal to the area of one half-cycle of voltage waveform divided by L .* Therefore, higher the inductance, lower the peak-to-peak ripple current in the inductor. This conclusion is independent of the exact shape of voltage waveform.

If the frequency of voltage waveform is increased without changing its amplitude and wave-shape, the half-cycle area decreases due to reduction in half-cycle duration. Then, the peak-to-peak ripple current will also decrease. *Therefore, higher the frequency of alternating voltage applied to an inductor, lower the peak-to-peak amplitude of the alternating component of inductor current.*

There can be a DC current through an inductor even when the applied voltage waveform is a pure alternating one. The amount of DC content depends upon the initial condition of the inductor and the instant at which the voltage waveform is switched on to the inductor.

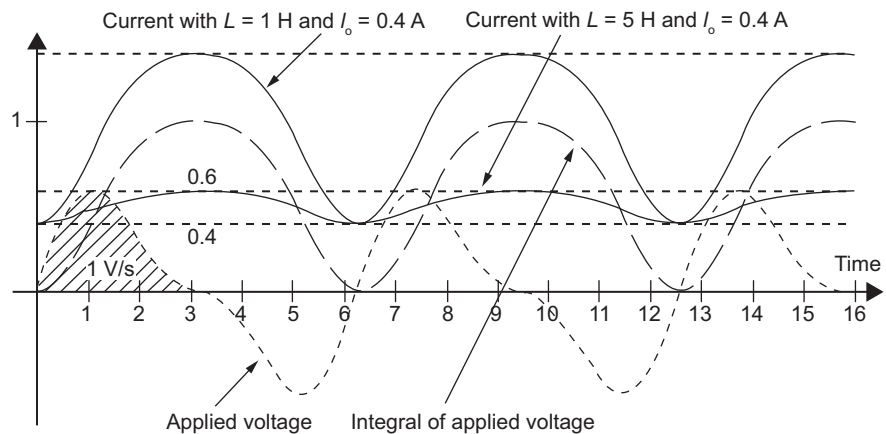


Fig. 3.2-3 Alternating Voltage Application to an Inductor

The DC content in inductor current is decided by two factors – the initial condition and the instant of application of the alternating voltage. Examine the integral of voltage waveform in Fig. 3.2-3. The voltage waveform was applied to the inductor at its zero-crossing. Therefore, its integral goes to a maximum value of 1 volt-sec in the first half-cycle and then returns to zero at the end of second half-cycle. It does not go negative. This area waveform divided by L will give us the current in the inductor with zero initial condition. Notice that the current will have a DC content since the voltage area waveform has a DC content. *Thus, the net DC content in the inductor current will be its initial condition value plus cyclic average of voltage area waveform divided by L .* Notice that the second contribution to DC content in the inductor current will depend on at which point in the voltage waveform we start applying it to the inductor. There exists one particular waveform position in any periodic voltage such that the DC contribution to inductor current will be zero if switching is done at that position.

When the applied voltage across an inductor is a periodic alternating waveform, the current in the inductor will contain an alternating component with the same period. The peak-to-peak amplitude of this alternating component will be directly proportional to half-cycle area of voltage waveform and inversely proportional to inductance value. It decreases with increase in frequency of the voltage.

Therefore, a large valued inductor in a circuit can absorb alternating voltages in the circuit without contributing significant amount of alternating currents to the circuit.

3.2.6 Inductor with Exponential and Sinusoidal Voltage Input

Consider an inductor of 1 H with a voltage $v_s(t) = e^{-t}$ V switched on to it from $t = 0$ onwards. We do not know what voltage was applied to inductor till $t = 0$. But the net effect of all that voltage which may have been applied to the inductor is condensed in the initial condition available to us. We assume an initial current of zero as in Fig. 3.2-4(a) and initial current of -0.5 A as in Fig. 3.2-4(b). Straightforward integration gives us $i(t) = I_0 + (1 - e^{-t})$ as the current in 1 H inductor. The applied voltage and current for the two initial condition values are shown in Fig. 3.2-4.

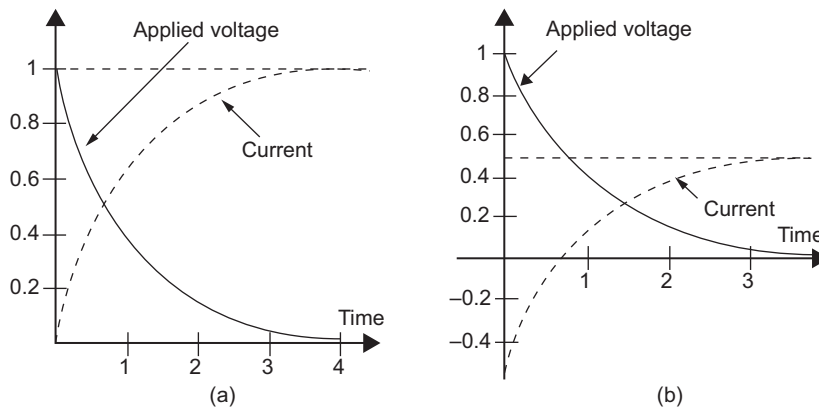


Fig. 3.2-4 Inductor with Exponential Voltage Applied to it

The inductor preserves the wave-shape of the input except for a DC offset, i.e., the current in the inductor is an exponential of same index as that of the applied voltage. This is due to the fact that exponential function does not change its shape on differentiation and integration. A sinusoidal function too has that property. Therefore, we expect the inductor current to be a sinusoid of same frequency as that of input when a sinusoidal voltage is applied to it. There may be DC offset in the current as in Fig. 3.2-5 that shows the current in a 1 H inductor with zero initial current when (a) $\sin t$ is applied (b) $\sin(t + \pi/4)$ is applied and (c) $\sin(t + \pi/2)$ is applied.

The alternating component of current indeed preserves the wave-shape of input. The DC offset is entirely due to the input since initial current was stated to be zero. Note that when a sinusoidal voltage is switched on at its zero-crossing, the resulting current is unipolar and reaches a peak which is twice that of its AC component amplitude, i.e., it has a DC content equal to AC amplitude. This is of course clearly seen by simple integration of $v_s(t)$ as below.

$$\begin{aligned} i(t) &= I_0 + \frac{1}{L} \int_0^t \sin(\omega t + \theta) dt \\ &= I_0 + \frac{1}{\omega L} [-\cos(\omega t + \theta)]_0^t \\ &= I_0 + \frac{1}{\omega L} [\cos \theta - \cos(\omega t + \theta)]. \end{aligned}$$

$I_0 = 0$ A, $\omega = 1$ rad/s and $L = 1$ H for the waveforms as shown in Fig. 3.2-5. Both the waveforms and the above equation make it clear that DC offset in the inductor

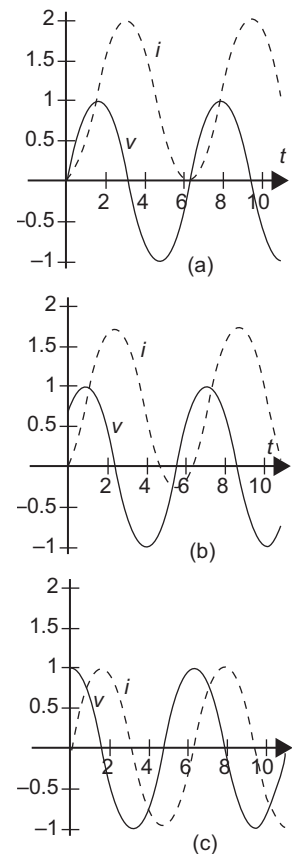


Fig. 3.2-5 Inductor with Sinusoidal Voltage Applied to It

current will be zero (if initial condition is zero) if the sinusoidal voltage is switched on at its peak.

Sinusoidal voltage is a special case of a general periodic alternating waveform. We expect the current peak-to-peak amplitude to go down with the frequency. The above equation shows that it does so. Moreover, when voltage across the inductor is a sine wave, its current waveform will be an inverted cosine wave. Corresponding positions in an inverted cosine wave will take place after $T/4$ s with respect to the sine wave. For example, the positive peak of current comes after $T/4$ s in time axis or $\pi/2$ rad in ωt axis the positive peak of voltage appears.

Inductor preserves the wave-shape for exponential and sinusoidal inputs.

The amplitude of current sinusoid in an inductor is inversely proportional to the product of frequency of applied voltage sinusoid and inductance value.

3.2.7 Linearity of Inductor

The flux linkage in an inductor and the current through it are related by a simple proportionality relationship. Hence, ψ versus i curve for a linear inductor is a *straight line* passing through origin. In that sense inductor is linear.

However, that is not what we mean when we say an electrical element is a linear element. *We call an electrical element linear when its voltage–current relationship satisfies two principles – the principle of homogeneity and the principle of additivity.* Principle of homogeneity requires that when input is scaled by a real constant the output also must get scaled by the same constant.

We treat $i(t)$ as input and $v(t)$ as output. This implies that we are applying a current source across inductor and observing the voltage appearing across the combination as output. The governing equation then is $v(t) = L di(t)/dt$. Obviously, when $i(t)$ is multiplied (scaled) by a real constant α , $v(t)$ also gets scaled by same number. Hence, the principle of homogeneity is satisfied.

Principle of additivity requires that when two inputs are applied simultaneously, the output observed is the sum of individual outputs observed when these inputs are applied individually. Let us say the voltage across inductor is $v_1(t)$ when a current source of $i_1(t)$ is applied to it and voltage across inductor is $v_2(t)$ when a current source of $i_2(t)$ is applied to it. Then the voltage will be $v_1(t) + v_2(t)$ when $i_1(t) + i_2(t)$ is applied if principle of additivity is satisfied. Obviously this is also true in the present case.

Therefore, the relation $v(t) = L \frac{di(t)}{dt}$ satisfies both principles.

Now, we consider voltage as input and current as output. This implies that we are applying a voltage source across the inductor and observing its current as output. The governing relationship in this case is

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt.$$

It may easily be verified that this relationship satisfies both the requirements. However, there is a caveat here. *We do not know $v(t)$ for $t < 0$.* Therefore, we write this relationship as

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt = \frac{1}{L} \int_{-\infty}^0 v(t) dt + \frac{1}{L} \int_0^t v(t) dt = I_0 + \frac{1}{L} \int_0^t v(t) dt$$

thereby absorbing all of $v(t)$ for $t < 0$ into a single number I_0 .

Now, if the portion of $v(t)$ that we apply, i.e., for $t \geq 0^+$ is multiplied by a real constant α , the inductor current will be

$$i(t) = I_0 + \frac{\alpha}{L} \int_0^t v(t) dt$$

and that is not α times the earlier current. Hence, the principle of homogeneity is not satisfied by the total current unless the initial condition is zero. Therefore, we have to make a qualified statement that the principle of homogeneity is satisfied by the component of current contributed by the applied voltage function. Similarly,

$$i(t) \text{ when } v_1(t) \text{ is applied } i_1(t) = I_0 + \frac{1}{L} \int_0^t v_1(t) dt$$

$$i(t) \text{ when } v_2(t) \text{ is applied } i_2(t) = I_0 + \frac{1}{L} \int_0^t v_2(t) dt$$

$$\begin{aligned} i(t) \text{ when } v_1(t) + v_2(t) \text{ is applied } i_{12}(t) &= I_0 + \frac{1}{L} \int_0^t (v_1(t) + v_2(t)) dt \\ &= I_0 + \frac{1}{L} \int_0^t v_1(t) dt + \frac{1}{L} \int_0^t v_2(t) dt \neq i_1(t) + i_2(t). \end{aligned}$$

Hence, the principle of additivity also is not satisfied by the total current unless the initial condition is zero. Therefore, we have to make a qualified statement that principle of additivity is satisfied by the component of current contributed by applied voltage function.

These two principles put together is called superposition principle. *A linear element is one that satisfies superposition principle.* Inductor is a linear element if it is understood that the superposition principle has to be applied to the current component which is produced by the applied voltage from $t = 0$ onwards. The initial current has to be excluded from the purview of superposition principle.

An inductor with zero initial current is a linear electrical element. An inductor with non-zero initial current is a linear element as far as the current component caused by applied voltage is concerned.

3.2.8 Energy Storage in an Inductor

An arbitrary $v(t)$ is applied across an inductor from $t = 0$ causing a current $i(t)$ through it. The source delivers power and energy to the inductor in this process. We will derive an expression for energy delivered to the inductor as a function of time, i.e., $E_L(t)$ and show that this energy is not dissipated by the inductor but stored by it.

$$\begin{aligned} E_L(t) - E_L(0) &= \int_0^t v(t)i(t) dt = L \int_0^t i(t) \frac{di(t)}{dt} dt = L \int_0^t i(t) di(t) = \frac{L}{2} \int_0^t d[i(t)]^2 = \frac{L[i(t)]^2 - [i(0)]^2}{2} \\ \therefore E_L(t) &= \frac{1}{2} L[i(t)]^2. \end{aligned}$$

Therefore, the change in energy delivered to an inductor over a time interval is the difference between the quantity $(1/2)Li^2$ evaluated at the end of the interval and at the beginning of the interval. The energy delivered till t is given by $(1/2)Li^2$, where i is the value of current at t .

Assume that we applied some $v(t)$ to L from $t = 0$ to t_0 and that initial condition of inductor was 0 A. At t_0 the current is I . Energy delivered to the inductor up to t_0 is $LI^2/2$ and all of that was delivered by the source connected since inductor had zero energy initially. At t_0 we remove the voltage source and short the inductor. That makes the voltage across inductor zero and therefore its current will continue at I . The power into or out of the inductor is zero since voltage across it is zero. Therefore, it will neither deliver nor take energy as long as it is kept shorted.

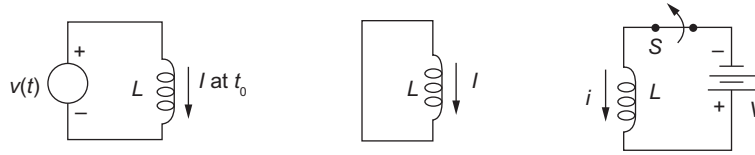


Fig. 3.2-6 On Energy Storage in an Inductor

The total energy delivered to an inductor carrying a current I is $(1/2)LI^2$ J. That this energy is stored in the magnetic field of the inductor is established here.

After some time we remove the short-circuit and connect the inductor to a DC voltage source of value V as shown in third circuit in Fig. 3.2-6. Notice that the polarity of applied voltage V is such that the current in the inductor goes down linearly with a slope of V/L A/s starting from I . It will take LI/V s for the current to reach zero. The switch S is a zero-current sensing switch and opens at the instant the circuit current touches zero thereby isolating the inductor from any further change in the current. Let us calculate the energy delivered to the DC source in this process.

$$i(t) = I - \frac{V}{L}t; t \text{ measured from the instant at which } V \text{ was connected to } L.$$

$$\text{Energy delivered to DC source} = \int_0^{LI/V} Vi(t)dt = \int_0^{LI/V} \left(VI - \frac{V^2 t}{L} \right) dt = \frac{1}{2} LI^2.$$

Hence, we see that the energy *delivered* by inductor *to* the DC source in this circuit is exactly equal to the energy *delivered* by the voltage source $v(t)$ *to* the inductor in the first circuit. Where was this energy when the inductor was in the second circuit? Therefore,

The total energy delivered to an inductor carrying a current I is $(1/2)LI^2$ J and this energy is stored in its magnetic field. The inductor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.

EXAMPLE: 3.2-1

The current in an inductor of 2 H is shown in Fig. 3.2-7. $v(t)$ does not contain any impulse. (i) What is the initial condition for inductor? (ii) Obtain $v(t)$ for 0 s to 9 s and plot it. (iii) Obtain the function $p(t)$ – the power delivered to the inductor – and plot it. (iv) Obtain the function $E(t)$ – the energy stored in the inductor – and plot it. Identify the time intervals during which the voltage source charges the inductor and discharges the inductor.

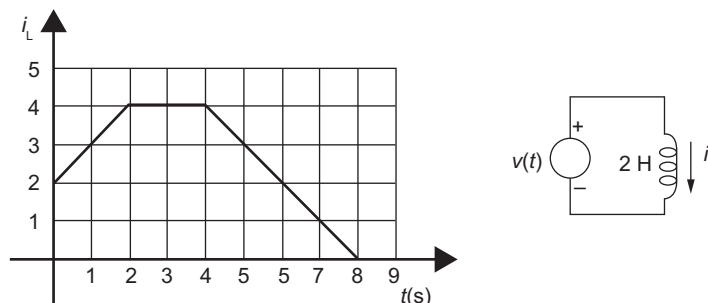


Fig. 3.2-7 Circuit and Waveform for Example 3.2-1

SOLUTION

Only an impulse voltage can change the inductor current over $[0^-, 0^+]$. The current at $t = 0^+$ is read from the given i_L waveform as 2 A. Therefore, the initial current in 2 H inductor at $t = 0^- = 2$ A.

We use $v(t) = L(di/dt)$ to work out the $v(t)$ waveform. $i(t)$ is a piecewise linear function. The slope of current is 1 A/s in the (0 s, 2 s) interval, 0 A/s in (2 s, 4 s) interval and -1 A/s in the (4 s, 8 s) interval. Note that all intervals are open intervals. This is so because the current waveform is not differentiable at the end points of the intervals and hence the endpoints will have to be excluded from the domain of the derivative function. We expect to observe jump discontinuities at those points in the $v(t)$ function. Using the slope values we can plot the function $v(t)$ as in Fig. 3.2-8.

The $v(t)$ waveform is discontinuous pulse waveform as expected. Note that the inductor accepts a *discontinuous voltage input* and generates a *continuous current* in response. This illustrates the ability of the inductor to keep a circuit current smooth.

The power and energy waveforms are calculated by

$$p(t) = v(t)i_L(t)$$

$$E(t) = E(0^+) + \int_{0^+}^t p(t)dt.$$

0^+ , 0^- and 0 are to be treated differently if there is an impulse voltage at $t = 0$. Generally, the right instant to use in the energy equation is 0^+ . If there is an impulse present at $t = 0$ it has to be accounted by suitably modifying the initial condition. We should avoid trying to integrate the product of impulse and a step discontinuity. That is why we use 0^+ as the lower limit of integration in the energy function. Impulse, if present, will result in a sudden change in initial condition over $[0^-, 0^+]$. We calculate the new initial condition at $t = 0^+$ and then evaluate the initial energy $E(0)$ as $0.5Li(0^+)^2$.

The power waveform in Fig. 3.2-8 shows that the source delivers power to the inductor during (0 s, 2 s) interval and source accepts power from inductor during (4 s, 8 s) interval. Inductor cannot dissipate energy. It can only store it. Therefore, when source delivers power to it, the energy storage in the inductor increases – this is called as *charging* an inductor. The opposite process in which the inductor delivers energy to some other element thereby reducing its stored energy level is called *discharging* an inductor. Thus, in this example, the voltage source charges the inductor during (0 s, 2 s) interval and discharges the inductor during (4 s, 8 s) interval.

The inductor had 4 J of energy to begin with. The source delivered 12 J of energy to it over (0 s, 2 s) interval, raising its energy storage to 16 J. The inductor gave all of 16 J to the source over (4 s, 8 s) interval and settled down at zero energy level. Therefore, the source received a net energy of 4 J from the inductor.

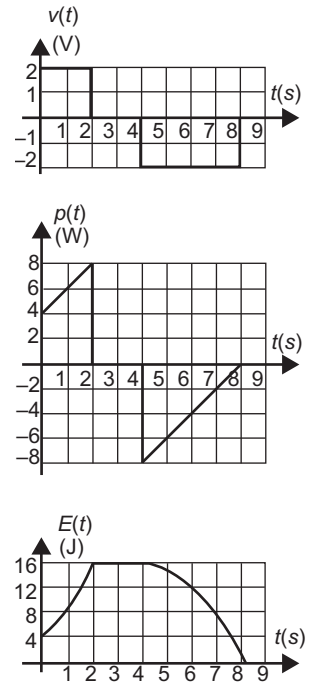


Fig. 3.2-8 Voltage, Power and Energy Waveforms in 2 H Inductor in Example 3.2-1

EXAMPLE: 3.2-2

The initial current in the 0.5 H inductor in Fig. 3.2-9 was 1 A at $t = 0^-$. Find the applied voltage $v(t)$ for 0 s to 9 s if the i_L waveform is as shown in Fig. 3.2-9.

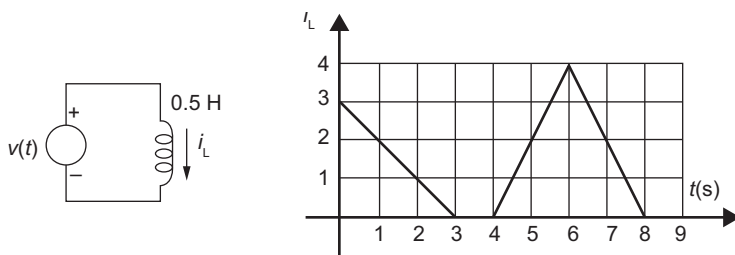


Fig. 3.2-9 Circuit and Waveform for Example 3.2-2

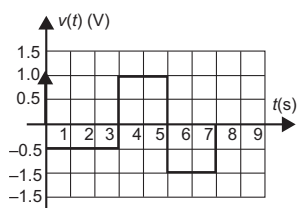


Fig. 3.2-10 Voltage Waveform Across 0.5 H Inductor for Example 3.2-2

SOLUTION

The value of current at $t = 0^+$ is different from the given initial condition of 1 A at $t = 0^-$. The current at $t = 0^+$ is seen to be 3 A from the given data. Hence, the flux linkage of inductor seems to change instantaneously from 0.5×1 Wb-T to 0.5×3 Wb-T i.e., a change by +1 Wb-T. This can happen only if a voltage area-content of 1 volt-sec gets dumped into the inductor at $t = 0$ instantaneously. Only a voltage impulse function of magnitude that is equal to 1 volt-sec can do this. Remember that *magnitude of impulse* is its area-content. Therefore, $v(t)$ should contain $\delta(t)$.

The current starts decreasing after $t = 0^+$ at the rate of -1 A/s and continues to fall at that rate in the interval (0 s, 3 s) till it reaches zero value.

After 3 s, it remains quiescent at 0 A. After that it rises with a rate of 2 A/s in (4 s, 6 s) interval and then falls at a rate of -2 A/s in (6 s, 8 s) interval. Multiplying the various values of slope of current with inductor value, we get the voltage waveform as in Fig. 3.2-10. Notice the impulse function at $t = 0$.

EXAMPLE: 3.2-3

The current through a 1 H inductor is shown in Fig. 3.2-11 for 0 s to 18 s interval. The applied voltage is known to be impulse-free. (i) What is the initial current in the inductor at $t = 0^-$? (ii) Obtain and plot the applied voltage $v(t)$ and the power delivered to the inductor, $p(t)$? (iii) What is the net energy delivered by the voltage source to the inductor?

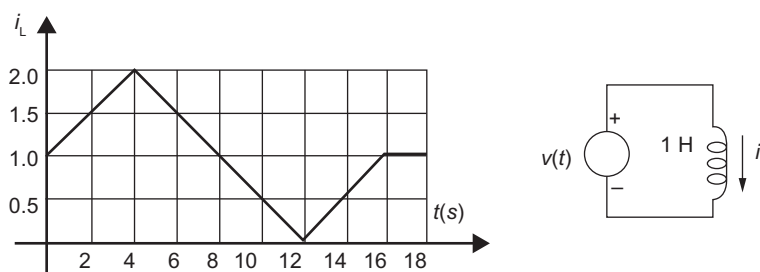


Fig. 3.2-11 Circuit and Waveform for Example 3.2-3

SOLUTION

The initial current at $t = 0^-$ is 1 A since only an impulse voltage can change the initial current instantaneously.

The value of di/dt is 0.25 A/s in (0 s, 4 s) interval, -0.25 A/s in (4 s, 12 s) interval, 0.25 A/s in (12 s, 16 s) interval and 0 A/s in (16 s, 18 s) interval. Therefore, $v(t)$ is 0.25 V in (0 s, 4 s) interval, -0.25 V in (4 s, 12 s) interval, 0.25 V in (12 s, 16 s) interval and 0 V in (16 s, 18 s) interval. The $v(t)$ waveform is shown in Fig. 3.2-12.

The power waveform is obtained by taking $v(t) i_L(t)$ product and plotting it. This is also shown in Fig. 3.2-12.

The inductor started with 1 A at $t = 0^+$ and ended with 1 A at 16 s. Therefore, the net change in stored energy of the inductor is zero. Hence, the net energy delivered by the voltage source to inductor must also be zero.

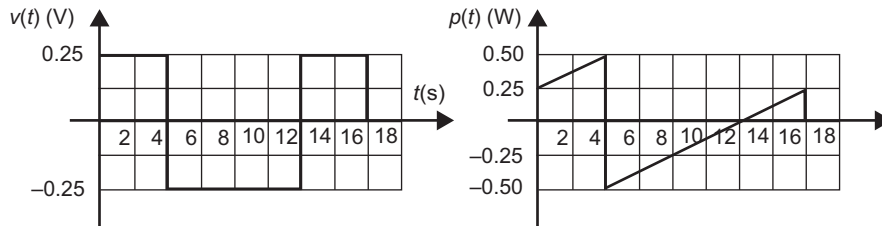


Fig. 3.2-12 Voltage and Power Waveforms for Example 3.2-3

EXAMPLE: 3.2-4

The voltage waveform applied to an inductor of 2 H is shown in Fig. 3.2-13. (i) Find the magnitude of current in the inductor at $t = 9$ s if the initial current is zero. (ii) What is the initial current if the current at 9 s is found to be 0 A? (iii) Plot current in the inductor in the first case.

SOLUTION

- (i) Inductor current at any instant is given by its initial current plus the volt-sec product divided by inductance value, where volt-sec is the area under the voltage curve from initial instant to the instant at which the current is calculated. The initial instant is 0 A here. Final instant is 9 s. The volt-sec available in this range is 25 volt-sec. Inductance value is 2 H. Initial current is 0. Therefore, the inductor current at $t = 9$ s is 12.5 A. In fact it is 12.5 A from $t = 6$ s onwards.
- (ii) 25 Wb-T of additional flux linkage will change the current of a 2 H inductor by 12.5 A. Hence, if the current at $t = 9$ s is observed to be 0 A, then the initial current must be -12.5 A.
- (iii) The equations for $v(t)$ in various time intervals are shown below.

$$v(t) = \begin{cases} 5t & \text{for } 0 \leq t < 1 \\ 5 & \text{for } 1 \leq t < 5 \\ 5 - 5(t - 5) & \text{for } 5 \leq t < 6 \\ 0 & \text{for } 6 \leq t \leq 9 \end{cases}$$

The inductor current can be found by integrating the $v(t)$ expression and dividing the integral by L .

$$i(t) = \begin{cases} 0 + 1.25t^2 & \text{for } 0 \leq t < 1 \\ 1.25 + 2.5(t - 1) & \text{for } 1 \leq t < 5 \\ 11.25 + 2.5(t - 5) & \text{for } 5 \leq t < 6 \\ 12.5 & \text{for } 6 \leq t \leq 9 \end{cases}$$

This is plotted in Fig. 3.2-14.

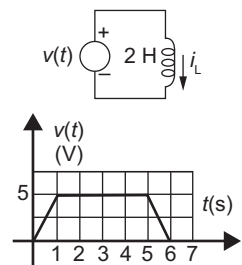


Fig. 3.2-13 Circuit and Waveform for Example 3.2-4

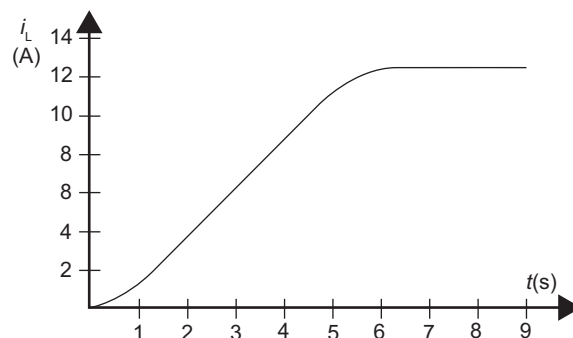


Fig. 3.2-14 Plot of Current in 2 H Inductor for Example 3.2-4

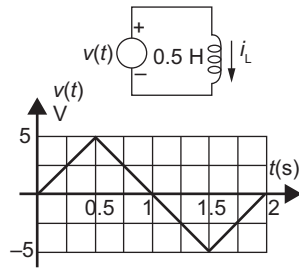


Fig. 3.2-15 Circuit and Waveform for Example 3.2-5

EXAMPLE: 3.2-5

The pulse voltage waveform as shown in Fig. 3.2-15 is applied to an inductor of 0.5 H with initial current of 1 A. (i) Find the inductor current and energy stored in the inductor at (a) $t = 1$ s (b) $t = 2$ s and (c) $t = 3$ s. (ii) What is the net energy delivered to inductance in the time interval (a) 0 s to 1 s (b) 1 s to 2 s and (c) 0 s to 2 s? (iii) What is the maximum value of inductor current and when does it occur?

SOLUTION

(i) The area under voltage waveform from 0 s to 1 s is 2.5 volt-sec. The value of inductance is 0.5 H. Therefore, the change in inductor current over this 1 s interval is $2.5/0.5 = 5$ A. Initial current was 1 A. There is no impulse voltage present at $t = 0$. Therefore, there is no instantaneous change in current at $t = 0$. Hence, the value of current at $t = 1$ s is $1 + 5 = 6$ A.

The area under voltage waveform from 0 s to 2 s is zero. Therefore, the net change in inductor current over this interval is zero. Hence, current at $t = 2$ s is $1 + 0 = 1$ A.

Only zero voltage is applied after 2 s. Hence the area under voltage waveform from 0 s to 3 s is again zero and hence current at $t = 3$ s is 1 A. In fact, current is at 1 A from 2 s onwards.

Energy storage in an inductor is given by $0.5 Li^2$ J. Hence energy storage in 0.5 H inductor at $t = 1$ s is 9 J, at $t = 2$ s is 0.25 J and at $t = 3$ s is 0.25 J.

(ii) Net energy delivered to inductor in the first 1 s = Energy storage in the inductor at $t = 1$ s minus energy storage in it at $t = 0$ s = $9 - 0.25 = 8.75$ J.

Net energy delivered to inductor in the interval between 1 s and 2 s = Energy storage in the inductor at $t = 2$ s minus energy storage in it at $t = 1$ s = $0.25 - 9 = -8.75$ J.

Net energy delivered to inductor in the interval between 0 s and 2 s = Energy storage in the inductor at $t = 2$ s minus energy storage in it at $t = 0$ s = $0.25 - 0.25 = 0$ J.

(iii) The area under voltage waveform keeps increasing in the interval $[0, 1]$ and hence inductor current keeps increasing in this interval. The area under voltage waveform starts decreasing after 1 s since voltage becomes negative from that point. Hence, $t = 1$ s is a time point at which the inductor current ceases to increase and starts to decrease. Hence, it must reach a local maximum there. It is a global maximum as well since the voltage does not become positive again. Value of this maximum current is 6 A. It occurs at $t = 1$ s.

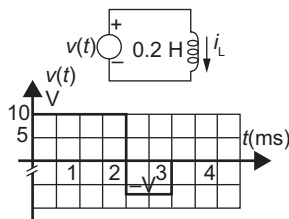


Fig. 3.2-16 Circuit and Waveform for Example 3.2-6

EXAMPLE: 3.2-6

The initial current in the inductor in the circuit as in Fig. 3.2-16 is 0.2 A in the direction shown. What should be V if the current in the inductor is to become 0 A at $t = 5$ ms?

SOLUTION

The area under voltage waveform from 0 s to 2 ms is 0.02 volt-sec. The value of inductance is 0.2 H. Hence, the change in inductor current over the first 2 ms will be $0.02/0.2 = 0.1$ A. Therefore, the current in 0.2 H inductor at $t = 2$ ms will be $0.2 + 0.1 = 0.3$ A.

$-V$ Volts is applied for 1 ms and 0 V is applied thereafter. Therefore, volt-sec added to the inductor will be -0.001 V volt-sec for all $t \geq 3$ ms. We want the current to become zero at 5 ms. This is possible only if it already becomes zero at 3 ms due to the -0.001 V volt-sec dumped into the inductor during 2 ms to 3 ms interval.

The required flux linkage change is $-0.2 \text{ H} \times 0.3 \text{ A} = -0.06 \text{ Wb-T}$. Change in flux linkage in an inductor is equal to the volt-sec dumped into it. Therefore, V has to be $0.06/0.001 = 60$ V.

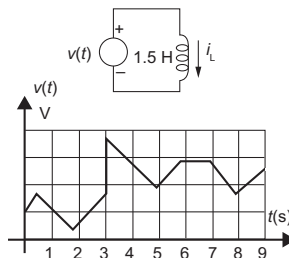


Fig. 3.2-17 Circuit and Waveform for Example 3.2-7

EXAMPLE: 3.2-7

An arbitrary time-varying voltage waveform is applied across 1.5 H with zero initial conditions as in Fig. 3.2-17. The current through the inductance is found to be 7.5 A at 7 s. (i) What must be the DC voltage that should be used to replace this source such that

the current through the inductor will be the same at 7 s? (ii) If this replacement is carried out will the current in the inductor be same in the two cases at $t = 9$ s?

SOLUTION

- (i) The *change* in inductor flux linkage over any time interval is equal to the area under voltage waveform during that interval. The *change* in flux linkage over 7 s interval is $1.5 \text{ H} \times (7.5 - 0) = 11.25 \text{ Wb-T}$. Therefore, the volt-sec during that period must be 11.25 volt-sec. If a constant voltage is to provide this much volt-sec in 7 s its value must be $11.25/7 = 1.607 \text{ V}$.
- (ii) If this replacement is carried out, the inductor current will be same in the two cases at 7 s. This is what can be asserted from the data provided. Since the $v(t)$ waveform is unknown we cannot expect its area at any particular time instant to be equal to the area under a constant function. There is no reason why it cannot be so. In short, in the absence any additional data on $v(t)$ we cannot make any predictions about equality of currents at 9 s in the two cases.

EXAMPLE: 3.2-8

A periodic voltage waveform is applied across an inductor of value 0.1 H from $t = 0$ s as in Fig. 3.2-18. The current in the inductor is found to vary periodically between 1 A and 5 A. (i) What is the full-cycle average value of the applied voltage waveform? (ii) What is the half-cycle average of the applied voltage waveform? (iii) What was the initial current in the inductor? (iv) Find V_p .

SOLUTION

- (i) The current is stated to be periodic. Therefore, it must either be a pure alternating waveform or such an alternating waveform plus a DC offset. Differentiation of a pure alternating waveform gives another pure alternating waveform. Differentiation of a DC term can give only zero. Hence, the derivative of inductor current will not contain DC term. Derivative of current multiplied by inductance value is the voltage across the inductor. Therefore, voltage across the inductor will not have a DC value. But the DC content in a periodic waveform is nothing but its average over a cycle period. Therefore, this voltage waveform has a full-cycle average of 0 V.
- (ii) Half-cycle average of alternating voltage = Half-cycle area/half the time period. Half-cycle area of the alternating voltage is given by *change* in flux linkage of inductor between the maximum and minimum current values. It is $(5 - 1) \text{ A} \times 0.1 \text{ H} = 0.4 \text{ Wb-T}$ in this case. Therefore, the half-cycle area of voltage waveform is 0.4 volt-sec and its half-cycle average value is $0.4 \text{ volt-sec}/4 \text{ s} = 0.1 \text{ V}$.

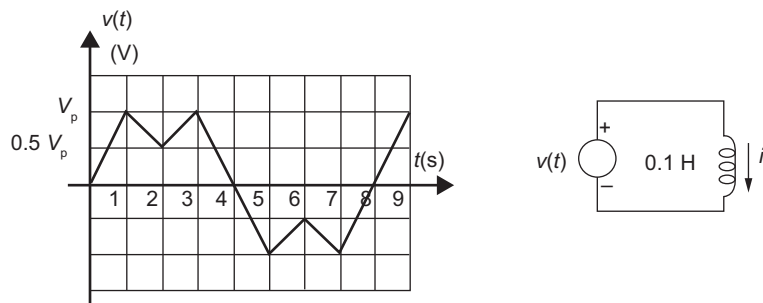


Fig. 3.2-18 Circuit and Waveform for Example 3.2-8

- (iii) The inductor current is periodic between 1 A and 5 A. There is no impulse content in the applied voltage. Hence, its initial current must have been 1 A.
- (iv) The half-cycle area in terms of V_p is $(0.5 V_p + 0.5 V_p + 0.25 V_p) \times 2 = 2.5 V_p$ volt-sec. This must be equal to 0.4 volt-sec.
- Therefore, $V_p = 0.4/2.5 = 0.16$ V.

3.3 SERIES CONNECTION OF INDUCTORS

A single equivalent inductor can replace many inductors connected in series for specific analysis purposes. We look into series equivalent and constraints on it in this section.

3.3.1 Series Connection of Inductors with Same Initial Current

We consider series connection of n inductors which have no mutual magnetic coupling among them. Let the inductance values be $L_1, L_2, \dots, L_n$. We assume that they have the same initial current in the same direction at $t = 0^-$. Let the applied voltage be $v(t)$ and the current in the series combination be $i(t)$ (Fig. 3.3-1).

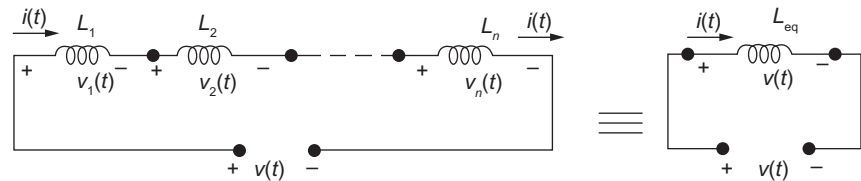


Fig. 3.3-1 Series Connection of n Inductors

Series Connection of Inductors

Series connection of many inductors without mutual coupling and with same initial currents can be replaced by a single inductor of inductance value equal to sum of inductance values of all the inductors in series.

Total voltage in a series combination divides in proportion to inductance value across various inductors.

Total flux linkage developed in a series combination is distributed in proportion to inductance value in various inductors.

Applying KVL along with the element equation of inductor, we get

$$\begin{aligned}
 v(t) &= v_1(t) + v_2(t) + \dots + v_n(t) \\
 &= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \dots + L_n \frac{di(t)}{dt} \\
 &= (L_1 + L_2 + \dots + L_n) \frac{di(t)}{dt} \\
 &= L_{eq} \frac{di(t)}{dt}, \text{ where } L_{eq} = L_1 + L_2 + \dots + L_n
 \end{aligned}$$

Thus, a series connection of n inductors may be replaced by an equivalent inductor with an inductance value equal to sum of the inductance values of n inductors as far as the v - i relationship is concerned. The total applied voltage across the combination is shared by the various inductors in direct proportion to inductance value. i.e.,

$$\begin{aligned}
 v(t) &= v_1(t) + v_2(t) + \dots + v_n(t) \\
 v_1(t) : v_2(t) : \dots : v_n(t) &= L_1 : L_2 : \dots : L_n \\
 v_j(t) &= \frac{L_j}{L_{eq}} v(t) \text{ for } j = 1 \dots n.
 \end{aligned}$$

It can be seen that sum of flux linkages in the individual inductors is same as the flux linkage of the equivalent inductor. Thus, flux linkage is shared in proportion to the inductance value. Similarly, the total energy stored in all inductors put together is the same as the energy storage calculated using equivalent inductance value.

Thus, the series equivalent of n inductors is 'equivalent' with respect to $v-i$ relation, flux linkage and stored energy. Note that the participating inductors should not have mutual magnetic coupling. And that they must have the same initial current.

3.3.2 Series Connection with Unequal Initial Currents

What happens if they have different initial currents? We take up a simple situation of two inductors L_1 and L_2 in series with initial currents of I_1 and I_2 , respectively. They have been connected in series at $t = 0$ and the terminals of the series combination are shorted to allow the initial current to flow. What is the common initial current at $t = 0^+$?

The current in both inductors must be same at $t = 0^+$ – otherwise KCL gets violated. Therefore, it looks as if the current in both inductors will have to change instantaneously. But that requires impulse voltage.

Impulse voltage may not be a problem in this circuit since inductor can support impulse voltages. Let us assume that an impulse voltage of magnitude k appeared across L_2 at $t = 0$. Then $-k\delta(t)$ must have appeared across L_1 . Therefore, the current in L_1 will change to $I_1 - k/L_1$ from I_1 and current in L_2 will change to $I_2 + k/L_2$ from I_2 . We want $I_1 - k/L_1$ to be equal to $I_2 + k/L_2$. This gives us $k = L_1 L_2 (I_1 - I_2) / (L_1 + L_2)$. Therefore, the common current at $t = 0^+$ will be

$$I = \frac{L_1 I_1 + L_2 I_2}{L_1 + L_2}.$$

But the method given above for resolving the initial condition conflict is WRONG. It does not satisfy law of conservation of energy. Let us calculate the total stored energy at $t = 0^-$ and at $t = 0^+$. At $t = 0^-$ it is $0.5(L_1 I_1^2 + L_2 I_2^2)$ J and at $t = 0^+$ it is $0.5(L_1 I_1 + L_2 I_2)^2 / (L_1 + L_2)$ J. These two are not equal. Where did the mismatch energy go? There was no source in the circuit. Both conservation of energy and conservation of charge are unyielding physical laws – one cannot be compromised for the other. Therefore, the impulse function based attempt to resolve the initial current conflict will not work.

What is the correct solution to the problem? The fact of the matter is that this is one situation in which the idealised model of an inductance does not work for a physical inductor. A physical inductor has series resistance representing its winding resistance, a parallel resistance representing losses in its core (if an iron core is used) and a parallel capacitance representing the aggregate effect of inter-winding capacitance. An inductance is only a first level mathematical model for a physical inductor. There are application contexts in which we should be ready to drop the first level model for device and use higher level models that represent the physics of the device in more detail. The second level model for an inductor is shown in Fig. 3.3-2(b).

Now, even if the initial currents are unequal at the moment when they are put in series, the inductors have alternate paths available to them for diverting initial currents. But then, the circuit contains many components of different types and is no longer a single element circuit. Obviously, no series equivalent inductance can be thought of when two circuits like the one in Fig. 3.3-2(b) are put in series. In fact, we will see in a later chapter that connecting two inductors with different initial currents in series will result in high frequency oscillating currents and voltages everywhere in the circuit due to the winding capacitance of the inductors.

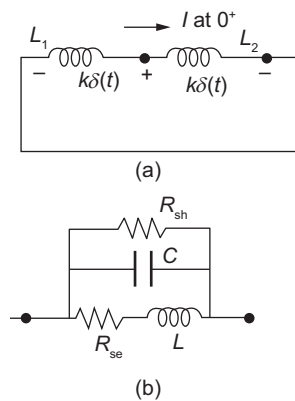


Fig. 3.3-2 (a) Series Connection with Unequal Initial Currents (b) Better Circuit Model for Inductor

The circuit problem arising out of series connection of inductors with unequal currents cannot be solved without bringing in losses and distributed capacitance in the inductors.

3.4 PARALLEL CONNECTION OF INDUCTORS

In this section, we develop the equivalent of n inductors in parallel. Here, we consider only *magnetically uncoupled* inductors. Further, we assume that all the inductors had zero initial current in them at the instant of paralleling. We will remove this restriction in the later part of this section.

3.4.1 Parallel Connection of Initially Relaxed Inductors

The *change* in inductor current over a time interval is given by the volt-sec applied to it during that time interval divided by value of inductance. We assumed zero initial currents at $t = 0$ and hence,

$$i_k(t) = 0 + \frac{1}{L_k} \int_{0^-}^t v(t) dt \text{ for } k = 1 \text{ to } n.$$

The volt-sec product applied to all the inductors will be the same since they all share a common voltage from $t = 0^-$ onwards. Hence, the currents in inductors will be *inversely proportional* to the inductance values. i.e.,

$$i_1(t) : i_2(t) : \dots : i_n(t) = \frac{1}{L_1} : \frac{1}{L_2} : \dots : \frac{1}{L_n}.$$

Further, applying KCL at the positive node of voltage source in the circuit as in Fig. 3.4-1, we get,

$$\begin{aligned} i(t) &= i_1(t) + i_2(t) + \dots + i_n(t) \\ &= \left[\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right] \int_{0^-}^t v(t) dt \\ &= \frac{1}{L_{eq}} \int_{0^-}^t v(t) dt. \end{aligned}$$

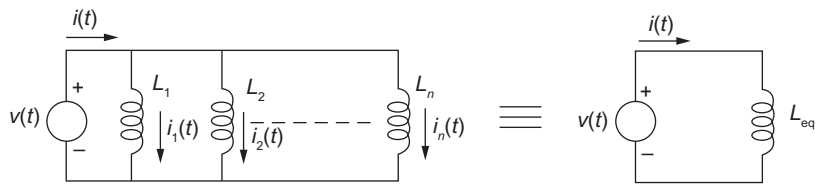


Fig. 3.4-1 Parallel Connection of n Inductors

Therefore, we see that a single inductor L_{eq} can represent the n inductors in parallel as far as the v - i relationship is concerned. This implies that the source will not be able to distinguish between the parallel combination of n inductors and a single inductor that is the parallel equivalent.

We assumed that all inductors started with zero initial current. Therefore, all of them had zero initial flux linkage and zero initial energy. The *change* in flux linkage in an inductance over a time interval is same as the volt-sec applied to it during that time interval. Since all of them started at zero flux linkage as per our assumption, it follows that all of them will have same flux linkage at all t . Hence *voltage, volt-sec and flux linkage are common variables in an initially relaxed parallel connection of inductors.*

Current sharing ratio in parallel connected initially relaxed inductors.

Now, we look at the stored energy picture. We know that the stored energy function $E(t)$ of an inductor $= 0.5 Li^2$, where i is the instantaneous current. However, we need another formulation for the same function now. We proceed as below.

$$\psi(t) = \psi(0^-) + \int_{0^-}^t v(t) dt = \text{initial flux linkage} + \text{volt-sec added}$$

$$\text{Also, } \psi(t) = Li(t)$$

$$\therefore E(t) = \frac{1}{2} L[i(t)]^2 = \frac{1}{2} \psi(t)i(t) = \frac{1}{2L} (\text{volt-sec})^2 \text{ (if initial flux linkage is zero).}$$

Hence, the stored energy at any instant in an inductor is proportional to the square of volt-sec product dumped into it till that instant provided the inductor was initially relaxed. Therefore, in the parallel connection of inductors under consideration, the stored energy in the inductors will be inversely proportional to their inductance values. They all share the same volt-sec product at the same instant.

$$\begin{aligned} E_{\text{total}}(t) &= E_1(t) + E_2(t) + \dots + E_n(t) \\ &= \frac{1}{2} \left[\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right] [\text{volt-sec}]^2 \\ &= \frac{1}{2L_{\text{eq}}} [\text{volt-sec}]^2 \\ &= \text{Energy stored in an initially relaxed inductor } L_{\text{eq}}. \end{aligned}$$

Expression for energy storage in an inductor in terms of volt-sec.

Therefore, the equivalent inductance L_{eq} gives the correct value for stored energy. Note that we have shown this only for a parallel connection of initially relaxed inductors.

3.4.2 Parallel Connection of Inductors with Initial Energy

We discuss the general case of parallel connection of inductors with non-zero initial current in detail in this section.

Let a parallel connection of two inductors be formed along with rest of the circuit at $t = 0$ as in Fig. 3.4-2 and let the initial currents in the inductors be I_1 and I_2 in L_1 and L_2 , respectively. We assume that there are no impulse voltages around. Therefore, the value of $i(t)$ at $t = 0^+$ must be $I_1 + I_2$.

The two inductors share the same volt-sec product from $t = 0^-$ onwards and hence they have the same *change* in flux linkage from $t = 0^-$ onwards. Therefore, they will have *change* in current proportional to $1/L$ since *change* in flux linkage $= L \times \text{change in current}$. Therefore, the parallel connection will behave as if it is a single inductor of value $L_{\text{eq}} = L_1 L_2 / (L_1 + L_2)$ as far as the current *changes* are concerned.

Since the *change* in total current gets shared in the inductors in inverse proportion to their inductance values, we have

$$\begin{aligned} \text{change in } i_1(t) &= \text{change in } i(t) \times \frac{1/L_1}{1/L_1 + 1/L_2} \\ &= \text{change in } i(t) \times \frac{L_2}{L_1 + L_2} \\ \text{change in } i_2(t) &= \text{change in } i(t) \times \frac{L_1}{L_1 + L_2}. \end{aligned}$$

Now, we calculate the *change* in total energy of the two inductors and verify whether L_{eq} will predict the same *change* in energy.

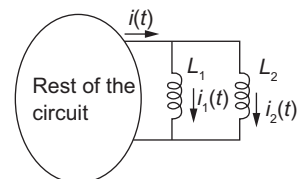


Fig. 3.4-2 Parallel Connection of Two Inductors with Initial Energy

Expression for change in Total Energy using equivalent inductance

Initial Current in $L_{eq} = I_1 + I_2$

Current at $t = I_1 + I_2 + \Delta i$

$$\begin{aligned}\therefore \Delta E_{total} &= \frac{L_{eq}}{2} [(I_1 + I_2 + \Delta i)^2 - (I_1 + I_2)^2] \\ &= \frac{L_1 L_2}{2(L_1 + L_2)} [(\Delta i)^2 + 2(I_1 + I_2)\Delta i].\end{aligned}$$

Expression for change in Total Energy using individual inductance values

Initial Current in $L_1 = I_1$

Current in L_1 at $t = I_1 + \frac{L_2}{L_1 + L_2} \Delta i$

$$\therefore \Delta E_1 = \frac{L_1}{2} \left[\left(I_1 + \frac{L_2}{L_1 + L_2} \Delta i \right)^2 - (I_1)^2 \right] = \frac{L_1}{2} \left[\left(\frac{L_2}{L_1 + L_2} \Delta i \right)^2 + 2 \frac{L_2}{L_1 + L_2} I_1 \Delta i \right]$$

$$\text{Similarly, } \Delta E_2 = \frac{L_2}{2} \left[\left(\frac{L_1}{L_1 + L_2} \Delta i \right)^2 + 2 \frac{L_1}{L_1 + L_2} I_2 \Delta i \right]$$

$$\Delta E_{total} = \Delta E_1 + \Delta E_2$$

$$= \frac{1}{2} \frac{L_1 L_2}{L_1 + L_2} [(\Delta i)^2 + 2(I_1 + I_2)\Delta i].$$

Parallel Connection of Inductors

A single inductor L_{eq} can replace a set of n inductors without mutual coupling connected in parallel as far as changes in flux linkages, changes in currents and changes in stored energy are concerned.

$$\frac{1}{L_{eq}} = \left[\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right]$$

However, the total flux linkage in each inductor and total flux linkage in L_{eq} will not be the same unless all inductors had the same flux linkage at $t = 0^-$.

Similarly, the total stored energy in the system will not be the same as the total stored energy in L_{eq} unless all inductors had same flux linkage at $t = 0^-$.

If the inductors started with different flux linkages at $t = 0^-$, circulating currents will be set up in local loops formed by the inductors, thereby trapping a portion of total initial energy and hiding that portion of initial energy from rest of the circuit forever.

Thus, both approaches lead to the same total *change* in total stored energy. Therefore L_{eq} is equivalent to parallel connected inductors for *change* in total stored energy too. However, the distribution of *change* in energy in individual inductors is *not* as per reciprocal of inductance values.

Now, we come to the total flux linkage of each inductor. All the inductors get the same volt-sec in them from $t = 0^-$ onwards since voltage is a common variable in parallel connection. Therefore, they will have same change in flux linkage after $t = 0^-$. However, if they start with different initial flux linkages, i.e., if $L_1 I_1 \neq L_2 I_2$, then $L_1 I_1 + \Delta \psi \neq L_2 I_2 + \Delta \psi$ and they will never have equal flux linkage. Also, the total flux linkage predicted by L_{eq} as $L_1 L_2 (I_1 + I_2 + \Delta i) / (L_1 + L_2)$ will also be not equal to any of the above two values. Thus, the three values of flux linkages are different unless the inductors started with same flux linkage (which includes zero flux linkage as a special case).

Similar conclusions can be drawn on total energy stored in the system and in the equivalent inductor. They are equal only if $L_1 I_1 = L_2 I_2$. Therefore, L_{eq} cannot replace a parallel combination for total flux linkage studies and total stored energy studies.

The initial current in L_{eq} is $I_1 + I_2$. This gives us an impression that initial stored energy is $0.5 L_{eq} (I_1 + I_2)^2$ J. But we know that the total initial stored energy is $0.5 (L_1 I_1^2 + L_2 I_2^2)$. These two quantities will be equal only if the inductors had same initial flux linkage, i.e., only if $L_1 I_1 = L_2 I_2$.

If we make two circuits – one containing L_1 and L_2 in parallel and another containing a single inductor of value $L_1 L_2 / (L_1 + L_2)$ – and put them in black boxes and apply same voltage source to both, we will not be able to distinguish between them by observing the current response. Both circuits will show identical initial currents and identical currents at any t . Thus, we expect both circuits to carry same initial energy. We proceed to check it out by extracting the initial energy from the black boxes. How do we do that? We connect a DC voltage of V V with such a polarity that the current will tend to go down. We will disconnect the source the moment current touches zero. In this process we expect all the initial stored energy to go out into the DC voltage source and we expect the black boxes to reach a

peaceful state of zero energy. It will take $L_{eq}(I_1 + I_2)/V$ s for this to happen. The black box containing L_{eq} will deliver $0.5 L_{eq}(I_1 + I_2)^2$ J of energy to the DC source and reach a zero energy state. But what about the other one?

We dumped $-V \times L_{eq}(I_1 + I_2)/V = -L_{eq}(I_1 + I_2)$ volt-sec into both L_1 and L_2 . Therefore, the current in L_1 becomes $I_1 - L_{eq}(I_1 + I_2)/L_1 = (L_1 I_1 - L_2 I_2)/(L_1 + L_2)$. Similarly, the current in L_2 becomes $I_2 - L_{eq}(I_1 + I_2)/L_2 = (-L_1 I_1 + L_2 I_2)/(L_1 + L_2)$. Notice that the two currents form a circulating current in the local loop formed by the two inductors. The individual currents in the two inductors do not go to zero when the current flowing out of the parallel combination goes to zero. There will be some stored energy associated with the circulating current in both inductors that is trapped in the loop forever. This energy *cannot* be taken out whatever we do to the parallel combination. The currents in two parallel inductors can never go to zero simultaneously unless they started with same initial flux linkage since they always undergo identical changes in flux linkage after they get connected in parallel.

EXAMPLE: 3.4-1

Three inductors of 0.1 H, 0.05 H and 0.15 H with equal initial current of 1.5 A are connected in series at $t = 0$. $v(t)$ from $t = 0$ is $0.3\delta(t) + v_s(t)$, where $v_s(t)$ is not known. $i(t)$ is found to be 3 A in the same direction as that of initial current flow at $t = 5$ s and $v(t)$ at that instant is found to be 6 V. (i) Find the voltage across each inductor and flux linkages in them at $t = 5$ s. (ii) The average value of $v_s(t)$ in the first 5 s. (iii) Stored energy in the circuit and in each inductor at $t = 5$ s. (iv) Total energy delivered by the applied voltage source in 5 s and average power delivered by the source over 5 s?

SOLUTION

The inductors are in series and they have same initial current. Hence, they can be replaced by a single inductor of $0.1 + 0.05 + 0.15 = 0.3$ H inductor as shown in Fig. 3.4-3.

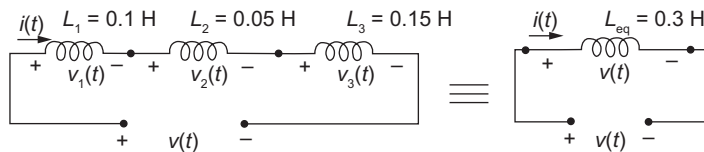


Fig. 3.4-3 Circuit for Solving Example 3.4-1

- (i) Inductors in series share instantaneous voltage in proportion to their inductance values. Hence, 6 V gets distributed in the ratio 0.1:0.05:0.15, i.e., 2:1:3. Therefore, 0.1 H gets 2 V across it, 0.05 H gets 1 V across it and 0.15 H gets 3 V across it at $t = 5$ s.

Total flux linkage in the combination at $t = 5$ s is $L_{eq}i = 0.3 \text{ H} \times 3 \text{ A} = 0.9 \text{ Wb-T}$. Total flux linkage gets distributed in proportion to inductance value in a series combination. Hence, 0.9 Wb-T gets distributed in the ratio 2:1:3 in the three inductors. Hence, at $t = 5$ s the 0.1 H inductor will have 0.3 Wb-T flux linkage in it, 0.05 H will have 0.15 Wb-T flux linkage in it and 0.15 H will have 0.45 Wb-T in it.

- (ii) The total volt-sec dumped into effective inductance of 0.3 H in 5 s = 0.3 volt-sec from the impulse component + area of $v_s(t)$ from $t = 0$ s to $t = 5$ s. But, the total volt-sec dumped must be equal to flux linkage change. This change in flux linkage is $0.3 \text{ H} \times (3 \text{ A} - 1.2 \text{ A}) = 0.54 \text{ Wb-T}$.

Therefore, $0.54 = 0.3 + \text{area of } v_s(t) \text{ in } [0 \text{ s}, 5 \text{ s}]$.

Therefore, area of $v_s(t)$ in $[0 \text{ s}, 5 \text{ s}] = 0.24 \text{ volt-sec}$.

Therefore, average value of $v_s(t)$ over $[0 \text{ s}, 5 \text{ s}] = 0.24 \text{ V-s}/5 \text{ s} = 0.48 \text{ V}$.

- (iii) The total stored energy in a series combination will be distributed among the inductors in proportion to inductance value. The total energy storage at $t = 5$ s will be $0.5 \times 0.3 \times 3^2 = 1.35 \text{ J}$. Stored energy will be 0.45 J in 0.1 H inductor, 0.225 J in 0.05 H inductor and 0.675 J in 0.15 inductor.

(iv) Initial stored energy in the system was $0.5 \times 0.3 \times 1.2^2 = 0.216$ J. Total stored energy at $t = 5$ s is 1.35 J.

Therefore, the total energy delivered by voltage source in the first 5 s = $1.35 - 0.216 = 1.134$ J.

Therefore, the average power delivered in the first 5 s = $1.134 \text{ J} / 5 \text{ s} = 226.8 \text{ mW}$.

EXAMPLE: 3.4-2

The three inductors – 0.02 H, 0.05 H and 0.0333 H – with same initial current of 1.5 A in Fig. 3.4-4 are connected in parallel. All the three inductors have initial current of 1.5 A from top to down at $t = 0^-$. The applied voltage is $v_s(t)$ and has no impulse content. The total current into the parallel combination is found to be 9 A in the same direction as the initial current at $t = 5$ ms. (i) Find the trapped energy in the parallel combination. (ii) Find the current through each inductor and flux linkages in them at $t = 5$ ms. (iii) The average value of $v_s(t)$ in the first 5 ms. (iv) Stored energy in the circuit and in each inductor at $t = 5$ ms. (v) Total energy delivered by the applied voltage source in 5 ms and average power delivered by the source over 5 ms?

SOLUTION

The circuit containing three inductors and its equivalent circuit are shown in Fig. 3.4-4. The initial currents in various paths at $t = 0^-$ are marked within parenthesis. The initial values remain the same at $t = 0^+$ too since applied voltage has no impulse content.

(i) Initial stored energy = Sum of initial energy in the three inductors = $0.5 \times 0.1033 \times 1.5^2 = 116.21$ mJ. But the initial energy as per the equivalent circuit is $0.5 \times 0.01 \times 4.5^2 = 101.25$ mJ. The difference between these two will be the trapped energy. The total energy that is actually stored in the three inductors put together and the total energy storage according to equivalent circuit will be different by this amount at all t . Trapped energy in this circuit is $116.21 - 101.25 \approx 15$ mJ.

(ii) $i(t)$ at 5 ms is 9 A. Therefore, the change in current is $9 - 4.5 = 4.5$ A. The change in current gets shared among the three inductors in proportion to $1/L$ value. Therefore, the change in total current gets divided in the three inductors in the ratio 50:20:30 or 5:2:3. Therefore, the change in current of 0.02 H = $4.5 \times 5/(5 + 2 + 3) = 2.25$ A. Similarly the currents in the other two inductors will change by 0.9 A and 1.35 A, respectively. The change current in all three inductors flow in the same direction as the initial current. Therefore, the total current in the 0.02 H = $1.5 + 2.25 = 3.75$ A, in 0.05 H = $1.5 + 0.9 = 2.4$ A and in 0.0333 H = $1.5 + 1.35 = 2.85$ A.

The flux linkages in the inductors at 5 ms can be worked out by multiplying $i(t)$ by L values. They are 75 mWb-T in 0.02 H, 120 mWb-T in 0.05 H and 95 mWb-T in 0.0333 H. The same values can be obtained by another method. The change in flux linkage will be correctly predicted by equivalent circuit – it will be $4.5 \text{ A} \times 0.01 \text{ H} = 45 \text{ mWb-T}$. This change in flux linkage is applicable to all the three inductors. Hence, the total flux linkages in them can be obtained by adding this change amount to the initial flux linkages in them. Initial flux linkages were 30 mWb-T, 75 mWb-T and 50 mWb-T in 0.02 H, 0.05 H and 0.0333 H, respectively. Adding 45 mWb-T to each we get the total flux linkages as 75 mWb-T, 120 mWb-T and 95 mWb-T, respectively.

(iii) Average value of source voltage in 5 ms = (change in flux linkage of effective inductor) divided by 5 ms = $45 \text{ mWb-T} / 5 \text{ ms} = 45 \text{ mV-s} / 5 \text{ ms} = 9 \text{ V}$.

(iv) Stored energy in the 0.02 H inductor at 5 ms = $0.5 \times 0.02 \times 3.75^2 = 140.63$ mJ. Corresponding values for 0.05 H and 0.0333 H are 144 mJ and 135.24 mJ, respectively.

Total stored energy at 5 ms = $140.63 + 144 + 135.24 \approx 420$ mJ.

We can get the same results by another method. The change in total stored energy will be predicted correctly by the equivalent circuit. This value is $(0.5 \times 0.01 \times 9^2 - 0.5 \times 0.01 \times 4.5^2) = 303.75$ mJ. Adding the change to initial stored energy of 116.21 mJ, we get the total stored energy in the system at 5 ms as ≈ 420 mJ. The equivalent circuit predicts the total energy to be $0.5 \times 0.01 \times 9^2 = 405$ mJ. Notice that this is less than the actual total stored energy of ≈ 420 mJ by ≈ 15 mJ, which is the amount of trapped energy in the system as we have seen earlier.

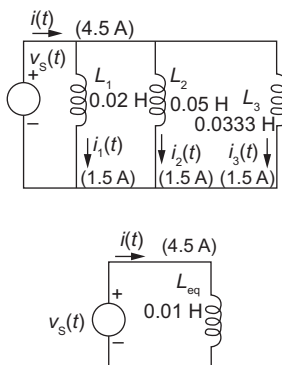


Fig. 3.4-4 Circuits for Solving Example 3.4-2

(v) Total energy delivered by $v_s(t)$ in 5 ms = change in total stored energy = 303.75 mJ.
 Average power delivered by voltage source during first 5 ms = 303.75 mJ/5 ms = 60.75 W.

3.5 THE CAPACITOR

The physical basis of the two-terminal element, called Capacitor, had been dealt with in Chap. 1 in detail. We revise it briefly.

Sources of e.m.f. present in circuits establish potential differences across various physical devices connected by creating charge distributions in them. This charge distribution present in any physical device involved in a circuit is static in the case of DC circuits. And it is quasi-static in the case of circuits containing time-varying sources of e.m.f. Therefore, there will be charge segregation and charge storage between any two nodes supporting a potential difference in a circuit. The ratio of charge to potential difference, i.e., the charge required to be present at the terminals of a device for it to have 1 V potential difference across itself, is termed as its capacitance. Thus, any physical two-terminal element will have a capacitance value associated with it along with other parameters representing other physical processes in it.

However, some two-terminal elements are intentionally designed to require more charge per unit potential difference than other kinds of two-terminal elements. In addition, they are designed in such a way that the electric field produced by charge distribution at their terminals will be confined to immediate vicinity of that device itself and will not produce potential differences in other devices. Such a two-terminal element specially designed to enhance the capacitive effect and to minimise resistive and inductive effects is called a *lumped capacitor*. When such lumped capacitors are present in a circuit we usually ignore the small capacitance invariably present across other lumped elements like two-terminal resistor and two-terminal inductor as a first level approximation.

The charge stored in a *linear* capacitor is proportional to the potential difference across it. The value of proportionality constant is called *capacitance value* of the capacitor. We use the same symbol C to represent the capacitor and its capacitance value simultaneously, i.e., in the context of an expression or calculation C stands for capacitance value and it is a pointer to the capacitor element in other contexts. Its unit is C/V or A-s/V, which is given a special name 'Farad' and is abbreviated by 'F'.

The graphic symbol, variable polarity as per passive sign convention and the element relations are shown below.

$$\begin{array}{l}
 \begin{array}{c} + \\ \xrightarrow{v(t)} \\ i(t) \end{array} \parallel \begin{array}{c} C \\ - \end{array} \\
 Q(t) = Cv(t), \text{ where } Q(t) = \text{instantaneous charge storage in } C \\
 i(t) = C \frac{dv(t)}{dt}; v(t) = \frac{1}{C} \int_{-\infty}^t i(t) dt \text{ and } Q(t) = \int_{-\infty}^t i(t) dt.
 \end{array}$$

Let us study in detail about the element relation of a capacitor.

The current through a capacitor is proportional to the rate of change of voltage across it. The voltage across the capacitor is proportional to the area under the current waveform, i.e., the A-s product (or Coulombs) applied through it from $t = -\infty$, where $t = -\infty$ has to be understood as the moment this capacitor came into being.

We notice that the element relationship of a capacitor is similar to that of an inductor. Only the role of voltage and current in the relationship has been interchanged.

Therefore, we need not enter into a detailed discussion on the implications of the element relationship of capacitor. Such a discussion will be analogous to the line of reasoning we employed in the case of inductor. Hence, we list the implications without detailed explanations.

Inductor is an element that accumulates flux linkage (volt-sec or Wb-T) and makes its response variable, i.e., current, proportional to the accumulated flux linkage.

Capacitor is an element that accumulates charge (amp-sec or C) and makes its response variable, i.e., voltage, proportional to the accumulated charge.

Relations among charge, current and voltage of a capacitor.

Restatement of v - i relation of a capacitor

The change in voltage across a capacitor is proportional to area-content under current waveform that is applied to it.

The time required to change a capacitor voltage by a given amount is inversely proportional to the average value of current applied to it.

Voltage in a capacitor can not change instantaneously unless an impulse current is applied or supported in the circuit.

The relationship $i(t) = C \frac{dv(t)}{dt}$ implies that:

1. *Instantaneous voltage across a capacitor cannot be predicted from instantaneous value of current through it.*
2. *If instantaneous value of current is positive the capacitor voltage will be increasing at that instant and if it is negative the voltage will be decreasing at that instant.*
3. *When current through a capacitor crosses zero in the downward direction, its voltage attains a local maximum and when it crosses zero in the upward direction, the capacitor voltage attains a local minimum.*
4. *Current through a capacitor with a constant voltage across it is zero.*
5. *Capacitor preserves the wave-shape for exponential and sinusoidal inputs.*
6. *The amplitude of voltage sinusoid in a capacitor is inversely proportional to the product of frequency of applied current sinusoid and capacitance value.*

The relationship $v(t) = \frac{1}{C} \int_{-\infty}^t i(t) dt = \frac{1}{C} \underbrace{\int_{-\infty}^{0^-} i(t) dt}_{V_0} + \frac{1}{C} \int_{0^-}^t i(t) dt = V_0 + \frac{1}{C} \int_0^t i(t) dt$ implies that:

1. *Change in capacitor voltage over $[t_1, t_2]$, $\Delta v = (\text{Area under capacitor current over } [t_1, t_2])/C$.*
2. *$v(t)$ at $t = t_2$ is $v(t)$ at $t = t_1$ plus Δv .*
3. *$v(t) = V_0 + (\text{Area under capacitor current over } [0, t])/C$,*

where V_0 is the voltage across the capacitor at $t = 0$ and is called initial condition of the capacitor:

The relationships $C \times \Delta v = \int_{t_1}^{t_2} i(t) dt$ and $i_{av} \text{ in } [t_1, t_2] = \frac{\int_{t_1}^{t_2} i(t) dt}{(t_2 - t_1)} = \frac{C \Delta v}{(t_2 - t_1)}$ implies that:

1. *The amount of voltage change required in a capacitor decides the area-content under current waveform to be applied to it to bring about the change. The time allowed to bring about it decides the average current to be applied. Thus, rapid change in the capacitor voltage calls for large amplitude current through it.*
2. *Voltage in a capacitor cannot change instantaneously unless an impulse current is applied or supported in the circuit.*
3. *Unit Impulse Current will have an area-content of unity since it is a unit impulse. Thus, unit impulse current will deposit 1 C of charge in a capacitor over $[0^-, 0^+]$, i.e., instantaneously. Therefore, the voltage across a capacitor C changes instantaneously by $1/C$ V when the circuit applies or supports a unit impulse current through it.*
4. *Therefore, if a circuit does not apply or support impulse current, the voltage across capacitors in that circuit will be continuous functions of time. Capacitors absorb rapid variations in circuit currents and tend to keep circuit voltages smooth.*

When the applied current through a capacitor is a periodic alternating waveform, the voltage across the capacitor will contain an alternating component with the same period. The peak-to-peak amplitude of this alternating component will be directly proportional to half cycle area of current waveform and inversely proportional to capacitance value. It decreases with increase in frequency of the current.

Therefore, a large valued capacitor in a circuit can absorb alternating currents in the circuit without contributing significant amount of alternating voltages to the circuit. A large valued capacitor can hold the potential difference across two points in a circuit at a reasonably constant level even when large amplitude alternating currents flow through them.

There can be a DC voltage across a capacitor even when the applied current waveform is a pure alternating one. The amount of DC content depends upon the initial condition of the capacitor and the point at which the current waveform is switched on to the capacitor.

A capacitor with zero initial voltage is a linear electrical element. A capacitor with non-zero initial voltage is a linear element as far as the voltage component caused by applied current is concerned.

The total energy delivered to a capacitor carrying a voltage V across it is $(1/2)CV^2$ J and this energy is stored in its electric field.

Stored energy in a capacitor is also given by $(1/2 C)Q^2$ J and $QV/2$ J.

The capacitor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.

EXAMPLE: 3.5-1

The voltage observed across the $10\text{ }\mu\text{F}$ capacitor is shown in the Fig. 3.5-1. Find the current source function $i_s(t)$ if the initial voltage across the capacitor was zero.

SOLUTION

The voltage across capacitor undergoes a sudden jump by 1 V at $t = 0$. This is possible only if a charge of $10\text{ }\mu\text{F} \times 1\text{ V} = 10\text{ }\mu\text{C}$ is dumped on the capacitor instantaneously at $t = 0$. Therefore, $i_s(t)$ must contain $10^{-5}\delta(t)$.

Similarly, at $t = 1\text{ }\mu\text{s}$ the capacitor voltage again jumps up by 1 V . This calls for another $10\text{ }\mu\text{C}$ to be dumped on the capacitor at $t = 1\text{ }\mu\text{s}$. Therefore, $i_s(t)$ must contain $10^{-5}\delta(t - 10^{-5})$.

The capacitor voltage jumps by -1 V at $t = 2\text{ }\mu\text{s}$. This calls for $-10^{-5}\delta(t - 2 \times 10^{-5})$ in $i_s(t)$. Proceeding this way we get the following waveform for $i_s(t)$ (Fig. 3.5-2).

The unit given in the vertical axis is Amperes. However, when impulse content is indicated in a waveform the value read from vertical axis must be interpreted as magnitude of area-content and unit must be suitably re-interpreted as coulombs (C) or amp-seconds (A-s).

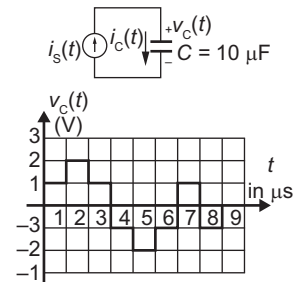


Fig. 3.5-1 Circuit and Waveform for Example 3.5-1

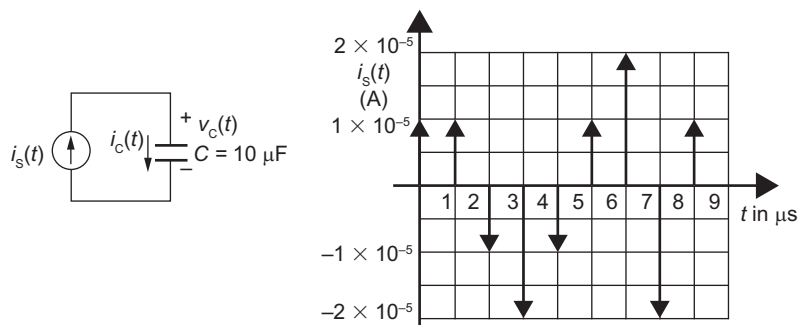


Fig. 3.5-2 Solution for Current Source Function for Example 3.5-1

EXAMPLE: 3.5-2

The voltage across a $1000\text{ }\mu\text{F}$ capacitor with zero initial voltage at $t = 0^-$ is given in Fig. 3.5-3. Find (i) the applied current waveform, (ii) waveforms of power and energy delivered by the current source, (iii) time intervals during which the capacitor is delivering energy to the source and (iv) the net energy delivered by capacitor to the source. Also, explain why the energy delivered by the source never becomes negative in this example.

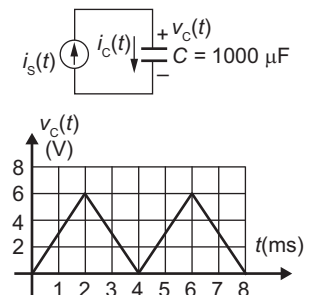


Fig. 3.5-3 Circuit and Waveform for Example 3.5-2

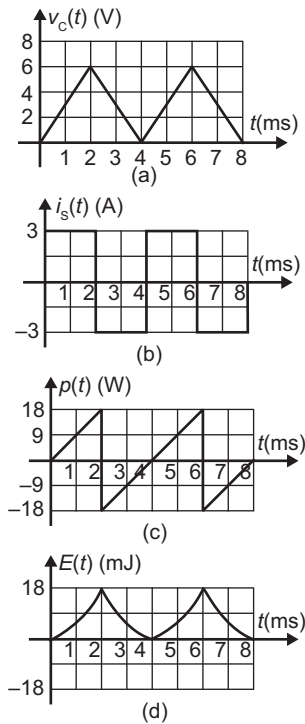


Fig. 3.5-4 Waveforms for Example 3.5-2 (a) Capacitor Voltage (b) Capacitor Current (c) Power Delivered by Source (d) Energy Delivered by Source

SOLUTION

(i) The current through a capacitor is given by the first derivative of voltage scaled by the capacitance value of the capacitor. The voltage waveform contains four straight-line segments followed by zero value. The slopes of voltage in various intervals are as follows.

Value of slope of capacitor voltage in $[0^+, 2^-] = 3 \text{ V/ms}$

Value of slope of capacitor voltage in $[2^+, 4^-] = -3 \text{ V/ms}$

Value of slope of capacitor voltage in $[4^+, 6^-] = 3 \text{ V/ms}$

Value of slope of capacitor voltage in $[6^+, 8^-] = -3 \text{ V/ms}$

Value of slope of capacitor voltage in $[8^+, 9] = 0 \text{ V/ms}$

Multiplying these values by 1 mF we get 3 A , -3 A , 3 A , -3 A and 0 A as the value of current in the five intervals. Hence, the current source function will be a rectangular pulse waveform as shown in Fig. 3.5-4(b).

(ii) The power delivered by the current source will be given by the product of $v_c(t)$ and $i_s(t)$. It will have a waveform containing straight-line segments since $v_c(t)$ contains straight-line segments and the waveform of $i_s(t)$ is a symmetric rectangular pulse waveform. The power waveform is shown in Fig. 3.5-4(c). The power delivered by the source alternates between positive and negative values.

Energy delivered by the current source is given by the running integral of power waveform from 0^+ . The waveform of delivered energy is shown in Fig. 3.5-4(d). It is always positive.

(iii) The capacitor is delivering energy to the current source when the power delivered by the current source shows a negative value.

Hence, during $[2^+, 4^-]$ and $[6^+, 8^-]$ (values indicating time in ms) time intervals the capacitor delivers energy to the current source.

(iv) The capacitor had an initial voltage of 0 V and it ends up with 0 V . Therefore, the net change in stored energy of capacitor is zero. There is no other element in the circuit that can store or dissipate energy. Hence, the net energy delivered by the current source also must be zero.

The capacitor in this example started with zero voltage initially. Hence the initial energy stored in it is zero. Capacitors can only store energy and they cannot generate or dissipate energy. They can store energy temporarily and give it back to other elements later. Therefore, energy function of a capacitor is always zero or positive-valued.

Electrical elements with an energy function, $E(t) = \int_{-\infty}^t v(t)i(t)dt$ that is ≥ 0 for all t

are called passive elements and capacitor is one such passive element.

A capacitor can give back more energy to a source than it received from it, even temporarily, only if it already had some energy in store before the source started acting on it. In this example, C had no such initial energy. Hence, the source cannot receive more than what it produced. Therefore, the value of energy delivered by the source will never be negative in this circuit.

EXAMPLE: 3.5-3

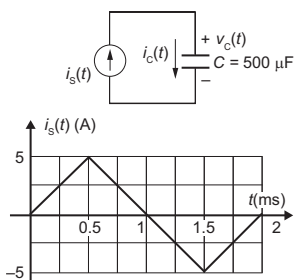


Fig. 3.5-5 Circuit and Waveform for Example 3.5-3

A pulse current waveform shown in Fig. 3.5-5 is used to charge a capacitor of 0.5 mF with zero initial voltage. $i_s(t)$ is zero after 2 ms . (i) Find voltage developed across the capacitor at (a) $t = 1 \text{ ms}$ (b) $t = 2 \text{ ms}$ (c) $t = 4 \text{ ms}$. (ii) What is the maximum voltage across the capacitor and when does it occur? (iii) What is the net energy delivered by source to the capacitor?

SOLUTION

(i) The change in voltage across a capacitor over a time interval is equal to the amp-sec product dumped into it during that time interval divided by capacitance value.

The area under $i_s(t)$ for $[0^+, 1 \text{ ms}] = 2.5 \text{ mC}$

The area under $i_s(t)$ for $[0^+, 2 \text{ ms}] = 0 \text{ mC}$

The area under $i_s(t)$ for $[0^+, 4 \text{ ms}] = 0 \text{ mC}$ since $i_s(t)$ is zero after 2 ms.

Initial voltage across capacitor is given to be zero.

Therefore, the voltage across capacitor at 1 ms = $0 + 2.5 \text{ mC}/0.5 \text{ mF} = 5 \text{ V}$

Voltage across capacitor at 2 ms = $0 + 0 = 0 \text{ V}$

Voltage across capacitor at 4 ms = $0 + 0 = 0 \text{ V}$

- (ii) The current employed to charge the capacitor is positive in the interval $[0^+, 1 \text{ ms}]$ and hence the voltage across capacitor increases in this interval. The current changes polarity at 1 ms and becomes a discharging current from then on. Therefore, 1 ms is the time instant at which the capacitor voltage reaches a local maximum. In this case it is a global maximum as well since the current never becomes positive after 1 ms. Hence, the maximum value of capacitor voltage is 5 V and it occurs at 1 ms.
- (iii) The capacitor started with zero voltage and ends up with zero voltage. Therefore, change in stored energy in the capacitor is zero. And hence, the net energy delivered by the source to capacitor is zero.

EXAMPLE: 3.5-4

Initial voltage at $t = 0^-$ across the $200 \mu\text{F}$ capacitor in Fig. 3.5-6 is -50 V . The current source is zero after 3 ms. Find (i) the voltage across the capacitor at 2 ms, energy stored in it at 2 ms and the energy delivered by the current source in the first 2 ms, (ii) the value of I if the capacitor voltage is to become zero at 4 ms and (iii) the net energy delivered by current source with the above value of I .

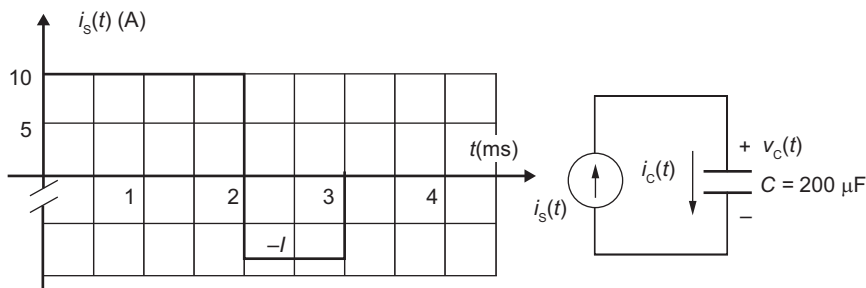


Fig. 3.5-6 Circuit and Waveform for Example 3.5-4

SOLUTION

- (i) The area under current waveform over 0 ms to 2 ms interval is 20 mC. Hence the change in capacitor voltage is $20 \text{ mC}/0.2 \text{ mF} = 100 \text{ V}$. Initial voltage is -50 V . Therefore, the voltage across capacitor at $t = 2 \text{ ms}$ is $-50 + 100 = 50 \text{ V}$.

Energy stored in a capacitor is given by $0.5 CV^2 \text{ J}$. Therefore, the energy stored in capacitor at $t = 2 \text{ ms}$ is 250 mJ.

Energy delivered by current source in the first 2 ms = change in stored energy in the capacitor over first 2 ms. Initial stored energy in the capacitor is $0.5 \times 0.2 \text{ mF} \times (-50 \text{ V}) \times (-50 \text{ V}) = 250 \text{ mJ}$. Stored energy at 2 ms = $0.5 \times 0.2 \text{ mF} \times 50 \text{ V} \times 50 \text{ V} = 250 \text{ mJ}$. Therefore, the change in stored energy over first 2 ms is 0. And hence, the energy delivered by current source over this time interval is zero.

The capacitor voltage starts at -50 V and increases linearly with a slope of 50 V/ms for the first 2 ms. Therefore, the voltage is linear from -50 V to $+50 \text{ V}$, crossing

0 at 1 ms. The power delivered by current source is negative in the first 1 ms and positive in the second 1 ms. The capacitor empties its initial energy into the current source in the first 1 ms and the current source puts it back on the capacitor over the next 1 ms duration.

- (ii) The capacitor has 50 V across it at 2 ms. Hence a charge of $0.2 \text{ mF} \times 50 \text{ V} = 10 \text{ mC}$ has to be removed at a constant rate of $I \text{ C/s}$ over [2 ms, 3 ms]. Therefore, I has to be $10 \text{ mC}/1 \text{ ms} = 10 \text{ C/s} = 10 \text{ A}$. With this value of I , the capacitor voltage becomes zero at 3 ms and remains at zero thereafter since $i_s(t)$ is given to be zero after 3 ms.
- (iii) With $I = 10 \text{ A}$ the capacitor voltage becomes 0 V at 3 ms. Therefore, the final stored energy is zero and initial stored energy is 250 mJ. Thus, the change in stored energy is -250 mJ. There are no other energy storage elements or energy dissipating elements in the circuit. Hence, this value must be the net energy delivered by the current source. It is negative implying that the source received 250 mJ of energy from the capacitor. It is so because the capacitor started with 250 mJ in it and ended with 0 J.

EXAMPLE: 3.5-5

The switch S in the circuit as in Fig. 3.5-7(a) is ideal and capacitor is initially uncharged. The switch S is closed at $t = 0$. (i) What is the current delivered by the source? (ii) Express the voltage across capacitor as a function of time. (iii) What is the energy delivered by the voltage source? (iv) What is the energy stored in the capacitor? (v) Are the two energy values equal? If not, what happens to the mismatch energy?

SOLUTION

- (i) The voltage across the capacitor has to *change instantaneously* in this circuit since KVL has to be obeyed at $t = 0^+$. But instantaneous change in capacitor voltage will require an impulse current with an area content equal to the change in charge required. The change in stored charge required here is $100 \text{ V} \times 1000 \text{ } \mu\text{F} = 100 \text{ mC}$. Therefore, the current that flows in the circuit will be $0.1 \delta(t)$.
- (ii) The voltage across capacitor was zero till $t = 0^-$. It will be 100 V at and after $t = 0^+$. It is discontinuous at $t = 0$.

$$v_C(t) = \begin{cases} 0 & \text{for } t = 0^- \\ \text{undefined} & \text{for } t = 0 \\ 100 & \text{for } t \geq 0^+ \end{cases}$$

We will be tempted to write this as $v_C(t) = 100 u(t)$, where $u(t)$ is the unit step function. But we do not know the capacitor voltage before $t = 0^-$. So we cannot express it this way.

- (iii) Energy delivered by the source = Voltage $\times$ Charge transported across the voltage = $100 \text{ V} \times 0.1 \text{ C} = 10 \text{ J}$. It can also be calculated as the area under the power waveform. The power delivered as a function of time is $100 \times 0.1 \delta(t) = 10 \delta(t)$. Area under this waveform is 10.
- (iv) Energy stored in the capacitor = $0.5 \times 0.001 \text{ F} \times (100 \text{ V})^2 = 5 \text{ J}$.
- (v) Energy delivered by the source and energy stored in the capacitor are not equal.

The source delivered 5 J extra. This is a circuit in which the 'capacitance' model for a physical capacitor is not adequate. A physical capacitor is a physical device that is represented by the mathematical model of 'capacitance'. This model is only a first level model that ignores the second level details of the physical device. Any electrical element will have all the three effects – resistance, capacitance and inductance – involved in it. Therefore, a capacitor has resistance and inductance associated with it. The resistance of metal foil used in the capacitor and the resistance of connecting leads contribute a series resistor to the model of capacitor. Leakage of current through the imperfect insulator used as the dielectric in the capacitor contributes a shunt resistor in the second level model for a capacitor. And the magnetic flux produced within the capacitor when current flows in its leads

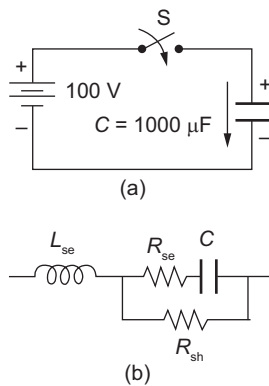


Fig. 3.5-7 (a) Circuit for Example 3.5-5
(b) A More Accurate Model for a Capacitor

and foil contribute an inductance too. Thus, a more accurate model for a physical capacitor is as shown in Fig. 3.5-7(b). R_{se} is usually in $m\Omega$ range, R_{sh} is usually in $k\Omega$ – $M\Omega$ range and L_{se} is in nH – μH range.

We will see in a later chapter that this detailed model will predict transient currents and voltages which oscillate at a high frequency when a DC voltage is suddenly switched on to an uncharged capacitor. The two resistors that are invariably present in any physical capacitor will take care of the extra energy delivered by the voltage source – they dissipate it.

The circuit problem arising out of switching a voltage source on to an uncharged capacitor requires a more detailed circuit model for correct solution.

EXAMPLE: 3.5-6

A periodic current waveform is applied to a capacitor of value 0.1 F from $t = 0\text{ s}$ as in Fig. 3.5-8. The voltage across the capacitor is found to vary periodically between 1 V and 5 V . (i) What is the full-cycle average value of the applied current waveform? (ii) What is the half-cycle average of the applied current waveform? (iii) What was the initial voltage in the capacitor? (iv) Find I_p .

SOLUTION

- (i) The voltage is stated to be periodic. Therefore, it must either be a pure alternating waveform or such an alternating waveform plus a DC offset. Differentiation of a pure alternating waveform gives another pure alternating waveform. Differentiation of a DC term can give only zero. Hence, the derivative of capacitor voltage will not contain a DC term. Derivative of voltage multiplied by capacitance value is the current through the capacitor. Therefore, current through the capacitor will not have a DC value. But the DC content in a periodic waveform is nothing but its average over a cycle period. Therefore, this current waveform has a full-cycle average of 0 A .
- (ii) Half-cycle average of alternating current = Half-cycle area/half the time period. Half-cycle area of the alternating current is given by *change in charge of capacitor* between the maximum and minimum voltage values. It is $(5 - 1)\text{ V} \times 0.1\text{ F} = 0.4\text{ C}$ in this case. Therefore, the half-cycle area of current waveform is 0.4 C and its half-cycle average value is $0.4\text{ C}/4\text{ s} = 0.1\text{ A}$.
- (iii) The capacitor voltage is periodic between 1 V and 5 V . There is no impulse content in the applied current. Hence, its initial voltage must be 1 V .
- (iv) The half-cycle area in terms of I_p is $= (0.5 I_p + 0.5 I_p + 0.25 I_p) \times 2 = 2.5 I_p\text{ C}$. This must be equal to 0.4 C .
Therefore, $I_p = 0.4/2.5 = 0.16\text{ A}$.

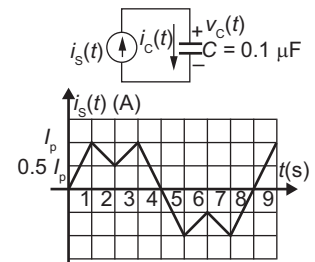


Fig. 3.5-8 Circuit and Waveform for Example 3.5-6

3.6 SERIES CONNECTION OF CAPACITORS

Practical capacitors come in standard sizes with standard voltage, current and capacitance ratings. Series or parallel connection of such capacitors is almost always required in practical applications in order to meet the application requirements on voltage, current or capacitance rating. Such series or parallel connections of capacitors can often be represented as a single equivalent capacitor for analysis purposes provided we pay careful attention to initial conditions of the capacitors involved in the connection. We look at series connection of capacitors in this section.

3.6.1 Series Connection of Capacitors with Zero Initial Energy

Let n capacitors with capacitance values $C_1, C_2, \dots, C_n$ be with zero initial voltage and *no mutual electrostatic coupling* among them be connected in series as shown in Fig. 3.6-1.

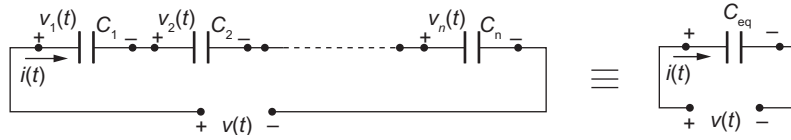


Fig. 3.6-1 Series Connection of Capacitors and Equivalent Capacitor

Current and charge are the common variables in a series connection of capacitors.

The *change* in the voltage across a capacitor is proportional to the amp-sec product (i.e., charge) dumped into it and the proportionality constant is the reciprocal of capacitance value. Once the capacitors have been connected in series, they must have a common current and hence all of them will get only the same amp-sec product (i.e., charge) dumped on them subsequently. *Therefore, the change in the capacitor voltages across the capacitors in a series combination due to $i(t)$ will be in proportion to reciprocal of their capacitance values.* If all the capacitors had zero initial voltage at the instant of connection, then capacitor voltages (not the *change* in them alone) themselves will be in proportion to reciprocal of capacitance values. Thus, $v(t)$ gets distributed among the capacitors as per the following relation.

$$\text{Ratio of distribution of } v(t) = \frac{1}{C_1} : \frac{1}{C_2} : \frac{1}{C_3} : \dots : \frac{1}{C_n}$$

$$\therefore v_j(t) = \frac{1/C_j}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n}} v(t) = \frac{1/C_j}{\sum_{k=1}^n 1/C_k} v(t).$$

$$\text{If there are only two capacitors, } v_1(t) = \frac{C_2}{C_1 + C_2} v(t) \text{ and } v_2(t) = \frac{C_1}{C_1 + C_2} v(t).$$

Note that the smaller capacitors will take larger share of voltage in a series connection of the capacitors.

Assume that this series connection is being driven by a current source $i(t)$ across its terminals from $t = 0$ onwards. The voltage $v(t)$ developed across the current source terminals will be

$v(t) = \frac{1}{C_1} \int_{-\infty}^t i_1(t) dt + \frac{1}{C_2} \int_{-\infty}^t i_2(t) dt + \dots + \frac{1}{C_n} \int_{-\infty}^t i_n(t) dt$, where $i_1(t), i_2(t) \dots i_n(t)$ are the currents flowing through the various capacitors from $t = -\infty$ onwards. But $i_1(t) = i_2(t) = \dots = i_n(t) = i(t)$ after $t = 0^-$. Therefore,

$$v(t) = \left[\frac{1}{C_1} \int_{-\infty}^{0^-} i_1(t) dt + \frac{1}{C_2} \int_{-\infty}^{0^-} i_2(t) dt + \dots + \frac{1}{C_n} \int_{-\infty}^{0^-} i_n(t) dt \right] + \left[\sum_{k=1}^n \frac{1}{C_k} \right] \int_{0^-}^t i(t) dt.$$

The quantity inside the first square bracket pair is nothing but the sum of initial voltage of the capacitors. Therefore,

$$v(t) = \sum_{k=1}^n V_k(0^-) + \left[\sum_{k=1}^n \frac{1}{C_k} \right] \int_{0^-}^t i(t) dt = V_{eq}(0^-) + \left[\frac{1}{C_{eq}} \right] \int_{0^-}^t i(t) dt, \quad (3.6-1)$$

$$\text{where } \frac{1}{C_{eq}} = \sum_{k=1}^n \frac{1}{C_k} \text{ and } V_{eq}(0^-) = \sum_{k=1}^n V_k(0^-).$$

Hence, we can replace the series connection by a single capacitor of value C_{eq} with an initial voltage of $V_{eq}(0^-)$ defined as in Eqn. 3.6-1. When all the capacitors are initially relaxed, the effective capacitor will also be initially relaxed. Else, the algebraic sum of initial voltages across the capacitors will give the initial voltage across the effective capacitor.

In summary, when n capacitors with *zero initial condition* are connected in series, one capacitor with its value as defined in Eqn. 3.6-1 can be used to replace the series combination as far as v - i relationship, stored charge and stored energy are concerned.

Series Connection of Initially Relaxed Capacitors

n capacitors with zero initial condition are connected in series can be replaced by an equivalent C_{eq} given by

$$\frac{1}{C_{eq}} = \sum_{k=1}^n \frac{1}{C_k} \text{ as far as}$$

v - i relationship, charge and stored energy are concerned.

All capacitors as well as effective capacitor will have same charge at all time instants.

Individual capacitor voltage will be as per the ratio of reciprocal of capacitance values.

Total stored energy will be distributed in various capacitors as per the ratio of reciprocal of capacitance values.

All capacitors as well as effective capacitor will have equal charge at all t . Individual capacitor voltages will be as per the ratio of reciprocal of capacitance values. Total stored energy will be distributed in various capacitors as per the ratio of reciprocal of capacitance values.

3.6.2 Series Connection of Capacitors with Non-zero Initial Energy

We identify two cases in this situation. In the first case, we consider a situation in which all the capacitors had *same initial charge in polarity and magnitude* at the instant they were put in series. If they had same initial charge to begin with, they will continue to have equal charge even after they have been connected in series. This is so because current is a common variable in a series connection. Therefore, the individual capacitor voltages will be inversely proportional to the capacitance values initially and subsequently. Therefore, the series connection can be replaced by an effective capacitor of value as per Eqn. 3.6-1 and with an initial voltage as per Eqn. 3.6-1. The equivalent capacitor will be equivalent in $v-i$ relationship, in charge and in *total* stored energy. Total voltage and *total* stored energy will get distributed in individual capacitors in inverse proportion to capacitance values.

In the second case, we consider a set of capacitors with arbitrary initial charges. If they had unequal initial charges they will also continue to have unequal charges subsequently. Only *change* in charge for various capacitors will be equal in this case.

Therefore, in this case, the effective capacitor will describe only the *change* in capacitor voltages correctly. The equivalent capacitor is equivalent with respect to $\Delta v(t) - i(t)$ relationship, *change* in charge and *change* in total stored energy. *Change* in charge will be equal in all capacitors. *Change* in total voltage is distributed as *changes* in individual capacitor voltages in inverse proportion to capacitance values. *Change* in total stored energy calculated from equivalent circuit will be correct. This *change* in total stored energy is equal to sum of *changes* in stored energy of individual capacitors. But the distribution is *not* inversely proportional to capacitance values.

The initial voltage across the equivalent capacitor will be given by the algebraic sum of the initial voltages of individual capacitors in this case too. *But the initial stored energy that is calculated from equivalent capacitor will be less than the actual initial stored energy in the system.* The difference will represent the portion of initial energy that gets trapped and hidden in the system. This portion is trapped forever in the series connection and cannot be taken out by other elements.

A simple example will show that there can be trapped energy in a series connection. Consider two capacitors with equal magnitude and opposite polarity initial voltages. When we connect them in series, the initial voltage observed at terminals of series combination will be zero. We cannot take out the initial energy though we know that it is present. In this case, the entire initial energy becomes trapped energy.

The amount of trapped energy in a series connection is given by:

$$\text{Trapped Energy} = \sum_{k=1}^n \frac{1}{2} C_k [V_k(0^-)]^2 - \frac{1}{2} C_{eq} \left[\sum_{k=1}^n V_k(0^-) \right]^2 \text{ J.}$$

A single capacitor C_{eq} can replace a set of n capacitors connected in series as far as changes in charge, changes in voltage and changes in total stored energy are concerned.

$$\frac{1}{C_{eq}} = \left[\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n} \right]$$

However, the total charge in each capacitor and total charge in C_{eq} will not be the same unless all capacitors had the same charge at $t = 0^-$.

Similarly, the total stored energy in the system will not be the same as the total stored energy in C_{eq} unless all capacitors had same charge at $t = 0^-$.

If the capacitors started with different charges at $t = 0^-$, a portion of total initial energy is trapped in the circuit and gets hidden from rest of the circuit forever.

Series Connection of Capacitors with Initial Energy

A single capacitor C_{eq} given by

$$\frac{1}{C_{eq}} = \sum_{k=1}^n \frac{1}{C_k} \text{ can}$$

replace a set of n capacitors connected in series as far as changes in charge, changes in voltage and changes in total stored energy are concerned.

However, the total charge in each capacitor and total charge in C_{eq} will not be the same unless all capacitors had the same initial charge.

Similarly, the total stored energy in the system will not be the same as the total stored energy in C_{eq} unless all capacitors had same initial charge.

If the capacitors started with different initial charge, a portion of total initial energy is trapped in the circuit and gets hidden from rest of the circuit forever.

EXAMPLE: 3.6-1

Charging a large number of capacitors in parallel from a low voltage DC source and later reconnecting them in series after removing the source is a technique that is employed in High Voltage Engineering for producing high DC voltages. Solid state switches are used for forming parallel and series combinations of capacitors for this purpose. Usually capacitors with same capacitance value are employed. Assume that the capacitors used in such a system have a tolerance band of $\pm 20\%$ in their capacitance value. Consider such a system employing eight capacitors of $2 \mu\text{F}$ each. The charging voltage is 400 V . Assume that four of them had capacitance value at lower end of tolerance band and the remaining four had capacitance value at higher end of the band. (i) What is the initial energy content in the series combination? (ii) If the high voltage obtained from the series combination is used to deliver a pulse of energy to a test specimen, what is the maximum energy that is available for this purpose?

SOLUTION

Let the nominal value of capacitors be represented by C . Then, four capacitors are of $0.8 C$ farads and remaining four are of $1.2 C$ farads. The effective capacitor is then $0.8 C/4$ and $1.2 C/4$ in series $= 0.12 C$ farads. The initial voltage across the effective capacitor $= 8 \times 400 \text{ V} = 3200 \text{ V}$.

- (i) Initial energy of the series combination $= 0.5 C (4 \times 0.8 \times 400^2 + 4 \times 1.2 \times 400^2) = 640,000 C \text{ J}$. Substituting $C = 2 \mu\text{F}$, Initial energy $= 1.28 \text{ J}$
(ii) Initial energy calculated from equivalent circuit $= 0.5 C \times 0.12 \times 3200^2 = 614,400 C \text{ J}$. Substituting for $C = 1.229 \text{ J}$. Therefore, the maximum energy available for specimen testing is 1.229 J i.e., about 96% of total stored energy. The remaining 4% is trapped in the series combination and will not be available to external elements.

3.7 PARALLEL CONNECTION OF CAPACITORS

A single equivalent capacitor can replace many capacitors connected in parallel for specific analysis purposes. We look into parallel equivalent and constraints on it in this section.

We consider parallel connection of n capacitors *that have no mutual electrostatic coupling* among them. Let the capacitance values be $C_1, C_2, \dots, C_n$. We assume that they have the same initial voltage with the same polarity at $t = 0^-$. Let the applied current be $i(t)$ and the voltage across the parallel combination be $v(t)$ (Fig. 3.7-1).

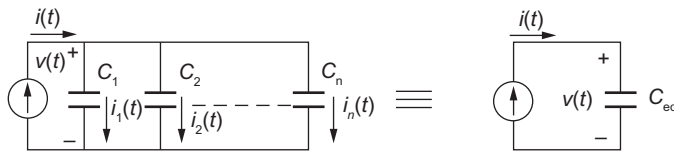


Fig. 3.7-1 Parallel Connection of n Capacitors

Applying KCL along with the element equation of capacitor, we get

$$\begin{aligned}
 i(t) &= i_1(t) + i_2(t) + \dots + i_n(t) \\
 &= C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + \dots + C_n \frac{dv(t)}{dt} \\
 &= (C_1 + C_2 + \dots + C_n) \frac{dv(t)}{dt} \\
 &= C_{eq} \frac{dv(t)}{dt}, \text{ where } C_{eq} = C_1 + C_2 + \dots + C_n.
 \end{aligned}$$

Parallel Connection of Capacitors with same Initial Voltage

Parallel connection of n capacitors with same initial condition value can be replaced by an equivalent

$$\text{capacitor } C_{eq} = \sum_{k=1}^n C_k$$

as far as v - i relationship is concerned.

Total applied current into the combination is shared by the various capacitors in direct proportion to the capacitance value.

Total charge developed across the combination is shared by the various capacitors in direct proportion to the capacitance value.

Total energy storage in the combination is distributed among the various capacitors in direct proportion to the capacitance value.

Thus, a parallel connection of n capacitors may be replaced by an equivalent capacitor with a capacitance value equal to sum of the capacitance values of n capacitors as far as the v - i relationship is concerned. The total applied current into the combination is shared by the various capacitors is directly proportional to the capacitance value. i.e.,

$$\begin{aligned} i(t) &= i_1(t) + i_2(t) + \dots + i_n(t) \\ i_1(t) : i_2(t) : \dots : i_n(t) &= C_1 : C_2 : \dots : C_n \\ i_j(t) &= \frac{C_j}{C_{eq}} i(t) \text{ for } j = 1 \dots n. \end{aligned}$$

It can be seen that sum of charges in the individual capacitors is same as the charge of the equivalent capacitor. Thus, charge is shared in proportion to the capacitance value. Similarly, the total energy stored in all capacitors put together is the same as the energy storage calculated using equivalent capacitance value.

Thus, parallel equivalent of n capacitors is 'equivalent' with respect to v - i relation, charge and stored energy. Note that the participating capacitors should not have mutual electrostatic coupling. And that they must have the same initial voltage.

If they have unequal initial voltage, high frequency oscillating currents and voltages appear in the local loops formed when they are put in parallel due to the inductive elements that are always present in any circuit. We need better model for a physical capacitor to handle this kind of problem analytically. (See the discussion on a similar problem we had in connecting inductors in series when they have different initial current values in Sect. 3.3).

EXAMPLE: 3.7-1

$C_1 = 6 \mu\text{F}$, $C_2 = 3 \mu\text{F}$ and $C_3 = 3 \mu\text{F}$ in the circuit as in Fig. 3.7-2. C_1 has 5 V across it at $t = 0^-$. C_2 and C_3 have 3 V across them at $t = 0^-$. The polarity of initial voltages is same as that of voltage variables identified in the circuit. The current source function $i_1(t)$ is zero after $3 \mu\text{s}$. (i) Find the voltage across all the capacitors at $5 \mu\text{s}$. (ii) Find the stored energy in the individual capacitors and total stored energy at $4 \mu\text{s}$. (iii) Is there any energy trapped in the system? If yes, how much?

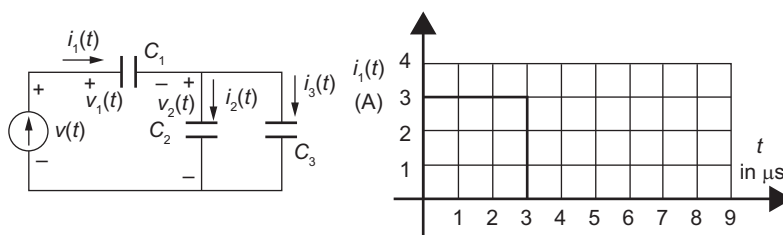


Fig. 3.7-2 Circuit and Waveform for Example 3.7-1

SOLUTION

(i) The effective value of C_2 and C_3 in parallel is $6 \mu\text{F}$ and this $6 \mu\text{F}$ is in series with C_1 which is also of $6 \mu\text{F}$. Therefore, the effective capacitance of the entire circuit is $3 \mu\text{F}$. The initial voltage across the equivalent capacitor is $5 \text{ V} + 3 \text{ V} = 8 \text{ V}$. The area under input current is $9 \mu\text{C}$ for all $t \geq 3 \mu\text{s}$. Therefore, the net amp-sec dumped into the equivalent capacitor at $5 \mu\text{s}$ is $9 \mu\text{C}$. This produces a change in voltage by $9 \mu\text{C} / 3 \mu\text{F} = 3 \text{ V}$.

Change in voltage in a series combination of capacitors is shared by the capacitors in inverse proportion to the capacitance values. Therefore, change in voltage of $C_1 = (1/C_2 // C_3) / (1/C_1 + 1/C_2 // C_3)$ times $3 \text{ V} = 1.5 \text{ V}$. Similarly change in $v_2(t) = 1.5 \text{ V}$. Therefore, $v_1(t)$ at $5 \mu\text{s} = 5 \text{ V} + 1.5 \text{ V} = 6.5 \text{ V}$ and $v_2(t)$ at $5 \mu\text{s} = 3 \text{ V} + 1.5 \text{ V} = 4.5 \text{ V}$.

(ii) The capacitor voltages at $4\ \mu\text{s}$ are the same as at $5\ \mu\text{s}$ since the current source value is zero after $3\ \mu\text{s}$. Therefore, the stored energy in various capacitors can be calculated using the voltage values.

Stored energy in $C_1 = 0.5 \times 6 \times 6.5^2\ \mu\text{J} = 126.75\ \mu\text{J}$.

Stored energy in $C_2 = 0.5 \times 3 \times 4.5^2\ \mu\text{J} = 30.375\ \mu\text{J}$.

Stored energy in $C_3 = 0.5 \times 3 \times 4.5^2\ \mu\text{J} = 30.375\ \mu\text{J}$.

Total stored energy = $187.5\ \mu\text{J}$.

(iii) The initial charge in C_1 was $30\ \mu\text{C}$ and in $C_2 // C_3$ was $18\ \mu\text{C}$. Thus, initial charges in the two capacitors in series combination were not equal. Hence, there will be trapped energy in the system. Initial stored energy = $0.5 \times 6 \times 5^2 + 0.5 \times 3 \times 3^2 + 0.5 \times 3 \times 3^2 = 102\ \mu\text{J}$.

Initial energy in the equivalent capacitor = $0.5 \times 3 \times 8^2 = 96\ \mu\text{J}$.

Therefore, trapped energy = $102 - 96 = 6\ \mu\text{J}$.

3.8 SUMMARY

- A linear resistor obeys Ohm's Law and is a memoryless element. The power delivered to a positive resistor is a non-negative function of time. Resistors in series have a common current and share the total voltage and power in proportion to their resistance values. Resistors in parallel have a common voltage and share the total current and power in proportion to their conductance values.
- Inductor is an element that *accumulates* flux linkage (volt-sec or Wb-T) and makes its response variable, i.e., current, proportional to the accumulated flux linkage. Capacitor is an element that *accumulates* charge (amp-sec or C) and makes its response variable, i.e., voltage, proportional to the accumulated charge.
- The voltage across an inductor at t is proportional to the rate of change of current through it at t . The current through the inductor is proportional to the area under the voltage waveform, i.e., the volt-sec product (or Wb-T) applied across its terminals from $t = -\infty$ to t .
- Instantaneous current in an inductor cannot be predicted from instantaneous value of voltage across it. If instantaneous value of voltage is positive the inductor current will be increasing at that instant and if it is negative the current will be decreasing at that instant.
- When voltage across an inductor crosses zero in the downward direction, its current attains a local maximum, and, when it crosses zero in the upward direction the inductor current attains a local minimum. Voltage across an inductor carrying a constant current is zero.
- Current in an inductor cannot change instantaneously unless an impulse voltage is applied or supported in the circuit. The current in an inductor L changes instantaneously by $1/L\ A$ when the circuit applies or supports a *unit impulse voltage* across it.
- An inductor with a large inductance value can absorb alternating voltages in a circuit without contributing significant amount of alternating currents to the circuit.
- The total energy delivered to an inductor carrying a current I is $(1/2)LI^2\ J$ and this energy is stored in its magnetic field. The inductor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.
- A single inductor $L_{eq} = L_1 + L_2 + \dots + L_n$ can replace a set of n inductors connected in series. The total applied voltage, total flux linkage and total stored energy are shared by the various inductors in direct proportion to the inductance values.
- A single inductor L_{eq} can replace a set of n inductors connected in parallel as far as *changes* in flux linkages, *changes* in currents and *changes* in stored energy are concerned.

$$\frac{1}{L_{eq}} = \left[\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right]$$
- The current through a capacitor at any instant is proportional to the rate of change of voltage across it at that instant. The voltage across the capacitor at any instant is proportional to the area under the current waveform, i.e., the amp-sec product (or C) applied through it from $t = -\infty$ to that instant.
- Instantaneous voltage across a capacitor cannot be predicted from instantaneous value of current through it. If instantaneous value of current is positive, the capacitor voltage will be increasing at that instant, and, if it is negative the voltage will be decreasing at that instant.
- When current through a capacitor crosses zero in the downward direction, its voltage attains a local maximum and when it crosses zero in the upward direction, the capacitor voltage attains a local minimum. Current through a capacitor with a constant voltage across it, is zero.
- Voltage in a capacitor cannot change instantaneously unless an impulse current is applied or supported in the circuit. Unit impulse current will deposit 1 Coulomb of charge in a capacitor instantaneously. Therefore, the voltage across a capacitor C changes instantaneously by $1/C\ V$ when the circuit applies or supports a *unit impulse current* through it.

- A large capacitor can absorb alternating currents in a circuit without contributing significant amount of alternating voltages in the circuit.
 - The total energy delivered to a capacitor carrying a voltage V across it is $(\frac{1}{2})CV^2$ J and this energy is stored in its electric field. Stored energy in a capacitor is also given by $(\frac{1}{2})CQ^2$ J and $QV/2$ J. The capacitor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.
 - A single capacitor C_{eq} can replace a set of n capacitors connected in series as far as *changes* in charge, *changes* in voltage and *changes* in total stored energy are concerned.
- $$\frac{1}{C_{eq}} = \left[\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n} \right]$$
- A single capacitor $C_{eq} = C_1 + C_2 + \dots + C_n$ can replace a set of n capacitors connected in parallel. The total charge, total current and total stored energy are shared by the various capacitors in direct proportion to capacitance value in a parallel connection of capacitors.

3.9 QUESTIONS

[Passive sign convention is assumed throughout]

1. What is meant by *linearity* of an electrical element? Show that a resistor satisfying Ohm's law is a linear element.
2. What are series equivalent and parallel equivalent of n equal resistors?
3. Show that a resistor in parallel with a short-circuit is a short-circuit.
4. Show that a resistor in series with an open-circuit is an open-circuit.
5. Show that the parallel equivalent of a set of resistors will be less than the resistor with the least value among them.
6. How many different values of resistance can be obtained by using five resistors of equal value in series-parallel combinations? Enumerate them.
7. Explain why an inductor needs an initial condition specification whereas a resistor does not.
8. The voltage across a 0.1 H inductor is seen to be 7.5 V at $t = 7$ ms. What is the current in the inductor at that instant?
9. The voltage across a 0.1 H inductor is seen to be a constant at 10 V between 10 ms and 15 ms. The current through the inductor was 0.3 A at 12 ms. What is the current at 13.5 ms?
10. The area under voltage waveform applied to a 10 mH inductor is 5 mV-s between 7 ms and 9 ms. If the current at 7 ms was 1 A how much is it at 9 ms?
11. An inductor of 0.2 H has current of 2 A at $t = 0^-$ in it. The voltage applied across it is $3\delta(t - 2)$. Find the current in it (a) at 1 s (b) at 3 s.
12. An inductor of 2 H undergoes a flux linkage change of 7 Wb-T between 15 s and 17 s. What is the average voltage applied to the inductor during that interval?
13. Two identical inductors L_1 and L_2 undergo a flux linkage change by 10 Wb-T. L_1 takes 2 s for this change and L_2 takes 20 s. What is the ratio of average voltage applied to the inductors during the relevant intervals?
14. A 10 H has an initial energy equivalent to the energy consumed by a 40 W lamp in 1 h. Find the initial current in the inductor.
15. A DC voltage source of 24 V is switched on to an initially relaxed inductor of 4 H through a 48 A fuse. Assume that the fuse acts instantaneously when current through it touches 48 A. How much time do we have to open the switch before the fuse blows?
16. A DC source of 12 V is switched on to an inductor of 0.5 H at $t = 0$. The current in it is found to be 0 A at 5 s. Was there any initial stored energy in the inductor? If yes, how much?
17. A symmetric triangular voltage waveform with a peak-to-peak value of 20 V and frequency 1 kHz is applied to an inductor from 0 s onwards. The inductor was carrying an initial current of 10 A. The inductor current is found to vary within $\pm 3\%$ of its initial current subsequently. What is the value of inductance?
18. Two inductors of 1 H and 1.8 H with initial currents of 5 A and 2 A, respectively are connected in parallel. How much energy can be taken out from this parallel combination?
19. Three inductors are connected in series and the current in the circuit is found to vary at the rate of 7 A/s at an instant when the applied voltage was at 14 V. The value of voltage measured across the third inductor at the same instant was 4 V. What is the value of the third inductor?
20. Two inductors with zero initial energy were paralleled at $t = 0$ and a voltage source was applied across them. The rate of change of source current at 2 s is 5 A/s and the source voltage at that time was 2.5 V. It was also found that the first inductor had a stored energy that is twice that of the second inductor. Find the inductance values.
21. How much time is required to charge a 10 mF capacitor with an initial voltage of -100 V to +100 V using a DC current source of value 10 mA?
22. The voltage rating of a 10 μ F capacitor is 100 V. It is being charged by a 100 μ A pulse current source. Its initial voltage was -75 V. What is the maximum pulse width that the current source can have if we do not want to end up with a blown capacitor?
23. The DC power supply in a PC uses 470 μ F capacitor across its DC output. The DC output value is normally 320 V. The PC can function without rebooting till the DC voltage across falls

to 220 V. If the PC takes a constant current of 0.5 A from the DC output while it is functioning, find out how long it will continue running after the AC mains goes off.

24. A 5 mF capacitor undergoes a change in its voltage by 25 V in 10 ms. What is the average value during that interval of the current source output used to charge this capacitor?
25. A symmetric triangular current waveform with a peak-to-peak value of 20 mA and frequency 10 kHz is applied to a capacitor from 0 s onwards. The capacitor was carrying an initial voltage of 10 V. The capacitor voltage is found to vary within $\pm 5\%$ of its initial voltage subsequently. What is the value of capacitance?

26. A DC current source of 12 mA is switched on to a capacitor of 0.5 mF at $t = 0$. The voltage in it is found to be 0 V at 5 s. Was there any initial stored energy in the capacitor? If yes, how much?
27. A sinusoidal current source $2 \sin 200t$ A is applied to three capacitors – 0.1 mF, 0.2 mF and 0.05 mF in series. What is the peak-to-peak voltage developed across the combination? Which capacitor has highest peak-to-peak voltage across it?
28. Three capacitors – 10 μ F, 22 μ F and 33 μ F – are in parallel. The circuit is driven by a AC current source $10 \cos 300t$ A. What is the peak-to-peak voltage developed across the combination?

3.10 PROBLEMS

(Assume zero initial condition unless specified otherwise. Passive sign convention is assumed.)

1. A voltage source is first connected across A–B in the circuit as in Fig. 3.10-1. Later it is moved and connected across C–D. Find the ratio of power delivered by the voltage source in the two cases.

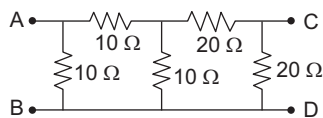


Fig. 3.10-1

2. Find i_x in Fig. 3.10-2.

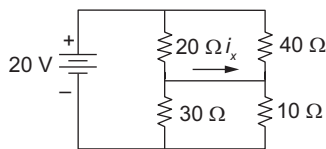


Fig. 3.10-2

3. The power dissipated in the 4 Ω resistor is 1 W in the circuit as in Fig. 3.10-3. Find the power dissipated in the 3 Ω resistor, power delivered by the DC source and the value of source voltage.

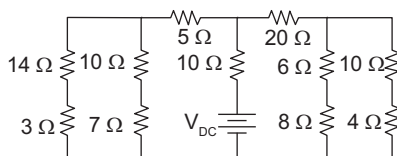


Fig. 3.10-3

4. A voltage of 5 V is to be produced from a DC voltage source of 24 V by a potential divider arrangement. The 5 V output

will be loaded by a resistance in the range 1 k Ω –10 k Ω . The output voltage should not vary by more than 2% when the load varies in this range. Design the potential divider such that the no-load power dissipation in it is at minimum possible value.

5. The power dissipated in 15 Ω resistor is 15 W and power dissipated in R_2 is 5 W in the circuit as in Fig. 3.10-4. (i) Find R_1 and R_2 . (ii) Solve the circuit completely and mark voltage, current and power dissipated for all elements.

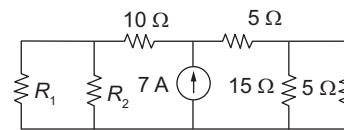


Fig. 3.10-4

6. (i) Find the value of R in the circuit as in Fig. 3.10-5 such that R dissipates 200 W of power. (ii) What is the additional resistor to be connected in parallel to R such that total power dissipated by R and the additional resistor will be 400 W?

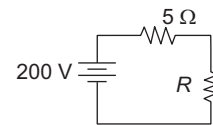


Fig. 3.10-5

7. What must be the value of R such that i_x is zero in the circuit as in Fig. 3.10-6.

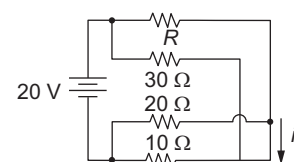


Fig. 3.10-6

8. What must be the value of R in Fig. 3.10-7 such that shorting A to B will not affect the current through R ?

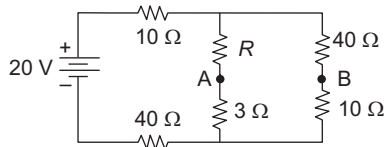


Fig. 3.10-7

9. The *unit step function* was described in Eqn. 3.2-4. The function $u(t - t_0)$ is termed as a *delayed unit step function*.

$$u(t - t_0) = \begin{cases} 0 & \text{for } (t - t_0) \leq 0^- \\ \text{undefined for } (t - t_0) = 0 \\ 1 & \text{for } (t - t_0) \geq 0^+ \end{cases}$$

Therefore, the jump from 0 to 1 takes place at t_0 i.e., after a delay of t_0 s. What is the applied voltage function across an inductor of 0.5 H if its current is $2u(t - 2)$?

10. Another standard test signal frequently used in Electrical and Electronics Engineering is the *unit ramp function* defined as

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

(i) Show that unit ramp is the integral of unit step function.
(ii) Find the voltage across a 0.25 H inductor when the current is $(5 + 25t)$ for $t \geq 0^+$ and 0 A for $t \leq 0^-$.

11. The voltage applied across an inductor of 0.3 H is a rectangular pulse of height 10 V and duration 30 ms. The pulse starts at $t = 20$ ms. (i) Express the voltage waveform in terms of scaled and delayed unit step functions. (ii) Obtain an expression for the current in the inductor as a function of time for $t \geq 10$ ms if the initial current at 10 ms is 1 A. (iii) Plot the current in inductor, power delivered to the inductor and energy storage in inductor as functions of time.
12. The value of $i_L(t)$ is found to be 10 A at 18 s in the circuit as in Fig. 3.10-8. Find the ratio of initial energy storage in the inductor to stored energy in it at 17 s.

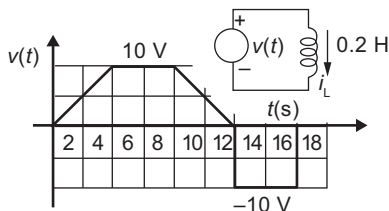


Fig. 3.10-8

13. A periodic ramp voltage waveform applied across the 2 mH inductor as in Fig. 3.10-9 from $t = 0$. (i) What must be the initial current and initial energy storage in the inductor such that there is no DC component in the inductor after $t = 0$? (ii)

Calculate and plot the inductor current and stored energy for one period with initial current at the value calculated above. (iii) Find the average power delivered by the source, the average being taken over a cycle.

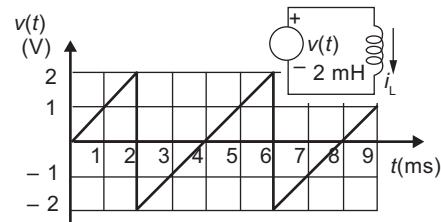


Fig. 3.10-9

14. A sinusoidal voltage $v_s(t) = 10 \sin(400t + \pi/3)$ is applied to a 25 mH inductor from $t = 0$. (i) Plot the current, power and stored energy in the inductor as functions of time for one period of input voltage. (ii) What is the DC content in the inductor current? (iii) What is the initial current to be specified at $t = 0^-$ such that the DC content in the inductor current will be zero? (iv) What is the frequency of power variation with this value of initial current?
15. A voltage source $v_s(t) = 5e^{0.5t} [u(t) - u(t - 2)]$ V is connected across an inductor of 0.5 H. The initial current specified at $t = 0^-$ is 2 A. (i) Plot the applied voltage and inductor current for $t = 0$ to 3 s. (ii) What is the value of current and stored energy in the inductor at $t = 4$ s?
16. An arbitrary voltage is applied across a 0.4 H inductor from $t = 0$. The current in inductor was observed to be 3 A at 1.5 s. The stored energy in it was found to quadruple in the next 0.2 s. Explain why the applied voltage could not be less than 6 V during the entire interval [1.5 s, 1.7 s].
17. An inductor used as a smoothing inductor in a DC power supply is expected to carry a DC current of 10 A. At the same time a periodic voltage with a waveform as shown in Fig. 3.10-10 will be applied across it. Find the minimum value of inductance such that the current in inductor will not get perturbed by more than $\pm 2\%$ of its DC value. Assume that the value of DC current given includes the DC component due to this periodic voltage also.

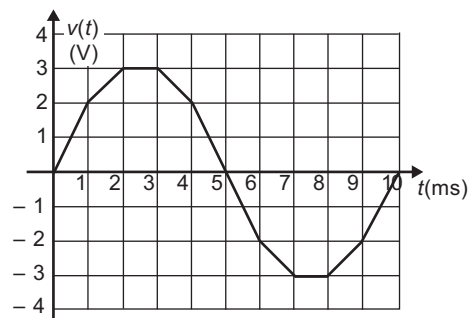


Fig. 3.10-10

18. The switch S in Fig. 3.10-11 starts at position-1 at $t = 0$ with initial current in 1 mH at zero. It remains at position-1 for 1 ms and then goes to position-2 and remains there for 0.1 ms . Then, it is brought back to position-1 and the entire cycle is repeated. (i) Find the value of V_{DC} such that the inductor current reaches zero just prior to the switch going back to position-1. (ii) With this value of V_{DC} , calculate and plot v_L , i_L , i_s and i_{DC} for two switching cycles. (iii) Calculate the average values of v_L , i_L , i_s , i_{DC} and power flow from 10 V source to the second source. Averaging is to be done over a switching cycle. (iv) Assume that the first switching cycle was $1.5\text{ ms}/0.1\text{ ms}$ and all the subsequent cycles were $1\text{ ms}/0.1\text{ ms}$. Repeat step (iii) for a cycle after the first one. (v) Suggest a method to control the power flow to the second source.

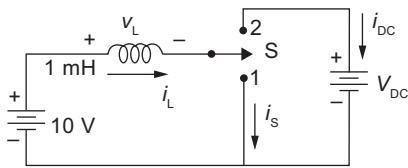


Fig. 3.10-11

19. The switch S in Fig. 3.10-12 starts at position-1 at $t = 0$ with initial current in 1 mH at zero. It remains at position-1 for 1 ms and then goes to position-2 and remains there for 0.02 ms . Then, it is brought back to position-1 and the entire cycle is repeated. (i) Find the value of V_{DC} such that the inductor current reaches zero just prior to the switch going back to position-1. (ii) With this value of V_{DC} , calculate and plot v_L , i_L , i_{s1} and i_{s2} for two switching cycles. (iii) Calculate the average values of v_L , i_L , i_{s1} , i_{s2} and power flow from 10 V source to the second source. Averaging is to be done over a switching cycle.

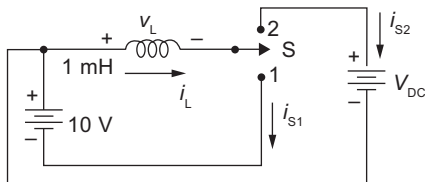


Fig. 3.10-12

20. All the inductors in the circuit as in Fig. 3.10-13 had zero initial energy. The applied source voltage is $v_s(t) = (10 \sin 2000\pi t)u(t)\text{ V}$. Find and plot the waveform of current drawn from the source, current in L_4 and voltage across L_4 .

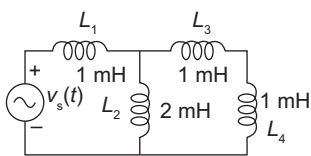


Fig. 3.10-13

21. Three inductors of 0.1 H , 0.05 H and 0.15 H with equal initial current of 3 A are connected in series at $t = 0$. $v(t)$ from $t = 0^-$ is $0.5\delta(t - 2) + v_s(t)$, where $v_s(t)$ is not known. The total flux linkage in the circuit is found to double in the first 3 s and $v(t)$ at $t = 3\text{ s}$ is found to be 12 V . Find (i) the voltage across each inductor and flux linkages in them at $t = 3\text{ s}$ (ii) the average value of $v_s(t)$ in the first 3 s (iii) stored energy in the circuit and each inductor at $t = 3\text{ s}$ and (iv) total energy delivered by the applied voltage source in 3 s and average power delivered by the source over 3 s ?
22. Three inductors -0.02 H , 0.05 H and 0.0333 H – with same initial current of 2 A are connected in parallel and a voltage of $0.1\delta(t - 0.002) + v_s(t)$ is applied across them. The flux linkage in the 0.05 H inductor is found to be 0.3 Wb-T at $t = 5\text{ ms}$. Find (i) the trapped energy in the parallel combination, (ii) the current through each inductor and flux linkages in them at $t = 5\text{ ms}$, (iii) the average value of $v_s(t)$ in the first 5 ms , (iv) stored energy in the circuit and in each inductor at $t = 5\text{ ms}$ and (v) total energy delivered by the applied voltage source in 5 ms and average power delivered by the source over 5 ms ?
23. What is the applied current into a capacitor of 0.02 F if its voltage is $5u(t - 3)$?
24. What must be the charging current function if the voltage across an initially uncharged $10\text{ }\mu\text{F}$ capacitor is to vary as $(5 + 2t)u(t)\text{ V}$?
25. The current applied into a capacitor of 10 mF is a rectangular pulse of height 10 A and duration 25 ms . The pulse starts at $t = 10\text{ ms}$. (i) Express the current waveform in terms of scaled and delayed unit step functions. (ii) Obtain voltage across the capacitor as a function of time for $t \geq 10\text{ ms}$ if the initial voltage at 10 ms is -10 V . (iii) Plot the voltage across capacitor, power delivered to the capacitor and energy storage in it as functions of time.
26. The value of $v_C(t)$ is found to be 10 V at 18 ms in the circuit as in Fig. 3.10-14. Find the ratio of initial energy storage in the capacitor to stored energy in it at 17 ms .

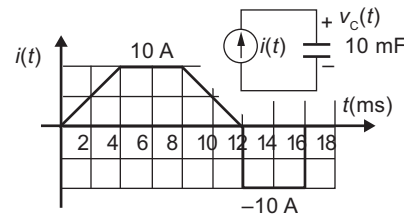


Fig. 3.10-14

27. The periodic ramp current waveform in Fig. 3.10-15 is applied into a 100 pF capacitor from $t = 0$. (i) What must be the initial voltage and initial energy storage in the capacitor such that there is no DC component in the capacitor voltage after $t = 0$? (ii) Calculate and plot the capacitor voltage and stored energy for one period with initial voltage at the value calculated above. (iii) Find the average power delivered by the source, the average being taken over a cycle.

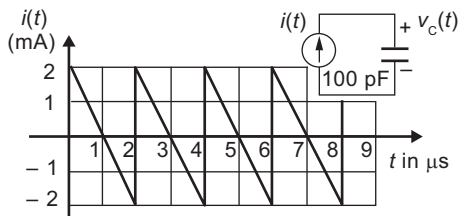


Fig. 3.10-15

28. A sinusoidal current $i_s(t) = 2 \sin(400t + \pi/3)$ A is applied to a $20 \mu\text{F}$ capacitor from $t = 0$. (i) Plot the voltage, power and stored energy in the capacitor as functions of time for one period of input current. (ii) What is the DC content in the capacitor voltage? (iii) What is the initial voltage to be specified at $t = 0^-$ such that the DC content in the capacitor voltage will be zero? (iv) What is the frequency of power variation with this value of initial voltage?
29. A current source $i_s(t) = 5e^{1.5t} [u(t) - u(t-1)]$ mA is connected across capacitor of $470 \mu\text{F}$. The initial voltage specified at $t = 0^-$ is 2 V. (i) Plot the applied current and capacitor voltage for $t = 0$ to 2 s. (ii) What is the value of voltage and stored energy in the capacitor at $t = 4$ s?
30. An arbitrary current is applied to a 0.1 F capacitor from $t = 0$. The voltage across capacitor was observed to be 10 V at 2 s. The stored energy in it was found to quadruple in the next 0.5 s. Explain why the applied current could not have been less than 2 A during the entire interval [2 s, 2.5 s].
31. The circuit as in Fig. 3.10-16 shows an idealised model for the output of a DC power supply. The output is taken across the bulk capacitor C . Load on the supply is modelled as a current source of value i_L . i_L has a DC component and pure AC component. Its AC component is shown in the same figure. The voltage across the capacitor is found to have a DC component of 12 V and rest of the power supply circuit is in effect delivering a constant current of 5 A to the capacitor node. (i) What must be the value of DC component in load current i_L ? (ii) What must be the minimum value of C such that the peak-to-peak ripple voltage at output will be less than 2% of its DC value?

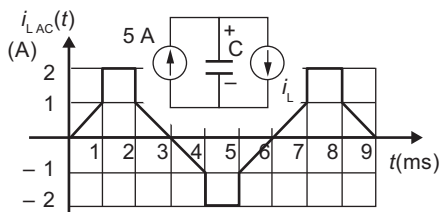


Fig. 3.10-16

32. The capacitance values of C_1 , C_2 , C_3 and C_4 in Fig. 3.10-17 are $20 \mu\text{F}$, $10 \mu\text{F}$, $20 \mu\text{F}$ and $20 \mu\text{F}$, respectively. They had zero initial energy at $t = 0^-$. The applied source is $(200 \sin 300t)u(t)$ V.

Find and plot (i) the current drawn from the source and (ii) the voltage across C_4 and current through it.

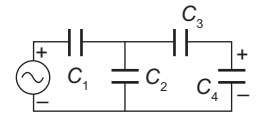


Fig. 3.10-17

33. The source in the circuit in Fig. 3.10-18 is $(2 \cos 300t)u(t)$ A. Find and plot (i) the voltage across the source and (ii) the voltage across C_4 and current through it. $C_1 = C_3 = C_4 = 10 \mu\text{F}$, $C_2 = 5 \mu\text{F}$.

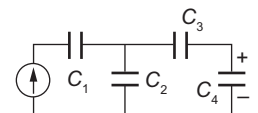


Fig. 3.10-18

34. All the three capacitors in Fig. 3.10-19 had equal initial voltages at $t = 0^-$. The source current is zero for $t > 8$ ms. The voltage across the $10 \mu\text{F}$ capacitor is observed to be 4 V at $t = 2$ ms. (i) What was the initial voltage across the capacitors and what were the initial stored energy in them? (ii) Calculate and plot the voltage across the capacitors and the voltage across the current source as functions of time for 0–9 ms range. (iii) Calculate the total energy delivered by the current source, total energy dissipated in the resistor and change in stored energy of all the capacitors.

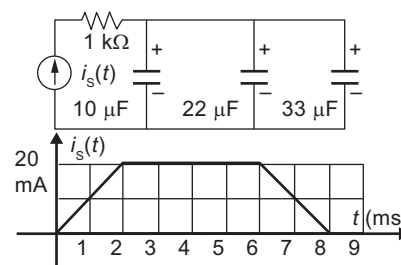


Fig. 3.10-19

35. Three capacitors – 0.02 mF , 0.05 mF and 0.0333 mF – with same initial voltage of 10 V are connected in series. The applied current source is $i_s(t)$ mA, where $i_s(t)$ is not known. The charge in the 0.05 mF capacitor is found to be 0.3 mC at $t = 3$ s. Find (i) the trapped energy in the series combination, (ii) the voltage across each capacitor and charges in them at $t = 3$ s, (iii) the average value of $i_s(t)$ in the first 3 s, (iv) stored energy in the circuit and in each capacitor at $t = 3$ s and (v) total energy delivered by the applied current source in 3 s and average power delivered by the source over 3 s?

Part Two

Analysis of Memoryless Circuits

Nodal Analysis and Mesh Analysis of Memoryless Circuits

CHAPTER OBJECTIVES

- *Introduce the circuit analysis problem and explain the constraints that exist in various equation sets.*
- *Define node voltage variable and develop nodal analysis technique for memoryless circuits containing resistors, dependent voltage and current sources and independent voltage and current sources.*
- *Introduce Nodal Conductance Matrix Y and its properties.*
- *Illustrate nodal analysis technique through a series of solved examples.*
- *Define mesh current variable and develop mesh analysis technique for memoryless circuits containing resistors, dependent voltage and current sources and independent voltage and current sources.*
- *Introduce Mesh Resistance Matrix Z and its properties.*
- *Illustrate mesh analysis technique through a series of solved examples.*
- *Show that any voltage variable or current variable in a memoryless circuit can be expressed as a linear combination of independent voltage and current source functions.*

The reader is expected to become proficient in setting up node equations and mesh equations for circuits containing resistors, dependent sources and independent sources and arrive at a circuit solution after a thorough study of this chapter.

INTRODUCTION

A set of memoryless elements interconnected and driven by independent voltage sources and/or current sources results in a memoryless circuit. We deal with two systematic procedures for analysing such circuits in this chapter. These analysis procedures are called *nodal analysis* and *mesh analysis*. Nodal analysis procedure is applicable to any circuit. However, mesh analysis procedure is applicable to only a subclass of circuits called *planar networks*.

These two analysis procedures are developed and illustrated in the context of memoryless circuits in this chapter. However, they are very well applicable to dynamic circuits as well. We will extend these procedures to dynamic circuits in later chapters.

The Four Types of Linear Dependent Sources

A Voltage-Controlled Voltage Source (VCVS) outputs a voltage equal to $k_v v_x$ V, where k_v is a dimensionless constant and v_x is the voltage across some other element in the circuit.

A Current-Controlled Voltage Source (CCVS) outputs voltage equal to $k_i i_x$ V, where k_i is the transfer-resistance and i_x is the current through some other element in the circuit.

A Current-Controlled Current Source (CCCS) produces a current equal to $k_i i_x$ A, where k_i is a dimensionless constant and i_x is the current through some other element in the circuit.

A Voltage-Controlled Current Source (VCCS) produces a current equal to $k_v v_x$ A, where k_v is the transfer-conductance and v_x is the voltage across some other element in the circuit.

A linear time-invariant memoryless circuit can contain linear resistors (i.e., resistors that obey Ohm's Law) and linear dependent sources.

In addition to linear resistors and linear dependent sources, the circuit will also contain independent voltage sources and independent current sources. Independent voltage sources and current sources are *non-linear elements*.

4.1 THE CIRCUIT ANALYSIS PROBLEM

Let such a memoryless circuit contain b such elements interconnected to form n nodes and l loops. Each two-terminal element has two variables associated with it – a voltage and a current variable. Thus, there are $2b$ variables to be solved for in the circuit analysis problem. Each element is also associated with a terminal voltage–current relationship that yields an equation called the *element equation*. Ohm's law gives the element equation of a resistor. The element equation of a linear dependent source is given in the form indicated in the side box.

Thus, we have b element equations available and we need b more equations to solve for $2b$ variables.

These b equations are obtained by applying KVL and KCL constraints in the circuit. We get a KCL equation at each node, and hence, there are n KCL constraint equations available. However, we had noted earlier that only $n - 1$ of them will be independent. Thus, we need another set of $b - n + 1$ independent equations to solve the circuit. These are provided by the KVL constraints. There are l KVL constraint equations available, where l is the number of loops in the circuit. The number of loops in the circuit will invariably be greater than $b - n + 1$. Not all of the l KVL equations are useful. Some of them can be obtained from others by forming linear combinations. However, it will be possible to find an independent set of $b - n + 1$ equations from l equations. Moreover, the maximum number of equations that can be present in an independent set of equations drawn from these l -loop KVL equations will be exactly $b - n + 1$. However, the choice of equations is not unique. That is, it will be possible to find many sets of independent equations, each set containing $b - n + 1$ equations from the set of l KVL equations. Consider the circuit shown in Fig. 4.1-1.

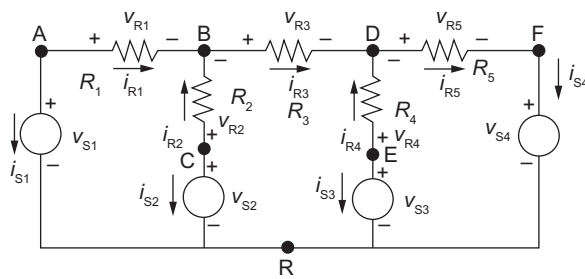


Fig. 4.1-1 A Memoryless Circuit with all Element Variables Identified

This circuit has nine elements, seven nodes and six loops. The element variables are marked in the circuit diagram. The direction of current in each resistor is arbitrary. However, once the current direction is chosen, the voltage polarity will follow the passive sign convention being observed. In the case of voltage sources, the polarity of the voltage variable is usually selected to coincide with the stated polarity of the source and the current direction is set as per the passive sign convention.

The KCL equations at all the seven nodes are listed below. We follow a certain sign convention in writing the KCL equation at a node. We write the KCL equation as *sum of currents leaving the chosen node is equal to zero*.

$$\begin{array}{lcl}
 \text{Node-A} & i_{S1} + i_{R1} & = 0 \\
 \text{Node-B} & -i_{R1} - i_{R2} + i_{R3} & = 0 \\
 \text{Node-C} & i_{S2} + i_{R2} & = 0 \\
 \text{Node-D} & -i_{R3} - i_{R4} + i_{R5} & = 0 \\
 \text{Node-E} & i_{S3} + i_{R4} & = 0 \\
 \text{Node-F} & i_{S4} - i_{R5} & = 0 \\
 \text{Node-R} & -i_{S1} - i_{S2} - i_{S3} - i_{S4} & = 0
 \end{array} \left. \vphantom{\begin{array}{lcl}} \right\} \text{for all time } t$$

Obviously, this set of seven equations do not form an independent set since the KCL equation at Node-R – we will call the KCL equation at a node as the *node equation* at that node from this point onwards – can be obtained by adding the KCL equations at all the other nodes. Any set of node equations containing six equations will be an independent set of equations in this circuit.

We usually assign one of the nodes in the circuit as a *reference node*, in the sense that voltages of the other nodes will be defined and measured with respect to this node. Any node can be set as a reference node in theory; but in practice the choice will be obvious since there will be a node which forms a common point of reference for applying inputs and measuring outputs. If such a choice is not obvious, the practical convention is to set that node which has the maximum number of elements connected to it as the reference node. *The KCL equation for reference node is dropped and the KCL equations at the remaining nodes are accepted as a set of $(n - 1)$ independent equations in circuit analysis.*

The Reference Node in a circuit.

Now, we shift our attention to the KVL equations written for the six loops. Here too, we follow a certain convention for writing the KVL equations. We start at the leftmost corner of a loop and traverse it in the clockwise direction until we get back to the starting point. As we go along, we enter the element voltages in the equation with the polarity that we see first. That is, if we meet an element voltage at its positive polarity first, we enter that voltage variable with a positive sign, and, if we meet an element voltage at its negative polarity first, we enter that variable with a negative sign.

$$\begin{array}{lcl}
 \text{Loop RABCR} & -v_{S1} + v_{R1} - v_{R2} + v_{S2} & = 0 \\
 \text{Loop RCBDER} & -v_{S2} + v_{R2} + v_{R3} - v_{R4} + v_{S3} & = 0 \\
 \text{Loop REDFR} & -v_{S3} + v_{R4} + v_{R5} + v_{S4} & = 0 \\
 \text{Loop RABDER} & -v_{S1} + v_{R1} + v_{R3} - v_{R4} + v_{S3} & = 0 \\
 \text{Loop RCBDFR} & -v_{S2} + v_{R2} + v_{R3} + v_{R5} + v_{S4} & = 0 \\
 \text{Loop RABDFR} & -v_{S1} + v_{R1} + v_{R3} + v_{R5} + v_{S4} & = 0
 \end{array} \left. \vphantom{\begin{array}{lcl}} \right\} \text{for all time } t$$

Obviously, these six equations do not form an independent set. For example, the first three will add up to yield the last one. The first two will add up to yield the fourth one. The sum of the second and the third will be the fifth equation. Thus, the last three are not independent equations. The first three are independent since each contains at least one voltage variable that does not figure in the other two. Hence, we may accept the first three as the independent set of three KVL equations. However, there are other possible choices too. For instance, the first two and the last will form an independent set of three equations.

Kirchhoff's Voltage Law applied to all loops in a circuit results in a dependent set of equations.

Thus, we have six independent KCL equations and three independent KVL equations making up nine equations involving 18 variables – nine current variables and nine voltage variables. The remaining nine equations come from the element equations. The complete set of 18 equations needed to solve for 18 variables are listed below:

$$\begin{array}{l}
 i_{S1} + i_{R1} = 0; -i_{R1} - i_{R2} + i_{R3} = 0; i_{S2} + i_{R2} = 0 \\
 -i_{R3} - i_{R4} + i_{R5} = 0; i_{S3} + i_{R4} = 0; i_{S4} - i_{R5} = 0 \\
 -v_{S1} + v_{R1} - v_{R2} + v_{S2} = 0 \\
 -v_{S2} + v_{R2} + v_{R3} - v_{R4} + v_{S3} = 0 \\
 -v_{S3} + v_{R4} + v_{R5} + v_{S4} = 0 \\
 v_{R1} = R_1 i_{R1}; v_{R2} = R_2 i_{R2}; v_{R3} = R_3 i_{R3}; v_{R4} = R_4 i_{R4}; v_{R5} = R_5 i_{R5} \\
 v_{S1} = v_1(t); v_{S2} = v_2(t); v_{S3} = v_3(t); v_{S4} = v_4(t)
 \end{array}$$

Nodal Analysis and Mesh Analysis

Nodal Analysis uses the KVL and element equations to eliminate voltage variables, reduces the number of pertinent variables to $(n - 1)$ node voltage variables and uses the KCL equations to solve for these node voltage variables.

Mesh Analysis uses KCL and element equations to eliminate current variables, reduces the number of pertinent variables to $(b - n + 1)$ mesh current variables and uses the KVL equations to solve for these variables.

$v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$ are the time-functions which describe the voltage delivered by the independent voltage sources.

Thus, we have 18 equations for 18 unknowns. They come in three sets – the first set consists of $(n - 1)$ KCL equations, the second set contains $(b - n + 1)$ KVL equations and the third set contributes b element equations. Can we simplify this problem and reduce the number of variables we have to deal with? This is where the systematic procedures we set out to develop in this chapter come into focus.

We develop the method of Nodal Analysis first through a series of examples that follow.

4.2 NODAL ANALYSIS OF CIRCUITS CONTAINING RESISTORS WITH INDEPENDENT CURRENT SOURCES

The example circuit that we employ to discuss nodal analysis is shown in Fig. 4.2-1. It has nine elements and four nodes. It is driven by three independent current sources I_1 , I_2 and I_3 . All the element voltages and current variables adhering to passive sign convention have been identified, though, not labelled. The labelling scheme is the same as the one we employed in the previous section.

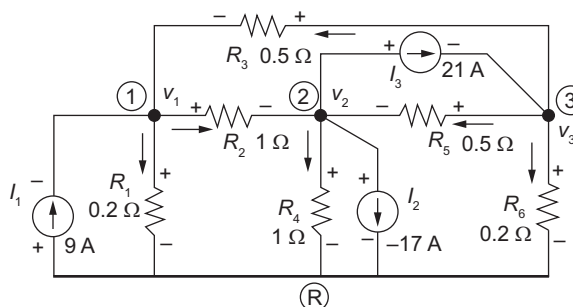


Fig. 4.2-1 Example Circuit with Resistors and Independent Current Sources for Nodal Analysis

The node that has the largest number of elements connected to it is taken as reference node-R and is indicated by a *thick* line in the diagram.

This circuit has a solution. Certain finite voltages will exist across elements and certain finite currents will flow through them. We can measure the voltage across elements by connecting a voltmeter across them. Now, assume that the resistor R_6 is removed. Obviously, the circuit will have a different solution, but note that with R_6 removed, there is no element connected directly from node-3 to the reference node. However, we can still connect a voltmeter between node-3 and the reference node and get a finite reading that indicates the voltage of node-3 with respect to the reference node. This measured voltage is *not* the voltage across any element (we are assuming that R_6 is removed). Thus, we see that each node in the circuit will have a voltage difference with respect to the reference node irrespective of whether that voltage can be identified as the voltage across some element or the other. These voltages are called *node voltages*.

In the present circuit, we observe that all the three node voltage variables are identifiable as element voltages, but this need not be the case always.

If the node voltages are known, the element voltages may be calculated as linear combinations of the node voltages. After all, an element has to be connected between two nodes. If one of them is a reference node itself, then, the element voltage is equal to the node voltage of the other node with a + or – sign. If both nodes are different from the reference node, then, the element voltage will be the difference between the node voltages at those two nodes.

'Node voltage' is the voltage of a node in a circuit with respect to a chosen reference node in the circuit.

All the element voltages may be calculated as a linear combination of node voltages.

Three node voltage variables are identified and labelled as v_1 , v_2 and v_3 in the circuit. Since reference node is the node with respect to which the other node potentials are defined, no node voltage variable needs to be assigned for it. Its node voltage is zero by definition.

Now, apply KVL to get equations relating the element voltages to node voltages. For example, consider the loop formed by R-Node-1-Node-2-R. KVL in this loop gives,

$$\begin{aligned} -v_1 + v_{R2} + v_2 &= 0 \\ \therefore v_{R2} &= v_1 - v_2 \end{aligned}$$

All element voltages may be related to the three node voltages in this manner by inspection.

$$\begin{aligned} v_{R1} &= v_1 \\ v_{R2} &= v_1 - v_2 \\ v_{R3} &= v_3 - v_1 \\ v_{R4} &= v_2 \\ v_{R5} &= v_3 - v_2 \\ v_{R6} &= v_3 \\ v_{I1} &= -v_1 \\ v_{I2} &= v_2 \\ v_{I3} &= v_2 - v_3 \end{aligned} \quad (4.2-1)$$

We observe that all element voltages can be obtained either as some node voltage straightaway or as a difference between two node voltages. Hence, a set of $(n - 1)$ node voltages, defined with respect to the reference node, is a sufficient set of voltage variables for determining b element voltages. Notice that we are using KVL equations to reduce the number of pertinent voltage variables to $(n - 1)$ from b .

The KCL equations remain. We use them to solve for these node voltages. However, KCL equations are written in terms of currents. This is where the element equations come in. We substitute element equations in KCL equations as and when we write KCL equations in order to substitute for element currents in terms of element voltages. Of course, element voltages will be expressed in terms of node voltages. Thus, writing node equations at the $n - 1$ nodes (because only $n - 1$ KCL equations are independent) involves two mental operations for each element – obtaining element voltage in terms of node voltages and replacing current variable by voltage variable with the help of the element equation. We illustrate this for node-1 in the circuit in Fig. 4.2-1.

We write KCL at this node by equating the sum of currents *going away from the node* to zero.

$$\begin{aligned} i_{R1} + i_{R2} - i_{R3} - I_1 &= 0 \\ \text{i.e., } \frac{v_{R1}}{R_1} + \frac{v_{R2}}{R_2} - \frac{v_{R3}}{R_3} - I_1 &= 0 \\ \text{i.e., } G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) &= I_1, \end{aligned}$$

Node voltage of a reference node is zero by definition.

Sign convention for writing nodal equations.

where G represents the conductance value ($1/R$) of the corresponding resistance. Now, we will skip writing the first two steps that we do usually; we perform them mentally and write the third equation straightaway for nodal analysis. With our convention of assigning a *positive sign for current flowing away* from the node, the node voltage variable at the node where KCL is being written will appear with a positive sign and the other node voltage variables will appear with a negative sign in the equation. The *net current delivered by current sources* to that node will appear with a *positive sign on the right side* of the equation. The remaining two node equations for this circuit are:

$$\begin{aligned} G_2(v_2 - v_1) + G_5(v_2 - v_3) + G_4 v_2 &= -I_2 - I_3 \\ G_3(v_3 - v_1) + G_5(v_3 - v_2) + G_6 v_3 &= I_2 \end{aligned}$$

We can solve for v_1 , v_2 and v_3 using these three equations listed below:

$$\begin{aligned} G_1 v_1 + G_2(v_1 - v_2) + G_3(v_1 - v_3) &= I_1 \\ G_4 v_2 + G_2(v_2 - v_1) + G_5(v_2 - v_3) &= -I_2 - I_3 \\ G_6 v_3 + G_3(v_3 - v_1) + G_5(v_3 - v_2) &= I_2 \end{aligned} \quad (4.2-2)$$

The element voltages can be determined subsequently by using Eqn. 4.2-1 and the element currents can be obtained by using element equations in the last step.

We followed a certain convention in writing the node equations in Eqn. 4.2-2. Adhering to such a convention has yielded certain symmetry in these equations. Let us express these equations in matrix notation to see the symmetry clearly.

$$\begin{aligned} (G_1 + G_2 + G_3)v_1 - G_2 v_2 - G_3 v_3 &= I_1 \\ -G_2 v_1 + (G_2 + G_4 + G_3)v_2 - G_5 v_3 &= -I_2 - I_3 \\ -G_3 v_1 - G_5 v_2 + (G_3 + G_5 + G_6)v_3 &= I_2 \end{aligned}$$

$$\text{i.e., } \begin{bmatrix} (G_1 + G_2 + G_3) & -G_2 & -G_3 \\ -G_2 & (G_2 + G_4 + G_3) & -G_5 \\ -G_3 & -G_5 & (G_3 + G_5 + G_6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad (4.2-3)$$

$$\text{i.e., } \mathbf{YV} = \mathbf{CU},$$

where $\mathbf{Y}$ is called the *Nodal Conductance Matrix*, $\mathbf{V}$ is called the *Node Voltage Vector*, $\mathbf{U}$ is called the *Input Vector* and $\mathbf{C}$ is called the *Input Matrix*. Note that the order of nodal conductance matrix is $(n - 1) \times (n - 1)$ and that it is *symmetric* (see side-box).

The right-hand side product $\mathbf{CU}$ is a column vector of the *net current injected by the current sources at the corresponding nodes*.

Now, we can write down this matrix equation by inspection after skipping all the intermediate steps. The following matrix equation results in the case of the example we considered in this section.

$$\begin{aligned} G_1 &= \frac{1}{0.2 \, \Omega} = 5S; \quad G_2 = \frac{1}{1 \, \Omega} = 1S; \quad G_3 = \frac{1}{0.5 \, \Omega} = 2S; \\ G_4 &= \frac{1}{1 \, \Omega} = 1S; \quad G_5 = \frac{1}{0.5 \, \Omega} = 2S; \quad G_6 = \frac{1}{0.2 \, \Omega} = 5S; \end{aligned} \quad (4.2-4)$$

$$\begin{bmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -21 - (-17) \\ 21 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 21 \end{bmatrix}$$

Solving the matrix equation by Cramer's rule, we get,

$$\begin{aligned} v_1 &= \frac{\begin{vmatrix} 9 & -1 & -2 \\ -4 & 4 & -2 \\ 21 & -2 & 9 \end{vmatrix}}{\begin{vmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{vmatrix}} = \frac{9(36 - 4) + 1(-36 + 42) - 2(8 - 84)}{8(36 - 4) + 1(-9 - 4) - 2(2 + 8)} = \frac{446}{223} = 2 \, \text{V} \\ v_2 &= \frac{\begin{vmatrix} 8 & 9 & -2 \\ -1 & -4 & -2 \\ -2 & 21 & 9 \end{vmatrix}}{\begin{vmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{vmatrix}} = \frac{8(-36 + 42) - 9(-9 - 4) - 2(-21 - 8)}{223} = \frac{223}{223} = 1 \, \text{V} \\ v_3 &= \frac{\begin{vmatrix} 8 & -1 & 9 \\ -1 & 4 & -4 \\ -2 & -2 & 21 \end{vmatrix}}{\begin{vmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{vmatrix}} = \frac{8(84 - 8) + 1(-21 - 8) + 9(2 + 8)}{8(36 - 4) + 1(-9 - 4) - 2(2 + 8)} = \frac{669}{223} = 3 \, \text{V} \end{aligned}$$

Nodal Conductance Matrix

The nodal conductance $\mathbf{Y}$ matrix of an n -node circuit containing only resistors and independent current sources is a symmetric $(n - 1) \times (n - 1)$ matrix.

The diagonal element of $\mathbf{Y}$ matrix, y_{ii} , is the sum of conductances connected at the node- i .

The off-diagonal element of $\mathbf{Y}$ matrix, y_{ij} , is the negative of the sum of all conductances connected between node- i and node- j .

There can be more than one conductance connected between two nodes. Then, they will be in parallel and they will add in y_{ij} . That is why y_{ij} should be the negative of the sum of all conductances connected between node- i and node- j .

The element voltages can be calculated by using Eqn. 4.2-1 or by inspection. Similarly, the element currents may be obtained by inspection. The complete solution is shown in the circuits appearing in Fig. 4.2-2.

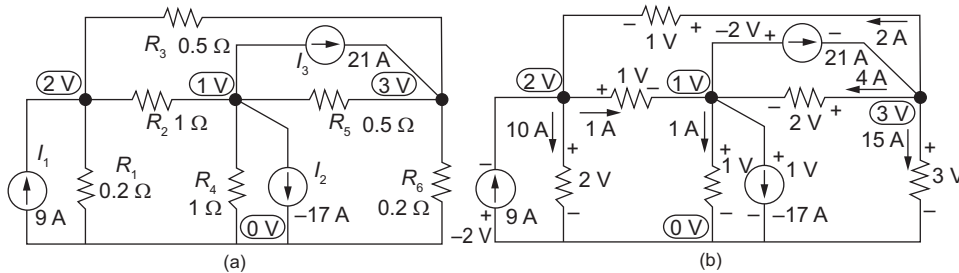


Fig. 4.2-2 Nodal Analysis Solution for Circuit in Fig. 4.2-1

This section has shown that an n -node circuit containing only linear resistors and independent current sources will have a nodal representation given by $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix of order $(n - 1) \times (n - 1)$, $\mathbf{V}$ is the node voltage column vector of order $(n - 1) \times 1$, $\mathbf{U}$ is the source current column vector of order $n_{cs} \times 1$ and $\mathbf{C}$ is the input matrix of order $(n - 1) \times n_{cs}$. n_{cs} is the number of independent current sources in the circuit.

y_{ii} = sum of all conductances connected at i^{th} node

y_{ij} = negative of sum of all conductances connected between i^{th} node and j^{th} node

$c_{ij} = \begin{cases} 0 & \text{if } j^{th} \text{ current source is not connected at } i^{th} \text{ node} \\ 1 & \text{if } j^{th} \text{ current source is delivering current into } i^{th} \text{ node} \\ -1 & \text{if } j^{th} \text{ current source is drawing current from } i^{th} \text{ node} \end{cases}$

Equivalently, the matrix product $\mathbf{CU}$ may be replaced by a column vector which contains the *net current delivered to a node by all current sources connected at that node*. The nodal conductance matrix will be symmetric for this kind of circuit. The node voltage vector is obtained by Cramer's rule or by matrix inversion as $\mathbf{V} = \mathbf{Y}^{-1}\mathbf{CU}$.

Element voltages and currents may be obtained in terms of node voltages by subsequent inspection. For instance, the left end of R_2 is at 2 V with respect to the reference node and the right end is at 1 V. Hence, v_{R2} is $2 - 1 = 1$ V and its current is $1 \text{ V} / 1 \Omega = 1 \text{ A}$.

4.3 NODAL ANALYSIS OF CIRCUITS CONTAINING INDEPENDENT VOLTAGE SOURCES

We extend the nodal analysis technique that we developed in the previous section to circuits containing independent voltage sources, in addition to independent current sources, in this section.

An independent voltage source can appear in three positions in a circuit. It may appear from a node to the reference node. Secondly, it may appear between two nodes at which more than two elements are incident. Thirdly, it may appear in series with some resistor. We deal with the first two cases in this section and take up the third case in the next section.

An independent voltage source connected straight across a node and the reference node *constrains* the node voltage variable at that node to be the same as the source function of the source. This results in a reduction in the number of node voltage variables by one.

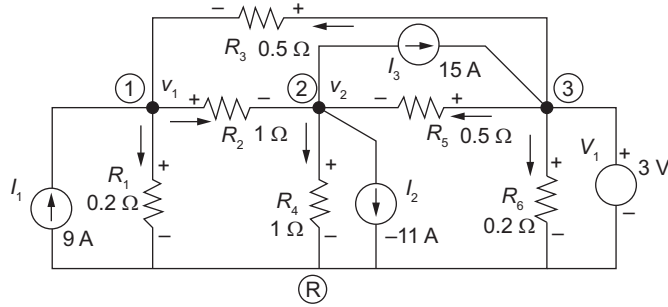


Fig. 4.3-1 Circuit Showing a Node Voltage Getting Fixed by a Voltage Source

The circuit in Fig. 4.3-1 shows the first case.

We identify the reference node and assign node voltage variables to other nodes as the first step in the nodal analysis of circuits. The reference node and the remaining three nodes are identified in the circuit in Fig. 4.3-1. However, only two node voltage variables are assigned; the third node voltage variable is not shown as assigned. The reason is obvious. The independent voltage source connected directly from that node to the reference node *fixes* or *constrains* that node voltage to be at the source value. Thus, the number of node variables to be solved for has come down to two from three. This, in general, is the effect of voltage sources in a circuit as far as nodal analysis is concerned. Each such voltage source imposes one constraint on the degree of freedom the circuit has and reduces the number of node voltage variables by one.

Thus, we have only two unknown node voltage variables in this 3-node-plus reference node problem. We write the two node equations as below:

$$\begin{aligned} G_1 v_1 + G_2(v_1 - v_2) + G_3(v_1 - V_1) &= I_1 \\ G_4 v_2 + G_2(v_2 - v_1) + G_5(v_2 - V_1) &= -I_2 - I_3 \end{aligned} \quad (4.3-1)$$

We do not have to write the third node equation since we know that $v_3 = V_1$. Let us cast these equations in matrix form.

Y matrix can be obtained by inspection after deactivating all the independent sources in the circuit.

$$\begin{bmatrix} (G_1 + G_2 + G_3) & -G_2 \\ -G_2 & (G_2 + G_4 + G_5) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 + G_3 V_1 \\ -I_2 - I_3 + G_5 V_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & G_3 \\ 0 & -1 & -1 & G_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_1 \end{bmatrix} \quad (4.3-2)$$

The matrix equation has come out in the form of $\mathbf{YV} = \mathbf{CU}$ again. We note that the **Y** matrix is symmetric. But now, the **U** vector has four entries – three current source values and one voltage source value. We note carefully that the **C** matrix has suitable entries that convert the volts units into amps. Thus, the matrix product **CU** is a column vector of currents, since a voltage multiplied by conductance is a current.

Moreover, we observe that the form and entries of **Y** matrix on the left-hand side of the equation cannot depend on the particular values of I_1 , I_2 , I_3 and V_1 that happen to be present in the circuit. The matrix must be the same for any numerical value for these four inputs. Therefore, **Y** matrix entries must be the same for a case where all these inputs are zero-valued. But, an independent current source of zero value is an open-circuit and an independent voltage source with a zero value is a short-circuit. Therefore, it should be possible to write down that matrix by inspection after we deactivate all the independent sources in the circuit. The deactivated circuit is shown in Fig. 4.3-2.

The resistor R_6 simply goes out of the picture due to the short-circuit across it. R_5 gets connected between node-2 and the reference node. R_3 gets connected between node-1 and the reference node. We write the **Y** matrix of this circuit by using the rules we developed in the previous section.

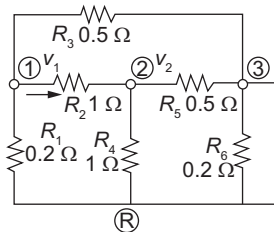


Fig. 4.3-2 Circuit in Fig. 4.3-1 with all Independent Sources Deactivated

$$\begin{aligned} y_{11} &= G_1 + G_2 + G_3 \\ y_{12} &= y_{21} = -G_2 \\ y_{22} &= G_2 + G_4 + G_5 \end{aligned}$$

We find that this is the same as the $\mathbf{Y}$ matrix in Eqn. 4.3-2. Thus, we conclude that *an independent voltage source directly at a node in a circuit results in a reduction of node voltage variables by one and in a reduction of order of nodal conductance matrix by one. The nodal conductance matrix remains symmetric.* The nodal conductance matrix may be found out by using the deactivated circuit, if necessary. However, the node equations will have to be written in order to get the right-hand side of Eqn. 4.3-2.

The resistor R_6 disappearing altogether from the analysis is an interesting aspect. We observe that the value of conductance of this resistor does not appear anywhere in Eqn. 4.3-2. Thus, it has no effect on the node voltages. We may even remove it when we write the equations for node voltage variables. Why is it so? For the simple reason that the voltage source has fixed the node potential at the third node and the presence or absence of R_6 has no effect on that aspect. Since node potentials decide the voltage across all elements and currents through them, the rest of the circuit is totally unaffected by the value of R_6 . The only element that gets affected by R_6 (apart from itself!) is the voltage source. Its current will have a component due to R_6 . *We may treat a resistor across an independent voltage source as a trivial element.*

Substituting the values for conductances and source functions in Eqn. 4.3-2, we get,

$$\begin{bmatrix} 8 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 9 + 2 \times 3 \\ -(-11) - 15 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \end{bmatrix}$$

and the solution is $v_1 = 2$ V and $v_2 = 1$ V.

Now, the voltages across all the resistors and the current sources may be found by inspection. Similarly, the currents through the resistors can be found by applying Ohm's law to them. Currents through current sources are already known; what remains is the current in the voltage source. An extra step has to be applied for finding the current in V_1 . This is the price that the voltage source demands from us for fixing the node-3 potential for us.

The extra step involves applying KCL at node-3. Note that we did not use the KCL equation at node-3 in the process of solving for node voltage variables. We use it now. All the other currents leaving or entering node-3 are known. Only the current through V_1 is unknown. Hence, it can be found.

$$G_3(V_1 - v_1) + G_5(V_1 - v_2) + G_6V_1 - I_3 + i_{V_1} = 0$$

Substituting values,

$$\begin{aligned} 2(3 - 2) + 2(3 - 1) + 5 \times 3 - 15 + i_{V_1} &= 0 \\ \therefore i_{V_1} &= -6 \text{ A} \end{aligned}$$

Note that the trivial element, R_6 , gets accounted now. Thus, the current in the 3 V source is -6 A into the positive terminal, i.e., 6 A out of the positive terminal. Fig. 4.3-3 shows the complete solution of the circuit.

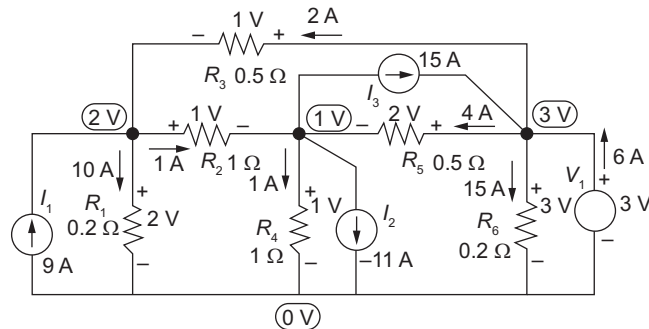


Fig. 4.3-3 Complete Solution for the Circuit in Fig. 4.3-1

Next, we consider a case in which an independent voltage source appears across two nodes other than the reference node (see the circuit in Fig. 4.3-4).

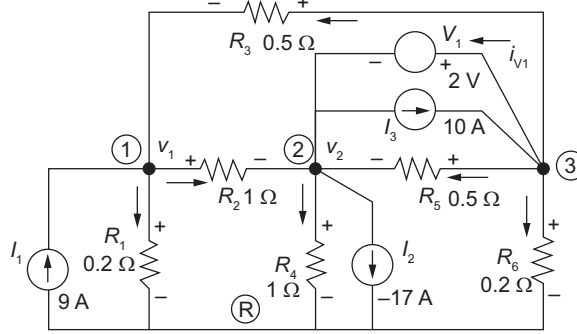


Fig. 4.3-4 Circuit with an Independent Voltage Source Across Two Nodes

An independent voltage source connected between two non-reference nodes imposes a *constraint* between the two node voltages. This results in a reduction in the number of node voltage variables by one.

The independent voltage source V_1 constrains the voltage at node-3 to be above the voltage at node-2 by V_1 V. If we know one of them, we can find the other. Hence, only one of them – either v_2 or v_3 – needs to be assigned. Node-1 is not constrained in any way, and hence, needs to be assigned a node voltage variable. Hence, v_1 and v_2 are assigned as shown. It could have been v_1 and v_3 with no difference to the final solution. Thus, here too, we find that an independent voltage source imposes a constraint on node voltage variables and reduces the number of node voltage variable by one.

We write the KCL equation (or node equation) at all the nodes except at the reference node. We make use of the constraint equation $v_3 = v_2 + V_1$, wherever we need v_3 , in the process of writing node equations at node-1 and node-2.

The resulting equations are:

$$\begin{aligned} \text{Node-1: } & G_1 v_1 + G_2(v_1 - v_2) + G_3(v_1 - v_2 - V_1) = I_1 \\ \text{Node-2: } & G_4 v_2 + G_2(v_2 - v_1) - G_5 V_1 - i_{V_1} = -I_2 - I_3 \\ \text{Node-3: } & G_6(v_2 + V_1) + G_3(v_2 + V_1 - v_1) + G_5 V_1 + i_{V_1} = I_3 \end{aligned}$$

We get rid of i_{V_1} by adding the last two equations and end up with two equations for two unknowns.

$$\begin{aligned} \text{Node-1: } & G_1 v_1 + G_2(v_1 - v_2) + G_3(v_1 - v_2 - V_1) = I_1 \\ \text{Node-2 + Node-3: } & G_4 v_2 + G_2(v_2 - v_1) + G_6(v_2 + V_1) \\ & + G_3(v_2 + V_1 - v_1) = -I_2 \end{aligned} \quad (4.3-3)$$

Note that the resistor R_5 and the current source I_3 , which are directly across the voltage source, disappear from the equations. They become *trivial elements*. They do not affect the node voltages, and hence, the voltage and current of other elements. They affect only the current through the voltage source.

The Eqn. 4.3-3 can be expressed in matrix form as:

$$\begin{bmatrix} (G_1 + G_2 + G_3) & -(G_2 + G_3) \\ -(G_2 + G_3) & (G_2 + G_3 + G_4 + G_6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} G_3 V_1 + I_1 \\ -(G_6 + G_3) V_1 - I_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & G_3 \\ 0 & -1 & 0 & -(G_6 + G_3) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_1 \end{bmatrix}$$

The matrix equation has come out in the form of $\mathbf{YV} = \mathbf{CU}$ again. We note that the $\mathbf{Y}$ matrix is symmetric and the $\mathbf{Y}$ matrix can be obtained from the deactivated circuit as in the earlier case.

Substituting values for conductances and the source functions, we get,

$$\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 9 + 2 \times 2 \\ -(-17) - 7 \times 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$$

The solution is $v_1 = 2$ V and $v_2 = 1$ V. The current through the voltage source can be obtained by applying KCL at node-2 or node-3. Using node-3, we get,

$$G_6(v_2 + V_1) + G_3(v_2 + V_1 - v_1) + G_5V_1 + i_{V_1} = I_3$$

$$5(1 + 2) + 2(1 + 2 - 2) + 2 \times 2 + i_{V_1} = 10 \Rightarrow i_{V_1} = -11 \text{ A}$$

Note that the trivial elements R_5 and I_3 have come back in this equation. The complete solution is shown in Fig. 4.3-5.

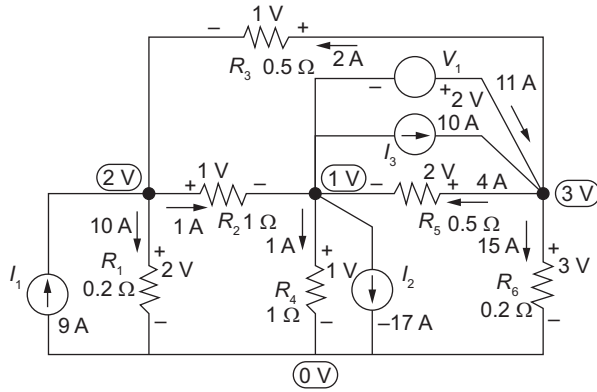


Fig. 4.3-5 Complete Solution for the Circuit in Fig. 4.3-4

The two previous examples have demonstrated that:

- (i) An independent voltage source imposes a constraint on the node voltage variables and reduces their number by one.
- (ii) A node voltage variable *need not be* assigned at a node if an independent voltage source determines that node voltage directly or indirectly through another node voltage variable assigned to another node.
- (iii) Node equation at a node where the node voltage is fixed directly by an independent voltage source connected from that node to the reference node *is not required for solving the other node voltage variables*.
- (iv) Node equations at two nodes with an independent voltage source between them have to be added to get a combined node equation that will be useful in solving the circuit analysis problem.
- (v) The nodal analysis formulation results in an equation $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix of a reduced order circuit resulting from deactivating all independent sources in it. The input vector $\mathbf{U}$ contains all the independent current source functions and the independent voltage source functions. The solution for the node voltage vector $\mathbf{V}$ can be written as $\mathbf{Y}^{-1}\mathbf{CU}$. This indicates that each node voltage (and hence, all element voltages and currents) can be expressed as a linear combination of input source functions, i.e., $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$, where x is some node voltage variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit conductances and connection details. Some of the a 's and b 's

may turn out to be zero for certain choices of x . For instance, a_3 is zero if $x = v_1$ in the example we just concluded, but it is non-zero if $x = i_{V_1}$.

- (vi) The current through the independent voltage sources can be found only after the node voltage variables are solved. Finding the current through an independent voltage source requires the application of KCL at one of the end nodes of that voltage source.

We know that an n -node circuit has only $n - 1$ node voltage variables. Each independent voltage source reduces the number of node voltage variables to be solved for by one. Then, what happens if there are $n - 1$ independent voltage sources in the circuit? This leads us to our next example. Consider the following example circuit in Fig. 4.3-6 that has all the three node voltage variables constrained by three independent voltage sources.

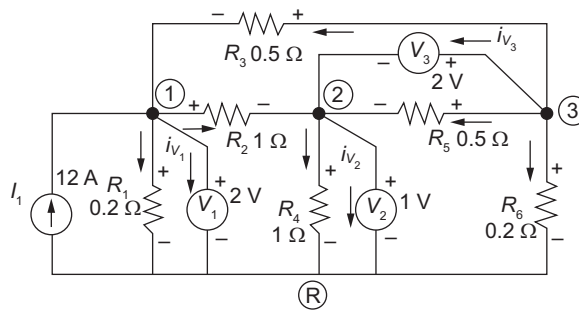


Fig. 4.3-6 A Fully Constrained Nodal Analysis Example Circuit

Although all the three nodes are identified in the diagram, no node voltage variable is assigned to them. This is because the voltage source V_1 fixes the first node potential, the voltage source V_2 fixes the second node potential and sources V_2 and V_3 together fix the third node potential. Hence, the circuit solution is $v_1 = 2$ V, $v_2 = 1$ V and $v_3 = 3$ V. Now, all resistor voltages/currents and voltages across current sources can be worked out by inspection. i_{V_1} , i_{V_2} and i_{V_3} can be found by applying KCL at the three nodes.

$$\text{KCL at node-1} \rightarrow G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) + i_{V_1} = I_1.$$

Substituting numerical values and solving for i_{V_1} , we get $i_{V_1} = 3$ A. Further,

$$\text{KCL at node-2} \rightarrow G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_3) + i_{V_2} - i_{V_3} = 0$$

$$\text{KCL at node-3} \rightarrow G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - v_2) + i_{V_3} = 0$$

We eliminate i_{V_3} by adding these two equations to get $G_4 v_2 + G_2 (v_2 - v_1) + G_6 v_3 + G_3 (v_3 - v_1) + i_{V_2} = 0$. Substituting numerical values and solving for i_{V_2} , we get, $i_{V_2} = -17$ A. Substituting this value in the node equation at node-2, we solve for i_{V_3} to get, $i_{V_3} = -21$ A. The complete solution is shown in Fig. 4.3-7.

An n -node circuit with $(n - 1)$ independent voltage sources in it can be a fully constrained circuit.

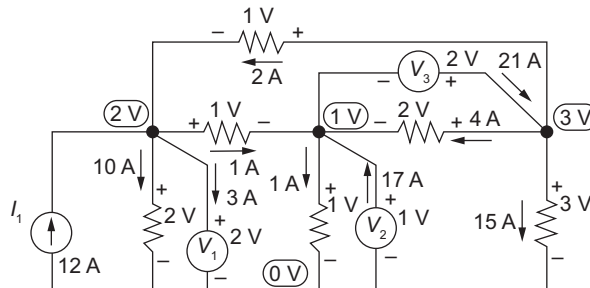


Fig. 4.3-7 Solution for Circuit Shown in Fig. 4.3-6

This circuit had no free node voltage variable. The three independent voltage sources suitably connected had fixed the node voltage variables already. *Thus, there was no circuit analysis problem to begin with.*

Three independent voltage sources in a four-node circuit need not necessarily result in a fully constrained circuit. For instance, assume that the source V_3 is shifted and connected between node-1 and node-2. We note that the three independent voltage sources will then form a closed loop in which the KVL equation will lead to an inconsistent equation $-2 - 2 + 1 = 0$. Practically this means that all the three sources are getting shorted together with large currents in them. But if we insist on modelling the sources as independent ones, the circuit can have no solution when the KVL equation in a loop leads to an inconsistent equation. Thus, the value of V_3 has to be changed to -1 V if it has to be connected between node-1 and node-2, to satisfy the KVL equation. Now, the node voltage variable at node-3 is not constrained by these three sources, and hence, will have to be obtained by nodal analysis. However, the reader may verify that there is no way to determine the currents in the three voltage sources uniquely. There are infinite possible sets of values for these three currents. Thus, the circuit cannot be solved uniquely. Similar conclusions will follow for a case with more than $(n - 1)$ independent voltage sources in an n -node circuit – either the circuit cannot be solved due to inconsistencies in KVL equations or the circuit cannot be solved uniquely.

The maximum number of independent voltage sources that can be present in an n -node circuit with a unique solution is $(n - 1)$.

4.4 SOURCE TRANSFORMATION THEOREM AND ITS USE IN NODAL ANALYSIS

We continue the development of nodal analysis technique for circuits containing independent voltage sources in this section by taking up the case where an independent voltage source is connected in series with a resistor. This case can indeed be solved by the procedure developed in the previous section. However, there is a better way. We need the *Source Transformation Theorem* to understand this method.

4.4.1 Source Transformation Theorem

It was pointed out in Chap. 1 that practical voltage sources can often be modelled as ideal independent voltage source in series with a resistance. Similarly, practical current sources can be modelled by an ideal independent current source in parallel with a resistance. Consider a pair of such practical sources delivering power to identical load resistors as shown in the circuits of Fig. 4.4-1(a) and (b). $v(t)$ is the voltage appearing across the load resistor and $i(t)$ is the current through it in both cases.

Applying Ohm's law and KVL we get $v(t) = v_s(t) - R_{sv}i(t)$ in the circuit in Fig. 4.4-1(a) and applying the current division principle and Ohm's law, we get, $v(t) = \frac{R_L R_{si}}{R_{si} + R_L} i_s(t) = R_{si} i_s(t) \left[1 - \frac{R_{si}}{R_{si} + R_L} \right] = R_{si} i_s(t) - R_{si} i(t)$ in the circuit in Fig. 4.4-1(b).

What are the conditions under which the voltage developed across the load resistor is the same in both cases?

The required conditions are (i) $R_{sv} = R_{si} = R_s$ and (ii) $v_s(t) = R_s i_s(t)$. If these two conditions are satisfied, the sources are indistinguishable from their terminal behaviour. That is, if the source is put inside a black box and an effort is made to determine whether it is a voltage source or a current source by measuring $v(t)$ and/or $i(t)$ for various values of R_L , such an effort will fail. The two sources in Fig. 4.4-1 are completely equivalent as far as their effect on external element is concerned and one may replace the other, provided they satisfy the two conditions listed above. Note that the equivalence is only with respect to what

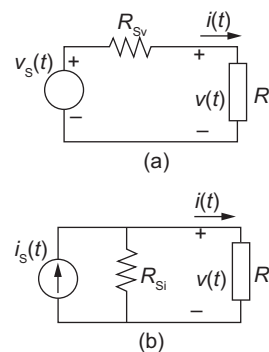


Fig. 4.4-1 Practical Voltage and Current Sources Supplying Power to Identical Load Resistors

Statement of
Source
Transformation
Theorem.

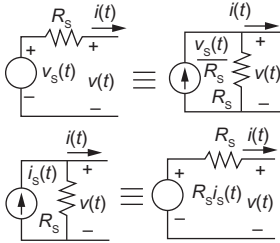


Fig. 4.4-2 Source
Equivalence Between
Voltage and Current
Sources

happens to the external element. They are *not equivalent* as far as what happens inside the source is concerned. For instance, the power dissipation in R_s is not the same in the two sources for the same value of R_L .

Source Transformation Theorem

Source Transformation Theorem states that a voltage source with source function $v_s(t)$ in series with a resistance R_s can be replaced by a current source with source function $i_s(t) = v_s(t)/R_s$ in parallel with a resistance R_s without affecting any voltage/current/power variable external to the source. The direction of current source is such that current flows out of the terminal at which the positive of the voltage source is presently connected.

Similarly, a current source with source function $i_s(t)$ in parallel with a resistance R_s can be replaced by a voltage source with source function $v_s(t) = R_s i_s(t)$ in series with a resistance R_s without affecting any voltage/current/power variable external to the source. The polarity of voltage source is such that it tends to establish a current in the external circuit in the same direction as in the case when the current source is acting.

The reasoning employed in arriving at this theorem is equally valid in the case of dependent sources. Hence, Source Transformation Theorem is applicable to dependent sources also.

Figure 4.4-2 states the Source Transformation Theorem graphically.

4.4.2 Applying Source Transformation Theorem in Nodal Analysis of Circuits

The application of this theorem in nodal analysis of circuits containing independent voltage sources is illustrated below. Consider the circuit in Fig. 4.4-3(a). The circuit has five nodes including the reference node. All the nodes are identified. We observe that the voltage source V_3 is in series with resistor R_6 . This combination may be replaced by a current source $I_2 = V_3/R_6$ in parallel with R_6 as shown in Fig. 4.4-3(b). This results in the elimination of node-4. The circuit in Fig. 4.4-3(b) is a 4-node circuit with two independent voltage sources (V_1 and V_2) constraining its node voltage variables. Thus, there is only one node voltage variable.

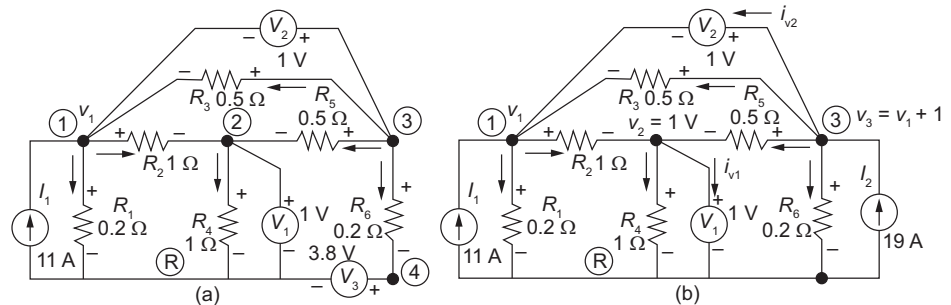


Fig. 4.4-3(a) Nodal Analysis Example Circuit (b) Circuit after Node Reduction by Source Transformation

We start assigning node voltage variables from the leftmost node. The first node shown in the circuit of Fig. 4.4-3(b) is unconstrained, and therefore, we assign the node voltage variable v_1 to that node. That fixes the node voltage at node-3 as $v_1 + V_2$. Hence, a new node voltage variable is not needed at node-3. Node-2 potential is directly constrained by the source V_1 , and hence, a node voltage variable is not needed there.

We need to write the node equations at node-1 and node-3 and combine them to get an equation in the single variable v_1 . The node equation at node-2 is not needed for solving node potentials since it is a directly-constrained node.

$$\text{KCL at node-1} \rightarrow G_1 v_1 + G_2(v_1 - V_1) - G_3 V_2 - i_{V_2} = I_s$$

$$\text{KCL at node-3} \rightarrow G_6(v_1 + V_2) + G_3 V_2 + G_5(v_1 + V_2 - V_1) + i_{V_2} = I_2$$

Adding these two equations, we get,

$$G_1 v_1 + G_2 (v_1 - V_1) + G_6 (v_1 + V_2) + G_5 (v_1 + V_2 - V_1) = I_1 + I_2 = I_1 + G_6 V_3$$

Expressing this in matrix form, we get,

$$[(G_1 + G_2 + G_5 + G_6)][v_1] = \begin{bmatrix} 1 & (G_2 + G_5) & -(G_5 + G_6) & G_6 \end{bmatrix} \begin{bmatrix} I_1 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

We note that the *trivial element* R_3 has no role in deciding the node voltages. Further, we note that the node equation has the format $\mathbf{YV} = \mathbf{CI}$, where the $\mathbf{Y}$ matrix is of 1×1 form and is the nodal conductance matrix of the deactivated circuit. On deactivating the circuit in Fig. 4.4-3(a) by replacing voltage sources with short-circuits and current sources with open-circuits, a simple circuit containing four resistors – R_1 , R_2 , R_c and R_e – will result.

Substituting the numerical values and solving for v_1 , we get, $13v_1 = 26 \Rightarrow v_1 = 2$ V. Therefore, $v_3 = 3$ V.

Now, we fit these values of node voltages into the circuit in Fig. 4.4-3(a) and obtain the voltage across the resistors and current sources and currents through resistors by inspection.

i_{Y_1} is obtained by applying KCL at node-2 of the original circuit.

$$\begin{aligned} G_2(v_2 - v_1) + G_4 v_2 + G_5(v_2 - v_3) + i_{V_1} &= 0 \\ \text{i.e., } 1(1 - 2) + 1(1) + 2(1 - 3) + i_{V_1} &= 0 \\ \Rightarrow i_{V_1} &= 4 \text{ A} \end{aligned}$$

i_{V_2} is obtained by applying KCL either at node-1 or at node-3 of the original circuit. Choosing node-1,

$$\begin{aligned} G_2(v_1 - v_2) + G_1 v_1 + G_3(v_1 - v_3) - iV_2 &= I_1 \\ \text{i.e., } 1(2 - 1) + 5(2) + 2(2 - 3) - iV_2 &= 11 \\ \Rightarrow iV_2 &= -2 \text{ A} \end{aligned}$$

The current delivered by V_3 is the same as the current in R_6 . Hence, $i_{V_3} = -4$ A

The complete solution is shown in Fig. 4.4-4.

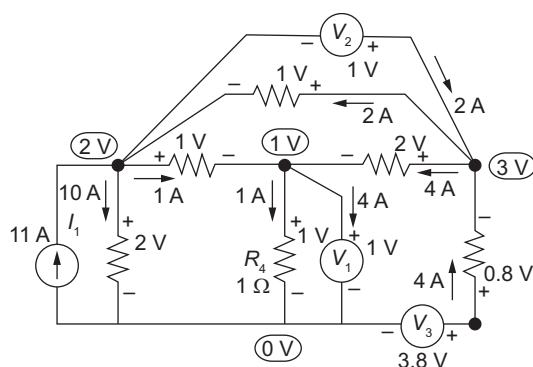


Fig. 4.4-4 Complete Solution for the Circuit in Fig. 4.4-3(a)

4.5 NODAL ANALYSIS OF CIRCUITS CONTAINING DEPENDENT CURRENT SOURCES

Dependent current sources do not pose a problem for nodal analysis. Independent current sources appear on the right-hand side of node equations. However, dependent current sources affect the coefficients of node voltage variables on the left-hand side of node equations.

The controlling variable of a linear dependent current source will be a voltage or a current existing elsewhere in the circuit, but any voltage or current variable in the circuit can be expressed in terms of node voltage variables. Hence, the dependent current source function can be expressed in terms of node voltage variables. Therefore, dependent current sources will affect the coefficients of node equation, i.e., they will change the nodal conductance matrix. We will see that they destroy the symmetry of the nodal conductance matrix.

We develop the nodal analysis procedure for these kinds of circuits through two examples. The first example has node voltages that are not constrained by independent voltage sources and the second one has node voltage variables constrained by independent voltage source.

EXAMPLE: 4.5-1

Solve the circuit in Fig. 4.5-1 (a) completely.

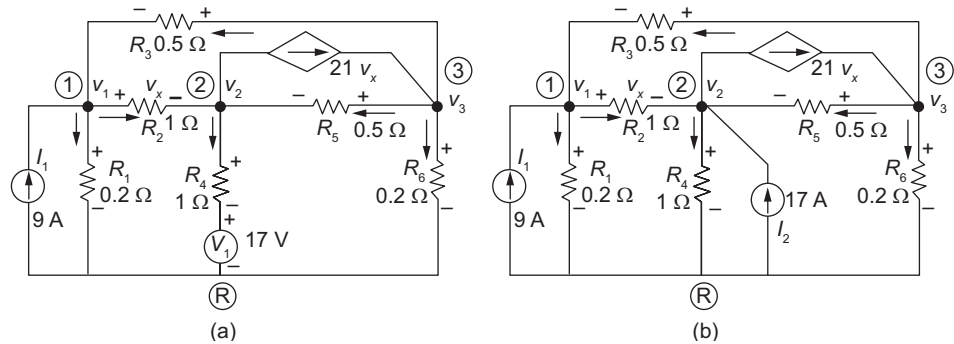


Fig. 4.5-1 (a) Circuit for Example 4.5-1 (b) Circuit after Node Reduction by Source Transformation

SOLUTION

Step-1: Look for independent voltage sources in series with resistors and apply source transformation to such combinations.

There is one such combination in this circuit. It is V_1 in series with R_4 . Applying source transformation to this combination results in an independent current source of 17 A in parallel with R_4 as shown in the circuit of Fig. 4.5-1 (b).

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

Now, all the three non-reference nodes in the circuit of Fig. 4.5-1 (b) are unconstrained nodes, and hence, we assign three node voltage variables v_1 , v_2 and v_3 – as shown in the figure.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the previous step and rewrite the source functions of dependent sources in terms of node voltage variables.

v_x is the controlling variable in this circuit, but v_x is the voltage across R_2 and $= v_1 - v_2$. Therefore, the current source function is $k(v_1 - v_2)$, where $k = 21$.

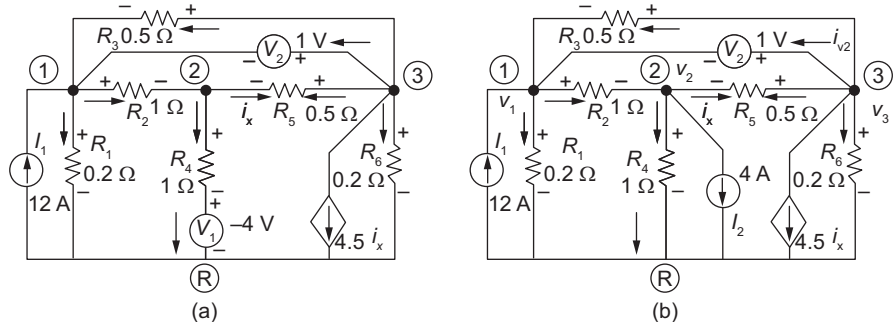


Fig. 4.5-3(a) Circuit for Example 4.5-2(b) Circuit after Node Reduction by Source Transformation

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

We start at the leftmost node of the circuit in Fig. 4.5-3(b) and assign a node voltage variable v_1 there since that node is not directly constrained by a voltage source. Moving to node-2, we see that the node voltage at that node cannot be obtained from the already assigned variable v_1 and that there is no direct constraint at that node. Hence, we assign a node voltage variable v_2 at that node. Now, the node voltage at node-3 can be obtained as $v_1 + V_2$ and a node voltage variable is not needed at that node. Therefore, there are only two node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the previous step and rewrite the source functions of dependent sources in terms of node voltage variables.

i_x is the controlling variable in this circuit, but $i_x = G_5[v_2 - (v_1 + V_2)]$. Therefore, the current source function is $kG_5[v_2 - (v_1 + V_2)]$, where $k = 4.5$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where the voltage sources are connected directly to the reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below:

$$\text{Node-1: } G_1 v_1 + G_2(v_1 - v_2) - G_3 V_2 - i_{V_2} = I_1$$

$$\text{Node-2: } G_4 v_2 + G_2(v_2 - v_1) + G_5(v_2 - v_1 - V_2) = -I_2 = G_4 V_1$$

$$\text{Node-3: } G_6(v_1 + V_2) + G_3 V_2 + G_5(v_1 + V_2 - v_2) + kG_5(v_2 - v_1 - V_2) + i_{V_2} = 0$$

Combining the node equations at node-1 and node-3 to eliminate i_{V_2} ,

$$\text{Node-1 + Node-3: } G_1 v_1 + G_2(v_1 - v_2) + G_6(v_1 + V_2) + G_5(v_1 + V_2 - v_2) + kG_5(v_2 - v_1 - V_2) = I_1$$

$$\text{Node-2: } G_4 v_2 + G_2(v_2 - v_1) + G_5(v_2 - v_1 - V_2) = -I_2 = G_4 V_1$$

Casting these equations in matrix form, we get,

$$\begin{bmatrix} (G_1 + G_2 + G_6 + (1-k)G_5) & -(G_2 + (1-k)G_5) \\ -(G_2 + G_5) & (G_2 + G_4 + G_5) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(G_6 + (1-k)G_5) \\ 0 & G_4 & G_5 \end{bmatrix} \begin{bmatrix} I_1 \\ V_1 \\ V_2 \end{bmatrix}$$

This equation is in the form $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix, but, the nodal conductance matrix is now *asymmetric* and cannot be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as linear combinations of independent source functions.

Substituting the numerical values, we get,

$$\begin{bmatrix} 4 & 6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -2 \end{bmatrix}$$

Solving for the voltage vector by Cramer's rule, $v_1 = 2 \text{ V}$, $v_2 = 1 \text{ V}$ and $v_3 = v_1 + v_2 = 3 \text{ V}$.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and currents through resistive elements can be obtained by inspection. The currents through independent voltage sources in series with resistors can also be obtained at this stage.

Step-6: Use appropriate node equations to solve for currents through the remaining independent voltage sources.

The current through the independent voltage source V_2 has to be determined. We use the node equation at node-1 for this purpose.

$$G_1 v_1 + G_2 (v_1 - v_2) - G_3 v_2 - i_{v_2} = I_1$$

$$10 + 1 - 2 - i_{v_2} = 12 \Rightarrow i_{v_2} = -3 \text{ A}$$

The complete solution is shown in Fig. 4.5-4.

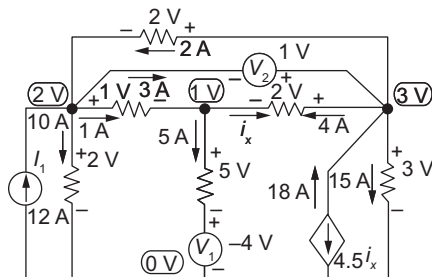


Fig. 4.5-4 Complete Solution for Circuit in Example 4.5-2

4.6 NODAL ANALYSIS OF CIRCUITS CONTAINING DEPENDENT VOLTAGE SOURCES

A dependent voltage source can appear in three positions in a circuit. It may appear from a node to the reference node. Secondly, it may appear between two nodes where more than two elements are incident. Thirdly, it may appear in series with some resistor. The following example shows a circuit that has a dependent voltage source in series with a resistor.

EXAMPLE: 4.6-1

Solve the circuit in Fig. 4.6-1 (a).

SOLUTION

Step-1: Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation to such combinations.

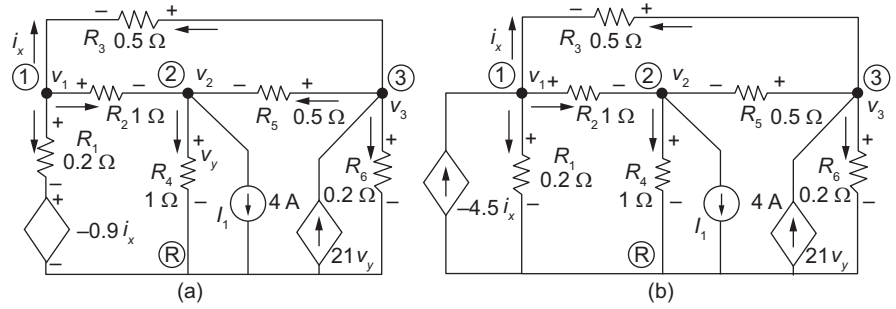


Fig. 4.6-1 (a) Circuit for Example 4.6-1 (b) Reduced Circuit after Source Transformation

There is one such combination in this circuit. It is $-0.9i_x$ in series with R_1 . Applying source transformation to this combination results in a dependent current source of $-0.9G_1i_x$ A in parallel with R_1 as shown in the circuit of Fig. 4.6-1(b).

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

We start at the leftmost node of the circuit in Fig. 4.6-1(b) and assign a node voltage variable v_1 there since that node is not directly constrained by a voltage source. Moving to node-2, we see that the node voltage at that node cannot be obtained from the already assigned variable v_1 and that there is no direct constraint at that node. Hence, we assign a node voltage variable v_2 at that node. The node voltage at node-3 cannot be obtained from v_1 and v_2 . Hence, we assign a node voltage variable v_3 at that node. Therefore, there are three node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the previous step and rewrite the source functions of dependent sources in terms of node voltage variables.

i_x is the controlling variable for the dependent current source at node-1 in the circuit of Fig. 4.6-1(b). But $i_x = G_3[v_1 - v_3]$. Therefore, the current source function is $k_1G_1G_3[v_1 - v_3]$, where $k_1 = -0.9$.

v_y is the controlling variable for the dependent current source at node-3. But $v_y = v_2$. Therefore, the current source function at node-3 is k_2v_2 , where $k_2 = 21$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where the voltage sources are connected directly to the reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below:

$$\text{Node-1: } G_1v_1 + G_2(v_1 - v_2) + G_3(v_1 - v_3) + k_1G_1G_3(v_1 - v_3) = 0$$

$$\text{Node-2: } G_4v_2 + G_2(v_2 - v_1) + G_5(v_2 - v_3) = -I_1$$

$$\text{Node-3: } G_6v_3 + G_3(v_3 - v_1) + G_5(v_3 - v_2) - k_2v_2 = 0$$

Substituting the numerical values and casting these equations in matrix form, we get,

$$\begin{bmatrix} 17 & -1 & -11 \\ -1 & 4 & -2 \\ -2 & -23 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} [4]$$

This equation is in the form $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix, but the nodal conductance matrix is now *asymmetric* and cannot be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as linear combinations of independent source functions.

Solving for the voltage vector by Cramer's rule, $v_1 = 2$ V, $v_2 = 1$ V and $v_3 = 3$ V.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and the currents through resistive elements can be obtained by inspection. The currents through independent voltage sources in series with resistors can also be obtained at this stage.

The complete solution is shown in Fig. 4.6-2.

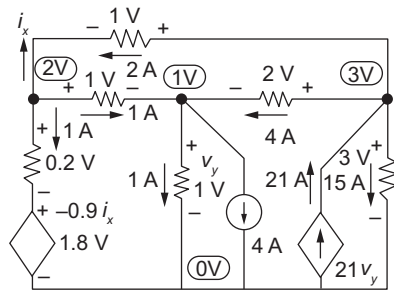


Fig. 4.6-2 Complete Solution for Circuit in Fig. 4.6-1(a)

EXAMPLE: 4.6-2

Solve the circuit in Fig. 4.6-3(a) in Example 4.6-2 by nodal analysis.

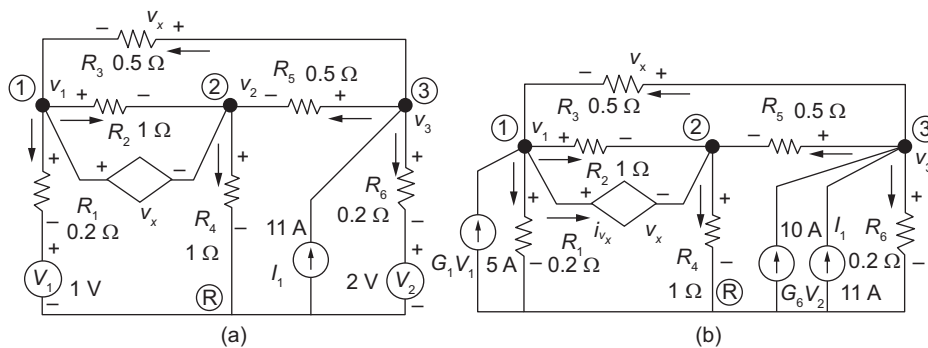


Fig. 4.6-3(a) Circuit for Example 4.6-2(b) Circuit after Node Reduction by Source Transformation

SOLUTION

Step-1: Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation to such combinations.

There are two such combinations in this circuit. They are V_1 in series with R_1 and V_2 in series with R_6 . Applying source transformation to these combinations results in the circuit in Fig. 4.6-3(b).

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

We start at the leftmost node of the circuit in Fig. 4.6-3(b) and assign a node voltage v_1 there since that node is not directly constrained by a voltage source to the reference node. Moving to node-2, we see that the voltage at that node can be obtained from the already assigned variable v_1 by adding $-v_x$. Hence, we do not assign

EXAMPLE: 4.6-3

Solve the circuit in Fig. 4.6-5 by nodal analysis.

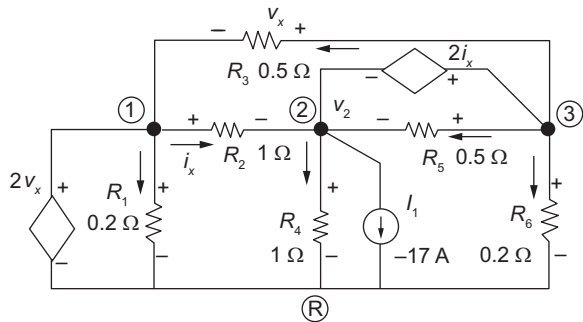


Fig. 4.6-5 Circuit for Nodal Analysis in Example 4.6-3

SOLUTION

Node-1 is constrained by the dependent source to the reference node, and hence, no node voltage variable can be assigned there. Node-2 is assigned a node voltage variable v_2 . Then, the node voltage variable gets fixed as $v_2 + 2I_x$ through the dependent voltage source connected between node-2 and node-3. Thus, this circuit has only one node voltage variable to be solved for.

The node equation at node-1 is not required for determining node voltages. However, it will be needed later for determining the current through the dependent voltage source connected at that node.

$$v_x = v_2 + 2i_x - 2v_x \text{ and } i_x = 2v_x - v_2$$

On solving these two equations, we get, $v_x = v_2$ and $i_x = v_2$.

The KCL equations at node-2 and node-3 can be combined to form a single equation in v_2 .

This combined equation will be $v_2 + (v_2 - 2v_x) + 5(v_2 + 2i_x) + 2(v_2 + 2i_x - 2v_x) = 17$. Substituting for v_x and i_x in terms of v_2 , we get, $17v_2 = 17 \Rightarrow v_2 = 1 \text{ V}$.

Now, $v_x = v_2 = 1$ V and $i_x = 1$ A. Therefore, the node voltages are 2 V, 1 V and 3 V, respectively, at node-1, node-2 and node-3. Then, the node equation at node-1 can be employed to find the current into the positive terminal of dependent source as -9 A. The node equation at node-3 is used to determine the current into the positive terminal of second dependent source as -21 A. The complete solution is shown in Fig. 4.6-6.

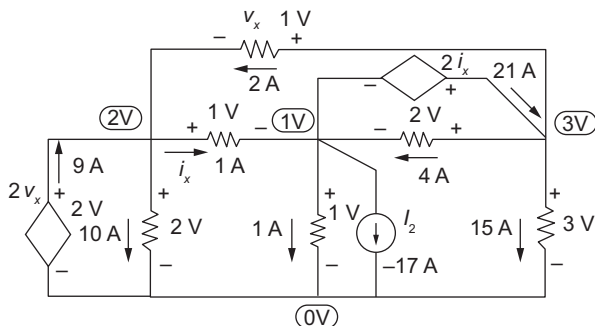


Fig. 4.6-6 Solution for Circuit in Example 4.6-3

Nodal Analysis Procedure

Step-1: Assign reference current directions and reference polarities for voltages of all elements as per the passive sign convention. Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation on such combinations to convert them into current sources in parallel with resistors. This is called 'node reduction'. The resulting circuit is referred to as the reduced circuit.

Step-2: Select a reference node. Assign node voltage variables at those nodes where the node voltage variable is not decided directly by a voltage source (independent or dependent source) or indirectly by already assigned node voltage variables and voltage source functions.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the previous step and express the source functions of all dependent sources in terms of node voltage variables.

Step-4: Prepare the node equations for the reduced circuit. Ignore node equation at nodes where voltage sources (independent or dependent sources) are connected directly to the reference node. Combine (add) the node equations at the end nodes of voltage sources (independent or dependent sources) connected between two non-reference nodes. The number of equations at the end of this step will be equal to number of nodes minus number of irreducible voltage sources.

continued

Step-5: Solve for node voltage variables by elimination technique. The equations may also be expressed as a matrix equation and solved by using Cramer's rule or matrix inversion.

Step-6: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and voltages across current sources. This is done by applying KVL in various loops in the circuit along with Ohm's law for linear resistors.

Step-7: Use appropriate node equations to solve for currents through the independent and dependent voltage sources.

We have completed the development of nodal analysis technique for memoryless circuits containing linear resistors, four kinds of linear dependent sources, independent voltage sources and independent current sources.

The *general nodal analysis procedure* that has emerged is summarised in the side-box.

4.7 MESH ANALYSIS OF CIRCUITS WITH RESISTORS AND INDEPENDENT VOLTAGE SOURCES

Mesh Analysis uses the KCL equations and the element equations to eliminate variables, reduces the number of pertinent variables to $(b - n + 1)$ mesh current variables and uses the second set of equations (KVL equations) to solve for these variables.

Mesh analysis is applicable only to *planar networks*. A planar network is one that can be drawn on a plane without any component crossing over another. Consider the two circuits shown in Fig. 4.7-1.

The circuit in Fig. 4.7-1(a) is non-planar since it cannot be drawn on a plane surface without crossovers. The circuit in Fig. 4.7-1(b) is planar, though there appears to be a crossover due to the way it is drawn in Fig. 4.7-1. However, it can be redrawn to avoid the crossover.

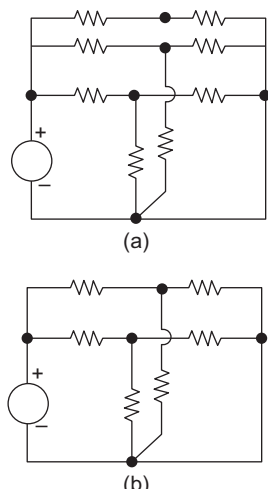


Fig. 4.7-1 (a) A Non-Planar Circuit
(b) A Planar Circuit that Appears to be Non-Planar

4.7.1 Principle of Mesh Analysis

A circuit with n nodes, b elements and l loops will have n KCL equations, l KVL equations and b element equations involving $2b$ element variables. Only $(n - 1)$ KCL equations of n will be linearly independent. Only $(b - n + 1)$ KVL equations of l such equations will be linearly independent. Any set of KCL equations written for $(n - 1)$ nodes of the circuit will form a linearly independent set. However, any set of $(b - n + 1)$ equations drawn from the set of l voltage equations need not form a linearly independent set.

In node analysis, KVL equations are used to show that all element voltages can be expressed in terms of $(n - 1)$ node voltages and KCL equations along with element relations are used subsequently to set up $(n - 1)$ node equations needed for determining the node voltages.

Analogously, we try to use the KCL equations to show that all element currents can be expressed in terms of a reduced set of $(b - n + 1)$ specially defined currents called *mesh currents*. Subsequently, we set up $(b - n + 1)$ KVL equations involving these currents to determine them.

Which ones among the $(b - n + 1)$ loops do we choose for writing these KVL equations? The loops have to be chosen in such a way that the KVL equations will form an independent set. Two equations are necessarily independent if both equations contain terms that are exclusive to them. Consider the following equations.

$$v_1 - v_2 + v_3 - 2 = 0 \text{ and } -v_1 + v_2 + v_4 - 7 = 0$$

Obviously, no combinations like $a(v_1 - v_2 + v_3 - 2) + b(-v_1 + v_2 + v_4 - 7)$ can become equal to zero for all *time* for any combination of values for a and b , for the simple reason that v_3 and v_4 cannot be eliminated. v_3 is present in only one equation, and v_4 too, is present only in one equation.

Thus, a sufficient, but not a necessary condition, for a set of linear equations to form an independent set is that each equation should have at least one variable that does not appear in any other equation in the set.

Refer to the circuit in Fig. 4.7-2.

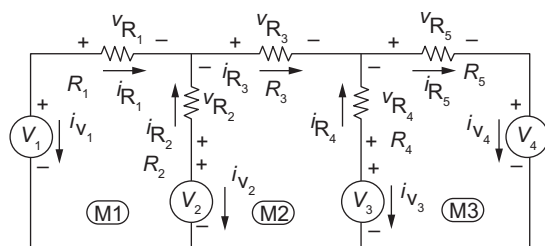


Fig. 4.7-2 Circuit for Illustrating Mesh Analysis

The KVL equations for the three *windows* designated as M_1 , M_2 and M_3 in this circuit are given below:

$$\begin{aligned}\text{Window } M_1: & -V_1 + v_{R_1} - v_{R_2} + V_2 = 0 \\ \text{Window } M_2: & -V_2 + v_{R_2} + v_{R_3} - v_{R_4} + V_3 = 0 \\ \text{Window } M_3: & -V_3 + v_{R_4} + v_{R_5} + V_4 = 0\end{aligned}$$

These equations form an independent set of $(b - n + 1) = (9 - 7 + 1) = 3$ KVL equations for the circuit. They are independent since each equation contains at least one element voltage variable that is not present in the other two. This is because the element R_1 is *completely owned* by the first window, the element R_3 is *completely owned* by the second window and the element R_5 is *completely owned* by the third window. If an element is not *shared* among many windows and is *completely owned* by a particular window, its voltage variable will appear only in the KVL equation of that particular window. That KVL equation cannot be generated by linearly combining KVL equations for the other windows.

Thus, the KVL equations for the windows of a circuit will be the required set of independent equations, provided each window contains at least one element that is exclusively owned by it. The windows in a planar circuit are called *Meshes*. It is possible to prove that, in a planar connected network containing b -elements and n -nodes, there will be exactly $(b - n + 1)$ meshes.

However, is it not possible for a mesh to have no element that is completely owned by it? It is possible (see Fig. 4.7-3). The mesh M_2 does not own any element exclusively.

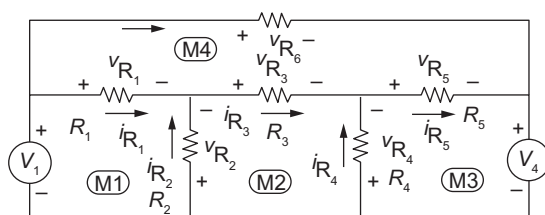


Fig. 4.7-3 Circuit with a Mesh that Does Not Own Any Element

However, the four mesh KVL equations will be independent in this case too. This is because no linear combination of three equations for the other three meshes can eliminate v_{R_6} or V_1 or V_4 .

Hence, we accept the fact that KVL equations for $(b - n + 1)$ meshes in a planar circuit will form a linearly independent set of equations. This will be proved in Chapter 17.

Let us take the issue of defining $(b - n + 1)$ special currents that can be used to express all the element currents in the circuit. We can think of categorising all the elements that participate in a mesh into two groups – the first group containing those elements which appear only in that mesh and the second group containing those elements shared by this

The concept of *complete ownership* of an element.

Meshes are windows in a planar circuit.

There will be exactly $(b - n + 1)$ meshes in a planar network. KVL equations for these meshes will form linearly independent set of $(b - n + 1)$ equations.

'Mesh current variable' introduced.

mesh with other meshes. Obviously, the same current flows through all the elements belonging to the first group. They are in a series combination. Thus, each mesh will have a clearly identifiable current that flows in all the elements completely owned by that mesh. (The case of a mesh that does not own any element completely will be discussed in the next subsection.) This current, *with clockwise direction assumed*, is defined as *mesh current* for that mesh and is used as the describing variable in Mesh Analysis just as *node voltage* was used as the describing variable in Nodal Analysis.

With reference to Fig. 4.7-2, we can derive the following equations by applying KCL for various nodes in this circuit.

$$\begin{aligned}i_{V_1} &= -i_{R_1} \\i_{R_2} &= i_{R_3} - i_{R_1} \\i_{V_2} &= i_{R_1} - i_{R_3} \\i_{R_4} &= i_{R_5} - i_{R_3} \\i_{V_3} &= i_{R_3} - i_{R_5} \\i_{V_4} &= i_{R_5}\end{aligned}$$

Thus, all the remaining currents can be expressed in terms of three currents $-i_{R_1}$, i_{R_3} and i_{R_5} . These three currents will be the mesh currents in this circuit.

However, a separate symbolic representation of mesh current is used in circuit analysis in order to emphasise the clockwise direction of flow in the definition of mesh current and to highlight the point that mesh current is a current that is common to all elements in the mesh. This is shown in Fig. 4.7-4.

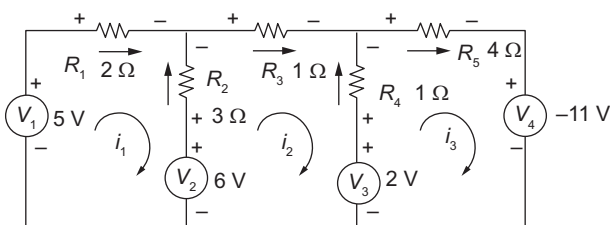


Fig. 4.7-4 Circuit for Mesh Analysis

Relation between
mesh currents and
element currents.

The clockwise arrow and the symbol beside it in every mesh stand for the mesh current variable. The mesh current magnitude itself gives the magnitude of current for all the elements owned by that mesh. If the assumed current direction in such an element coincides with that of mesh current, the element current is the same as the mesh current. If the assumed current direction in such an element is opposite to that of mesh current, the element current is the same as the negative of mesh current. If an element is shared by two meshes, its current is given by the difference between the two mesh currents with due attention placed on current directions.

The procedure of mesh analysis is illustrated using the circuit in Fig. 4.7-4 as an example. Three mesh currents i_1 , i_2 and i_3 are assigned to the three meshes in a clockwise direction as shown. The KVL equations for the three meshes are written now with element equations employed to convert the voltage variables into mesh current variables.

We follow a convention in writing these KVL equations. *We start at the leftmost corner of the mesh and traverse the mesh in a clockwise direction. We enter the voltages that we meet with in a sum. A voltage is entered in the sum with the same polarity as its first polarity marking that we meet during our traversal – if we meet its positive polarity first, we enter it with a positive sign and if we meet its negative polarity first, we enter it with a negative sign.*

Sign convention
to be followed in
preparing mesh
equations.

The mesh equation for the first mesh is derived below.

The first voltage that we meet is that of V_1 . We meet its negative polarity first. Therefore, $-V_1$ enters the equation. Then, we see the voltage across R_1 with positive polarity first. The current through it is the same as the mesh current i_1 , and hence, $R_1 i_1$ enters the equation. The next voltage we meet with is that of R_2 with its negative polarity first. The current through R_2 is $i_2 - i_1$ in the direction assumed for it in the diagram. Hence, $-R_2(i_2 - i_1)$ enters the equation. The last voltage we meet with in the first mesh is that of V_2 , positive polarity coming first. Hence, $+V_2$ enters the equation. Thus, the mesh equation for the first mesh is:

$$\begin{aligned} -V_1 + R_1 i_1 - R_2(i_2 - i_1) + V_2 &= 0 \\ \text{i.e., } (R_1 + R_2)i_1 - R_2 i_2 &= V_1 - V_2 \end{aligned}$$

With our convention of traversing a mesh in a clockwise direction, the mesh current variable in the mesh where KVL is being applied will appear with a positive sign and the other mesh current variables will appear with a negative sign in the equation. The *net rise in voltage contributed by all the independent voltage sources* in that mesh will appear with a *positive sign on the right side* of the equation. The remaining two mesh equations for this circuit are:

$$\begin{aligned} R_2(i_2 - i_1) + R_3 i_2 - R_4(i_3 - i_2) &= V_2 - V_3 \\ \text{i.e., } -R_2 i_1 + (R_2 + R_3 + R_4)i_2 - R_4 i_3 &= V_2 - V_3 \\ R_4(i_3 - i_2) + R_5 i_3 &= V_3 - V_4 \\ \text{i.e., } -R_4 i_2 + (R_4 + R_5)i_3 &= V_3 - V_4 \end{aligned}$$

We can solve for i_1 , i_2 and i_3 using these three equations listed below:

$$\begin{aligned} (R_1 + R_2)i_1 - R_2 i_2 + 0i_3 &= V_1 - V_2 \\ -R_2 i_1 + (R_2 + R_3 + R_4)i_2 - R_4 i_3 &= V_2 - V_3 \\ 0i_1 - R_4 i_2 + (R_4 + R_5)i_3 &= V_3 - V_4 \end{aligned}$$

We followed a certain convention in writing these mesh equations in Eqn. 4.2-2. Adhering to such a convention has resulted in certain symmetry in these equations. Let us express these equations in matrix notation to see the symmetry clearly.

$$\begin{bmatrix} (R_1 + R_2) & -R_2 & 0 \\ -R_2 & (R_2 + R_3 + R_4) & -R_4 \\ 0 & -R_4 & (R_4 + R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

$$\text{i.e., } \mathbf{Z}\mathbf{I} = \mathbf{D}\mathbf{U},$$

where $\mathbf{Z}$ is called the *Mesh Resistance Matrix*, $\mathbf{I}$ is called the *Mesh Current Vector*, $\mathbf{U}$ is called the *Input Vector* and $\mathbf{D}$ is called the *Input Matrix*.

Note that the order of Mesh Resistance Matrix is $(b - n + 1) \times (b - n + 1)$ and that it is *symmetric*.

The right-hand side product $\mathbf{D}\mathbf{U}$ is a column vector of *net voltage rise imparted to the mesh by the independent voltage sources in the corresponding meshes*.

Now, we can write down this matrix equation by inspection after skipping all the intermediate steps. The following matrix equation results in the case of the example we considered in this section.

$$R_1 = 2\Omega; R_2 = 3\Omega; R_3 = 1\Omega; R_4 = 1\Omega; R_5 = 4\Omega; V_1 = 5\text{ V}; V_2 = 6\text{ V}; V_3 = 2\text{ V}; V_4 = -11\text{ V}$$

$$\begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 13 \end{bmatrix}$$

Illustration of procedure to write a mesh equation.

Mesh Resistance Matrix

The mesh resistance matrix $\mathbf{Z}$ of an n -node, b -branch circuit containing only resistors and independent voltage sources is a symmetric $(b - n + 1) \times (b - n + 1)$ matrix.

The diagonal element of matrix $\mathbf{Z}$, z_{ii} , is the sum of resistances appearing in the mesh- i .

The off-diagonal element of matrix $\mathbf{Z}$, z_{ij} , is the negative of sum of all resistances appearing in common between mesh- i and mesh- j . There can be more than one resistance shared between two meshes. Then they will be in series and they will add in z_{ij} . That is why z_{ij} should be the negative of sum of all resistances shared by mesh- i and mesh- j .

Solving the matrix equation by Cramer's rule, we get, $i_1 = 1$ A, $i_2 = 2$ A and $i_3 = 3$ A. The element currents can be obtained by applying KCL at various nodes in the circuit. This can be done by inspection. Resistor voltages may then be obtained by applying Ohm's law. The complete solution of the circuit is shown in Fig. 4.7-5.

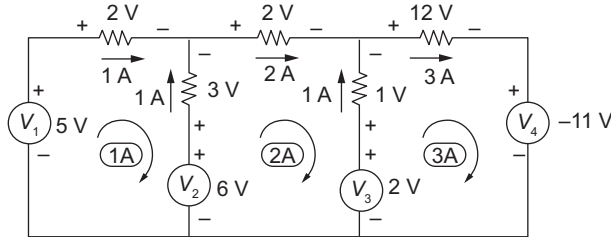


Fig. 4.7-5 Complete Mesh Analysis Solution for the Circuit in Fig. 4.7-4

This section has shown that an n -node, b -element circuit containing only linear resistors and independent voltage sources will have a mesh representation given by $\mathbf{ZI} = \mathbf{DU}$, where $\mathbf{Z}$ is the Mesh Resistance Matrix of order $(b - n + 1) \times (b - n + 1)$, $\mathbf{I}$ is the mesh current column vector of order $(b - n + 1) \times 1$, $\mathbf{U}$ is the source voltage column vector of order $n_{vs} \times 1$ and $\mathbf{D}$ is the input matrix of order $(b - n + 1) \times n_{vs}$. n_{vs} is the number of independent voltage sources in the circuit.

z_{ii} = sum of all resistances appearing in the i^{th} mesh

z_{ij} = negative of sum of all resistances common to i^{th} mesh and j^{th} mesh

$$d_{ij} = \begin{cases} 0 & \text{if } j^{th} \text{ voltage source is not present in the } i^{th} \text{ mesh} \\ 1 & \text{if } j^{th} \text{ voltage source provides a voltage rise in the } i^{th} \text{ mesh} \\ -1 & \text{if } j^{th} \text{ voltage source provides a voltage drop in the } i^{th} \text{ mesh} \end{cases}$$

Equivalently, the matrix product $\mathbf{DU}$ may be replaced by a column vector which contains the *net voltage rise contributed to a mesh by all voltage sources participating in that mesh*. The Mesh Resistance Matrix will be symmetric for this kind of circuit. The mesh current vector is obtained by Cramer's rule or by matrix inversion as $\mathbf{I} = \mathbf{Z}^{-1} \mathbf{DU}$.

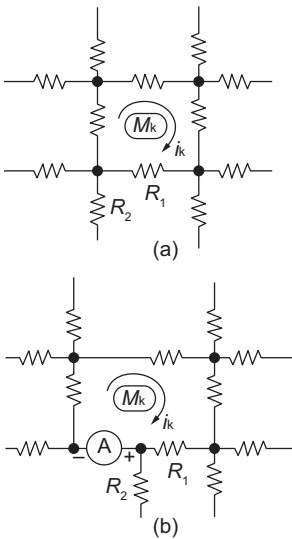


Fig. 4.7-6 Circuit for Illustrating Measurability of Mesh Currents

4.7.2 Is Mesh Current Measurable?

Mesh current of a mesh in a planar circuit is related to the current that flows in the series combination of all those elements that participate only in that mesh *if such elements are present in that mesh*. In such cases a mesh current is indeed a physical quantity and it can be measured. One can always introduce an ammeter in series with an element that appears only in the concerned mesh and measure the mesh current flowing in that mesh.

But what if there is no wholly owned element in a particular mesh? For instance consider the mesh marked as M_k in part of a large circuit, shown in Fig. 4.7-6.

This mesh in circuit Fig. 4.7-6(a) has no element wholly owned by it. The mesh current i_k assigned to this mesh cannot be identified as the current flowing through any of the circuit element appearing in the mesh. But let us try to create a *wholly owned* element in this mesh without affecting the circuit solution in any manner. Assume that R_2 is a member of only one mesh. Then nothing prevents us from changing our viewpoint to that expressed by the circuit in Fig. 4.7-6(b). Here we have introduced an additional node at the junction between R_2 and R_1 and introduced a short-circuit element in between the

new node and the old one. Introduction of a shorting link causes no change in the circuit variables anywhere in the circuit. However, now we identify this newly introduced short-circuit element as the element *exclusively owned* by mesh M_k and identify the mesh current variable i_k as the current that flows in this element. We can introduce an ammeter there as shown in Fig. 4.7-6(b) and measure i_k .

However, even this technique will fail to make the mesh current variable i_k measurable if R_2 is a member of yet another mesh. If the entire periphery of the mesh M_k is shared by some mesh or other, then, a short-circuit element introduced anywhere in the periphery will be shared by some other mesh.

Thus, we conclude that there can be meshes in which mesh current can not be identified as the current flowing in any element in that mesh. Therefore, in general, mesh current is a ‘fictitious current’ that is not measurable directly. It is a ‘fictitious current’ that can be thought of as ‘flowing around the periphery of the mesh’. Element currents are measurable. Each current is combination of two ‘peripheral currents’ or mesh currents. But these peripheral currents are not always measurable.

However, the KVL equations written for meshes in a planar circuit will form a set of $(b - n + 1)$ independent equations quite independent of whether the mesh currents are measurable or not. We show this in the last chapter of this book.

4.8 MESH ANALYSIS OF CIRCUITS WITH INDEPENDENT CURRENT SOURCES

An independent current source that appears in parallel with a resistor can be converted into an independent voltage source in series with a resistor by applying source transformation theorem. This measure will reduce the number of meshes in the circuit by one and increase the number of nodes in the circuit by one. Such current sources in a circuit do not pose any special problem for the mesh analysis procedure.

An independent current source may also appear in series with another element. Such a current source cannot be converted into an equivalent voltage source. This kind of current source will impose constraints on the mesh current variables in the circuit.

If an independent current source that cannot be equivalenced to a voltage source participates in only one mesh, then, that mesh current is *fixed* by that current source function. That is, that mesh current is no more a variable. Thus, the degree of freedom of the circuit decreases by one and the number of mesh current variables to be determined comes down by one. Moreover, one does not need the mesh equation for that mesh to solve for the remaining mesh current variables. However, that mesh equation will have to be used for finding the voltage across the current source after all the other mesh current variables have been obtained.

If two meshes in a circuit share a current source that cannot be equivalenced to a voltage source, it constrains the mesh currents in those two meshes to have a difference decided by the current source function. This constraint equation of the form, $i_i - i_j = i_s(t)$, where $i_s(t)$ is the value of current source, supplies one equation for finding the two mesh currents. Hence, we will not need both the two-mesh equations obtained by applying KVL in the two meshes. In fact, both of them will contain the voltage across the current source as a term and we will have to eliminate that voltage term by adding the two mesh equations to obtain a combined mesh equation. Thus, for this kind of a current source connection, we get one equation from the constraint imposed by the current source and a second one by adding the two mesh equations for the meshes. Finally, one of those two mesh equations will have to be used to determine the voltage across the current source, after all mesh current variables have been solved for.

These aspects of mesh analysis are illustrated through a series of examples that follow.

An irreducible independent current source completely owned by a mesh in a circuit makes the mesh current for that mesh equal to the source function, and thereby, reduces the number of independent mesh currents by one.

An irreducible independent current source shared by two meshes in a circuit imposes a constraint involving the two mesh currents, and thereby, reduces the number of independent mesh currents by one.

EXAMPLE: 4.8-1

Apply mesh analysis procedure on the circuit in Fig. 4.8-1 (a).

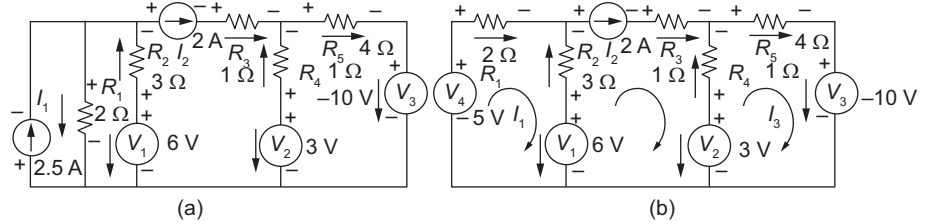


Fig. 4.8-1 (a) Circuit for Example 4.8-1 (b) Circuit after Mesh Reduction by Source Transformation

SOLUTION

All the circuit variables are identified by following the passive sign convention. The labelling of variables will be v or i , with the name of the element as a subscript.

Step-1: If there are current sources that appear directly across resistors, convert them into equivalent voltage sources in series with resistors. This is called 'mesh reduction'.

There is one such combination in the circuit of Fig. 4.8-1 (a). It is I_1 (2.5 A) in parallel with R_1 (2 Ω). We replace these with a voltage source of value $R_1 I_1$ (5 V) in series with R_1 (2 Ω) with a positive polarity at the top as shown in the circuit of Fig. 4.8-1 (b).

The second current source I_2 in the circuit of 4.8-1 (a) cannot be reduced this way.

Step-2: Assign mesh current variables in the reduced circuit, starting with the leftmost mesh. Assign a mesh current variable to a mesh only if its mesh current is not constrained directly by a current source and its mesh current is not decided by other mesh current variables already assigned along with current source functions.

There are three meshes in the reduced circuit. The first mesh has no current source directly constraining its mesh current. No other mesh was assigned a mesh current variable, and hence, this mesh current cannot get decided by other already assigned mesh current variables. Therefore, we assign a mesh current variable to this mesh and call it i_1 .

Moving on to the second mesh, we observe that the independent current source I_2 directly constrains its mesh current to be I_2 A (2 A). Therefore, we do not assign any mesh current variable to this mesh.

There is no current source in the third mesh. Its mesh current cannot be obtained from the already assigned mesh current variables. Therefore, we assign a mesh current variable to this mesh and call it i_3 .

Thus, this circuit has two mesh current variables $-i_1$ and i_3 – to be solved for.

Step-3: Prepare the mesh equations by applying KVL in meshes, starting at the left bottom corner and traversing the mesh in a clockwise direction. Ignore the meshes that are directly constrained by the current sources. If two meshes share a current source, then, add the mesh equations for those two meshes to generate a new equation that will be used in the solution process. The number of equations at the end of this step will be equal to the number of meshes – number of irreducible independent current sources.

The two mesh equations are:

$$\begin{aligned} -V_4 + R_1 i_1 - R_2 (i_2 - i_1) + V_1 &= 0 \text{ for mesh-1} \\ \text{i.e., } (R_1 + R_2) i_1 - R_2 i_2 + V_4 - V_1 &= R_2 I_2 + R_1 I_1 - V_1 \\ -V_2 + R_4 (i_3 - I_2) + R_5 i_3 + V_3 &= 0 \text{ for mesh-3} \\ \text{i.e., } (R_4 + R_5) i_3 &= R_4 I_2 + V_2 - V_3 \end{aligned}$$

These mesh equations are expressed in matrix form below:

$$\begin{bmatrix} R_1 + R_2 & 0 \\ 0 & R_4 + R_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_3 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & -1 & 0 & 0 \\ 0 & R_4 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

The third mesh current gets constrained to be $I_1 + I_2 = 3$ A. Therefore, there is no need to assign mesh current variables in the second and third meshes. The mesh current variable i_1 is assigned to the first mesh as shown in Fig. 4.8-3.

The mesh equation for the first mesh is $5i_1 = 4\text{ V} + 3\Omega \times 2\text{ A} - 5\text{ V} = 5\text{ V} \Rightarrow i_1 = 1\text{ A}$. The complete solution is shown in Fig. 4.8-4.

We have followed the passive sign convention throughout. Hence, the power dissipated in each resistor is the product of the voltage across it and the current through it. The total power dissipated in the circuit is found by adding up this product for all the resistors.

$$\begin{aligned} \therefore \text{Total power dissipated in the circuit} &= 2\text{ V} \times 1\text{ A} + 3\text{ V} \times 1\text{ A} + 2\text{ V} \times 2\text{ A} \\ &\quad + 1\text{ V} \times 1\text{ A} + 12\text{ V} \times 3\text{ A} \\ &= 46\text{ W} \end{aligned}$$

Power delivered by an element is equal to the negative of power dissipated in it. Power dissipated in an element is the product of voltage and current as per the passive sign convention. Hence, the power delivered by a source is negative of vi product, with v and i marked according to the passive sign convention.

$$\begin{aligned} \therefore \text{Power delivered by } V_1 &= -(4\text{ V} \times -1\text{ A}) = 4\text{ W} \\ \text{Power delivered by } V_2 &= -(5\text{ V} \times -1\text{ A}) = 5\text{ W} \\ \text{Power delivered by } V_3 &= -(-11\text{ V} \times 3\text{ A}) = 33\text{ W} \\ \text{Power delivered by } I_1 &= -(-1\text{ V} \times 2\text{ A}) = 2\text{ W} \\ \text{Power delivered by } I_2 &= -(-2\text{ V} \times 1\text{ A}) = 2\text{ W} \end{aligned}$$

The total delivered power is equal to the total dissipated power.

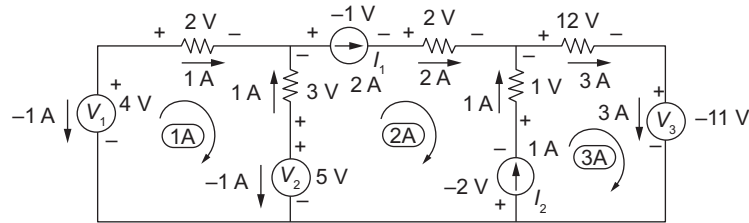


Fig. 4.8-4 Circuit Solution for Example 4.8-2

The two previous examples have demonstrated that:

- (i) An independent current source imposes a constraint on the mesh current variables and reduces their number by one.
- (ii) A mesh current variable *need not be* assigned for a mesh if an independent current source determines that mesh current directly or indirectly through another mesh current variable assigned to another mesh.
- (iii) Mesh equation for a mesh, in which the mesh current is fixed directly by an independent current source appearing only in that mesh, *is not required for solving other mesh current variables*.
- (iv) Mesh equations for two meshes that share an independent current source among them have to be *added* to get a combined mesh equation that will be useful in solving the circuit analysis problem.
- (v) The mesh analysis formulation results in an equation $\mathbf{ZI} = \mathbf{DU}$, where $\mathbf{Z}$ is the Mesh Resistance Matrix of a reduced order circuit resulting from deactivating all independent sources in it. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the mesh current vector $\mathbf{I}$ can be written as $\mathbf{Z}^{-1}\mathbf{DU}$. This indicates that each mesh current (and hence all element voltages and currents) can be expressed as a linear combination of input source functions, i.e., $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$, where x is some mesh current variable or

element current/voltage variable and a 's and b 's are coefficients decided by circuit resistances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .

- (vi) The voltage across independent current sources can be found only after the mesh current variables are solved. Finding the voltage across an independent current source requires the application of KVL in a mesh containing that current source.

We had observed that an n -node, b -element circuit has only $(b - n - 1)$ mesh current variables and that each independent current source reduces the number of mesh current variables to be solved for by one. Then, what happens if there are $(b - n - 1)$ independent current sources in the circuit? And, what if there are more than that many independent current sources? This leads us to our next example. Consider the following example circuit that has all the three mesh current variables constrained by three independent current sources.

EXAMPLE: 4.8-3

Solve the circuit in Fig. 4.8-5 by mesh analysis.

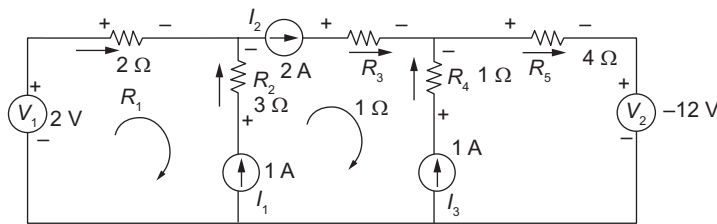


Fig. 4.8-5 Circuit for Example 4.8-3

SOLUTION

The independent current source I_2 constrains the second mesh current to be equal to 2 A. This results in the first mesh current getting constrained by the equation $i_1 - i_2 = -1$ A, resulting in $i_1 = 1$ A. The independent current source I_3 along with the current source I_2 imposes a constraint on i_3 , resulting in $i_3 = 3$ A. Thus, all the three mesh current variables are constrained by the independent current sources. There is no mesh current variable to be determined.

If the sources in this circuit are deactivated, the resulting network will have *no closed loops*. This is yet another feature of a fully constrained circuit.

Elements other than the current sources have nothing to do with mesh currents in this circuit. They affect only the voltages that appear across the current sources. These voltages can be found by applying KVL in the meshes.

Applying KVL in the first mesh, we get,

$$-2\text{ V} + 2\Omega \times 1\text{ A} - 3\Omega \times (2\text{ A} - 1\text{ A}) - v_{I_1} = 0 \Rightarrow v_{I_1} = -3\text{ V}$$

Applying KVL in the third mesh, we get,

$$v_{I_3} + 1\Omega \times (3\text{ A} - 2\text{ A}) + 4\Omega \times 3\text{ A} - 12\text{ V} = 0 \Rightarrow v_{I_3} = -1\text{ V}$$

Applying KVL in the second mesh, we get,

$$v_{I_1} + 3\Omega \times (2\text{ A} - 1\text{ A}) + v_{I_2} + 1\Omega \times 2\text{ A} - 1\Omega \times (3\text{ A} - 2\text{ A}) - v_{I_3} = 0; v_{I_2} = -2\text{ V}$$

The currents through resistors, the voltages across them and the currents through voltage sources can be found by inspection. The complete solution is marked in Fig. 4.8-6.

Three independent current sources in a three-mesh circuit need not necessarily result in a fully constrained circuit. For instance, assume that the source I_3 is shifted and connected in series with R_1 in Fig. 4.8-5. We note that the three independent current sources will have to satisfy the condition that $I_3 + I_1 - I_2 = 0$ due to the KCL constraint at the node between R_1 and R_2 . With the values used in the present example, this

An n -node, b -element circuit with $(b - n + 1)$ independent current sources in it can be a fully constrained circuit.

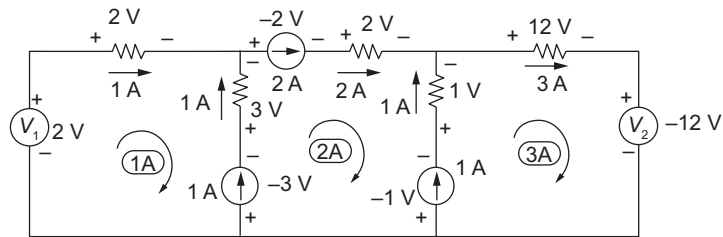


Fig. 4.8-6 Complete Solution for Circuit in Example 4.8-3

The maximum number of independent current sources that can be present in an n -node, b -element circuit with a unique solution is $(b - n - 1)$.

constraint is satisfied. However, the reader may verify that it will be impossible to determine the voltages that appear across the current sources in this case, though the mesh currents can be determined. The circuit will have many solutions.

The constraint above need not necessarily be satisfied by any three arbitrary current sources. If they do not, then they are trying to violate KCL. That will imply that the circuit has no solution. In practice, there will be a solution since each practical current source will have some finite resistance across it. But then, the circuit becomes a different one.

Similar conclusions will follow for a case with more than $(b - n + 1)$ independent current sources in an n -node, b -element circuit – either the circuit cannot be solved due to inconsistencies in KCL equations or the circuit cannot be solved uniquely.

4.9 MESH ANALYSIS OF CIRCUITS CONTAINING DEPENDENT SOURCES

Dependent voltage sources do not pose any problems for mesh analysis. Independent voltage sources appear on the right-hand side of mesh equations. However, dependent voltage sources affect the coefficients of mesh current variables on the left-hand side of mesh equations.

The controlling variable of a linear dependent voltage source will be a voltage or a current existing elsewhere in the circuit. However, any voltage or current variable in the circuit can be expressed in terms of mesh current variables. Hence, the dependent voltage source function can be expressed in terms of mesh current variables. Therefore, dependent voltage sources will affect the coefficients of mesh equation, i.e., they will change the mesh resistance matrix. We will see that they destroy the symmetry of the mesh resistance matrix.

Dependent current sources place constraints on mesh current variables; the same way independent current sources do. They cause a reduction in the number of mesh currents to be determined in the circuit. They too affect the coefficients of mesh equations and make mesh resistance matrix asymmetric.

Dependent sources can make the matrix $\mathbf{Z}$ asymmetric.

EXAMPLE: 4.9-1

Apply mesh analysis to the circuit in Fig. 4.9-1.

SOLUTION

Step-1: Carry out mesh reduction by employing source transformation, if relevant.

There are no current sources appearing in parallel with any resistor. Hence, no mesh reduction is possible.

Step-2: Identify the meshes and assign mesh current variables.

There are three meshes and there is no current source to impose any constraints on them. Hence, all the three meshes are assigned mesh current variables.

Step-3: Identify the controlling variable of dependent sources in terms of mesh current variables and express dependent source functions in terms of mesh current variables.

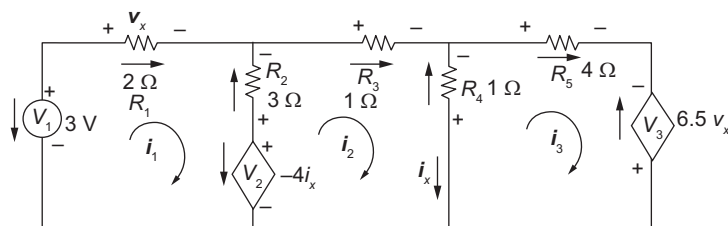


Fig. 4.9-1 Circuit for Example 4.9-1

i_x is the controlling variable of V_2 . i_x is the current flowing in R_4 from top to bottom, and hence, it is equal to $(i_2 - i_3)$. Therefore, the source function of V_2 is $k_1(i_2 - i_3)$, where $k_1 = -4$.

v_x is the controlling variable for V_3 . v_x is the voltage across R_1 . Therefore, $v_x = R_1 i_1$ and the source function of V_3 is $k_2 R_1 i_1$, where $k_2 = 6.5$.

Step-4: Prepare the mesh equations and solve them.

The mesh equations are written with the dependent source functions expressed in terms of mesh current variables.

$$\text{Mesh-1: } -V_1 + R_1 i_1 - R_2(i_2 - i_1) + k_1(i_2 - i_3) = 0$$

$$\text{i.e., } (R_1 + R_2)i_1 - (R_2 - k_1)i_2 - k_1 i_3 = V_1$$

$$\text{Mesh-2: } -k_1(i_2 - i_3) + R_3(i_2 - i_1) + R_4(i_2 - i_3) = 0$$

$$\text{i.e., } -R_2 i_1 + (R_2 + R_3 + R_4 - k_1)i_2 - (R_4 - k_1)i_3 = 0$$

$$\text{Mesh-3: } R_4(i_3 - i_2) + R_5 i_3 - k_2 R_1 i_1 = 0$$

$$\text{i.e., } -k_2 R_1 i_1 - R_4 i_2 + (R_4 + R_5)i_3 = 0$$

These equations are expressed in matrix form below:

$$\begin{bmatrix} (R_1 + R_2) & -(R_2 - k_1) & -k_1 \\ -R_2 & (R_2 + R_3 + R_4 - k_1) & -(R_4 - k_1) \\ -k_2 R_1 & -R_4 & (R_4 + R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [V_1]$$

Note the asymmetry in mesh resistance matrix.

Substituting the numerical values and solving for mesh currents by Cramer's rule, we get,

$$\begin{bmatrix} 5 & -7 & 4 \\ -3 & 9 & -5 \\ -13 & -1 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow i_1 = 1 \text{ A, } i_2 = 2 \text{ A and } i_3 = 3 \text{ A}$$

Step-5: Apply KCL at various nodes of the circuit to find all the element currents and resistor voltages.

Consider R_4 . Applying KCL at the node formed by R_3 , R_4 and R_5 , we get the current flowing from top to bottom in R_4 as $i_2 - i_3$. However, the reference direction that was chosen for current in R_4 is from bottom to top. Hence, current in $R_4 = -(i_2 - i_3)$, in the direction marked in Fig. 4.9-1. The value of the current is 1 A.

This makes the value of i_x equal to -1 A, and hence the dependence voltage source V_2 source function becomes $-4 \times -1 = 4$ V.

Consider R_1 . The reference direction for its current, as marked in Fig. 4.9-1, is from left to right and is in the same direction as the first mesh current. R_1 is exclusively owned by the first mesh, and hence, the current through it is i_1 itself. The value is 1 A. That makes the voltage across R_1 equal to 2 V. Moreover, the value of v_x is also 2 V. Therefore, the source function of the dependent voltage V_3 is $6.5 \times 2 = 13$ V.

The remaining voltage and current variables also can be evaluated similarly. The complete circuit solution is shown in Fig. 4.9-2.

Mesh Analysis Procedure

Step-1: Assign reference current directions and reference polarities for voltages of all elements as per the passive sign convention. If there are current sources that appear directly across resistors, convert them into equivalent voltage sources in series with resistors. This is called 'mesh reduction'. The resulting circuit is referred to as the reduced circuit.

Step-2: Assign mesh current variables in the reduced circuit, starting with the leftmost mesh. Assign a mesh current variable to a mesh only if its mesh current is not constrained directly by a current source (independent or dependent) and its mesh current is not decided by other mesh current variables already assigned along with current source functions.

Step-3: Prepare the mesh equations by applying KVL in meshes, starting at the left bottom corner and traversing the mesh in a clockwise direction. Ignore the meshes that are directly constrained by current sources (independent or dependent). If two meshes share a current source, then, add the mesh equations for those two meshes to generate a new equation that will be used in the solution process. The number of equations at the end of this step will be equal to the number of meshes minus the number of irreducible current sources.

Step-4: Identify the controlling variable of dependent sources in terms of mesh current variables and express the dependent source functions in terms of mesh current variables.

continued

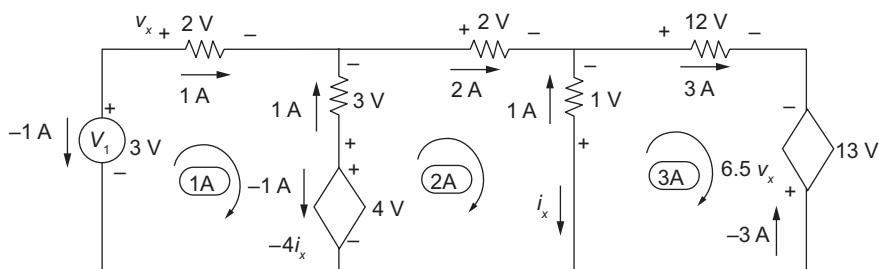


Fig. 4.9-2 Complete Mesh Analysis Solution for Example 4.9-1

EXAMPLE: 4.9-2

Apply mesh analysis to the circuit in Fig. 4.9-3.

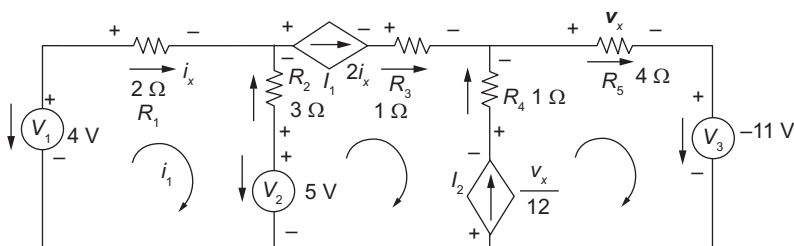


Fig. 4.9-3 Circuit for Example 4.9-2

SOLUTION

Step-1: Carry out mesh reduction by employing source transformation, if relevant.

There are no current sources appearing in parallel with any resistor. Hence, no mesh reduction is possible.

Step-2: Identify meshes and assign mesh current variables.

The second mesh current is directly constrained by the dependent current source I_1 and the third mesh current is indirectly constrained by $i_3 = I_2 + I_1$. Hence, there is only one mesh current variable and that is i_1 in the first mesh.

Step-3: Identify the controlling variable of dependent sources in terms of mesh current variables and express dependent source functions in terms of mesh current variables.

i_x is the controlling variable of I_1 . i_x is the current flowing in R_1 from left to right, and hence, it is equal to i_1 . Therefore, the source function of I_1 is $2i_1$ A.

v_x is the controlling variable for I_2 . v_x is the voltage across R_5 . Therefore, $v_x = R_5(I_2 + I_1) = R_5(I_2 + 2i_1) = R_5(2i_1 + v_x/12) = (8i_1 + v_x/3)$
 $\therefore v_x = 12i_1$ and the source function of $I_2 = i_1$ A.

Step-4: Prepare the mesh equations and solve them.

The mesh equation for the first mesh is written with the dependent source functions expressed in terms of mesh current variables.

$$-4 + 2i_1 - 3(2i_1 - i_1) + 5 = 0; \text{ i.e., } i_1 = 1$$

Therefore, the mesh currents are $i_1 = 1$ A, $i_2 = 2$ A and $i_3 = 3$ A.

Step-5: Apply KCL at various nodes of the circuit to find all the element currents and resistor voltages.

Consider R_4 . Applying KCL at the node formed by R_3 , R_4 and R_5 , we get the current flowing from top to bottom in R_4 as $i_2 - i_3$, but the reference direction that was chosen for current in R_4 is from bottom to top. Hence, current in $R_4 = -(i_2 - i_3)$ in the direction marked in Fig. 4.9-3. The value of the current is 1 A.

Currents through other resistors and voltage sources may be obtained in a similar manner.

Step-6: Apply KVL in various meshes to obtain the voltage across the current sources. Apply KVL in the third mesh first.

$$v_{I_2} + 1 \times (3 - 2) + 4 \times 3 + (-11) = 0 \Rightarrow v_{I_2} = -2 \text{ V}$$

Now, apply KVL in the second mesh.

$$-5 + 3 \times (2 - 1) + v_{I_1} + 1 \times 2 - 1(3 - 2) - v_{I_2} = 0 \Rightarrow v_{I_1} = -1 \text{ V}$$

The complete circuit solution is shown in Fig. 4.9-4.

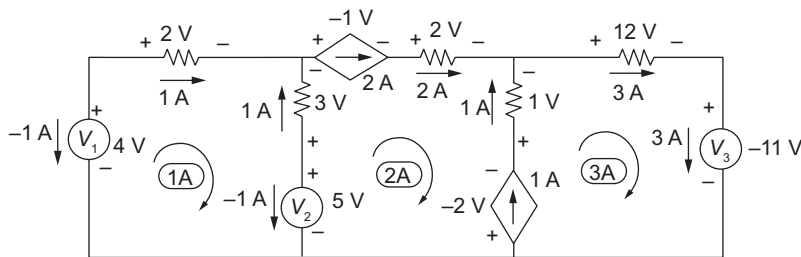


Fig. 4.9-4 Complete Mesh Analysis Solution for the Circuit in Example 4.9-2

Step-5: Solve the mesh equations by elimination technique. The equations may also be expressed as a matrix equation and solved using Cramer's rule or matrix inversion.

Step-6: Use the mesh current values and apply KCL at various nodes in the original circuit to obtain element currents and voltages for all resistive elements (use Ohm's law) and current through voltage sources.

Step-7: Use the element voltages calculated in the above step and apply KVL in various meshes in the original circuit to obtain element voltages for all current sources.

We have completed the development of mesh analysis technique for *planar* memoryless circuits containing linear resistors, four kinds of linear dependent sources, independent voltage sources and independent current sources.

The *general mesh analysis procedure* that has emerged is summarised in the side-box.

4.10 SUMMARY

- This chapter dealt with two systematic procedures of solving the circuit analysis problem in the case of memoryless circuits. Memoryless circuits contain linear resistors, linear dependent sources and independent sources.
- Circuit analysis problem for an n -node, b -element circuit involves the determination of b element voltage variables and b element current variables, given the source functions of all independent sources that are present in the circuit.
- Node voltage* is the voltage of a node in a circuit with respect to a chosen reference node in the circuit. In Nodal Analysis, KVL equations are used to show that all element voltages can be expressed in terms of $(n - 1)$ node voltages and KCL equations along with element relations are used subsequently to set up the $(n - 1)$ node equations needed for determining the node voltages. Nodal Analysis is applicable to any circuit that has a unique solution.
- Mesh current* is a current that flows in a clockwise direction in an element or in a set of series connected elements (including suitably located short-circuit element) that participate only in the concerned mesh. In Mesh Analysis, the KCL equations are used to show that all element currents can be expressed in terms of a reduced set of $(b - n + 1)$ specially defined currents called *mesh currents*. Subsequently, $(b - n + 1)$ KVL equations involving these currents are set up to determine them. Mesh Analysis is applicable only to circuits that are planar.
- Source Transformation Theorem is a valuable aid in both analysis procedures. It states that a voltage source $v_s(t)$ in series with a resistance R_s can be replaced with a current source $i_s(t) = v_s(t)/R_s$ in parallel with R_s without affecting any voltage/current/power variable external to the source. The direction of the current source is such that current

flows out of the terminal at which the positive of the voltage source is presently connected.

- An n -node circuit containing only linear resistors and independent current sources will have a nodal representation given by $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix of order $(n - 1) \times (n - 1)$, $\mathbf{V}$ is the node voltage column vector of order $(n - 1) \times 1$, $\mathbf{U}$ is the source current column vector of order $n_{cs} \times 1$ and $\mathbf{C}$ is the input matrix of order $(n - 1) \times n_{cs}$. n_{cs} is the number of independent current sources in the circuit. $\mathbf{CU}$ may be replaced by a column vector which contains the *net current delivered to a node by all independent current sources connected at that node*. The *Nodal Conductance Matrix* will be *symmetric* for this kind of circuit.
- The nodal analysis formulation, in general, results in an equation $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix of the reduced order circuit (if node reduction is possible) resulting from deactivating all independent sources in it. $\mathbf{Y}$ will be *symmetric* if there are no dependent sources in the circuit. Dependent sources can make $\mathbf{Y}$ *asymmetric*. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the node voltage vector $\mathbf{V}$ can be written as $\mathbf{Y}^{-1}\mathbf{CU}$.
- Each node voltage (and hence all element voltages and currents) can be expressed as a linear combination of input source functions, i.e., $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$, where x is some node voltage variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit conductances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .

- An n -node, b -element circuit containing only linear resistors and independent voltage sources will have a mesh representation given by $\mathbf{ZI} = \mathbf{DU}$, where $\mathbf{Z}$ is the mesh resistance matrix of order $(b - n + 1) \times (b - n + 1)$, $\mathbf{I}$ is the mesh current column vector of order $(b - n + 1) \times 1$, $\mathbf{U}$ is the source voltage column vector of order $n_{vs} \times 1$ and $\mathbf{D}$ is the input matrix of order $(b - n + 1) \times n_{vs}$. n_{vs} is the number of independent voltage sources in the circuit. $\mathbf{DU}$ may be replaced by a column vector which contains the *net voltage rise contributed to a mesh by all voltage sources participating in that mesh*. The *mesh resistance matrix* will be *symmetric* for this kind of circuit.
- The mesh analysis formulation, in general, results in an equation $\mathbf{ZI} = \mathbf{DU}$, where $\mathbf{Z}$ is the mesh resistance matrix of a reduced order circuit (if mesh reduction is possible) resulting from deactivating all independent sources in it. $\mathbf{Z}$ will be *symmetric* if there are no dependent sources in the circuit. Dependent sources can make $\mathbf{Z}$ *asymmetric*. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the mesh current vector $\mathbf{I}$ can be written as $\mathbf{Z}^{-1}\mathbf{DU}$.
- Each mesh current (and hence all element voltages and currents) can be expressed as a linear combination of input source functions, i.e., $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$, where x is some mesh current variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit resistances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .

4.11 PROBLEMS

- Find the power delivered by the -7 A current source in the circuit in Fig. 4.11-1 by nodal analysis.

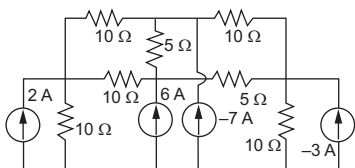


Fig. 4.11-1

- The node voltages in the circuit in Fig. 4.11-2 are marked in the figure. Find the values for R_1 , R_2 , R_3 and I .

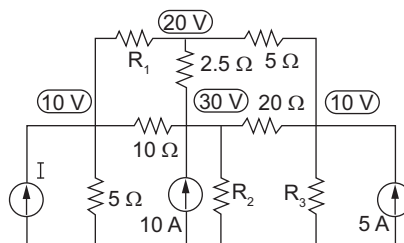


Fig. 4.11-2

- The $\mathbf{Y}$ matrix of the circuit in Fig. 4.11-3 is given below. Find the values of all resistances in the circuit.

$$Y = \begin{bmatrix} 0.10 & -0.02 & -0.04 \\ -0.02 & 0.06 & -0.02 \\ -0.04 & -0.02 & 0.12 \end{bmatrix} S$$

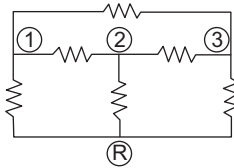


Fig. 4.11-3

4. (i) Express all the currents marked with arrows in the circuit in Fig. 4.11-4 as linear combinations of I_1 , I_2 and I_3 by nodal analysis. (ii) Solve for I_1 , I_2 and I_3 such that currents through R_2 , R_3 and R_6 are zero. With these values of current sources find the currents through remaining resistors.

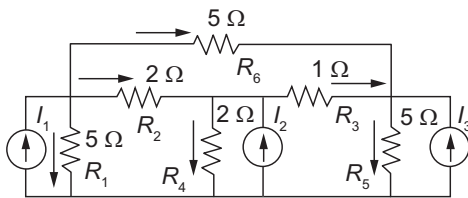


Fig. 4.11-4

5. (i) Express all the currents marked with arrows in the circuit in Fig. 4.11-5 as linear combinations of V_1 , V_2 and I by nodal analysis. (ii) Solve for V_1 , V_2 and I such that currents through R_1 , R_5 and R_6 are zero. With these values of current sources, find the currents through the remaining resistors and the node voltages with respect to the bottommost node.

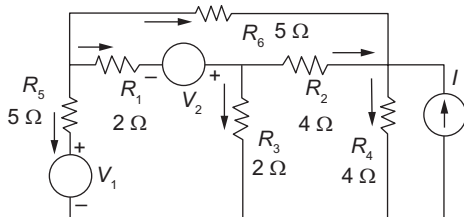


Fig. 4.11-5

6. Find V_1 , V_2 and V_3 such that $v_1 = 10$ V, $v_2 = 10$ V and $v_3 = 20$ V in the circuit in Fig. 4.11-6. With these values of V_1 , V_2 and V_3 , find the power delivered by all voltage sources and power dissipated by all resistors. Use nodal analysis.

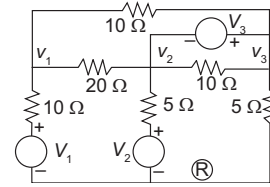


Fig. 4.11-6

7. (i) The nodal conductance matrix of the circuit in Fig. 4.11-7 is given as:

$$Y = \begin{bmatrix} 0.7 & -0.3 & 0.13 \\ -0.2 & 0.8 & -0.53 \\ 0.1 & -0.6 & 0.7 \end{bmatrix} S$$

Find k_1 , k_2 and k_3 . (ii) Solve the circuit completely by nodal analysis, if $I = 1$ A.

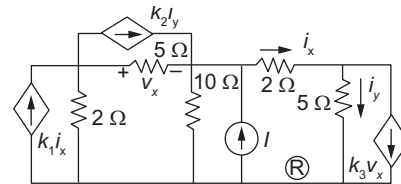


Fig. 4.11-7

8. Find k_1 , k_2 and k_3 such that the Y for the circuit in Fig. 4.11-8 is lower triangular. Find the power delivered by independent sources and dependent sources. Use nodal analysis.

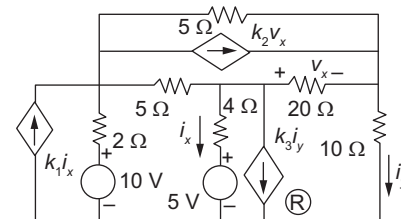


Fig. 4.11-8

9. Find all dependent source coefficients such that the Y matrix of the circuit in Fig. 4.11-9 is diagonal.

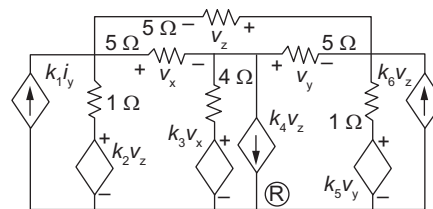


Fig. 4.11-9

10. Find k such that v is zero in the circuit in Fig. 4.11-10. Solve the circuit completely with this value of k . Use nodal analysis.

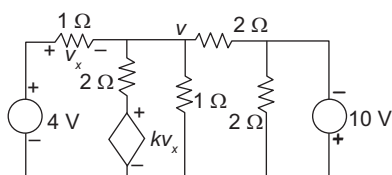


Fig. 4.11-10

11. Find k such that v is zero in the circuit in Fig. 4.11-11. Solve the circuit completely for this value of k . Use nodal analysis.

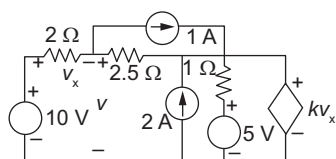


Fig. 4.11-11

12. Find the node voltages and resistor currents in the circuit in Fig. 4.11-12 by nodal analysis.

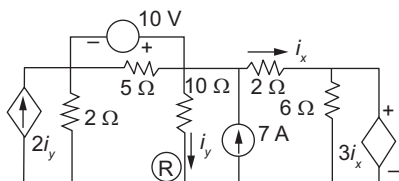


Fig. 4.11-12

13. The mesh resistance matrix of the circuit in Fig. 4.11-13 is given below. Find the values of all resistances in the circuit.

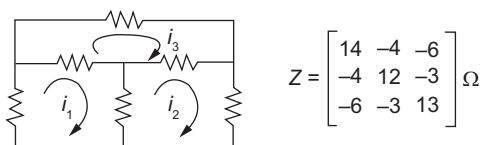


Fig. 4.11-13

14. Express all the resistor currents in the directions as marked in the form linear combinations of V_1 , V_2 and V_3 for the circuit shown in Fig. 4.11-14. Use mesh analysis.

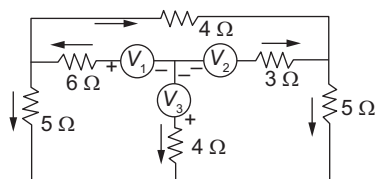


Fig. 4.11-14

15. (i) Express v in the circuit in Fig. 4.11-15 as a linear combination of V_1 , V_2 and I . (ii) Find I such that $v = 0$, if $V_1 = V_2 = 5$ V. (iii) Solve the circuit completely with these source values. Use mesh analysis.

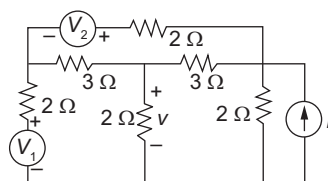


Fig. 4.11-15

16. All resistors in the circuit in Fig. 4.11-16 are of 2Ω . Find currents in all the resistors and voltages across current sources by mesh analysis.

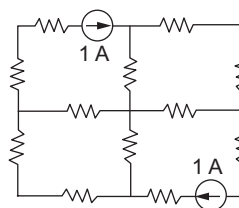


Fig. 4.11-16

17. Find the current delivered to the 12.5 V source, power delivered by all sources and power dissipated in all the resistors by mesh analysis for the circuit in Fig. 4.11-17.

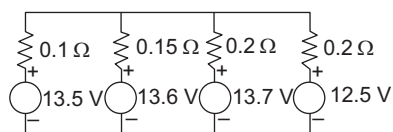


Fig. 4.11-17

18. Solve the above problem (Problem 17) using nodal analysis.

19. Solve Problem 5 using mesh analysis.

20. Solve the circuit in Fig. 4.11-18 completely.

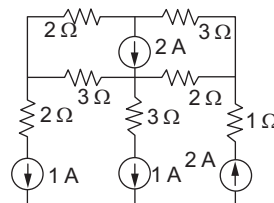


Fig. 4.11-18

21. Can the circuit in Fig. 4.11-19 be solved uniquely? If yes, find the solution. If no, find at least two solutions.

22. Solve Problem 11 using mesh analysis.

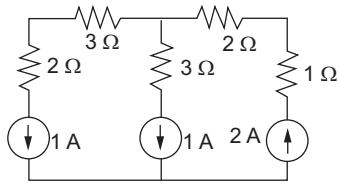


Fig. 4.11-19

23. Express the resistor currents as linear combinations of V_1 , V_2 in the circuit in Fig. 4.11-20. Use mesh analysis.

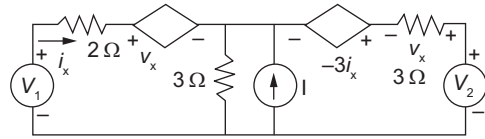


Fig. 4.11-20

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Circuit Theorems

CHAPTER OBJECTIVES

- *Derive Superposition Theorem from the property of linearity of elements.*
- *Explain the two key theorems – Superposition Theorem and Substitution Theorem in detail.*
- *Derive other theorems like Compensation Theorem, Thevenin's Theorem, Norton's Theorem, Reciprocity Theorem and Maximum Power Transfer Theorem from these two key principles.*
- *Provide illustrations for applications of circuit theorems in circuit analysis through solved examples.*
- *Emphasise the use of Compensation Theorem, Thevenin's Theorem and Norton's Theorem in circuits containing dependent sources as a pointer to their applications in the study of Electronic Circuits.*

This Chapter identifies the Substitution Theorem and Superposition Theorem as the two key theorems and shows how the other theorems may be extracted from them.

INTRODUCTION

The previous chapter showed that:

- (1) All the element voltages and element currents in a circuit can be obtained from its node voltages. The node voltages are governed by a matrix equation $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{V}$ is the node voltage column vector, $\mathbf{Y}$ is the nodal conductance matrix of the circuit, $\mathbf{U}$ is the input column vector containing source functions of all *independent* voltage sources and current sources in the circuit and $\mathbf{C}$ is the input matrix. The values of conductances in the circuit and values of coefficients of linear dependent sources in the circuit decide the elements of $\mathbf{Y}$ -matrix. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources *can* make $\mathbf{Y}$ -matrix asymmetric. The $\mathbf{C}$ -matrix, in general, contain 0, 1, -1 and conductance values as well as dependent source coefficients.
- (2) An alternative formulation is given by a matrix equation $\mathbf{ZI} = \mathbf{DU}$, where $\mathbf{I}$ is the mesh current column vector, $\mathbf{Z}$ is the mesh resistance matrix of the circuit,

Why Circuit Theorems?

Circuit Analysis involves determination of element voltages and currents in all elements of the circuit using element equations and interconnection equations. Kirchhoff's Current Law equations at all nodes and Kirchhoff's Voltage Law equations in all loops along with element $v-i$ relationship equations will yield the necessary set of equations.

However, we need systematic procedures for exploiting these equations. Node analysis and Mesh analysis were two such systematic procedures we have taken for detailed study in the last chapter. In this chapter, we discuss some circuit theorems and circuit transformations that increase our efficiency in solving circuits. Moreover, they render further insight into certain features of a linear circuit. These theorems constitute a basic set of tools that enhance the analyst's efficiency in solving circuits.

$\mathbf{U}$ is the input column vector containing source functions of all *independent* voltage sources and current sources in the circuit and $\mathbf{D}$ is the input matrix. All the element voltages and element currents in a circuit can be obtained from its mesh currents. The values of resistances in the circuit and values of coefficients of linear dependent sources in the circuit decide the elements of $\mathbf{Z}$ -matrix. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources *can* make $\mathbf{Z}$ -matrix asymmetric. The $\mathbf{D}$ -matrix, in general, contain 0, 1, -1 and resistance values as well as dependent source coefficients.

- (3) Any response variable in a circuit (i.e., any element voltage or current or combination thereof) can be expressed as a *linear combination* of source functions of independent voltage sources and independent current sources present and active in the circuit. i.e., $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$, where x is some chosen response variable and a and b are coefficients decided by circuit conductance/resistance, dependent source coefficients and interconnection details. Some of the a and b may turn out to be zero for certain choices of x .

These three basic properties of a circuit comprising linear elements are recast into various useful theorems that simplify the circuit analysis procedure in practical contexts. That is, the so-called circuit theorems are more or less restatements of these basic facts. These three observations were arrived at by analysis of memoryless circuits. However, we will show in later chapters that they are true for dynamic circuits too. Therefore, all important circuit theorems that we arrive in this chapter will be valid for dynamic circuits.

The crucial fact the reader should keep in mind is that we are stating nothing more in almost all circuit theorems than what is already contained in the three properties described above.

5.1 LINEARITY OF A CIRCUIT AND SUPERPOSITION THEOREM

Consider a purely resistive circuit driven by two independent current sources and an independent voltage source as shown in Fig. 5.1-1.

The mesh equations for circuit as in Fig. 5.1-1(b) in matrix form can be derived as,

$$\begin{bmatrix} 5 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V_1 \end{bmatrix}.$$

Solving the matrix equation, we get,

$$\begin{aligned} i_1 &= \frac{3}{13} I_1 - \frac{1}{13} I_2 - \frac{2}{13} V_1 \\ i_2 &= \frac{1}{13} I_1 - \frac{5}{26} I_2 + \frac{3}{26} V_1. \end{aligned}$$

Now, any element voltage or current can be expressed in terms of these two mesh currents. For example, consider the current in resistor in the central limb in the direction as shown in circuit in Fig. 5.1-1(a). This current is obtained by applying KCL at the junction between the three resistors and is $i_1 - i_2 = \frac{2}{13} I_1 + \frac{3}{26} I_2 - \frac{7}{26} V_1$.

Consider the current in the 1Ω across the current source I_1 . This is obtained by applying KCL at the node, where I_1 , 2Ω and 1Ω are connected together. The current in 2Ω is the same as that of the first mesh current in circuit as in Fig. 5.1-1(b) and hence the current in $1 \Omega = I_1 - i_1 = \frac{10}{13} I_1 + \frac{1}{13} I_2 + \frac{2}{13} V_1$.

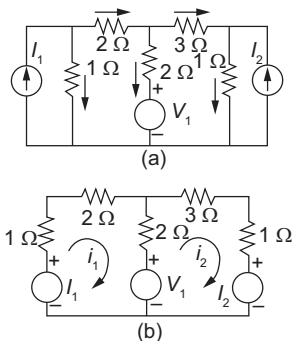


Fig. 5.1-1 (a) A Circuit with Three Independent Sources
(b) Circuit After Source Transformation

Currents in all the elements can be worked out in a similar manner. These currents are marked in Fig. 5.1-2 using a notation where the three numbers in parenthesis show the coefficients of I_1 , I_2 and V_1 , respectively. Or, they can be interpreted as the current components when all the three sources have *unit values*. Consider the numbers marked for the voltage source V_1 . It is $\left(\frac{2}{13}, \frac{3}{26}, -\frac{7}{26}\right)$. This means that I_1 contributes $\frac{2}{13}$ A per unit amp to this current, I_2 contributes $\frac{3}{26}$ A per unit amp to this current and V_1 contributes $-\frac{7}{26}$ A per unit volt to this current.

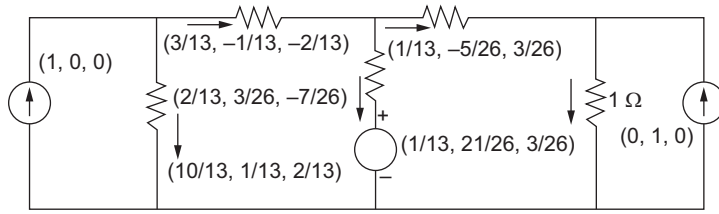


Fig. 5.1-2 Currents in Various Elements in the Circuit in Fig. 5.1-1 (a)

Thus, each element current (and also element voltage) is made up of three components. Each independent source contributes one component to each element current and voltage. The components contributed by all the independent sources add together to form the total response.

That the total currents in elements and total voltage across them will satisfy KCL and KVL respectively is expected. In fact, this is the basis for node analysis and mesh analysis. However, what is not so obvious is that *components contributed by a particular independent source to all element currents will satisfy KCL at all nodes without depending in any way on the components provided by other independent sources*. Similarly, *components contributed by a particular independent source to all element voltages will satisfy KVL in all loops without depending in any way on the components provided by other independent sources*. This may be verified easily in the example that was analysed in this section.

The implication from this observation is that the components contributed by a particular independent source to circuit variables are the same as the solution of the circuit when that source is acting alone without the other sources present. Or, in other words, the contribution of a particular source to a circuit variable does not change when some other source(s) is acting simultaneously.

This can also be understood in a different way. We had seen that all mesh current variables and node voltage variables (and hence all voltage variables and current variables in the circuit) for a circuit can be expressed as linear combinations of independent source functions. The solution for mesh currents in the circuit in Fig. 5.1-1 was,

$$\begin{aligned} i_1 &= \frac{3}{13} I_1 - \frac{1}{13} I_2 - \frac{2}{13} V_1 \\ i_2 &= \frac{1}{13} I_1 - \frac{5}{26} I_2 + \frac{3}{26} V_1. \end{aligned}$$

Thus, there are three contributions in each variable. The coefficient involved in each contribution is a constant. Its value does not depend on the particular values that the sources happen to assume. It depends only on the resistance values, structure of the circuit and the location where the particular independent source is connected. Therefore, we expect the coefficients of I_1 to remain at $\frac{3}{13}$ and $\frac{1}{13}$ in i_1 and i_2 respectively, whatever be the values

The contribution of a particular source to a circuit variable does not change when some other source(s) is acting simultaneously along with it.

I_2 and V_1 happen to have. And, we choose to think of the situation when I_2 and V_1 are zero-valued. Then, the circuit has only one source and it will produce $i_1 = \frac{3}{13}I_1$ and $i_2 = \frac{1}{13}I_1$ since the coefficients are independent of source values. Similarly, the circuit will have $i_1 = -\frac{1}{13}I_2$ and $i_2 = \frac{5}{26}I_2$ when $I_1 = V_1 = 0$ and $i_1 = -\frac{2}{13}V_1$ and $i_2 = \frac{3}{26}V_1$ when $I_1 = I_2 = 0$.

Thus, we can view the solution of the circuit when I_1 , I_2 and V_1 act simultaneously, as the sum or *superposition* of solution of three identical circuits with only one of the sources taking a non-zero value in each circuit. *A current source with zero-value is an open-circuit and a voltage source with zero-value is a short-circuit.* Hence, in the first circuit we replace I_2 by an open-circuit and V_1 by a short-circuit and solve it to get the first contribution due to I_1 . In the second circuit we replace I_1 by an open-circuit and V_1 by a short-circuit and solve it to get the second contribution due to I_2 . And in the third circuit we replace I_1 and I_2 by an open-circuits and solve it to get the third contribution due to V_1 . We add the contributions to get the solution for the original circuit in which all the three sources act simultaneously. *We can do this only because the contributions from various sources stand segregated in the form of a linear combination with no interaction among them.*

A multi-source circuit problem can be split up into many single-source circuit problems.

Thus, a multi-source circuit problem can be split up into many single-source circuit problems and the response for any circuit variable in the multi-source circuit can be found as the *superposition* (i.e., sum) of responses for same circuit variable in all those single-source circuits. In doing so, we will be constructing the final solution by piecing together the individual contributions from independent sources.

To systematise this further, we note that the contribution from any particular independent source to a particular response variable is proportional to the source function value. We term this proportionality constant as a *coefficient of contribution*. In $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$, where x is some circuit response variable and $I_1, I_2, \dots$ are the independent current source functions and $V_1, V_2, \dots$ are the independent voltage source functions, the a and b are the proportionality constants or the so-called coefficients of contribution. They are the ones that matter; and not the particular values of source functions. Each coefficient can be interpreted as the *contribution to the circuit variable due to the unit value of a particular input source* – i.e., contribution per unit input. If we know the *contribution per unit input* for each source, we can find the contribution due to that source by a simple *scaling* operation that involves multiplying *contribution per unit input* by the source function value.

Now, we are ready to state the different forms of *Superposition Theorem*.

Superposition Theorem Form-1

‘The response of any circuit variable in a multi-source linear memoryless circuit containing “n” independent sources can be obtained by adding the responses of the same circuit variable in n single-source circuits with i^{th} single-source circuit formed by keeping only i^{th} independent source active and all the remaining independent sources deactivated’.

Deactivation of an independent current source is achieved by replacing it with an open-circuit and deactivation of an independent voltage source is achieved by replacing it with a short-circuit. *Dependent sources are not to be treated as sources while applying Superposition Theorem. They will be present in all the single-source component circuits.*

The principle embodied in the above can also be stated in the following manner.

Superposition Theorem Form-2

‘The response of any circuit variable x in a multi-source linear memoryless circuit containing “n” independent sources can be expressed as $x(t) = \sum_{i=1}^{i=n} a_i U_i(t)$, where $U_i(t)$ is the source function of i^{th} independent source (can be a voltage source or current source)

Superposition Theorem – First form.

Superposition Theorem – Second form.

and a_i is its “coefficient of contribution”. The coefficient of contribution has the physical significance of contribution per unit input’.

The coefficient of contribution, a_p , which is a constant for a time-invariant circuit, can be obtained by solving for $x(t)$ in a single-source circuit in which all independent sources other than the i^{th} one are deactivated by replacing independent voltage sources with short-circuits and independent current sources with open-circuits.

But, why should a linear combination $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$ be found term by term always? Is it possible to get it in subsets that contain more than one term? The third form of *Superposition Theorem* states that it can be done.

Superposition Theorem Form-3

‘The response of any circuit variable in a multi-source linear memoryless circuit containing “ n ” independent sources can be obtained by adding responses of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between such active source subsets among them’.

5.1.1 Linearity of a Circuit

Why did the memoryless circuits we have been dealing with till now obey superposition principle? The elements of memoryless circuits were constrained to be linear time-invariant elements. We used only *linear* resistors and *linear* dependent sources. The v - i relations of all those elements obey superposition principle. As a result, all KCL and KVL equations in nodal analysis and mesh analysis had the form of linear combinations. Such KVL and KCL equations lead to nodal conductance matrix (and mesh resistance matrix) that contain only constants in the case of a time-invariant circuit (i.e., resistances are constants and coefficients of dependent sources are also constants). Similarly, the input matrix ($\mathbf{C}$ in nodal analysis and $\mathbf{D}$ in mesh analysis) will contain only constants in the case of circuits constructed using linear time-invariant elements. Thus, the solution for node voltage variables and mesh current variables will come out in the form of linear combination of independent source functions. And, after all Superposition Theorem is only a restatement of this fact. Therefore, Superposition Theorem holds in the circuit since we used only linear elements in constructing it except for independent sources which are non-linear. Hence, we conclude that *a memoryless circuit constructed from a set of linear resistors, linear dependent sources and independent sources (they are non-linear elements) results in a circuit which obeys Superposition Theorem and hence, by definition, is a linear circuit.*

Linearity of a circuit element and linearity of a circuit are two different concepts. An element is linear if its v - i relationship obeys principle of homogeneity and principle of additivity. A circuit is linear, if all circuit variables in it, without any exception, obey principle of homogeneity and principle of additivity, i.e., the principle of superposition. It may appear intuitively obvious that a circuit containing only linear elements will turn out to be a linear circuit. But, note that we used non-linear elements – independent sources are non-linear elements – and hence, it is not so apparent. The preceding discussion offers a plausibility reasoning to convince us that a circuit containing linear elements and independent sources will indeed be a linear circuit. But the mathematical proof for this apparently straightforward conclusion is somewhat formidable.

Linearity and Superposition appear so natural to us. But the fact is that most of the practical electrical and electronic circuits are non-linear in nature. Linearity, at best, is only an approximation that circuit analysts employ to make the analysis problem more tractable. We illustrate why Superposition Theorem does not hold for a circuit containing a non-linear element by an example. The circuit is shown in Fig. 5.1-3(a). The resistor R is a non-linear one with a v - i relation given by $v = 2i^2$ for $i \geq 0$ and $-2i^2$ for $i < 0$.

Superposition
Theorem – Third form.

Linearity of a Circuit

Linearity of a circuit element and linearity of a circuit are two different concepts.

A circuit is called *linear* if its solution obeys superposition principle. This is why we stated the Superposition Theorem with the adjective *linear* behind ‘circuit’. Whether we view the statements on Superposition Theorem as a definition of linearity of a circuit or as an assertion of an important property of linear circuits is matter of viewpoint.

There is indeed a bit of circularity in *Linearity* and *Superposition Principle*.

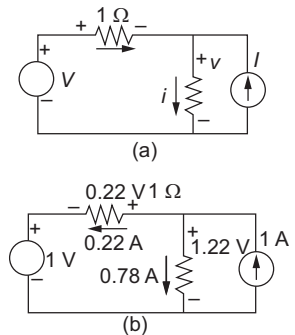


Fig. 5.1-3 (a) A Circuit Containing a Non-linear Resistor (b) Circuit Solution for $V = 1\text{ V}$ and $I = 1\text{ A}$

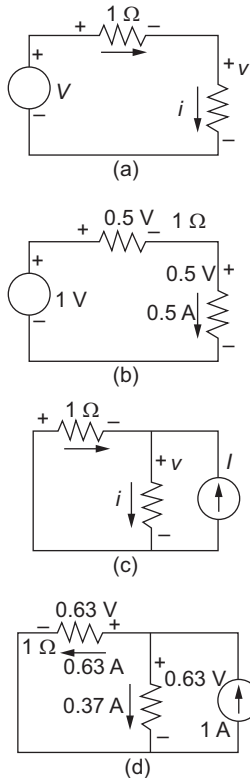


Fig. 5.1-4 Circuits with one Independent Source Acting at a time and Circuit Solution

The circuit is solved by writing the KVL equation in the first mesh. We first make use of KCL at the current source node to obtain the current through the $1\ \Omega$ resistor as $i - I$ A. Then, KVL in the first mesh gives

$$-V + (i - I) + 2i^2 = 0 \Rightarrow i = 0.25 \left[\sqrt{1 + 8(V + I)} - 1 \right] \text{ A.}$$

The value of this current for $V = 1\text{ V}$ and $I = 1\text{ A}$ is 0.78 A . Corresponding voltage across the non-linear resistor is $= 2i^2 \approx 1.22\text{ V}$ and the remaining circuit variables can now be obtained easily. The complete solution is marked in circuit as in Fig. 5.1-3(b).

We find the circuit solution when the independent sources act one by one. Figure 5.1-4 shows the relevant sub-circuits and the solution.

The circuit in Fig. 5.1-4(a) is solved by using the KVL equation $-V + i + 2i^2 = 0$. The solution for i will be $i = 0.25 \left[\sqrt{1 + 8V} - 1 \right] \text{ A}$. The solution for a case with $V = 1\text{ V}$ and $I = 1\text{ A}$ is marked in circuit as in Fig. 5.1-4(b).

The circuit as in Fig. 5.1-4(c) is solved by KCL equation at the current source node $2i^2 + 2i - I = 0$. The solution for i will be $i = 0.25 \left[\sqrt{1 + 2I} - 1 \right] \text{ A}$. The solution for a case with $V = 1\text{ V}$ and $I = 1\text{ A}$ is marked in circuit as in Fig. 5.1-4(d).

We observe that the current through the non-linear resistor when both the sources act simultaneously is 0.78 A , whereas the sum of responses from two circuits (Figs. 5.1-4(a) and (c)) is $0.5\text{ A} + 0.37\text{ A} = 0.87\text{ A}$. Thus, Superposition does not work in this circuit.

In general, $0.25 \left[\sqrt{1 + 8(V + I)} - 1 \right] \neq 0.25 \left[\sqrt{1 + 8V} - 1 \right] + 0.5 \left[\sqrt{1 + 2I} - 1 \right]$ and hence this circuit does not obey Superposition theorem. We also note that it is not possible to identify the contributions from the independent voltage source and independent current source separately when the two sources act simultaneously. We may try expanding the $\sqrt{1 + 8(V + I)}$ term in the solution for i in binomial series. Then we get,

$$i = \left[(V + I) - 0.25(V + I)^2 + \dots \right] = V + I - 0.25V^2 - 0.25I^2 - 0.5VI + \dots$$

Thus, i is decided by V and I through their higher powers along with first power terms. Higher power terms cannot satisfy superposition principle. Moreover, there are cross product terms like VI , V^2I , VI^2 etc., in the expression. We cannot ascribe such terms to voltage source or current source exclusively. We may take the view that they are the contributions from current source. In that case we have to admit that the contribution from the current source to the current i depends on whether the other source is active or not. And that kind of dependence results in non-adherence to superposition principle. Thus, we conclude that, non-linear elements in a circuit results in the circuit response failing to meet superposition principle due to (i) independent sources contributing to response variables through their higher powers and (ii) independent sources contributing jointly to response variables through cross product terms.

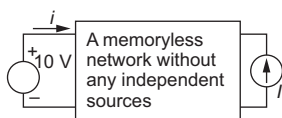


Fig. 5.1-5 Circuit for Example 5.1-1

EXAMPLE: 5.1-1

A memoryless circuit containing no independent sources inside is driven by an independent voltage source and an independent current source from outside as shown in Fig. 5.1-5. The current delivered by the voltage source is found to be 1 A when the current source is disconnected, 2 A when the current source is delivering 10 A into the circuit and 0.5 A when the current source is taking out 10 A from the circuit. Is the memoryless circuit a linear one?

SOLUTION

Let us assume that the circuit is linear. Then the current delivered by the 10 V source can be expressed as a linear combination of the two source functions.

i.e., $i = a \times 10 + bI$. The source function value of voltage source has been substituted in this equation. i is given to be 1 A when $I = 0$ and 2 A when $I = 10$ A. Therefore, $a = 0.1$ A/V and $b = 0.1$ A/A if the circuit is linear.

Then, when $I = -10$ A, the current delivered by the voltage source must be $0.1 \times 10 - 0.1 \times 10 = 0$ A. But it is stated that the current observed under this condition is 0.5 A. Hence, the circuit within the box is not linear.

EXAMPLE: 5.1-2

A certain resistor R in a linear memoryless network driven by two independent sources as shown in Fig. 5.1-6 is found to dissipate 36 W when only the voltage source is acting and 64 W when only the current source is acting. Find the power dissipated in the resistor when both sources are acting simultaneously. Is the answer unique?

SOLUTION

Since the network is linear, the current through the resistor R can be expressed as a linear combination of V and I .

$$\therefore i = aV + bI$$

The power dissipated, i.e., $i^2 R$ is given as 36 W when $I = 0$ and 64 W when $V = 0$.

$$\therefore P_v = (aV)^2 R = 36 \text{ and } P_i = (bI)^2 R = 64$$

$$\therefore (aV) = \sqrt{\frac{P_v}{R}} \text{ and } (bI) = \sqrt{\frac{P_i}{R}}$$

The power that will be dissipated when both sources are acting simultaneously is given by,

$$\begin{aligned} P_{vi} &= (aV + bI)^2 R \\ &= (aV)^2 R + (bI)^2 R + 2(aV)(bI)R \\ &= P_v + P_i + 2\sqrt{\frac{P_v}{R}}\sqrt{\frac{P_i}{R}}R \\ &= P_v + P_i + 2\sqrt{P_v P_i} = 36 + 64 + 2\sqrt{36 \times 64} = 196 \text{ W.} \end{aligned}$$

Note that the power dissipated when both the sources are acting is not the sum of powers dissipated when one source is acting at a time. i.e., *the power is not a superposable quantity*. The reason is very simple – $(i_1 + i_2)^2 \neq i_1^2 + i_2^2$ and $(v_1 + v_2)(i_1 + i_2) \neq v_1 i_1 + v_2 i_2$.

The power calculated as 196 W is not a unique answer. Power dissipated in a resistor when a certain current is flowing through it is independent of direction of the current since power depends on square of the resistor current. We have reflexively assumed that both a and b are positive or negative. But we have to account for the possibility of a and b having opposite signs – i.e., the possibility of two current contributions cancelling each other partially. This possibility is taken into account by modifying the total power equation as $P_{vi} = P_v + P_i \pm 2\sqrt{\frac{P_v}{R}}\sqrt{\frac{P_i}{R}}R$. Hence, the second possible value of power when both sources are acting simultaneously is 4 W.

Additional information in the form of current values or voltage values will be needed to decide between 196 W and 4 W.

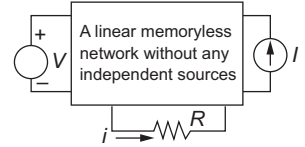


Fig. 5.1-6 Circuit for Example 5.1-2

Power is not a superposable quantity.

EXAMPLE: 5.1-3

The source function values for the three independent sources as in Fig. 5.1-7 are $V = 10$ V, $I_1 = 1$ A and $I_2 = 2$ A. The current in the resistor R is seen to be 1 A when the two current sources are switched off and 1.5 A when only I_2 is switched off and 2 A when all the three sources are active. Find what voltage must be applied by the voltage source if the current in R is to become zero with no change in current source values?

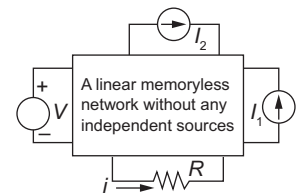


Fig. 5.1-7 Circuit for Example 5.1-3

SOLUTION

The current through R can be written as a linear combination of three source functions.

$$\text{i.e., } i = aV + bI_1 + cI_2$$

Substituting the data stated in the problem, we get three equations in three unknowns as below.

$$1 = 10a$$

$$1.5 = 10a + b$$

$$2 = 10a + b + 2c$$

Solving this system of equations, $a = 0.1$, $b = 0.5$ and $c = 0.25$.

$$\therefore i = 0.1V + 0.5I_1 + 0.25I_2$$

The required voltage to make i zero with $I_1 = 1$ A and $I_2 = 2$ A is obtained by

$$0 = 0.1V + 0.5 + 0.5 \Rightarrow V = -10 \text{ V.}$$

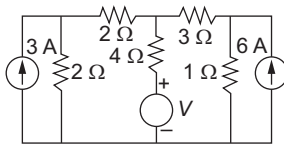


Fig. 5.1-8 Circuit for Example 5.1-4

EXAMPLE: 5.1-4

Find the value of V in the circuit in Fig. 5.1-8 such that the voltage source delivers zero power to the circuit by using Superposition theorem.

SOLUTION

The circuit is a linear one. Therefore, the current delivered by the voltage source can be expressed as a linear combination of the three source functions. Power delivered by the voltage source will be zero if the current delivered by it is zero. Thus, we want a value of V such that $aV + bI_1 + cI_2 = 0$, where a , b and c are the per unit contributions to current delivered by the voltage source from the three source functions.

We determine the three contributions first by solving three single-source circuits as shown in Fig. 5.1-9.

$$\text{The value of } i \text{ in circuit in Fig. 5.1-9(a) is found as } -3A \times \frac{2}{2 + (2 + 4 // (3 + 1))} \times \frac{3 + 1}{4 + (3 + 1)} = -0.5 \text{ A}$$

$$\text{The value of } i \text{ in circuit in Fig. 5.1-9(b) is found as } -6A \times \frac{1}{1 + (3 + 4 // (2 + 2))} \times \frac{2 + 2}{4 + (2 + 2)} = -0.5 \text{ A}$$

$$\text{The value of } i \text{ in circuit in Fig. 5.1-9(c) is found as } \frac{V}{4 + (2 + 2) // (3 + 1)} = \frac{V}{6} \text{ A}$$

Therefore, the current delivered by voltage source when all the three sources are active is $= -0.5 - 0.5 + \frac{V}{6} = \frac{V - 6}{6}$ A.

Therefore, the value of V such that the current (and power) delivered by the voltage source be zero $= 6$ V.

We note that we did not have to resort to node analysis or mesh analysis in solving the three single-source circuits shown in Fig. 5.1-9.

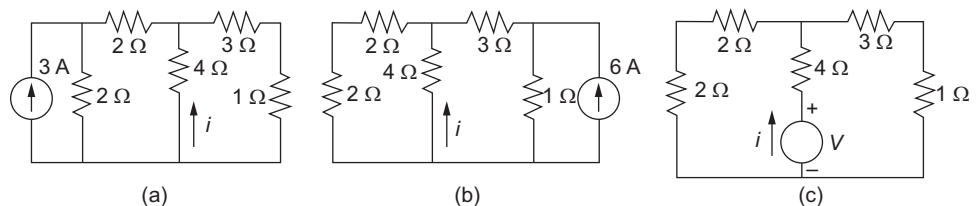


Fig. 5.1-9 Sub-circuits for Applying Superposition Theorem in Example 5.1-4

EXAMPLE: 5.1-5

Find the power dissipated in the resistor R_2 in the circuit in Fig. 5.1-10 by applying Superposition theorem.

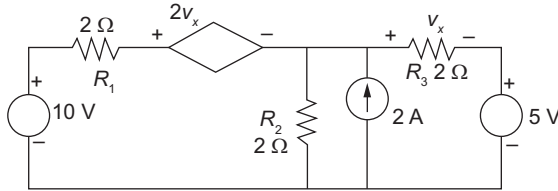


Fig. 5.1-10 Circuit for Example 5.1-5

SOLUTION

First, let us find the current through R_2 by applying Superposition Theorem. The single-source circuit when the first voltage source is acting is shown in Fig. 5.1-11.

This circuit is solved by KVL in the first mesh. Let the first mesh current be i_1 . Then, $v_x = 2/2 i_1 = i_1$ itself.

$$-10 + 2i_1 + 2i_1 + i_1 = 0 \Rightarrow i_1 = 2 \text{ A and hence, } i = 1 \text{ A.}$$

The single-source circuit when the second voltage source is acting alone is shown in circuit as in Fig. 5.1-12(a).

This circuit is solved by mesh analysis. The controlling variable v_x of the dependent source is $2i_2$. The mesh equations are

$$\begin{aligned} 2i_1 + 4i_2 + 2(i_1 - i_2) &= 0 \\ 2(i_2 - i_1) + 2i_2 + 5 &= 0. \end{aligned}$$

Simplifying the equations leads to $2i_1 + i_2 = 0$ and $-2i_1 + 4i_2 = -5$. Solving these we get $i_1 = 0.5 \text{ A}$ and $i_2 = -1 \text{ A}$. Therefore, $i = i_1 - i_2 = 1.5 \text{ A}$.

The circuit in Fig. 5.1-12(b) shows the single-source circuit when only the current source is acting. This circuit is solved by nodal analysis. The node voltage v_1 is assigned as shown in the circuit diagram. The controlling variable of dependent source is same as v_1 . Therefore, the potential at the right end of R_1 is $3v_1$ with respect to the reference node. Writing KCL at the top node, $0.5v_1 + 0.5v_1 + 0.5 \times 3v_1 = 2 \Rightarrow 2.5v_1 = 2 \Rightarrow v_1 = 0.8 \text{ V}$. Therefore, $i = 0.4 \text{ A}$.

Therefore, the current in R_2 when all the three sources are acting simultaneously is $i = 1 + 1.5 + 0.4 = 2.9 \text{ A}$.

$$\text{Therefore, the power dissipated in } R_2 = 2 \times 2.9^2 = 16.82 \text{ W.}$$

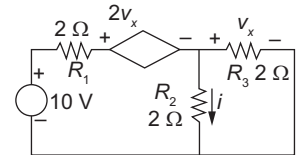


Fig. 5.1-11 Circuit with only the First Voltage Source Acting in Example 5.1-5

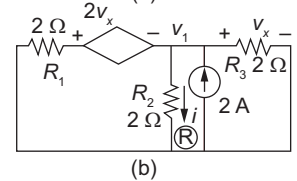
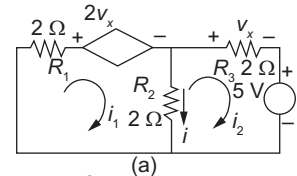


Fig. 5.1-12 Circuits with only (a) the Second Voltage Source Acting and (b) only the Current Source is Acting in Example 5.1-5

5.2 STAR-DELTA TRANSFORMATION THEOREM

We observe from the examples on application of Superposition Theorem in Sect. 5.1 that the single-source circuits that need to be solved in that context may require us to use nodal analysis and mesh analysis often. However, we can expect to avoid these procedures in the case of circuits involving only resistors and independent sources. We will be able to solve the single-source circuits by employing series and parallel equivalents repeatedly. But, there is one pair of resistor connections that will not yield to this kind of approach. For instance, consider the problem of finding the current through the resistor R in circuit in Fig. 5.2-1(a) by applying superposition principle. The relevant single-source circuits are shown in Figs. 5.2-1(b) and (c).

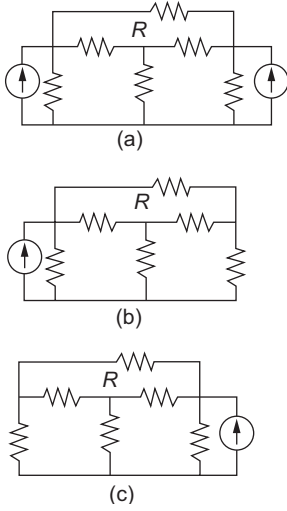


Fig. 5.2-1 A Circuit in Which Star-Delta Transformation will be Helpful

The logic behind Star-Delta transformation.

This problem cannot be solved by series-parallel equivalents. The T-shaped (also called Y-shaped or Star-connected) resistor network containing three resistors makes it impossible to apply series-parallel reduction. Equivalently, the three outer resistors which are connected in Π -form (also called Delta-connected, Mesh-connected, etc.,) makes it impossible to apply series-parallel reduction.

It turns out that a Y-connected set of three resistors can be replaced with a delta-connected set of three resistors *without any circuit variable outside these three resistors* getting affected. Similarly, a set of three resistors connected in delta can be replaced with a set of resistors connected in Star without affecting the circuit solution in the remaining portion of the circuit. We develop equations for this transformation in this section.

First, we consider Star to Delta transformation. We want the two resistor networks as shown in Fig. 5.2-2 to be equivalent with respect to the external network.

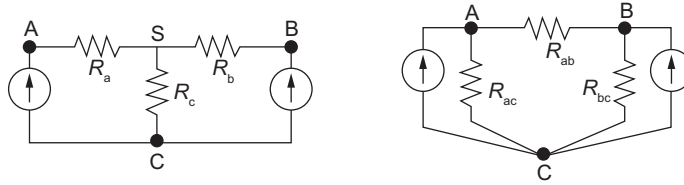


Fig. 5.2-2 Circuits Related to Star-Delta Transformation

The net effect of the external network on the star connected resistor may be modelled by two current sources driving it as shown. If the second network that is connected in delta produces same node voltages at node-A and node-B with respect to node-C when driven by the same two current sources as in the star network, the external circuit solution will not be affected. This is because the node voltage variables in a circuit decide all other voltages and currents in a circuit. If the node voltage variables are not affected, then, no voltage or current in the circuit gets affected. Therefore, we can derive the values for resistors in delta network in terms of resistor values in star network by imposing the condition that v_A and v_B with respect to v_C must be the same in both circuits when driven by the same current sources.

Let I_1 and I_2 be the current source functions. Then, v_A and v_B in star circuit is given by,

$$\begin{bmatrix} G_a & 0 & -G_a \\ 0 & G_b & -G_b \\ -G_a & -G_b & G_a + G_b + G_c \end{bmatrix} \begin{bmatrix} v_A \\ v_B \\ v_S \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix}$$

We do not need the node voltage at the v_S . We eliminate it easily since there is no current source injection at that node.

$$\begin{aligned} -G_a v_A - G_b v_B + (G_a + G_b + G_c) v_S &= 0 \\ \therefore v_S &= \frac{G_a}{G_a + G_b + G_c} v_A + \frac{G_b}{G_a + G_b + G_c} v_B. \end{aligned}$$

Substituting the expression for v_S in the first two node equations, rearranging terms and expressing the final equations in matrix form, we get,

$$\begin{bmatrix} \frac{G_a(G_b + G_c)}{(G_a + G_b + G_c)} & \frac{-G_a G_b}{(G_a + G_b + G_c)} \\ \frac{-G_a G_b}{(G_a + G_b + G_c)} & \frac{G_b(G_a + G_c)}{(G_a + G_b + G_c)} \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (5.2-1)$$

Now, we write the node equation for the delta-connected network as,

$$\begin{bmatrix} G_{ac} + G_{ab} & -G_{ab} \\ -G_{ab} & G_{ab} + G_{bc} \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (5.2-2)$$

Both the equations, Eqn. 5.2-1 and 5.2-2, should result in same v_A and v_B for network equivalence. This requires that the two nodal conductance matrices be equal. Therefore,

$$\begin{aligned} G_{ab} &= \frac{G_a G_b}{G_a + G_b + G_c} \\ G_{ac} &= \frac{G_a (G_b + G_c)}{G_a + G_b + G_c} - \frac{G_a G_b}{G_a + G_b + G_c} = \frac{G_a G_c}{G_a + G_b + G_c} \\ G_{bc} &= \frac{G_b (G_a + G_c)}{G_a + G_b + G_c} - \frac{G_a G_b}{G_a + G_b + G_c} = \frac{G_b G_c}{G_a + G_b + G_c}. \end{aligned} \quad (5.2-3)$$

Expressing this in terms of resistances,

$$\begin{aligned} R_{ab} &= \frac{R_a R_b + R_b R_c + R_c R_a}{R_c} \\ R_{ac} &= \frac{R_a R_b + R_b R_c + R_c R_a}{R_b} \\ R_{bc} &= \frac{R_a R_b + R_b R_c + R_c R_a}{R_a}. \end{aligned} \quad (5.2-4)$$

Expressions for determining Delta network resistors from Star network resistors.

Equation 5.2-4 shows how the resistance values for equivalent delta may be calculated from resistance values used in star network.

Figure 5.2-3 shows the star-delta transformation in a way that makes the symmetry in the equations evident.

The Delta-Star transformation may similarly be derived using an approach based on mesh analysis. Assume that a pair of independent voltage sources drives both circuits. Then the currents drawn from the sources must be the same in both circuits. The delta network will have three meshes; however, the central mesh has no voltage source in it and hence its mesh current can be eliminated by using the technique we employed in this section. The details of this derivation are skipped and the final result is given as,

$$R_a = \frac{R_{ab} R_{ac}}{R_{ab} + R_{bc} + R_{ac}}, R_b = \frac{R_{ab} R_{bc}}{R_{ab} + R_{bc} + R_{ac}}, R_c = \frac{R_{ac} R_{bc}}{R_{ab} + R_{bc} + R_{ac}}. \quad (5.2-5)$$

Expressions for determining Star network resistors from Delta network resistors.

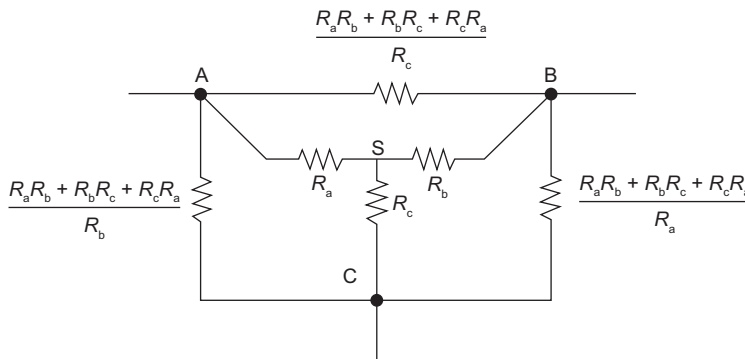


Fig. 5.2-3 Star to Delta Transformation Equations

Equation 5.2-5 shows how the resistance values of the equivalent star network may be obtained from the delta network resistances and Fig. 5.2-4 shows in such a manner that makes the symmetry in these equations evident.

Expressions for determining Star network resistors from Delta network resistors.

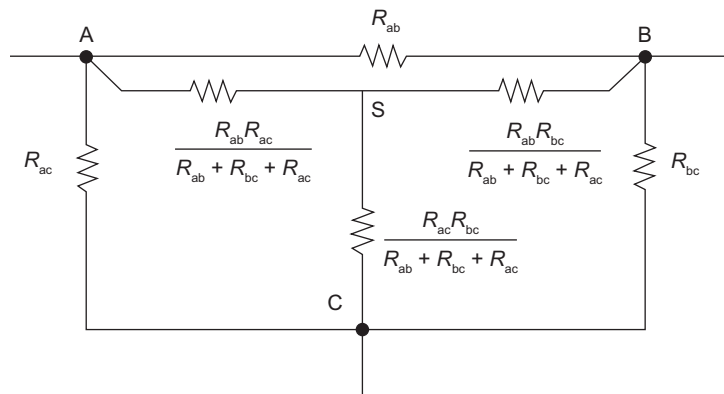


Fig. 5.2-4 Delta to Star Transformation Equations

Star-Delta (also called $Y-\Delta$ or $T-\Pi$ transformation) transformation is widely employed in analysis of three-phase AC circuits. A case of special interest is that of equal resistances in all limbs of star or delta. The transformation equations for this special case are shown in Fig. 5.2-5.

A special case of equal resistors in Delta or Star.

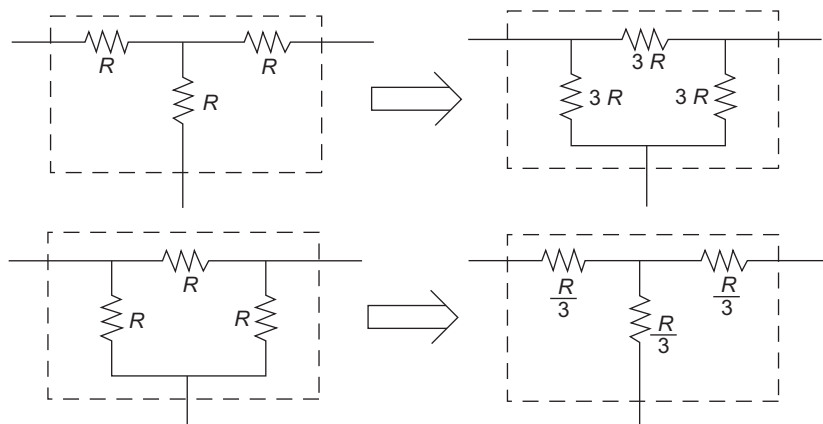


Fig. 5.2-5 A Special Case of Star-Delta Transformation and Delta-Star Transformation

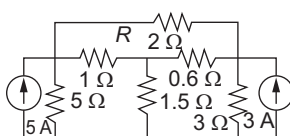


Fig. 5.2-6 Circuit for Example 5.2-1

EXAMPLE: 5.2-1

Find the power dissipated in the resistor R in circuit in Fig. 5.2-6.

SOLUTION

The single-source circuits required for applying superposition principle are shown in Figs. 5.2-7(a) and (c).

In Fig. 5.2-7, the circuits (b) and (d) show the circuits (a) and (c) respectively, after star-delta transformation on the inner star-connected resistors of $1\ \Omega$, $0.6\ \Omega$ and $1.5\ \Omega$. Taking positive direction of current flow to be from left to right in R , the current flowing in R in circuit (b) is $5\text{ A} \times \frac{5/5}{5/5 + (2/2 + 3/3)} \times \frac{2}{2+2} = 1.25\text{ A}$. Similarly, the current flow as in circuit (d) is $-3\text{ A} \times \frac{3/3}{3/3 + (2/2 + 5/5)} \times \frac{2}{2+2} = -0.45\text{ A}$.

Therefore, the current through R when both sources are acting will be $1.25 - 0.45 = 0.8\text{ A}$.

Therefore, power dissipation in $R = 0.8^2 \times 2 = 1.28\text{ W}$.

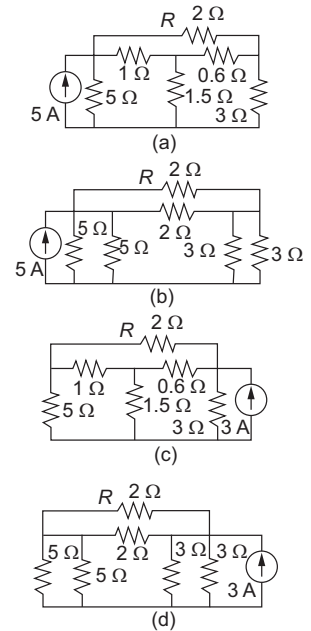


Fig. 5.2-7 Single-source Circuits for Applying Superposition Theorem for Example 5.2-1

5.3 SUBSTITUTION THEOREM

Consider the three-mesh circuit shown in Fig. 5.3-1. Mesh analysis reveals that the mesh currents are 1 A , 2 A and 3 A as shown in the figure. Two nodes **a** and **a'** have been identified in the circuit and the current crossing the node **a** from left to right is marked as 2 A . The voltage of **a** with respect to **a'** is calculated to be 1 V and is marked in the figure.

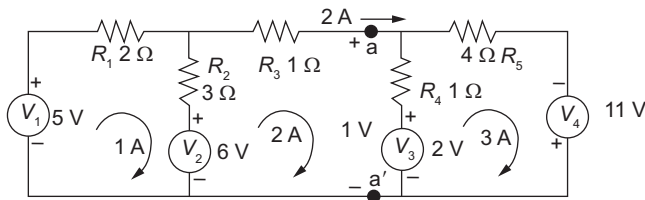


Fig. 5.3-1 A Three-mesh Circuit with Two Nodes – **a** and **a'** – Identified

Now, we add two current sources between the two nodes, **a** and **a'**, as shown in Fig. 5.3-2. The current sources have equal and opposite currents of 2 A magnitude.

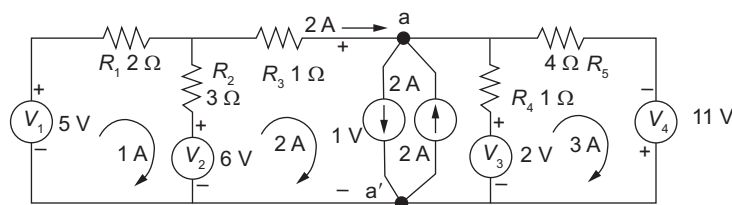


Fig. 5.3-2 Circuit in Fig. 5.3-1 with Two Current Sources Added

We have not changed the KCL equation at node **a** and node **a'**. The mesh introduced in this step is a trivial one. Hence, the circuit solution everywhere will remain the same as before. Now, a pair of nodes **b** and **b'** is introduced that are connected to **a** and **a'**, respectively by shorting links as in Fig. 5.3-3.

Application of KCL at node **a** and **a'** show that there is no current flow in the two shorting links.

We note that the current flows in the shorting link **a–b** and **a'–b'** are zero. Therefore, breaking the shorting links should not affect the circuit solution in both parts of the original circuit (Fig. 5.3-4).

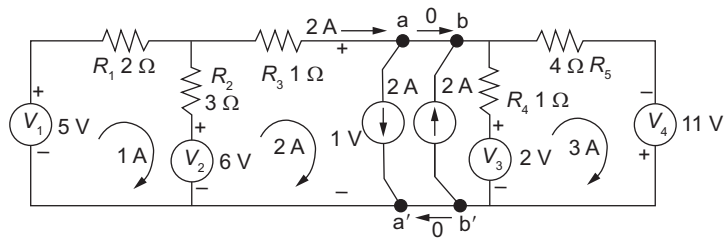
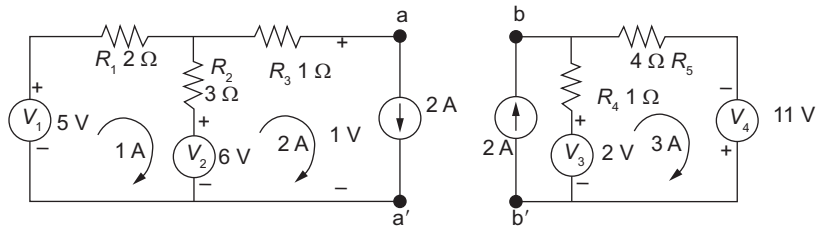
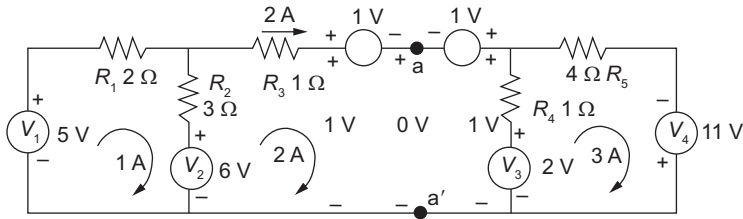
Fig. 5.3-3 Additional Node pair **b** and **b'** Identified in Circuit of Fig. 5.3-1

Fig. 5.3-4 The Original Circuit Separated into Two Parts Without the Solution in Either Part Getting Affected

Thus, as far as the first part is concerned, we have been able to replace or *substitute* the second part with a current source, which has a value exactly equal to the current drawn by the second part of the circuit from first part of the circuit, without any circuit variable in the first part undergoing any change. Similar statement can be framed for the second part of the circuit.

Now, let us go back to Fig. 5.3-1 and add two independent voltage sources instead of current sources as shown in Fig. 5.3-5. The KVL in the second mesh is not affected and hence, the circuit solution remains the same. However, we have reduced the voltage between **a** and **a'** to zero.

Fig. 5.3-5 Circuit in Fig. 5.3-1 with Two Equal and Opposite Voltage Sources Introduced in Series at Node **a**

If **a** and **a'** are at the same potential, they can be joined together. If they can be joined together, the two parts of the circuit are connected at only one point and hence they cannot affect each other in any way. We can very well draw them as separate circuits without a common touch point as shown in Fig. 5.3-6.

Thus, as far as the first part is concerned, we have been able to replace or *substitute* the second part with a voltage source, which has a value exactly equal to the voltage that was impressed on the second part of the circuit by the first part, without any circuit variable in the first part undergoing any change. Similar statement can be framed for the second part of the circuit.

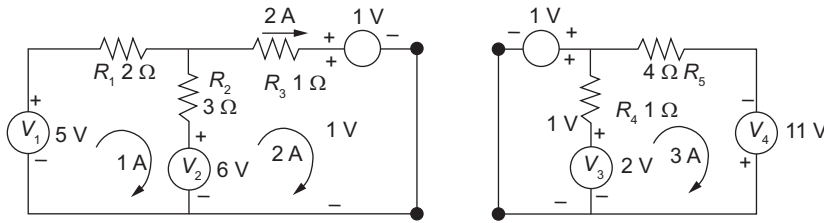


Fig. 5.3-6 Original Circuit is Separated into Two Parts Without the Circuit Solution in Either Part Getting Affected

If we go one step further, we end up in trouble. We extract the first part from Fig. 5.3-4 and apply the same reasoning we employed to arrive at the two parts in that figure to arrive at the circuit shown in of Fig. 5.3-7(b).

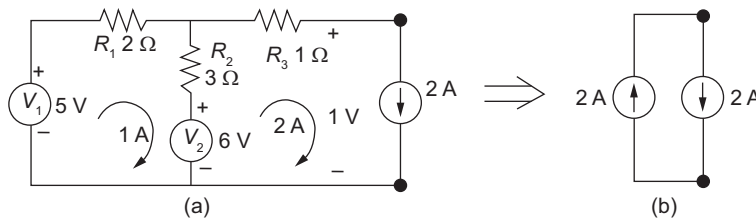


Fig. 5.3-7 The Result of Stretching an Idea too Much

That circuit in Fig. 5.3-7(b) has no unique solution since the voltage across the current sources can now be any value without violating any circuit laws.

Thus, there must be some constraints to be satisfied by a circuit if *substitution* of a part by a current source of value equal to the current drawn by it (or by a voltage source of value equal to voltage appearing across it) is not to affect the circuit solution in the remaining part. *The constraint is that the original circuit must have a unique solution and the circuit after substitution also must have a unique solution.* Linear circuits usually have unique solution – i.e., the currents and voltages everywhere are *uniquely* decided by values of independent sources and the circuit structure – except in some trivial and avoidable situations like ideal independent voltage sources in parallel or ideal independent current sources in series etc.

However, note that only KCL and KVL-based arguments are used to arrive at the validity of *substitution*. We did not make use of element relations at all. Hence, the arguments are valid for any circuit – linear or non-linear. *Substitution theorem* is more general than *Superposition theorem*. The constraint of *unique solution* assumes particular significance in the case of non-linear circuits since there are non-linear elements that have multi-valued v - i relationships. A tunnel diode, a uni-junction transistor etc., are some examples.

There is another constraint to be satisfied before *substitution* can be done in a circuit. Consider the situation where the controlling variable of a dependent source is in the part that was subjected to substitution with the dependent source output connected in the other part. Obviously that will not work. Therefore, if there are dependent sources in the part of the circuit that is being substituted by an independent current source or voltage source, both the controlling variable and the dependent source must be within that part of the circuit. Similarly, if there is magnetic coupling in the part of the circuit being substituted, all coils belonging to the magnetically coupled system must be within that part of the circuit. This constraint may alternatively be stated as – *there should not be any interaction between the*

A constraint to be satisfied by a circuit so that Substitution Theorem can be applied to it.

Another constraint to be satisfied by a circuit so that Substitution Theorem can be applied to it.

But, what is the use of a theorem that wants us to solve a circuit first and then replace part of the circuit by a source that has a value depending on the solution of the circuit? Obviously, such a theorem will not help us directly in solving circuits.

The significance of this theorem lies in the fact that it can be used to construct theoretical arguments that lead to other powerful circuit theorems that indeed help us to solve circuit analysis problems in an elegant and efficient manner.

Moreover, it does find application in circuit analysis in a slightly disguised form. We take up that disguised form of Substitution Theorem in Sect. 5.4.

part of the circuit that is being substituted and the remaining circuit except through the pair of terminals at which they are interconnected.

Subject to the constraints on *unique solution* and *interaction only through the connecting terminals*, we state the *Substitution theorem* as below (Fig. 5.3-8).

Let a circuit with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let $v(t)$ be the voltage that appears at the terminals between N_1 and N_2 and let $i(t)$ be the current flowing into N_2 from N_1 . Then, the network N_2 may be replaced by an independent current source of value $i(t)$ connected across the output of N_1 or an independent voltage source of value $v(t)$ connected across the output of N_1 without affecting any voltage or current variable within N_1 provided the resulting network has unique solution.

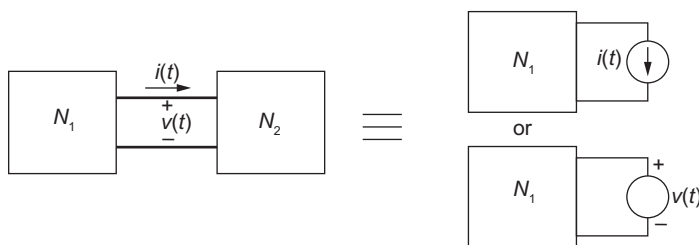


Fig. 5.3-8 The Substitution Theorem

5.4 COMPENSATION THEOREM

The circuit in Fig. 5.4-1(a) has a resistor marked as R . It has a nominal value of $2\ \Omega$. Mesh analysis was carried out to find the current in this resistor and the current was found to be 1 A as marked in the circuit as in Fig. 5.4-1(a).

Now, let us assume that the resistor value changes by ΔR to $R + \Delta R$. Correspondingly all circuit variables change by small quantities as shown in Fig. 5.4-1(b). The current through that resistor will also change to $i + \Delta i$. We can conduct a mesh analysis once again and get a new solution. However, we can do better than that. We can work out *changes* in variables everywhere by solving a single-source circuit and then construct the circuit solution by adding *change* to the initial solution value.

We apply Substitution theorem on the first circuit with R as the element that is being substituted and on the second circuit with $R + \Delta R$ as the part that is being substituted by an independent voltage source. The voltage source in the first circuit must be $Ri\text{ V}$ and the voltage source in the second circuit must be $(R + \Delta R)(i + \Delta i)\text{ V}$.

$$(R + \Delta R)(i + \Delta i) = Ri + (R + \Delta R)\Delta i + i\Delta R \text{ (Fig. 5.4-2).}$$

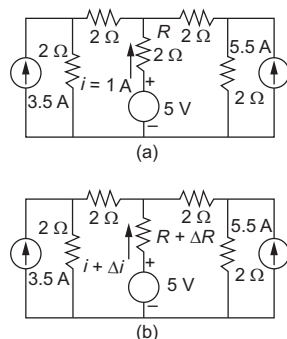


Fig. 5.4-1 Circuit to Illustrate Compensation Theorem

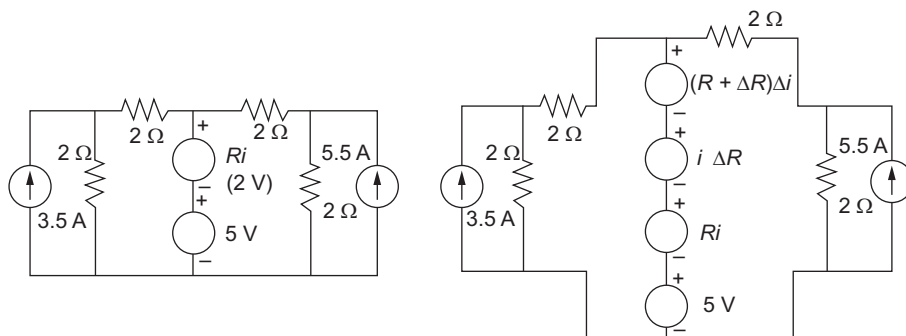


Fig. 5.4-2 Circuits After Applying Substitution Theorem

Now, we solve the second circuit by applying superposition principle by taking the 5 V and Ri V sources along with the two current sources together first and deactivating the remaining two voltage sources. The solution we get will be the same as the solution of the original circuit since the second circuit with the $i\Delta R$ V source and the $(R+\Delta R)\Delta i$ V source deactivated is the same as the first circuit. We already know the solution, and this is the initial solution.

We have to solve the circuit with the two sources – the $i\Delta R$ V source and the $(R+\Delta R)\Delta i$ V source to get the second component of complete solution for second circuit. This circuit is shown in Fig. 5.4-3(a). The solution of this circuit must give the *changes* in all circuit variables due the change in R since the initial values of variables are given by the solution contributed by the other sources. Therefore, the current through central branch in the circuit in Fig. 5.4-3(a) must be Δi .

We note that the voltage of the voltage source $(R+\Delta R)\Delta i$ in circuit in Fig. 5.4-3(a) is exactly the same as the voltage drop that will be produced by a resistor of value $(R+\Delta R)$ since the current in that branch is Δi . That is, the voltage source of value $(R+\Delta R)\Delta i$ can be thought of as the result of a substitution operation on a resistor of value $(R+\Delta R)$ in that path. We reverse this substitution and replace the voltage source by the resistor in Fig. 5.4-3(b). Solving circuit (b) will give us the *change* in all circuit variables due to a change in R . Adding the initial values to change values will give us the final solution. The circuit in Fig. 5.4-3(b) is a single-source circuit with only one voltage source of value = (change in component value) $\times$ (initial current through that component).

Let us assume that $\Delta R = 0.1 \Omega$. Then, the source value is $0.1 \Omega \times 1 \text{ A} = 0.1 \text{ V}$. Therefore,

$$\Delta i = -\frac{0.1}{2.1 + (2+2)/(2+2)} = -0.0244 \text{ A and } (i + \Delta i) = (1 - 0.0244) \text{ A} = 0.9756 \text{ A}.$$

Reader may note that we used Superposition theorem along with Substitution theorem to arrive at this result. Hence, Compensation theorem is a specialised form of Substitution theorem for a *Linear Circuit*.

Compensation Theorem:

In a linear memoryless circuit, the change in circuit variables due to change in one resistor value from R to $R + \Delta R$ in the circuit can be obtained by solving a single-source circuit analysis problem with an independent voltage source of value $i\Delta R$ in series with $R+\Delta R$, where i is the current flowing through the resistor before its value changed (Fig. 5.4-4).

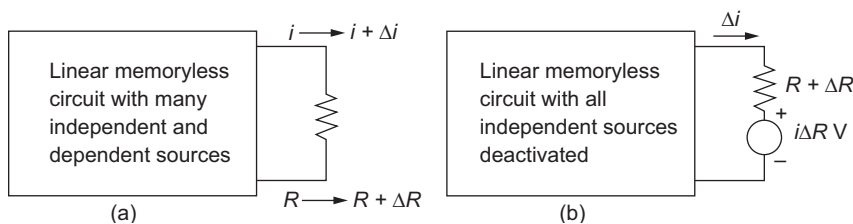


Fig. 5.4-4 The Compensation Theorem

The theorem can be extended to include dependent source coefficients too. Changes in circuit variables due to simultaneous changes in many circuit parameters can be obtained by repeated application of Compensation theorem or as a solution of a multi-source *change* circuit in which each parameter change is taken into account by a voltage source of suitable value.

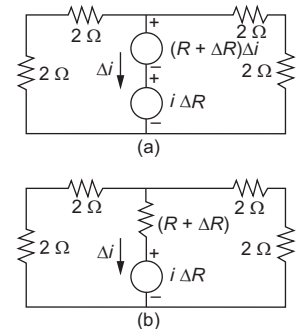


Fig. 5.4-3 (a) Circuit for Obtaining Changes in Variables (b) After Replacing Voltage Source by Resistor

Compensation Theorem stated here is a specialised form of Substitution Theorem for Linear Circuits.

5.5 THEVENIN'S THEOREM AND NORTON'S THEOREM

The problem of solving a circuit with different load networks connected to the same delivery network arises in Electrical and Electronics Engineering quite often. We do not want to write the same node equations or mesh equations of the delivery network whenever the load network undergoes some change and solve the circuit in its entirety again and again. Thevenin's Theorem and Norton's Theorem help us to avoid this kind of wasted effort and become efficient in solving circuits. They help us to conduct node analysis or mesh analysis of the delivery network and replace it with a simple equivalent circuit for further analysis when different load networks are connected to it. These are two tools indispensable to a circuit analyst.

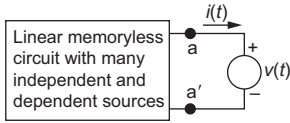


Fig. 5.5-1 A Memoryless Network Terminated in a Voltage Source at its Output Terminals

We are applying Superposition Theorem here.

Consider a memoryless network shown in Fig. 5.5-1 containing linear resistors, linear dependent sources and independent sources with a pair of terminals identified as the output terminals of the network. The network interacts with the external world only through this pair of terminals. No parameter inside the circuit changes, but different load networks may get connected to the circuit at its output terminals. Assume that an independent voltage source of source function $v(t)$ is connected across the output terminals of the network. With no loss of generality, we assume further that the voltage source negative terminal, i.e., a' , is taken as the reference node for writing the node equations of the circuit. We are interested in the behaviour of the current $i(t)$ delivered by the circuit to the terminating voltage source versus the source function $v(t)$.

We recollect that any circuit variable in a linear circuit can be expressed as a linear combination of all the independent source functions in the circuit. Hence, $i(t)$ in this circuit can be expressed as

$$i(t) = \left[\begin{aligned} &(a_1 v_{S_1}(t) + (a_2 v_{S_2}(t) + \dots + a_{n_v} v_{S_{n_v}}(t))) \\ &+ (b_1 i_{S_1}(t) + b_2 i_{S_2}(t) + \dots + b_{n_i} i_{S_{n_i}}(t)) \end{aligned} \right] + a_0 v(t).$$

This current has two components – one contributed by all independent current sources and voltage sources within the circuit; and the second contributed by the independent voltage source connected from outside, i.e., $v(t)$. The functions $v_{S_1}(t) \dots$ represent the source functions of independent voltage sources within the circuit and the functions $i_{S_1}(t) \dots$ represent the source functions of independent current sources within the circuit. n_v and n_i are the number of independent voltage sources and current sources, respectively within the circuit. The contribution coefficients $a_0, a_1, a_2, \dots$ and $b_1, b_2, \dots$ etc., are determined by the circuit parameters. They may be found by node analysis or mesh analysis.

The source functions and contribution coefficients are fixed by the circuit and the only aspect of the circuit that can change is the network that gets connected at the output terminals. Hence, we represent the terms within the square brackets in the expression for $i(t)$ as a fixed function of time that does not depend on what is connected at the output and term it as $i_{SC}(t)$. Therefore,

$$i(t) = i_{SC}(t) + a_0 v(t),$$

$$\text{where } i_{SC}(t) = \sum_{i=1}^{n_v} a_i v_{S_i}(t) + \sum_{i=1}^{n_i} b_i i_{S_i}(t). \quad (5.5-1)$$

This equation can be interpreted in an interesting manner if a_0 can be written as $-G_o$. The number a_0 can be obtained by finding the current *delivered* to the voltage source when all the independent sources are set to zero. If the circuit contains only resistors, then, the current will actually be delivered to the circuit and hence a_0 will be a negative number, making G_o a positive number. The possibility of a_0 assuming a positive value does exist if there are dependent sources within the circuit. Therefore, G_o is positive for

a purely resistive network, whereas it could be negative for a circuit containing dependent sources.

$$\therefore i(t) = i_{sc}(t) - G_0 v(t). \quad (5.5-2)$$

This equation suggests that the current $i(t)$ is as if it comes from an independent current source of source function $i_{sc}(t)$ that is in parallel with a resistance of $R_0 = 1/G_0$ (Fig. 5.5-2).

How do we get the source function $i_{sc}(t)$? $i(t) = i_{sc}(t)$, when $v(t) = 0$. Therefore, we can find $i_{sc}(t)$ by finding the current that flows out into a short-circuit that is put across its output. This is the reason why we used 'SC' as the subscript for this current source function.

And, how do we find out the value of R_0 ? If we can reduce $i_{sc}(t)$ to zero and apply a non-zero $v(t)$, the ratio of current drawn from $v(t)$ to the voltage $v(t)$ will be R_0 . We can reduce $i_{sc}(t)$ to zero by deactivating all the independent sources within the circuit. Thus, we see that, R_0 is nothing but the equivalent resistance of the deactivated network from terminals **a**–**a'**.

We conclude that a linear memoryless circuit containing resistors, dependent sources and independent sources may be replaced by a current source $i_{sc}(t)$ in parallel with a resistance R_0 when it is terminated in an independent voltage source, where $i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_0 is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

The circuit was terminated in an independent voltage source $v(t)$ until now. We now choose to view that voltage source as the result of a *Substitution Operation*. That is, this voltage source came up there because we *substituted* a part of the original network by an independent voltage source by invoking *Substitution Theorem*. We note that *Substitution Theorem* does not require the part of the circuit that is being substituted to be linear. Now, we bring that part of the circuit back and dispense with the independent voltage source $v(t)$. We will keep in mind that the circuit must meet all those constraints that *Substitution Theorem* calls for. We are now ready to state *Norton's Theorem*.

Norton's Theorem:

Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent current source of value $i_{sc}(t)$ in parallel with a resistance R_0 without affecting any voltage or current variable within N_2 provided the resulting network has unique solution.

$i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_0 is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

This equivalent circuit for N_1 is called its *Norton's Equivalent* (Fig. 5.5-3).

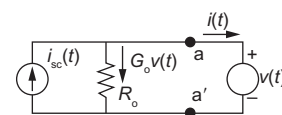


Fig. 5.5-2 A Circuit that follows Eqn. 5.5-2

Interpretation for $i_{sc}(t)$ in Fig. 5.5-2.

Interpretation for R_0 in Fig. 5.5-2.

Statement of Norton's Theorem.

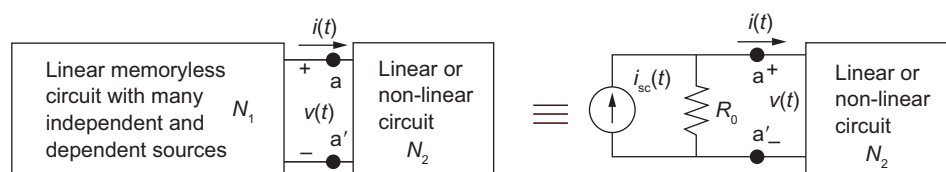


Fig. 5.5-3 Norton's Theorem and Norton's Equivalent

A similar argument after terminating the network N_1 in an independent current source of source function $i(t)$ will lead us to the conclusion that it may be replaced by an independent voltage source $v_{oc}(t)$ in series with a resistance R_o without affecting the circuit solution in N_2 . $v_{oc}(t)$ in this case will be the voltage generated across **a–a'** by all the independent sources within the network N_1 when the output terminals are kept open. Therefore, it is called the open-circuit voltage. R_o will again be the equivalent resistance of the deactivated network seen from **a–a'**. The resulting equivalent circuit for N_1 is called its *Thevenin's Equivalent*.

Thevenin's Equivalent may also be derived from Norton's equivalent by applying Source Transformation Theorem.

Thevenin's Theorem:

Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent voltage source of value $v_{oc}(t)$ in series with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution.

$v_{oc}(t)$ is the voltage that will appear across the terminals when they are kept open and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

This equivalent circuit for N_1 is called its *Thevenin's Equivalent* (Fig. 5.5-4).

Statement of
Thevenin's Theorem.

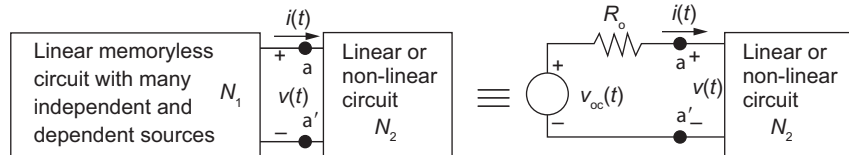


Fig. 5.5-4 Thevenin's Theorem and Thevenin's Equivalent

Compensation theorem is a special form of Substitution theorem for Linear Circuits. Norton's theorem and Thevenin's theorem are two other kinds of links between Substitution theorem and Superposition theorem. The network N_1 has to be linear since we used the idea of linear combination (i.e., Superposition theorem) in replacing it by equivalents. N_2 can be non-linear since it is subjected to only Substitution theorem and not to Superposition theorem.

EXAMPLE: 5.5-1

Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.5-5 with respect to the terminals 'a' and 'b'.

SOLUTION

Step-1: Find the open-circuit voltage across 'a–b'

This step may require nodal analysis or mesh analysis in general. But in simple resistive circuits like this, one may use superposition principle and solve for the required voltage. The two single-source circuits needed for this are shown in Fig. 5.5-6.

The contribution to v_{oc} from the 6 A current source

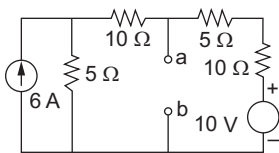
$$= (10 + 5) \times \frac{5}{5 + (10 + 5 + 10)} \times 6 = 15 \text{ V.}$$


Fig. 5.5-5 Circuit for
Example 5.5-1

The contribution to v_{oc} from the 10 V voltage source = $\frac{10+5}{10+5+10+5} \times 10 = 5$ V.

Therefore, $v_{oc} = 15 + 5 = 20$ V.

Step-2: Find Thevenin's Equivalent Resistance R_o .

The deactivated circuit is shown in Fig. 5.5-7.

The equivalent resistance seen from a-b = $(10 + 5) // (10 + 5) = 7.5 \Omega$

Therefore, the Thevenin's equivalent and Norton's (determined by applying source transformation on Thevenin's equivalent) are as shown in Fig. 5.5-8.

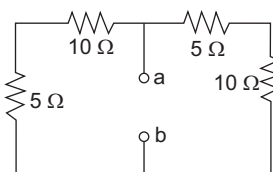


Fig. 5.5-7 The Deactivated Circuit for Determining R_o

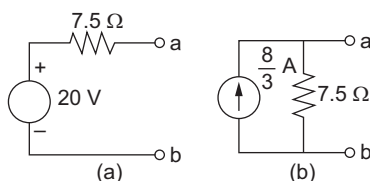


Fig. 5.5-8 (a) Thevenin's Equivalent and (b) Norton's Equivalent for Example 5.5-1

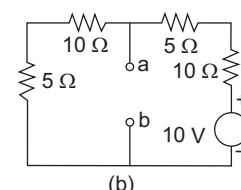
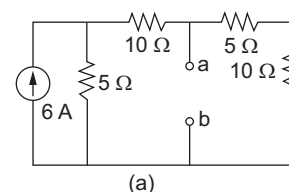


Fig. 5.5-6 Single-source Circuits for Finding Contributions to Open-circuit Voltage Across a-b

Thevenin's equivalent and Norton's equivalent are equivalent only as far as the circuit variables in the network that is connected across them. They are not equivalents as far as the circuit variables in the network they replace are concerned. We agree not to seek any information on the circuit variables inside the network that was replaced by equivalent whenever we use such equivalents. For instance, *the power dissipated in R_o is not the power actually dissipated in the network that is replaced by Thevenin's equivalent.*

Determining R_o for resistive circuits is simple. Series-parallel equivalents and star-delta transformation will be useful. But these are not helpful in the case of circuits containing dependent sources since the deactivated source will still contain them. Special procedures are needed in the case of such circuits.

5.6 DETERMINATION OF EQUIVALENTS FOR CIRCUITS WITH DEPENDENT SOURCES

Method-1:

- (i) Find v_{oc} by nodal analysis or mesh analysis or superposition principle.
- (ii) Find i_{sc} by nodal analysis or mesh analysis or superposition principle.
- (iii) Obtain R_o by $R_o = \frac{v_{oc}}{i_{sc}}$.

Method-2:

- (i) Find v_{oc} by nodal analysis or mesh analysis or superposition principle.
- (ii) Assume that a current source of 1 A is applied to the output terminals of the deactivated network such that the current flows into the network at the first terminal. Carry out a node or mesh analysis and find out the voltage appearing at first terminal with respect to second terminal. The numerical value of this voltage gives the value of R_o .
- (iii) Determine i_{sc} by $i_{sc} = \frac{v_{oc}}{R_o}$.

Method-3:

- (i) Find i_{sc} by nodal analysis or mesh analysis or superposition principle.
- (ii) Assume that a voltage source of 1 V is applied to the output terminals of the deactivated network with positive polarity at the first terminal. Carry out a node or mesh analysis and find out the current flowing into that terminal. The value of this current gives the value of $G_o = 1/R_o$.
- (iii) Determine v_{oc} by $v_{oc} = R_o i_{sc}$.

EXAMPLE: 5.6-1

Find the Thevenin's equivalent of the circuit in Fig. 5.6-1 with respect to the terminals 'a' and 'b' and thereby find the ratio of $v_x(t)$ to $v_s(t)$ when a resistor of $2\text{ k}\Omega$ is connected across the output. The circuit is the low-frequency signal model for a RC-coupled Common Emitter Amplifier using a bipolar junction transistor.

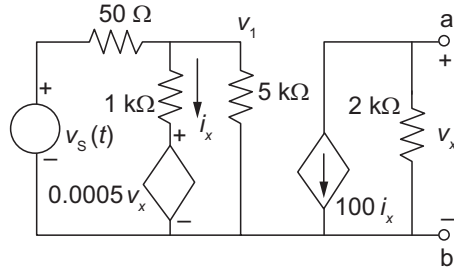


Fig. 5.6-1 Circuit for Example 5.6-1

SOLUTION

The first method is used in this example. Let 'b' be the reference node and let the node voltage at top end of $5\text{ k}\Omega$ be v_1 as marked in the figure. Then,

$$\begin{aligned} i_x &= 1 \times 10^{-3} (v_1 - 0.0005v_x) \text{ and } v_x = -2 \times 10^5 i_x = -200(v_1 - 0.0005v_x) \\ &= -200v_1 + 0.1v_x \\ \therefore 0.9v_x &= -200v_1 \Rightarrow v_x = -222.2v_1 \\ \therefore 0.0005v_x &= -0.11v_1 \end{aligned}$$

Now writing KCL at the node where v_1 is assigned,

$$\begin{aligned} 0.2 \times 10^{-3} v_1 + (1 - (-0.11)) \times 1 \times 10^{-3} v_1 + 0.02(v_1 - v_s(t)) &= 0 \\ \text{i.e., } 0.02131v_1 &= 0.02v_s(t) \\ \therefore v_1 &= 0.9385v_s(t). \end{aligned}$$

Since $v_x = -222.2v_1$, $v_x = -208.5v_s(t)$

Therefore, $v_{oc} = -208.5v_s(t)$.

When the terminals a-a' are shorted, $v_x = 0$ and therefore the dependent voltage source at the input side is zero-valued. The value of v_1 under this condition is given by, $v_1 = \frac{5k // 1k}{50 + 5k // 1k} v_s(t) = 0.9434v_s(t)$.

Now the current i_x is $0.9434 \times 10^{-3} v_s(t)$ and hence the current in the dependent source at the output side is $0.09434v_s(t)$. All this current flows out of 'a' to 'b' through the short-circuit. Hence, $i_{sc}(t) = -0.09434v_s(t)$. Therefore,

$$R_o = \frac{v_{oc}}{i_{sc}} = \frac{208.5v_s(t)}{0.09434v_s(t)} = 2.21\text{ k}\Omega.$$

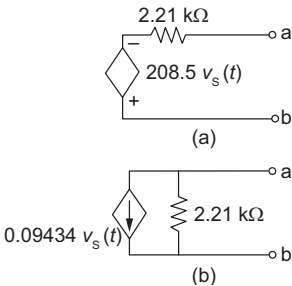


Fig. 5.6-2 (a) Thevenin's Equivalent Circuit for the Amplifier in Example 5.6-1 (b) Norton's Equivalent Circuit

The two equivalent circuits are shown in Fig. 5.6-2.

If a load resistance of $2\text{ k}\Omega$ is connected at the output, the output voltage will be $-208.5 v_s(t) \times \frac{2}{2 + 2.21} = -99.05 v_s(t)$. Hence, the ratio between output and input (i.e., the gain of the amplifier) is -99.05 .

EXAMPLE: 5.6-2

The equivalent circuit of DC current source realised using a transistor and few resistors is shown in Fig. 5.6-3. The design is expected to deliver -2 mA at 'a'. Find the Norton's equivalent circuit for this current source design.

SOLUTION

We find the short-circuit current at output first. The circuit for this is shown in Fig. 5.6-4.

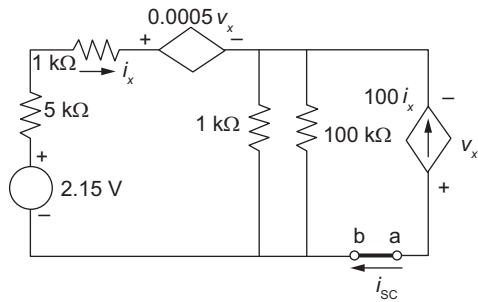


Fig. 5.6-4 Circuit for Determining Short-circuit Current for Example 5.6-2

We solve this circuit by mesh analysis. The current into $1\text{ k}\Omega/100\text{ k}\Omega$ ($= 0.99\text{ k}\Omega$) is $101i_x$. We will assume that the unit of i_x is in mA. Therefore, the KVL in the first mesh is

$$\begin{aligned} -2.15 + 6 \times i_x + 0.0005 \times -(0.99 \times 101i_x) + 0.99 \times 101i_x &= 0 \\ \text{i.e., } 105.94i_x = 2.15 \Rightarrow i_x &= 0.0203\text{ mA} \\ \text{Therefore, } i_{sc} = -100i_x &= -2.03\text{ mA}. \end{aligned}$$

To find R_o

Let us assume that we are injecting 1 mA into the network from terminal 'a' after deactivating the circuit. We determine the voltage v_{ab} . This voltage directly gives R_o in $\text{k}\Omega$ units. Refer to the circuit in Fig. 5.6-5(a).

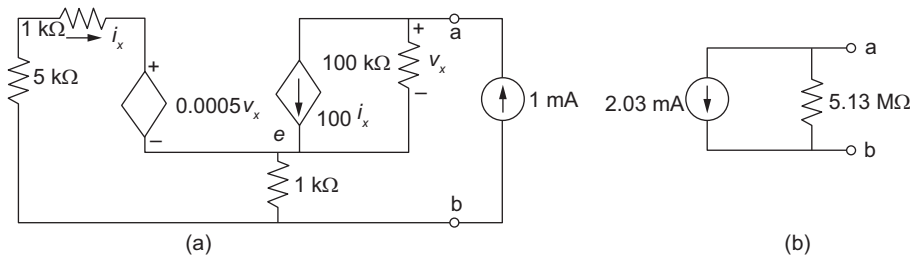


Fig. 5.6-5 (a) Circuit for Determining Thevenin's Equivalent Resistance in Example 5.6-2 (b) The Norton's Equivalent Circuit Required

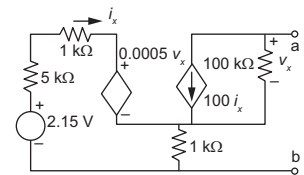


Fig. 5.6-3 Circuit for Example 5.6-2

Let i_x be in mA. The current in $100\text{ k}\Omega$ resistor is $1 - 100i_x$ (by applying KCL at node-a) and therefore $v_x = 100 - 10^4 i_x$ V. The current into $1\text{ k}\Omega$ resistor at node-e is $(1 + i_x)$ mA and hence voltage of node-e with respect to node-b is $(1 + i_x)$ V. Now applying KVL in the first mesh,

$$5i_x + i_x + 0.0005 \times (100 - 10^4 i_x) + (1 + i_x) = 0, \therefore i_x = -0.5025\text{ mA}$$

$$\therefore v_x = 100 - 10^4 i_x = 5125\text{ V and } \therefore v_{ab} = v_x + v_e = 5125 + (1 - 0.5025) = 5125.5\text{ V}$$

$$\Rightarrow R_o = 5125.5\text{ k}\Omega.$$

If we certainly apply 1 mA into the output of the transistor circuit, either the current source we are applying will fail to function as a current source or the transistor will fail on over-voltage. But then, this is only a 'thought experiment' aimed at evaluating R_o . The Norton's equivalent circuit for this current source design is shown in Fig. 5.6-5(b).

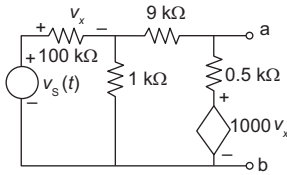


Fig. 5.6-6 Circuit for Example 5.6-3

EXAMPLE: 5.6-3

Find the Thevenin's equivalent of the circuit in Fig. 5.6-6 with respect to 'a' and 'b'.

SOLUTION

We find the open-circuit voltage across 'a-b' first. Assume two mesh currents i_1 (mA) and i_2 (mA) in the clockwise direction in the first and second mesh, respectively. The two mesh equations are,

$$100i_1 + (i_1 - i_2) = v_s(t)$$

$$(i_2 - i_1) + 9i_2 + 0.5i_2 + 1000 \times 100i_1 = 0.$$

Solving these two equations we get, $i_1 = \frac{v_s(t)}{9624.7}$ and $i_2 = -0.9895 v_s(t)$.

$$\therefore v_{oc} = 0.5i_2 + 1000 \times 100i_1 = 9.9v_s(t).$$

To find R_o

We assume that 1 V is applied across 'a-b' after deactivating the circuit and find the current drawn by the circuit from this 1 V source. The value of current gives G_o . The circuit required to solve for R_o is shown as circuit in Fig. 5.6-7(a).

The current drawn from 1 V has two components – one flowing into the $9\text{ k}\Omega$ resistor and the second flowing into $0.5\text{ k}\Omega$ path. The first component is $1/(9 + 1/100) = 1/9.99 = 0.1\text{ mA}$. The value of v_x is given by applying voltage division principle as $-1 \times (100/1)/(9 + 100/1) = -0.099\text{ V}$. Therefore, the voltage across $0.5\text{ k}\Omega$ resistor is $1 - (-0.099 \times 1000) = 100\text{ V}$. Therefore, the current flowing into $0.5\text{ k}\Omega$ path is $100/0.5 = 200\text{ mA}$. Then, the total current drawn from 1 V source is 200.1 mA and the value of G_o is 0.2 S . Therefore, the value of R_o is $5\text{ }\Omega$.

The Thevenin's equivalent for the circuit is shown in Fig. 5.6-7(b).

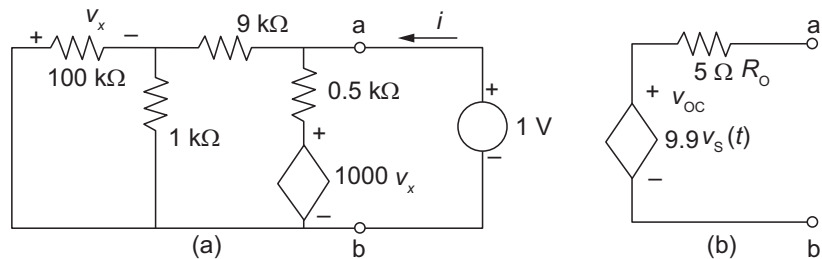


Fig. 5.6-7 (a) Circuit for Determining R_o for Example 5.6-3 (b) The Thevenin's Equivalent

5.7 RECIPROCITY THEOREM

The nodal conductance matrix and mesh resistance matrix of a memoryless circuit without any dependent sources in it (i.e., a pure resistive circuit) are symmetric matrices. Reciprocity Theorem for resistive circuits is a restatement of this fact.

Let us consider a pure resistive circuit with only one independent current source driving it as shown in circuit of Fig. 5.7-1(a).

A current source of value I is connected across a node-pair 'i' and 'j'. Two other nodes – 'k' and 'm' – form a node-pair across which the voltage can be measured. There is no other independent source or dependent source inside the circuit in the box.

The nodal analysis formulation of this circuit will result in a matrix equation $\mathbf{YV} = \mathbf{I}$, where $\mathbf{Y}$ is a symmetric nodal conductance matrix, $\mathbf{V}$ is a column vector of node voltages and $\mathbf{I}$ is the column vector containing the net current injection at nodes. In this case $\mathbf{I}$ will contain I in the i^{th} row and $-I$ in the j^{th} row. All other entries will be zero. Let $\mathbf{A} = \mathbf{Y}^{-1}$. Then, $\mathbf{A}$ will be a symmetric matrix since $\mathbf{Y}$ is a symmetric matrix. We can write the node voltage vector in terms of $\mathbf{A}$ as $\mathbf{V} = \mathbf{AI}$. But $\mathbf{I}$ contain non-zero entries only in the i^{th} row and in the j^{th} row. Therefore, the node voltages v_k and v_m can be written as,

$$\begin{aligned} v_k &= a_{ki}I - a_{kj}I \\ v_m &= a_{mi}I - a_{mj}I, \end{aligned}$$

where a_{ki} is the element in k^{th} row and i^{th} column. The other 'a' values also have same interpretation. We can now express the voltage between the two nodes as

$$v_{km} = v_k - v_m = [(a_{ki} + a_{mj}) - (a_{kj} + a_{mi})]I.$$

Therefore, the ratio of response measured to excitation applied is,

$$= [(a_{ki} + a_{mj}) - (a_{kj} + a_{mi})]. \quad (5.7-1)$$

Now, consider the circuit in Fig. 5.7-1(b). The location of excitation and response are interchanged. The current source is applied across the node-pair 'k' and 'm' and the voltage response is measured between the node-pair 'i' and 'j'.

Now the current injection vector $\mathbf{I}$ will have non-zero entries only in k^{th} row ($= I$) and in m^{th} row ($= -I$). Therefore, we can express the node voltages at node-i and node-j as,

$$\begin{aligned} v_i &= a_{ik}I - a_{im}I \\ v_j &= a_{jk}I - a_{jm}I \end{aligned}$$

and the voltage between the two nodes as

$$v_{ij} = v_i - v_j = [(a_{ik} + a_{jm}) - (a_{jk} + a_{im})]I.$$

Therefore, the ratio of response measured to excitation applied is,

$$= [(a_{ik} + a_{jm}) - (a_{jk} + a_{im})]. \quad (5.7-2)$$

$\mathbf{A}$ is a symmetric matrix. Therefore, $a_{ik} = a_{ki}$, $a_{mj} = a_{jm}$, $a_{jk} = a_{kj}$ and $a_{im} = a_{mi}$. Therefore, the ratios given by Eqns. 5.7-1 and 5.7-2 are equal.

Does it matter when the two ratios in Eqns. 5.7-1 and 5.7-2 were calculated? For instance, can we calculate the ratio in Eqn. 5.7-1 at t and the other ratio at a different instant t' ? The answer, of course, is yes – provided the entries in $\mathbf{A}$ (i.e., $\mathbf{Y}^{-1}$) matrix are time-invariant quantities. Hence, the circuit has to be a 'linear time-invariant resistive' for Reciprocity Theorem to work.

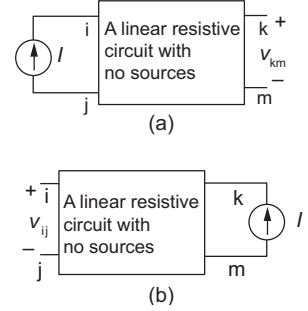


Fig. 5.7-1 Circuits to Illustrate Reciprocity Theorem

First form of Reciprocity Theorem

The ratio of voltage measured across a pair of terminals to the excitation current applied at another pair of terminals is invariant to an interchange of excitation terminals and

Statement of first form of Reciprocity Theorem.

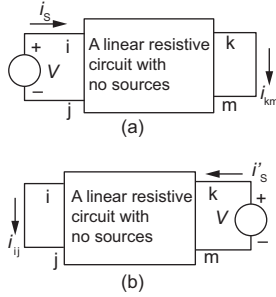


Fig. 5.7-2 Illustrating Second Form of Reciprocity Theorem

response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.

The second form can be obtained by considering a dual situation shown in Fig. 5.7-2.

It is possible to show that the ratio $\frac{i_{km}}{V}$ is same as $\frac{i'_{ij}}{V}$. It is easy to show this for a planar network using a mesh analysis formulation and exploiting the symmetry properties of mesh resistance matrix. In that case, we view 'i, j, k and m' as mesh identifiers. One has to view the voltage source as participating in i^{th} and j^{th} meshes and the shorting link participating in k^{th} and m^{th} meshes and use an argument similar to the one we used in the case of first form of Reciprocity Theorem. However, $\frac{i_{km}}{V}$ will be equal to $\frac{i'_{ij}}{V}$ even for a non-planar resistive network and mesh analysis does not help us with non-planar networks.

It is possible to show that $\frac{i_{km}}{V}$ will be equal to $\frac{i'_{ij}}{V}$ using nodal analysis formulation too. In that case, we view i, j, k and m as node identifiers. We view the voltage source as connected between i^{th} and j^{th} nodes and shorting link between k^{th} and m^{th} nodes in circuit of Fig. 5.7-2(a). Then, we impose the constraints that $v_i - v_j = V$ and $v_k - v_m = 0$ with a current injection of i_s at i^{th} node, $-i_s$ at j^{th} node, $-i_{km}$ at k^{th} node and i_{km} at m^{th} node. This will result in two equations in two unknowns i_{km} and i_s . We solve for i_{km} . The procedure is repeated for circuit in Fig. 5.7-2(b) and solution for i'_{ij} is obtained. Note that i_s will not be the same as i'_s . Comparison of expressions for i_{km} and i'_{ij} for the same applied voltage will reveal that they are equal due to symmetry of Y^{-1} matrix.

We skip the details and state the second form of Reciprocity Theorem.

Second form of Reciprocity Theorem

The ratio of current measured in a short-circuit across a pair of terminals to the excitation voltage applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.

The third and last form of this theorem can be obtained by considering the circuits shown in Fig. 5.7-3.

We calculate the ratio $\frac{v_{km}}{V}$ in the circuit in Fig. 5.7-3(a) first.

$$\begin{aligned} \text{Node voltage at node-i} &= v_i = a_{ii} i_s - a_{ij} i_s \\ \text{Node voltage at node-j} &= v_j = -a_{ji} i_s + a_{jj} i_s \\ \text{Node voltage at node-k} &= v_k = a_{ki} i_s - a_{kj} i_s \\ \text{Node voltage at node-m} &= v_m = a_{mi} i_s - a_{mj} i_s \end{aligned}$$

The node voltages at node-i and node-j are constrained to have a difference of V .

$$\therefore a_{ii} i_s - a_{ij} i_s - (-a_{ji} i_s + a_{jj} i_s) = V \Rightarrow i_s = \frac{V}{a_{ii} + a_{jj} - a_{ij} - a_{ji}}$$

Substituting this expression for i_s in the equations for v_k and v_m we get the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals as

$$\frac{v_{km}}{V} = \frac{(a_{ki} - a_{mi}) + (a_{mj} - a_{kj})}{a_{ii} + a_{jj} - a_{ij} - a_{ji}} \quad (5.7-3)$$

Now we calculate $\frac{i'_{ij}}{I}$ in the circuit in Fig. 5.7-3(b).

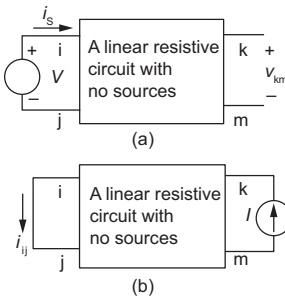


Fig. 5.7-3 Circuit Illustrating Third Form of Reciprocity Theorem

Statement of second form of Reciprocity Theorem.

$$\begin{aligned} v_i &= a_{ii}i_{ij} - a_{ij}i_{ij} + a_{ik}I - a_{im}I \\ v_j &= -a_{ji}i_{ij} + a_{jj}i_{ij} + a_{jk}I - a_{jm}I \end{aligned}$$

But $v_i = v_j$.

$$\therefore 0 = (a_{ii}i_{ij} - a_{ij}i_{ij} + a_{ik}I - a_{im}I) - (-a_{ji}i_{ij} + a_{jj}i_{ij} + a_{jk}I - a_{jm}I).$$

Solving this, we get the ratio of current measured in short-circuit across the first pair of terminals to the current source applied across the second pair of terminals as

$$\frac{i_{ij}}{I} = \frac{(a_{ik} - a_{im}) + (a_{jm} - a_{jk})}{a_{ii} + a_{jj} - a_{ij} - a_{ji}} \quad (5.7-4)$$

Comparing the two expressions in Eqn. 5.7-3 and Eqn. 5.7-4 and using the symmetry of $\mathbf{A}$ (i.e., the inverse of nodal conductance matrix) we see that the two ratios are equal.

Third form of Reciprocity Theorem

The ratio of current measured in a short-circuit across first pair of terminals to the excitation current applied at the second pair of terminals is same as the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals in the case of a linear time-invariant resistive circuit with no independent sources inside. (Refer Fig. 5.7-3 for polarity of currents and voltages)

Statement of third form of Reciprocity Theorem.

Reciprocity Theorem is not used in routine circuit analysis as frequently as Superposition theorem and Thevenin's and Norton's theorems are. However, it comes in handy in the analysis of two-port networks. Sometime it helps to ease measurement issues in circuits. It is also valid for circuits containing linear inductor, capacitors and mutual inductors. We will prove that when we take up dynamic circuits for detailed study in later chapters.

Note that the key to Reciprocity theorem is that (i) the nodal conductance matrix $\mathbf{Y}$ (and mesh resistance matrix $\mathbf{Z}$) of the circuit must be time-invariant and *symmetric* and (ii) excitation should be applied only at terminals identified, i.e., there should not be independent sources present within the circuit. $\mathbf{Y}$ and $\mathbf{Z}$ matrices of a circuit containing *linear two-terminal time-invariant resistors* will be symmetric. Hence, such circuits will obey all the three forms of Reciprocity Theorem unconditionally.

Dependent sources, even if they are linear, bilateral and time-invariant, *can* make these matrices asymmetric. *But they need not do so always.* There can be dependent sources in the circuit and yet the circuit may have symmetric $\mathbf{Y}$ and $\mathbf{Z}$ matrices. Reciprocity Theorem will hold for such circuits too, as in Example 5.7-1

EXAMPLE: 5.7-1

Show that Reciprocity Theorem is valid for the circuit in Fig. 5.7-4.

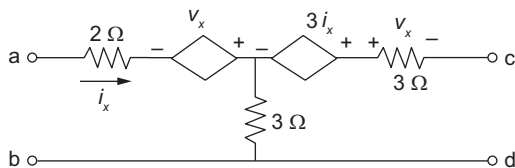


Fig. 5.7-4 Circuit for Example 5.7-1

SOLUTION

We find the mesh resistance matrix of the circuit first and verify whether it is symmetric and time-invariant. We know that the mesh resistance matrix of a circuit can be found from its deactivated version. Since excitation can be applied only across 'a-b' and 'c-d' we short these two ports (since for mesh analysis voltage source is the excitation source) and get the circuit in Fig. 5.7-5. The mesh currents are identified in it.

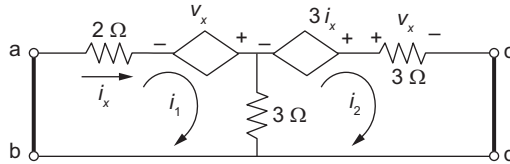


Fig. 5.7-5 Circuit to Obtain $\mathbf{Z}$ Matrix for Example 5.7-1

The mesh equations are written for the two meshes after observing that $i_x = i_1$ and $v_x = 3i_2$.

$2i_1 - 3i_2 + 3(i_1 - i_2) = 0$ and $3(i_2 - i_1) - 3i_1 + 3i_2 = 0$ are the mesh equations. Therefore, the mesh resistance matrix $\mathbf{Z}$ is,

$$\mathbf{Z} = \begin{bmatrix} 5 & -6 \\ -6 & 6 \end{bmatrix} \text{ and it is a symmetric time-invariant matrix. Therefore, Reciprocity}$$

theorem will be valid in the circuit.

We verify the first form by using the circuit configurations shown in Eqn. 5.7-4. A 1 A independent current source is used to drive the 'a-b' terminal pair first and the voltage v_{cd} is noted. Then the same current source is used to drive the 'c-d' terminal pair and the voltage v_{ab} is noted. We expect to see that $v_{ab} = v_{cd}$.

The second mesh current in circuit in Fig. 5.7-6(a) is zero. First mesh current is 1 A. Applying KVL in the second mesh, $-3 \times 1 - 3 \times 1 + 3 \times 0 + v_{cd} = 0 \Rightarrow v_{cd} = 6 \text{ V}$.

The first mesh current in circuit in Fig. 5.7-6(b) is zero. Second mesh current is -1 A. Applying KVL in the first mesh, $-v_{ab} + 2 \times 0 - (3 \times -1) + 3 \times 1 = 0 \Rightarrow v_{ab} = 6 \text{ V}$.

Thus, we see that first form of the theorem holds in this circuit. It may be verified in a similar manner that the other two forms are also valid in this circuit.

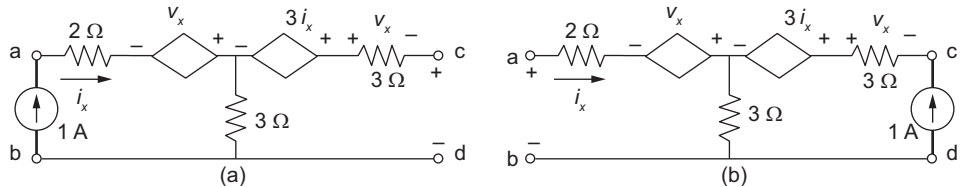


Fig. 5.7-6 Circuits for Verifying Reciprocity Theorem for Example 5.7-1

5.8 MAXIMUM POWER TRANSFER THEOREM

All electrical and electronic circuits fall under one of the three broad categories – power generation and delivery circuits, power conditioning circuits and signal generation and conditioning circuits.

In a power delivery context, one part of the circuit acts as a power source and delivers power to the other part of the circuit. In the process of delivering power to load part of the circuit, the source part of the circuit ends up dissipating some of the power within itself. This compromises the efficiency of power delivery as well as the power availability to the load at the same time. Hence, the power delivery capability of source part of the circuit for a

given load circuit is of crucial practical significance – both in high-power electrical circuits (kW to 100s of MW) and low-power electronic circuits (pW to 100s of W). We address the issue of power delivery capability of a source circuit in this section.

Figure 5.8-1 shows a linear time-invariant memoryless circuit containing one or more independent DC sources delivering power to a load circuit which may be linear or non-linear. It is assumed that the constraints required for applying Thevenin's theorem are satisfied by the entire circuit – that is, the circuits in Fig. 5.8-1(a) and (b) have unique solution and there is no interaction between the delivery circuit and load circuit other than through the common terminals. Then, we can replace the power delivery circuit by its Thevenin's equivalent comprising an open-circuit voltage in series with the Thevenin's equivalent resistance.

We state the *Maximum Power Transfer Theorem* when the power delivery circuit is a linear time-invariant memoryless circuit containing one or more independent DC sources.

'The power delivered by a linear time-invariant memoryless circuit containing independent DC sources is a maximum of $\frac{v_{oc} i_{sc}}{4}$ W when it is delivering $\frac{i_{sc}}{2}$ to the load, where v_{oc} is the open-circuit voltage in its Thevenin's equivalent and i_{sc} is the short-circuit current in its Norton's equivalent'.

$$p = vi = [v_{oc} - R_o i]i,$$

$$\therefore \frac{dp}{di} = v_{oc} - 2R_o i.$$

Equating the derivative of power with respect current to zero, we get the condition for maximum power transfer as $i = \frac{v_{oc}}{2R_o} = \frac{i_{sc}}{2}$ (since $R_o = \frac{v_{oc}}{i_{sc}}$). The value of maximum power transferred to load circuit is $= \left(v_{oc} - \frac{R_o i_{sc}}{2} \right) \times \frac{i_{sc}}{2} = \frac{v_{oc} i_{sc}}{4}$ W. This transfer will take place with a load voltage of $\frac{v_{oc}}{2}$ V and load current of $\frac{i_{sc}}{2}$ A. The internal dissipation inside the power delivery circuit under this condition will be the same as the power transferred and the efficiency of power delivery will be 50%.

If the load circuit is a single resistor of value R_L , then the condition for maximum power transfer reduces to $R_L = R_o$ and the maximum power transferred will be $\frac{v_{oc}^2}{4R_L}$ W.

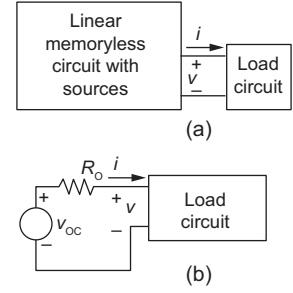


Fig. 5.8-1 (a) The Power Delivery Context (b) Power Delivery Circuit Replaced by its Thevenin's Equivalent

Statement of Maximum Power Transfer Theorem.

EXAMPLE: 5.8-1

A DC voltage source of 200 V with an internal resistance of $2\ \Omega$ delivers power to another DC source in series with a resistor R . The value of this DC source is 50 V and the two sources oppose each other in the circuit. Find the value of R and the power transferred to load circuit if maximum power transfer is to take place.

SOLUTION

Maximum power transfer takes place when the source is delivering half its short-circuit current, i.e., 100 A in this case.

The voltage across the load circuit under this current flow has to be half of the open-circuit voltage of the source.

Therefore, the 50 V source and R together should absorb 100 V when 100 A is flowing through them. Then R must be $0.5\ \Omega$.

The maximum power transferred will be $100\text{ V} \times 100\text{ A} = 10\text{ kW}$ of which 5 kW will go into the 50 V DC source and 5 kW will be dissipated in $0.5\ \Omega$ resistor. The internal dissipation of the 200 V source will be 10 kW.

It should be obvious that we do not intend to load an electrical power source to the maximum power transfer level since that will result in an inefficient operation. But, a comparison between the maximum power that the source can deliver and the power actually required in the load will immediately reveal to us the level of efficiency we can hope for in the delivery system. However, when efficiency is of no concern and signal strength is of prime concern, we take care to achieve maximum power transfer condition in the circuit. This is relevant in low-power electronic circuits. The source signal may be weak and it may come through a high R_o . R_L may be very small compared to the internal resistance. Then some kind of matching circuit that makes the low R_L appear high and equal to R_o to the source will be interposed between the source and the load.

5.9 MILLMAN'S THEOREM

Consider a set of n independent voltage sources, each in series with a resistance representing its internal resistance. Assume that these n sources are connected in parallel and are supplying a common load. We wish to replace these n sources by a single independent voltage source in series with a resistance. The relevant circuit is shown in Fig. 5.9-1.

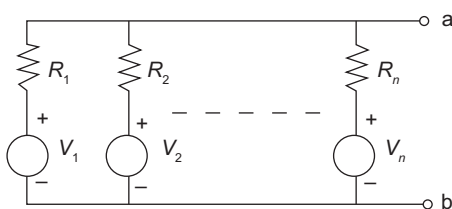


Fig. 5.9-1 Practical Voltage Sources Connected in Parallel

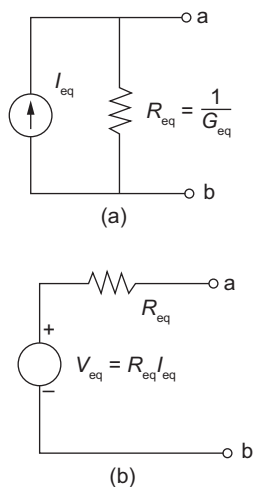


Fig. 5.9-3 (a) Current Source Equivalent (Norton's Equivalent) (b) Voltage Source Equivalent (Thevenin's Equivalent) for the Circuit in Fig. 5.9-1

We can achieve our objective by applying Source Transformation theorem repeatedly. Each voltage source is replaced by a current source in parallel with a resistor across the terminal pair 'a-b'. The current source will have a value equal to the voltage of the voltage source multiplied by the conductance of resistor in series with it. The circuit after this replacement is shown in Fig. 5.9-2.

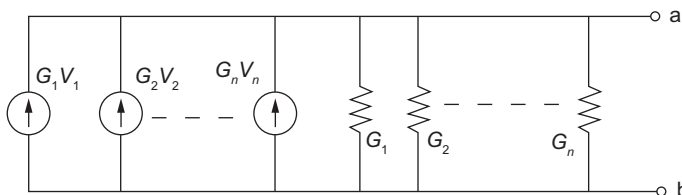


Fig. 5.9-2 Circuit in Fig. 5.9-1 After Source Transformation

$G_1, G_2, \dots, G_n$ are the conductance of the resistors. Current sources in parallel can be replaced by a single current source with a source function equal to the sum of source functions of each current source. Conductance in parallel can be replaced by a single conductance of value equal to the sum of conductance. This circuit reduction process results in a single current source in parallel with a single resistance (Fig. 5.9-3(a)). This can be replaced by a single voltage source in series with a single resistance as shown in Fig. 5.9-3(b).

Thus, n ideal independent voltage sources of voltage values $V_1, V_2, \dots, V_n$ each in series with a resistance, delivering power to a common load in parallel, can be replaced by a single ideal independent voltage source in series with a resistance. The value of voltage source is given by,

$$V_{\text{eq}} = \frac{\sum_{i=1}^n G_i V_i}{\sum_{i=1}^n G_i}; R_{\text{eq}} = \frac{1}{\sum_{i=1}^n G_i},$$

where $G_i = \frac{1}{R_i}$ for $i = 1$ to n .

This is known as Millman's Theorem. Millman's theorem is only a restatement of Source Transformation Theorem that is valid under a special context.

5.10 SUMMARY

- This chapter dealt with some circuit theorems that form an indispensable tool set in circuit analysis. Many of them were stated for linear time-invariant memoryless circuits. However, they are of wider applicability and will be extended to circuits containing inductors, capacitors and mutually coupled inductors in later chapters.
- Superposition theorem* is applicable only to linear circuits. It states that 'the response of any circuit variable in a multi-source *linear memoryless* circuit containing n independent sources can be obtained by adding the responses of the same circuit variable in n single-source circuits with i^{th} single-source circuit formed by keeping only i^{th} independent source active and all the remaining independent sources deactivated'.
- A more general form of Superposition Theorem states that 'the response of any circuit variable in a multi-source *linear memoryless* circuit containing n independent sources can be obtained by adding responses of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between such active source-subsets among them'.
- Substitution theorem* is applicable to any circuit satisfying certain stated constraints. Let a circuit with unique solution be represented as interconnection of the two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let $v(t)$ be the voltage that appears at the terminals between N_1 and N_2 and let $i(t)$ be the current flowing into N_2 from N_1 . Then, the network N_2 may be replaced by an independent current source of value $i(t)$ connected across the output of N_1 or an independent voltage source of value $v(t)$ connected across the output of N_1 without affecting any voltage or current variable within N_1 provided the resulting network has unique solution.
- Compensation theorem* is applicable to linear circuits and states that 'in a linear memoryless circuit, the *change* in circuit variables due to *change* in one resistor value from R to $R + \Delta R$ in the circuit can be obtained by solving a single-source circuit analysis problem with an independent voltage source of value $i\Delta R$ in series with $R + \Delta R$, where i is the current flowing through the resistor before its value changed'.
- Thevenin's and Norton's Theorems are applicable to linear circuits. Let a network with unique solution be represented as interconnection of the two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent voltage source of value $v_{\text{oc}}(t)$ in series with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution. $v_{\text{oc}}(t)$ is the voltage that will appear across the terminals when they are kept open and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals. This equivalent circuit for N_1 is called its *Thevenin's equivalent*.
- Let a network with unique solution be represented as interconnection of the two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent current source of value $i_{\text{sc}}(t)$ in parallel with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution. $i_{\text{sc}}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals. This equivalent circuit for N_1 is called its *Norton's equivalent*.
- Reciprocity theorem* is applicable to linear time-invariant circuits with no dependent sources.

- First form of Reciprocity theorem states that ‘the ratio of voltage measured across a pair of terminals to the excitation current applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside’.
- Second form of Reciprocity theorem states that ‘the ratio of current measured in a short-circuit across a pair of terminals to the excitation voltage applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside’.
- Third form of Reciprocity theorem states that ‘the ratio of current measured in a short-circuit across first pair of

terminals to the excitation current applied at the second pair of terminals is the same as the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals in the case of a linear time-invariant resistive circuit with no independent sources inside’.

- *Maximum power transfer theorem* is applicable to linear time-invariant circuit under steady-state conditions and states that ‘the power delivered by a linear time-invariant memoryless circuit containing independent DC sources is maximum of $\frac{v_{oc} i_{sc}}{4}$ when it delivers $\frac{i_{sc}}{2}$ to the load, where v_{oc} is the open-circuit voltage in its Thevenin’s equivalent and i_{sc} is the short-circuit current in its Norton’s equivalent’.

5.11 PROBLEMS

1. Find the change in voltage across $20\ \Omega$ resistor in Fig. 5.11-1 when the current source value increases by 0.5 A .

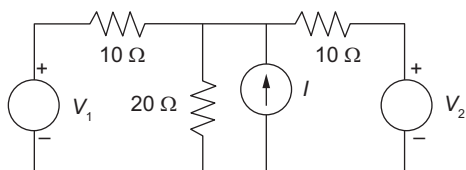


Fig. 5.11-1

2. $V_1 = V_2 = 6\text{ V}$ in the circuit in Fig. 5.11-2. (i) If v_x is found to be 8 V , find the value of current source. (ii) If V_1 increases by 1 V now and I does not change, what should be the change in V_2 such that v_x does not change?

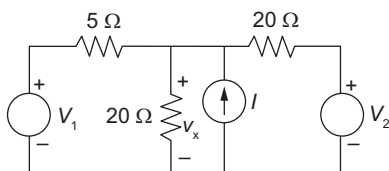


Fig. 5.11-2

3. (i) Find R in the circuit in Fig. 5.11-3 if $\frac{\partial v_x}{\partial I_1}$ is found to be 7.5 V/A . (ii) With this value of R , determine $\frac{\partial v_x}{\partial I_2}$ and $\frac{\partial v_x}{\partial V_1}$. (iii) If I_1 and I_2 change at the rate of 0.5 A/s what should be $\frac{dV_1}{dt}$ such that $\frac{\partial v_x}{\partial t} = 0\text{ V/s}$.

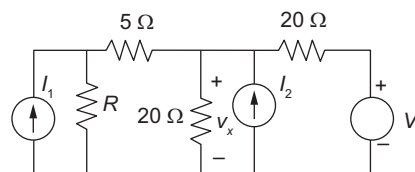


Fig. 5.11-3

4. The voltage v_x in the circuit in Fig. 5.11-4 is found to be zero when $V_2 = -2 V_1$. dv_x/dt is found to be -0.6 V/s when both source voltages change at the rate of 1 V/s . Find at least two sets of values for the three resistors that will explain these observations.

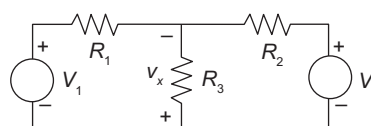


Fig. 5.11-4

5. If the two current sources are equal at all instants of time what should be $\frac{dv_s(t)}{di_s(t)}$ such that $\frac{di_x(t)}{dt} = 0$ in the circuit in Fig. 5.11-5? The value of R is $1\ \Omega$.

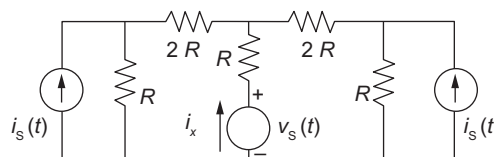


Fig. 5.11-5

6. Find (i) voltage across the $10\ \Omega$ and (ii) power dissipated in all resistors by using superposition principle in the circuit in Fig. 5.11-6.

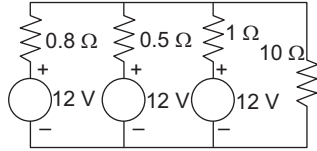


Fig. 5.11-6

7. Find (i) i_x and (ii) power dissipated in all resistors by using superposition principle in the circuit in Fig. 5.11-7.

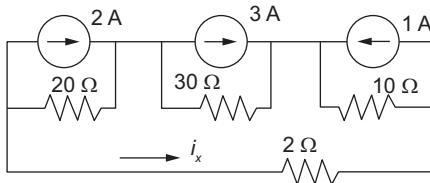


Fig. 5.11-7

8. Find the time instant at which v_x crosses zero first time after $t = 0$ in the circuit in Fig. 5.11-8.

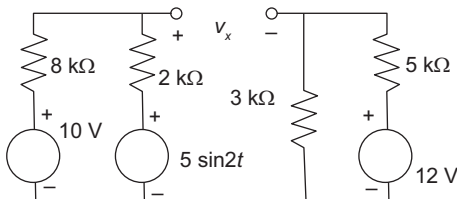


Fig. 5.11-8

9. Show that $v_o = -(v_1 + v_2 + v_3)$ and $v_x \approx 0$ as $k \rightarrow \infty$ in the circuit in Fig. 5.11-9. This is an *inverting summer circuit* that can be realised using an electronic amplifier with high gain.

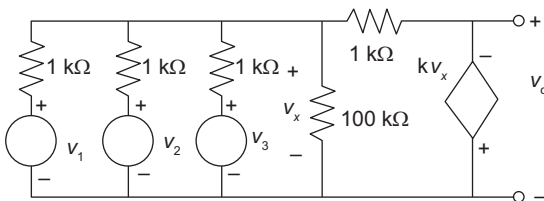


Fig. 5.11-9

10. Show that $v_o = v_1 - v_2$ and $v_x \approx 0$ as $k \rightarrow \infty$ in the circuit in Fig. 5.11-10.

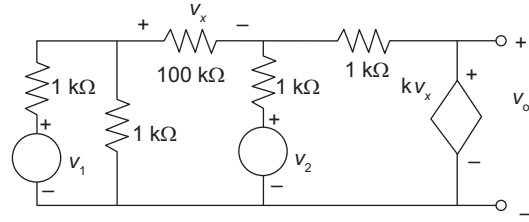


Fig. 5.11-10

11. Find v_x in the circuit in Fig. 5.11-11 by employing superposition theorem and star-delta transformation.

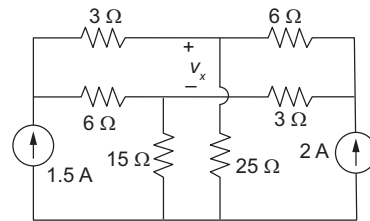


Fig. 5.11-11

12. Find i_x in the circuit in Fig. 5.11-12 by applying superposition theorem and star-delta transformation.

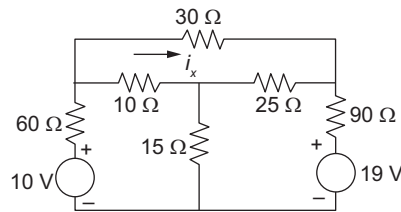


Fig. 5.11-12

13. With reference to Fig. 5.11-10, if all the $1\ \text{k}\Omega$ resistors can have values between $0.9\ \text{k}\Omega$ and $1.1\ \text{k}\Omega$ due to manufacturing tolerances find an expression for maximum deviation in v_o from the expected $v_o = (v_1 - v_2)$ in terms of v_1 and v_2 by applying compensation theorem.

14. (i) Find an expression for v_x in the circuit in Fig. 5.11-13 in terms of v_1 and v_2 by using superposition theorem. (ii) If all resistors with nominal value of $1\ \text{k}\Omega$ can have values between $0.98\ \text{k}\Omega$ and $1.02\ \text{k}\Omega$ due to manufacturing tolerances or mismatches between devices, find an expression for deviation in v_x from the expression arrived at in the first step by employing compensation theorem. The circuit is an approximate model of a *differential amplifier* using two bipolar junction transistors.

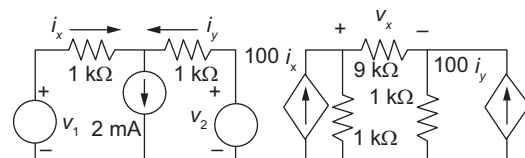


Fig. 5.11-13

15. (i) Find the value of R in the circuit in Fig. 5.11-14 if i_x is to be zero. (ii) If i_x was seen to be 0.01 mA find the value of R by using compensation theorem.

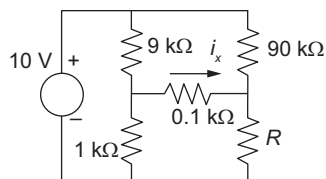


Fig. 5.11-14

16. Find the Thevenin's equivalent of the circuit in Fig. 5.11-15 with respect to terminals 'a' and 'b'.

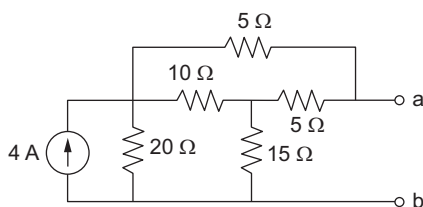


Fig. 5.11-15

17. What should be the value of R if the current through it is to be 1 A in the direction shown in the circuit in Fig. 5.11-16?

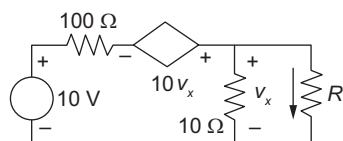


Fig. 5.11-16

18. Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.11-17 with respect to terminal pair 'a-b'.

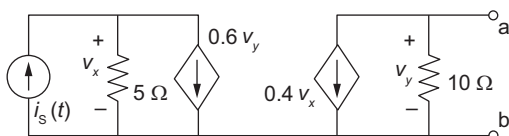


Fig. 5.11-17

19. Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.11-18.

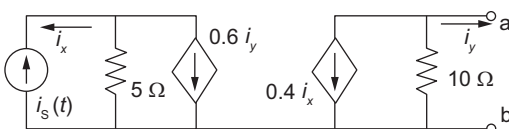


Fig. 5.11-18

20. Find the Thevenin's equivalent and Norton's equivalent for the circuit in Fig. 5.11-19 with respect to terminal pair 'a-b'.

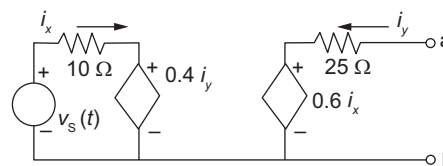


Fig. 5.11-19

21. Find the Thevenin's equivalent and Norton's equivalent for the circuit in Fig. 5.11-20 with respect to terminal pair 'a-b'.

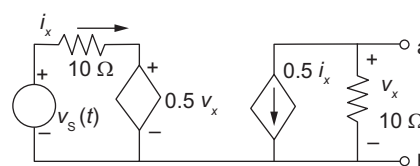


Fig. 5.11-20

22. Approximate equivalent circuit of a *Unity Gain Buffer Amplifier* (also called *Voltage Follower*) using a high gain differential amplifier (an electronic amplifier that amplifies the difference between two voltages) is shown in Fig. 5.11-21. The $200 \text{ k}\Omega$ represents the input resistance of the amplifier and $1 \text{ k}\Omega$ represents the output resistance of the amplifier. k is the gain of the amplifier. (i) Obtain the Thevenin's equivalent of the circuit in terms of $v_s(t)$ and k . (ii) Show that the Thevenin's equivalent resistance (R_o) approaches zero, $v_{oc}(t)$ approaches $v_s(t)$ and the ratio $v_s(t)/i_s(t)$ (i.e., input resistance of the circuit) approaches infinity as $k \rightarrow \infty$.

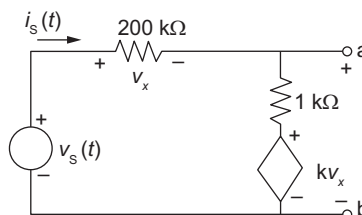


Fig. 5.11-21

23. The circuit in Fig. 5.11-22 shows the approximate equivalent circuit of the so-called *common-base amplifier* using a single bipolar junction transistor. Find the Thevenin's equivalent of the amplifier across the terminals marked 'a' and 'b'.

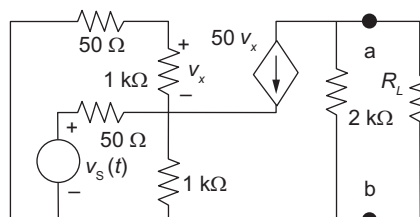


Fig. 5.11-22

24. The circuit in Fig. 5.11-23 shows the approximate equivalent circuit of the so-called *common-emitter amplifier with unbypassed emitter* using a single bipolar junction transistor. Find the Thevenin's equivalent of the amplifier across the terminals marked 'a' and 'b'.

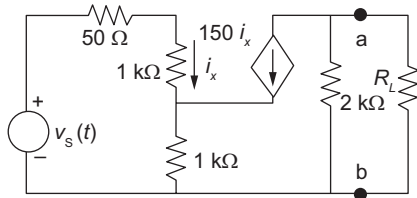


Fig. 5.11-23

25. With reference to the circuit in Fig. 5.11-24, $V = 10$ V, $I = 1$ A and $R = 10$ Ω . The value of i_x is found to be 0.5 A. The corresponding value with $R = 20$ Ω , $V = 20$ V and $I = -2$ A is -2 A. Find the value of i_x when $R = 5$ Ω , $V = 10$ V and $I = 2$ A. (Hint: Find Thevenin's equivalent using superposition principle.)

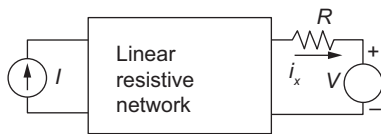


Fig. 5.11-24

26. With reference to the circuit in Fig. 5.11-25, $V = 10$ V, $I = 1$ A and $R = 10$ Ω . The value of v_x is found to be 10 V. The corresponding value with $R = 20$ Ω , $V = 20$ V and $I = -4$ A is 0 V. Find the value of v_x when $R = 5$ Ω , $V = 10$ V and $I = 2$ A. (Hint: Find Norton's equivalent using superposition principle.)

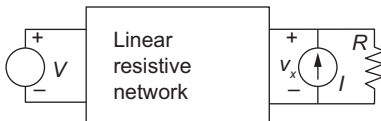


Fig. 5.11-25

27. Using the data shown in the first circuit in Fig. 5.11-26 find I_1 for second circuit. Can I_2 be found using the given data? (Hint: Reciprocity theorem + Superposition theorem.)

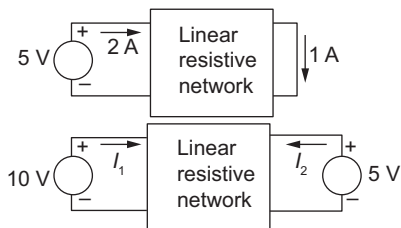


Fig. 5.11-26

28. Using the data shown in the first circuit in Fig. 5.11-27 find v_1 for the second circuit. Can v_2 be found using the given data? (Hint: Reciprocity theorem + Superposition theorem.)

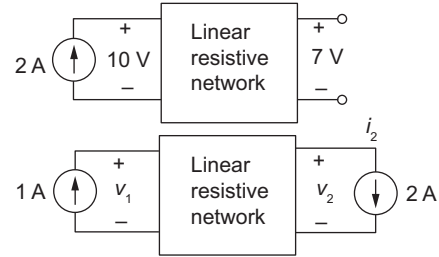


Fig. 5.11-27

29. Show that the circuit in Fig. 5.11-28 is reciprocal with respect to terminal pairs 'a-b' and 'c-d'.

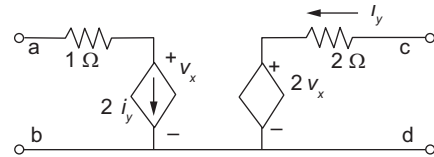


Fig. 5.11-28

30. Using the data shown in the first circuit in Fig. 5.11-29 find v_1 for the second circuit. Can i_2 be found using the given data? (Hint: Reciprocity theorem + Superposition theorem.)

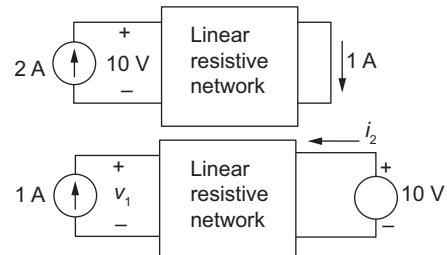


Fig. 5.11-29

31. (i) Find the value of R for maximum power transfer into it and the value of power transferred to it in the circuit in Fig. 5.11-30. (ii) Calculate the power loss in all resistors in the power delivery circuit and find efficiency of power transfer with half of the value calculated in the first step for R .

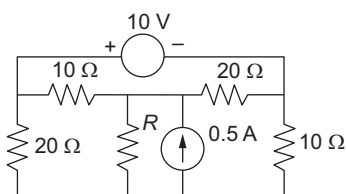


Fig. 5.11-30

32. A resistor of $20\ \Omega$ connected across 'a-b' in the circuit of Fig. 5.11-31 draws maximum power from the circuit and the power drawn is 100 W. (i) Find the value of R and I_1 . (ii) With $20\ \Omega$ across 'a-b' find the value of I_1 such that power transferred to it is 0 W.

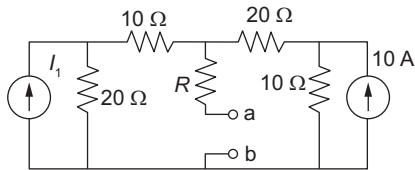


Fig. 5.11-31

33. Find the value of R for maximum power transfer in the circuit in Fig. 5.11-32 and the ratio v_x/v_s with this value of R .

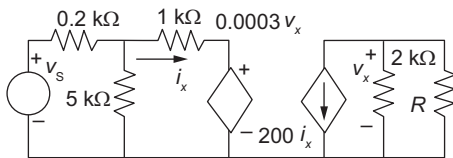


Fig. 5.11-32

34. A composite load consisting of a resistor R in parallel with a 6 V DC source in series with $3\ \Omega$ is connected across terminal pair 'a-b' in the circuit in Fig. 5.11-33. (i) Find the value of R such that maximum power is delivered to the load circuit. (ii) Find the current in the 6 V source under this condition.

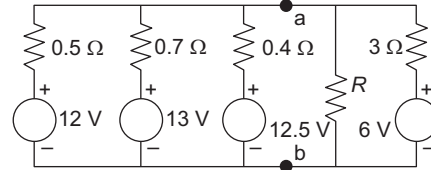


Fig. 5.11-33

35. Find R such that maximum power is transferred to the load connected to the right of 'a-b' in the circuit in Fig. 5.11-34.

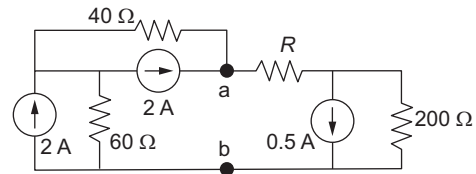


Fig. 5.11-34

The Operational Amplifier as a Circuit Element

CHAPTER OBJECTIVES

- To introduce Operational Amplifier (Opamp) as a circuit element and explain its features.
- To develop and illustrate the analysis of Opamp circuits using the Ideal Operational Amplifier (IOA) model.
- To explain the principles of operation of commonly employed linear Opamp circuits.
- To extend the IOA Model to include offset and input bias current effects.
- To illustrate the effect of voltage, current and slope limits at the Opamp output on circuit performance.

The introductory part of the chapter uses a MOSFET amplifier as an example to develop various concepts like bias point, small-signal and large-signal operation, linear and non-linear distortion, role of DC power supply, output limits, etc., in the context of amplifiers.

INTRODUCTION

We have developed certain powerful procedures of analysis and a set of powerful tools in the form of circuit theorems for memoryless circuits in the last two chapters. Memoryless circuits contain linear resistors and linear dependent sources and are driven by independent voltage sources and independent current sources. We continue our discussion on such circuits by introducing a very popular circuit element called Operational Amplifier (Opamp). It is an electronic amplifier that can be modelled by a voltage-controlled voltage source (VCVS).

We are familiar with four kinds of dependent sources – VCVS, CCVS, VCCS and CCCS. All these dependent sources are employed in modelling various kinds of electronic amplifiers. In fact, any interaction between two circuit variables that do not pertain to the same electrical element can be modelled by dependent sources. Coupled coils are modelled using dependent sources occasionally. However, the most frequent application of dependent source models occurs in modelling electronic devices and systems. Electronic amplifiers

form a sub-class of such systems. Opamp is an important electronic amplifier that senses a voltage difference and amplifies it into a voltage at the output.

We introduce this important circuit element and develop a method for analysing memoryless circuits containing Opamps in this chapter.

6.1 IDEAL AMPLIFIERS AND THEIR FEATURES

Amplification of a signal involves sensing a *signal* across a pair of terminals (if it is a *voltage signal*) or through a pair of terminals (if it is a *current signal*) and producing an identical copy of this signal across (or through) a second pair of terminals with only a *scaling* by a *real number* >1 and a possible *change in dimensions* (for example, the sensed signal may be a voltage and the amplified copy may be a current) effected.

There can be four types of amplifiers depending on the nature of the input and output signal, they are: (i) Voltage Amplifier, which senses a voltage signal and provides a voltage output, (ii) Current Amplifier, which senses a current signal and provides a current signal as its output, (iii) Trans-conductance Amplifier, which senses a voltage signal and delivers a dependent current as its output and (iv) Trans-resistance Amplifier, which senses a current signal and provides a dependent voltage as its output. All these four types of amplifiers can be linear or non-linear. We deal with the linear amplifiers, almost exclusively, in this chapter.

Amplifiers are represented by a pair of equivalent circuits – one for the input port (a port is a pair of terminals in this context) and one for the output port.

The *input equivalent* in the case of an amplifier that senses a voltage signal is the circuit that we must connect across the element whose voltage is being sensed as the amplifier input. The *input equivalent* in the case of an amplifier that senses a current signal is the circuit that we must connect in series with the element whose current is being sensed as the amplifier input. In both cases, the *input equivalent* is nothing but the Thevenin's equivalent (or Norton's equivalent) of the amplifier, as seen from its input terminals. It is possible that the connections implemented at the output side of the amplifier will influence the parameters in the *input equivalent* of the amplifier. However, in practice, it is seen that this influence is marginal, and hence, negligible. Therefore, in practice, the *input equivalent* ends up as a resistor in most cases.

The *output equivalent* in the case of an amplifier that provides a voltage output is its Thevenin's equivalent seen from the output terminals. It is the Norton's equivalent seen from the output terminals in the case of an amplifier with current output.

An ideal amplifier provides an input resistance of such a value that it does not affect the element that is sensed in any manner. And, an ideal amplifier provides an output that does not get affected in any manner, whatever be the element that is connected across the output terminals.

This implies that the *input equivalent* of an ideal amplifier that senses a voltage must be an *open-circuit* and the *input equivalent* of an ideal amplifier that senses a current must be a *short-circuit*. Moreover, the *output equivalent* of an amplifier that provides a voltage output must be a pure dependent voltage source with zero resistance in series and that of an amplifier which provides a current output must be a pure dependent current source with an open-circuit in parallel.

Figure 6.1-1 shows the equivalent circuits for the four types of amplifiers, using dependent sources to model them. The resistance that appears at the *input equivalent* of an amplifier is called the *input resistance* of the amplifier and the resistance that appears at the *output equivalent* is called the *output resistance* of the amplifier. Thus, *input resistance* is the Thevenin's equivalent resistance seen from the input terminals and *output resistance* is the Thevenin's equivalent resistance seen from the output terminals. An ideal voltage-sensing amplifier will have *infinite input resistance* and an ideal current-sensing amplifier will have *zero input resistance*. An ideal voltage-output amplifier will have *zero output resistance* and an ideal current-output amplifier will have *infinite output resistance*.

Four kinds
of Amplifiers.

Input equivalent
and output equivalent
of amplifiers.

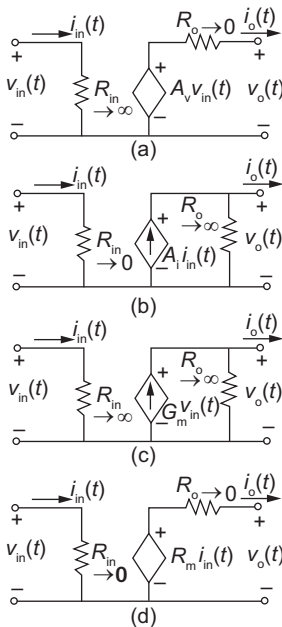


Fig. 6.1-1 Equivalent Circuits of Amplifiers
(a) Voltage Amplifier
(b) Current Amplifier
(c) Trans-conductance Amplifier
(d) Trans-impedance Amplifier

The parameters A_v (voltage gain), A_i (current gain), G_m (trans-conductance) and R_m (trans-resistance) are real constants. Therefore, the waveshape of the input and output signals will be the same in the idealised models of amplifiers. Moreover, the gain realised in ideal amplifiers is not dependent on what is connected at the output, since ideal amplifiers are represented by pure sources at the output.

Thus ideal amplifiers (i) do not affect the circuit in any way at the input connection, (ii) do not get affected in their characteristics by the load connected to them and (iii) preserve the waveshape presented to them.

However, practical amplifiers will fail on all the three aspects. They will have finite and non-zero input resistance, thereby, drawing current in parallel or developing voltage drop in series at the input connection. They have non-zero and finite output resistance, thereby, producing a load-dependent gain at the output. They have capacitance at various locations inside them, leading to different gain values for different frequencies of input signal. This kind of differential treatment to sinusoids with different frequencies leads to loss of waveshape. Therefore, the waveshape of output in a practical amplifier will be different from that of the input. This difference can only be minimised by design and cannot be eliminated altogether.

6.1.1 Ground in Electronic Amplifiers

The equivalent circuits shown in Fig. 6.1-1 show that there is a common node (represented by the bottom line in all four circuits) between the input side and the output side. This is not always so. However, the absence of a common terminal between the input side and the output side does not mean that the two sides are not conductively coupled at all. They may be conductively coupled or conductively uncoupled. The equivalent circuit of a *differential amplifier* illustrates the case where there is *conductive* coupling, but both input terminals are different from the common reference node in the circuit.

The input terminals in the circuit of Fig. 6.1-2(a) do not share a common node with the output terminals, but the voltages applied to both the input terminals are referred to a third terminal that is one of the output terminals. This kind of input-output configuration is called *differential input-single-ended output*. It is also possible to make an amplifier in which the output is taken across a pair of terminals different from the reference node. However, the point is that there has to be one node in a circuit such that all voltages may be referred to that node. This common node (similar to the reference node in nodal analysis) used to define a reference point for voltage measurement in the circuit is called the *ground* in the electronic circuit. It has nothing to do with physical earth. The symbol for *ground* is shown in the circuit in Fig. 6.1-2(b).

The input side and the output side of an amplifier need not necessarily have any conductive coupling between them. For instance, one side may be coupled to the other side by electromagnetic coupling using transformers or by optical coupling using optoelectronic devices. In such a case, the reference nodes for the input and the output side will be two nodes with no *conductive path* between them. Voltages in the input part of such a circuit can be measured only with respect to *input ground* and voltages in the output part of the circuit can be measured only with respect to *output ground*.

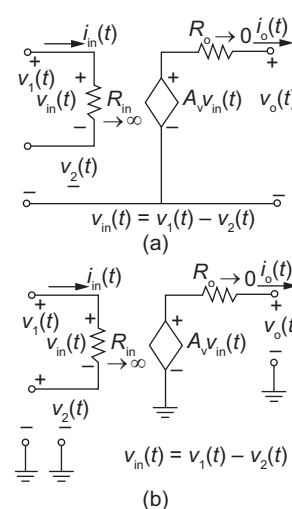


Fig. 6.1-2 A Differential Amplifier with Conductive Coupling Between Input and Output

6.2 THE ROLE OF DC POWER SUPPLY IN AMPLIFIERS

Figure 6.2-1 shows a voltage amplifier with input resistance R_{in} and output resistance R_o driven by a voltage source $v_s(t)$ delivering its output to a load resistance of R_L .

Let P_{in} be the power drawn from the input source and P_L be the power delivered to the load resistance R_L . Then,

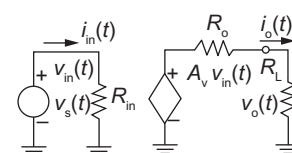


Fig. 6.2-1 Voltage Amplifier with Load

$$v_o(t) = \frac{A_v R_L}{R_L + R_o} v_s(t)$$

$$\therefore P_L = \frac{[v_o(t)]^2}{R_L} = \frac{A_v^2 R_L}{[R_L + R_o]^2} [v_s(t)]^2$$

$$\text{and } P_{in} = \frac{[v_s(t)]^2}{R_{in}}$$

$$\therefore \frac{P_L}{P_{in}} = \frac{A_v^2 R_L R_{in}}{[R_L + R_o]^2}$$

For practical values of the parameters ($A_v \gg 1$, $R_{in} \gg R_L$, $R_o \ll R_L$), this ratio will be $\gg 1$. Thus, the load gets much more power than what is drawn from the signal source. In fact, the power gain $\rightarrow \infty$ in the case of an ideal amplifier. Where does this extra amount of power come from? It comes from the DC power supply used in the amplifier. *Thus, amplifier is a circuit that draws power from the DC supply and modulates this power into a waveshape that is provided by the signal source and delivers it to the load resistance with the desired waveshape.*

6.2.1 Linear Amplification in Electronic Amplifiers

We consider an electronic amplifier using a device called a MOSFET (Metal-Oxide Semiconductor Field Effect Transistor) to bring out the process of linear amplification and to arrive at other roles of DC power supply in amplifiers. We cannot get into the details of the device physics here. However, the characteristic curves shown in Fig. 6.2-2 are sufficient for our discussion.

The symbol of MOSFET is shown in Fig. 6.2-2. We will consider it as a three-terminal circuit element for our limited scope in this section. The three terminals, called *drain*, *source* and *gate*, are identified with letters D, S and G, respectively. The *gate* of a MOSFET draws negligible current, and hence, the current flowing into D flows out of S too. This current is the *drain current* i_{ds} of the device. The value of i_{ds} in the device for a given value of *drain-source voltage* v_{ds} is governed by the *gate-source voltage* v_{gs} of the device. Hence, this is a *voltage-controlled device* and v_{gs} is the *controlling voltage*.

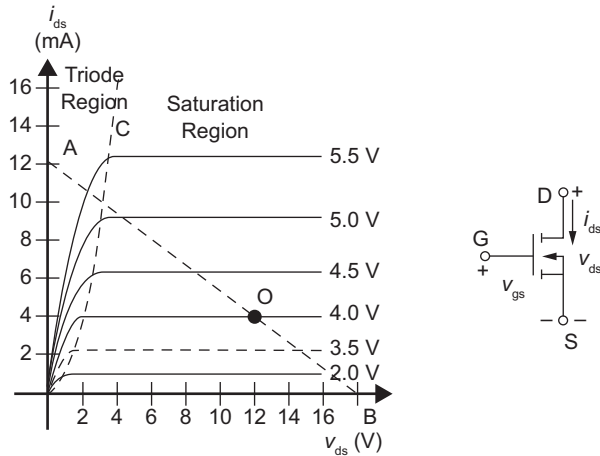


Fig. 6.2-2 Symbol of a MOSFET and Its Characteristic Curves

The characteristics shown in Fig. 6.2-2 display the variation of i_{ds} versus v_{ds} , with v_{gs} as a parameter. The device does not conduct at all until v_{gs} reaches a value V_T , called the *threshold voltage* of the device. After v_{gs} crosses V_T , the drain current increases with v_{ds} and reaches a constant value beyond a certain value of v_{ds} . The region of operation where the device current is a constant for a particular v_{gs} is called the *saturation region* and the region of operation where the drain current varies with the drain-source voltage for a fixed gate-source voltage is called the *triode region*. These operating regions are marked in Fig. 6.2-2 and C denotes the curve that demarcates these two regions.

The equations governing the drain current in a MOSFET in the two regions of operation are given below:

$$\text{Triode region: } i_{ds} = k \left[2(v_{gs} - V_T)v_{ds} - v_{ds}^2 \right], 0 \leq v_{ds} \leq (v_{gs} - V_T) \quad (6.2-1)$$

$$\text{Saturation region: } i_{ds} = k(v_{gs} - V_T)^2, 0 \leq (v_{gs} - V_T) \leq v_{ds}$$

The characteristic curves shown in Fig. 6.2-2 are for a MOSFET, where $k = 1 \text{ mA/V}$ and $V_T = 2 \text{ V}$. The v_{gs} value relevant for each curve is marked above the curves. Note that equal increments in v_{gs} do not lead to equal increments in the drain current in the saturation region. This is due to the square non-linearity present in the equations governing i_{ds} .

An amplifier is constructed using this MOSFET as shown in the circuit in Fig. 6.2-3(a). Consider the operation of the circuit when the signal $v_s(t)$ is zero. The DC source in the gate circuit will provide 4 V to the gate-source. With $k = 1 \text{ mA/V}$ and $V_T = 2 \text{ V}$, this will result in 4 mA of drain current. 4 mA of current through a $1.5 \text{ k}\Omega$ resistor results in a 6 V drop across it, and hence, the value of v_{ds} is 12 V. This 12 V is absorbed by the 12 V DC source to reduce the output voltage $v_o(t)$ to zero. The ordered pair (I_{ds}, V_{ds}) , where I_{ds} and V_{ds} are the drain current and drain-source voltage, respectively, with the signal set to zero, is called the *Quiescent Operating Point (Q-point)* of the device or *Bias Point* of the device. *One of the major roles of DC power supply in amplifiers is to set up the bias point of devices at proper locations in their operating plane to ensure satisfactory performance as a linear amplifier.*

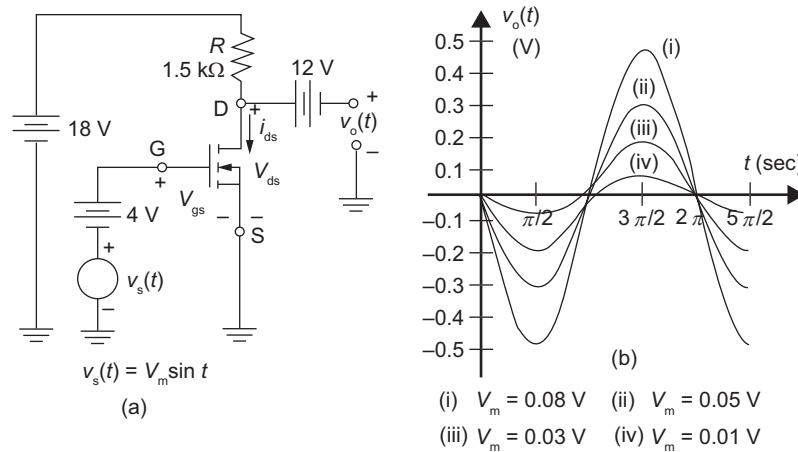


Fig. 6.2-3 (a) A MOSFET Amplifier (b) Output Waveform for Various Input Amplitude

How do we proceed to find $v_o(t)$ when a signal $v_s(t) = V_m \sin t \text{ V}$ is applied in series with a 4 V DC source in the gate circuit? Strictly speaking, we must express v_{gs} as $4 + V_m \sin t$ and substitute in Eqn. 6.2-1 along with the equation $v_{ds} = 18 - 1.5i_{ds}$ and solve for i_{ds} and v_{ds} in order to find $v_o(t)$. But the equation to be used depends on the operating region of MOSFET and we do not know where it will operate before we solve the problem! Hence, the procedure will be, to assume that it operates in one region, use the corresponding equation, solve for i_{ds} and obtain v_{ds} , thereby, and check whether the values of v_{gs} and the calculated v_{ds} justify the

assumed region of operation. If it does not, then, repeat the process after assuming the other region as the region of operation.

The equation $v_{ds} = 18 - 1.5i_{ds}$ is plotted as a straight line AOB in Fig. 6.2-2. Obviously, the operating point of MOSFET has to lie on one of its characteristic curves as well as on this line. If sufficiently large number of curves for a large set of v_{gs} values is available, the values of v_{ds} for various v_{gs} values can be read from the intersection of i_{ds} versus v_{ds} curve and the straight line AOB, and the waveshape of $v_o(t)$ can be constructed, thereby.

Let us assume that the signal is *small* – i.e., $V_m \ll V_{gs}$, where V_{gs} is the bias point value of v_{gs} . Then, we may represent v_{gs} as $V_{gs} + \Delta v_{gs}$, where $\Delta v_{gs} = V_m \sin t$ is a *small increment* or a *small-signal* around the bias point value of v_{gs} , i.e., around V_{gs} (4 V in the present case). Similarly, $i_{ds} = I_{ds} + \Delta i_{ds}$ where $I_{ds} = 4$ mA in the present case. Moreover, $v_{ds} = V_{ds} + \Delta v_{ds}$ and $v_o = V_o + \Delta v_o$, where $V_{ds} = 12$ V and $V_o = 0$ V in this example.

Substituting these in the expression for i_{ds} when the MOSFET is in the saturation region, we get,

$$\begin{aligned} I_{ds} + \Delta i_{ds} &= k[V_{gs} + \Delta v_{gs} - V_T]^2 \\ &= k(V_{gs} - V_T)^2 \left[1 + \frac{\Delta v_{gs}}{V_{gs} - V_T} \right]^2 \\ &\approx k(V_{gs} - V_T)^2 \left[1 + \frac{2\Delta v_{gs}}{V_{gs} - V_T} \right] \text{ for } \Delta v_{gs} \ll V_{gs} - V_T \end{aligned}$$

But $k(V_{gs} - V_T)^2$ is the bias point drain current itself, i.e., I_{ds} . Let us define $\frac{2I_{ds}}{(V_{gs} - V_T)}$ as g_m , the *trans-conductance* of the MOSFET. Note that g_m depends on the bias point.

$$\begin{aligned} \therefore I_{ds} + \Delta i_{ds} &\approx I_{ds} + g_m \Delta v_{gs} \\ \therefore \Delta i_{ds} &\approx g_m \Delta v_{gs} \end{aligned}$$

g_m in the present instance is calculated as 4 mA/V.

Now, $V_{ds} + \Delta v_{ds} = 18 - 1.5(I_{ds} + \Delta i_{ds})$ but $18 - 1.5I_{ds}$ is V_{ds} (12 V) itself. $\therefore \Delta v_{ds} = -1.5\Delta i_{ds}$, where Δi_{ds} is assumed to be in mA.

$$\begin{aligned} v_o &= v_{ds} - 12 \\ \therefore v_o &= V_o + \Delta v_o \\ &= V_{ds} + \Delta v_{ds} - 12 \\ &= \Delta v_{ds} \\ &= -1.5\Delta i_{ds} \\ &= -1.5g_m \Delta v_{gs} \\ &= -6\Delta v_{gs} \\ &= -6V_m \sin t \text{ V.} \end{aligned}$$

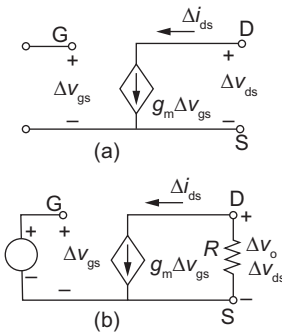


Fig. 6.2-4 (a) Small-signal Equivalent of a MOSFET (b) Small-signal Equivalent of a MOSFET Amplifier

Hence, we see that if the signal is a *small-signal*, the output is *proportional to the input*. That is, approximately linear amplification takes place in the amplifier if the magnitude of the signal is such that the operating point of the device is moved only by a small distance around its original bias point. As a consequence, we can represent the amplification process that takes place under *small-signal* or *incremental* conditions by a *small-signal linear equivalent circuit* using dependent source/s and solve for the signal quantities using this equivalent circuit. The parameters of the equivalent circuit will be strongly dependent on the bias point values of device voltages and currents.

The small-signal equivalent of a MOSFET is shown in the circuit in Fig. 6.2-4(a) and the small-signal equivalent of the amplifier using MOSFET is shown in the circuit in Fig. 6.2-4(b).

When v_{gs} increases, the drain current increases. This results in increased voltage drop in the drain resistance leading to a decrease in voltage at the drain. This explains the negative sign in the gain of the amplifier in this example. The gain is $-g_m R = -6$ V/V.

6.2.2 Large-signal Operation of Amplifiers

Figure 6.2-3 shows the output waveform of the amplifier for various values of amplitude of input sine wave. $v_o(t)$ was first calculated without employing the assumption of small-signal. Then, it was calculated using the linear model developed above. The result obtained from the linear model is shown in dotted curves. These waveforms show that the agreement between the exact result from Eqn. 6.2-1 and the approximate result from the linear model is satisfactory. Notice that the largest amplitude of input was 0.08 V, which is only 4% of $V_{gs} - V_T$ value at bias point.

The same calculations are repeated for another set of amplitude values and the resulting output waveforms are plotted in Fig. 6.2-5.

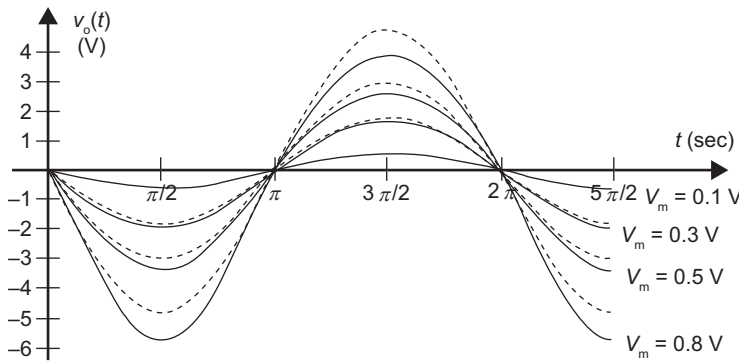


Fig. 6.2-5 Output Waveforms of MOSFET Amplifier Showing Non-linear Distortion

The dotted curves represent the waveforms predicted by the linear model and solid curves are the result of exact calculation using Eqn. 6.2-1. The deviation between the two is tolerable up to $V_m = 0.3$ V, i.e., for signals up to 15% of bias value of $V_{gs} - V_T$. When signal excursions are beyond that, we observe that the linear model is an overestimate in the negative half-cycle of the input and it is an underestimate in the positive half-cycle (remember polarity inversion). More importantly, we observe that the output waveform becomes (i) *progressively more and more non-sinusoidal* and (ii) *progressively more and more asymmetric about time-axis*, as signal amplitude increases. This departure of waveshape of the output from that of the input is due to the non-linear relationship between i_{ds} and v_{gs} of a MOSFET. When we lose the waveshape on amplifying a signal, we state that the signal underwent *waveform distortion*, and, when that distortion is due to non-linearity of devices, we call it *non-linear distortion*. Thus, *large-signal operation of amplifiers usually results in non-linear distortion*.

6.2.3 Output Limits in Amplifiers

As the signal voltage amplitude is increased further, there comes a situation where the MOSFET finds itself with a $v_{gs} < V_T$ during the negative half-cycle of input. This happens when $V_m \geq 2$ V in the example we have been using in this section. Then, the MOSFET remains *cut-off* with zero drain current for the duration for which $v_{gs}(t) \leq 2$ V and consequently the output voltage stays put at $18 - 12 = 6$ V. That is, the output voltage does not respond any further than +6 V in the positive half-cycle (during negative half-cycle of input) and the waveform remains clipped at 6 V as long as v_{gs} is ≤ 2 V.

The MOSFET enters the triode region of operation beyond a certain value of input during the positive half-cycle of a large amplitude input signal. This value may be calculated as 1.15 V in the example. The device is in the triode region for inputs greater than this value.

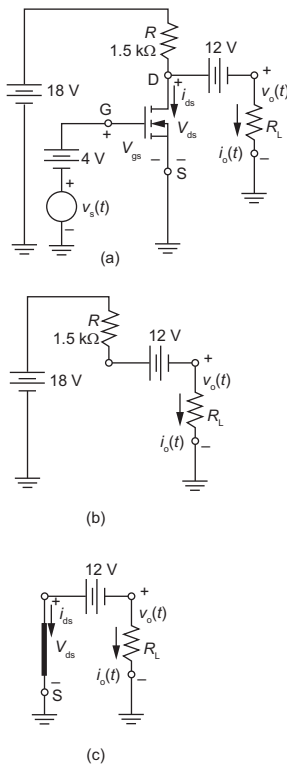


Fig. 6.2-6 (a) MOSFET Amplifier with Load (b) Circuit Under Cut-off Condition (c) Circuit Approximation When MOSFET is in Triode Region and is Conducting Heavily

The response of drain current to gate-source voltage decreases when the device is at the triode region due to bunching of characteristic curves in that region. This leads to a gradual clipping of the negative half-cycle of output waveform. In the limit, when the input signal goes to a very large value, v_{ds} across the device approaches zero and $v_o(t)$ approaches -12 V.

Thus, clean clipping takes place in the positive half-cycle of the output at 6 V and rounded clipping takes place in the negative half-cycle of the output at -12 V. Therefore, the maximum positive and negative peaks at output the terminal is $+6$ V and -12 V, respectively. The *maximum peak-to-peak swing at output* is $6 - (-12) = 18$ V which is the same as the DC supply voltage. Also, note that the swing is not symmetrical. We get more swing available in one direction than in the other. The bias point has to be fixed suitably to make the swing of 18 V symmetrical around zero level.

Thus, the DC supply voltage fixes the maximum peak-to-peak voltage swing available at the output of an amplifier. This quantity can never exceed the supply voltage. If the amplifier is overdriven by a large-signal, the effective gain of amplifier for large values of input goes to zero.

DC power supplies, along with the bias point, will also decide the maximum current that the amplifier can deliver to the load. There was no load resistance connected at the output of the amplifier in the example we considered until now. Let us add a load resistance R_L at its output now as shown in the circuit in Fig. 6.2-6(a).

The device reaches cut-off condition with zero drain current when the input signal is large and negative. The equivalent circuit under this condition is shown in the circuit of Fig. 6.2-6(b) and it is clear from the equivalent circuit that the clipping level in the positive half-cycle of output gets modified to $6 \times R_L / (R + R_L)$ V. The maximum current that the amplifier can deliver to the output is $6/R$, which is equal to I_{ds} and this current will be delivered only when the load is shorted. Thus, in this amplifier, the bias point current sets a maximum value for the current that can be delivered at the output during the positive half-cycle.

The device enters the triode region and conducts heavily with negligible drain-source voltage, when the input is large and positive. The corresponding output equivalent circuit is shown in the circuit in Fig. 6.2-6(c). There is no limit (except the maximum current capability of the device) for the current that can be delivered at the output under this condition, provided the device does not enter saturation mode. However, the output current will be limited in both directions in amplifier designs employing symmetry and the limits will be set by the bias point values.

Parasitic capacitance is always present across any two terminals of any electrical circuit. In addition, the load itself may have a shunt capacitance. The limited availability of output current from an amplifier assumes special significance under capacitive loading conditions. It leads to *rate limitations* on the amplifier output. The rate of change of voltage across a capacitor is directly proportional to the current flow through it. Hence, the maximum rate of change of output voltage under capacitive loading conditions gets limited by the value of capacitance and output current limits. Thus, the DC supply, along with the bias point design, sets definite limits on the maximum rate of change of output voltage that may appear at its output. The amplifier will not be able to follow the rapidly changing input waveforms due to this rate limitation. When called upon to do so, it does its best and moves its output at the maximum rate that it is capable of. The maximum rate at which an amplifier output voltage can change is termed as its *slew rate*. The *slew rate* may be different for the two directions of change.

We conclude this section by reiterating the major points.

- DC power supplies are needed to fix the Quiescent Operating Point of the devices in an electronic amplifier. The Q-point (along with loading details) will decide the gain of the amplifier, the maximum peak-to-peak swing at output and the maximum symmetrical swing at the output before clipping takes place at any one end. Maximum peak-to-peak swing cannot exceed the value of power supply voltage.
- Maximum current available at the output of an amplifier is limited and the positive and the negative limits are also set by DC supplies through bias point details.

- Maximum rate of change that the output voltage of an amplifier can exhibit is also limited. The limits are again set by DC supplies through the bias point of devices.
- DC power supplies are needed in amplifiers to function as the source of power delivered to the output.
- Linear models using dependent sources can be used to model electronic amplifiers only if the operating point of various devices in the amplifier undergoes only *small* perturbations (typically <15%) around their Q-points.
- The output waveshape will suffer non-linear distortion when the input to an amplifier is a large-signal. There may be *clipping distortion* too if the input signal is large enough. Distortions can also occur due to the amplifier getting into *current-limited* or *rate-limited* modes of operation. Linear models using dependent sources *cannot predict* these phenomena. Linear models *will not be valid* if the amplifier is in *large-signal mode*, *voltage-limited mode*, *current-limited mode* or *slope-limited (rate-limited) mode*. These modes of operation will be collectively termed as *non-linear range of operation*.

6.3 THE OPERATIONAL AMPLIFIER

Ideal amplifiers are physically unrealisable. Operational Amplifiers try to approach the ideal amplifiers.

Operational Amplifier is a multi-stage high gain voltage-to-voltage amplifier with a differential input and a single-ended output. It is an integrated circuit package. We use the short name *Opamp* for Operational Amplifier in this book.

It is a *differential amplifier* and has two signal input terminals. It amplifies the difference between the voltages applied at the input terminals. These two input voltages as well as the output voltage are referred to a common *ground* terminal.

Ground in an Opamp is not a terminal or a pin of the Opamp. It is a node outside the Opamp. In Opamps working from a single DC power supply, the *ground* is commonly assigned to the negative of the DC source. In Opamps working from a *balanced dual DC power supply*, the *ground* is commonly taken as the midpoint of the dual supply. The output voltage is measured between a single output terminal from the Opamp and the *ground* terminal that is outside the Opamp package.

Various integrated circuit design techniques are employed to minimise the non-linear distortion in the output signal arising out of the non-linear transfer characteristics of transistors. However, even an Opamp cannot do away with voltage, current and rate limits. Non-linear distortion arising out of *clipping* due to one or more of these limits takes place in Opamp circuits when they are overdriven or overloaded. Thus, linear models are applicable to Opamps as long as they are not operating in any of the *limited modes*. That is, *small-signal* approximation is not needed for analysing Opamp circuits using linear models. However, *voltage saturation*, *current saturation* or *slope saturation* should not take place in the circuit if linear models are to yield the correct solution.

An ideal voltage amplifier is expected to have infinite input resistance, zero output resistance and infinite bandwidth, i.e., it does not differentiate between two sinusoidal signals of the same amplitude but different frequencies and provides same gain to both. *Bandwidth is a measure of variation of gain with frequency of an applied sinusoid.* A practical Opamp has a very large input resistance (in $M\Omega$'s) and a small output resistance (in ohms) and it has finite bandwidth. Therefore, if the input applied to it is a mixture of sinusoids, it will offer different gains for different sinusoidal components at different frequencies. It will also add a phase shift to the output, which will also depend on the frequency. These two kinds of differential treatment to sinusoids of different frequencies lead to a difference in the waveshape of the output compared to that of the input. This is *distortion*, but it is not due to non-linearity in transistors. Therefore, it is not *non-linear* distortion.

The previous section showed that an amplifier cannot escape from the limits on its output voltage, limits on its output current and limits on the rate of change of output voltage. Neither can it escape from the non-linear distortion under large-signal conditions.

An Opamp design aims at the following:

- (i) Minimise the non-linear distortion until it is forced into one or more of limited modes – voltage-limited mode, current-limited mode or slope-limited mode.
- (ii) Make the voltage limits (called *saturation voltages*) symmetrical about the middle level of DC power supply.
- (iii) Make the peak-to-peak voltage swing at the output terminal approach the total DC supply voltage employed.
- (iv) Make the current limits symmetrical.
- (v) Make the slope-limits (*slew rate*) more or less equal, for increasing output and decreasing output.

Features of an Ideal Operational Amplifier (IOA)

Ideal Opamp, obviously, cannot be made. However, it provides a benchmark for evaluating a practical Opamp.

An IOA is a voltage-to-voltage differential amplifier with infinite input resistance, infinite gain, infinite bandwidth and zero output resistance. An IOA has the following features:

1. Input resistance, $R_i \rightarrow \infty$
2. Output resistance, $R_o \rightarrow 0$
3. Voltage gain, $A_v \rightarrow \infty$
4. Bandwidth, $bw \rightarrow \infty$
5. Common Mode Rejection Ratio (CMRR) $\rightarrow \infty$
6. No voltage, current and slope limits.
7. Zero offsets and zero input bias currents.

It is due to the gain of Opamp becoming a function of frequency and that happens because of capacitance of transistors. This distortion takes place even when the Opamp is in the *linear range* of operation. Therefore, it is called *linear distortion*.

Non-linear distortion changes the waveshape of the output even when the input is a single frequency sinusoid, but linear distortion does not do so. Linear distortion changes the waveshape of the output only when the input is *not* a single frequency sine wave, but a mixture of sine waves of different frequencies. An ideal Opamp does not produce any linear distortion.

The voltage gain of a practical Opamp is very large — typically hundreds of thousands. We hardly ever need that kind of a gain in any practical application. Thus, we seldom find an Opamp being used as an amplifier without some other components (usually resistors) limiting the gain of the overall amplifier circuit to the required value (likely to be in the range 1–100). Why do we make a circuit with a huge gain and kill its gain when it is used for amplification purposes? The simple answer is that the huge gain of Opamp is the currency that we pay for improvements in other performance measures of the overall amplifier circuit. We gain on the other performance parameters by paying out gain.

For instance, the input resistance of the overall amplifier can be increased and its output resistance decreased by sacrificing the gain. Its bandwidth can be increased and non-linear distortion can be decreased by sacrificing the Opamp gain.

The circuit technique that we use in order to bring about this trade-off between gain of Opamp and performance of the overall amplifier circuit in which the Opamp is embedded is called *negative feedback*. We will take it up in a later section. However, we note here that higher the Opamp gain, better the advantages that accrue from employing negative feedback around it.

An Opamp is expected to produce zero output when both its input terminals are connected to the same voltage with respect to ground. However, practical Opamps do produce a small output under this condition. The corresponding gain is called the *common-mode gain*. The gain registered by the Opamp when it is driven by two equal, but opposite sources, at its input terminals is called the *differential gain*. The ratio of differential mode gain to common mode gain is defined as its '*Common Mode Rejection Ratio*' and is usually quoted in decibels (dB). Decibel value of a quantity is obtained by calculating 20 times the logarithm of the quantity with 10 as the base of logarithm.

An Opamp is expected to produce zero output when both its input terminals are connected to ground. However, practical Opamps do produce non-zero output under this condition. Thus, practical Opamps exhibit *output offset*.

The need for establishing proper bias points for transistors in an amplifier was already discussed. Biasing of transistors has to take place within an Opamp too. As a result, some DC currents will flow out of (or into) the signal input terminals in certain Opamp designs. These currents are called *input bias currents*.

6.3.1 The Practical Operational Amplifier

The most popular general-purpose Opamp, the $\mu A 741$, is available as a DIP (Dual-in-Line) package of $\frac{1}{4}$ inch $\times$ $\frac{3}{8}$ inch size. It has 8 pins. The pin details and circuit symbol of Opamp are shown in Fig. 6.3-1. It is a dual supply Opamp and the midpoint of the supply connection is taken as the ground usually.

This Opamp has a differential gain of 250,000 and a CMRR of 80 dB. This implies that its common mode gain is indeed negligible. Its input resistance is about 2 M Ω and output resistance is about 75 Ω .

With a supply voltage of ± 12 V, its output saturation limits are 10.6 V and -11 V. It has a bandwidth of ≈ 4 Hz (very small) and a slew rate of 0.5 V/ μ s. Its output current is limited at ± 20 mA with a supply voltage of ± 12 V.

Its input bias currents, called I_{B+} and I_{B-} , are ≈ 100 nA and the difference ($I_{B+} - I_{B-}$) between them can be ± 20 nA. This difference is called *input bias offset current*.

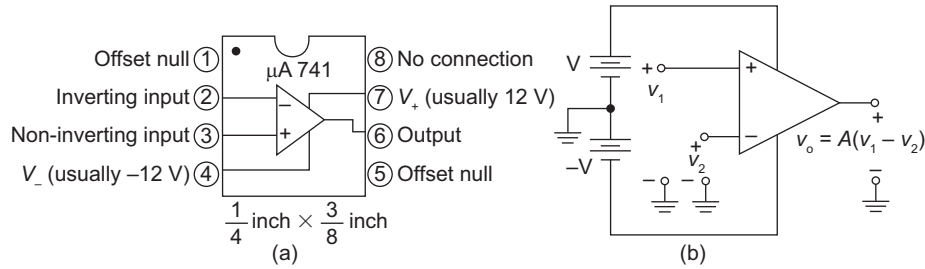


Fig. 6.3-1 (a) Top View of $\mu\text{A 741}$ IC (b) Circuit Symbol and Voltage Polarities for an Opamp

The output of this Opamp will be at one of the saturation limits when both inputs are grounded. The offset is so high. Hence, the offset voltage for Opamps is specified at the input indirectly rather than at the output directly. Applying a differential voltage across the inverting and non-inverting inputs of the Opamp can null the output offset. The differential voltage that has to be applied across the two input terminals in order to bring the output to zero with respect to ground is defined as the *input offset voltage* and denoted by v_{io} . The value for $\mu\text{A 741}$ is about 2 mV. The polarity of this voltage can be decided only by experiment on the particular piece of IC that is being used.

The circuit in Fig. 6.3-1(b) shows the power supply connections in detail. However, these connections are usually suppressed in circuit diagrams involving Opamps. It is understood implicitly that the Opamps are powered properly in circuits. This is permissible since we hardly apply KCL at power supply nodes of Opamps in analysing a circuit containing Opamps. We do not apply KCL at the ground node too, but that is because of the fact that we always assign the role of reference node to the ground node in Opamp circuits when we prepare the node equations of such circuits.

6.4 NEGATIVE FEEDBACK IN OPERATIONAL AMPLIFIER CIRCUITS

Consider the circuit in Fig. 6.4-1(a). We ignore the offsets in output in our analysis in this section. Therefore, the output in the circuit in Fig. 6.4-1(a) is given by $v_o = A(v_1 - v_2)$, where A is the differential mode voltage gain of the Opamp. The value of A for $\mu\text{A 741}$ Opamp is around 250,000. Let us assume that the power supply used is ± 12 V in all the three circuits. The voltage saturation levels will be taken approximately equal to the supply voltage of 12 V. Actually, it is 10.6 V and -11 V; we ignore the difference.

Hence, the first circuit will saturate when the differential input voltage $v_d = v_1 - v_2$ goes out of the range $(-48 \mu\text{V}, +48 \mu\text{V})$. We usually do not need this much of gain. If $v_d = v_1 - v_2$ is in mV or V range, the output will spend most of the time in the non-linear range under the saturated condition. For instance, say $v_1 = 0.0048 \sin 2\pi t$ and $v_2 = 0$, then the output will be an almost clean square wave of period 1 s and amplitude of 12 V. The Opamp will be in the linear range of operation only for about 6.4 ms in 1 s. It will be in a saturated condition for the remaining duration. Thus, this mode of using Opamp, called the *open loop* mode, is not really useful for linear amplification purposes.

Consider the circuit in Fig. 6.4-1(b). Here, the Opamp is embedded in a resistive network that generates interaction between the input side of Opamp with its own output side. This interaction takes place through the resistor chain R_1 and R_2 . They loop back the output of the Opamp to its own input. When there is such 'looping back' of output of an Opamp to its input side in a circuit, we state that *the circuit employs feedback*. The *sense of feedback* can be negative or positive. We decide the *sense of feedback* in the circuit by analysing the circuit after assuming that its source is reduced to zero. The resulting circuit is shown in Fig. 6.4-1(c). We expect the output to be at zero since the Opamp was assumed to be free of offset.

Input Offset
Voltage of an
Opamp.

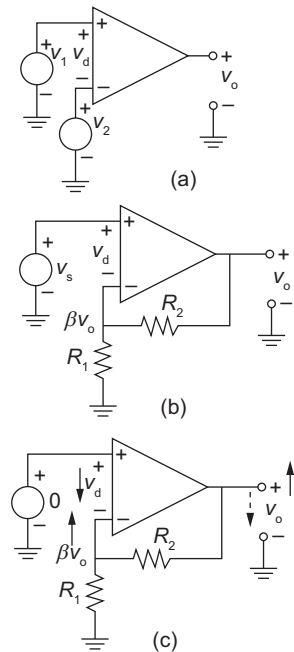


Fig. 6.4-1 (a) Opamp on Open Loop (b) Opamp Embedded in a Resistive Network (c) Opamp Circuit Illustrating Negative Feedback

The feedback action, negative feedback, positive feedback.

Those Pins can't be Mixed up!

A simple interchange of the roles of inverting input terminal and non-inverting input terminal of an Opamp in the circuit changes the nature of feedback in the circuit and affects the function and operation of the circuit significantly.

One has to be very careful in drawing the circuit diagrams containing an Opamp. The inverting and non-inverting terminals of Opamps have to be marked properly without fail.

Now, let us analyse the process that takes place in the system when the output was momentarily disturbed by some kind of electromagnetic pickup from some neighbouring circuit. Assume that the output voltage increased from zero level. This increase is felt at the inverting input of Opamp through the R_1 – R_2 chain. The current drawn by the inverting the input of Opamp loads this potential divider. However, the Opamp has a very large input resistance, and hence, the R_1 – R_2 potential divider may be considered unloaded. Therefore, the potential that appears at the inverting input is βv_o , where β is the feedback factor and is equal to $R_1/(R_1 + R_2)$. Therefore, when the output increases, the inverting input potential with respect to the ground node also increases. This leads to a reduction in the differential voltage v_d that appears across the Opamp input terminals. A reduction of differential input of Opamp is followed by a reduction of its output voltage itself. Thus, we see that an inadvertent increase in Opamp output goes through the feedback loop to generate a corrective action that tends to restore the output to its pre-disturbed condition. When a feedback results in this kind of a corrective action, we term it as *negative feedback* or *degenerative feedback*.

Note that if we had connected the resistor chain at the non-inverting input of Opamp and the input source to the inverting input terminal, a regenerative action, instead of a corrective action, would have taken place. An increase in output would have resulted in an increase in differential input voltage and that would have resulted in a further increase in Opamp output. All circuits are under constant disturbance from other circuits. Therefore, an Opamp circuit with this kind of a feedback connection will find itself going to one of the saturation levels due to some initial pickup voltage at output getting encouragement from the feedback to grow further in the same direction that it started with. This kind of feedback is called *positive feedback* or *regenerative feedback*. Positive feedback usually takes the Opamp output to saturation condition as soon as it is switched on. Obviously, positive feedback Opamp circuits cannot function as linear amplifiers.

6.5 THE PRINCIPLES OF 'VIRTUAL SHORT' AND 'ZERO INPUT CURRENT'

We continue with the analysis of the circuit in Fig. 6.4-1(b). We have ascertained that the feedback involved in this circuit is negative in sense. Hence, we expect the output to be zero when the input is zero. Now, we set out to find the output in terms of input, when the input is non-zero. Assume that the Opamp is ideal. Hence, its input resistance is an open-circuit, output resistance is a short-circuit and its gain is infinity. The gain is taken as A first and is moved to infinity at the end of the circuit solution. The circuit is redrawn with the Opamp replaced by its ideal equivalent circuit as in the circuit in Fig. 6.5-1(b). We derive the equation for the output as below:

$$v_d = v_s - \beta v_o, \text{ where } \beta = \frac{R_1}{R_1 + R_2}$$

$$v_o = A v_d = A(v_s - \beta v_o)$$

$$\therefore v_o = \frac{A}{1 + A\beta} v_s \text{ and } v_d = \frac{1}{1 + A\beta} v_s$$

We observe that the differential input v_d is only $\frac{1}{1 + A\beta}$ times that of what it would

have been had the source been applied directly to Opamp as in the circuit in Fig. 6.5-1(a). Since the Opamp gain is very large for a practical Opamp and is infinity for an ideal Opamp, we evaluate the limit of these expressions as $A \rightarrow \infty$. We get,

$$v_o = \lim_{A \rightarrow \infty} \frac{A}{1 + A\beta} v_s = \frac{1}{\beta} v_s = \left[1 + \frac{R_2}{R_1} \right] v_s$$

$$\text{and } v_d = \lim_{A \rightarrow \infty} \frac{1}{1 + A\beta} v_s = 0$$

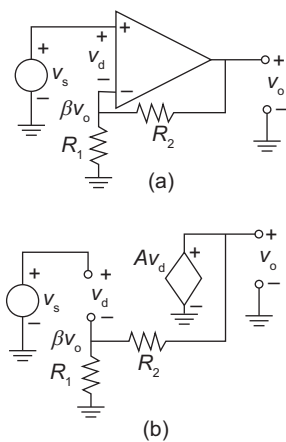


Fig. 6.5-1 (a) A Single Opamp Amplifier Circuit (b) Circuit with Opamp Replaced by its Equivalent Circuit

Hence, the gain of overall amplifier goes to $1 + R_2/R_1$. It is decided entirely by external components. And, the differential input voltage goes to zero.

Why does the differential input voltage go to zero? If the Opamp is in the linear range, its differential input voltage has to be equal to its output divided by the gain. The negative feedback present in the circuit resists any change in the output. Consider the situation when a certain voltage is suddenly applied to the input. Then, the differential voltage increases suddenly since the Opamp will take a little time to adjust its output. The large differential voltage causes the Opamp output to increase. Increasing Opamp output reduces the differential input voltage through the feedback mechanism. Finally, a steady state comes up in the circuit when the output value is such that the difference between the source voltage and the fed back voltage is exactly equal to the output divided by gain. The circuit attains equilibrium under that condition. Any deviation from this equilibrium condition will be corrected by the negative feedback action. Since the gain is large, it requires only a small differential voltage to remain at this equilibrium. For instance, let v_s be 0.1 V, $A = 250,000$ and $\beta = 0.1$. Then, $v_d = 0.1 \times 1/(1 + 25000) = 4 \mu\text{V}$ and $v_o = 0.1 \times 250000/(1 + 25000) = 999960 \mu\text{V} \approx 1 \text{ V}$. It requires only $4 \mu\text{V}$ of v_d to justify 1 V of output since the gain is 250,000. Now, if the gain is increased further, the value of v_d goes down further and v_o approaches closer to 1 V. In the limit when gain goes to infinity, v_d goes to zero and v_o goes to 1 V.

But, will v_d be zero if v_s is 10 V? No, since the amplifier will saturate and will be in the non-linear range of operation. The large gain that is effective in the linear range of operation is not available when the Opamp is operating in voltage-limited, current-limited or slope-limited modes of operation. Hence, we may conclude that the differential voltage across the non-inverting input terminal and the inverting input terminal of an Opamp is arbitrarily close to zero if the Opamp is under negative feedback and is in its linear range of operation. Thus, the two input terminals, though not connected together, are virtually at the same potential under these conditions and behave as if they are shorted together. Therefore, there is a *virtual short* across the input terminals of an Opamp working in its linear range of operation in a negative feedback circuit.

The input resistance of an ideal Opamp is infinite, and hence, the Opamp does not draw any current at its input terminals. The input resistance of a practical Opamp is large and the current drawn by the input terminals is usually negligible compared to the currents elsewhere in the circuit. This remains true even when the Opamp is in its non-linear range of operation. This principle is called 'the zero input current principle'.

Thus, from the point of view of input currents drawn by the ideal Opamp, its input terminals represent an open-circuit and from the point of view of differential input voltage, the same two terminals represent a short-circuit. This model for an Opamp is called the *Ideal Opamp Model* (IOA Model). *It is emphasised again that IOA model will lead to a correct analysis only if the Opamp is in a negative feedback circuit and is working in its linear range of operation.*

6.6 ANALYSIS OF OPERATIONAL AMPLIFIER CIRCUITS USING THE IOA MODEL

The principles enunciated in the preceding section can be used to develop a much simplified analysis procedure for circuits containing Opamps. The procedure is outlined below:

1. Ascertain whether the circuit has a negative or positive feedback. If it is a positive feedback circuit, the IOA model cannot be used for its analysis. Only the principle of zero input current will be applicable for such circuits. Other analysis procedures using nodal analysis or mesh analysis along with zero input current principle will be needed in this case.

The Virtual Short Principle

The input terminals of an ideal Opamp in a negative feedback circuit behave as if there is short-circuit across them.

The input terminals of a practical Opamp with a large gain behave as if there is short-circuit across them, provided the Opamp is in a negative feedback circuit and it is operating in its linear range.

The Zero Input Current Principle

'The input terminals of an ideal Opamp do not draw any current from the circuit in which the Opamp is embedded.'

The input terminals of a practical Opamp with a large input resistance draw negligible current from the circuit in which the Opamp is embedded.

The currents drawn by the input terminals are usually negligible in both linear and non-linear ranges of operation.

2. Prepare node equations at all nodes except the ground node. Ground node is taken as the reference node for nodal analysis. Use the principle of zero input current for writing the node equations.
3. Apply the principle of virtual short on all Opamps in the circuit to reduce the number of node equations and solve the reduced set of equations.

This procedure is illustrated in the case of various Opamp circuits in the remainder of this section. These circuits not only serve as illustrations for the technique of analysis but also introduce the reader to Opamp circuits that are frequently employed in analog signal processing applications. In fact, the popularity of these circuits provides the motivation for inclusion of Opamp circuits in a book on basic circuit theory.

6.6.1 The Non-Inverting Amplifier Circuit

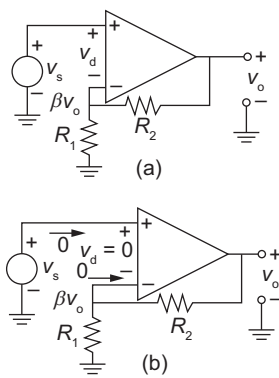


Fig. 6.6-1 (a) The Non-Inverting Amplifier Circuit (b) Application of IOA Model to the Amplifier Circuit

The first circuit is called *Non-Inverting Amplifier* or *Amplitude Scalar*. We have already used this circuit in the preceding sections. It is shown in the circuit in Fig. 6.6-1(a). The circuit in Fig. 6.6-1(b) shows the application of the principles of zero input current and virtual short.

We do not need any node equation here. The potential divider formed by $R_1 - R_2$ is unloaded at its output and is driven by an ideal dependent voltage source with zero series resistance. Therefore, the potential at inverting input is βv_o , where $\beta = R_1/(R_1 + R_2)$. The potential at non-inverting input is v_s . It will be v_s even if the source has a series resistance. This is due to the fact that the current in input line is zero. Now, we apply the principle of virtual short to equate these two potentials. Both potentials are referred to the ground node.

$$\therefore v_s = \beta v_o \Rightarrow v_o = \frac{v_s}{\beta} = \left[1 + \frac{R_2}{R_1} \right] v_s. \text{ Therefore, the gain of non-inverting amplifier}$$

$= 1 + \frac{R_2}{R_1}$. It has infinite input resistance and zero output resistance as per IOA model.

Practically, input resistance and output resistance of Opamp will limit these.

6.6.2 The Voltage Follower Circuit

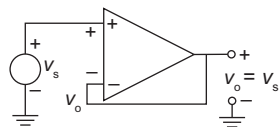


Fig. 6.6-2 The Voltage Follower Circuit

This circuit is a special case of non-inverting amplifier circuit. R_2 is kept at zero value and R_1 is made an open-circuit. The resulting gain will be unity. The unity gain amplifier is used as a buffer amplifier to interconnect circuits without interfering with each other. When a second circuit is connected as a load to the first circuit, the performance of the first circuit is affected by the input characteristics of the second circuit and the performance of the second circuit is affected by the output characteristics of the first circuit. A unity gain buffer amplifier interfaces the two circuits by providing an infinite load resistance to the first circuit and an ideal voltage source with zero series resistance to the second circuit. Voltage follower circuit shown in Fig. 6.6-2 is used in buffering applications.

6.6.3 The Inverting Amplifier Circuit

This amplifier is shown in Fig. 6.6-3.

Assume that the input is grounded and the output voltage deviates upwards from zero due to some temporary disturbance. Then, the voltage fed back to the inverting terminal increases, thereby, effecting a decrease in the differential input voltage. This leads to a corrective action at the output. Hence, there is a negative feedback in this circuit.

The non-inverting input is grounded. Therefore, by virtual short principle, the potential at the inverting input is also zero. That is, the inverting terminal is *virtually grounded*. Hence,

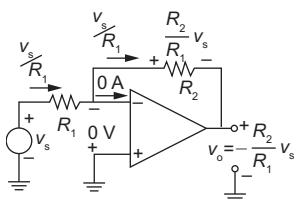


Fig. 6.6-3 The Inverting Amplifier Circuit

the current in R_1 has to be v_s/R_1 A. Thus, the Opamp with grounded non-inverting terminal converts the input voltage source into a current source with the conversion factor decided by R_1 .

The input current into the inverting input terminal is zero by zero input current principle. Therefore, the current in R_2 has to be v_s/R_1 A by KCL at the inverting input. This current flow results in a voltage drop of $v_s R_2/R_1$ V with the polarity shown in Fig. 6.6-3. Applying KVL in a loop starting at a non-inverting input terminal and ending at the ground node, we get,

$$0 + \frac{R_2}{R_1} v_s + v_o = 0$$

Therefore, $v_o = -\frac{R_2}{R_1} v_s$. The gain of the amplifier is $-R_2/R_1$. The output undergoes a sign change with respect to the input, and hence the name *Inverting Amplifier*. The input resistance of the amplifier is R_1 . The output resistance of this amplifier is zero as per IOA model and is small-valued in practice.

6.6.4 The Inverting Summer

The circuit of *Inverting Summer* is shown in Fig. 6.6-4.

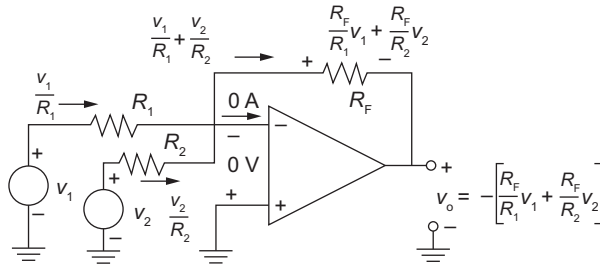


Fig. 6.6-4 The Inverting Summer Amplifier Circuit

The inverting input terminal is at the virtual ground. Therefore, the currents through R_1 and R_2 are v_1/R_1 and v_2/R_2 , respectively. These two currents add at the inverting node and flow into the feedback resistor R_F , producing a voltage drop of $v_1(R_F/R_1) + v_2(R_F/R_2)$ across it. Therefore, the output voltage is $-[v_1(R_F/R_1) + v_2(R_F/R_2)]$.

The circuit acts as an inverting summer if $R_1 = R_2 = R_F = R$. Then, the output voltage will be $-(v_1 + v_2)$.

6.6.5 The Non-Inverting Summer Amplifier

The *non-inverting summer* amplifier with three inputs is shown in Fig. 6.6-5.

The gain from non-inverting input terminal to the output terminal will be $1 + (k-1)R/R = k$. We need to find the potential at the non-inverting input terminal in terms of v_1 , v_2 and v_3 , in order to obtain an expression for v_o in terms of the source voltages. The node potential at the non-inverting input is marked as v . Applying KCL at this node and using the principle of zero input current, we get the following node equation, where the G represent the conductance values.

$$G_1(v - v_1) + G_2(v - v_2) + G_3(v - v_3) = 0$$

$$\text{i.e., } v[G_1 + G_2 + G_3] = G_1v_1 + G_2v_2 + G_3v_3$$

Adding Additional Input Lines to Inverting Summer

More than two sources can be connected to the summer in the same manner.

Essentially, the virtual ground at the inverting input terminal results in a voltage to current conversion in all source lines. The inverting input acts as a summing node where all the input line currents get added.

The sum current gets pushed into the feedback resistor since Opamp input terminal does not draw any current.

Adding Additional Input Lines to Non-Inverting Summer

Note that the gain for each source voltage is dependent on the conductance values of resistors in *all* source lines. Hence, addition of an extra source will affect the gain for all other sources. The inverting summer circuit does not suffer from this shortcoming; thanks to the inverting input terminal acting as a *summing node for currents*.

If one can spend two Opamps, an inverting summer followed by an inverting amplifier with a gain of -1 is a better solution than the non-inverting summer.

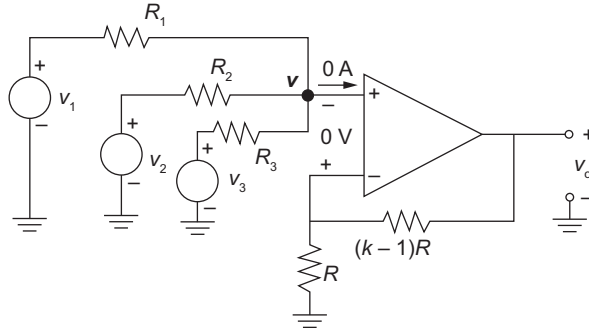


Fig. 6.6-5 The Non-Inverting Summer Amplifier Circuit

$$\therefore v = \frac{G_1 v_1 + G_2 v_2 + G_3 v_3}{G_1 + G_2 + G_3}$$

$$\text{Therefore, the output, } v_o = \frac{kG_1}{G_1 + G_2 + G_3} v_1 + \frac{kG_2}{G_1 + G_2 + G_3} v_2 + \frac{kG_3}{G_1 + G_2 + G_3} v_3$$

If there are n inputs connected at the input, the equation for v_o gets generalised to

$$v_o = \frac{k}{\sum_{j=1}^n G_j} \sum_{i=1}^n G_i v_i \text{ V.}$$

If all the input resistors are equal and $k = n$, then the circuit performs the addition function without any gain,

$$\text{i.e., } v_o = \sum_{i=1}^n v_i \text{ V.}$$

6.6.6 The Subtractor Circuit

The *Subtractor Circuit* is shown in Fig. 6.6-6.

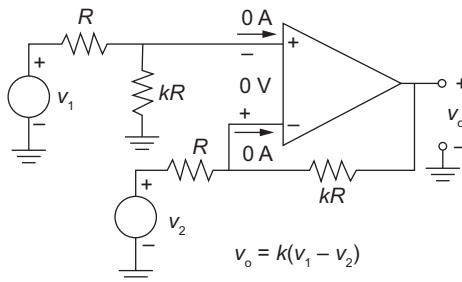


Fig. 6.6-6 The Subtractor Circuit Using a Single Opamp

The principle of zero input current is used to arrive at the conclusion that the potential dividers connected at both input terminals are unloaded. Therefore, the potential of non-inverting input terminal is $kv_1/(k+1)$.

The potential at the inverting input terminal is the same as the potential at the non-inverting input terminal by the principle of virtual short. Therefore, the potential at the inverting terminal is $kv_1/(k+1)$ V with respect to the ground node.

Now, we write the KCL at the inverting input terminal and make use of the principle of zero input current while writing down the equation. The node equation is obtained as:

$$G\left(\frac{kv_1}{k+1} - v_2\right) + \frac{G}{k}\left(\frac{kv_1}{k+1} - v_o\right) = 0.$$

Solving for v_o , we get, $v_o = k(v_1 - v_2)$. $k = 1$, if all resistors chosen are equal.

This circuit is also called *differential amplifier*. The input resistance offered to the sources depends on the value of R used in the circuit. The performance of this circuit in amplifying the voltage difference is satisfactory for less stringent applications. However, when the difference between two voltages with a large common mode voltage (i.e., average of v_1 and v_2) is to be amplified, the common mode performance of this circuit is far from satisfactory. Also, the circuit is further unsuited if these voltage sources come with large series resistance values – i.e., if they are high resistance sources. These two conditions often appear in applications in the fields of instrumentation and measurement. The high input resistance, low common mode gain differential amplifier designed to amplify μVs or mVs of voltage differences embedded in Volts of common mode level is called an *Instrumentation Amplifier*.

There are many configurations possible for an *Instrumentation Amplifier*. The next sub-section looks at a popular design.

6.6.7 The Instrumentation Amplifier

A popular instrumentation amplifier using two Opamps is shown in Fig. 6.6-7.

The first two Opamps provide the required gain $(2k + 1)$ to the differential voltage $(v_1 - v_2)$ and a gain of 1, when $v_1 = v_2$. Let us see how.

The potentials at the non-inverting input terminals of the first and second Opamps are v_1 and v_2 , with respect to the ground node, respectively. This is true quite independent of the values of source resistance, since there is no current flow in the non-inverting input lines in the Opamps. Virtual short across the input terminals carry these voltages to two ends of the middle resistor in the three-resistor chain strung across the output terminals of these two Opamps. That sets up a current of $(v_1 - v_2)/R$ in that resistor, as shown in Fig. 6.6-7.

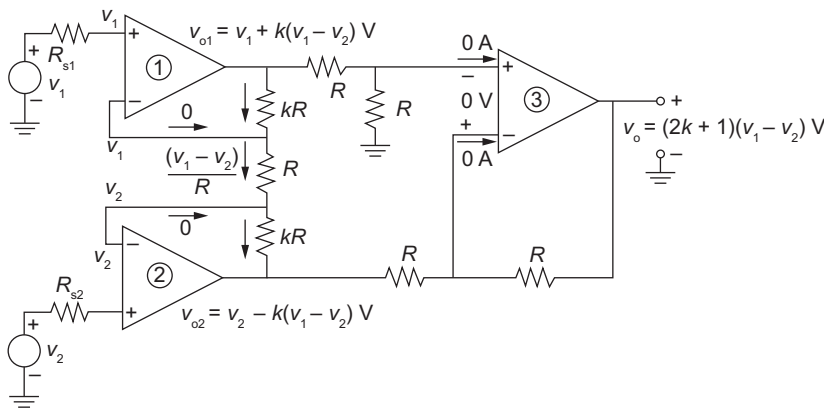


Fig. 6.6-7 An Instrumentation Amplifier Using Two Opamps

The currents in the remaining two resistors in the three-resistor chain are also same at $(v_1 - v_2)/R$, since the Opamp inverting terminal lines do not contribute any current. Now, the output voltage of the two Opamps may be calculated as:

$$v_{o1} = v_1 + kR \times \frac{(v_1 - v_2)}{R} = v_1 + k(v_1 - v_2) \quad \text{and} \quad v_{o2} = v_2 - kR \times \frac{(v_1 - v_2)}{R} = v_2 - k(v_1 - v_2)$$

Common Mode Gain of the Subtractor Circuit

The circuit is expected to give zero output if $v_1 = v_2$. That is, its common mode gain is expected to be zero. However, in practice it isn't. The reason is the invariably present deviations in the resistance value of resistors from their nominal values. Resistors have a tolerance factor.

If the actual values of resistance of resistors is such that the ratio between the two resistors connected to inverting terminal is different from the ratio of resistors connected at the non-inverting terminal, the common mode gain of the circuit will not be zero.

It can be shown that the common mode gain of the circuit is directly proportional to the fractional tolerance of resistors by using Compensation Theorem or otherwise.

These voltages form the two inputs of a single-Opamp subtractor circuit with unity gain. Therefore, the final output taken across the output of the third Opamp with respect to the ground node is given by $v_o = v_{o1} - v_{o2} = (2k + 1)(v_1 - v_2)$.

It may be verified that $v_{o1} = v_1$ and $v_{o2} = v_2$ when $v_1 = v_2$, quite independent of the resistance values of the resistors in the three-resistor chain. This is so because, when $v_1 = v_2$, there is no current flow in them at all. Thus, the first stage comprising two Opamps contributes the required differential gain without affecting the common mode gain. The second stage contributes only unity gain to $(v_{o1} - v_{o2})$. The common mode rejection in the entire amplifier is governed by the common mode gain in the subtractor circuit due to resistor tolerances. The common mode gain in the subtractor circuit is proportional to its differential mode gain and resistor fractional tolerance value. Therefore, the common mode gain of the subtractor circuit can be kept at a much lower level now since the differential gain that this stage has to contribute is only unity.

Thus, the instrumentation amplifier in Fig. 6.6-7 offers a very large input resistance to the input sources and has a much improved common mode rejection as compared to the simple subtractor circuit that was discussed in the preceding sub-section.

6.6.8 Voltage to Current Converters

Some loads call for a current drive rather than a voltage drive. An Opamp circuit that can convert a given voltage source into a current source is required for driving such loads. We distinguish between two kinds of loads here — a floating load and a grounded load. A floating load is one that has no terminal connected to the ground node. A grounded load is one that has one of its terminals connected to the ground node.

The Opamp circuits that can convert a given voltage source v_s into a current source of value v_s/R A into a floating load are shown in Fig. 6.6-8. The Fig. in circuit 6.6-8(a) uses inverting configuration and the circuit in 6.6-8(b) uses non-inverting configuration.

The grounded non-inverting terminal in the circuit in 6.6-8(a) conducts the ground potential to the inverting input terminal by virtue of the virtual short existing between them. Therefore, the input voltage v_s is converted into a current of v_s/R in the line. This current is pushed into the load from left to right since the Opamp input does not take any current. Essentially, the circuit is the same as the inverting amplifier. However, here we do not design it for gain. Rather, we design it for a known current to flow in an arbitrary floating load. The load is shown as a resistor in the circuits in Fig. 6.6-8(a) and (b). However, it should be kept in mind that the load could be any network containing passive and active components including inductors and capacitors. If the load is not a simple resistor R_L , the equation for the output voltage will be different, but the current in the load is decided by v_s and R and is independent of v - i characteristic of the load circuit.

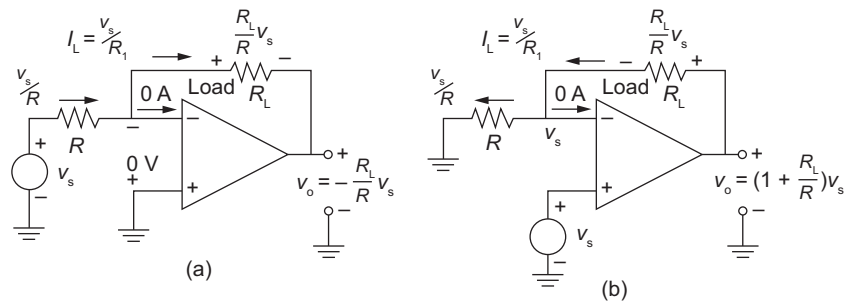


Fig. 6.6-8 Voltage to Current Conversion for Floating Loads (a) Inverting Configuration (b) Non-inverting Configuration

However, we have used the IOA model for this analysis. This model gives a correct result only if the principle of virtual short holds in the circuit. That will be true only if the Opamp is operating in the linear range. Hence, the current in the load is independent of v - i characteristic of the load if, and only if, (i) the voltage developed across the output is below saturation levels at all t , (ii) the load current is below the current limit values for the Opamp at all t and (iii) the rate of change of output voltage developed across the Opamp output is lower than the slew rate of the Opamp. We have to know the v - i characteristic of the load to ascertain compliance with these three conditions. If the Opamp is in one of the three non-linear modes at any t , then the load current at that t will not be independent of the load characteristic.

The circuit in Fig. 6.6-8(b) can be used to reverse the direction of current flow in the floating load. It has an added advantage in that the input resistance offered to the voltage source is infinite, and hence, no current is drawn from the source..

The Opamp circuit for driving a grounded load is shown in Fig. 6.6-9

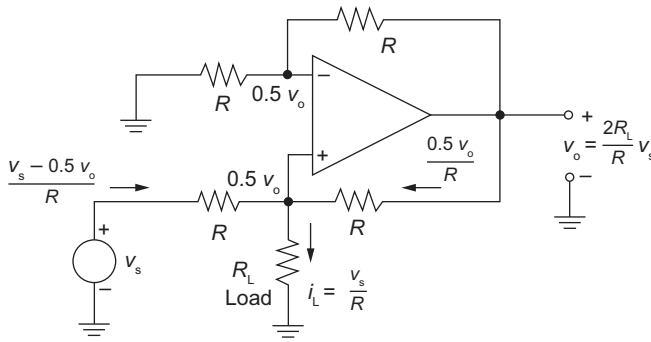


Fig. 6.6-9 Voltage to Current Converter for a Grounded Load

The inverting input does not draw any current from the R - R potential divider connected from the output to ground. The output terminal of an ideal Opamp is a pure voltage source without any series resistance. Therefore, the potential at the inverting input is $0.5v_o$.

The virtual short across the input terminals of the Opamp transfers this voltage to the non-inverting input. Now, the currents in R in the source line is

$(v_s - 0.5v_o)/R$ and the current in the resistor R in the line between the output and the non-inverting terminal is $0.5v_o/R$, with both currents flowing towards the non-inverting input terminal. Therefore, KCL at that node yields $i_L = v_s/R$. Thus, the current in the load is independent of load v - i characteristic.

If the load is a resistor R_L , then the voltage across it is $\frac{R_L}{R}v_s$ and this must be equal to $0.5v_o$. Therefore,

$$v_o = \frac{2R_L}{R}v_s.$$

The current in the load will remain independent of the v - i characteristic of the load only if (i) Opamp output current ($2i_L R_L/R$) is below the output current limit of Opamp (ii) v_o is below clipping levels of Opamp and (iii) the first derivative of v_o is less than the slew rate of Opamp.

The load can be any network containing passive and active components (even non-linear elements), provided the resulting circuit has a unique solution and v_s and load v - i characteristics are such that the conditions stated above are satisfied.

6.7 OFFSET MODEL FOR AN OPERATIONAL AMPLIFIER

We had neglected the offsets that are invariably present in any Practical Opamp (POA) circuit until now. We take up the analysis of offset performance of Opamp circuits in this section and develop a model to account for offsets in Opamps.

Consider the circuit in Fig. 6.7-1(a).

The output is usually at the saturation level when both the input terminals of a POA are shorted to ground. However, it is not possible to predict whether the output is at a positive saturation level (V_{sat}) or at a negative saturation level ($-V_{sat}$). An experiment will be required to find that. Two Opamps of the same type; for instance $2\ \mu A741$ ICs, need not show the same saturation level under the input-short condition.

We also observe that two DC currents $-I_{B+}$ and I_{B-} flow into the non-inverting and inverting terminals, respectively, from the ground node. They are called *input bias currents* and they flow to establish a proper quiescent point for the transistors in the input differential amplifier stage inside the Opamp.

Now, consider the circuit in Fig. 6.7-1(b). Here, we introduce a voltage source in series in the non-inverting input line and vary it such that the output reaches 0 V with respect to the ground node. We cannot decide the polarity or magnitude of the voltage required for this before we conduct an experiment with the particular unit of Opamp we plan to use in a circuit. This is because we cannot predict the output before we observe it experimentally. If we observe a positive saturation in the circuit in Fig. 6.7-1(a), then, we must connect the voltage source in the circuit in Fig. 6.7-1(b) with its negative polarity connected at the non-inverting input terminal. Similarly, if we observe a negative saturation at the output in the circuit in Fig. 6.7-1(a), we must connect the voltage source in the circuit in Fig. 6.7-1(b) with its positive polarity at the non-inverting terminal.

The magnitude of the voltage source between the input terminals required to null the output is defined as the 'input offset voltage' of the Opamp and is designated by v_{io} . Manufacturers quote typical values for this voltage in the data sheets of Opamps.

An ideal Opamp will show zero output voltage and zero DC currents in the input lines in the circuit in Fig. 6.7-1(a). What are the additional elements that must be added to an IOA such that the observations in an experiment similar to the one indicated by the circuit in Fig. 6.7-1(a) will be the same for a practical Opamp and the IOA with the added elements? The observations will have to be that (i) I_{B+} and I_{B-} flow into the box containing the IOA and the added elements at non-inverting and inverting terminals, respectively, (ii) the output voltage is saturated and (iii) a voltage source of magnitude v_{io} injected in the non-inverting input line will bring the output back to zero. The elements that are to be added to an IOA to justify the observations on offset behaviour of a practical Opamp are shown in the circuit in Fig. 6.7-1(c).

Note that the polarity of v_{io} source in the circuit in Fig. 6.7-1(c) is opposite to that of v_{io} source in the circuit in Fig. 6.7-1(b). The nodes that are within the *dotted* rectangular box in the circuit in Fig. 6.7-1(c) are inaccessible. The accessible nodes are brought outside the box. Now, if the non-inverting and inverting terminals that are accessible from outside the box are grounded, we will observe I_{B+} flowing from ground to the accessible non-inverting terminal and I_{B-} flowing from ground to the accessible inverting terminal. We will also observe the accessible output terminal to be at the same saturation level observed in the circuit in Fig. 6.7-1(a). Moreover, if we introduce a voltage source in series with the accessible non-inverting terminal and vary it to null the output, we will observe that the magnitude and polarity required are the same as in the circuit in Fig. 6.7-1(b). Thus, the circuit in Fig. 6.7-1(c) is indistinguishable from the circuit in Fig. 6.7-1(a) by external measurements. Therefore, we may replace a practical Opamp with the circuit in Fig. 6.7-1(c) in Opamp circuits to analyse the impact of Opamp offsets and bias currents on the circuit performance. The circuit in Fig. 6.7-1(c) is called the *Offset Model for a Practical Opamp*. Note that the Opamp within this model is an ideal Opamp.

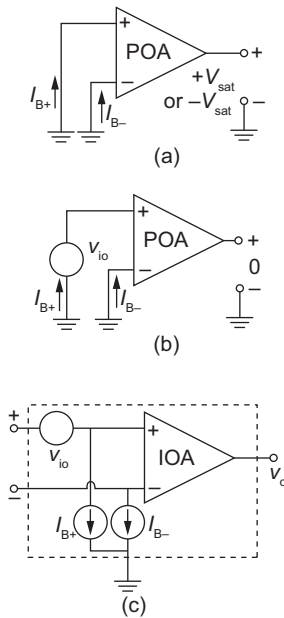


Fig. 6.7-1 (a) Offset behaviour of operational amplifiers (b) Input offset voltage definition (c) Offset Model

EXAMPLE: 6.7-1

Derive an expression for output voltage of a non-inverting amplifier including offsets.

SOLUTION

The non-inverting amplifier circuit employing a practical Opamp and the circuit with the Opamp replaced by its offset model are shown as the circuit in Fig. 6.7-2(a) and the circuit in Fig. 6.7-2(b).

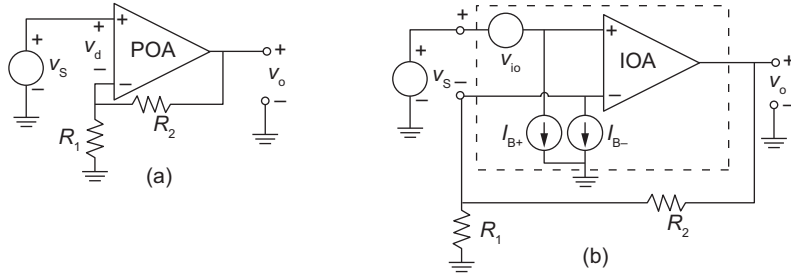


Fig. 6.7-2 (a) Non-Inverting Amplifier (b) Opamp Replaced by its Offset Model

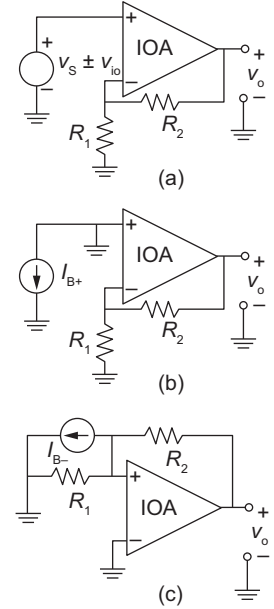


Fig. 6.7-3 Sub-Circuits for Applying Superposition Theorem in Example 6.7-1

The circuit in Fig. 6.7-2(b) is a circuit driven by four independent sources. We use Superposition Theorem to obtain the output. The sub-circuits needed to solve for the output components are shown in Fig. 6.7-3.

The two voltage sources have been combined into a single source of $v_s \pm v_{io}$ V. The polarity of the input offset voltage is unknown, and hence, the $\pm$ sign.

The output in the circuit in Fig. 6.7-3(a) is $\left(1 + \frac{R_2}{R_1}\right)v_s \pm \left(1 + \frac{R_2}{R_1}\right)v_{io}$. The circuit in Fig. 6.7-3(b) generates zero output since the ground connection at the non-inverting terminal makes it a non-inverting amplifier with zero input. The circuit in Fig. 6.7-3(c) is an inverting amplifier with a gain of $-R_2/R_1$ driven by a voltage source of value equal to $-I_{B-}R_1$. Applying source transformation on the current source and parallel resistance to verify this the output from this circuit is R_2I_{B-} V.

$$v_o = \left(1 + \frac{R_2}{R_1}\right)v_s \pm \left(1 + \frac{R_2}{R_1}\right)v_{io} + R_2I_{B-}$$

This equation shows that both the input signal and the input offset voltage get the same gain. Hence, the percentage error in output is virtually governed by the ratio v_{io}/v_s . We will need Opamps that are designed specially for low values of input offset voltage if we are to amplify millivolts and microvolts of input.

We note from the output equation that the input bias current flowing in circuit resistances lead to an offset voltage component at the output. The reader may verify that the term R_2I_{B-} may be cancelled fully by a term $-R_2I_{B+}$ if a resistor of value $R_1//R_2$ is introduced in the line between v_s and the non-inverting input terminal and if $I_{B+} = I_{B-}$. The two bias currents are more or less equal in practice, and hence, this extra resistor is usually put in series with the source in non-inverting amplifier applications – especially if value of R_2 is in 100s of k Ω .

EXAMPLE: 6.7-2

Derive an expression for output voltage of an inverting amplifier with Opamp offsets and input bias currents taken into account.

SOLUTION

The inverting amplifier circuit is shown in Fig. 6.7-4(a) and the circuit after replacing the Opamp with its offset model is shown in Fig. 6.7-4(b).

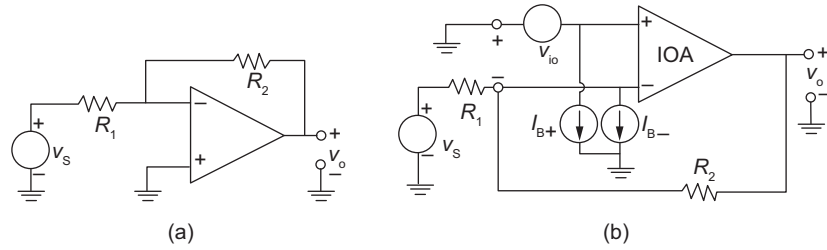


Fig. 6.7-4 (a) The Inverting Amplifier Circuit (b) Circuit After the Opamp is Replaced with its Offset Model

We apply superposition theorem to obtain the output voltage and obtain the output components by inspection.

When v_{io} is acting alone, the circuit becomes a non-inverting amplifier of gain $(1 + R_2/R_1)$. Hence, the output component due to input offset voltage is $\pm (1 + R_2/R_1)v_{io}$.

When v_s is acting alone, the circuit is the usual inverting amplifier with an output of $-(R_2/R_1)v_s$.

When I_{B+} is acting alone, it gets shorted by the short-circuit that replaces the v_{io} source. Therefore the circuit becomes a non-inverting amplifier with zero input. Therefore, the component contributed by I_{B+} to output is zero.

The circuit is an inverting amplifier with a gain of $-R_2/R_1$ and an input of $-R_1 I_{B-}$ V when I_{B-} is acting alone. (Apply source transformation theorem). Hence, the contribution of I_{B-} to output is $R_2 I_{B-}$ V.

$$\therefore v_o = -\frac{R_2}{R_1} v_s \pm \left(1 + \frac{R_2}{R_1}\right) v_{io} + R_2 I_{B-} V.$$

The reader may verify that introducing a resistor of value R_1/R_2 in between the non-inverting input terminal and the ground node will result in partial or full cancellation of the term $R_2 I_{B-}$ in the output.

6.8 EFFECT OF NON-IDEAL PROPERTIES OF OPAMP ON CIRCUIT PERFORMANCE

A practical Opamp has finite input resistance and non-zero output resistance. It has finite gain and finite bandwidth. It has non-zero common mode gain. Its output voltage and output current are limited to some finite values that depend on the supply voltage used. Its output is incapable of changing at a rate faster than a finite rate that is specified as its slew rate. It has non-zero output offset and non-zero input bias currents.

These non-idealities in the practical Opamp result in a degraded performance in circuits that have embedded Opamps. We consider the impact of some of these non-ideal properties on the performance of non-inverting and inverting amplifiers made using a practical Opamp in this section. The procedure to take offsets and input bias currents into account was explained in detail in the preceding section. Hence, offset and input bias current issues are not taken up in this section again.

EXAMPLE: 6.8-1

Derive an expression for output voltage in a non-inverting amplifier using an Opamp that has finite gain $A = 100,000$, finite $R_i = 2 \text{ M}\Omega$ and non-zero $R_o = 75 \Omega$. The amplifier uses $R_2 = 0.9 \text{ M}\Omega$ and $R_1 = 0.1 \text{ M}\Omega$.

SOLUTION

The amplifier and its equivalent circuit are shown in the circuits in Figs. 6.8-1(a) and (b), respectively.

We solve the problem by finding the node voltage v_x first. We express v_d as $v_s - v_x$ and write the node equation at the node where v_x is assigned.

$$\frac{1}{R_i}(v_x - v_s) + \frac{1}{R_1}v_x + \frac{1}{R_2 + R_o}(v_x - A(v_s - v_x)) = 0$$

$$\text{Solving for } v_x, v_x = \frac{\frac{1}{R_i} + \frac{A}{R_2 + R_o}}{\frac{1}{R_i} + \frac{1}{R_1} + \frac{(A+1)}{R_2 + R_o}} v_s.$$

Substituting the numerical values, we get, $v_x = 0.9999 v_s$. Therefore, $v_d = 0.0001 v_s$.

The current in R_o is $100000v_d - 0.9999v_s$ divided by 900075Ω . Therefore, it is equal to $9.9993 \times 10^{-6} v_s$ A.

Therefore, the voltage drop in $R_o = 75 \times 9.9993 \times 10^{-6} v_s = 7.5 \times 10^{-4} v_s$ V.

$\therefore v_o = 10v_s - 7.5 \times 10^{-4} v_s \approx 10 v_s$ V. This is the same as the output predicted by the IOA model.

Let us repeat the calculations by assuming $A = 1000$, $R_i = 200 \text{ k}\Omega$ and $R_o = 1 \text{ k}\Omega$. Now, the node voltage $v_x = 0.9901 v_s$, the differential input voltage $v_d = 0.0099 v_s$ and $v_o = 9.86 v_o$. Thus, the gain will deviate by 1.4% away from its expected value of 10.

In general, the results predicted by the IOA model will be sufficiently accurate if the gain realised in the circuit is below 1% of the Opamp gain and the resistors used in the circuit are much higher than the Opamp output resistance and much lower than the Opamp input resistance.

A thumb rule for choosing the resistor values in a circuit containing Opamps and resistors may be arrived at as a result of these calculations on commonly used Opamp circuits.

The design rule for choosing the values for resistors in an Opamp circuit is that all resistors must be chosen to lie between $R_i/25$ and $25R_o$, where R_i and R_o are the input and output resistance of Opamps used in the circuit.

Voltage saturation at the output of an Opamp and the consequent clipping of output waveform are easy to understand. However, clipping at a level lower than the voltage saturation limit may take place under current-limited operation of Opamps. The next example illustrates this issue.

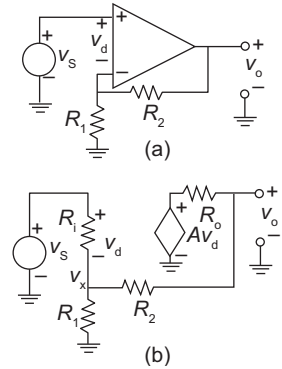


Fig. 6.8-1 (a) Non-Inverting Amplifier (b) Equivalent Circuit of Non-Inverting Amplifier

EXAMPLE: 6.8-2

The Opamp used in an inverting amplifier (Fig. 6.8-2) employs ± 12 V supply. The output saturation limit of the Opamp at this power supply level is ± 10 V. The output current of Opamp is limited to ± 20 mA with a supply voltage of ± 12 V. The feedback resistance draws negligible current from the output and the gain of the amplifier is -10 . Obtain the output of the amplifier if the input is a sine wave of 1 V amplitude and 10 Hz frequency and the load connected at the output is (i) $10 \text{ k}\Omega$ and (ii) 250Ω .

SOLUTION

(i) The gain of the amplifier is -10 . The input is $v_s(t) = 1 \sin 20\pi t$ V. Therefore, the output will be $v_o(t) = -10 \sin 20\pi t$ V if the Opamp does not enter the non-linear range of operation at any instant. The peak voltage of the expected output is 10 V and this is just about equal to the voltage saturation limits. Therefore, clipping will not take place on this count. The maximum current that will be drawn by the $10 \text{ k}\Omega$ load will be $10 \text{ V}/10 \text{ k}\Omega = 1 \text{ mA}$ and that is well below the output current limit of Opamp. Therefore, the output in this case will be a pure sine wave given by $v_o(t) = -10 \sin 20\pi t$ V.

(ii) Clipping cannot take place in this case too due to the output voltage trying to exceed the saturation limits. However, if the output is really $-10 \sin 20\pi t$ V, then the load resistor will draw a current of $10/0.25 = 40 \text{ mA}$ at the peak of sine wave, but the Opamp output current is limited at $\pm 20 \text{ mA}$. The load resistor of 250Ω will draw 20 mA when the voltage across it is 5 V. This will happen at the 30° position on the sine wave. Thus, the output voltage will follow a sinusoid of 10 V amplitude until the 30° position, then, remain clipped at 5 V for the entire 30° to 150° range and again follows a sinusoidal variation for 150° to 180° in a half-cycle. Thus, output shows a clipping level of ± 5 V for two-thirds of cycle period.

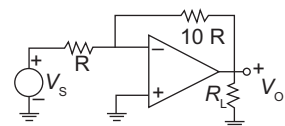


Fig. 6.8-2 The Inverting Amplifier in Example 6.8-2

The Opamp is in its non-linear range of operation during the period for which the output remains clipped. The principle of virtual short does not hold during such periods, and hence, there will be non-negligible voltage across the input terminals of Opamp during such periods. The input current into the inverting and non-inverting terminals remains zero even when the Opamp is working in the non-linear range. Hence, the potential at the inverting input terminal can be found by the superposition theorem and voltage division principle as $\frac{10}{11}V_s + \frac{1}{11}v_o$. The differential input voltage is equal to the negative of the voltage at the inverting input.

$$\therefore V_d = \begin{cases} 0 & \text{when } v_o \text{ is along } 10 \sin 20\pi t \\ -\frac{10}{11} \sin 20\pi t - \frac{1}{11} v_o & \text{when } v_o \text{ is clipped.} \end{cases}$$

The output voltage $v_o(t)$ in this case is shown in Fig. 6.8-3(a) and the Fig. 6.8-3(b) shows the differential voltage across Opamp input terminals in comparison with the input signal.

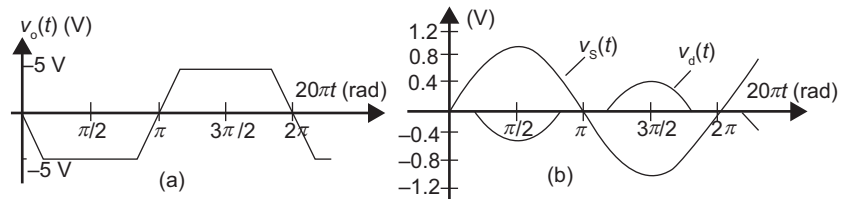


Fig. 6.8-3 (a) Output of an Inverting Amplifier Showing Voltage Clipping due to Current Limiting (b) The Differential Input Voltage under Current-Limited Operation

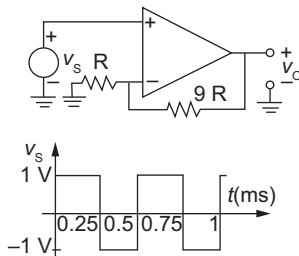


Fig. 6.8-4 Amplifier Circuit and Input Waveform in Example 6.8-3

EXAMPLE: 6.8-3

A square wave of 1 V amplitude and 2 kHz frequency is applied to a non-inverting amplifier of gain 10 made using an Opamp that has a slew rate of ± 0.5 V/ μ s. (See Fig. 6.8-4) Plot the output waveform and the waveform of differential input voltage of Opamp. Assume that the voltage saturation at the output takes place above 10 V.

SOLUTION

At $t = 0$, the input voltage suddenly changes from 0 to 1 V. The rate of change of input at this instant is infinite. The IOA model for Opamp will predict a sudden change in the output from zero to 10 V, but the Opamp has a limited rate of change capability. Therefore, its output cannot change instantaneously. Hence, its output remains at zero at the instant after $t = 0$. This results in the differential voltage across Opamp input terminals attaining a level of 1 V. This large differential voltage will compel the Opamp output to rise to the maximum possible rate that it is capable of, i.e., at 0.5 V/ μ s. The output voltage rises linearly at this rate from zero towards 10 V. As the output climbs, the differential input voltage decreases linearly since $v_d = v_s - 0.1v_o$. The differential input voltage approaches zero and the Opamp enters the linear range of operation when the output approaches 10 V. This happens at $t = 10/0.5 = 20 \mu$ s.

The output remains at 10 V for $t = 20 \mu$ s to 250 μ s. At 250 μ s, the input suddenly changes to -1 and the differential input voltage changes suddenly from 0 to -2 V. This large negative differential compels the Opamp output to move in the negative direction at the maximum rate that it is capable of, i.e., at -0.5 V/ μ s. The differential voltage approaches zero from -2 V linearly as the output moves from 10 V to -10 V linearly at the rate of -0.5 V/ μ s. The Opamp enters the linear range again when its output voltage reaches -10 V. This happens at $20/0.5 = 40 \mu$ s after the 250 μ s point. Subsequently, the output remains at -10 V and the differential input voltage remains at 0 V for the period from 290 μ s to 500 μ s.

The waveforms of output and differential input voltage are shown in Fig. 6.8-5.

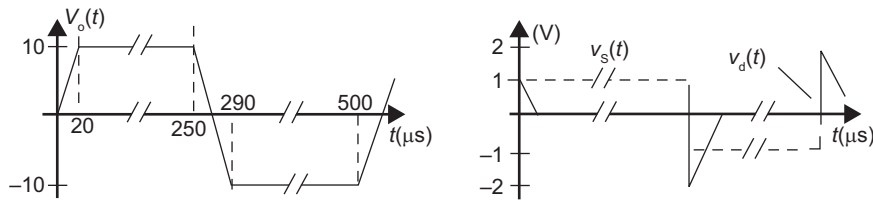


Fig. 6.8-5 Waveforms Illustrating the Effect of Limited Slew Rate of Opamp in Example 6.8-3

EXAMPLE: 6.8-4

A voltage follower circuit made using $\mu\text{A} 741$ IC is driven by $v_s(t) = V_m \sin \omega t$ V. It has a slew rate of ± 0.5 V/ μs . Saturation levels at output are ± 10 V. Find the maximum frequency that the input can have such that the output will swing from -10 V to $+10$ V without any waveform distortion. Assume that the voltage follower is unloaded.

SOLUTION

Two kinds of distortion can come up in this application. The first is due to linear distortion that is caused by the finite bandwidth of the Opamp. We have not discussed this aspect in this chapter. We are not yet ready to do that and we will take up this bandwidth issue in later chapters.

The second kind of distortion is due to slew rate restriction. The output $v_o(t) = V_m \sin \omega t$. Therefore, its rate of change is $\omega V_m \sin \omega t$ V/s. This will have a maximum value of ωV_m and that has to be less than 0.5 V/ μs . Solving for ω with $V_m = 10$ V, we get $\omega = 50,000$ rad/s. Hence, the maximum frequency that the input can have without resulting in an output restricted by slew rate is ≈ 8 kHz.

6.9 SUMMARY

- This chapter introduced the Operational Amplifier as a circuit element and developed a model called the 'Ideal Operational Amplifier Model' (IOA model) for analysing circuits containing Opamps.
- Various factors that set limits to the output voltage, output current and rate of change of output voltage in voltage amplifiers were discussed with a MOSFET amplifier taken as an example to develop the related concepts. Various causes of waveform distortion in voltage amplifiers were also examined using this MOSFET amplifier as an example. This was necessary to set the background for the study of Opamps and to understand some limitations that they suffer from in circuit applications. The role of DC power supplies in setting the output limits is explained.
- Operational Amplifier is a multi-stage high gain voltage-to-voltage amplifier with a differential input and a single-ended output.* It is a differential amplifier and has two signal input terminals. It amplifies the difference between the voltages applied at the input terminals. These two input voltages as well as the output voltage are referred to a common ground terminal.
- Ground in an Opamp is not a terminal or a pin of the Opamp. It is a node outside the Opamp.* In Opamps working from a single DC power supply, the ground is commonly assigned to the negative of the DC source. In Opamps working from a balanced dual DC power supply, the ground is commonly taken as the midpoint of the dual supply.
- An Ideal Operational Amplifier (IOA) is a voltage-to-voltage differential amplifier with infinite input resistance, zero output resistance, infinite differential gain, zero common mode gain, infinite bandwidth, no output current and rate restrictions, no offsets and no input bias currents.
- A simple interchange of the roles of inverting input terminal and non-inverting input terminal of an Opamp in circuit changes the nature of the feedback in the circuit and affects the functioning and operation of the circuit significantly. One has to be very careful in drawing the circuit diagrams containing an Opamp. The inverting and non-inverting terminals of Opamps have to be marked properly without fail.
- The principle of virtual short* – The input terminals of an ideal Opamp in a negative feedback circuit behave as if there is a

short-circuit across them. The input terminals of a practical Opamp with a large gain behave as if there is a short-circuit across them, provided the Opamp is in a negative feedback circuit and it is operating in its linear range.

- *The principle of zero input current* – The input terminals of an ideal Opamp do not draw any current from the circuit in which the Opamp is embedded. The input terminals of a practical Opamp with a large input resistance draw negligible current from the circuit in which the Opamp is embedded. The currents drawn by the input terminals are usually negligible in both linear and non-linear ranges of operation.
- These two principles can be used to analyse Opamp circuits quickly, provided the Opamp is in a negative feedback circuit and is in its linear range of operation.
- Offsets and input bias currents in a practical Opamp can be included in the IOA model by adding a voltage source equal to the input offset voltage and two current sources, which are equal to the input bias currents, to the IOA. This model is called the ‘Offset Model’ of an Opamp. Using this model, the problem of including offsets and bias currents in the analysis reduces to that of applying superposition principle along with IOA model based analysis, employing the principles of virtual short and zero input current.

6.10 QUESTIONS

1. A voltage amplifier made using transistors uses a 15 V power supply. Explain why its peak-to-peak output when driven by a sinusoidal source has to be less than 15 V.
2. Explain why the Thevenin’s equivalent of the output side of a voltage amplifier will have a non-zero Thevenin’s equivalent resistance.
3. A transistor voltage amplifier uses 10 V power supply. It has a gain of 100 and is loaded by a 1 k Ω resistor. Input applied to the amplifier is 0.05 V DC. Explain why the current drawn from the power supply has to be more than 2.5 mA.
4. A transistor voltage amplifier uses 24 V power supply. It has a gain of 50 and is loaded by a 120 Ω resistor. Input applied to the amplifier is 0.24 V DC. The current drawn from the power supply is seen to be 150 mA. What is the power efficiency of the amplifier at this operating condition?
5. A voltage amplifier using a MOSFET has a bias point of (5 mA, 7 V). It uses a power supply of 12 V. It has a gain of 10. It is driven by a sine wave of 0.35 V amplitude and 100 Hz frequency. Explain why the output will not be a clean sine wave at 100 Hz.
6. An Opamp-based amplifier uses ± 15 V power supply and is driven by a 1 V amplitude sine wave. The amplifier has a gain of 20. What is the amplitude of the output?
7. The Opamp μA 741 has a current limit of ± 25 mA at ± 15 V supply level. This Opamp is used to make an inverting amplifier of gain 10 and is driven by a 0.5 V amplitude sinusoidal source at 100 Hz frequency. The output of the amplifier gets shorted to the ground node accidentally. Plot the waveform of the current flowing in the short-circuit.
8. An Opamp has a CMRR of 100 dB and its differential gain is 100,000. What is its common mode gain?
9. Explain why a practical Opamp can never produce a square wave output?
10. Explain why offset is specified at input rather than at the output in an Opamp? What is the meaning of ‘input offset voltage’?
11. Input bias currents of an Opamp are currents that flow in input lines. Explain how they produce *voltage offsets* in amplifiers constructed using Opamp?
12. Explain how the slew rate of an Opamp will get affected if a capacitor is connected across its output as a load.
13. Explain why the Opamp circuit in Fig. 6.10-1 does not function as a voltage follower.
14. Show that the feedback in a *subtractor circuit* is degenerative.
15. Show that the Instrumentation Amplifier described in the text is a negative feedback amplifier.
16. Show that the overall feedback in the voltage to current converter for a grounded load is negative in sense for a resistive load.
17. A voltage follower using μA 741 is driven by a 10 V amplitude sine wave at 1 MHz. Explain why the output is more or less triangular.

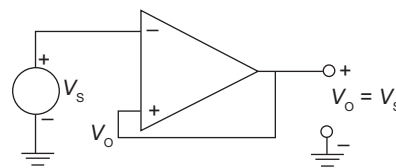


Fig. 6.10-1

6.11 PROBLEMS

(Opamps are μA 741 units unless specified otherwise. μA 741 has $R_i = 2\text{ M}\Omega$, $R_o = 75\text{ }\Omega$, $v_{io} = 2\text{ mV}$, $I_{B+} = I_{B-} = 100\text{ nA}$, $V_{sat} = \pm 10\text{ V}$, current limit $= \pm 20\text{ mA}$ and slew rate $= 0.5\text{ V}/\mu\text{s}$.)

- Design an Opamp circuit to produce $v_o(t) = 3 + 7 \sin 200\pi t\text{ V}$ using $\pm 12\text{ V}$ power supply and a signal source $v_s(t) = 0.5 \sin 200\pi t\text{ V}$.
- Find i_L , i_s , i_o and v_o in the voltage to current converter in Fig. 6.11-1 designed to produce a constant current in a $10\text{ }\Omega$ load.

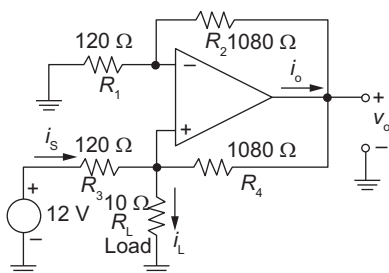


Fig. 6.11-1

- Repeat Problem 2 if $R_1 = R_2 = R_3 = R_4 = 120\text{ }\Omega$. (Hint: The Opamp works in the current-limited mode.)
- (i) Show that the current in the load resistance R_L in the circuit in Fig. 6.11-2 is independent of R_L if $(R_3 + R_4) \gg R_s/R_L$. (ii) Find i_L , i_s , i_o and v_o with the component values shown.

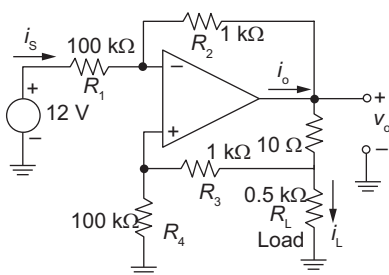


Fig. 6.11-2

- The resistors in the subtractor circuit shown in Fig. 6.11-3 have fractional tolerance of δ . Derive an expression for CMRR of this subtractor circuit in terms of δ .

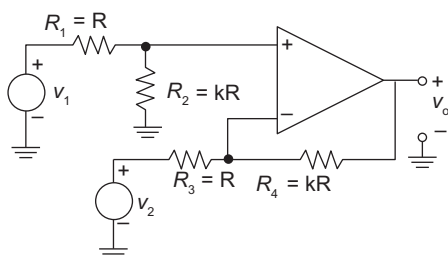


Fig. 6.11-3

- Show that $v_o(t) = (1 + R_2/R_1)(v_1 - v_2)$ in the circuit in Fig. 6.11-4.

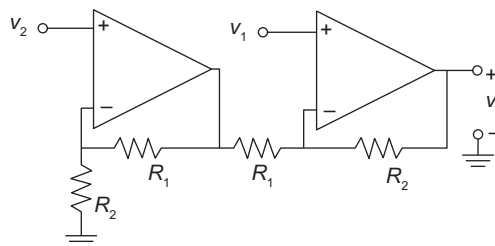


Fig. 6.11-4

- The signal $v_s(t)$ in the circuit in Fig. 6.11-5 is $V_m \sin \omega t$. The Opamp used has saturation levels of $\pm 10\text{ V}$, output current limit at 10 mA and a slew rate of $1\text{ V}/\mu\text{s}$. (i) Find the maximum value of V_m such that there will be no clipping in the output. (ii) Find the maximum value of ω with this value of V_m such that input pins of the Opamp remain virtually shorted throughout the input cycle.

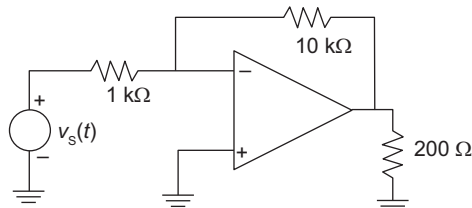


Fig. 6.11-5

- Draw the complete offset model for the amplifier circuit given in the Fig. 6.11-6. The input offset voltage of the Opamps is $v_{io} = 10\text{ mV}$, $I_{B+} = 500\text{ nA}$ and $I_{B-} = 350\text{ nA}$. Calculate the output voltage at the output of the two Opamps when the input is grounded through a $500\text{ k}\Omega$ resistance.

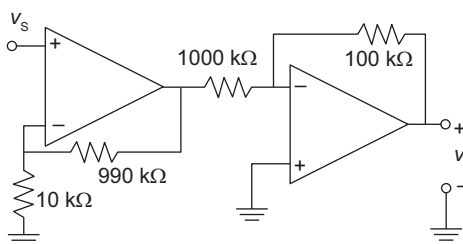


Fig. 6.11-6

- (i) Show that the overall feedback is of degenerative nature in the circuit in Fig. 6.11-7. (ii) Derive expressions for gains at the outputs of both Opamps. (iii) Evaluate the gain v_o/v_s with

$R_4 = 99R$, $R_3 = 100R$ and $R_2 = 9R_1$. (iv) Design the circuit using $\mu A 741$ IC with the above constraints on resistances. (v) Calculate the offset voltages at the two outputs with the above design and using $v_{io} = 4$ mV, $I_{B+} = 100$ nA and $I_{B-} = 80$ nA.

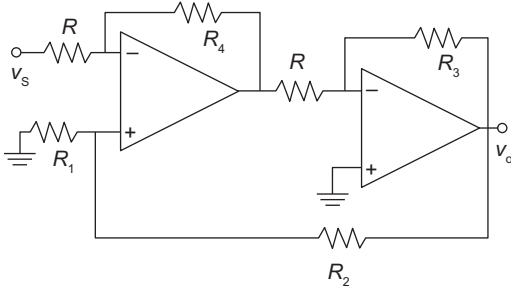


Fig. 6.11-7

10. The output of an Opamp on the open loop was found to be at a positive saturation level when its inverting and non-inverting terminals were grounded. This Opamp is used to construct a non-inverting amplifier of gain 100 using two resistors of 10 k Ω and 990 k Ω . The Opamp has input bias currents of 200 nA. When the amplifier was driven by $v_s(t) = 0.01 \sin 200\pi t$ V, the output was found to have a DC component of 0.5 V. Find v_{io} of the Opamp.
11. The switches in the circuit in Fig. 6.11-8 are electronically controlled and the b represent their state. If b_i is 1, the corresponding switch is closed and if it is 0, the switch is open. (i) Find the value of k such that the output voltage is 7.5 V, when $b_0 = b_1 = b_2 = b_3 = 1$. (ii) Show that if ' $b_3b_2b_1b_0$ ' represent a 4-digit binary number, $2v_o$ represents its decimal equivalent.

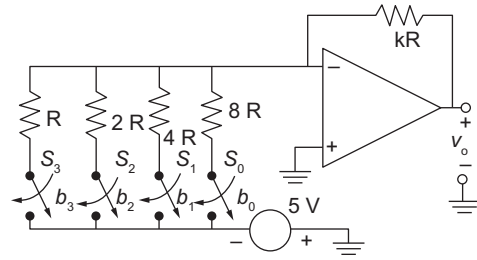


Fig. 6.11-8

12. (i) Show that the net feedback in the circuit in Fig. 6.11-9 is degenerative. (ii) Obtain the output of Opamp by using Ideal Opamp Model and nodal analysis.

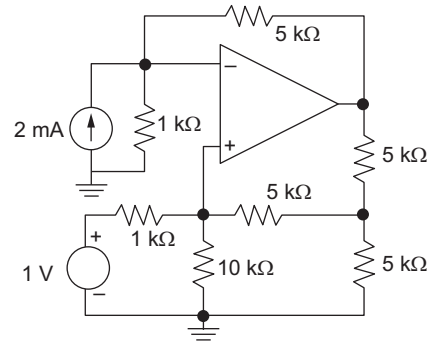


Fig. 6.11-9

13. An Opamp with $A = 2,500$, $R_i = 1$ M Ω , $R_o = 1$ k Ω is used to design an inverting amplifier with an expected gain of -1 . The resistors used are $R_1 = R_2 = 1$ M Ω . Find the gain that will be realised in the circuit.

Part Three

Sinusoidal Steady-State in Dynamic Circuits

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Power and Energy in Periodic Waveforms

CHAPTER OBJECTIVES

- To explain the need for sinusoidal waveforms and the importance of sinusoidal analysis.
- To explain the concepts of phase, phase difference, phase lag/lead, phase delay/advance, time delay/advance etc., in the context of sinusoidal waveforms.
- To introduce instantaneous power, cyclic average power and average power in periodic waveforms.
- To define effective value of periodic waveforms and illustrate rms calculations with examples.
- To develop and explain the power superposition principle and to emphasise the pitfalls in applying this principle.
- To develop an expression for effective value of composite periodic waveform and point out pitfalls in applying the result.

This chapter places special emphasis on the difference between phase lag/lead and phase delay/advance. Cyclic average power and average power are carefully distinguished.

INTRODUCTION

Electrical power generation, transmission and distribution employ sinusoidal voltage and current waveforms to carry power. That, in itself, is a sufficient reason for a detailed discussion on steady-state analysis of circuits containing R , L , C and M .

A sinusoidal waveform is completely specified by three parameters. For example, if $v_s(t) = A \sin(\omega t + \theta)$ V, this waveform is completely specified by three numbers – A , ω and θ . Therefore, sending a pure sine wave from a transmitter to a receiver in a communications context is pointless because such a waveform cannot carry any information other than that is contained in just three numbers. And, for that matter, no waveform that is known completely beforehand can carry any information. Certain degree of uncertainty in the waveform to be transmitted is a precondition for information transmission from one location to another. Hence, the correct mathematical description of an information-bearing signal can only be a statistical description. But, despite this, the entire area of Electronics and Communication Engineering relies heavily on sinusoidal analysis of circuits and systems.

This raises two questions – (i) Why did the Electrical Power Industry prefer sinusoidal waveform to any other waveform? (ii) Why does Electronics and Communications Engineering concern itself with sinusoidal analysis though a single-frequency sinusoid is hardly ever employed in a communication system?

These two questions are answered first in this chapter. Subsequently, the concepts of instantaneous power, average power, effective value of waveforms etc., are developed for sinusoidal waveforms as well as for other periodic waveforms.

7.1 WHY SINUSOIDS?

An Electrical Power System, like any other electrical circuit, contains passive electrical elements like resistors, inductors, capacitors, mutually coupled coils and active elements in the form of independent voltage sources and current sources. The elements in a power system are modelled by linear time-invariant elements to a first degree of approximation. Resistors produce voltage drops across them that are proportional to the current flowing through them. Inductors demand a voltage that is proportional to the rate of change of current through them. Capacitors demand a voltage across them that are proportional to the integral of current through them.

Electrical power system is usually voltage-driven. That is, independent voltage sources connected at various points in the system serve as sources of power in the system. The voltage sources connected at various source nodes (i.e., the generating stations) drive currents through the series interconnections in the system, get modified by the voltage drops produced across various series path elements and appear as load voltage at load nodes in a modified form. Shunt elements connected from various nodes to the reference node in the system also influence this process of transformation of source voltages into load voltages.

Voltage drops across various elements thus modify the load voltage with respect to source voltage. These voltage drops are decided by a *scaling* of current by resistance value in the case of a resistor. It is decided by *derivative of current* in the case of an inductor and by *integral of current* in the case of a capacitor.

A time-function retains its wave-shape when multiplied by a constant. But, in general, it does not maintain its wave-shape on *differentiation* and *integration*. Therefore, it follows that, in general, voltages and currents at various locations in an interconnected electrical network will have different wave-shape even if all sources in the network have same wave-shape.

That would surely complicate things in an Electrical Power System. In fact, it will not be a viable system at all. That prompts us to raise the question – is there any wave-shape that will be *invariant* to time-domain differentiation and integration?

A generalised exponential function, $Ae^{\alpha t}$, has this property, as may be verified easily. The value of α can be complex. If α is real and positive, it represents a growing exponential and obviously is not suited in an electrical system that is expected to operate steadily for extended duration. If α is real and negative, it represents a real decaying waveform that tapers down to zero sometime. An electrical system excited by a set of such sources will settle down ultimately to a state in which all voltages and currents everywhere will be zero. Obviously, such source waveforms cannot help the system to deliver power to loads in a steady manner for extended duration. If $\alpha = \gamma + j\omega$, the exponential function is a complex function of time and is given by $Ae^{\gamma t}\cos\omega t + jAe^{\gamma t}\sin\omega t$. We cannot generate an imaginary waveform in a physical system. But that problem can be solved by generating $0.5(Ae^{\alpha t} + Ae^{-\alpha t})$ which is equal to $Ae^{\gamma t}\cos(\omega t)$. Here, the same objections raised for a real exponential will hold if γ is non-zero. Therefore, we conclude that $0.5(Ae^{j\omega t} + Ae^{-j\omega t}) = A \cos\omega t$ or $-j0.5(Ae^{j\omega t} - Ae^{-j\omega t}) = A \sin\omega t$ are the possible choices for electrical power system source functions.

Changes in wave-shape can occur in linear time-invariant circuits containing dynamic elements.

We observe that a *constant time-function* (i.e., DC voltages and currents) is a special case of $Ae^{\alpha t}$ with $\alpha = 0$ and hence must satisfy the wave-shape preservation requirement. Thus, it must be possible to generate, transmit and deliver steady power to loads in an electrical system by means of DC sources. It is indeed possible and that was how Electrical Power Industry started in late 19th century. But the problem associated with DC system was total inflexibility with respect to voltage and current levels used in the system. For instance, if the customer had to be given 220 V at his premises, the generators had to generate 220 V and all the interconnection system had to work at that voltage level. As the load level increases, the current flow everywhere become excessive and generation/transmission become inefficient due to resistive loss everywhere in the system. It would have been very convenient if generation could be done at a voltage level economical from the point of view of electrical machine design and operation. Similarly, it would have been convenient if the transmission of power through transmission lines could be done at high voltage level so that the current level and consequently loss in lines would decrease. But this calls for generation at low voltage level, transmission at high voltage level and consumption at low voltage level. An efficient ‘voltage level conversion unit’ is required for this. Such voltage level conversion equipment for DC at high power levels was simply not available at the initial stages of evolution of power systems in the late 19th century and early 20th century. And the same task turned out to be very easy in the case of a sinusoidal voltage system. A two-winding transformer can easily and efficiently change voltage levels in a system in the case of sinusoidal voltages. Thus, economic generation and transmission of huge quantities of electrical power became possible with sinusoidal voltage system and transformers. This is another reason why Electrical Power Industry is linked to sinusoidal waveforms inalienably.

Advances in an area called Power Electronics resulted in power system loads becoming non-linear in nature progressively from early 1980s. Power Electronic Equipment (AC to DC converters, thyristorised DC motor drives, variable speed AC drives, uninterruptible power supplies, HVDC transmission systems etc.,) process the sinusoidal system power using non-linear devices for a variety of purposes. Non-linear loads draw *non-sinusoidal* currents from sinusoidal voltages. Non-sinusoidal current waveforms, subjected to differentiation and integration in various circuit elements, result in non-sinusoidal voltage drops across them. These non-sinusoidal voltage drops, in combination with the sinusoidal voltages produced at various generating stations, result in a non-sinusoidal voltage at load nodes in a power system. This is called the *Power System Harmonics Problem*. Another important feature of sine waves helps in analysing electrical systems that have non-sinusoidal periodic waveforms.

A broad class of periodic non-sinusoidal waveforms can be expressed as an infinite sum of sinusoidal waveforms with frequencies that are integer multiples of frequency of the periodic wave. For instance, a ± 1 volt square wave with 1 cycle per second frequency can be expressed as $= (4/\pi)[\sin 2\pi t + (1/3)\sin 6\pi t + (1/5)\sin 10\pi t + \dots + (1/n)\sin 2n\pi t + \dots]$. Infinite terms are needed; but the amplitude of sine wave decreases as the order of the term increases. It is possible to truncate this kind of series expansion of a periodic waveform to obtain reasonably accurate results in circuit analysis problems. Thus, series expansion of non-sinusoidal periodic waveforms in terms of sinusoidal waveforms and Superposition Theorem will help us to solve a circuit driven by such periodic waveforms provided we know how to solve it for a sinusoidal waveform. Thus, even a power system under the influence of non-linear loads can be analysed by sinusoidal analysis techniques with the help of this kind of series expansion for non-sinusoidal periodic waveforms.

Further, it turns out that, this series expansion of a periodic waveform leads to an expansion of even *aperiodic waveforms* in terms of sinusoidal waveforms as a limiting case. The series expansion of periodic waveform referred here is called *Fourier Series* and the expansion of an arbitrary aperiodic waveform in terms of sinusoidal functions is called *Fourier Transforms*. We deal with them extensively in later chapters of this book.

Thus, sinusoidal signals have gained their pre-eminent position in Electrical Power Engineering since they belong to a special class of time-functions that preserve their wave-shape on differentiation and integration.

A linear network excited by sinusoidal sources of a particular frequency will have sinusoidal voltages and currents at the same frequency everywhere in the system in the long run.

Why not DC instead of Sinusoids?

Why Sinusoids?

Sinusoidal waveforms and sinusoidal steady-state analysis of circuits is of great importance to electrical and electronic circuits due to the following reasons:

- Sinusoidal waveforms preserve their wave-shape in linear circuits.
- Sinusoidal waveform render the voltage levels in electrical and electronic systems flexible so that optimisation of sub-system performance in various parts of the system by choosing a suitable voltage level in that part of the system becomes possible. Transformers help us to realise this voltage flexibility.
- Periodic non-sinusoidal waveforms as well as a broad class of aperiodic waveforms can be expressed as a sum of sinusoids by Fourier series and Fourier transforms. Hence, circuit solution with such input waveforms can be obtained with relative ease if solution for a sinusoidal input is known.

Fourier Series and Fourier Transforms along with Superposition Theorem help us to solve an electrical circuit driven by sources with arbitrary source functions by solving it for sinusoidal excitation. This is yet another factor that leads to pre-eminence of sinusoidal waveforms in circuit analysis. In fact, this is *the* reason why Electronics and Communication Engineers use sinusoidal analysis.

7.2 THE SINUSOIDAL SOURCE FUNCTION

The sinusoidal voltage source function is generated in Synchronous Generators in Electrical Power Systems and by low power electronic oscillator circuits in Electronic Systems Communication Systems and Instrumentation/Measurement Systems. A source will have to be switched on at some point of time. Switching on a source may be a simple affair of switching on the DC power supply as in the case of an electronic sinusoidal oscillator circuit or switching on the AC mains to a function generator in the laboratory. It may be a complicated affair involving a sequence of steps as in the case of bringing a generator in a Nuclear Power Station online.

Moreover, the sinusoidal voltage source may not start producing a sinusoidal output as soon as it is powered up. Usually it goes through a transient period during which its output builds up. During this period, the output will not be a pure sine wave. After an initial period of adjustments, it starts delivering sinusoidal output to whatever that is connected at its output.

Even if it produces a sine wave right from the instant at which it is powered up, it may not start at zero position in the waveform or at peak position in the waveform.

We assume, in this section, that the sinusoidal sources have been powered up in the past and have become steady. Moreover, we assume that they started at zero position on a sine wave when they were switched on. The concepts we evolve are not really dependent on these assumptions. The assumptions are made only to render clarity to the discussion that follow.

7.2.1 Amplitude, Period, Cyclic Frequency and Angular Frequency

Consider a single sinusoidal voltage source that was powered up earlier. We start observing the waveform of the voltage output in an oscilloscope from a particular point of time. We assign zero value to the time variable at that instant. Thus, the source was powered up earlier with respect to the instant at which we start observing it. The observed waveform is shown in Fig. 7.2-1. The time variable t is used in the horizontal axis.

The maximum positive value attained by the waveform is seen to be 10 V and the maximum negative value attained is also 10 V. This quantity is called the *amplitude* of the

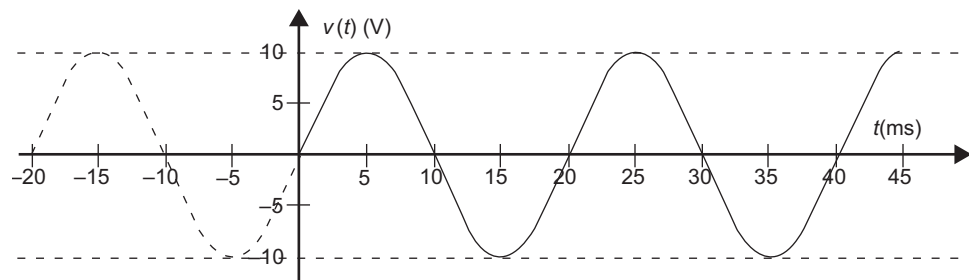


Fig. 7.2-1 Waveform of a Sinusoidal Voltage

sinusoidal waveform. It so happened that the waveform was crossing zero from negative value to positive value at $t = 0$. This zero crossing is called the *positive-going zero crossing*. The zero crossing that happens when the voltage is crossing over from positive value to negative value is termed as *negative-going zero crossing*. That $t = 0$ happens to be a positive-going zero crossing is the result of a coincidence. As a result, we are now free to write the voltage waveform that we observe from $t = 0$ onwards as $v(t) = 10 \sin \omega t$. We need to work out the meaning and value of ω .

We observe from Fig. 7.2-1 that the sinusoidal voltage completes one full cycle of variation in 20 ms. That is, if we start at any t and move through the waveform till we reach $t + 20$ ms, we will find that the instantaneous voltage at $t + 20$ ms is the same as the instantaneous voltage at t . Moreover, the shape of voltage variation in any $(t, t + 20)$, $(t + 20, t + 40)$ interval is same as in the interval $(t, t + 20)$, where unit of time is in ms and n is a positive integer. Thus, the wave-shape is repetitive with its basic repeating unit decided by any 20 ms interval. That is, the waveform is periodic from the instant we start observing it. The *period* of this waveform is 20 ms. In general, period of a periodic waveform is the time interval needed to complete one full cycle of the waveform. The symbol, T is used to represent the period of a periodic waveform. In other words, it is the width of the basic repeating unit of the periodic waveform in the time-axis.

The number cycles of variation that the waveform goes through in one second is defined as its *cyclic frequency*. The qualifier *cyclic* is often dropped when there is no cause for ambiguity or when the unit employed makes it clear that it is cyclic frequency that is being referred to. The unit of *cyclic frequency* is 'cycles-per-second' and is given a name Hertz. Hertz is written in short form as Hz. The shortened form 'cps' is also used to designate the unit of cyclic frequency. The cyclic frequency of the waveform in Fig. 7.2-1 is $1/20 \text{ ms} = 50 \text{ Hz}$. *Cyclic frequency* is usually indicated by the symbol f .

A sinusoidal function of an angle is periodic with a period of 2π rad. Thus, the argument of the sinusoidal function in a sinusoidal waveform will go through an increment of 2π rad in one period. Therefore, the increment in the argument of the trigonometric function in one second will be $2\pi/T$ rad, where T is the period of waveform ($= 1/f$). This quantity, which represents the *rate of change of angle argument of the sinusoidal function with respect to time* is defined as the *angular frequency* or *radian frequency* of the sinusoidal waveform and is usually represented by the symbol ω (lower case omega). The unit of *angular frequency* is radians/sec, abbreviated as rad/s. Thus,

$$f = \frac{1}{T} \text{ Hz}, \omega = \frac{2\pi}{T} = 2\pi f \text{ rad/s}, T = \frac{2\pi}{\omega} = \frac{1}{f} \text{ s}.$$

The sinusoidal waveform of voltage source $v(t) = 10 \sin 100\pi t$ shown against t in Fig. 7.2-1 is redrawn against the angular argument ωt in Fig. 7.2-2.

A sinusoidal source voltage waveform that undergoes a positive-going zero crossing at $t = 0$ can be expressed as $v(t) = A \sin \omega t = A \sin(2\pi/T)t = A \sin 2\pi f t$ V, where A is its

Period T of a sinusoidal waveform is the width of the basic repeating unit of the periodic waveform in the time-axis.

Cyclic frequency f defined.

Angular frequency ω defined as rate of change of angular argument of the sinusoidal waveform with respect to time.

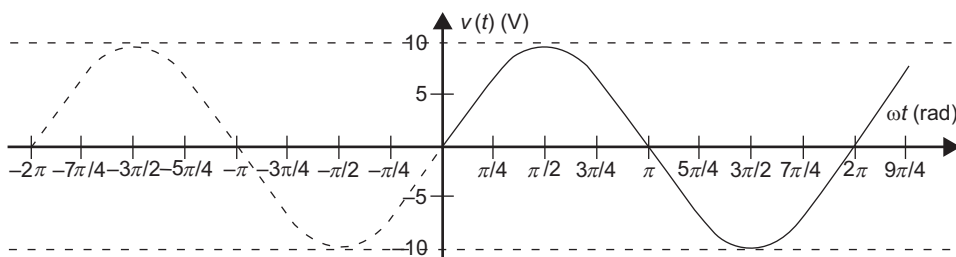


Fig. 7.2-2 Sinusoidal Waveform $v(t)$ Plotted Against ωt

amplitude, T is its period in seconds, f is its cyclic frequency in s^{-1} (Hertz, Hz) and ω is its radian frequency or angular frequency in radians/seconds (rad/s) unit.

7.2.2 Phase of a Sinusoidal Waveform

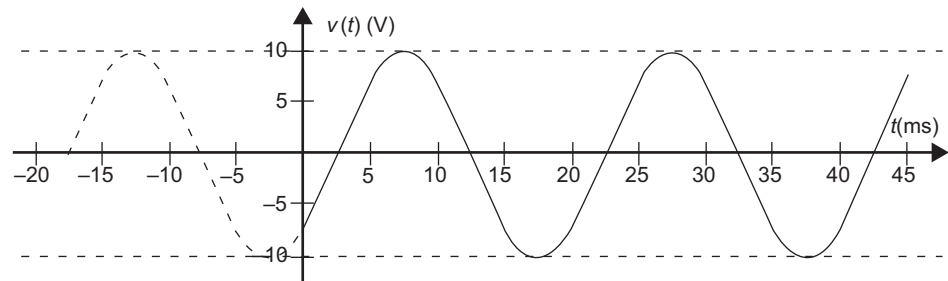


Fig. 7.2-3 Sinusoidal Waveform with a Non-zero Phase

Consider the observed sinusoidal waveform shown in Fig. 7.2-3. This waveform may be interpreted as a sinusoidal source powered up 17.5 ms prior to the start of observation. Assume that the source started delivering the sine wave output from zero value. The instantaneous value observed at the starting instant of observation is not zero, but -7.07 V. The amplitude is observed to be 10 V and the frequency is 50 Hz. Thus, we can express this waveform as $v(t) = 10 \sin(100\pi t - \theta)$, where θ is an angle to be determined from the observed amplitude and instantaneous value at $t = 0$. The value at $t = 0$ is $10 \sin\theta$ and this is seen to be -7.07 V. Therefore, $\theta = 45^\circ = \pi/4$ rad.

$$\therefore v(t) = 10 \sin\left(100\pi t - \frac{\pi}{4}\right).$$

This waveform is plotted against ωt in Fig. 7.2-4. Note that the instantaneous value of $v(t)$ at $\omega t = 100\pi t = \pi/4$ rad is zero as it should be.

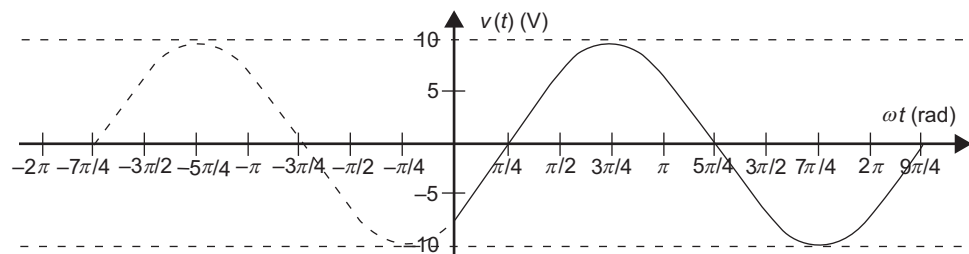


Fig. 7.2-4 A Sinusoidal Waveform with Non-zero Phase Plotted Against ωt

Phase of a Sinusoid

The quantity θ in $v(t) = A \sin(\omega t + \theta)$ is defined as the *phase* of the sinusoidal function.

Phase has to be in radians when a trigonometric function is evaluated.

The argument of trigonometric function in the expression for a sinusoidal waveform is always in radians. However, it is permitted to write the angle θ in degrees provided the symbol of degree is clearly shown. That is, $v(t)$ can be written as $10 \sin(100\pi t - 45^\circ)$ V, but not as $10 \sin(100\pi t - 45)$. In the second case, 45 will be interpreted with radian unit. Moreover, though the form $10 \sin(100\pi t - 45^\circ)$ V is permitted, 45° has to be converted into radians before subtracting from the value of $100\pi t$ for some particular t before evaluating

the sine function. In short, the form $10 \sin(100\pi t - 45^\circ)$ is allowed only as a *notation* and not for *function evaluation*.

7.2.3 Phase Difference Between Two Sinusoids

Consider the situation in Fig. 7.2-5. Two observers – A and B – use XY-Recorders A and B, respectively to record the output from two sinusoidal voltage sources $v_1(t)$ and $v_2(t)$. A closes the two switches onto its recorder at $t = 0$. t represents the time-axis chosen by A. B closes the switches onto its recorder at $t' = 0$. t' represents the time-axis chosen by B. $v_1(t)$ has an amplitude of V_{m1} and $v_2(t)$ has an amplitude of V_{m2} .

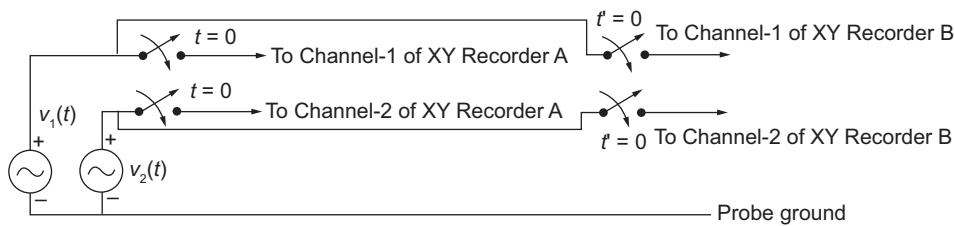


Fig. 7.2-5 Simultaneous Observation of Two Sinusoidal Sources by Two Observers with Different Starting Instants for Observation

The waveforms recorded by Observer A are shown in Fig. 7.2-6(a) by the *solid* curve. The *dotted* curve shows the sinusoidal variation of sources prior to recording and will not show up in the recorder output. The waveforms in Fig. 7.2-6(a) are normalised with respect to their respective amplitude values to obtain the waveforms in Fig. 7.2-6(b).

Two pairs of *similarly located waveform points within a cycle period* are located in the wave-shape of $v_1(t)/V_{m1}$ and $v_2(t)/V_{m2}$ as shown in Fig. 7.2-6(b). Observer A notes that similarly located points in the two waveforms are separated by 45° angle argument. A also notes that points on $v_2(t)$ come *after* (in a visual sense) similarly located points on $v_1(t)$.

Phase Difference Between Sinusoids

The angular difference between similarly located points within a cycle period on two normalised sinusoidal waveforms (normalised with respect to their respective amplitude values) with same frequency is defined as the *phase difference* between them.

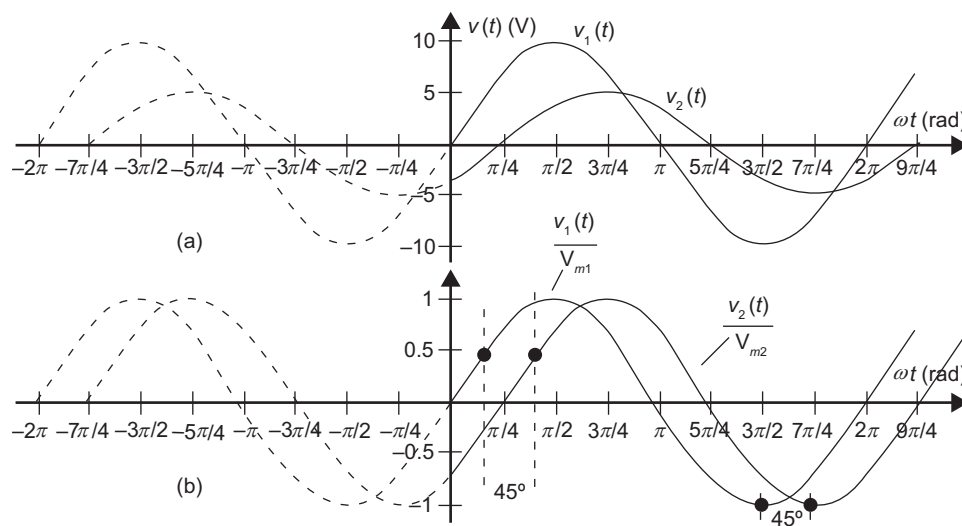


Fig. 7.2-6 Waveform Observation by Observer A

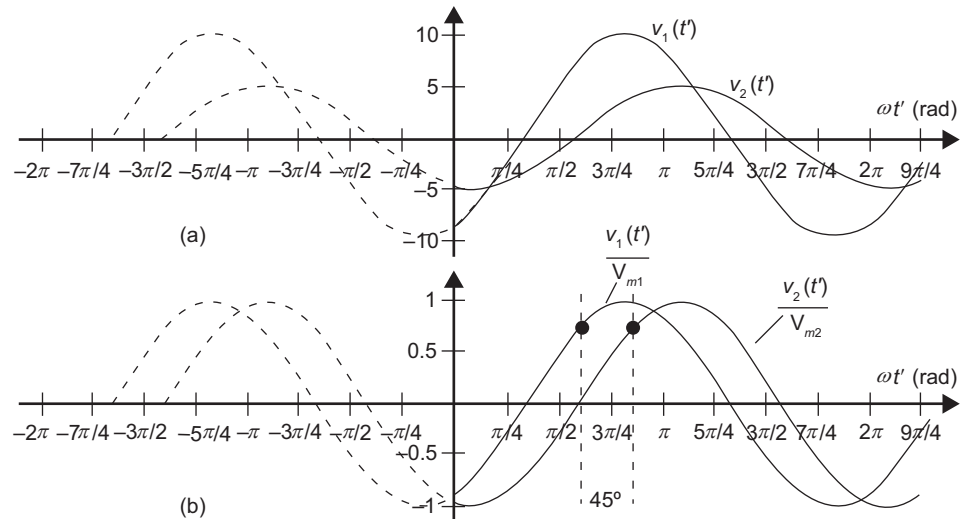


Fig. 7.2-7 Waveform Observations by Observer B

Similar observations recorded by B in $\omega t'$ axis are shown in Fig. 7.2-7.

B measures 45° angular separation between similarly located points on normalised $v_1(t')$ and $v_2(t')$. Moreover, B observes that points on $v_2(t')$ that come *after* similarly located points on $v_1(t')$.

The angular difference between similarly located points within a cycle period on two normalised sinusoidal waveforms (normalised with respect to their respective amplitude values) with same frequency is defined as the phase difference between them. The phase difference between two sinusoids is independent of choice of origin in t or ωt axis. The precedence relationship (i.e., which comes after (in a visual sense) which) between them in t or ωt axis too, is independent of choice of origin.

However, based on the observed amplitudes, values at origin and the position of first zero-crossing, A will write the sinusoidal functions as $v_1(t) = 10\sin 100\pi t$ V and $v_2(t) = 5\sin(100\pi t - 45^\circ)$ V. But, B will conclude that $v_1(t') = 10\sin(100\pi t' - 60^\circ)$ V and $v_2(t') = 5\sin(100\pi t' - 105^\circ)$ V. Thus, the *phase* of $v_1(t)$ is 0° and *phase* of $v_2(t)$ is -45° as far as A is concerned. And they are -60° and -105° , respectively as from B's point of view.

The phase of a sinusoidal waveform depends on the choice of origin in t or ωt axis. Phase difference between two sinusoidal waveforms at same frequency does not.

When a waveform point on a sinusoidal function $v_2(t)$ appears *after* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ with *same frequency*, $v_2(t)$ is said to *lag* $v_1(t)$ in phase and the corresponding phase difference between them is called a *lag phase angle* under this condition.

Similarly, when a waveform point on a sinusoidal function $v_2(t)$ appears *before* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ with *same frequency*, $v_2(t)$ is said to *lead* $v_1(t)$ in phase and the corresponding phase difference between them is called a *lead phase angle* under this condition.

It must be obvious that if $v_2(t)$ lags $v_1(t)$, then, $v_1(t)$ must necessarily *lead* $v_2(t)$. Moreover, if $v_2(t)$ lags $v_1(t)$, then, $v_2(t')$ will also lag $v_1(t')$, where t' is a new time variable as a result of a different choice of origin.

7.2.4 Lag or Lead?

Similarly located points on two sinusoidal waveforms with same frequency have to be located within a period of the waveforms. But this leads to two choices for locating the

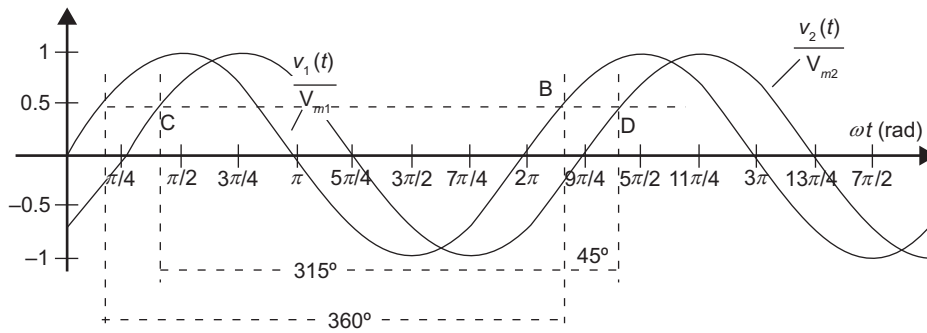


Fig. 7.2-8 Relationship Between Phase Lag and Phase Lead

point on the second waveform after having chosen a point on the first waveform (Fig. 7.2-8).

We locate the point B on normalised $v_1(t)$ first. We are free to locate the similarly located point on normalised $v_2(t)$ on either side of B within a span of 2π rad or 360° . This gives us two choices – point C and point D on the second waveform. If we choose point C, we can conclude that $v_2(t)$ *leads* $v_1(t)$ by 315° . If we choose point D, we conclude that $v_2(t)$ *lags* $v_1(t)$ by 45° . Therefore, a lag angle of θ rad and a lead angle of $(2\pi - \theta)$ rad mean the same. As a convention, we favour the angle that turns out to be less than 180° (or π rad). Thus, in Fig. 7.2-8, we will term it as a lag angle of 45° .

7.2.5 Phase Lag/Lead Versus Time Delay/Advance

It does leave a certain ambiguity about which waveform is *after* the other. If we accept the point C on $v_2(t)$ in Fig. 7.2-8 as the point corresponding to point B on $v_1(t)$, we conclude that $v_2(t)$ comes *before* $v_1(t)$. Similarly, If we accept the point D on $v_2(t)$ as the point corresponding to point B on $v_1(t)$, we conclude that $v_2(t)$ comes *after* $v_1(t)$. However, note carefully that we had been careful to keep the precedence relationships – *before* and *after* – only in relation to our visual perception of the waveform plots. We have not yet ascribed temporal significance to these terms. That is, we have *not* stated until now that if a waveform $v_2(t)$ comes *after* $v_1(t)$ in a *visual* sense, then, $v_2(t)$ started later than $v_1(t)$ in time. In other words, we have not correlated the *phase difference* between two waveforms with *time delay* or *time advance* between them. The term ‘*phase lag*’ tends to give us an impression that the waveform that *lags* behind suffered some *time delay* with respect to the other waveform. But this impression can be wrong. Similarly, the waveform that *leads* ahead of another waveform did not necessarily start earlier. The reader is cautioned against equating a ‘*phase lag*’ with a ‘*time delay*’ and a ‘*phase lead*’ with a ‘*time advance*’ indiscriminately. There are situations in which a ‘*phase lag (lead)*’ implies a ‘*time delay (advance)*’ – in that case, we will term the phase lag (lead) as ‘*phase delay (advance)*’. And there are situations in which *lag/lead* cannot be uniquely correlated to *delay/advance* in time-domain.

Consider the two waveforms $v_1(t)$ and $v_2(t)$ in Fig. 7.2-9(a). Additional information in the form of *dotted* curves is also shown in Fig. 7.2-9(a). The frequency of waveform is 50 Hz. The waveforms in Fig. 7.2-9(a) show that the source $v_1(t)$ started generating a sinusoidal voltage at 20 ms before the observation started and $v_2(t)$ started only 2.5 ms later. It is also clear that $v_2(t)$ lags $v_1(t)$ by 45° . Thus, a *time delay* of 2.5 ms has resulted in a *phase lag* of 45° . Since we know from the additional information provided in the form of dotted curves that the phase difference between two sources resulted from a time delay, we can term this 45° phase lag as a 45° *phase delay* too. Obviously the following relation between *time delay* and *phase delay* holds.

Phase Lag/Lead versus Time Delay/Advance

Phase lag/lead between various voltages and currents in a Power System has profound implications in the economic operation of the system.

Phase delay and time delay between various sinusoidal waveforms in an electronic system or communication system has great significance in terms of waveform distortion and loss of information contained in a waveform.

Therefore, Electrical Power Engineers pay a great deal of attention to *phase lag/lead* between waveforms whereas Electronics and Communication Engineers place even higher emphasis on *phase delay/advance* and *time delay/advance* between waveforms.

Hence, these terms are discussed in detail in this section.

Relationship between phase lag, phase delay and time delay.

phase delay ϕ in radians = $\omega \times$ time delay t_d in seconds
 i.e., $\phi = \omega t_d$

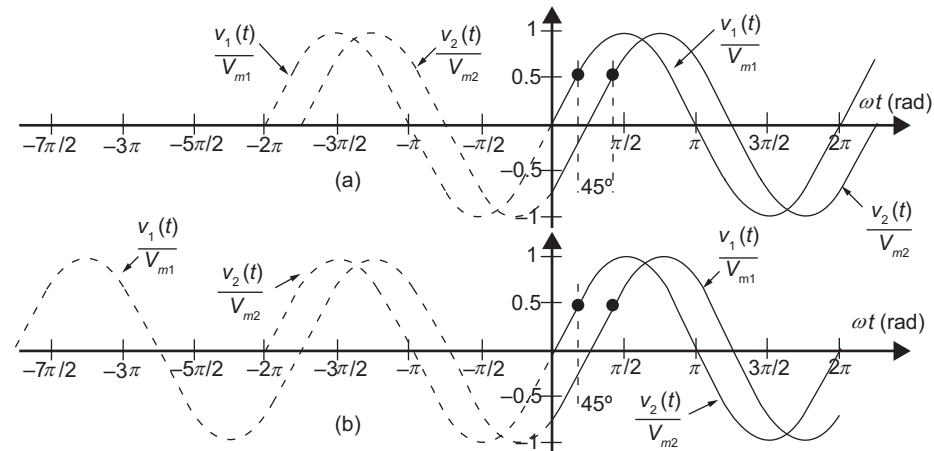


Fig. 7.2-9 Illustrating Phase Delay and Time Delay

Similar statement can be arrived in the case of *time advance* also – provided that we know from information other than we obtained from observing the two waveforms from $t = 0$ that the observed phase difference is due to a *time advance*.

phase advance ϕ in radians = $\omega \times$ time advance t_d in seconds
 i.e., $\phi = \omega t_d$

Now, consider the waveforms in Fig. 7.2-9(b). Here $v_1(t)$ started 17.5 ms earlier than $v_2(t)$. But if we go only by the observation from $t = 0$ onwards, we will conclude that $v_2(t)$ *leads* $v_1(t)$ by 45° or equivalently $v_2(t)$ *lags* $v_1(t)$ by 315° , and, as per the agreed convention, we will settle for ' $v_2(t)$ *leads* $v_1(t)$ by 45° '. But, in the light of the additional information given in the form of *dotted* curves, a translation to the effect that $v_2(t)$ started 2.5 ms earlier than $v_1(t)$ will be in error. Actually $v_2(t)$ started 17.5 ms (315°) after $v_1(t)$ and hence *phase delay* of $v_2(t)$ is 315° and *time delay* of $v_2(t)$ is 17.5 ms with respect to $v_1(t)$. The conclusion from observation from $t = 0$ onwards can also be stated as $v_1(t)$ *lags* $v_2(t)$ by 45° . Again, a translation to the effect that $v_1(t)$ started 2.5 ms after $v_2(t)$ started is wrong in the light of additional information given. Actually $v_1(t)$ started 17.5 ms before $v_2(t)$ started and hence $v_1(t)$ has a *phase advance* of 315° and a *time advance* of 17.5 ms with respect to $v_2(t)$.

The additional information needed to decide *time delay/advance* from *phase lag/lead* is not usually available in the case of multiple sinusoidal source waveforms in a complex electrical system. But then, we do not usually need the *time delay/advance* information in Electrical Power Systems.

There is one situation in which this additional information needed is invariably available. Consider a situation in which a sinusoidal voltage source is applied to a linear electrical network at $t = 0$. The circuit variables respond to this excitation and assume pure sinusoidal variation at the same frequency as that of sinusoidal excitation in the long run. There will be a definite phase difference between a response variable (may be a current in some element or a voltage across some element) and the source function. *No physical system can produce a response before the excitation is applied to it.* Response always *follows* the excitation in a *physical system* and cannot precede excitation. This intuitively obvious fact is known as the '*law of causality*' for physical systems. Thus, law of causality of physical systems effectively states that the response will be *delayed* with respect to excitation.

Additional information is required to translate an observed *phase lag/lead* relationship between two sinusoidal waveforms into a *phase delay/advance* (equivalently, *time delay/advance*) relationship between them.

'Phase lag' is not necessarily a 'phase delay' and 'phase lead' is not necessarily a 'phase advance'.

'Phase lag' does not necessarily imply 'time delay' and 'phase lead' does not necessarily imply 'time advance'.

EXAMPLE: 7.2-1

Two sinusoidal waveforms, $x(t)$ and $y(t)$, recorded from $t = 0$ are shown in Fig. 7.2-10. $x(0) = -2.571$ and $y(0) = 1.25$ (i) Express $x(t)$ and $y(t)$ as sine functions and identify their amplitude, period, cyclic frequency, radian frequency, phase and phase difference. (ii) If no additional information is available, list all possible time delay/advance relations between the two assuming that both started at positive-going zero-crossing position. (iii) If $x(t)$ is voltage waveform that was powered up and applied to a linear electrical circuit long back in the past and $y(t)$ is the current flow in some element in that circuit, find the time delay/advance between the two and phase delay/advance between the two.

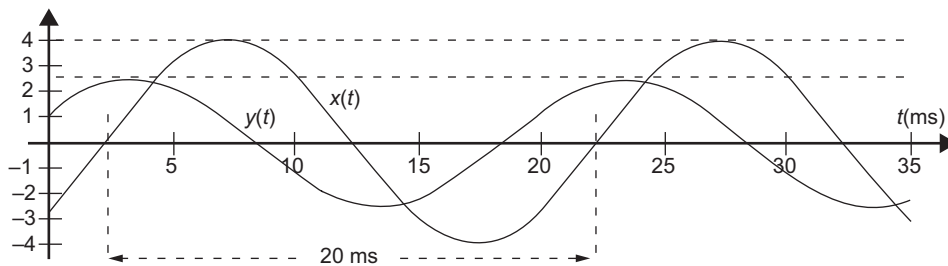


Fig. 7.2-10 Sinusoidal Waveforms for Example 7.2-1

SOLUTION

(i) The period of both waveforms, $T = 20$ ms. Therefore, $f = 50$ Hz and $\omega = 100\pi$ rad/s. Amplitude of $x(t)$ is 4 and amplitude of $y(t)$ is 2.5.

Therefore, $x(t) = 4\sin(100\pi t + \theta_x)$ and $y(t) = 2.5\sin(100\pi t + \theta_y)$, where θ_x and θ_y are the phases of $x(t)$ and $y(t)$, respectively. The values of $x(t)$ and $y(t)$ at $t = 0$ are given as -2.571 and 1.25 , respectively.

$$\therefore \sin\theta_x = -2.571/4 = -0.6428 \Rightarrow \theta_x = -40^\circ \text{ or } -140^\circ.$$

The choice between -40° and -140° is made by observing that if it is -40° , the first zero-crossing of the waveform that takes place at $\omega t = -\theta_x$ will take place within the first quarter cycle and if it is -140° the first zero-crossing after $t = 0$ will take place after first quarter cycle. In the present case $x(t)$ crosses zero within first 5 ms (first quarter cycle) and hence $\theta_x = -40^\circ$.

$$\therefore x(t) = 4\sin(100\pi t - 40^\circ).$$

Similarly, $\sin\theta_y = 1.25/2.5 = 0.5 \Rightarrow \theta_y = 30^\circ \text{ or } 150^\circ$. Now the first zero-crossing will take place at $\omega t = 150^\circ$ position (i.e., in the second quarter cycle) if $\theta_y = 30^\circ$ and it will take place at $\omega t = 30^\circ$ position (i.e., in the first quarter cycle) if $\theta_y = 150^\circ$. Since it takes place within the second quarter cycle in the present case, $\theta_y = 30^\circ$.

$$\therefore y(t) = 2.5\sin(100\pi t + 30^\circ).$$

The phase difference between the two waveforms has a magnitude of $30^\circ - (-40^\circ) = 70^\circ$ and $x(t)$ lags $y(t)$ by 70° . Equivalently, $y(t)$ leads $x(t)$ by 70° . Equivalently, $x(t)$ leads $y(t)$ by 290° and $y(t)$ lags $x(t)$ by 290° .

(ii) 70° phase difference translates to $20 \times 70/360 \approx 3.89$ ms time interval. The possibilities are:

(a) $y(t)$ started $(3.89 + 20n)$ ms before $x(t)$ and therefore $y(t)$ has a phase advance of $(70^\circ + n360^\circ)$ with respect to $x(t)$, where $n = 0, 1, 2, 3, \dots$. This may also be restated as $x(t)$ has a phase delay of $(70^\circ + n360^\circ)$ with respect to $y(t)$.

(b) $x(t)$ started $(16.11 + 20n)$ ms before $y(t)$ and therefore $x(t)$ has a phase advance of $(290^\circ + n360^\circ)$ with respect to $y(t)$, where $n = 1, 2, 3, \dots$. This may be restated as $y(t)$ has a phase delay of $(290^\circ + n360^\circ)$ with respect to $x(t)$.

Phase Lag and Time Delay in Circuits

The response sinusoid in an electrical circuit will always be delayed with respect to the excitation sinusoid quite regardless of whether the phase difference is a lag angle or lead angle. A phase lead that the response variable exhibits with respect to excitation variable in a physical electrical network has to be understood as a phase delay that is more than π rad and a time delay that is more than half-period.

The reader is cautioned against the commonly made mistake of assuming that the apparent phase lead exhibited by a response variable implies a phase advance or time advance.

(iii) Now, $x(t)$ is the cause and $y(t)$ is the effect in a physical electrical circuit. Therefore, by law of causality, $y(t)$ can have only a phase delay with respect to $x(t)$. Therefore, $y(t)$ has a phase delay of $(290^\circ + n360^\circ)$ with respect to $x(t)$.

However, this does not mean that the circuit waited for $(16.11 + 20n)$ ms after the voltage waveform was applied to it doing nothing in that interval and then started producing a sinusoidal current in the element under consideration. What happens in the electrical circuit is that, as soon as the voltage waveform applied, the circuit starts a mixed response that includes even non-sinusoidal terms caused by the electrical inertia of the circuit. This period is called the transient period. The non-sinusoidal components in the response die down to zero as time progresses. After a sufficiently long duration decided by circuit parameters, a steady-state comes up in the circuit in which all the response variables become pure sinusoidal waveforms at the same frequency as that of excitation. And, by that time the current $y(t)$ would have acquired a steady phase delay of $(290^\circ + n360^\circ)$ with respect to the applied voltage $x(t)$. The value of n that is applicable can be obtained only if the circuit is known in detail and a full circuit solution is obtained.

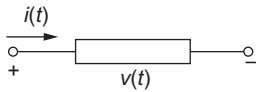


Fig. 7.3-1 A Two-terminal Element with Voltage and Current Marked as per Passive Sign Convention

Instantaneous power delivered to an element is defined as $p(t) = v(t) i(t)$.

Total energy dissipated in the element from infinite past to t .

7.3 INSTANTANEOUS POWER IN PERIODIC WAVEFORMS

A waveform $v(t)$ is said to be periodic with a periodicity of T s if $v(t + nT) = v(t)$ for all integer values of n and for all t . This implies that it must be possible to identify a basic section of the waveform that lasts for T s and that repeats to infinite extent into the past and into the future. Thus, a waveform is strictly periodic only if it is ever existent. But, in practice waveforms are switched on at some definite time instant. Such switched waveforms cannot be called periodic waveforms in the strict sense of definition of periodicity. However, we can view them as periodic waveforms for circuit analysis purposes provided we focus our attention to time instants located far away from the instant at which the waveform was switched on.

Consider a two-terminal electrical element with the current and voltage variables marked as per *passive sign convention* in Fig. 7.3-1.

The voltage difference v_{AB} between two points A and B is the work to be done in moving $+1$ C of charge from B to A. Energy has to be spent in carrying charge from a lower potential point to higher potential point. Similarly, energy is released when a charge is allowed to fall through a higher potential point to lower potential point. The amount of charge that went through the element from a higher potential point to lower potential point in one second is given by $i(t)$. Therefore, the product of $v(t)$ and $i(t)$ must be the energy released into element in one second. The rate of change of energy is defined as *instantaneous power* and denoted by $p(t)$.

Therefore, instantaneous power delivered to a two-terminal element, $p(t) = v(t) i(t)$, where $v(t)$ and $i(t)$ are the element variables defined as per passive sign convention.

Then, energy delivered to a two-terminal element is obviously given by

$$E(t) = \int_{-\infty}^t p(t) dt = \int_{-\infty}^t v(t) i(t) dt = E(0) + \int_0^t v(t) i(t) dt,$$

where $E(0)$ is the total energy dissipated in the element from infinite past to $t = 0$.

And the relation between the energy function $E(t)$ and the instantaneous power $p(t)$ is given by $p(t) = \frac{dE(t)}{dt}$.

Let $E_i(t)$ be the energy dissipation function (i.e., the net energy delivered from $-\infty$ to t) of the i^{th} element in a b -element electrical circuit. Then the total energy dissipation

function of the circuit is $E_T(t) = \sum_{i=1}^b E_i(t)$. The circuit considered as a whole is an isolated system and the *total energy in an isolated system is a constant* by Conservation of Energy.

$$\therefore \sum_{i=1}^b E_i(t) = A \text{ Constant.}$$

This implies that if some elements of the circuit are receiving energy, then, some other elements must be losing energy.

Since the statement of conservation of energy is true on an instant to instant basis, we can differentiate both sides of the above equation with respect to time.

$$\therefore \frac{dE_T(t)}{dt} = 0 \Rightarrow \frac{dE_1(t)}{dt} + \frac{dE_2(t)}{dt} + \dots + \frac{dE_b(t)}{dt} = 0.$$

But each term of the type $\frac{dE_i(t)}{dt}$ can be interpreted as the *instantaneous power delivered to the i^{th} circuit element*.

$$p_1(t) + p_2(t) + \dots + p_b(t) = 0 \Rightarrow v_1(t)i_1(t) + v_2(t)i_2(t) + \dots + v_b(t)i_b(t) = 0.$$

Therefore, *sum of the instantaneous power delivered to all the elements in a circuit will be zero*. Equivalently, *sum of the instantaneous power delivered by all the elements in a circuit will be zero*.

This implies that in any circuit some elements will be delivering positive power and the remaining elements will be receiving positive power at all instants of time. The sum of positive powers delivered will be equal to the sum of positive powers received (or consumed) at all t . This is the *principle of conservation of instantaneous power* which is valid for all isolated circuits containing arbitrary kind of elements.

Isolated circuit is the one that has no interaction with environment. If a circuit is coupled to some other elements (electromagnetic coupling, optical coupling, thermal coupling, etc.), then the coupling also has to be modelled into the circuit before calling it an isolated system.

Energy conservation in an isolated circuit.

Principle of Conservation of Instantaneous Power

The sum of instantaneous power delivered to all elements in an isolated circuit will be zero.

i.e., $p_1(t) + p_1(t) + \dots + p_b(t) = 0$, where b is the total number of elements in the circuit.

EXAMPLE: 7.3-1

A DC voltage of 10 V is switched on to a resistor of 10Ω at $t = 0$. Find and plot $p(t)$ and $E(t)$ for $t \geq 0$.

SOLUTION

$p(t) = \frac{V^2}{R} = 10 \text{ W}$ for $t \geq 0$ and $E(t) = \int_0^t 10 dt + 0 = 10t \text{ J}$. The waveforms of applied instantaneous power and the energy dissipated are shown in Fig. 7.3-2.

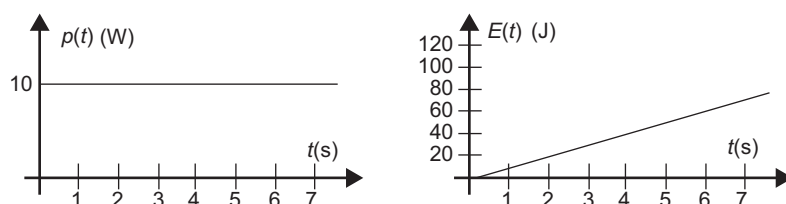


Fig. 7.3-2 Instantaneous Power and Energy for Example 7.3-1

EXAMPLE: 7.3-2

A symmetric square wave of 100 Hz frequency and ± 10 V amplitude is switched on to a resistor of $10\ \Omega$ at $t = 0$ at the positive-going zero crossing of the square wave. Find and plot the instantaneous power and energy dissipation in the resistor.

SOLUTION

$$p(t) = v(t) i(t) = [v(t)]^2/R$$

When a symmetric square wave is squared it becomes a constant quite irrespective of its frequency. Therefore, $[v(t)]^2 = 100$ for $t \geq 0$.

$\therefore p(t) = 10\text{ W}$ for $t \geq 0$ and $E(t) = 10t\text{ J}$. The plots are same as in Fig. 7.3-2.

Therefore, we conclude that applying a symmetric square wave voltage of amplitude V and frequency f to a resistive load has the same effect as applying a DC voltage of V volts to it as far as the instantaneous power and energy dissipation function are concerned.

EXAMPLE: 7.3-3

A periodic rectangular pulse voltage shown in Fig. 7.3-3 is applied to a resistor of $1\ \Omega$. Find and plot instantaneous power and energy dissipation function.

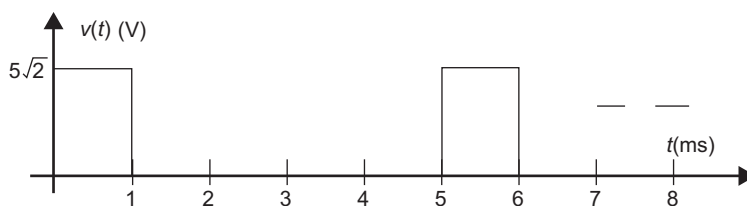


Fig. 7.3-3 The Applied Voltage Waveform for Example 7.3-3

SOLUTION

The waveform of $p(t)$ is obtained by squaring the waveform of $v(t)$ and dividing by 1. $E(t)$ is obtained by integrating (i.e., evaluating the area under $p(t)$ curve) the power waveform. The two plots are shown in Fig. 7.3-4 for the time range $[0\text{ ms}, 30\text{ ms}]$.

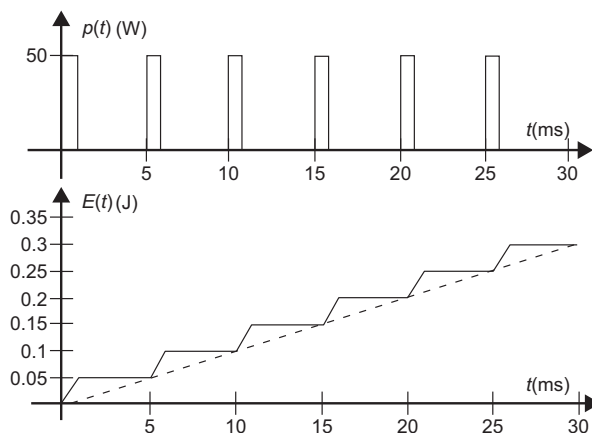


Fig. 7.3-4 Instantaneous Power and Energy Plots for Example 7.3-3

Energy delivery to the load takes place through power pulses in this example. As a consequence, the energy dissipated in the resistor rises non-uniformly with time. However, $E(t)$ is either increasing or constant at all t consistent with the fact that $p(t)$ never goes negative. The plot of $E(t)$ versus t shows a *dotted* line. This line is obtained by joining the points representing the total energy dissipation in the resistor at the end of a cycle. It turns out to be *straight* line with a slope of $0.01 \text{ J/ms} = 10 \text{ W}$. The power pulse located within a cycle of $p(t)$ has an amplitude of 50 W and duration of 1 ms . The cycle duration is 5 ms . Hence, the energy delivered to the resistor in each cycle is $50 \text{ W} \times 0.001 \text{ s} = 0.05 \text{ J}$. The same energy would have been delivered to the resistor in 5 ms if a constant power of 10 W were available for the entire cycle. We note that 10 W is the average of instantaneous power over one cycle period.

A line joining the end-of-cycle energy values in the energy versus time diagram will have a slope that is equal to the average value of instantaneous power over a cycle.

EXAMPLE: 7.3-4

$v(t) = 10\sqrt{2} \sin t \text{ V}$ is applied to a 10Ω resistor from $t = 0$ onwards. Plot $p(t)$ and $E(t)$ for $t \geq 0$.

SOLUTION

The waveform of $p(t)$ is obtained by squaring the waveform of $v(t)$ and dividing by 10 . $E(t)$ is obtained by integrating (i.e., evaluating the area under $p(t)$ curve) the power waveform.

$$\therefore p(t) = 20 \sin^2 t = 10 - 10 \cos 2t \text{ for } t \geq 0$$

$$E(t) = \int_0^t [10 - 10 \cos 2t] dt + 0 = 10t - 5 \sin 2t.$$

Therefore, the instantaneous power has a frequency that is double that of the frequency of voltage. Moreover, $p(t)$ contains a DC component. The plots of $p(t)$ and $E(t)$ are shown in Fig. 7.3-5 for $0 \leq t \leq 4\pi \text{ s}$.

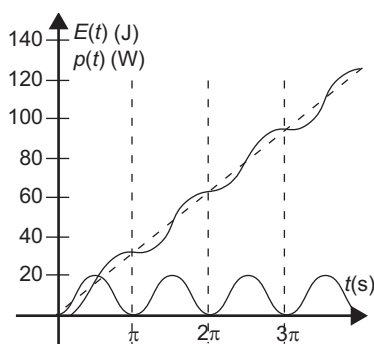


Fig. 7.3-5 Plots of Instantaneous Power and Energy for Example 7.3-4

The cycle period of power waveform is $\pi \text{ s}$. The values of $E(t)$ at integer multiples of cycle period of power waveform are joined by a *dotted* line as shown in Fig. 7.3-5. This line turns out to be a *straight* line with a slope of 10 W . The average value of $p(t)$ over one cycle of $\pi \text{ s}$ is also 10 W , as shown in the following derivation.

$$\text{Average of } p(t) \text{ over a cycle} = \frac{\text{Area of } p(t) \text{ over a cycle}}{\text{Period of cycle}}$$

$$= \frac{\int_0^\pi (10 - 10 \cos 2t) dt}{\pi} = 10 - \frac{5}{\pi} \sin 2t \Big|_0^\pi = 10 \text{ W}.$$

The energy function in this example is a monotonically increasing function of t . Next example considers a situation in which the energy function is not monotonic on t .

EXAMPLE: 7.3-5

A sinusoidal voltage source $v(t) = 10\sqrt{2} \sin t$ V is connected across a sinusoidal current source $i(t) = \sqrt{2} \sin(t - 60^\circ)$ A at $t = 0$. $i(t)$ flows out of the positive terminal of $v(t)$. Find the instantaneous power delivered by the voltage source to the current source and the corresponding energy function and plot them.

SOLUTION

Let $p(t)$ be the instantaneous power delivered by the voltage source to the current source. Then,

$$\begin{aligned} p(t) &= v(t)i(t) \\ &= 20 \sin t \sin(t - 60^\circ) \text{ W} \\ &= 10[\cos 60^\circ - \cos(2t - 60^\circ)] \text{ W} = [5 - 10 \cos(2t - 60^\circ)] \text{ W}. \end{aligned}$$

Energy is obtained by integrating this function over $t = 0$ to $t = t$.

$$E(t) = \int_0^t [5 - 10 \cos(2t - 60^\circ)] dt = [5t - 5 \sin(2t - 60^\circ) - 4.33] \text{ J}.$$

The waveforms of voltage, current, power and the plot of energy function are shown in Fig. 7.3-6.

Voltage and current have a frequency of $\frac{1}{2\pi}$ Hz whereas the power waveform has a frequency of $\frac{1}{\pi}$ Hz. Power waveform is bipolar. The voltage source delivers energy to the current source for some time during a cycle and the current source delivers energy to the voltage source for the remaining time. However, the net energy delivered in one cycle is from voltage source to current source. The energy plot shows that $E(t)$ is non-monotonic on t . This must be so because the power waveform is bipolar.

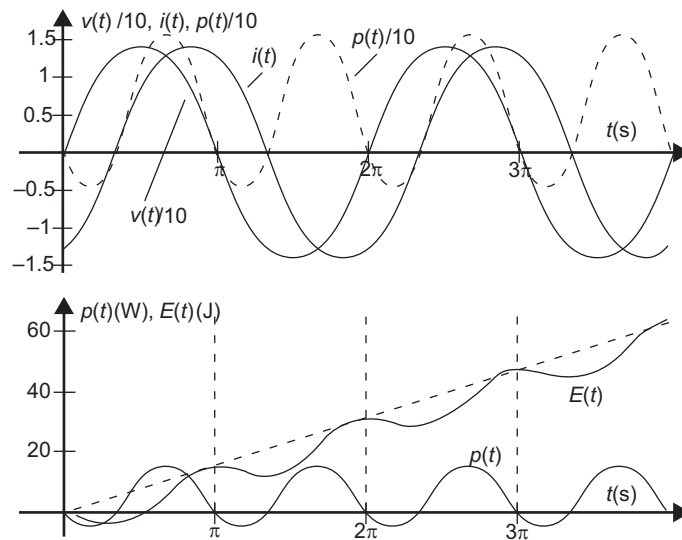


Fig. 7.3-6 Waveforms and Plots for Example 7.3-5. Upper Traces show Voltage, Current and Power

The line joining the energy values at the end of cycles of power is seen to be a straight line with a slope of 5 W. The average value of $p(t)$ over a cycle must then be 5 W. It is verified to be 5 W in the derivation that follows.

$$\begin{aligned} \text{Average of } p(t) \text{ over a cycle} &= \frac{\text{Area of } p(t) \text{ over a cycle}}{\text{Period of cycle}} \\ &= \frac{\int_0^\pi [5 - 10 \cos(2t - 60^\circ)] dt}{\pi} = 5 - \frac{5}{\pi} \sin(2t - 60^\circ) \Big|_0^\pi = 5 \text{ W}. \end{aligned}$$

EXAMPLE: 7.3-6

An electrical element draws a current of $i(t) = -\sqrt{2} \cos 100\pi t$ A from a sinusoidal voltage source of $v(t) = 10\sqrt{2} \sin 100\pi t$ V. Find and plot instantaneous power delivered to the load and energy delivered to it as functions of time.

SOLUTION

$p(t) = -20 \sin 100\pi t \cos 100\pi t = -10 \sin 200\pi t$ W and $E(t) = -0.05(1 - \cos 200\pi t)$ J.

Power waveform has 100 Hz frequency whereas the voltage and current waveforms have 50 Hz frequency. The period of the power waveform is 10 ms.

Average value of power over a cycle = $\int_0^{10 \text{ ms}} -10 \sin 200\pi t \, dt = 0$. Therefore, the energy function cannot be a growing function. The net change in $E(t)$ over a cycle of power waveform will be zero. It has to be an alternating function of time with a possible average content, indicating that there is no net energy transfer from source to load over a cycle of power waveform. This is confirmed by the plots shown in Fig. 7.3-7.

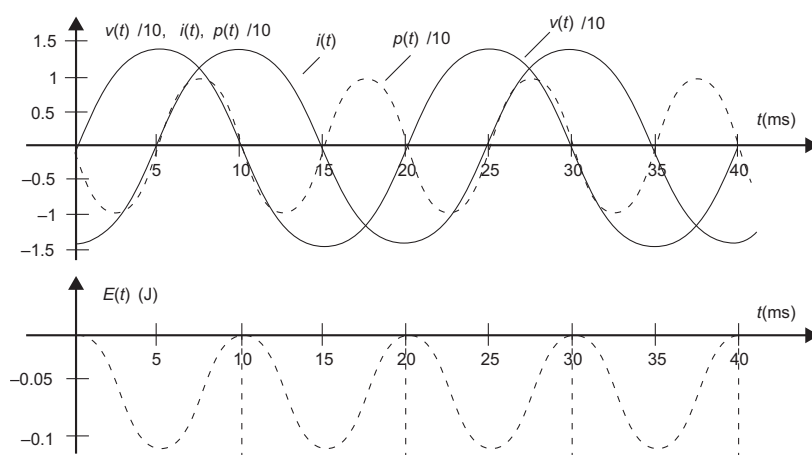


Fig. 7.3-7 Waveforms and Plots for Example 7.3-6

7.4 AVERAGE POWER IN PERIODIC WAVEFORMS

The following points emerge from the discussion on instantaneous power in Sect. 7.3.

- Instantaneous power delivered to an element is a non-constant function of time in general.
- If the voltage across the element and the current through the element are periodic waveforms with period T and zero average value over a cycle period, the instantaneous power will be a periodic waveform with period $0.5T$ and may have a non-zero average value over its cycle period of $0.5T$.
- The energy delivered to the element will also be a function of time. The value of total energy delivered to the element at the end-of-cycle points will fall on a straight line with a slope equal to the average value of instantaneous power waveform over its cycle period.
- If the instantaneous power waveform is unipolar, the energy function will be monotonic on t . If instantaneous power waveform is bipolar, the energy function will be non-monotonic.

A 100 W incandescent lamp draws about 0.615 A peak sinusoidal current when a 325 V peak sinusoidal voltage at 50 Hz is applied across it. This results in a $p(t) = 200 \sin^2 200\pi t$ W

of instantaneous power in it. $p(t)$ can also be expressed as $p(t) = 100 - 100 \cos 200\pi t$ W. Thus, the instantaneous power varies from 0 to 200 W and goes back to zero in 10 ms. Does the lamp filament respond to this power input variation?

It does respond to the power variation. However, 10 ms is too small a time interval for significant variations in the temperature of the filament to take place. This is due to the thermal capacity of the lamp system that tends to behave like thermal inertia when it comes to changes in lamp temperature. Thus, though the lamp responds to both the 100 W constant component and the 100 W cosine term in instantaneous power, the response of temperature variable to the 100 W cosine term is very small compared to the response to 100 W constant term. Thus, the lamp temperature and hence its light output is more or less constant in the long run (after about 50 to 100 ms of switching on in this context) corresponding to the 100 W constant term in $p(t)$, with the 100 W rippling term contributing only a negligible amplitude oscillation in them. And, persistence of human vision virtually blots out even this small rippling component in light output. Hence, we do not get to see the small amount of light flicker that takes place invariably due to the power changing periodically with a period of 10 ms. If we apply a 5 Hz voltage instead of 50 Hz voltage to a lamp, the oscillating component in lamp light output will be much more in amplitude and the frequency of this oscillating component will be 10 Hz. We will experience a prominent flicker in the lamp light output.

Thus, we conclude that, if the frequency of pulsation in instantaneous power delivered to an electrical load (lamps, heating element, motors etc.,) is sufficiently high, the output from the load (light, temperature, torque and speed etc.,) will be constant in the long run. The magnitude of this constant output is decided by the average of instantaneous power over its cycle period. The load will ignore the pure alternating component/s in instantaneous power. Therefore, the average value of $p(t)$, averaged over its cycle period, is a much more relevant quantity in practice.

The Cycle Average Power in the context of periodic waveforms is defined as the cycle average of instantaneous power over a cycle of instantaneous power and is denoted by P .

'Cycle average power' P defined.

$$\text{i.e., } P = \frac{1}{\text{Period}} \int_{\text{over one period of } p(t)} p(t) dt = \frac{2}{T} \int_t^{t+0.5T} v(t)i(t)dt,$$

where T is the period of the periodic voltage and current waveforms and t is any arbitrary time instant after $t = 0$. $p(t)$ is assumed to be zero for $t < 0$. In practice, the integration is carried out from the beginning of a power cycle to the end of that cycle.

The *Average Power* contained in an instantaneous power waveform and the *Cycle Average Power* are two different concepts altogether. First of all, there can be a cycle average power only if $p(t)$ is a periodic waveform. But the average power can be defined and calculated for any $p(t)$. Let $v(t)$ be the voltage across an element and $i(t)$ be the current through it over an interval of time denoted by $[t_1, t_2]$. Then the energy delivered to the element during this interval is given by the area under $p(t) = v(t)i(t)$ from t_1 to t_2 . Then, the *Average Power* (P_{av}) delivered during this interval is the value of constant power that would have delivered the same amount of energy to the element in the interval between t_1 to t_2 .

Average power and Cycle average power.

$$\therefore P_{av} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} v(t)i(t)dt.$$

The instant t_1 is usually chosen to be the instant at which the voltage was applied to the element and the instant t_2 is the instant at which the supply to the electrical element was switched off. If the two instants are chosen this way, the P value gives the average rate at which the energy was delivered to the load element during the entire period of connection.

A simple relation exists between *Average Power* (P_{av}) and *Cyclic Average Power* (P) in the context of periodic voltages and currents. Let $v(t) = V_m \sin \omega t$ V and

$i(t) = I_m \sin(\omega t + \theta) A$. Let the interval duration $t_2 - t_1$ extend over a large number of cycles of $p(t)$. The instantaneous power in this case will be periodic with a period of $0.5T$. But $t_2 - t_1$ may not be an integer multiple of $0.5T$. Hence, we express $t_2 - t_1$ as $0.5nT + 0.5kT$, where k is a real number between -1 and 1 and n is an integer.

$$\begin{aligned}
 p(t) &= V_m I_m \sin \omega t \sin(\omega t + \theta) = \frac{V_m I_m}{2} [\cos \theta - \cos(2\omega t + \theta)] \\
 \therefore P &= \frac{2}{T} \int_0^{0.5T} p(t) dt = \frac{\pi}{\omega} \int_0^{\pi} \frac{V_m I_m}{2} [\cos \theta - \cos(2\omega t + \theta)] dt \\
 &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2 \times 2\omega} \sin(2\omega t + \theta) \Big|_0^{\pi} = \frac{V_m I_m}{2} \cos \theta \\
 P_{av} &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t) dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{V_m I_m}{2} [\cos \theta - \cos(2\omega t + \theta)] dt \\
 &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2 \times 2\omega(t_2 - t_1)} \sin(2\omega t + \theta) \Big|_{t_1}^{t_2} \\
 &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2\pi(n+k)} \sin[\pi n + \pi k] \cos[2\omega(t_1 + t_2) + \theta].
 \end{aligned}$$

We used $t_2 - t_1 = 0.5nT + 0.5kT$ in the last step. $\sin(\pi n + \pi k)$ is zero if there are integer periods of length $0.5T$ within $t_2 - t_1$. Else, the magnitude of second term in the expression for P_{av} is bound by $V_m I_m / 2\pi(n+k)$. For a sufficiently large n , i.e., if the length of the interval over which the average power is calculated is very large compared to the period of instantaneous power waveform, then, the second term becomes negligible. The first term is the same as cycle average power. Therefore, *Average Power*, $P_{av} = \text{Cycle Average Power}$, P , if the waveforms last for sufficient duration compared to their period. If $v(t)$ and $i(t)$ persist for more than 20 or more cycles, the error in taking P as P_{av} will be $<1\%$.

Average Power, P_{av} is approximately equal to Cycle Average Power, P , if the waveforms last for sufficient duration compared to their period.

EXAMPLE: 7.4-1

The periodic 50 Hz voltage waveform applied across a 5Ω resistor is shown in Fig. 7.4-1. (i) Find the average power delivered to the load. (ii) Find the DC voltage that will deliver the same average power. (iii) Find the amplitude of a 50 Hz sinusoidal voltage that will deliver the same average power.

SOLUTION

(i) The power waveform will be described by the following equation for the first period.

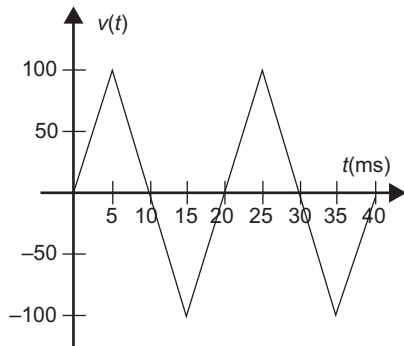


Fig. 7.4-1 Applied Voltage Waveform in Example 7.4-1

$$p(t) = \begin{cases} 2000(200t)^2 & \text{for } 0 \leq t \leq 5 \text{ ms} \\ 2000(2 - 200t)^2 & \text{for } 5 \text{ ms} < t \leq 10 \text{ ms} \end{cases}$$

Energy delivered in one cycle of $p(t)$ = The area under $p(t)$ over one cycle (10 ms) of $p(t) = 2 \times$ area under $p(t)$ for 0 to 5 ms interval

$$= 2 \int_0^{0.005} 8 \times 10^7 t^2 dt = 2 \times 8 \times 10^7 \times 0.005^3 \div 3 = 6.67 \text{ J.}$$

Therefore, Cycle Average Power = $6.67 \text{ J} / 0.01 \text{ s} = 667 \text{ W}$.

Therefore, $P_{av} = 667 \text{ W}$.

(ii) The required DC voltage V is such that $V^2/5 = 667$. Therefore, $V = 57.74 \text{ V}$.

(iii) Let $v(t) = V_m \sin 100\pi t$. Then, $i(t) = 0.2 V_m \sin 100\pi t$ and $p(t) = 0.2 V_m^2 \sin^2 \omega t \text{ W}$. Therefore, cycle average power = $0.2 V_m^2 / 2 = 0.1 V_m^2$. This has to be 667 W . Therefore, the required amplitude of sinusoidal voltage is, $V_m = 81.67 \text{ V}$.

EXAMPLE: 7.4-2

A 12 V battery delivers power to a power electronic system through its internal resistance of 0.05Ω as shown in Fig. 7.4-2. The figure also shows that the current drawn by the load is a pulsed current with period of $T = 1 \text{ ms}$. The load draws the current in such a way that the average value of the current is kept constant at 10 A while d and I vary under different operating conditions. Calculate the average power delivered by the source, average power delivered to load and efficiency for (i) $I = 100 \text{ A}$, (ii) $I = 40 \text{ A}$, (iii) $I = 12 \text{ A}$ and (iv) $I = 10 \text{ A}$.

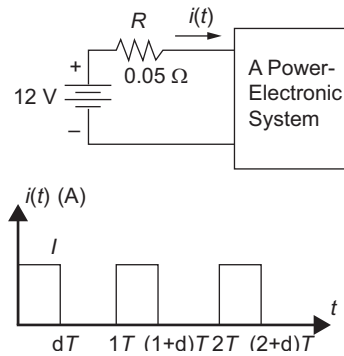


Fig. 7.4-2 Circuit and Waveform for Example 7.4-2

SOLUTION

Since the average value of $i(t)$ is kept constant, Id must be constant.

Average power delivered by the source

$$P_s = \frac{1}{0.001} \int_0^{0.001} 12 \times i(t) dt = 1000 \int_0^{0.001d} 12I dt = 12Id \text{ W} = 120 \text{ W.}$$

This value is independent of I and d as long as the average value of current is kept at 10 A. The power delivered by a *constant voltage source* is equal to the product of source voltage and the cycle average value of the current.

Average power dissipated in the internal resistance, P_R ,

$$\begin{aligned} P_R &= \frac{1}{0.001} \int_0^{0.001} 0.05 \times [i(t)]^2 dt \\ &= \frac{1}{0.001} \int_0^{0.001d} 0.05 \times I^2 dt = 0.05 I^2 d = 0.05 \times 10 \times I = 0.5 I \text{ W.} \end{aligned}$$

We have used the fact that $I d = 10$ A in the previous step.

(i) $I = 100$ A. Therefore, $d = 10/100 = 0.1$.

$$P_R = 0.5 \times 100 = 50 \text{ W}, P_S = 100 \text{ W}$$

$$\therefore \text{Load Power, } P_L = 100 - 50 = 50 \text{ W}$$

$$\text{Efficiency} = 50\%$$

(ii) $I = 40$ A. Therefore, $d = 10/40 = 0.25$

$$P_R = 0.5 \times 40 = 20 \text{ W}, P_S = 100 \text{ W}$$

$$\therefore \text{Load Power, } P_L = 100 - 20 = 80 \text{ W}$$

$$\text{Efficiency} = 80\%$$

(iii) $I = 12$ A. Therefore, $d = 10/12 = 0.833$

$$P_R = 0.5 \times 12 = 6 \text{ W}, P_S = 100 \text{ W}$$

$$\therefore \text{Load Power, } P_L = 100 - 6 = 94 \text{ W}$$

$$\text{Efficiency} = 94\%$$

(iv) $I = 10$ A. Therefore, $d = 10/10 = 1$, $i(t)$ becomes a constant current of 10 A.

$$P_R = 0.5 \times 10 = 5 \text{ W}, P_S = 100 \text{ W}$$

$$\therefore \text{Load Power, } P_L = 100 - 5 = 95 \text{ W}$$

$$\text{Efficiency} = 95\%$$

EXAMPLE: 7.4-3

A DC voltage source of V volts delivers a current $i(t)$ to an external load through its internal resistance of $R \Omega$. Show that, among the infinite possible periodic waveforms for $i(t)$ with the same value of cycle average, $i(t) = \text{constant}$ is the waveform that results in minimum loss and maximum efficiency.

SOLUTION

Let I be the cycle average value of a periodic $i(t)$. Then $i(t)$ can be expressed as a constant plus a pure alternating component as $i(t) = I + i_{AC}(t)$, where $i_{AC}(t)$ is bipolar and has equal areas under positive and negative half-cycle. Let P_S be the average power delivered by the source, P_L be the average power delivered to the load and P_R be the average power dissipated in R . Average powers are equal to the corresponding cycle average powers if $i(t)$ lasts for a long time compared to its period. Then,

$$P_S = \frac{1}{T} \int_0^T V [I + i_{AC}(t)]^2 dt$$

where T is the period of $i_{AC}(t)$. The cycle average value of a pure alternating waveform with equal magnitude areas under its positive and negative half-cycles will be zero. Therefore, $P_S = VI$ W.

$$P_R = \frac{1}{T} \int_0^T R [I + i_{AC}(t)]^2 dt = \frac{1}{T} \int_0^T R I^2 dt + \frac{1}{T} \int_0^T 2R I i_{AC}(t) dt + \frac{1}{T} \int_0^T R [i_{AC}(t)]^2 dt.$$

Since the instantaneous power in R contains a term proportional to $i_{AC}(t)$, its basic period is T and not $0.5T$. Hence, the integration in the above step is from 0 to T . The second integral goes to zero since cycle average of a pure alternating component is zero. Therefore,

The optimum way to draw power from a DC source is by drawing a DC current. Similarly, the optimum way to charge a battery is by delivering a constant current into it. No other current waveform is as energy efficient as the DC current waveform when drawing power from a DC voltage source or delivering power to it.

$$P_R = \frac{1}{T} \int_0^T R I^2 dt + \frac{1}{T} \int_0^T R [i_{AC}(t)]^2 dt.$$

The quantity $[i_{AC}(t)]^2$ is always positive and hence the second integral in the last equation will be positive. Therefore,

$P_R \geq \frac{1}{T} \int_0^T R I^2 dt$ with the equality sign applicable only when $i_{AC}(t) = 0$ for all t – i.e., only when $i(t) = I$, a constant.

Therefore, for a given amount of average current drawn from a DC source, the power loss in the internal resistance and connecting link resistance will be a minimum when the energy is drawn at a constant rate – that is, power delivered is kept constant by keeping current constant.

EXAMPLE: 7.4-4

An AC voltage source (i.e., a sinusoidal source) $v(t) = 750 \sin 100\pi t$ V delivers 1 kW of average power to a load through a resistor of 5Ω . This resistance is the sum of the internal resistance of the source and the resistance of cable connecting the load to the source. Assume that the load draws a current $i(t) = I_m \sin(100\pi t + \theta)$ and that I_m and θ can be varied keeping the power delivered by the source at 1 kW itself. Find the amplitude of current, power dissipated in the series resistor and the efficiency of power transfer when (i) $\theta = -80^\circ$ (ii) $\theta = 45^\circ$ (iii) $\theta = -30^\circ$ and (iv) $\theta = 0^\circ$.

SOLUTION

Let $v(t) = V_m \sin \omega t$ and $i(t) = I_m \sin(\omega t + \theta)$. Then the average power delivered by the source $= \frac{V_m I_m}{2} \cos \theta$ W. (We have derived this a number of times in earlier sections).

The average power delivered depends on magnitude of phase difference between voltage and current waveforms. But it does not depend on the sign of phase difference.

The power delivered by the source is kept constant at 1000 W in this example. Therefore, $I_m \cos \theta = 2000/750 = 3.077$ A.

Let P_R be the average power dissipated in the resistor. Then,

$$\begin{aligned} P_R &= \frac{2}{T} \int_0^{0.5T} R [i(t)]^2 dt = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} R I_m^2 \sin^2(\omega t + \theta) dt \\ &= \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} R I_m^2 \left[\frac{1}{2} - \frac{1}{2} \cos(2\omega t + 2\theta) \right] dt = \frac{R I_m^2}{2} \text{ W.} \end{aligned}$$

But $I_m = 3.077/\cos \theta$. Therefore, $P_R = 4.734 R / \cos^2 \theta$. Now we evaluate the numbers for the various cases.

- (i) $\theta = -80^\circ$. Then, $\cos \theta = 0.1736$, $\therefore I_m = 3.077/0.1736 = 17.72$ A, $\therefore P_R = 4.734 \times 5/0.1736^2 = 785.4$ W, $P_S = 1000$ W, $\therefore P_L = 214.6$ W and efficiency = 21.46%.
- (ii) $\theta = 45^\circ$. Then, $\cos \theta = 0.707$, $\therefore I_m = 3.077/0.707 = 4.35$ A, $\therefore P_R = 4.734 \times 5/0.707^2 = 47.3$ W, $P_S = 1000$ W, $\therefore P_L = 952.7$ W and efficiency = 95.27%.
- (iii) $\theta = -30^\circ$. Then, $\cos \theta = 0.866$, $\therefore I_m = 3.077/0.866 = 3.55$ A, $\therefore P_R = 4.734 \times 5/0.866^2 = 31.6$ W, $P_S = 1000$ W, $\therefore P_L = 968.4$ W and efficiency = 96.84%.
- (iv) $\theta = 0^\circ$. Then, $\cos \theta = 1$, $\therefore I_m = 3.077$ A, $\therefore P_R = 4.734 \times 5/1^2 = 23.7$ W, $P_S = 1000$ W, $\therefore P_L = 976.3$ W and efficiency = 97.63%.

Increasing magnitude of phase difference between voltage and current makes power transfer to the load more and more inefficient.

EXAMPLE: 7.4-5

Show that the fixed amount of average power delivered by a sinusoidal voltage source through its internal resistance and connecting link resistance to a load will reach the load with minimum loss and maximum efficiency when the current drawn by the load has zero phase difference with respect to the source voltage.

SOLUTION

Let $v(t) = V_m \sin \omega t$ and $i(t) = I_m \sin(\omega t + \theta)$. Let the total series resistance in the path be R .

Then the average power delivered by the source $= \frac{V_m I_m}{2} \cos \theta$ W. The average power dissipated in R is $0.5 R I_m^2$ W (Example 7.4-4).

But the power delivered by the source is stated to be fixed. Let this fixed value be P W. Then $0.5 V_m I_m \cos \theta = P$. Therefore, $I_m = 2P / (V_m \cos \theta)$. Therefore, the average power loss in R ,

$$P_R = 0.5 R I_m^2 = \frac{2RP^2}{V_m^2 \cos^2 \theta}.$$

The minimum of this loss takes place when $\cos^2 \theta$ is a maximum – i.e., $\theta = 0^\circ$ or 180° . $\theta = 180^\circ$ is relevant when the load is delivering power to source.

Therefore, when a sinusoidal voltage source is delivering power to load, a given amount of source power is transferred to load with minimum losses and maximum efficiency when the load draws current from source at zero phase difference at the source terminal.

This fact has great significance in Electrical Power Distribution systems and Electricity tariff structure.

7.5 EFFECTIVE VALUE (RMS VALUE) OF PERIODIC WAVEFORMS

The cycle average power delivered to the same resistance R by different voltage or current waveforms forms the basis for comparison of their effectiveness in delivering useful power to a load. We can define a measure of effectiveness of a given voltage or current waveform by calculating the cyclic average power that will be delivered to a resistance R and by answering the question – what is the value of a DC voltage or current that will result in same average power in R ? The measure defined this way is called the *Root-Mean-Square Value (rms value)* of the waveform. It is also called the *Effective Value* of the waveform.

The assumption of $R = 1 \Omega$ is a matter of convenience. It could have been $R \Omega$ and that will not make any difference since R will get cancelled when the heating effects are equated.

We can develop an expression for rms value of a periodic waveform as shown below. Let $x(t)$ be a periodic waveform with period of T s. We find out the cyclic average power as if $x(t)$ is a voltage signal applied across 1Ω .

$$P = \frac{1}{T} \int_0^T \frac{[x(t)]^2}{1} dt.$$

Let X_{rms} be the value of DC quantity that will produce same power in 1Ω . Then,

$$\frac{1}{T} \int_0^T [x(t)]^2 dt = (X_{\text{rms}})^2$$

$$\therefore X_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T [x(t)]^2 dt}. \quad (7.5-1)$$

Therefore, finding out rms value of a waveform involves *squaring* the waveform, finding out the *mean* (i.e., average over a cycle) of the *squared waveform* over a cycle and then finding out *root* of the *mean*, and hence the name – *root mean square*.

Now, we can express the average power delivered to a resistance of $R \Omega$ by a periodic voltage waveform $v(t)$ applied to it as $P = (V_{\text{rms}})^2 / R$ W. The average power delivered to a resistance of $R \Omega$ by a periodic current $i(t)$ through it is $P = R(I_{\text{rms}})^2$ W.

Effective Value or RMS Value of a Waveform

It is the value of DC quantity that will produce the same heating effect as that produced by the waveform when it is applied as a voltage across a resistance of 1Ω or as a current through a resistance of 1Ω .

Expression for effective value of a periodic waveform.

7.5.1 RMS Value of Sinusoidal Waveforms

Let $v(t) = V_m \sin \omega t$ V. We find its rms value as,

RMS value of a sinusoidal waveform.

$$V_{\text{rms}} = \sqrt{\frac{2\pi}{\omega} \int_0^{\frac{2\pi}{\omega}} V_m^2 \sin^2 \omega t \, dt} = \sqrt{\frac{2\pi}{\omega} \int_0^{\frac{2\pi}{\omega}} \frac{1}{2} V_m^2 [1 - \cos^2 \omega t] \, dt} = \sqrt{\frac{1}{2} V_m^2} = \frac{V_m}{\sqrt{2}}.$$

Thus, a sinusoidal voltage is only as effective as a DC voltage that is 70.7% of its peak value as far as its heating capability is concerned.

Let $v(t) = V_m \sin \omega t$ V and $i(t) = I_m \sin(\omega t + \theta)$ A be the voltage across and current through an electrical element as per passive sign convention. Then, we know that the average power delivered to the element is given by $0.5 V_m I_m \cos \theta$ W. Now, we can express this power equation in terms of rms values of voltage and current.

$$P = V_{\text{rms}} I_{\text{rms}} \cos \theta \text{ W.} \quad (7.5-2)$$

Two other measures are useful in the context of periodic waveforms. They are *cycle average* and *cycle average of absolute value of the waveform*. We denote them by V_{cav} and V_{caav} , respectively.

Average power in terms of effective value of voltage and current waveforms under sinusoidal steady-state.

$$V_{\text{cav}} = \frac{1}{T} \int_0^T v(t) \, dt \text{ and } V_{\text{caav}} = \frac{1}{T} \int_0^T |v(t)| \, dt. \quad (7.5-3)$$

Note that both the definitions of rms value in Eqn. 7.5-1 and average values in Eqn. 7.5-3, permit us to carry out the integration over any interval of width equal to the period of the waveform. It does not have to be between 0 and T .

A *pure alternating waveform* will have a cycle average of zero.

The cycle average of absolute value of a pure alternating waveform that has identical positive half-cycle width and negative half-cycle width will be same as the average of positive half-cycle over $0.5T$. Hence, the cycle average of absolute value is also called *half-cycle average* in the case of such a waveform. A sinusoidal waveform is one such waveform. The V_{caav} value for a sinusoidal waveform $v(t) = V_m \sin \omega t$ is obtained below.

Cycle average and cycle average of absolute value of a periodic waveform.

$$V_{\text{caav}} = \frac{2}{T} \int_0^{0.5T} v(t) \, dt = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} V_m \sin \omega t \, dt = \frac{V_m}{\pi} \cos \omega t \Big|_{\frac{\pi}{\omega}}^0 = \frac{2V_m}{\pi}.$$

Half-cycle average of a sinusoidal waveform.

The ratio between rms value and half-cycle average value of a periodic waveform with equal half-cycle widths and zero cycle average (i.e., a pure alternating waveform) is defined as its *form factor*. The form factor of a pure sinusoidal waveform = $\pi/2\sqrt{2} = 1.11$.

Form factor of a periodic waveform.

The ratio between peak value and rms value of an alternating waveform with equal half-cycle widths and zero cycle average is defined as its *crest factor*. The crest factor of a pure sinusoidal waveform is $\sqrt{2}$.

EXAMPLE: 7.5-1

If $v(t)$ is a pure alternating periodic waveform with period T and rms value V_{rms} , find the rms value of a new periodic waveform $v_1(t) = V + v(t)$, where V is a constant quantity. Thereby find the rms value of $10 + 10 \sin 100\pi t$ V.

SOLUTION

The component $2Vv(t)$ represents a pure alternating component with zero cycle average since $v(t)$ is pure alternating waveform and V is a constant. Hence, the integration of this term over one cycle period yields zero integral.

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T [V + v(t)]^2 dt} = \sqrt{\frac{1}{T} \int_0^T [V^2 + (v(t))^2 + 2Vv(t)] dt}$$

$$\therefore V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T [V^2 + (v(t))^2] dt} = \sqrt{V^2 + V_{\text{rms}}^2}$$

Therefore, the rms value of $10 + 10\sin 100\pi t = \sqrt{10^2 + \left(\frac{10}{\sqrt{2}}\right)^2} = \sqrt{150} = 12.25 \text{ V}$.

EXAMPLE: 7.5-2

If $v(t)$ is a periodic waveform with period T , rms value V_{rms} and cycle average value V_{cav} , find the rms value and cycle average value of a new periodic waveform $v_1(t) = V + v(t)$, where V is a constant quantity.

SOLUTION

$v(t)$ has a non-zero cycle average value. Therefore, it can be written as $V_{\text{cav}} + v_{\text{AC}}(t)$, where $v_{\text{AC}}(t)$ is a pure alternating waveform with zero cycle average. Using the result we arrived at in Example 7.5-1,

$$V_{\text{rms}}^2 = V_{\text{cav}}^2 + V_{\text{AC rms}}^2$$

$$\therefore V_{\text{AC rms}}^2 = V_{\text{rms}}^2 - V_{\text{cav}}^2$$

$$\text{Now, } v_1(t) = V + v(t) = V + V_{\text{cav}} + v_{\text{AC}}(t).$$

Using the result from Example 7.5-1,

$$V_{\text{rms}}^2 = (V + V_{\text{cav}})^2 + V_{\text{AC rms}}^2$$

Substituting for $V_{\text{AC rms}}^2$

$$V_{\text{rms}}^2 = (V + V_{\text{cav}})^2 + (V_{\text{rms}}^2 - V_{\text{cav}}^2)$$

$$= V^2 + 2VV_{\text{cav}} + V_{\text{rms}}^2$$

$$\therefore V_{\text{rms}} = \sqrt{V^2 + 2VV_{\text{cav}} + V_{\text{rms}}^2}$$

The cycle average of $v_1(t) = V_{\text{cav}} + V$.

EXAMPLE: 7.5-3

Find the rms value of a symmetric square wave of $\pm 10 \text{ V}$ and 50 Hz frequency.

SOLUTION

When a symmetric square wave of amplitude V is squared, the resulting waveform is the same as squaring a DC voltage of V volts. Therefore, rms value of a symmetric square wave is same as the amplitude of the wave. Hence, the rms value of square wave voltage in this example is 10 V.

EXAMPLE: 7.5-4

Find the r.m.s value, half-cycle value, crest factor and form factor for a symmetric triangular waveform shown in Fig. 7.5-1.

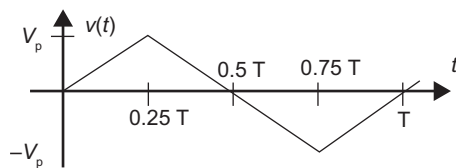


Fig. 7.5-1 Symmetric Triangle Waveform for Example 7.5-4

SOLUTION

Due to symmetry, the squared waveform needs to be integrated only from 0 to 0.25T. The integral for 0 to T range is four times this value.

$$V_{\text{rms}} = \sqrt{\frac{4}{T} \int_0^{0.25T} \left(\frac{4V_p t}{T} \right)^2 dt} = \sqrt{\frac{64V_p^2}{3T^3} \times (0.25T)^3} = \frac{V_p}{\sqrt{3}} \text{ V.}$$

$$V_{\text{caav}} = \frac{2}{T} \int_0^{0.5T} v(t) dt = 2 \times \frac{2}{T} \int_0^{0.25T} \frac{4V_p t}{T} dt = \frac{V_p}{2} \text{ V.}$$

$$\therefore \text{Form factor} = \frac{V_p / \sqrt{3}}{V_p / 2} = \frac{2}{\sqrt{3}} = 1.1547$$

$$\text{Crest factor} = \frac{V_p}{V_p / \sqrt{3}} = \sqrt{3} = 1.732.$$

EXAMPLE: 7.5-5

Find an expression for rms value of the periodic pulse train shown in Fig. 7.5-2.

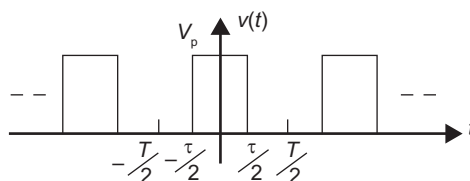


Fig. 7.5-2 Periodic Waveform for Example 7.5-5

SOLUTION

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt} = \sqrt{\frac{1}{T} \int_{-0.5T}^{0.5T} [V_p]^2 dt} = V_p \sqrt{\frac{\tau}{T}} \text{ V.}$$

Note the limits of integration.

EXAMPLE: 7.5-6

Let $v(t) = V_m \sin \omega t$ V. A new voltage waveform is generated from this waveform by a process called *half-wave rectification*, which is represented mathematically as

$$v_r(t) = \begin{cases} 0 & \text{if } v(t) < 0 \\ v(t) & \text{if } v(t) \geq 0. \end{cases}$$

Find the rms value and cycle average value of $v_r(t)$.

SOLUTION

The area under squared $v_r(t)$ over one T will be half the area under squared $v(t)$ since one half-cycle is missing in $v_r(t)$. Therefore, rms value of $v_r(t)$ will be $\frac{1}{\sqrt{2}}$ times the rms value of $v(t)$. Therefore, rms value of $v_r(t) = 0.5 V_m$ V.

Half-cycle average of $v(t)$ is $2 V_m / \pi$. Therefore, half-cycle area $= \frac{2 V_m}{\pi} \times \frac{T}{2} = V_m T / \pi$. This area becomes the full cycle area in $v_r(t)$ since second half cycle is zero-valued. Therefore, cycle average of $v_r(t) = \frac{V_m T / \pi}{T} = V_m / \pi$ V.

7.6 THE POWER SUPERPOSITION PRINCIPLE

Current variables and voltage variables in a *linear* electric circuit obey the Superposition Theorem. However, instantaneous power in a linear circuit does not follow superposition principle. Consider a circuit driven by two sources $v_{s1}(t)$ and $v_{s2}(t)$ and let the corresponding response components for the voltage across a particular element be $v_{11}(t)$ and $v_{12}(t)$. The corresponding response components of current through that element are $i_{11}(t)$ and $i_{12}(t)$ as per the passive sign convention. Then,

The instantaneous power delivered to the element when $v_{s1}(t)$ is acting alone,

$$p_1(t) = v_{11}(t) i_{11}(t)$$

The instantaneous power delivered to the element when $v_{s2}(t)$ is acting alone,

$$p_2(t) = v_{12}(t) i_{12}(t)$$

The instantaneous power delivered to the element when $v_{s1}(t)$ and $v_{s2}(t)$ are acting together,

$$p(t) = [v_{11}(t) + v_{12}(t)] [i_{11}(t) + i_{12}(t)] = v_{11}(t) i_{11}(t) + v_{12}(t) i_{12}(t) + [v_{11}(t) i_{12}(t) + v_{12}(t) i_{11}(t)] \\ = p_1(t) + p_2(t) + \text{Cross Power Terms}$$

$p(t)$ should have been equal to $p_1(t) + p_2(t)$ if instantaneous power obeys superposition principle. Therefore, instantaneous power does not follow superposition principle in a linear circuit. This conclusion is general in nature and there are no exceptions.

However, there are exceptions to this rule when it comes to *average power*. Let us confine to the case of a periodic waveforms that are applied to circuits for sufficient duration such that average power can be taken as cyclic average power. Let the average power delivered to the element in question when $v_{s1}(t)$ is acting alone be P_1 and the corresponding value when $v_{s2}(t)$ is acting alone be P_2 . We assume that $v_{s1}(t)$ and $v_{s2}(t)$ have a least common period of T s. This common period need not be the period of any one of them. For instance, let the period of $v_{s1}(t)$ be 1 s and that of $v_{s2}(t)$ be 1.25 s. Then the value of T will be 5 s since there will be an integer number of cycles of both in a 5 s period. The lowest interval that will contain integer number of both the cycles is 5 s. Cyclic average power can be found by integrating over multiple cycles provided the averaging is done over the same number of cycle periods. Therefore, taking T as the integration interval is permitted in the evaluation of P_1 and P_2 .

Instantaneous power does not follow superposition principle in a linear circuit.

$$P_1 = \frac{1}{T} \int_0^T v_{11}(t) i_{11}(t) dt; \quad P_2 = \frac{1}{T} \int_0^T v_{12}(t) i_{12}(t) dt;$$

The average power delivered to the element when both sources are acting together, P , is given by

$$\begin{aligned} P &= \frac{1}{T} \int_0^T [v_{11}(t) + v_{12}(t)][i_{11}(t) + i_{12}(t)] dt \\ &= \frac{1}{T} \int_0^T [v_{11}(t)i_{11}(t) + v_{12}(t)i_{12}(t) + v_{12}(t)i_{11}(t) + v_{11}(t)i_{12}(t)] dt \\ &= P_1 + P_2 + \frac{1}{T} \int_0^T v_{12}(t)i_{11}(t) dt + \frac{1}{T} \int_0^T v_{11}(t)i_{12}(t) dt. \end{aligned} \quad (7.6-1)$$

Hence, in general, average power does not follow superposition principle. However, unlike in the case of instantaneous power, the cross power terms in the average power can go to zero under special conditions.

Consider a case where $v_{s1}(t)$ and $v_{s2}(t)$ are two sinusoidal waveforms at different angular frequencies of ω_1 and ω_2 . Note that one of them can be zero as a special case indicating a DC or steady input. Assume that $\omega_1 < \omega_2$. In this case, T will be an integer multiple of $2\pi/\omega_1$ and another integer multiple of $2\pi/\omega_2$. i.e., $T = k_1 \frac{2\pi}{\omega_1} = k_2 \frac{2\pi}{\omega_2}$ and $\frac{k_1}{k_2} = \frac{\omega_1}{\omega_2}$, where k_1 and k_2 are

two integers. It is possible to find a value for T satisfying this condition only if the ratio ω_1/ω_2 is a rational number.

The circuit is linear. Therefore, $v_{11}(t)$ and $i_{11}(t)$ will be sinusoids at ω_1 rad/s and $v_{12}(t)$ and $i_{12}(t)$ will be sinusoids at ω_2 rad/s. Therefore, both cross power terms $-v_{11}(t)i_{12}(t)$ and $v_{12}(t)i_{11}(t)$ – will involve products of two sinusoidal waveforms of different frequencies – ω_1 and ω_2 . By applying trigonometric identities they may be expressed as sinusoidal waveforms of sum frequency and difference frequency. Therefore, $v_{11}(t)i_{12}(t) + v_{12}(t)i_{11}(t)$ will yield two sinusoids – one with a frequency of $\omega_1 + \omega_2$ rad/s and the second with a frequency of $\omega_2 - \omega_1$ rad/s.

Both these sinusoidal waveforms will have integer number of cycles within an interval of length T as shown next.

$$\text{Period of } \omega_1 + \omega_2 \text{ rad/s component } T_s = \frac{2\pi}{\omega_1 + \omega_2}$$

$$\frac{T}{T_s} = \frac{k_1 2\pi / \omega_1}{2\pi / (\omega_1 + \omega_2)} = k_1 + k_1 \frac{\omega_2}{\omega_1} = k_1 + k_2 = \text{an integer} \left(\because \frac{k_1}{k_2} = \frac{\omega_1}{\omega_2} \right).$$

$$\text{Period of } \omega_2 - \omega_1 \text{ rad/s component } T_d = \frac{2\pi}{\omega_2 - \omega_1}$$

$$\frac{T}{T_d} = \frac{k_1 2\pi / \omega_1}{2\pi / (\omega_2 - \omega_1)} = -k_1 + k_1 \frac{\omega_2}{\omega_1} = k_2 - k_1 = \text{an integer} \left(\because \frac{k_1}{k_2} = \frac{\omega_1}{\omega_2} \right).$$

Therefore, the area under these two sinusoidal waveforms over an interval of T s long will be zero. Thus, the cross power terms in the instantaneous power contribute only zero to the average power in this case. Therefore, $P = P_1 + P_2$. We can extend this analysis to a case involving many sinusoidal sources of different frequencies easily and arrive at the *Power Superposition Principle* stated below.

The average power delivered to an element in a linear circuit excited by sinusoidal sources of different frequencies (including DC, i.e., zero frequency) obeys superposition principle.

In general, average power does not follow superposition principle in a linear circuit.

The Power superposition principle.

7.6.1 RMS Value of a Composite Waveform

We will employ the superposition principle for average power to arrive at an expression for the rms value (i.e., the effective value) of a periodic waveform that is the sum of many sinusoidal periodic waveforms of different frequencies.

Let $v(t) = v_1(t) + v_2(t) + \dots + v_n(t)$ be a composite waveform comprising n *distinct frequency sinusoidal waveforms*. Note that the word *distinct* is used in the last sentence. No two components in the sum can have same frequency value. If there are two sources of same frequency, they have to be combined and treated as a single term.

Now imagine that we apply this signal as a voltage across $1\ \Omega$ resistor. The average power delivered to the resistor is found by applying power superposition principle as the sum of average power delivered to the same resistor when each source is acting alone. But that will be nothing but the square of rms value of each component waveform. The rms value of $v(t)$ is obtained by finding the square root of average power delivered to the $1\ \Omega$ resistor. Therefore,

$$V_{\text{rms}}^2 = V_{1\text{rms}}^2 + V_{2\text{rms}}^2 + \dots + V_{n\text{rms}}^2 \Rightarrow V_{\text{rms}} = \sqrt{V_{1\text{rms}}^2 + V_{2\text{rms}}^2 + \dots + V_{n\text{rms}}^2} \quad (7.6-2)$$

RMS Value of a Composite Waveform

The RMS value of a waveform that is the sum of many sinusoidal waveforms with *distinct frequencies* (including DC) is obtained by taking the square root of sum of squares of RMS values of individual components.

EXAMPLE: 7.6-1

The voltage applied across an electrical load circuit is $v(t) = 10 + 50\sin 100\pi t + 30\sin(150\pi t - 30^\circ)$ V and the current delivered to the load circuit is seen to be $i(t) = 2.5 + 5\sin(100\pi t - 45^\circ) + 1.5\sin(150\pi t - 80^\circ)$ A. (i) Find the average power delivered to the load circuit, rms value of applied voltage and load current. (ii) What is the period over which averaging was carried out in the last step?

SOLUTION

(i) All the three components in voltage waveform have distinct frequencies. Similarly, all the three components in current waveform too have distinct frequencies.

Therefore, average power delivered by applying power superposition principle is obtained as,

$$\begin{aligned} P &= 10 \times 2.5 + \frac{50}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \cos(0 - (-45^\circ)) + \frac{30}{\sqrt{2}} \times \frac{1.5}{\sqrt{2}} \cos(-30^\circ - (-80^\circ)) \\ &= 25 + 125 \cos 45^\circ + 22.5 \cos 50^\circ = 127.84 \text{ W.} \end{aligned}$$

The rms values of voltage and current waveforms are obtained by applying Eqn. 7.6-2.

$$V_{\text{rms}} = \sqrt{10^2 + \left(\frac{50}{\sqrt{2}}\right)^2 + \left(\frac{30}{\sqrt{2}}\right)^2} = 42.43 \text{ V}$$

$$I_{\text{rms}} = \sqrt{2.5^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{1.5}{\sqrt{2}}\right)^2} = 4.46 \text{ A}$$

$$\omega_1 = 100\pi \text{ and } \omega_2 = 150\pi.$$

$$(ii) \frac{\omega_1}{\omega_2} = \frac{100\pi}{150\pi} = \frac{2}{3} = \frac{k_1}{k_2}$$

The lowest values possible for k_1 and k_2 are 2 and 3, respectively.

$$\text{Therefore, } T = \frac{k_1 2\pi}{\omega_1} = \frac{2 \times 2\pi}{100\pi} = 0.04 \text{ s.}$$

Therefore, the period over which averaging was carried out in calculating average power and rms values is 40 ms.

EXAMPLE: 7.6-2

Find the rms value of a voltage waveform given by $v(t) = 10\sin(100\pi t - 30^\circ) + 5\cos(100\pi t + \pi/6) + 5\cos(150\pi t - 65^\circ) + 7\sin(300\pi t - \pi/3) - 7\cos(300\pi t - 45^\circ)$ V.

SOLUTION

The terms in the expression for $v(t)$ contain non-distinct frequencies. Terms with the same frequency will have to be combined into single term first.

$$\begin{aligned} & 10\sin(100\pi t - 30^\circ) + 5\cos(100\pi t + \pi/6) \\ &= \sin 100\pi t [10\cos 30^\circ - 5\sin \pi/6] + \cos 100\pi t [-10\sin 30^\circ + 5\cos \pi/6] \\ &= 6.16\sin 100\pi t - 0.67\cos 100\pi t \\ &= (6.16^2 + 0.67^2)^{0.5} \sin[100\pi t - \tan^{-1}(0.67/6.16)] \\ &= 6.196\sin(100\pi t - 6.2^\circ) \\ & 7\sin(300\pi t - \pi/3) - 7\cos(300\pi t - 45^\circ) \\ &= \sin 300\pi t [7\cos \pi/3 - 7\sin 45^\circ] + \cos 300\pi t [-7\sin \pi/3 - 7\cos 45^\circ] \\ &= -1.45\sin 300\pi t - 11.01\cos 300\pi t \\ &= -(1.45^2 + 11.01^2)^{0.5} \sin[300\pi t + \tan^{-1}(11.01/1.45)] \\ &= -11.105\sin(300\pi t + 82.5^\circ) \\ &= 11.105\sin(300\pi t - 97.5^\circ) \end{aligned}$$

$$\therefore v(t) = 6.196\sin(100\pi t - 6.2^\circ) + 5\cos(150\pi t - 65^\circ) + 11.105\sin(300\pi t - 97.5^\circ) \text{ V.}$$

$$V_{\text{rms}} = \sqrt{\left(\frac{6.196}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{11.105}{\sqrt{2}}\right)^2} = 9.66 \text{ V.}$$

Approximating average power by cyclic average power is valid only if the waveform lasts for duration much longer than the period over which averaging is carried out in cyclic average calculation.

This is the reason why the frequency information does not figure in the expressions for P and rms values.

The effective value of a waveform does not depend on frequency provided the waveform remains applied to the circuit for much longer than its period.

EXAMPLE: 7.6-3

The current through a $10\ \Omega$ resistor is seen to be $i(t) = 2 + 3\sin 500\pi t + 1.5\cos 501\pi t$ A. (i) Find the amplitude of a symmetric triangle wave current with 50 Hz frequency that will produce same heating in the resistor. (ii) Will the answer arrived at in (i) be correct if the current lasted only for 0.5 s?

SOLUTION

(i) The rms value of a triangular current with a peak value of I_p A will be $I_p/\sqrt{3}$. (Example 7.5-3). This must be equal to the rms value of $i(t)$.

$$I_{\text{rms}} = 2^2 + \left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{1.5}{\sqrt{2}}\right)^2 = 3.1 \text{ A} \Rightarrow \therefore I_p = 3.1 \times \sqrt{3} = 5.37 \text{ A.}$$

(ii) No. The period over which averaging is done in the calculation of rms value is 500 times the period of $\sin 500\pi t$. That is, $T = 500 \times 4 \text{ ms} = 2 \text{ s}$.

7.7 SUMMARY

- Sinusoidal waveforms are of great importance to the electrical and electronic circuits due to the following reasons: (i) They preserve their wave-shape in linear circuits. (ii) They render the voltage levels in electrical and electronic systems flexible so that optimisation of sub-system performance in various parts of the system by choosing a suitable voltage level in that part of the system becomes possible. Transformers help us to realise this voltage flexibility. (iii) Periodic non-sinusoidal waveforms as well as a broad class of aperiodic waveforms can be expressed as a sum of sinusoids by Fourier series and Fourier transforms. Hence, circuit solution with such input waveforms can be obtained with relative ease if solution for a sinusoidal input is known.
- A sinusoidal waveform of period T can be expressed as $v(t) = A \sin(\omega t + \theta) = A \sin(2\pi t/T + \theta) = A \sin(2\pi f t + \theta)$, where A is its *amplitude*, T is its *period* in seconds, $f = 1/T$ is its *cyclic frequency* in s^{-1} (Hertz, Hz) and $\omega = 2\pi f = 2\pi/T$ is its *radian frequency* or *angular frequency* in radians/sec (rad/s) unit. The quantity θ in $v(t) = A \sin(\omega t + \theta)$ is defined as the *phase* of the sinusoidal function.
- The angular difference between *similarly located points within a cycle period* on two normalised sinusoidal waveforms with same ω is defined as the *phase difference* between them. The *phase difference* between two sinusoids is independent of choice of origin in t or ωt axis. The precedence relationship

[i.e., which comes after which (in a visual sense)] between them in t or ωt axis too, is independent of choice of origin. However, the *phase* of a sinusoidal waveform depends on the choice of origin in t or ωt axis.

- When a waveform point on a sinusoidal function $v_2(t)$ appears *after* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ with same frequency, $v_2(t)$ is said to *lag* $v_1(t)$ in phase and $\theta_1 - \theta_2$ is called a *lag phase angle* under this condition.
- Similarly, when a waveform point on a sinusoidal function $v_2(t)$ appears *before* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ with same frequency, $v_2(t)$ is said to *lead* $v_1(t)$ in phase and $\theta_2 - \theta_1$ is called a *lead phase angle* under this condition.
- Phase lag* is not necessarily a *phase delay* and *phase lead* is not necessarily a *phase advance*. *Phase lag* does not necessarily imply *time delay* and *phase lead* does not necessarily imply *time advance*.
- Instantaneous power *delivered to* a two-terminal element, $p(t) = v(t) i(t)$, where $v(t)$ and $i(t)$ are the element variables defined as per passive sign convention. The sum of $p(t)$ delivered to all elements in an isolated circuit will be zero.
- $p(t)$ delivered to an element is a non-constant function of time in general. If $v(t)$ and $i(t)$ are periodic waveforms with period T and zero average value over a cycle period, $p(t)$ will be a periodic waveform with period $0.5T$ and can have a non-zero average value over its cycle period of $0.5T$.
- The energy delivered to the element will also be a function of time. The value of total energy delivered to the element at the end-of-cycle points will fall on a straight line with a slope equal to the average value of $p(t)$ waveform over its cycle period.
- The *Cycle Average Power* in the context of periodic waveforms is defined as the average of $p(t)$ over one cycle period and is denoted by P . *Average power*, P_{av} over a time interval $(t_2 - t_1)$ is defined as
$$P_{av} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} v(t) i(t) dt$$
 Average Power, $P_{av} = \text{Cycle Average Power}$, P , if the waveforms remain applied to the circuit for a large duration compared to their period.
- Effective value* or *rms value* of a waveform is the value of DC quantity that will produce the *same heating effect* as that produced by the waveform when it is applied as a voltage across a resistance of 1Ω or as a current through a resistance of 1Ω . If $x(t)$ is a periodic waveform with period T , its *rms value* is given by
$$X_{rms} = \sqrt{\frac{1}{T} \int_0^T [x(t)]^2 dt}$$
- Let $v(t) = V_m \sin \omega t$ V and $i(t) = I_m \sin(\omega t + \theta)$ A be the voltage across and current through an electrical element as per passive sign convention. Then, rms value of $v(t)$ is $V_m/\sqrt{2}$, rms value of current is $I_m/\sqrt{2}$ and the average power delivered to the element is $V_{rms} I_{rms} \cos \theta$ W.
- The *average power* delivered to an element in a linear circuit excited by sinusoidal sources of different frequencies (including DC, i.e., zero frequency) obeys superposition principle.
- Let $v(t) = v_1(t) + v_2(t) + \dots + v_n(t)$ be a composite waveform comprising n distinct frequency sinusoidal waveforms. Then,
$$V_{rms} = \sqrt{V_{1rms}^2 + V_{2rms}^2 + \dots + V_{nrms}^2}$$

7.8 QUESTIONS

- A periodic waveform is the sum of three sinusoidal waveforms of period T , $0.5T$ and $0.2T$, respectively. Will this periodic waveform preserve its wave-shape when differentiated and integrated?
- What is the amplitude of $v(t) = 5 \sin \omega t - 3 \cos \omega t$ and what is its phase in degrees with respect to $v_1(t) = 4 \sin(\omega t - \pi/5)$?
- Is the composite waveform $v(t) = 2 \sin \omega t + 3 \cos \sqrt{2} \omega t$ periodic? If yes, what is its period?
- Is the composite waveform $v(t) = 2 \sin 200\pi t + 4 \cos(200.0001\pi t - 45^\circ)$ periodic? If yes, what is its period and cyclic frequency?
- The voltage across a linear load is $v(t) = 100 \sin(100\pi t - 25^\circ)$ V. If the load current $i(t)$ is found to lag $v(t)$ by 36° and $i(0)$ is 2 A find $i(t)$ as a function of time.
- What is the phase of the first derivative of a sinusoidal waveform with respect to the original waveform?
- What is the phase of the indefinite integral of a sinusoidal waveform with respect to the original waveform?
- If $v_1(t) = V_1 \sin \omega t$ and $v_2(t) = V_2 \sin(\omega t + \theta)$ what is the value of θ such that (i) $v_1(t) + v_2(t)$ has a minimum amplitude, (ii) $v_1(t) + v_2(t)$ has a maximum amplitude and (iii) $v_1(t) + v_2(t)$ has an amplitude of $(V_1 + V_2)/2$?
- If $v_1(t) = V_1 \sin \omega t$, $v_2(t) = V_2 \sin(\omega t + \theta)$ and $v_3(t) = v_1(t) + v_2(t) = V_3 \sin(\omega t + \phi)$, find θ such that (V_1, V_2, V_3) is a Pythagorean triplet and find ϕ under this condition in terms of V_1 and V_2 .
- Derive an expression for phase angle of $v_2(t)$ with respect to $v_1(t)$ if $v_1(t) = a \sin \omega t + b \cos \omega t$ and $v_2(t) = c \sin \omega t + d \cos \omega t$.
- Show that average power in an isolated circuit is a conserved quantity.
- Show that cyclic average power in an isolated circuit is a conserved quantity.

13. An active circuit delivers $i(t) = 10 \sin 300\pi t$ A into the positive terminal of a 12 V battery. The battery has an internal resistance of 0.2Ω . (i) What is the average power delivered to the 12 V source inside the battery? (ii) What is the average power loss in the internal resistance of the battery?
14. The voltage across a linear load is $v(t) = 325 \sin(100\pi t - 25^\circ)$ V. If the load current $i(t)$ is found to lead $v(t)$ by 36° and the average power delivered to the load is 250 W, find $i(t)$ as a function of time.
15. Derive an expression for average power if $v(t) = a \sin \omega t + b \cos \omega t$ and $i(t) = c \sin \omega t + d \cos \omega t$ in terms of a, b, c and d .
16. A sinusoidal voltage source of 230 V rms delivers 1 kW of average power to a load. What is the minimum possible amplitude of load current?
17. What is the ratio between rms values of a symmetric square wave and a sinusoidal waveform with same amplitude?
18. What are the ratios between rms values and half-cycle average values of a symmetric triangle waveform and a sinusoidal waveform with same amplitude?
19. The rms value of a periodic waveform $v(t)$ with a frequency of 1 kHz is found to be 10 V. What is the rms value of $v_1(t)$ which has same amplitude and wave-shape as that of $v(t)$, but has a frequency of 10 Hz?
20. The rms value of $V_m \sin \omega t$ is $V_m/\sqrt{2}$ for any value of ω . Explain why ω value does not affect the r.m.s value.
21. If the average power delivered by a 50 Hz sinusoidal voltage to a resistor of 10Ω is 2 W, what is the average power delivered to the same resistor by another sinusoidal voltage source with same amplitude and 5 kHz frequency?
22. Show that if V_{rms} is the r.m.s value of a periodic waveform $v(t)$, the r.m.s value of $av(t)$ is aV_{rms} , where a is a real constant.
23. A sinusoidal voltage source $v(t) = 100 \sin 20\pi t$ is applied across a 10Ω resistor for 0.42 s from $t = 0$. (i) Find the cyclic average power. (ii) Find the average power delivered during the connection duration. (iii) Why are the two values different?
24. A non-sinusoidal periodic voltage waveform with r.m.s value of 100 V is applied to a 10Ω resistor. What is the average power delivered to the resistor?
25. If $v(t) = V_m \sin \omega t + 0.5 V_m \cos 3\omega t$ V and $i(t) = I_m \sin(\omega t - 45^\circ) - 2I_m \cos(3\omega t - 45^\circ)$ A are the terminal variables of a load, find the average power delivered to the load.
26. (a) If $v_1(t) = 5 \sin 100\pi t$ and $v_2(t) = 5 \sin 100\pi t$, find the rms value of $v_1(t) + v_2(t)$, average power that will be delivered to a 10Ω resistor by $v_1(t) + v_2(t)$, and the energy that will be delivered to 10Ω resistor in 0.5 s. (b) If $v_1(t) = 5 \sin 100\pi t$ and $v_2(t) = 5 \sin 100.000001\pi t$, repeat part (a). Can the energy delivered in 0.5 s found from average power in this case? Why are the rms values so different in (a) and (b) for such a small change in frequency of one component?
27. A current $i(t) = 10 \sin 200\pi t$ flows through a resistor of 5Ω from $t = 0$ to $t = 32$ ms. Explain why energy delivered to the resistor is *not equal* to average power multiplied by duration of connection. Calculate the actual value of energy dissipated in the resistor.
28. A sinusoidal voltage source $v(t) = V_m \sin \omega t$ V delivers a fixed amount of average power P through its internal resistance of R to a load. The load current has a general form of

$$i(t) = \sum_{n=1}^{\infty} a_n \sin n\omega t + b_n \cos n\omega t \text{ A.}$$
 Show that the fixed amount of average power delivered by the source reaches the load with maximum efficiency and minimum loss, when $i(t)$ is a pure sinusoidal waveform at ω rad/s with zero phase difference from $v(t)$, i.e., all a and b are zero-valued except a_1 .
29. Show that the rms value of a periodic waveform $x(t)$ and its absolute value $|x(t)|$ are equal.

7.9 PROBLEMS

1. Find the amplitude, frequency, angular frequency and phase of the following sinusoidal functions of time.
 - (i) $x(t) = 10 \sin 500\pi t - 5 \cos(500\pi t - \pi/5)$
 - (ii) $x(t) = 7 \sin(120\pi t - 50^\circ) + 5 \cos(120\pi t + 30^\circ)$
 - (iii) $x(t) = 7 \cos(50\pi t) + 2 \cos(50\pi t + 30^\circ) - 3 \sin(50\pi t + \pi/4)$
2. Find the phase of $x(t)$ and $y(t)$ with respect to a $\sin \omega t$ function, phase difference between $x(t)$ and $y(t)$ and the lead/lag relationship between them for the following.
 - (i) $x(t) = 3 \cos 57\pi t$
 $y(t) = 3 \sin(57\pi t - 15^\circ)$
 - (ii) $x(t) = 3 \cos(7\pi t - \pi/7)$
 $y(t) = 3 \sin(7\pi t + 15^\circ)$
 - (iii) $x(t) = 3 \cos(7\pi t - \pi/7)$
 $y(t) = 3 \sin(7\pi t + 15^\circ)$
 - (iv) $x(t) = \sin \pi t + 2 \cos(\pi t - \pi/3)$
 $y(t) = \cos \pi t - 2 \sin(\pi t + 15^\circ)$
3. If $x(t) = 2 \sin 50\pi t + 3 \cos 55\pi t + 3 \sin 60\pi t$, find the period and cyclic frequency of $x(t)$.
4. A linear circuit contains six two-terminal elements. The sum of instantaneous power in five of them at a particular instant is found to be 17 W. What is the nature of the remaining element and what is the instantaneous power delivered to it at that instant?
5. A linear element has $v(t) = 10 \sin 10\pi t$ V and $i(t) = 3 \cos(10\pi t - 45^\circ)$ A as per passive sign convention. (i) Is the element a resistor? If not, why? (ii) Can the element be a simple element like an inductor or a capacitor or must it be a composite element? (iii) Find the average rate of growth of energy dissipated in the element. (iv) Is this average rate accurate enough if the element remained powered up only for 0.43 s? If not, calculate the average rate at which energy was dissipated in it.
6. The square wave in Fig. 7.9-1 is the voltage across an element and the triangle wave is the current through it. (i) Find and plot

the instantaneous power delivered to the element and the energy delivered to it as functions of time for $t > 0$. (ii) Calculate the cycle average power.

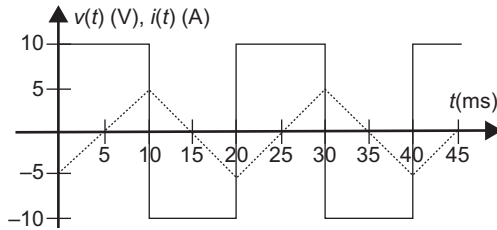


Fig. 7.9-1

7. Calculate the average power in an element with $v(t)$ same as the square wave in Fig. 7.9-1 and $i(t) = (5 + \text{the triangle wave in Fig. 7.9-1})$ for $t > 0$.
8. The periodic ramp voltage waveform in Fig. 7.9-2 is applied across a resistor of $10\ \Omega$. Find the average power dissipated in it.

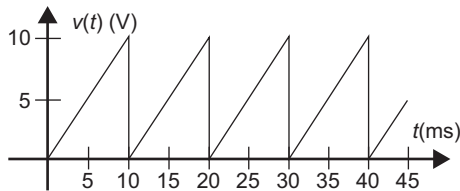


Fig. 7.9-2

9. The absolute value of $v(t)$ shown in Fig. 7.9-3 is applied to a $5\ \Omega$ resistor. Find the average power delivered to it.

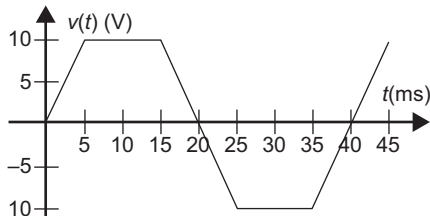


Fig. 7.9-3

10. A pure sinusoidal voltage $v(t) = 100 \sin 100\pi t$ V is applied across a $20\ \Omega$ resistor. (i) Find the amount of energy that is moving back and forth between the voltage source and the resistor. (ii) Find the time required such that this amount is less than 1% of the energy dissipated in the resistor.
11. A linear load draws $i(t) = 5 \sin 200\pi t$ A from a voltage source $v(t) = 200 \cos 200\pi t$ V. (i) Find the average power delivered to the load. (ii) Find the amount of energy that is moving back and forth between the voltage source and the load.
12. A linear composite load draws $i(t) = 5 \sin(100\pi t - 30^\circ)$ A from a voltage source $v(t) = 200 \sin 100\pi t$ V. (i) Find the average power delivered to the load. (ii) Find the amount of energy that is moving back and forth between the voltage source and the load. (iii) Find the time required such that this amount is less than 1% of the energy dissipated in the load.

13. A 48 V battery with a series resistance of $0.1\ \Omega$ delivers the current shown in Fig. 7.9-4 to a power electronic load through a connecting wire resistance of $0.05\ \Omega$. (i) Find the cycle average and rms value of $i(t)$. (ii) Find the average power delivered by the source and received by the load. (iii) What is the wave-shape and value of $i(t)$ that would have resulted in minimum loss in the system with source delivering same average power?

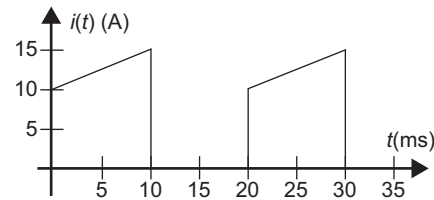


Fig. 7.9-4

14. The voltage appearing across a power electronic load and the current drawn by it are shown in Fig. 7.9-5. (i) Find the average power delivered to the load. (ii) Find the cycle average value and rms value of $v(t)$ and $i(t)$.

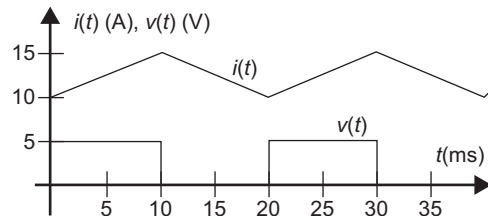


Fig. 7.9-5

15. Find the value of δ such that the periodic waveform shown in Fig. 7.9-6 will have the same rms value as that of a sinusoidal waveform with same peak voltage of V_p .

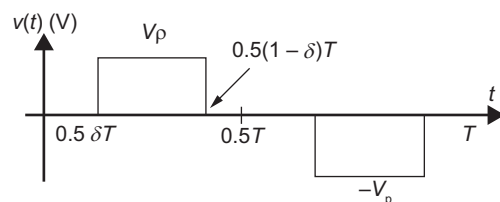


Fig. 7.9-6

16. Find the cycle average and rms value of the periodic waveform in Fig. 7.9-7.

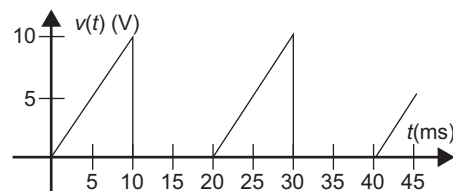


Fig. 7.9-7

17. If $v(t)$ across an element is $10 \sin 100\pi t$ V and $i(t)$ in that element is $2 + 3 \sin 200\pi t + 2 \cos 300\pi t$ A, find the average power delivered?
18. If $v(t)$ across an element is $5 + 10 \sin 100\pi t$ V and $i(t)$ in that element is $2 + 3 \sin 100\pi t + 2 \cos 100\pi t + 2 \sin 300\pi t$ A, find the average power delivered?
19. Calculate the form factor and crest factor for the waveform shown in Fig. 7.9-8.

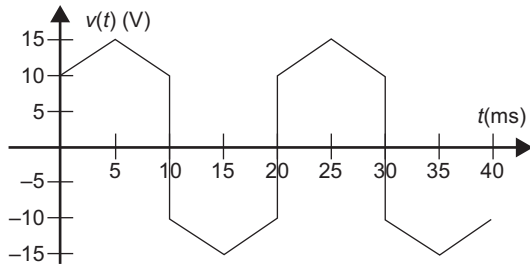


Fig. 7.9-8

20. A rectifier-type voltmeter reads the rms value of a sine wave by measuring the half-cycle average of the waveform and graduating the meter scale after accounting for the form factor of the waveform. What will be the meter reading if (i) a 10 V peak sinusoidal waveform is applied to it, (ii) a 10 V peak symmetrical square waveform is applied to it and (iii) a 10 V peak symmetric triangular waveform is applied to it?
21. Find the form factor and crest factor of the current waveform shown in Fig. 7.9-9.

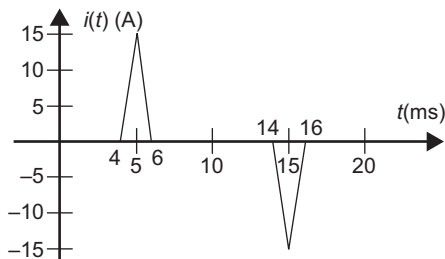


Fig. 7.9-9

22. The half-cycle average of a periodic voltage waveform is 25 V. Form factor is 1.2. Find the power delivered to a 10Ω resistor when this waveform is applied across it?
23. The cycle average of a periodic voltage waveform is 25 V. Its rms value is 35 V. Find the power delivered to a 5Ω resistor when this waveform is applied to it *after removing its DC component*?
24. Find the rms value of $v(t) = 20 + 20 \sin 100\pi t + 35 \cos(100\pi t - 35^\circ) + 23 \sin(250\pi t - 0.2\pi) + 21 \cos(250\pi t + 45^\circ)$ V.
25. If the current in an element as per passive sign convention is $i(t) = 3 + 2 \sin(100\pi t - 36^\circ) + 3.5 \cos(100\pi t - 71^\circ) + 1.15 \sin(250\pi t - 0.45\pi) + 1.05 \cos(250\pi t)$ A, when the voltage waveform in Problem 24 is applied across it, find the average power delivered to it.
26. One period of two periodic voltage waveforms $v_1(t)$ and $v_2(t)$ is shown in Fig. 7.9-10. Find the power delivered by $(v_1(t) + v_2(t))$ to a 10Ω resistor. Explain why average power does not satisfy superposition principle in this case.

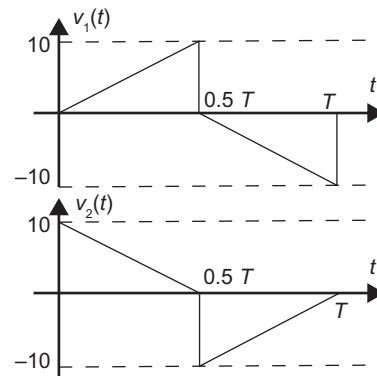


Fig. 7.9-10

27. $v_1(t)$ is a composite periodic waveform containing many sinusoidal waveforms of *distinct* frequencies. $v_2(t)$ is another composite periodic waveform containing many sinusoidal waveforms of *distinct* frequencies. $v_1(t)$ and $v_2(t)$ do not share a sinusoidal waveform component of same frequency. Show that the rms values of $[v_1(t) + v_2(t)]$ and $[v_1(t) - v_2(t)]$ will be the same.

The Sinusoidal Steady-State Response

CHAPTER OBJECTIVES

- *To define and explain the concept of sinusoidal steady-state.*
- *To develop a systematic procedure to analyse sinusoidal steady-state in circuits, in terms of steady-state solution for a complex exponential input function.*
- *To show how to use phasors, phasor equivalent circuits and phasor diagrams for solving circuits under the sinusoidal steady-state condition.*
- *To illustrate the application of circuit theorems in phasor equivalent circuits.*
- *To introduce complex power and its components and provide a detailed interpretation of power components.*
- *To introduce magnetically coupled circuits and explain the application contexts for linear perfectly coupled transformer and ideal transformer.*
- *To familiarise the reader with analyses strategies for sinusoidal steady-state response through a large number of solved examples.*

Basic familiarity with complex algebra is assumed in this Chapter. A detailed study of this chapter and the problems at the end is expected to prepare the reader for higher level courses on Electrical Machines, Electrical Power Systems, Electronic Circuits, etc.

INTRODUCTION

The last few chapters in this book dealt, almost exclusively, with memoryless circuits. We have, by now, developed methods of analyses for such circuits and a set of tools in the form of circuit theorems for such analyses. We did not pay particular attention to the nature of source functions in the analysis of memoryless circuits. We were ready to accommodate DC source functions or time-varying source functions – periodic or non-periodic. This was possible since a memoryless circuit responds to the *present value* of input and its response variables are quite independent of what the source function values were at prior instants. They do not *remember* the past inputs in any manner.

A Note on This Chapter

Strictly speaking, we should start with the transient response of circuits before we take up the study of steady-state response, since a circuit reaches steady-state for an input only by going through a transient period.

However, certain pedagogical constraints make it necessary to take up a detailed study of sinusoidal steady-state response of dynamic circuits even before a study of transient response can be attempted.

Sinusoidal steady-state is an important mode of operation in Electrical Circuits. A study of Electrical Machines, Power Systems, Electronic Amplifiers etc., will turn out to be difficult, if not impossible, unless the learner has a working knowledge of sinusoidal steady-state analysis of circuits.

The first course in Circuits is usually a part of the first-year Engineering curriculum and it is expected to introduce the students to sinusoidal steady-state analysis of circuits in order to prepare them for other EE courses in the second year. There is usually not enough time in the first year to cover the time-domain analysis of circuits before taking up sinusoidal steady-state analysis.

Moreover, the fact that the first year circuits course is usually offered for all disciplines and a detailed time-domain analysis of circuits may not exactly be needed for students of other disciplines, rules the choice of topics in this course. However, students of all disciplines need to be introduced to sinusoidal steady-state in circuits.

continued

The result of this *lack of memory* is that their response variables will have to vary in the same way the input source function varies with time. That is, if the only input to the circuit is a $\sin\omega t$ function, then all currents and voltage variables in a memoryless circuit will be $\sin\omega t$ functions – they cannot even be $\sin(\omega t + \theta)$ functions. Similarly, if the only input to such a circuit is a square wave at a particular frequency, then all response variables will have to be identically shaped square waves except for a change in amplitude.

We had dealt with the memory elements – inductor and capacitor – in single-element circuits in Chapter 3. Now, we bring them back into circuits and take up the time-domain analysis of circuits containing R , L , C , linear dependent sources and independent sources. Such circuits are called *dynamic circuits*, implying that the response at a particular time instant t depends not only on the value of inputs at that instant, but also on the value of inputs from the infinite past to the current instant. The energy storage elements – L and C – *remember* what was done to them from the infinite past to the current instant. Their ability to *remember* can be understood in terms of an inductor's ability to store flux linkage and a capacitor's ability to store charge. Alternatively, their memory-capability can be understood in terms of the inductor's capability to store magnetic energy and the capacitor's capability to store electrostatic energy.

The result of *memory* in some elements of the circuit is that the circuit response can have a different kind of time-variation as compared to the time-variation of input functions. For instance, if there is only one source and that varies as $\sin\omega t$, the response variables can be of $\sin(\omega t + \theta)$ type – that is, the response can contain $\cos\omega t$. If there is only one source and that is a square wave of a particular frequency, the response variables need not be square waves at all. In fact, they will assume quite complicated shapes in practice – but they will be periodic waveforms with the same period as that of the input square wave in the case of *linear dynamic circuits*.

In general, the response of a dynamic circuit to the application of an input at $t = 0$ will contain two components – the natural response component and the forced response component. The total response at any instant after $t = 0$ is a mixture of the two. The natural response component represents the reaction of *inertia* in the circuit against the compelling input source function. There is inertia in the circuit since there is *memory* in the circuit. *Memory* brings about *resistance to change*. The circuit adjusts its natural response component in such a way that no inductors or capacitors are required to change their initial energy storage suddenly as a result of the application of input. The natural response terms in a stable circuit usually die down with time, and hence, they are also called transient response terms. Only the forced response remains after the transient response terms die down to zero. Under this circumstance, the forced response is called the steady-state response if the notion of *steady-state* is applicable in relation to the input. The notion of *steady-state* is applicable only if the input function possesses some aspects that remain steady with time.

Sinusoidal steady-state in a dynamic circuit is a state when all the response variables (i.e., all element currents and element voltages) contain only one component with a sinusoidal waveshape that has the same frequency as that of the sinusoidal forcing function applied.

If there is more than one sinusoidal source, then all those sinusoidal sources must be at the same frequency for the concept of *sinusoidal steady-state* to be true. Sinusoidal steady-state, like any other steady-state, can come up in a circuit only after the circuit goes through the transient period that follows the application of sources. The transient period is deemed to have been completed when all the transient response terms, which usually have waveshapes different from that of the applied sinusoidal function, have died down to negligible levels.

The next section attempts to provide an overview of the transient response of circuits in order to set a background for taking up the study of sinusoidal steady-state. All the topics briefly touched upon in that section will be taken up in great detail in later chapters on time-domain analysis of dynamic circuits. The aim of the next section is only to understand the sinusoidal steady-state in the correct perspective.

8.1 TRANSIENT STATE AND STEADY-STATE IN CIRCUITS

The mesh equations (or node equations) needed to solve for the response of any circuit – memoryless or dynamic – are obtained by applying Kirchhoff's Voltage Law in meshes (or Kirchhoff's Current Law at nodes) along with the element equations. The element equation of an inductor is $v_L = L \frac{di_L}{dt}$ and that of a capacitor is $v_C = \frac{1}{C} \int_{-\infty}^t i_C dt$. Therefore, the mesh

equations (node equations) describing a *dynamic circuit* will be *integro-differential equations*. These equations are true for all t , since they are obtained from KVL and KCL that are true on an instant-to-instant basis. Therefore, both sides of such integro-differential equations can be differentiated or integrated with respect to time, if needed. We will be able to eliminate the integral terms in the equations by this technique. The resulting equations will be differential equations and the coefficients of the differential equations will be decided by the values of R , L , C and M . Since they are constants in the circuits we consider, the resulting differential equations will be a set of simultaneous linear differential equations with constant coefficients. We illustrate this in the two examples in the next sub-section.

Thus, it becomes necessary to deal with sinusoidal steady-state response before transient response in classrooms in many universities worldwide.

Dynamic circuits are quite unaware of this curricular constraint! They refuse to obey the input forcing function immediately and go through a transient response period before they agree to settle down to a steady-state. This fact must be kept in mind throughout this chapter.

8.1.1 Governing Differential Equation of Circuits – Examples

Consider the simple Series RC Circuit in Fig. 8.1-1(a). Assume that the capacitor was initially uncharged. The voltage across the capacitor is taken as the response variable. Then, the current through the resistor from left to right can be expressed as $(v_s - v_C)/R$. Writing KCL at the positive terminal of capacitor,

$$\begin{aligned} v_s - v_C &= 1 \times \frac{dv_C}{dt} \\ \therefore \frac{dv_C}{dt} + v_C &= v_s. \end{aligned}$$

Now we take up the second example circuit in Fig. 8.1-1(b) and write the two mesh equations as,

$$\frac{di_1}{dt} + i_1 - i_2 = v_s \quad (8.1-1)$$

$$\frac{di_2}{dt} + 2i_2 - i_1 = 0 \quad (8.1-2)$$

This is a set of two ordinary first order simultaneous differential equations with constant coefficients. The first mesh current variable i_2 is selected as the circuit response variable. We have to eliminate i_1 from these two equations and get a single differential equation for i_2 in order to find the differential equation describing the circuit. This may be done by using Eqn. 8.1-2 in Eqn. 8.1-1 as shown below:

$i_1 = \frac{di_2}{dt} + 2i_2$ from Eqn. 8.1-2. Substituting this in Eqn. 8.1-1, we get,

$$\frac{d}{dt} \left(\frac{di_2}{dt} + 2i_2 \right) + \left(\frac{di_2}{dt} + 2i_2 \right) - i_2 = v_s$$

Therefore, the describing differential equation for the circuit in Fig. 8.1-1(b) is

$$\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = v_s$$

These two examples help us to understand the following generalisation.

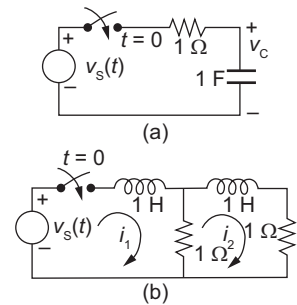


Fig. 8.1-1 Example Circuits for Illustrating Circuit Differential Equations

We accept the following from the basic courses in Mathematics. More detailed exposition on these issues will be provided in the later chapters on time-domain analysis of circuits.

1. The total solution of a linear, constant-coefficient differential equation contains two kinds of terms – the complementary solution terms and the particular integral terms.

2. Complementary solution terms are obtained by solving the differential equation with the right-hand side set to zero.

3. Particular integral terms are obtained by solving the differential equation with the forcing function on the right-hand side.

4. Complementary solution terms are of the type $Ae^{\alpha t}$, where A and α are to be solved for. An n^{th} order linear differential equation with constant coefficients will have n such solution terms. α 's can be obtained by substituting this solution in the differential equation with the right-hand side set to zero. A 's can be obtained by applying initial conditions for all derivatives of the dependent variable from zero to $(n-1)^{\text{th}}$ order on the total solution.

A dynamic circuit is described by a linear, constant-coefficient, ordinary differential equation for one chosen response variable. The coefficients will be decided by the circuit parameters. The right-hand side will contain the applied forcing function terms (including their derivatives in general).

8.1.2 Solution of the Circuit Differential Equation

The differential equation governing the circuit in Fig. 8.1-1(a) was $\frac{dv_C}{dt} + v_C = v_s$.

We try $Ae^{\alpha t}$ as the solution of $\frac{dv_C}{dt} + v_C = 0$. (See side-note)

On substituting the trial solution, we get,

$$A\alpha e^{\alpha t} + Ae^{\alpha t} = 0.$$

Since this has to be true for all t , we conclude that $\alpha + 1 = 0$, leading to $\alpha = -1$.

Now, we attempt to solve for the particular integral. We must specify v_s for that. Let us assume that $v_s = V$, a constant source – that is, a DC source. Then, the equation we have

to solve is given by $\frac{dv_C}{dt} + v_C = V$. The only possibility that a time-function and its first derivative can combine to yield a constant for all t occurs when the time-function is a *constant*. Therefore, we try a constant function as the trial solution. Since the first derivative of a constant is zero, the solution must be $v_C = V$.

Therefore, the total solution for v_C is $v_C = Ae^{-t} + V$ for $t > 0$.

Since the voltage across a capacitor cannot change instantaneously unless there is an impulse current flow in it, we expect the above expression to approach zero as $t \rightarrow 0$ from the right side. $\therefore 0 = Ae^{-0} + V \Rightarrow A = -V$

Therefore, the complete solution for the DC switching problem in the circuit in Fig. 8.1-1(a) is $v_C = V(1 - e^{-t})$ V.

The solution contains two terms; $-Ve^{-t}$ is the transient response term and V is the steady-state response term. The transient response term vanishes in about 5 s or so leaving only the steady-state term. DC steady-state, thus, comes up in this circuit in about 5 s after the application of DC voltage. The period during which the transient response term is active and non-negligible is called the *transient period*. The circuit reaches the steady-state only after the transient period is over.

The differential equation describing the circuit in Fig. 8.1-1(b) was

$\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = v_s$. Consider a DC switching problem with zero initial currents in the inductors in this case too.

The complementary solution terms are obtained by trying out $Ae^{\alpha t}$ in the homogeneous differential equation leading to $\alpha^2 + 3\alpha + 1 = 0$. Therefore, α has two values: -0.382 and -2.618 .

Therefore, the complementary solution is $A_1 e^{-0.382t} + A_2 e^{-2.618t}$.

The particular integral for DC switching problem is a constant in this case too. The value of that constant has to be V in order to satisfy the differential equation with V on the right-hand side. Therefore, the total solution is $i_2 = V + A_1 e^{-0.382t} + A_2 e^{-2.618t}$ A. We can find A_1 and A_2 by using the initial current values for the inductors if we desire to do so. However, those values are not important to us here. What is more important is the observation that the total solution again contains two sets of terms – one set which vanishes with time, and hence, transient in nature and the other which lasts even after the transients vanish. The transient response terms come from the complementary solution and the lasting component (i.e., the steady-state component) comes from the particular integral. The transient response terms vanish in a duration decided by the index in the exponential terms, which is in turn decided by the circuit parameters.

8.1.3 Complete Response with Sinusoidal Excitation

Now, consider the circuit in Fig. 8.1-1(a) with $v_s = V_m \cos \omega t$ V. This sinusoidal waveform is switched on to the circuit at $t = 0$.

The complementary solution is the same as before $-Ae^{-t}$. The particular integral has to be obtained from $\frac{dv_C}{dt} + v_C = V_m \cos \omega t$. This equation has to be true for all t . This can happen only if both sides of the equation are time functions with the same waveshape. Therefore, $\frac{dv_C}{dt} + v_C$ must have the same waveshape as that of $\cos \omega t$. This will imply that both v_C and $\frac{dv_C}{dt}$ must have a sinusoidal waveshape. Therefore, we can try $v_C = a \sin \omega t + b \cos \omega t$ as a trial solution. Substituting this trial solution in the differential equation and collecting terms, we get,

$$(a\omega + b)\cos \omega t + (a - b\omega)\sin \omega t = V_m \cos \omega t$$

This equation can be true for all t only if the coefficients of $\sin \omega t$ on both sides of the equation are equal and the coefficients of $\cos \omega t$ on both sides of the equation are equal.

$$\therefore a\omega + b = V_m \text{ and } a - b\omega = 0 \Rightarrow a = \frac{\omega V_m}{\omega^2 + 1} \text{ and } b = \frac{V_m}{\omega^2 + 1}.$$

$$\therefore \text{particular solution} = \frac{\omega V_m}{\omega^2 + 1} \sin \omega t + \frac{V_m}{\omega^2 + 1} \cos \omega t = \frac{1}{\sqrt{\omega^2 + 1}} \cos(\omega t - \tan^{-1} \omega)$$

The total solution is $v_C = Ae^{-t} + \frac{1}{\sqrt{\omega^2 + 1}} \cos(\omega t - \tan^{-1} \omega)$ for $t > 0$. This solution

must approach zero value as $t \rightarrow 0$ from the right side, since the initial value of voltage across the capacitor is zero. Therefore,

$$0 = Ae^{-0} + \frac{V_m}{\sqrt{\omega^2 + 1}} \cos(-\tan^{-1} \omega) \Rightarrow A = -\frac{V_m}{\sqrt{\omega^2 + 1}} \cos(-\tan^{-1} \omega)$$

$$\therefore v_C = \frac{V_m}{\sqrt{\omega^2 + 1}} \cos(\omega t - \tan^{-1} \omega) - \frac{V_m}{\sqrt{\omega^2 + 1}} \cos(-\tan^{-1} \omega) e^{-t} \text{ V.}$$

Once again, we see that the total response contains a transient term that vanishes as $t \rightarrow \infty$ and a steady-state term that persists. *The steady-state term is a sinusoidal waveform of the same frequency as that of the input sinusoid, but it has a phase difference with respect to the input sinusoid.* It is a phase lag. Both the amplitude and the phase of the steady-state response component depend on the angular frequency ω of the input sinusoid. The steady-state response is given by the particular integral and the transient response is contributed by the complementary function.

We can proceed the same way in the case of the circuit in Fig. 8.1-1(b) too. We solve only for the particular integral of the differential equation this time since we know that the transient response will vanish in this circuit. We try out the solution $i_2 = a \sin \omega t + b \cos \omega t$ in $\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = V_m \cos \omega t$ to find the sinusoidal steady-state solution for the second mesh current in the circuit when the circuit is driven by $V_m \cos \omega t$ V. Substituting the trial solution in the differential equation and collecting terms, we get,

$$(-\omega^2 b + 3\omega a + b)\cos \omega t + (-\omega^2 a - 3\omega b + a)\sin \omega t = V_m \cos \omega t$$

Equating the coefficients of $\sin \omega t$ and $\cos \omega t$ on both sides of the equation, we get, $(-\omega^2 b + 3\omega a + b) = V_m$ and $(-\omega^2 a - 3\omega b + a) = 0$

Features of Sinusoidal Steady-State

The steady-state response of a circuit variable in a linear, dynamic circuit under sinusoidal excitation is a sinusoidal waveform of the same frequency as that of the input.

The response will, in general, have a phase difference with respect to the input.

The amplitude and the phase of response under the steady-state condition will depend on the amplitude of the input and the angular frequency of the input sinusoid.

Solving for a and b , we get, $a = \frac{3\omega V_m}{(1-\omega^2)^2 + 9\omega^2}$ and $b = \frac{(1-\omega^2)V_m}{(1-\omega^2)^2 + 9\omega^2}$

Therefore, the sinusoidal steady-state response for i_2 in the circuit in Fig. 8.1-1(b) is

$$\begin{aligned} i_2(t) &= \frac{3\omega V_m}{(1-\omega^2)^2 + 9\omega^2} \sin \omega t + \frac{(1-\omega^2)V_m}{(1-\omega^2)^2 + 9\omega^2} \cos \omega t \\ &= \frac{V_m}{\sqrt{(1-\omega^2)^2 + 9\omega^2}} \cos(\omega t - \tan^{-1} \frac{3\omega}{1-\omega^2}) \end{aligned}$$

We observe again that the response is a pure sinusoidal waveform at the same frequency as that of the input sinusoid, but the response has a phase lag with respect to the input. Both the amplitude and the phase of response depend on the angular frequency ω of the input sinusoid.

8.2 THE COMPLEX EXPONENTIAL FORCING FUNCTION

The method outlined in the previous section to determine the sinusoidal steady-state response of a dynamic circuit can be summarised as below:

1. Use the mesh or nodal analysis to obtain integro-differential equations of the circuit.
2. Differentiate the equations again to eliminate integrals, if needed.
3. Choose one of the mesh currents (or node voltages) as the describing variable for the circuit. Eliminate all the other variables and obtain an n th order linear constant-coefficient differential equation describing the chosen circuit variable.
4. Assume a solution in the form $a \sin \omega t + b \cos \omega t$. Substitute the assumed solution in the differential equation. Equate the coefficients of cosine and sine on both sides of the equation. Solve for a and b using resulting equations.
5. Express the solution in the form of a single sinusoid with phase shift.
6. Find the other mesh currents (or node voltages) which were eliminated earlier by using the elimination equations in the reverse. Once all the mesh currents (node voltages) are available, any element variable can be obtained from them.

There is nothing wrong with this method except that it is going to be very tedious if the circuit contains more than two energy storage elements.

Hence, we look for a simpler and a more elegant method to obtain the sinusoidal steady-state. Euler's Identity, which relates a complex exponential time-function to trigonometric time-functions, is the key to this new method.

$$\text{Euler's Identity } e^{j\theta} = \cos \theta + j \sin \theta \quad (8.2-1)$$

By letting $\theta = \omega t$ and using Euler's Identity, we can express $e^{j\omega t}$ as $e^{j\omega t} = \cos \omega t + j \sin \omega t$ and by letting $\theta = -\omega t$ and using Euler's Identity, we can express $e^{-j\omega t}$ as $e^{-j\omega t} = \cos \omega t - j \sin \omega t$. Therefore,

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \text{ and } \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}. \quad (8.2-2)$$

$e^{j\omega t}$ is the *complex exponential function of unit amplitude*. Eqn. 8.2-2 expresses unit amplitude cosine and sine functions of time with an angular frequency of ω in terms of two complex exponential functions of time with ω and $-\omega$ in the indices of exponential functions. We can also express sine and cosine functions in terms of complex exponential function in another way too as in Eqn. 8.2-3.

$$\cos \omega t = \text{Re}[e^{j\omega t}] \text{ and } \sin \omega t = \text{Im}[e^{j\omega t}] \quad (8.2-3)$$

8.2.1 Sinusoidal Steady-State Response from Response to $e^{j\omega t}$

These two ways of expressing trigonometric functions in terms of complex exponential functions suggest the two methods to obtain the sinusoidal steady-state response in dynamic circuits.

The first method is to obtain the steady-state response to complex exponential inputs $e^{j\omega t}$ and $e^{-j\omega t}$ and obtain the steady-state response for $\cos\omega t$ as the sum of responses for $e^{j\omega t}$ and $e^{-j\omega t}$. However, this will be correct if, and only if, the particular integral of a linear constant-coefficient differential equation obeys the superposition principle. The mathematical theory of such differential equations assures us that it is indeed so.

The second method will be to obtain the steady-state response for a $\cos\omega t$ input as the *real part* of the steady-state response to a complex exponential function $e^{j\omega t}$ and the steady-state response for a $\sin\omega t$ input as its *imaginary part*. The underlying reason is that since $\cos\omega t$ is the *real part* of $e^{j\omega t}$, the response for $\cos\omega t$ must be the *real part* of the response for $e^{j\omega t}$. This looks intuitively evident, but this turns out to be true only for linear circuits. This will be true only if the *real part* of the input function does not affect the *imaginary part* of the response and the *imaginary part* of the input function does not affect the *real part* of the response. We employ the superposition principle for the particular integral of a linear constant-coefficient differential equation to verify that it is so in the case of such differential equations.

$e^{j\omega t}$ can be viewed as a linear combination of two input functions – $\cos\omega t$ multiplied by 1 added to $\sin\omega t$ multiplied by j . We can view the steady-state response of a linear circuit to $e^{j\omega t}$ input as the particular solution of the describing differential equation of the circuit to a composite input – $\cos\omega t$ multiplied by 1 added to $\sin\omega t$ multiplied by j . But, the particular solution obeys the superposition principle. Therefore, the steady-state response to $e^{j\omega t}$ = steady-state response to $\cos\omega t$ + j times the steady-state response to $\sin\omega t$. Therefore,

The real part of the steady-state response to $e^{j\omega t}$ = steady-state response to $\cos\omega t$ and the imaginary part of the steady-state response to $e^{j\omega t}$ = steady-state response to $\sin\omega t$.

Therefore, the second method for determining the sinusoidal steady-state response in terms of the steady-state response to complex exponential function will yield the correct result for linear circuits. (See side box)

The steady-state response component in linear time-invariant circuits obeys superposition principle.

8.2.2 Steady-State Solution to $e^{j\omega t}$ and the $j\omega$ Operator

Consider the describing differential equation for the second mesh current in the circuit in Fig. 8.1-1(b) with a unit amplitude complex exponential function driving the circuit. The equation is $\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = e^{j\omega t}$. This equation has to be true for all t and that will be possible only if both sides of the equation have the same waveshape in time. Therefore, the particular integral has to be chosen in such a way that, on substituting the trial solution in the differential equation, the left-hand side yields $(\cdot)e^{j\omega t}$, but this will imply that $\frac{d^2 i_2}{dt^2}$, $\frac{di_2}{dt}$ and i_2 must have $e^{j\omega t}$ in them. (In fact, there are special situations in circuits under which this is not strictly true. However, we leave special cases to the later chapters that deal extensively with time-domain analysis of circuits.) Therefore, we look for a function that has $e^{j\omega t}$ in its zeroth order derivative, first order derivative and second order derivative. The only function that comes to our mind is $e^{j\omega t}$ itself. Therefore, the trial solution for the particular integral is $Ae^{j\omega t}$, where A can now be a complex number.

Substituting this solution in the differential equation, we get,

$$(j\omega)^2 [Ae^{j\omega t}] + 3(j\omega)[Ae^{j\omega t}] + [Ae^{j\omega t}] = e^{j\omega t}$$

Since this has to be true for all t , we cancel out $e^{j\omega t}$ and get $[(j\omega)^2 + 3(j\omega) + 1]A = 1$. The solution is completed by solving for A .

The method based on Eqn. 8.2-2 and the superposition principle – we called it the first method – is the favourite method for an Electronics and Communication Engineer.

Electrical Power Engineers usually prefer the second method which is based on Eqn. 8.2-3 and the superposition principle.

Both the methods lead to the same result, of course. But, the analysis of sinusoidal steady-state response from the so-called

continued

phasor concept evolves from the second method.

Hence, we take up the second method for a detailed discussion in Section 8.3.

In this method, we obtain the sinusoidal steady-state response for $\cos\omega t$ by obtaining the *real part* of the steady-state response to $e^{j\omega t}$ input. The sinusoidal steady-state response for $\sin\omega t$ input is similarly the *imaginary part* of the steady-state response to $e^{j\omega t}$ input. A separate determination of response for the cosine and sine inputs is unnecessary. $\sin\omega t$ is the 90° phase delayed version of $\cos\omega t$, and therefore, it must be possible to obtain the steady-state response for $\sin\omega t$ input by delaying the steady-state response for $\cos\omega t$ input by 90° .

However, both methods based on applying $e^{j\omega t}$ will be meaningful only if finding the steady-state response to $e^{j\omega t}$ is simpler and more elegant than finding the steady-state response to a sinusoidal input straightaway.

$$i_2 = \frac{1}{(1-\omega^2) + 3j\omega} e^{j\omega t} = \left[\frac{1}{\sqrt{(1-\omega^2)^2 + 9j\omega^2}} e^{j\phi} \right] e^{j\omega t}, \text{ where } \phi = \tan^{-1} \frac{3\omega}{1-\omega^2}.$$

We note that each differentiation in time has been replaced by a multiplication by $(j\omega)$ in the equation determining the complex amplitude in the assumed particular solution. Once we appreciate this point, we can straightaway obtain the equation governing the complex amplitude A of the steady-state response to a complex exponential input by replacing every differentiation in the differential equation by a multiplication by $j\omega$ and solve for A easily.

Let us generalise this. We consider a linear circuit with one sinusoidal source at an angular frequency of ω driving it. If there are more sources, we employ the superposition principle and solve many single-source circuits. Hence, the basic problem is to solve the circuit for a single source. The most general differential equation governing a chosen response variable for a linear circuit is

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \quad (8.2-4)$$

where y is the chosen response variable, x is the input function and a and b are real positive constants decided by the circuit parameters. The order of the differential equation n is equal to the number of independent energy storage elements in the circuit.

If $x = e^{j\omega t}$, then $y = A e^{j\omega t}$, where A is a complex amplitude decided by the equation $A[(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0] = b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0$.

Therefore, the complex amplitude of the steady-state response to a unit amplitude complex exponential input comes out as a ratio of rational polynomials in the variable $j\omega$. The desired

$$\text{steady-state response is obtained as } y = \frac{\sum_{k=0}^m b_k (j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i (j\omega)^i} e^{j\omega t}.$$

Hence, solving for the steady-state response to a complex exponential function is much simpler and more elegant than solving for the steady-state response to a cosine or sine input function. Therefore, trying to obtain the sinusoidal steady-state response indirectly from a steady-state response to complex exponential input function is worthwhile.

8.3 SINUSOIDAL STEADY-STATE RESPONSE USING COMPLEX EXPONENTIAL INPUT

The method developed in the previous section for determining the sinusoidal steady-state response of a linear circuit for a sinusoidal input is summarised below:

1. Obtain the describing differential equation of the circuit in terms of a chosen describing variable y by nodal analysis or mesh analysis and by subsequent elimination of all variables other than the chosen one. The differential equation is put in the form shown in Eqn. 8.2-4.
2. Express the sinusoidal input as $x = X_m \cos(\omega t + \theta)$. This form is needed to accommodate even sine functions. It could be a current source or a voltage source and X_m represents the real amplitude of the function. Then, this function can be seen as the *real part* of $X_m e^{j(\omega t + \theta)}$.
3. Assume that the input is $x = X_m e^{j(\omega t + \theta)}$ instead of $x = X_m \cos(\omega t + \theta)$.
4. Let the steady-state solution for y for this input be $y_{ss} = Y e^{j(\omega t + \theta)}$, where Y is the *complex amplitude* of the response. **We use bold face italics for complex amplitudes.**

5. Obtain Y as $Y = X_m \times \frac{\sum_{k=0}^m b_k (j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} b_i (j\omega)^i}$ where a and b are as in Eqn. 8.2-4.

This complex number (which is a function of the real variable ω) can be expressed in the exponential form as $Y_m e^{j\phi}$, where Y_m is its magnitude and ϕ is its argument (i.e., angle). It can also be expressed in polar form as $Y_m \angle \phi$.

6. Now, $y_{ss} = Y_m e^{j\phi} e^{j(\omega t + \theta)} = Y_m e^{j(\omega t + \theta + \phi)}$.

7. The desired sinusoidal steady-state response is obtained by taking the *real part* of this. Hence, the sinusoidal steady-state response to $x = X_m \cos(\omega t + \theta)$ is $y = Y_m \cos(\omega t + \theta + \phi)$.

This procedure is applied to the circuit shown in Fig. 8.3-1.

The applied input is $v_s(t) = V_m \cos \omega t$ V. The second mesh current is the chosen circuit variable. Two mesh equations are written first.

$$\frac{di_1}{dt} + i_1 - i_2 = v_s \quad (8.3-1)$$

$$\frac{1}{1} \int_{-\infty}^t i_2 dt + 2i_2 - i_1 = 0 \quad (8.3-2)$$

Differentiating both sides of the second mesh equation with respect to time, we get,

$$2 \frac{di_2}{dt} + i_2 - \frac{di_1}{dt} = 0 \quad (8.3-3)$$

We need to eliminate i_1 from Eqn. 8.3-1 and Eqn. 8.3-3 to get a differential equation for i_2 .

$$\frac{di_1}{dt} = 2 \frac{di_2}{dt} + i_2 \quad (\text{from Eqn. 8.3-3})$$

$$\therefore \frac{d^2 i_1}{dt^2} = 2 \frac{d^2 i_2}{dt^2} + \frac{di_2}{dt}$$

Differentiating the Eqn. 8.3-1 with respect to time gives us,

$$\frac{d^2 i_1}{dt^2} + \frac{di_1}{dt} - \frac{di_2}{dt} = \frac{dv_s}{dt}$$

Substituting for $\frac{di_1}{dt}$ and $\frac{d^2 i_1}{dt^2}$, we get,

$$\frac{d^2 i_2}{dt^2} + \frac{di_2}{dt} + 0.5i_2 = 0.5 \frac{dv_s}{dt}$$

Now, we solve the steady-state response for $V_m e^{j\omega t}$. Let the response be $i_2 = Y e^{j\omega t}$. Then, substituting the solution in the differential equation above,

$$Y[(j\omega)^2 + (j\omega) + 0.5] e^{j\omega t} = 0.5(j\omega) V_m e^{j\omega t}$$

$$\therefore Y = \frac{0.5(j\omega)V_m}{(0.5 - \omega^2) + (j\omega)} = \frac{0.5\omega V_m}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} e^{j\phi}, \text{ where } \phi = \frac{\pi}{2} - \tan^{-1} \frac{\omega}{(0.5 - \omega^2)}.$$

$$\text{Therefore, the steady-state response for } V_m e^{j\omega t} = \frac{0.5\omega V_m}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} e^{j(\omega t + \phi)}$$

The desired sinusoidal steady-state response is obtained by taking the real part of this solution and is $= \frac{0.5\omega V_m}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} \cos \left[\omega t + \frac{\pi}{2} - \tan^{-1} \frac{\omega}{(0.5 - \omega^2)} \right]$.

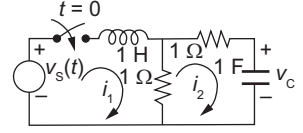


Fig. 8.3-1 A Two-Mesh Circuit for illustrating Sinusoidal Steady-State Solution

8.4 THE PHASOR CONCEPT

Solving for the particular integral of a differential equation for sinusoidal input has been rendered easy by the use of complex exponential function as shown in the previous sections. However, deriving the differential equation remains a tedious affair even now. Arriving at the proper elimination steps requires considerable ingenuity in the case of circuits containing many inductors and capacitors, but we can avoid all that. We proceed to see how.

The steady-state response of the second mesh current i_2 in the circuit in Fig. 8.3-1 for a complex exponential input of $V_m e^{j\omega t}$ was noted as $\frac{0.5\omega V_m e^{j\phi}}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} e^{j\omega t}$. We evaluate this for $\omega = 1$ rad/s. Then $i_2 = 0.447 V_m e^{-j26.6^\circ} e^{jt}$

The equation used for eliminating the first mesh current was $\frac{di_1}{dt} = 2 \frac{di_2}{dt} + i_2$. Substituting for i_2 , we get,

$$\frac{di_1}{dt} = j0.894 V_m e^{-j26.6^\circ} e^{jt} + 0.447 V_m e^{-j26.6^\circ} e^{jt} = (0.447 + j0.894) V_m e^{-j26.6^\circ} e^{jt}.$$

Integrating this equation gives us,

$$\begin{aligned} i_1 &= \frac{1}{j1} (0.447 + j0.894) V_m e^{-j26.6^\circ} e^{jt} \\ &= (0.894 - j0.447) V_m e^{-j26.6^\circ} e^{jt} \\ &= 1 e^{-j26.6^\circ} V_m e^{-j26.6^\circ} e^{jt} \\ &= V_m e^{-j53.2^\circ} e^{jt} \end{aligned}$$

We have got the two mesh currents now. Therefore, we can obtain all the circuit variables. For instance, the current in the common resistor is given by

$$\begin{aligned} i_1 - i_2 &= V_m (e^{-j53.2^\circ} - 0.447 e^{-j26.6^\circ}) e^{jt} = V_m (0.6 - j0.8 - 0.4 + j0.2) e^{jt} \\ &= V_m (0.2 - j0.6) e^{jt} = 0.632 V_m e^{-j71.6^\circ} e^{jt} \end{aligned}$$

The voltage across 1H inductor is given by

$$\frac{di_1}{dt} = \frac{d}{dt} (V_m e^{-j53.2^\circ} e^{jt}) = V_m j e^{-j53.2^\circ} e^{jt} = V_m e^{j90^\circ} e^{-j53.2^\circ} e^{jt} = V_m e^{j26.6^\circ} e^{jt}.$$

Based on the above example, we arrive at the following conclusion:

All element voltage variables and all element current variables in a linear dynamic circuit driven by a complex exponential function $X_m e^{j\omega t}$ will assume the form $(Y_m e^{j\phi}) e^{j\omega t}$ under the steady-state condition, where $(Y_m e^{j\phi})$ represents the relevant complex amplitude for the variables. Y_m will be proportional to X_m .

8.4.1 Kirchhoff's Laws in terms of Complex Amplitudes

Consider a mesh containing n elements in a general linear dynamic circuit. Let it be in the steady-state condition under complex exponential drive. Let the angular frequency of the complex exponential function be ω . Then, we can represent each element voltage variable, v_i for $i = 1$ to n , as $v_i = (V_{im} e^{j\phi_i}) e^{j\omega t}$ under steady-state condition. We assume (for simplicity) that we encounter the positive polarity of the voltage variable first when we traverse the mesh in a clockwise direction. Then, applying KVL in the mesh results in the following KVL equation:

$$\begin{aligned} (V_{1m} e^{j\phi_1}) e^{j\omega t} + (V_{2m} e^{j\phi_2}) e^{j\omega t} + \dots + (V_{nm} e^{j\phi_n}) e^{j\omega t} &= 0 \text{ for all } t \\ \text{i.e., } [(V_{1m} e^{j\phi_1}) + (V_{2m} e^{j\phi_2}) + \dots + (V_{nm} e^{j\phi_n})] e^{j\omega t} &= 0 \text{ for all } t \\ \text{i.e., } [(V_{1m} e^{j\phi_1}) + (V_{2m} e^{j\phi_2}) + \dots + (V_{nm} e^{j\phi_n})] &= 0. \end{aligned}$$

Thus, the KVL equation under the complex exponential steady-state response condition can be written entirely in terms of the *complex amplitudes* of the steady-state element voltages. The common complex exponential function format $e^{j\omega t}$ that appears in *all* the element voltages may be suppressed in KVL equations.

A similar conclusion can be stated for KCL equations at the nodes of a dynamic circuit under the complex exponential steady-state response condition.

The KVL equation in a mesh (KCL equation at a node) becomes a *mesh equation* (*node equation*) only when the *mesh currents* (*node voltages*) are used to replace the element voltages (currents). We need the *element equations* for this. Hence, we need to see how the *complex amplitudes* of element voltage and current are related to each other in the case of R , L , C etc.

8.4.2 Element Relations in terms of Complex Amplitudes

Consider a resistor. The passive sign convention is assumed throughout in the circuit. The element equation of a resistor is $v_R = Ri_R$.

Substituting $v_R = (V_{Rm} e^{j\phi_v})e^{j\omega t}$ and $i_R = (I_{Rm} e^{j\phi_i})e^{j\omega t}$, we get, $(V_{Rm} e^{j\phi_v})e^{j\omega t} = R(I_{Rm} e^{j\phi_i})e^{j\omega t}$, and hence, $V_{Rm} = RI_{Rm}$ and $\phi_v = \phi_i$. This implies that a resistor cannot bring about a phase difference between the current and the voltage variables. Using the bold face italic notation for the complex amplitude, we write the element relation of a resistor for complex amplitudes as below.

$$\mathbf{V}_R = R\mathbf{I}_R \quad (8.4-1)$$

The element equation of an inductor is $v_L = L \frac{di_L}{dt}$.

Substituting $v_L = (V_{Lm} e^{j\phi_v})e^{j\omega t}$ and $i_L = (I_{Lm} e^{j\phi_i})e^{j\omega t}$, we get, $(V_{Lm} e^{j\phi_v})e^{j\omega t} = L(j\omega)(I_{Lm} e^{j\phi_i})e^{j\omega t}$. This implies that $V_{Lm} = \omega LI_{Lm}$ and $\phi_v = \phi_i + 90^\circ$. Thus, an inductor scales the current amplitude by ωL and adds a 90° phase advance to the phase of the current to generate the voltage across it. Using the bold face italic notation for the complex amplitude, we write the element relation of an inductor for *complex amplitudes* as below:

$$\begin{aligned} \mathbf{V}_L &= (j\omega L)\mathbf{I}_L = (\omega L e^{j90^\circ})\mathbf{I}_L \\ \mathbf{I}_L &= \frac{1}{j\omega L}\mathbf{V}_L = \left[\frac{1}{\omega L} e^{-j90^\circ} \right] \mathbf{V}_L \end{aligned} \quad (8.4-2)$$

Thus, the inductor *operates* upon the *complex amplitude* of current with the *operator* $j\omega L$ to generate the voltage *complex amplitude* across itself.

The element equation of a capacitor is $i_C = C \frac{dv_C}{dt}$.

Substituting $v_C = (V_{Cm} e^{j\phi_v})e^{j\omega t}$ and $i_C = (I_{Cm} e^{j\phi_i})e^{j\omega t}$, we get, $(I_{Cm} e^{j\phi_i})e^{j\omega t} = C(j\omega)(V_{Cm} e^{j\phi_v})e^{j\omega t}$. This implies that $I_{Cm} = \omega C V_{Cm}$ and $\phi_i = \phi_v + 90^\circ$. Thus, a capacitor scales the voltage amplitude by ωC and adds a 90° phase advance to the voltage to generate the current through itself. Using the bold face italic notation for the complex amplitude, we write the element relation of a capacitor for *complex amplitudes* as below:

$$\begin{aligned} \mathbf{I}_C &= (j\omega C)\mathbf{V}_C = (\omega C e^{j90^\circ})\mathbf{V}_C \\ \mathbf{V}_C &= \frac{1}{j\omega C}\mathbf{I}_C = \left[\frac{1}{\omega C} e^{-j90^\circ} \right] \mathbf{I}_C \end{aligned} \quad (8.4-3)$$

Thus, the capacitor *operates* on the *complex amplitude* of voltage by the *operator* $j\omega C$ to generate the *complex amplitude* of current flowing through it.

The amplitude of current in an inductor is $1/\omega L$ times the amplitude of voltage and the current lags the voltage by 90° under the sinusoidal steady-state condition.

The amplitude of current in a capacitor is ωC times the amplitude of voltage and the current leads the voltage by 90° under the sinusoidal steady-state condition.

Phasors

Phasor is a complex number that gives the amplitude of the complex exponential function and the phase of the complex exponential function with the time-variation of the function understood as $e^{j\omega t}$. It can be used as a representation for a sinusoidal function.

The steps involved in forward and inverse *phasor* transformation are easy and can be done mentally with a little practice.

For instance, if $\mathbf{X} = 1 + j1$, then the magnitude is $\sqrt{2}$ and the angle is 45° and the time-domain waveform is $\sqrt{2} \cos(\omega t + 45^\circ)$, if we started with a cosine and it is $\sqrt{2} \sin(\omega t + 45^\circ)$, if we started with a sine. Of course, we need to know the value of ω in addition to $\mathbf{X}$.

The *phasor* representation of a sinusoidal waveform will not contain the information on frequency. Frequency has to be known separately.

8.4.3 The Phasor

Electrical Engineers decided long back that a new name is required for what we have been calling the *complex amplitude*. Complex amplitude is the number that gives the amplitude of the complex exponential function and the phase of the complex exponential function in the form of a single complex number. Its magnitude gives the amplitude of the signal and its angle gives the phase of the signal. The phase is referred to the standard $e^{j\omega t}$ reference function. Electrical Engineers call the complex amplitude a *phasor*.

Thus, *phasor* is just a new name for what we have understood until now as *complex amplitude*.

The process of starting with a sinusoidal function and ending up with its *phasor representation* is called *Phasor Transformation*. We summarise the steps involved in this transformation.

- Express the given sinusoidal function in the form $x(t) = X_m \cos(\omega t + \theta)$.
- Write $x(t)$ as the *real part* of $X_m e^{j(\omega t + \theta)}$.
- Suppress the qualifier *real part*.
- Suppress $e^{j\omega t}$ after noting the value of ω for later use.
- The resulting complex number $\mathbf{X} = X_m e^{j\theta}$ is the *phasor representation* for $x(t)$. The bold face italic notation stands for the magnitude and the angle together. The symbol $\bar{X}$ may be used in hand-written text.

Once we get the answer for a circuit analysis problem in the *phasor representation*, we need to go back to time-domain to get the time-domain output that we actually wanted. We start in the time-domain and want to end up in the time-domain and the nether world of *phasors* is only a temporary sojourn. The steps involved in the *inverse phasor transformation* are listed below:

- Obtain the magnitude X_m and angle θ of the *phasor* and put it in $X_m e^{j\theta}$ form.
- Multiply $X_m e^{j\theta}$ by $e^{j\omega t}$ and express it in $X_m e^{j(\omega t + \theta)}$ form.
- Get the *real part* as $X_m \cos(\omega t + \theta)$ by using Euler's Identity.

We are free to express the time-function as $x(t) = X_m \sin(\omega t + \theta)$. The *phasor* representation will remain the same, but the *imaginary part* of the complex exponential signals will represent the time-domain signals everywhere in the circuit.

8.5 TRANSFORMING A CIRCUIT INTO A PHASOR EQUIVALENT CIRCUIT

We have already seen that we can write the KVL and KCL equations directly in terms of *complex amplitudes* (i.e., phasors) and that there are well-defined relations between *complex voltage amplitude* (i.e., voltage phasors) and *complex current amplitude* (i.e., current phasors) for all two-terminal elements.

The ratio of *voltage phasor* to *current phasor* is equal to R in the case of a resistor. It is $j\omega L$ in the case of an inductor and it is $1/j\omega C$ in the case of a capacitor.

These facts suggest that we need not write down the mesh or node equations in the time-domain at all. We can write them in terms of mesh current phasors or node voltage phasors using the element relation that ties up the voltage phasor of the element to the current phasor of the element. The resulting equations will be algebraic equations involving phasors.

Thus, *phasor transformation* of all circuit variables results in a set of simultaneous algebraic equations rather than simultaneous differential equations. Eliminating the variables and solving for the desired circuit variable is far easier when we deal with algebraic equations than when we deal with differential equations.

The circuit that will help us deal with the steady-state response to complex exponential input as if it is a memoryless circuit is called the phasor equivalent circuit of the dynamic circuit.

8.5.1 Phasor Impedance, Phasor Admittance and Phasor Equivalent Circuit

First, we generalise the concept of resistance to accommodate a more general relationship between voltage and current than a simple scaling. The ratio between the voltage phasor and the current phasor at a pair of terminals will, in general, be a complex number indicating that the circuit connected between the pair of terminals is capable of scaling the amplitude and imparting a phase shift to one quantity with respect to the other. We define this ratio as the *driving-point phasor impedance* at the pair of terminals and represent it as $Z(j\omega)$. Its unit is Ohms (Ω). The reciprocal of the ratio – i.e., the ratio of current phasor to voltage phasor at a pair of terminals is defined as the *driving-point phasor admittance* at the terminals and is represented as $Y(j\omega)$. Its unit is Siemens or Mhos ($\mathfrak{S}$) (see Fig. 8.5-1).

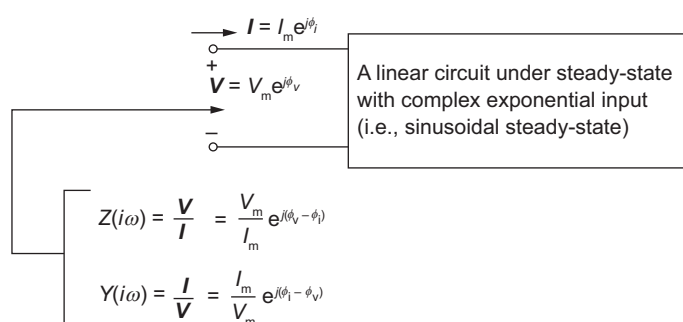


Fig. 8.5-1 Driving-Point Impedance and Admittance Under Sinusoidal Steady-State

Thus, if $V = V_m e^{j\phi_v}$ and $I = I_m e^{j\phi_i}$ are the voltage and current phasors as per the passive sign convention at a pair of terminals, then, the phasor impedance function $Z(j\omega) = \frac{V}{I} = \left[\frac{V_m}{I_m} \right] e^{j(\phi_v - \phi_i)}$ and the phasor admittance function $Y(j\omega) = \frac{I}{V} = \left[\frac{I_m}{V_m} \right] e^{j(\phi_i - \phi_v)}$. Note that both the impedance function and the admittance function are represented as functions of $j\omega$. They are, in fact, complex functions of a real variable ω and not complex functions of an imaginary variable $j\omega$ as indicated by the notation. The j in $j\omega$ serves to remind us that they are complex functions.

The magnitude of the complex Z gives the ratio of amplitudes of voltage phasor and current phasor. The angle of Z gives the angle by which the current phasor lags the voltage phasor.

The real part of Z is the *resistance* part of Z and the imaginary part of Z is defined as its *reactance* part. Similarly, the real part of Y is the *conductance* part of Y and the imaginary part of Y is defined as its *susceptance* part. Thus, $Z = R + jX$ and $Y = G + jB$, where X is the reactance and B is the susceptance. Reactance has Ohms (Ω) as its unit and susceptance has Siemens as its unit.

The *phasor equivalent circuit* is formed by carrying out the following steps:

1. Convert all sinusoidal sources at a single frequency ω into their phasor representations and mark them near the source symbols. There is no change in the graphic symbols used. Cosine function is assumed in time-domain by default.
2. Replace all passive elements by their phasor impedance/admittance and linear dependent sources by their phasor relations. The graphic symbols used for all elements will be the same in the original circuit and in its phasor equivalent circuit.

$Z(j\omega) = R$ and $Y(j\omega) = 1/R$ for a resistor of $R \Omega$.
 $Z(j\omega) = j\omega L$ and $Y(j\omega) = 1/j\omega L$ for an inductor of L H.
 $Z(j\omega) = 1/j\omega C$ and $Y(j\omega) = j\omega C$ for a capacitor of C F.

The procedure is illustrated for the circuit in Fig. 8.5-2.

The first source function $200 \sin 314t$ is expressed as $200 \cos(314t - 90^\circ)$. Then, the first source voltage phasor is $200e^{-j90^\circ}$ in the exponential form, $200\angle -90^\circ$ in the polar form and $0 - j200$ in the rectangular form.

The second source function $250 \sin(314t - 45^\circ)$ is expressed as $250 \cos(314t - 135^\circ)$. Then, the second source voltage phasor is $250e^{-j135^\circ}$ in the exponential form, $250\angle -135^\circ$ in the polar form and $-176.77 - j176.77$ in the rectangular form.

The value of $\omega = 314$ rad/s. Therefore, the 4 mH inductor will have an impedance of $j1.256 \Omega$, the 5 mH inductor will have an impedance of $j1.57 \Omega$ and the 10 mH inductor will have an impedance of $j3.14 \Omega$. The impedance of $100 \mu\text{F}$ capacitor will be $1/0.0314 = -j31.85 \Omega$.

We see that the impedances of inductors and capacitors are purely reactive. The reactance of an inductor is a positive quantity, whereas the reactance of a capacitor is a negative quantity. Similarly, the susceptance of an inductor is a negative quantity, whereas the susceptance of a capacitor is a positive quantity.

The phasor equivalent circuit of the circuit in Fig. 8.5-2 is now completed as in Fig. 8.5-3.

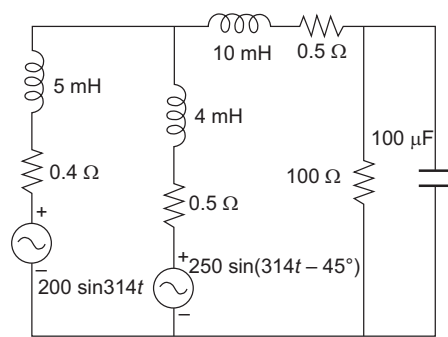


Fig. 8.5-2 Circuit for Illustrating Phasor Equivalent Circuit

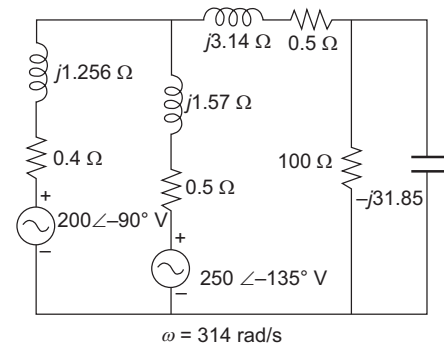


Fig. 8.5-3 Phasor Equivalent Circuit of Circuit in Fig. 8.5-2

8.6 SINUSOIDAL STEADY-STATE RESPONSE FROM PHASOR EQUIVALENT CIRCUIT

We now know how to convert the sinusoidal source functions to their phasor representations and how to construct the phasor equivalent circuits employing phasor impedances and phasor admittances for steady-state analysis of dynamic circuits for complex exponential inputs. Steady-state analyses for complex exponential inputs and steady-state analyses for sinusoidal inputs are effectively the same. Only the phasor transformation brings in the variation.

Systematic application of KVL and KCL, along with the element relationships that tie up the phasor voltage of an element to the phasor current through that element, will lead to the sinusoidal steady-state solution in circuits, in principle. However, systematic and routine analyses procedures would be quite welcome in this case too, as they were in the case of time-domain analyses of memoryless circuits. This prompts us to compare the time-domain analysis problem in memoryless circuits with the sinusoidal steady-state analysis problem in dynamic circuits. The aim of such a comparison is to determine whether we can employ the analyses methods we developed in the context of memoryless circuits to the sinusoidal steady-state analysis problem. Moreover, we would like to verify whether the circuit theorems developed in the context of memoryless circuits will hold good in the case of dynamic circuits in the sinusoidal steady-state.

8.6.1 Comparison Between Memoryless Circuits and Phasor Equivalent Circuits

1. The sources and circuit variables in a memoryless circuit are, in general, functions of time. DC resistive circuits form a sub-class of memoryless circuits in which all the sources are constant-valued. The source functions and circuit variables in a phasor equivalent circuit are *phasors* which are *complex amplitudes* of complex exponential functions of a common format $e^{j\omega t}$, where ω is the angular frequency of all sinusoidal sources active in the circuit. *A phasor equivalent circuit can be drawn only if all the sources are of the same frequency in that circuit.* If a dynamic circuit contains sinusoidal sources with different frequencies, phasor equivalent circuits for different frequencies have to be prepared separately and the superposition principle has to be used to combine the solution from various phasor equivalent circuits. Thus, a phasor equivalent circuit prepared for a particular value of ω is similar to a memoryless circuit driven by constant sources. The difference is that, in phasor equivalent circuits, the constant sources are complex-valued, whereas in memoryless circuits the constant sources are real-valued. Similarly, all circuit variables in a phasor equivalent circuit are complex-valued constant quantities in time, whereas they are real-valued constant quantities in time in the case of a DC memoryless circuit. Indeed, all the circuit variables in a dynamic circuit under the sinusoidal steady-state vary with time; but this time-variation has been absorbed in the term $e^{j\omega t}$ which is a common factor in all circuit variables and is suppressed in the phasor equivalent circuit.
2. The only passive element permitted in the memoryless circuit is the resistor. The voltage across a resistor is proportional to the current through it with a proportionality constant that is real-valued. Dependent sources with source functions of the form $y = kx$, where x is the controlling variable, y is the controlled variable and k is a real-valued proportionality constant, are allowed in a memoryless circuit. Thus, we see that all the elements permitted to be present in such a circuit (except independent sources) can only *scale* (i.e., result in a multiplication by a constant) the circuit variables. The resulting circuit equations will be *simultaneous algebraic equations* tying up all the instantaneous voltage and current variables (real-valued constant quantities in the case of DC excitation) in the circuit.
3. All kinds of linear passive elements, including dependent sources of the type $y = k_1x + k_2x' + k_3x'' + \dots$, where primed variable indicates derivatives, are permitted in a dynamic circuit. However, once a phasor equivalent circuit, using phasor impedances or phasor admittances for the elements, is constructed, only *scaling* of phasors through impedances and admittances can take place in the circuit. The resulting circuit equations will be *simultaneous algebraic equations* tying up the various phasor voltage and phasor current values (complex-valued constant quantities for a particular ω) in the phasor equivalent circuit.
4. Therefore, a memoryless circuit driven by DC sources and a dynamic circuit driven by sinusoidal sources of the same frequency under the sinusoidal steady-state will have *similarly structured equations* of analysis. The only difference is that the variables in a memoryless circuit are real-valued, whereas variables in a dynamic circuit under the sinusoidal steady-state are complex-valued.
5. Two circuit analysis techniques called *nodal analysis* and *mesh analysis* were developed for memoryless circuits in Chap. 4. We did not depend explicitly or implicitly on the ‘real-valued’ nature of the circuit variables to arrive at these analysis procedures in that chapter. Hence, those analysis techniques should remain valid even when the circuit variables and the element impedances turn out to be complex numbers.
6. Similarly, all the circuit theorems which depended on the properties of Nodal Conductance Matrix and Mesh Resistance Matrix and linearity for their proof should remain applicable to phasor equivalent circuits too, provided the criterion

A comparison between DC circuits and phasor equivalent circuits shows that all circuit theorems developed under the analysis of memoryless circuits can be applied in phasor equivalent circuits too.

Further nodal analysis and mesh analysis techniques developed for memoryless circuits apply to phasor equivalent circuits with no change except that the impedance $\mathbf{Z}$ takes the place of resistance R and admittance $\mathbf{Y}$ takes the place of conductance G .

of ‘sinusoidal steady-state’ is not in conflict with the underlying assumptions in the theorems in any manner.

7. A straightforward application of KVL and KCL will show that the series equivalent impedance of n phasor impedances connected in series will be given by $Z_{eq} = Z_1 + Z_2 + \dots + Z_n$ and that the parallel equivalent admittance of n phasor admittances connected in parallel will be equal to $Y_{eq} = Y_1 + Y_2 + \dots + Y_n$.
8. The nodal analysis and mesh analysis techniques developed for memoryless circuits apply to the phasor equivalent circuits with no change, except that the *impedance* Z takes the place of *resistance* R and *admittance* Y takes the place of *conductance* G . Nodal Conductance Matrix will be called Nodal Admittance Matrix Y_m and Mesh Resistance Matrix will be called Mesh Impedance Matrix Z_m in the sinusoidal steady-state analysis using phasor equivalent circuits. They will be symmetric complex matrices if the phasor equivalent circuit contains no dependent sources.

8.6.2 Nodal Analysis and Mesh Analysis of Phasor Equivalent Circuits – Examples

The nodal analysis and mesh analysis techniques for obtaining sinusoidal steady-state response quantities using phasor equivalent circuits are illustrated through some examples in this sub-section.

EXAMPLE: 8.6-1

Find the steady-state current and the average power dissipated in the resistor in an R – L series circuit with $R = 100\ \Omega$ and $L = 1\text{ H}$ when driven by a switched sinusoidal source $v_s(t) = 325 \sin 100\pi t\text{ V}$.

SOLUTION

The equation $v_s(t) = 325 \sin 100\pi t\text{ V}$, makes it clear that the sinusoidal source was switched on to the circuit only at $t = 0$. Hence, the steady-state situation will come up in the circuit only after some time and we should not expect the solution that we work out, based on the phasor equivalent circuit, to hold good during the immediate period after switching on the source.

The angular frequency of the source is $\omega = 100\pi\text{ rad/s}$. The value of reactance of the 1 H inductor at this angular frequency $= 100\pi \times 1 = 314.15\ \Omega$, and hence, the impedance of this inductor is $j314.15\ \Omega$. Note that reactance is a *real number*, whereas impedance is a *complex number*.

We need to represent the source function in cosine form first. $v_s(t) = 325 \sin 100\pi t = 325 \cos(100\pi t - 90^\circ)$. Therefore, the phasor representation of the source is $\mathbf{V}_s = 325\angle -90^\circ\text{ V}$. The circuit in the time-domain and the circuit in the phasor domain are shown in Fig. 8.6-1.

This is a single mesh circuit and the mesh current $\mathbf{I}_1$ is identified in the phasor equivalent circuit in Fig. 8.6-1(b). The mesh equation is obtained as

$$(100 + j314.15)\mathbf{I}_1 = 325\angle -90^\circ$$

Solving for $\mathbf{I}_1$, we get,

$$\mathbf{I}_1 = \frac{325\angle -90^\circ}{(100 + j314.15)} = \frac{325\angle -90^\circ}{329.7\angle 72.34^\circ} = 0.986\angle -162.34^\circ\text{ A}.$$

Going back to time-domain by inverse phasor transformation, we get,

$$\begin{aligned} i_1(t) &= 0.986 \cos(100\pi t - 162.34^\circ) \\ &= 0.986 \cos(100\pi t - 90^\circ - 72.34^\circ) \\ &= 0.986 \sin(100\pi t - 72.34^\circ)\text{ A}. \end{aligned}$$

The source voltage waveform and the circuit current waveform are shown in Fig. 8.6-2(a) and (b).

The current waveform, as drawn in Fig. 8.6-2(a), is wrong. Remember that we have obtained the sinusoidal steady-state solution only and not the complete circuit solution

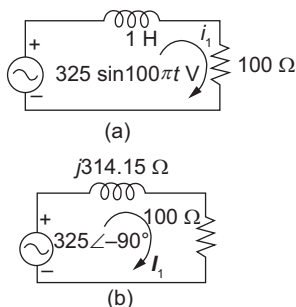


Fig. 8.6-1 (a) The Circuit in Time-Domain and (b) The Phasor Equivalent Circuit for Example 8.6-1

for all $t > 0$. Sinusoidal steady-state gets established only in the long run. The time taken for that will depend on the circuit parameters. We will learn how to estimate the time required for a given circuit to reach the sinusoidal steady-state in later chapters. We may accept the fact that it takes about $5L/R$ s (i.e., about 50 ms in this circuit) for an R - L circuit to reach the steady-state. Therefore, strictly speaking, the sinusoidal steady-state waveforms should be marked in time-axis as shown in Fig. 8.6-2(b). It is understood that the t used in the axis marking in this figure can have any value greater than 50 ms or so.

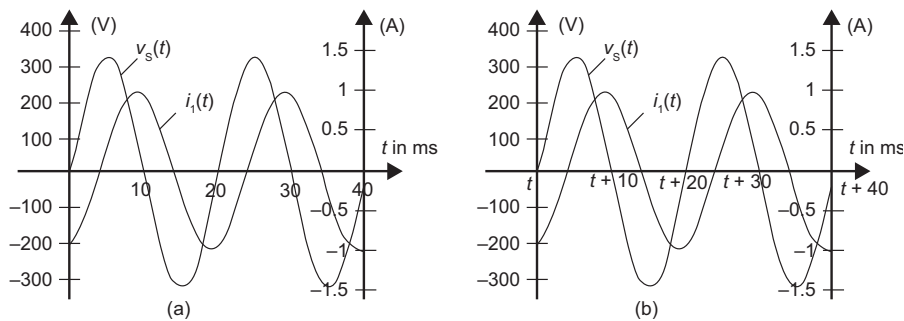


Fig. 8.6-2 Source Voltage and Circuit Current Waveforms in Example 8.6-1 with (a) Misleading Time-axis Marking (b) Correct Time-axis Marking

The waveform as shown in Fig. 8.6-2(a) is wrong from another point of view too. We remember that the voltage applied to the circuit was zero prior to $t = 0$. According to Fig. 8.6-2(a), the current suddenly changed from zero to a negative value at $t = 0$. It is true that this value of current will exist in the circuit whenever the voltage goes through a positive-going zero-crossing, once the circuit has reached the steady-state. But the current cannot do that at the first zero-crossing of the voltage itself, since it will be a violation of the law of causality. How did the circuit know while it was at $t = 0$ that the zero voltage that it is being subjected to at that instant is somehow different from the zero voltage that it was subjected to at the prior instants? Could it have anticipated that the voltage is going to rise and raised its current instantaneously based on its anticipation on what the voltage waveform is going to do in future, after $t = 0$, while it was at $t = 0$? No physical system can do that sort of a thing. *All physical systems are non-anticipatory.* The last sentence is yet another form of the law of causality. Hence, the current waveform as shown in Fig. 8.6-2(a) violates the law of causality.

We note from this example that (i) the impedance of an R - L circuit has a positive angle which is $\tan^{-1}(\omega L/R)$ in general (ii) the current in an R - L circuit lags the voltage waveform under the steady-state conditions by $\tan^{-1}(\omega L/R)$ in general.

$$\text{Average power delivered to resistor} = (I_{\text{rms}})^2 R = (0.986/\sqrt{2})^2 \times 100 = 48.6 \text{ W.}$$

Average power delivered to the resistor can also be calculated by calculating the power delivered by the voltage source minus the average power delivered to the inductor. The first quantity is given by $0.5V_m I_{\text{rms}} \cos \theta$, where θ is the phase angle by which the voltage phasor leads the current phasor. The angle in this case is $+72.34^\circ$. Therefore, average power delivered by the source is $0.5 \times 325 \times 0.986 \times \cos(72.34^\circ) = 48.6 \text{ W}$.

$$\text{The voltage phasor across the inductor} = j314.15 \times 0.986 \angle -162.34^\circ = 309.75 \angle -72.34^\circ \text{ V.}$$

$$\therefore \text{The voltage across inductor} = 309.75 \cos(100\pi t - 72.34^\circ) = 310.34 \sin(100\pi t + 17.66^\circ) \text{ V.}$$

$$\therefore \text{The phase angle between the inductor voltage and the current} = +17.66^\circ - (-72.34^\circ) = +90^\circ$$

This is the expected value since the voltage across an inductor is expected to lead its current under the sinusoidal steady-state. Since cosine of 90° is zero, the average power delivered to the inductor under the sinusoidal steady-state condition is zero. Therefore, the average power delivered to the resistor is the same as the average power delivered by the voltage source and is equal to 48.6 W.

EXAMPLE: 8.6-2

Find the steady-state current and the average power dissipated in the resistor in an R - C series circuit with $R = 100\ \Omega$ and $C = 1\ \mu\text{F}$ when driven by a switched sinusoidal source $v_s(t) = 325 \sin 100\pi t\ \text{V}$.

SOLUTION

The impedance of a capacitor at an angular frequency of $\omega\ \text{rad/s}$ is a purely reactive one of the form $0 + jX_C$ and is equal to $1/j\omega C$. Its reactance value X_C is $-1/\omega C$. The time-domain circuit and the phasor equivalent circuit are shown in Fig. 8.6-3.

This is a single mesh circuit and may be solved by using its mesh equation. However, we do not need even the mesh equation. It is a series connection of two impedances across a source voltage. The equivalent impedance is $R + 1/j\omega C = R - j/\omega C$. Hence, the impedance of an RC series circuit has a negative angle.

The current phasor in the circuit is given by the voltage phasor divided by the phasor impedance.

$$\text{Hence, } I_1 = \frac{V_s}{Z} = \frac{V_s}{R + \frac{1}{j\omega C}} = \frac{j\omega C V_s}{1 + j\omega RC} = \frac{V_s \omega C}{\sqrt{1 + (\omega RC)^2}} \angle(90^\circ - \tan^{-1} \omega RC)$$

Thus, a Series RC Circuit adds a positive phase angle to the voltage phasor in transforming it into a current phasor. The current in an RC circuit leads the voltage waveform by $90^\circ - \tan^{-1}(\omega RC)$.

We calculate the impedance in the present instance by substituting the relevant numbers.

$$\begin{aligned} Z &= 100 - j318.3 = 333.65 \angle -72.54^\circ\ \Omega. \\ \therefore I_1 &= 325 \angle -90^\circ \div 333.65 \angle -72.54^\circ = 0.975 \angle -17.46^\circ \\ \therefore i_1(t) &= 0.975 \cos(100\pi t - 17.46^\circ) = 0.975 \cos(100\pi t - 90^\circ + 72.54^\circ) \\ &= 0.975 \sin(100\pi t + 72.54^\circ)\ \text{A}. \end{aligned}$$

The circuit current *leads* the applied voltage by 72.54° .

$$\begin{aligned} \text{Average power delivered to resistor} \\ &= (I_{1\text{rms}})^2 R = (0.975/\sqrt{2})^2 \times 100 = 47.53\ \text{W}. \end{aligned}$$

Average power delivered to the resistor can also be determined by calculating the power delivered by the voltage source minus the average power delivered to the capacitor. The first quantity is given by $0.5 V_m I_m \cos \theta$, where θ is the phase angle by which the voltage phasor *leads* the current phasor. The angle in this case is 72.54° . Therefore, average power delivered by the source is $0.5 \times 325 \times 0.975 \times \cos(-72.54^\circ) = 47.53\ \text{W}$.

The voltage phasor across the capacitor $= -j318.3 \times 0.975 \angle -17.46^\circ = 310.34 \angle -107.46^\circ$.

$\therefore$ Voltage across capacitor $= 310.34 \cos(100\pi t - 107.46^\circ) = 310.34 \sin(100\pi t - 17.46^\circ)\ \text{V}$.

$\therefore$ The phase angle between the capacitor voltage and the current through it $= -17.46^\circ - 72.54^\circ = -90^\circ$.

This is the expected value since the voltage across a capacitor is expected to *lag* behind its current under the sinusoidal steady-state. Since cosine of -90° is zero, the average power delivered to the capacitor under the sinusoidal steady-state condition is zero. Therefore, the average power delivered to the resistor is the same as the average power delivered by the voltage source and is equal to $47.53\ \text{W}$.

Figure 8.6-4 shows the applied voltage waveform *lagging* behind the circuit current in (a) and the capacitor voltage *lagging* behind the capacitor by 90° in (b). Of course, a physical system can only delay the response with respect to input. Hence, the phase lead that the current in a capacitive circuit exhibits under the sinusoidal steady-state condition should not be understood as a time-advance. The apparent time-advance comes up only after the response undergoes a delay during the transient period.

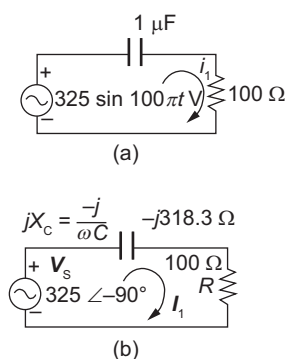


Fig. 8.6-3 (a) The RC Circuit in Time-Domain and (b) Its Phasor Equivalent Circuit in Example 8.6-2

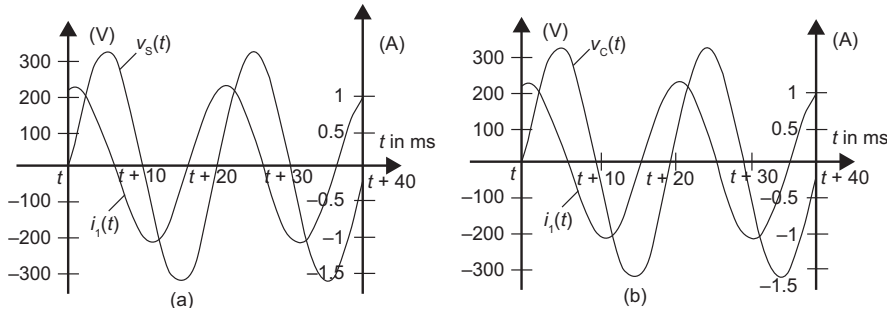


Fig. 8.6-4 (a) Applied Voltage and Current and (b) Capacitor Voltage and Current in Example 8.6-2

EXAMPLE: 8.6-3

Find the (i) source current (ii) source power (iii) output voltage and (iv) power delivered to $25\ \Omega$ resistor in the circuit in Fig. 8.6-5.

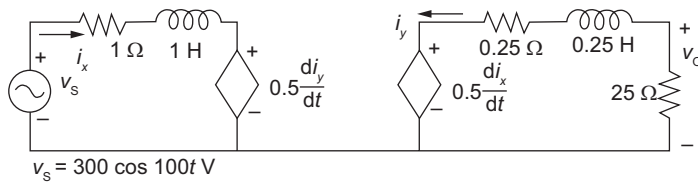


Fig. 8.6-5 Circuit for Example 8.6-3

Solution

The source voltage phasor is $300\angle 0^\circ\text{ V}$.

The dependent sources involve the first derivative of controlling currents. However, differentiation in time-domain is to be replaced by multiplication by $j\omega$ in the phasor equivalent circuit. The value of ω in this example is 100 rad/s . The phasor equivalent circuit of the circuit is given in Fig. 8.6-6.

The mesh equations are written as follows:

$$-300\angle 0^\circ + I_1 + j100I_1 + j50(-I_2) = 0$$

$$-j50I_1 + 0.25I_2 + j25I_2 + 25I_2 = 0$$

Recasting these equations in matrix form,

$$\begin{bmatrix} 1+j100 & -j50 \\ -j50 & 25.25+j25 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 300\angle 0^\circ \\ 0 \end{bmatrix}. \text{ Solving for } I_1 \text{ and } I_2, \text{ we get,}$$

$$I_1 = \frac{300\angle 0^\circ \times (25.25 + j25)}{(1+j100)(25.25+j25) - (j50)(j50)} = \frac{300\angle 0^\circ \times (25.25 + j25)}{(25.25 + j2550)}$$

$$= \frac{300\angle 0^\circ \times 35.53\angle 44.71^\circ}{2550.13\angle 89.43^\circ} = 4.18\angle -44.72^\circ\text{ A}$$

$$I_2 = \frac{300\angle 0^\circ \times j50}{2550.13\angle 89.43^\circ} = 5.88\angle 0.57^\circ\text{ A}$$

$$\therefore V_o = 25 \times 5.88\angle 0.57^\circ = 147\angle 0.57^\circ\text{ V.}$$

$$\therefore i_1(t) = 4.18 \cos(100t - 44.72^\circ)\text{ A.}$$

$$i_2(t) = 5.88 \cos(100t + 0.57^\circ)\text{ A.}$$

$$v_o(t) = 147 \cos(100t + 0.57^\circ)\text{ V.}$$

$$\text{Source power} = 0.5 \times 300 \times 4.18 \times \cos(0 - (-44.72^\circ)) = 445.5\text{ W.}$$

$$\text{Average power delivered to } 25\ \Omega = 0.5 \times 5.88 \times 5.88 \times 25 = 432.2\text{ W.}$$

Comments on Example: 8.6-3

The circuit in Fig. 8.6-6 is an equivalent circuit for a 2:1 two-winding transformer using dependent sources to model the mutual coupling between the windings.

Note that the output voltage is almost equal to half of the input voltage and almost in phase with it. The presence of the $1\ \Omega$ and $0.25\ \Omega$ resistors makes the output amplitude slightly less than 150 V and the phase of the output slightly different from zero. These resistors model the inevitably present winding resistances.

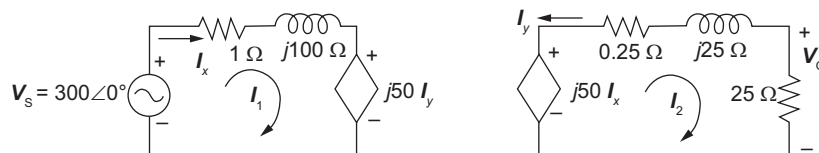


Fig. 8.6-6 The Phasor equivalent Circuit for the Circuit in Fig. 8.6-5

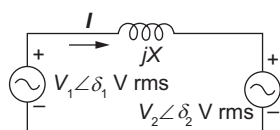


Fig. 8.6-7 A Synchronous Link

EXAMPLE: 8.6-4

A pair of AC voltage sources with the same frequency connected through an inductance is called a *synchronous link* in Electrical Power System Engineering. The sources are generating stations and the inductance is that of the high voltage transmission line that links up the stations. When two DC sources are linked together by means of a resistance, the higher voltage source sends power to the lower voltage source, but when two AC sources at the same frequency are linked together, it is not the magnitude of voltage that decides the magnitude and direction of power flow.

Show that in the synchronous link in Fig. 8.6-7, the *leading* voltage source sends average power to the *lagging* voltage source and that for a small phase difference between the two sources, the power exchanged is proportional to the phase difference in radians.

SOLUTION

There is no specific frequency given, however, the reactance value of the inductor is directly specified as X and the word *synchronous* implies that both sources are at the same frequency. The source voltages are specified as rms values and this is a common practice in Electrical Power System Engineering. Electronics and Communication Engineers prefer to specify the amplitude rather than the rms value. In this book, if rms value is specified, it will be explicitly mentioned after the unit of the quantity.

The current phasor I from the first source to the second source is determined below:

$$\begin{aligned}
 I &= \frac{V_1 \angle \delta_1 - V_2 \angle \delta_2}{jX} \\
 &= \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2) + j(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{jX} \\
 &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} - j \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \\
 &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} \angle 0^\circ - \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \angle 90^\circ \\
 \therefore i(t) &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} \cos \omega t - \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \cos(\omega t + 90^\circ) \\
 &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} \cos \omega t + \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \sin \omega t
 \end{aligned}$$

The source voltage time-function for the first source is $v_1(t) = \sqrt{2}V_1 \cos(\omega t + \delta_1)$

We find the average power delivered by the first source by taking the two terms in the current, one by one. The first component is a $\cos \omega t$ component and the phase angle by which the voltage leads this component is δ_1 . Therefore, average power delivered through this current, P_1 , is

$$\begin{aligned}
 P_1 &= 0.5 \times 2 \frac{V_1^2 \sin \delta_1 \cos \delta_1 - V_1 V_2 \sin \delta_2 \cos \delta_1}{X} \\
 &= \frac{V_1^2 \sin \delta_1 \cos \delta_1 - V_1 V_2 \sin \delta_2 \cos \delta_1}{X}
 \end{aligned}$$

The second current component is a $\sin \omega t$ i.e., a $\cos(\omega t - 90^\circ)$ component. The phase angle by which the voltage leads this component is $90^\circ + \delta_1$. Therefore, average power delivered through this current, P_2 , is

$$P_2 = 0.5 \times 2 \frac{V_1^2 \cos \delta_1 \cos(90^\circ + \delta_1) - V_1 V_2 \cos \delta_2 \cos(90^\circ + \delta_1)}{X}$$

$$= \frac{-V_1^2 \cos \delta_1 \sin \delta_1 - V_1 V_2 \cos \delta_2 \sin \delta_1}{X}$$

Adding P_1 and P_2 to get the total average power delivered by the first source,

$$P = \frac{V_1^2 \sin \delta_1 \cos \delta_1 - V_1 V_2 \sin \delta_2 \cos \delta_1}{X} + \frac{-V_1^2 \cos \delta_1 \sin \delta_1 + V_1 V_2 \cos \delta_2 \sin \delta_1}{X}$$

$$\therefore P = \frac{V_1 V_2 [\sin \delta_1 \cos \delta_2 - \cos \delta_1 \sin \delta_2]}{X} = \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2)$$

$$P \approx \frac{V_1 V_2}{X} (\delta_1 - \delta_2) \text{ if } (\delta_1 - \delta_2) \text{ in radians } \ll 1$$

Therefore, in the synchronous link shown in Fig. 8.6-7, average power flows from leading voltage source to lagging voltage source quite independent of the voltage magnitude relationship between them. Moreover, for a small phase difference between the two sources, the power flow is proportional to the phase difference in radians.

Active power flow equation for a synchronous link.

EXAMPLE: 8.6-5

The source current in the circuit in Fig. 8.6-8 is $i_s(t) = I_m \cos \omega t$ A. Find ω and k such that the current i_y is in phase with $i_s(t)$ and has the same amplitude as that of $i_s(t)$.

SOLUTION

The phasor equivalent circuit is shown in Fig. 8.6-9.

The dependent source current value is $= \frac{kR_a I_s}{R_a + R_b}$. This source is transformed into a dependent voltage source of value $\frac{kR_a R_c I_s}{R_a + R_b} = \alpha I_s$ in series with R_c . Source transformation theorem is applicable under the sinusoidal steady-state condition. Three meshes and

Comment on Example: 8.6-5

The circuit in Fig. 8.6-8 is the small-signal equivalent circuit used to analyse the conditions for sinusoidal oscillation to take place in a Transistor Phase Shift Oscillator circuit that is widely used to generate low power sinusoidal signals up to about 20 kHz frequency.

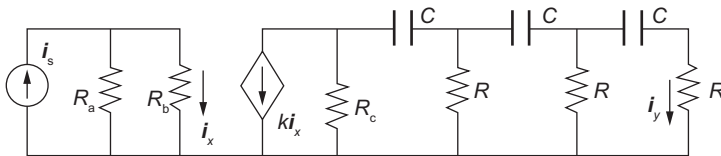


Fig. 8.6-8 Circuit for Example 8.6-5

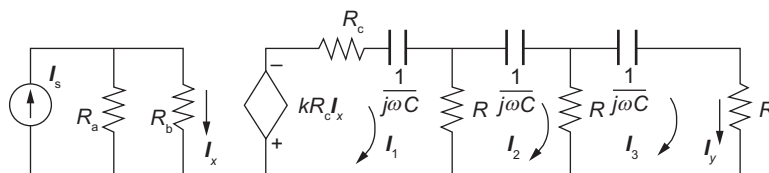


Fig. 8.6-9 The Phasor Equivalent Circuit for Circuit in Fig. 8.6-8

the mesh current phasors are as shown in Fig. 8.6-9. The mesh impedance matrix can be written by inspection. Hence, the mesh equations in matrix form will be as below:

$$\begin{bmatrix} R_c + R + \frac{1}{j\omega C} & -R & 0 \\ -R & 2R + \frac{1}{j\omega C} & -R \\ 0 & -R & 2R + \frac{1}{j\omega C} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -\alpha I_s \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the mesh impedance matrix after some simplification is

$$\Delta_z = \left[R^3 + 3R_c R^2 - \frac{R_c + 5R}{(\omega C)^2} \right] - j \left[\frac{R(6R + 4R_c)}{(\omega C)} - \frac{1}{(\omega C)^3} \right]$$

We need to solve for I_3 .

$$I_3 = \frac{-R^2 \alpha I_s}{\Delta_z}$$

We want the angle of I_3 to be the same as that of I_s . This is possible only if Δ_z is a negative real number because the numerator is a real negative number. Therefore, the imaginary part of Δ_z should go to zero at the ω we are looking for.

$$\text{Therefore, } \left[\frac{R(6R + 4R_c)}{(\omega C)} - \frac{1}{(\omega C)^3} \right] = 0$$

$$\Rightarrow \omega = \frac{1}{RC} \frac{1}{\sqrt{6 + 4\beta}}, \text{ where } \beta = \frac{R_c}{R}$$

We also want the magnitude of the two currents to be the same. Therefore,

$$\left[R^3 + 3R_c R^2 - \frac{R_c + 5R}{(\omega C)^2} \right] = -R^2 \alpha$$

$$\text{Substituting } \omega = \frac{1}{RC} \frac{1}{\sqrt{6 + 4\beta}}, \text{ where } \beta = \frac{R_c}{R} \text{ and } \alpha = \frac{kR_a R_c}{R_a + R_b} \text{ and simplifying, we get}$$

the required value of k as,

$$k = \frac{(R_a + R_b)}{R_b} \left[4 \frac{R_c}{R} + 23 + 29 \frac{R}{R_c} \right].$$

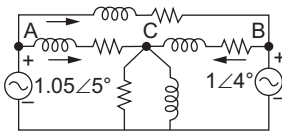


Fig. 8.6-10 Phasor Equivalent Circuit of a Simple Power System for Example 8.6-6

EXAMPLE: 8.6-6

The circuit in Fig. 8.6-10 is the phasor equivalent circuit of a small power system running at 50 Hz containing two generating stations and one load sub-station. The transmission lines connecting the generating stations to the load station and between them are modelled by a series R - L impedance.

A modern power system will have voltage values ranging from 230 V rms at the customer level to 100s of kVs at the transmission level. Similarly, the currents at various locations may have a spread of 1:1000 or even more. Thus, the range of numbers involved in a power system analysis problem is numerically inconvenient in the solution process. Power System Engineers have solved this problem by evolving a special kind of *normalisation* scheme for the quantities in the phasor equivalent circuit of a power system. This scheme is called *per unit* representation. We do not intend to go into the details of the *per unit* system. However, we note the fact that all nominal voltage values become values close to unity in this scheme. The values marked in the circuit in Fig. 8.6-10 is as per this scheme, but for our objective here, we may choose to view them as actual rms values themselves.

All the three lines have $0.02 + j0.1 \Omega$ impedance at 50 Hz. The load connected at C is 2Ω in parallel with $j2 \Omega$ inductive reactance.

Solve the given circuit for (i) all line phasor currents (ii) phasor voltage at load station C (iii) power delivered by sources at A and B and (iv) power delivered to the load at node C.

SOLUTION

This is a nodal analysis problem. The fact that two nodes are constrained by a voltage source does not prevent us from writing node equations at those nodes. We simply assume that the currents injected into node-A and node-B by their respective voltage sources are I_A and I_B and make use of these in nodal equations.

The admittance of the R-L impedance of lines is $1/(0.02 + j0.1) = 9.81\angle-78.7^\circ = 1.92 - j9.62$ S. The admittance of the parallel R-L connection at node-C is $0.5 - j0.5$ S. We write the node equations by inspection.

$$\begin{bmatrix} 3.84 - j19.24 & -1.92 + j9.62 & -1.92 + j9.62 \\ -1.92 + j9.62 & 3.84 - j19.24 & -1.92 + j9.62 \\ -1.92 + j9.62 & -1.92 + j9.62 & 4.34 - j19.74 \end{bmatrix} \begin{bmatrix} 1.05\angle5^\circ \\ 1\angle4^\circ \\ V_C \end{bmatrix} = \begin{bmatrix} I_A \\ I_B \\ 0 \end{bmatrix}$$

We have used the known voltage phasors at node-A and node-B. There is no source connected at node-C, and hence, there is no current injection at that node from the reference node. V_C can be obtained by using the third equation,

$$(-1.92 + j9.62)(1.05\angle5^\circ) + (-1.92 + j9.62)(1\angle4^\circ) + (4.34 - j19.74)V_C = 0.$$

Solving for V_C , we get,

$$V_C = 0.969 + j0.0574 = 0.971\angle3.39^\circ \text{ V rms}$$

Next, we find the currents injected by the voltage sources by making use of the value of V_C in the first two equations from the matrix nodal equation.

$$\begin{aligned} I_A &= (3.84 - j19.24)(1.05\angle5^\circ) + (-1.92 + j9.62)(1\angle4^\circ) + (-1.92 + j9.62)(0.971\angle3.39^\circ) \\ I_B &= (-1.92 + j9.62)(1.05\angle5^\circ) + (3.84 - j19.24)(1\angle4^\circ) + (-1.92 + j9.62)(0.971\angle3.39^\circ) \end{aligned}$$

Therefore,

$$I_A = 0.5037 - j0.1580 = 0.528\angle-17.42^\circ \text{ A rms}$$

$$I_B = 0.0095 - j0.2978 = 0.298\angle-88.2^\circ \text{ A rms}$$

$V_A = 1.05\angle5^\circ$ and $I_A = 0.528\angle-17.42^\circ$. Therefore, the phase angle by which the voltage leads the current is $5^\circ - (-17.42^\circ) = 22.42^\circ$. The average power delivered by the source at node-A = $1.05 \times 0.528 \times \cos 22.42^\circ = 0.513$ W.

$V_B = 1\angle4^\circ$ and $I_B = 0.298\angle-88.2^\circ$. Therefore, the phase angle by which the voltage leads the current is $4^\circ - (-88.2^\circ) = 92.2^\circ$. The average power delivered by the source at node-B = $1 \times 0.298 \times \cos 92.2^\circ = -0.0114$ W. We used $P = V_{\text{rms}} I_{\text{rms}} \cos \theta$ for these calculations, where θ is the phase angle by which the voltage phasor leads the current phasor.

Power delivered to the load at node-C = Power delivered to $2\ \Omega$ resistor + Average power delivered to $2\ \Omega$ reactance = $0.971 \times 0.971/2 + 0 = 0.471$ W. We used the expression $P = V_{\text{rms}}^2/R$ for this calculation.

The line currents are found by dividing the phasor difference between the voltage phasors at the two ends of the line by the line impedance.

$$\text{Line AC - from A to C} = [1.05\angle5^\circ - 0.971\angle3.39^\circ] \div (0.02 + j0.1) = 0.339 - j0.204 \text{ A rms}$$

$$\text{Line BC - from B to C} = [1\angle4^\circ - 0.971\angle3.39^\circ] \div (0.02 + j0.1) = 0.174 - j0.251 \text{ A rms}$$

$$\text{Line AB - from A to B} = [1.05\angle5^\circ - 1\angle4^\circ] \div (0.02 + j0.1) = 0.165 + j0.047 \text{ A rms}$$

Verification

The line currents from A to C and from A to B must add up to I_A . The line current from B to C minus current from A to B must be I_B . These are verified within numerical rounding errors.

The reader is encouraged to verify the power flow principle in synchronous links brought out in Example 8.6-4 with the results obtained in this example.

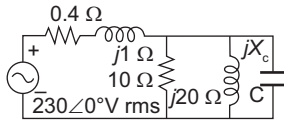


Fig. 8.6-11 Phasor Equivalent Circuit for Example 8.6-7

Capacitive Compensation

Connecting a capacitor across an inductive load results in (i) better load voltage, (ii) lower source current, (iii) lower phase difference between the source voltage and the source current and (iv) lower losses in feeder line. Thus, there is an overall improvement in system performance when the capacitor is connected across the inductive load.

We had shown in Example 7.4-5, in Section 7.4 of Chapter 7, that when a sinusoidal voltage source is delivering power to a load, a given amount of source power is transferred to the load with minimum loss and maximum efficiency when the load draws current from the source at zero phase difference at the source terminal. An inductive load draws a lagging current.

Connecting a capacitor across such a load makes the total current less lagging and closer to zero phase condition.

This method of improving the phase between the source voltage and the source current is called capacitive compensation of an inductive load.

EXAMPLE: 8.6-7

The circuit in Fig. 8.6-11 shows a 50 Hz, 230 V rms AC voltage source delivering power to a $10\ \Omega // j20\ \Omega$ inductive load through a feeder line of series impedance $0.4 + j1\ \Omega$. (i) Find the current phasor delivered by the source, load voltage phasor, average power delivered to the load and the efficiency of power delivery if the capacitor C is disconnected. (ii) Repeat part (i) with $X_c = -20\ \Omega$.

SOLUTION

(i) We use voltage division principle to determine the load voltage first. The parallel combination of $10\ \Omega$ and $j20\ \Omega$ is in series with $0.4 + j1\ \Omega$. Therefore, the voltage across the load, V_L is obtained as

$$\begin{aligned} V_L &= \frac{10 // j20}{0.4 + j1 + 10 // j20} \times 230\angle 0^\circ = \frac{j200}{-16 + j218} \times 230\angle 0^\circ \\ &= 0.915\angle -4.2^\circ \times 230\angle 0^\circ \\ &= 210.2\angle -4.2^\circ \text{ V rms} \end{aligned}$$

Now, the current delivered by the source is obtained as

$$I = \frac{230\angle 0^\circ - 210.2\angle -4.2^\circ}{0.4 + j1} = 20.2 - j12.0 = 23.5\angle -30.8^\circ \text{ A rms}$$

Power delivered to the load can be found in two ways. The first method is to apply $V_{\text{rms}} I_{\text{rms}} \cos\theta$ to the load voltage and the load current. The load voltage is $210.2\angle -4.2^\circ$ and the load current is $23.5\angle -30.8^\circ$. Therefore, $\theta = -4.2^\circ - (-30.8^\circ) = 26.6^\circ$ and $\cos\theta = 0.8942$. Therefore, average power delivered to load = $210.2 \times 23.5 \times 0.8942 = 4418 \text{ W}$.

The second method to find the load power is to find the power delivered to the $10\ \Omega$ resistor by applying $P = V_{\text{rms}}^2 / R$. The average power delivered to an inductance is zero. Therefore, the load power is the same as the average power in resistor. Therefore, the load power is $210.2^2 / 10 = 4418 \text{ W}$.

The average power delivered by the source is calculated as $230 \times 23.5 \times \cos 30.8^\circ = 4643 \text{ W}$.

Therefore, efficiency of power delivery = $100 \times 4418 / 4643 = 95.15\%$.

(ii) With X_c at $-20\ \Omega$, the reactance of an inductor and a capacitor in parallel cancel each other as can be seen from $j20 // -j20 = \frac{-400}{j20 - j20} = \text{open-circuit}$.

Therefore, the load circuit is effectively only a resistor of $10\ \Omega$.

$$\therefore V_L = \frac{10}{0.4 + j1 + 10} \times 230\angle 0^\circ = 220.14\angle -5.5^\circ \text{ V rms.}$$

Now, the current delivered by the source is obtained as

$$I = \frac{230\angle 0^\circ - 220.14\angle -5.5^\circ}{0.4 + j1} = 21.9 - j2.1 = 22.0\angle -5.5^\circ \text{ A rms.}$$

The average power delivered to load = $220.14^2 / 10 = 4846 \text{ W}$.

The average power delivered by source = $230 \times 22 \times \cos(5.5^\circ) = 5037 \text{ W}$.

Efficiency of power delivery = $100 \times 4846 / 5037 = 96.21\%$.

EXAMPLE: 8.6-8

The circuit shown in Fig. 8.6-12 is the small-signal equivalent circuit of a transistor amplifier for analysis of operation at high frequency. Find the gain, i.e., ratio of output voltage phasor to input voltage phasor at 1 MHz.

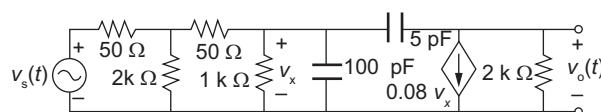


Fig. 8.6-12 Equivalent Circuit of a Transistor Amplifier at High Frequency

SOLUTION

As a first step, we convert the circuit to the left of the 100 pF capacitor to current source in parallel with a resistor by applying Norton's Theorem in time-domain itself. The resulting Norton's equivalent and the complete circuit with the Norton's equivalent in place are shown in Fig. 8.6-13.

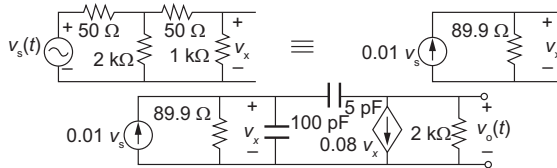


Fig. 8.6-13 Norton's Equivalent of Input Side of the Circuit in Fig. 8.6-12

Let $v_s(t) = V_m \cos \omega t$ V with $\omega = 2\pi \times 10^6$ rad/s. The admittance of 5 pF and 100 pF capacitors are calculated as $j3.1416 \times 10^{-5} \text{ S}$ and $j6.283 \times 10^{-4} \text{ S}$, respectively, at 1 MHz. The conductance of 89.9 Ω and 2 k Ω are 0.0111 S and 0.0005 S, respectively. The phasor equivalent circuit using these values is shown in Fig. 8.6-14. Two nodes and the corresponding voltage phasors and the ground node are also marked.

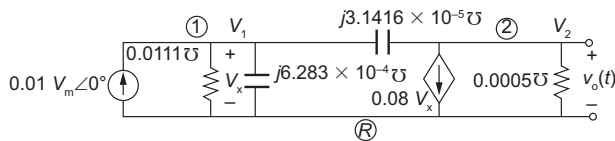


Fig. 8.6-14 The Phasor Equivalent Circuit at 1 MHz for the Circuit in Fig. 8.6-12

The node equation at the first node is

$$(0.0111 + j6.6 \times 10^{-4})\mathbf{V}_1 - j3.1416 \times 10^{-5}\mathbf{V}_2 = 0.01V_m \angle 0^\circ \quad (8.6-1)$$

We note that the controlling voltage phasor $\mathbf{V}_x$ is same as the node voltage phasor $\mathbf{V}_1$. Therefore, the node equation at the second node is

$$(0.08 - j3.1416 \times 10^{-5})\mathbf{V}_1 + (0.0005 + j3.1416 \times 10^{-5})\mathbf{V}_2 = 0 \quad (8.6-2)$$

We need to solve for only $\mathbf{V}_2$. Substituting for $\mathbf{V}_1$ in terms of $\mathbf{V}_2$ as determined from Eqn. 8.6-2 in Eqn. 8.6-1 yields the solution for $\mathbf{V}_2$.

$$\left[-(0.0111 + j6.6 \times 10^{-4}) \frac{(0.0005 + j3.1416 \times 10^{-5})}{(0.08 - j3.1416 \times 10^{-5})} - j3.1416 \times 10^{-5} \right] \mathbf{V}_2 = 0.01V_m \angle 0^\circ$$

Solving for $\mathbf{V}_2$, we get, $\mathbf{V}_2 = (-108.5 + j62.7)V_m$.

Therefore, gain at 1 MHz = $(-108.5 + j62.7) = -(108.5 - j62.7) = -125.3 \angle -30^\circ$. We express the gain in this way since this is an inverting amplifier and the overall negative sign accounts for that. Thus, the output sine wave of the amplifier lags by 30° at 1 MHz compared to the output when the input is at low frequencies of a few 100 Hz.

8.7 CIRCUIT THEOREMS IN SINUSOIDAL STEADY-STATE ANALYSIS

Sinusoidal steady-state, as we have defined, is a concept relevant only to a linear dynamic circuit. A non-linear circuit excited by a sinusoidal waveform may reach a steady-state; but that steady-state will not be a sinusoidal steady-state since the response of a non-linear circuit to a sinusoidal waveform may contain a distorted and non-sinusoidal content. Phasors and phasor equivalent circuits are applicable only if all response variables are sinusoidal waveforms with a frequency equal to that of the input, i.e., only if the dynamic circuit is linear.

A circuit has a phasor equivalent circuit only if it is a linear time-invariant circuit.

All circuit theorems we developed in Chapter 5, except the Maximum Power Transfer Theorem, are applicable to phasor equivalent circuits used to solve for sinusoidal steady-state variables in a linear dynamic circuit excited by sinusoidal sources with a common frequency. Thus, we may use Source Transformation Theorem, Superposition Theorem, Compensation Theorem, Thevenin's Theorem, Norton's Theorem and Reciprocity Theorem in phasor equivalent circuits. The Maximum Power Transfer Theorem needs some modification.

8.7.1 Maximum Power Transfer Theorem for Sinusoidal Steady-State Condition

Figure 8.7-1 shows a linear dynamic circuit containing one or more independent sinusoidal sources with a common frequency delivering power to a load circuit *which is also under the sinusoidal steady-state*. It is assumed that the constraints required for applying Thevenin's theorem are satisfied by the entire circuit – the circuits in Fig. 8.7-1(a) and (b) have a unique solution and there is no interaction between the delivery circuit and the load circuit other than through the common terminals. Then, we can replace the power delivery circuit by its Thevenin's equivalent comprising an open-circuit voltage in series with the Thevenin's equivalent impedance $R_s + jX_s$. The load voltage and the current phasors are identified.

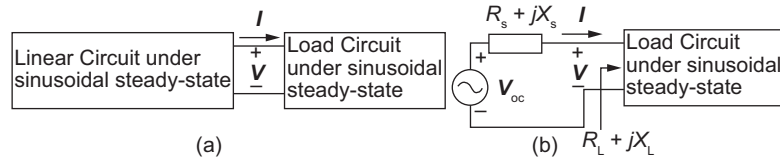


Fig. 8.7-1 (a) The Power Transfer Context (b) The Power Delivery Circuit Replaced by its Thevenin's Equivalent

We can assume, with no loss of generality, that the angle of V_{oc} phasor is zero, i.e., $V_{oc} = V_{ocm} \angle 0^\circ$. Let $I = I_m \angle \phi$ and $Z = Z \angle \theta = R_s + jX_s$. Therefore, $R_s = Z \cos \theta$ and $X_s = Z \sin \theta$. The load voltage phasor $V = V_{oc} - ZI = V_{ocm} - ZI_m \angle (\phi + \theta)$

Then, we may write the following time-domain expressions under the sinusoidal steady-state condition.

$$\begin{aligned} v_{oc}(t) &= V_{ocm} \cos \omega t \\ i(t) &= I_m \cos(\omega t + \phi) \\ v(t) &= V_{ocm} \cos \omega t - ZI_m \cos(\omega t + \phi + \theta) \end{aligned}$$

The average power delivered to the load circuit is the cycle average of $v(t) \times i(t)$.

$$v(t)i(t) = V_{ocm} I_m \cos \omega t \cos(\omega t + \phi) - ZI_m^2 \cos(\omega t + \phi) \cos(\omega t + \phi + \theta)$$

Hence the average power delivered to the load,

$$P_L = 0.5V_{ocm} I_m \cos \phi - 0.5ZI_m^2 \cos \theta.$$

We want to maximise this, but P_L is a function of I_m and ϕ , since both will be influenced by the load circuit. Therefore, we set the partial derivatives of P_L with respect to I_m and ϕ to zero.

$$\begin{aligned} \frac{\partial P_L}{\partial I_m} &= 0 \Rightarrow 0.5V_{ocm} \cos \phi - ZI_m \cos \theta = 0 \\ \frac{\partial P_L}{\partial \phi} &= 0 \Rightarrow 0.5V_{ocm} \sin \phi = 0 \end{aligned}$$

The second equation leads to $\phi = 0$ and with this value of ϕ , the first equation leads to

$$I_m = \frac{V_{ocm}}{2Z \cos \theta}, \text{ but } Z \cos \theta = R_s. \text{ Therefore, } I_m = V_{ocm} / 2R_s.$$

The condition $\phi = 0$ implies that the circuit has to be resistive for maximum power transfer to take place. Therefore, the effective reactance of the load at its terminals must be $-X_s$ such that it will cancel the Thevenin's reactance of the power delivery circuit and make the entire circuit resistive. Moreover, the condition that $I_m = V_{ocm}/2R_s$ implies that the effective resistance at the load terminals must be R_s such that the entire circuit becomes a resistor of $2R_s$ in series with V_{oc} . We now state the Maximum Power Transfer Theorem for circuits in sinusoidal steady-state.

Maximum average power is transferred to a load circuit from a power delivery circuit under the sinusoidal steady-state when the driving-point impedance, $Z_L = R_L + jX_L$, of the load is the conjugate of Thevenin's impedance, $Z_s = R_s + jX_s$ of the power delivery circuit. Therefore $R_L = R_s$ and $X_L = -X_s$ are the required conditions. The maximum average

power transferred under this condition will be $P_{Lmax} = \frac{(V_{ocm})^2}{4R_s} = \frac{(V_{rms})^2}{2R_s}$ W.

EXAMPLE: 8.7-1

(i) Find the Thevenin's equivalent across **a-b** in the circuit in Fig. 8.7-2. (ii) Find the voltage across a $500\ \Omega$ connected across **a-b**. (iii) Find the value of capacitive reactance to be connected across **a-b** if the magnitude of voltage across the $500\ \Omega$ load is to be raised to 132 kV rms . (iv) If **a-b** gets shorted, what is the current that flows through the short?

SOLUTION

(i) We use the superposition principle to obtain the V_{oc} phasor.

$$V_{oc} = 127\angle 6^\circ \times \frac{1.5 + j9}{3.5 + j21} + 127\angle 5^\circ \times \frac{2 + j12}{3.5 + j21} = 127\angle 5.43^\circ \text{ kV rms}$$

$$Z_{th} = (2 + j12) \parallel (1.5 + j9) = \frac{(2 + j12)(1.5 + j9)}{3.5 + j21} = 0.857 + j5.143\ \Omega.$$

The Thevenin's equivalent circuit for sinusoidal steady-state is shown in the circuit in Fig. 8.7-3(a).

(ii) The voltage phasor across the load of $500\ \Omega$

$$= 127\angle 5.43^\circ \times \frac{500}{500.857 + j5.143} = 126.77\angle 4.84^\circ \text{ kV rms}$$

(iii) We first determine a new Thevenin's equivalent of the circuit in Fig. 8.7-3(b) for further load connection across **a-b**. This equivalent circuit will have an open-circuit voltage phasor $= 126.77\angle 4.84^\circ \text{ kV rms}$ and Thevenin's equivalent impedance $= (0.857 + j5.143) \parallel 500 = 0.908 + j5.125\ \Omega$. Let $-jX_C$ be the capacitive reactance connected across the output of this new equivalent circuit now. The magnitude of voltage phasor across the capacitor has to be 132 kV rms .

$$\therefore \left| \frac{-jX_C}{0.908 + j(5.125 - X_C)} \right| = \frac{132}{126.77} = 1.0413$$

$$\text{i.e., } \frac{X_C^2}{0.908^2 + (5.125 - X_C)^2} = 1.0413^2 = 1.0843$$

There are two solutions for X_C : $2.7\ \Omega$ and $129.1\ \Omega$. The higher value is accepted. (Why?).

Therefore, the required reactance is $-129.1\ \Omega$ and the required capacitive reactance is $129.1\ \Omega$. Once we qualify a reactance by using the term *capacitive*, we do not have to include the negative sign.

(iv) The short-circuit current at **a-b** is $= 126.77\angle 4.84^\circ \div (0.908 + j5.125) = 126.77\angle 4.84^\circ \div 5.205\angle -80^\circ = 6.26\angle -75.16^\circ \text{ kA rms} = 24.36\angle -75.16^\circ \text{ kA rms}$.

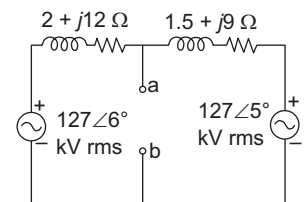


Fig. 8.7-2 Circuit for Example 8.7-1

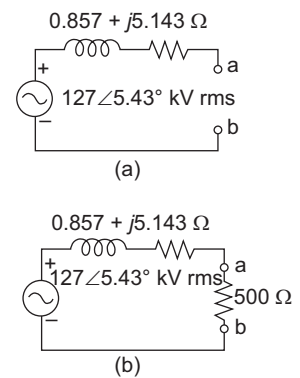


Fig. 8.7-3
(a) Thevenin's Equivalent Circuit for the Circuit in Fig. 8.7-2
(b) With $500\ \Omega$ Load

EXAMPLE: 8.7-2

Two terminals **a** and **b** are identified as output terminals of a linear circuit containing sinusoidal sources at a common frequency. The open-circuit voltage measured across **a–b** is seen to be 100 V rms. The voltage phasor across **a–b** goes down to 63.25 V rms when a $20\ \Omega$ resistor is connected at the output and it goes down to 44.71 V rms when a $10\ \Omega$ resistor is connected at the output. (i) Find the resistive load that will draw maximum average power from this circuit if a reactance can be introduced in series with the resistive load and can be varied to maximise the power. (ii) Find the resistive load that will draw maximum average power for this circuit if no reactance can be added in series with it.

SOLUTION

We have to find the Thevenin's equivalent of the circuit with respect to **a–b** first. Let the Thevenin's equivalent impedance be $R + jX\ \Omega$.

Then, the magnitude of voltage phasor across **a–b** when a resistor R_L is connected across the terminals is $\left| \frac{R_L}{R + R_L + jX} \right| |V_{oc}| = \frac{R_L}{\sqrt{(R + R_L)^2 + X^2}} |V_{oc}|$

$|V_{oc}|$ is given as 100 V rms. Also known are the rms output voltage values when $R_L = 10\ \Omega$ and $R_L = 20\ \Omega$. Substituting the numerical values, we get the following two equations for two unknowns, R and X .

$$\frac{20}{\sqrt{(20 + R)^2 + X^2}} = \frac{63.25}{100} = 0.6325 \text{ and } \frac{10}{\sqrt{(10 + R)^2 + X^2}} = \frac{44.71}{100} = 0.4471$$

Squaring both sides of the equations and carrying out the required algebraic manipulation, we get, $(20 + R)^2 + X^2 = 999.86$ and $(10 + R)^2 + X^2 = 500.25$. Subtracting the second equation from the first yields

$$(20 + R)^2 - (10 + R)^2 = 499.61 \text{ i.e., } (30 + 2R) \times 10 = 499.6 \Rightarrow R = 9.98\ \Omega \approx 10\ \Omega.$$

Using the value of $R = 10\ \Omega$ in $(10 + R)^2 + X^2 = 500.25$, we get $X = 10.01\ \Omega \approx 10\ \Omega$.

Thus, the Thevenin's equivalent of the circuit is $100\angle 0^\circ$ V rms in series with $(10 + j10)\ \Omega$.

(i) If a reactance can be put in series with the load resistance and the value of the reactance and the resistance can be independently adjusted, then, the maximum power transfer will take place under *conjugate impedance matching* condition. Therefore, R_L must be $10\ \Omega$ and X_L must be $-j10\ \Omega$ for maximum average power transfer to the load. The maximum power transferred under this condition will be $= R(V_{oc}/2R)^2 = V_{oc}^2/4R$ W, where V_{oc} is the rms open-circuit voltage. Therefore, the power transferred to $(10 - j10)\ \Omega$ is 250 W.

(ii) The condition for maximum power transfer has to be derived for this case. Let R_L be the load resistance. Then, the current phasor $= \frac{V_{oc}}{(R + R_L) + jX}$ and rms value (i.e., magnitude of the phasor, assuming V_{oc} is the rms open-circuit voltage) of current is $= \frac{V_{oc}}{\sqrt{(R + R_L)^2 + X^2}}$. Power in $R_L = R_L \times I_{rms}^2 = \frac{R_L V_{oc}^2}{(R + R_L)^2 + X^2}$. Value of R_L for maximising this quantity is found by equating its derivative with respect to R_L to zero.

$$\frac{dP}{dR_L} = 0 \Rightarrow (R + R_L)^2 + X^2 - 2R_L(R + R_L) = 0 \Rightarrow R_L = \sqrt{R^2 + X^2}.$$

Thus, the maximum power transfer takes place in a pure resistive load when the load resistance is equal to the magnitude of Thevenin's equivalent impedance of the power delivery circuit. The required load resistance in this example is $\sqrt{10^2 + 10^2} = 14.14\ \Omega$ and the power transferred is $= \frac{100^2}{(10 + 14.14)^2 + 10^2} \times 14.14 = 207\ \text{W}$.

8.8 PHASOR DIAGRAMS

We have understood a *phasor* as a *complex amplitude* of a complex exponential function that varies in time as per $e^{j\omega t}$ until now. We lend a little more colour to *phasors* in this section. We are motivated by uniform circular motion that is a part of school Physics. We take up a

time-domain signal $v_s(t) = V_m \cos \omega t u(t)$ represented by a phasor $V_s = V_m \angle 0^\circ$ and arrive at a *geometric interpretation* for the phasor.

Concept No. 1 – Consider a line of length V_m with an arrow at the end (instead of a stone at the end of a taut string) rotating at a constant angular velocity of ω rad/s in the counter-clockwise direction. Let the coordinates of the arrow-tip be represented as $x(t)$ and $y(t)$ in the horizontal and vertical directions in a right-handed Cartesian coordinate system as shown in Fig. 8.8-1(a).

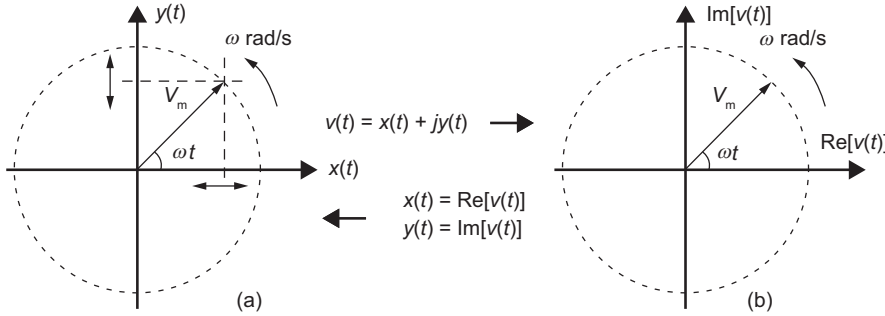


Fig. 8.8-1 (a) A Rotating Line of Length V_m in x - y Coordinate System
(b) A Time-Varying Complex Number in Complex Plane Representing a Complex Signal Constructed using Coordinates of Arrow-Tip in (a)

Assume that the line was collinear with x -axis at $t = 0$ and then started rotating at ω rad/s in the direction shown. Then, the angular position of the line in space is given by ωt rad measured in a counter-clockwise direction from the positive x -axis. The projection of the arrow-tip on the x -axis will then be $V_m \cos \omega t$ and the projection of the arrow-tip on the y -axis will be $V_m \sin \omega t$. Therefore, the signal we started with can be given a *geometric interpretation of horizontal projection of arrow-tip of a line of length V_m rotating in a counter-clockwise direction with a constant angular velocity of ω rad/s, starting from x -axis position, at $t = 0$.*

Concept No. 2 – Projections on both axes are functions of time. We define a composite function by using these two projection functions. We define a *complex function of time* $v(t) = [x(t) + jy(t)]u(t)$ by treating the horizontal projection as the *real part* of a complex number and the vertical projection as the *imaginary part* of the same complex number. This complex number can be represented as a point in a *complex plane*. As the line progresses in its rotation, the value of complex number, constructed as explained, too will change. Therefore, the point representing this number in the complex plane will also change with time. A complex number can be geometrically represented by a line with one end at the origin and with an arrow at the other end in the complex plane. When the complex number changes with time, the arrow-tip of the line representing the number in the complex plane will trace out a path in that plane. It must be evident in this case that when the rotating line moves in (a), the corresponding path traced out by the complex number in the complex plane in (b) will also be a circle of radius V_m and the arrow-tip will traverse this path with a constant angular velocity of ω rad/s.

Thus, a complex signal $v(t) = V_m (\cos \omega t + j \sin \omega t) u(t) = V_m e^{j\omega t} u(t)$ constructed from coordinates of the arrow-tip of a uniformly rotating line in space will be represented geometrically in the 'complex signal plane' by a directed line of length V_m rotating uniformly, starting from the real axis, in the plane. The values read on the axes of 'complex signal plane' at any instant t are the real and imaginary components of the complex number representing the signal value at that instant.

Concept No. 3 – Let $v_s(t) = V_m \cos(\omega t + \theta) u(t)$. Now, the arrow-tip of the rotating line in Fig. 8.8-1(a) will start at $(V_m \cos \theta, V_m \sin \theta)$ i.e., at an angular position of θ at $t = 0$

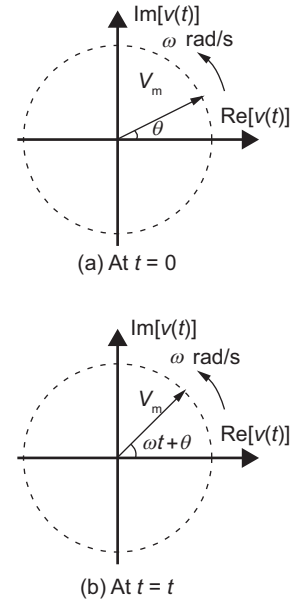


Fig. 8.8-2 Signal Positions for $V_m e^{j(\omega t + \theta)}$
(a) At $t = 0$ (b) At $t = t$

and will start rotating at ω rad/s from there. Therefore, its angular position at $t = t$ will be $(\omega t + \theta)$. The corresponding complex signal $V_m e^{j(\omega t + \theta)} u(t)$ positions in the complex signal plane are marked in Fig. 8.8-2 for $t = 0$ and $t = t$.

Concept No. 4 – Consider two signals $v_s(t) = V_m \cos(\omega t + \theta_v)$ and $i_s(t) = I_m \cos(\omega t + \theta_i)$ with the same angular frequency. Their representations in complex signal plane at $t = 0$ and at $t = t$ are shown in Fig. 8.8-3(a) and (b).

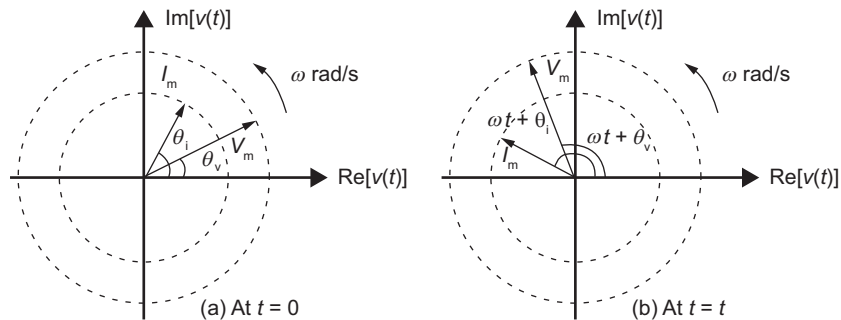


Fig. 8.8-3 Signal Positions of Two Complex Exponential Functions at (a) $t = 0$ (b) $t = t$

A Special Note on Phasors

Phasors are complex amplitudes used to represent sinusoidal quantities under the sinusoidal steady-state condition.

There should be only one value of angular frequency in the circuit. All sinusoidal sources must be at the same frequency. Therefore, a phasor diagram can be drawn only for a circuit that is in sinusoidal steady-state under the influence of one or more sinusoidal sources at the same angular frequency.

If there are different frequency sinusoids present in the same circuit, different phasor diagrams – one each for each frequency – should be drawn. The phasor solution arrived at from different phasor diagrams will have to be transformed into time-domain quantities using relevant angular frequency values before combining the solutions by invoking Superposition Theorem.

The directed lines of different lengths do change their angular positions with time; but they maintain a constant angular difference at all t . This constant angular difference is the value of angular difference they had at $t = 0$.

Therefore, a set of complex exponential signals, all with the same angular frequency but with different initial angular positions, will maintain their relative positions with respect to each other as they rotate in the counter-clockwise direction with a constant angular velocity of ω rad/s. Such a set of signals with the same angular frequency forms a ‘coherent group’ and always stays together with their relative positions unchanging.

Concept No. 5 – Thus, rotation aspect is common to all signals at the same angular frequency. This is a piece of information that we can supply at any time and does not have to be carried always in the diagram. What is of interest is the relative phase angles between members of a *coherent group* of complex exponential signals. Therefore, we may suppress the rotation of lines representing complex exponential signals in the complex signal plane – i.e., we may *freeze* the lines at their position at $t = 0$. This ‘freezing’ the signal lines at their initial position converts complex time-functions into complex numbers – i.e., *constant-valued* signals in *complex signal plane*. These *constant-valued* signals in *complex signal plane* are our *phasors*.

Thus, going from *phasor* to time-function involves ‘unfreezing’ the directed lines in the complex signal plane, allowing them to rotate in a counter-clockwise direction at a constant angular velocity of ω rad/s and extracting the horizontal projection of line end-points, i.e., extracting their real parts.

Concept No. 6 – A diagram depicting a group of coherent (i.e., of same angular frequency) complex exponential signals frozen at their initial position is called a *phasor diagram*. The angles measured in a counter-clockwise direction in a phasor diagram are *lead angles* and the angles measured in a clockwise direction in a phasor diagram are *lag angles*.

A phasor diagram shows the magnitude of phasors as the length of the directed arrows to some scale and the angle of phasors as angles measured from a reference phasor in a counter-clockwise direction. The reference phasor is aligned along the horizontal direction. As only the relative positions of various phasors in a circuit really matter, any one phasor may be taken as the reference phasor and the directions of all other phasors may be marked with respect to this reference phasor, provided the absolute phase of the reference phasor is preserved for later use. That is, a group of directed lines originating from the origin

may be rotated as a whole to a new position without affecting the relative positions among the members of the group.

Concept No. 7 – Multiplying a phasor by j or e^{j90° or $1\angle 90^\circ$ amounts to rotating it by 90° in the counter-clockwise direction in the phasor diagram. This amounts to converting a $\cos\omega t$ into $\cos(\omega t + 90^\circ) = -\sin\omega t$ in time-domain, and hence, is equivalent to introducing a phase lead of 90° . Multiplying a phasor by $-j$ or e^{-j90° or $1\angle -90^\circ$ amounts to rotating it by 90° in the clockwise direction in the phasor diagram. This amounts to converting a $\cos\omega t$ into $\cos(\omega t - 90^\circ) = \sin\omega t$ in time-domain, and hence, is equivalent to introducing a phase lag of 90° .

Concept No. 8 – Phasors in a phasor diagram can be added and subtracted by employing *parallelogram law of addition of complex numbers in a complex plane*. This law is the same as the law of addition of vectors in space coordinates.

EXAMPLE: 8.8-1

Draw the phasor diagram showing all voltage phasors and current phasors for a series RL circuit with $R = 6\ \Omega$, $X = 8\ \Omega$ at 50 Hz and $v_s(t) = 20 \cos 100\pi t$.

Solution

The impedance of the circuit, $\mathbf{Z} = (6 + j8)\ \Omega = 10\angle 53.13^\circ\ \Omega$. Applied voltage phasor $\mathbf{V}_s = 20\angle 0^\circ\ \text{V}$. The circuit current phasor $\mathbf{I} = 20\angle 0^\circ\ \text{V} \div 10\angle 53.13^\circ\ \Omega = 2\angle -53.13^\circ\ \text{A}$.

Voltage phasor across resistor $\mathbf{V}_R = 6\ \Omega \times 2\angle -53.13^\circ\ \text{A} = 12\angle -53.13^\circ\ \text{V}$. Voltage phasor across inductor $\mathbf{V}_L = j8\ \Omega \times 2\angle -53.13^\circ\ \text{A} = 16\angle 36.87^\circ\ \text{V}$.

We choose the applied voltage phasor as the reference phasor and align it along the horizontal direction. Different scaling for voltage phasor magnitudes and current phasor magnitudes will have to be employed when a phasor diagram shows voltage phasors and current phasors together.

The circuit and phasor diagrams are shown in Fig. 8.8-4.

Note that $\mathbf{V}_L$ and $\mathbf{V}_R$ add to form $\mathbf{V}_s$ by parallelogram law of addition. The inductor voltage is seen to lead the circuit current by 90° and the current lags the applied voltage by the impedance angle equal to 53.13° .

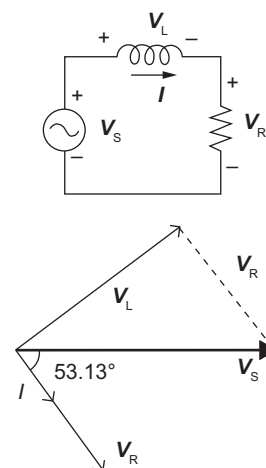


Fig. 8.8-4 An RL Circuit and its Phasor Diagram

EXAMPLE: 8.8-2

Draw the phasor diagram showing all voltage phasors and current phasors for a series RC circuit with $R = 6\ \Omega$, $X = -8\ \Omega$ at 50 Hz and $v_s(t) = 20 \cos 100\pi t$.

Solution

The impedance of the circuit $\mathbf{Z} = (6 - j8)\ \Omega = 10\angle -53.13^\circ\ \Omega$. Applied voltage phasor $\mathbf{V}_s = 20\angle 0^\circ$. The circuit current phasor $\mathbf{I} = 20\angle 0^\circ\ \text{V} \div 10\angle -53.13^\circ\ \Omega = 2\angle 53.13^\circ\ \text{A}$. Voltage phasor across resistor $\mathbf{V}_R = 6\ \Omega \times 2\angle 53.13^\circ\ \text{A} = 12\angle 53.13^\circ\ \text{V}$. Voltage phasor across inductor $\mathbf{V}_L = -j8\ \Omega \times 2\angle 53.13^\circ\ \text{A} = 16\angle -36.87^\circ\ \text{V}$. We choose the applied voltage phasor as the reference phasor and align it along the horizontal direction. The circuit and phasor diagrams are shown in Fig. 8.8-5.

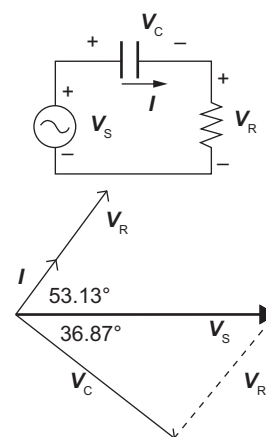


Fig. 8.8-5 The RC Circuit and its Phasor Diagram

EXAMPLE: 8.8-3

A series RLC circuit as shown in Fig. 8.8-6 is excited by a voltage source $v_s(t) = 100 \cos \omega t$ V. The inductive reactance at ω is $10\ \Omega$ and the capacitive reactance at ω is $10\ \Omega$. The resistor has a value of $1\ \Omega$. Draw the phasor diagram of the circuit under the sinusoidal steady-state condition.

SOLUTION

The impedance of the circuit at $\omega\ \text{rad/s}$, $\mathbf{Z} = 10 + j10 - j10 = 10\angle 0^\circ\ \Omega$
Applied voltage phasor, $\mathbf{V}_s = 100\angle 0^\circ\ \text{V}$

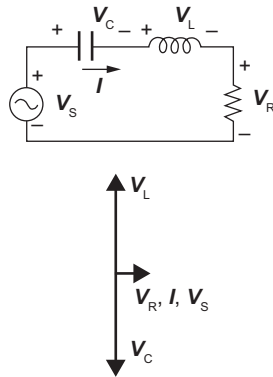


Fig. 8.8-6 An RLC Circuit and its Phasor Diagram

- $\therefore$ Circuit current phasor, $I = 100\angle 0^\circ$ A
- $\therefore$ Voltage phasor across R, $V_R = 100\angle 0^\circ$ V
- $\therefore$ Voltage phasor across L, $V_L = j10\ \Omega \times 100\angle 0^\circ$ A = $1000\angle 90^\circ$ V
- $\therefore$ Voltage phasor across C, $V_C = -j10\ \Omega \times 100\angle 0^\circ$ A = $1000\angle -90^\circ$ V

Note that the voltage across the capacitor and the inductor are in phase opposition. Therefore, they cancel each other completely, thereby, leaving the entire supply voltage to the resistor. Hence, the current under this condition is the maximum current that the circuit can have with given amplitude of applied voltage. This happens because the impedance of the capacitor and the inductor are equal in magnitude and opposite in sign. They cancel out, making the impedance of the circuit a minimum of R at this frequency. This is the *resonance* condition in a series RLC circuit.

EXAMPLE: 8.8-4

A sinusoidal current source $i_s(t) = 5 \cos(100\pi t - 45^\circ)$ A is applied across a parallel combination of an inductor, a resistor and a capacitor. Find the steady-state currents in elements and the voltage across the combination using the phasor diagram. The resistance value is $6\ \Omega$ and the values of the reactance of the inductor and the capacitor at $\omega = 100\pi$ rad/s are $8\ \Omega$ and $-4\ \Omega$, respectively.

SOLUTION

This example calls for the use of a phasor diagram to solve the circuit under the sinusoidal steady-state. Hence, the phasor quantities are unknown when we draw the phasor diagram. In this kind of a situation, the phasor that we choose as the reference phasor has to have the property that all the other phasors in the circuit can be worked out from this phasor employing KCL, KVL and element relationship. Applied voltage or current will not be suitable for this purpose. It has to be one of the response variables. But, if it is one of the response variables, its magnitude will be unknown, and hence, we cannot fix the scale in the phasor diagram. Thus, the phasor diagram is drawn by assigning an arbitrary, but known length, to the phasor that is chosen as the reference phasor. The scale in the diagram will emerge from the known applied voltage or current phasor once the diagram is completed according to KCL, KVL and element relationships.

We choose the current in the resistor as the reference phasor in this example. The circuit and the phasor diagram are shown in Fig. 8.8-7.

We have set I_R in the horizontal direction. This does not mean that $i_R(t)$ will be a $\cos 100\pi t$ wave. Once we solve the phasor diagram completely, the source current phasor will come out with some angle other than its actual angle of -45° . Then, we will rotate the entire phasor diagram such that I_s takes up -45° position in the diagram. The other phasors will then take up suitably shifted positions. The new angular positions can be calculated from the known angular position of I_s and the apparent angular position of I_s in the diagram shown in Fig. 8.8-7.

Let d be the length that we used to represent I_R . Then, the current through the capacitor, $I_C = R I_R / jX_C$, will have a magnitude of $6/4 = 1.5$ times that of I_R , and hence, needs a line of length $1.5d$ in the 90° position in the diagram. The current through inductor, $I_L = R I_R / jX_L$ will have a magnitude of $6/8 = 0.75$ times that of I_R , and hence, needs a line of length $0.75d$ in the -90° position in the diagram. Moving a copy of I_L to the tip of I_C takes us to $(I_C + I_L)$ and placing a copy of I_R at the tip of $(I_C + I_L)$ takes us to the tip of I_s phasor. Thus, we complete the diagram. Now, we either measure the length of I_s phasor or calculate it as $\sqrt{(0.75d)^2 + d^2} = 1.25d$ from the geometry of the figure. But, this must be equal to 5 A since the amplitude of $i_s(t)$ is stated to be 5 A. Therefore, the length of d stands for $5/1.25 = 4$ A. Therefore, the magnitude of I_R is 4 A, I_C is 6 A and I_L is 3 A.

The angle of I_s phasor in the phasor diagram is $\tan^{-1}(0.75) = 36.9^\circ$. But, we know that the actual angle of I_s phasor is -45° . Therefore, an angle of $-(45^\circ + 36.9^\circ) = -81.9^\circ$ will have to be added to all phasors in the diagram.

$\therefore I_s = 5\angle -45^\circ$ A, $I_R = 4\angle -81.9^\circ$ A, $I_C = 6\angle -8.1^\circ$ A and $I_L = 3\angle -171.9^\circ$ A and $V = 24\angle -81.9^\circ$ V is the phasor solution of the circuit.

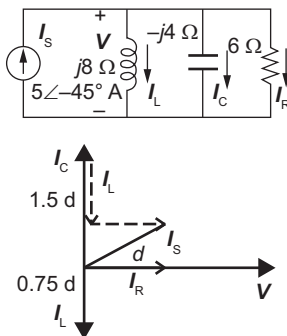


Fig. 8.8-7 Circuit and Phasor Diagram in Example 8.8-4

The corresponding time-domain functions are:

$$i_s(t) = 5 \cos(100\pi t - 45^\circ) \text{ A}, i_R(t) = 4 \cos(100\pi t - 81.9^\circ) \text{ A}, i_C(t) = 6 \cos(100\pi t + 8.1^\circ) \text{ A}, i_L(t) = 3 \cos(100\pi t - 171.9^\circ) \text{ A} \text{ and } v(t) = 24 \cos(100\pi t - 81.9^\circ) \text{ V}.$$

We could have used I_C or I_L or V as the reference phasor and developed the phasor diagram to arrive at the same solution. However, we could not have used I_s as the reference phasor as we would not have been able to proceed any further with that choice.

EXAMPLE: 8.8-5

Two impedances $Z_1 = 6 + j8 \Omega$ and $Z_2 = 8 - j6 \Omega$ are in parallel and the whole combination is in series with a third impedance $Z_3 = 5 + j5 \Omega$. The circuit is driven by a sinusoidal voltage source $v_s(t) = 50 \sin 100\pi t$. Solve the circuit by the phasor diagram method.

Solution

The circuit and the phasor diagrams are shown in Fig. 8.8-8.

We choose the current phasor I as the reference phasor.

The impedance values are converted to polar form as $Z_1 = 10 \angle 53.1^\circ \Omega$, $Z_2 = 10 \angle -36.9^\circ \Omega$ and $Z_3 = 7.07 \angle 45^\circ \Omega$. The parallel combination, $Z_1 // Z_2$ is $7 + j1 \Omega = 7.07 \angle 8.1^\circ$.

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I = (0.5 - j0.5) I = 0.707 \angle -45^\circ I$$

$$I_2 = \frac{Z_1}{Z_1 + Z_2} I = (0.5 + j0.5) I = 0.707 \angle 45^\circ I$$

We use a length d for I . Then, the length to be used for I_1 and I_2 are $0.707d$ and they are oriented at -45° and 45° respectively. Next, we draw the $V = 7.07 \angle 8.1^\circ I$ phasor at 8.1° with respect to the horizontal and use a convenient length d_1 if the length $7.07d$ is not suitable. The V_3 phasor is also of the same length since $V_3 = Z_3 I = 7.07 \angle 45^\circ I$, but it is to be drawn at a 45° position.

The phasors V and V_3 in addition as per parallelogram law should result in V_s . The length of V_s must be $2 \times d_1 \times \cos[(45^\circ - 8.1^\circ)/2] = 1.9 d_1$, but this length must stand for 50 V, and hence, d_1 must stand for 26.3 V. Therefore, magnitudes of V and V_3 are 26.3 V. Since $I = V/Z_3$, the magnitude of I will be 3.72 A. Now, magnitudes of I_1 and I_2 are 0.707 times the magnitude of I . Hence, their magnitude is 2.63 A.

The angle of V_s as per the phasor diagram is $8.1^\circ + (45^\circ - 8.1^\circ)/2 = 26.55^\circ$. Since $v_s(t) = 50 \sin 100\pi t$, the actual phase angle of V_s is -90° with respect to the standard cosine wave. Therefore, an angle of -116.55° has to be added to the angle of all phasors in the phasor diagram shown in Fig. 8.8-8.

Therefore, the sinusoidal steady-state solution of the circuit is obtained as:

$$V_s = 50 \angle -90^\circ, V = 26.3 \angle -108.45^\circ \text{ V}, V_3 = 26.3 \angle -71.55^\circ \text{ V}$$

$$I = 3.72 \angle -116.55^\circ \text{ A}, I_1 = 2.63 \angle -151.55^\circ \text{ A}, I_2 = 2.63 \angle -71.55^\circ \text{ A}$$

The time-domain functions are:

$$v_s(t) = 50 \cos(100\pi t - 90^\circ) = 50 \sin 100\pi t \text{ V}$$

$$v(t) = 26.3 \cos(100\pi t - 108.45^\circ) = 26.3 \sin(100\pi t - 18.45^\circ) \text{ V}$$

$$v_3(t) = 26.3 \cos(100\pi t - 71.55^\circ) = 26.3 \sin(100\pi t + 18.45^\circ) \text{ V}$$

$$i(t) = 3.72 \cos(100\pi t - 116.55^\circ) = 3.72 \sin(100\pi t - 26.55^\circ) \text{ A}$$

$$i_1(t) = 2.63 \cos(100\pi t - 151.55^\circ) = 2.63 \sin(100\pi t - 61.55^\circ) \text{ A}$$

$$i_2(t) = 2.63 \cos(100\pi t - 71.55^\circ) = 2.63 \sin(100\pi t + 18.45^\circ) \text{ A}$$

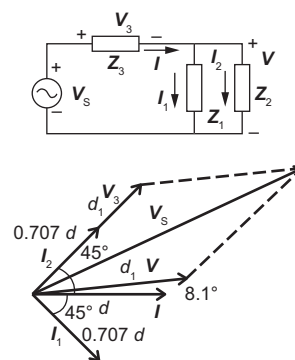


Fig. 8.8-8 Circuit and Phasor Diagram for Example 8.8-5

EXAMPLE: 8.8-6

Three sinusoidal voltage sources $v_1(t)$, $v_2(t)$ and $v_3(t)$ – with an angular frequency of 100π rad/s and amplitudes of 63 V, 52 V and 25 V, respectively, are connected in series along with a 10Ω resistor to form a closed loop. The voltage sources are connected in such a way that they aid each other in the loop. The current in the 10Ω resistor is found to be zero. Find $v_1(t)$, $v_2(t)$ and $v_3(t)$.

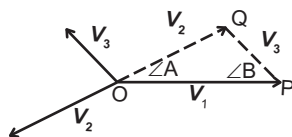


Fig. 8.8-9 Phasor Diagram in Example 8.8-6

Solution

The statement of the problem makes it clear that $v_1(t) + v_2(t) + v_3(t) = 0$. Therefore, the phasor diagram of the three voltage phasors will form a closed triangle. The phasor diagram is shown in Fig. 8.8-9.

The phasor diagram is drawn as follows. Choose a suitable scale and draw the line OP to represent magnitude of V_1 . With O as centre, draw a circle of radius 52 to scale. Draw another circle of radius 25 to scale with P as its centre. Let the two circles intersect at Q. They have to intersect; otherwise, the three voltages would not have added up to zero. Join QO and PQ. Create a copy of QO and move it to form V_2 . Similarly, create a copy of PQ and move it in parallel such that the non-arrow end comes to O to form V_3 .

Now, $\angle A$ and $\angle B$ can be measured from the diagram. Then $V_1 = 63\angle 0^\circ$, $V_2 = 52\angle -(180 - A)^\circ$ and $V_3 = 25\angle (180 - B)^\circ$.

The angles $\angle A$ and $\angle B$ can also be calculated by Law of Cosines.

$$25^2 = 63^2 + 52^2 - 2 \times 63 \times 52 \times \cos A \Rightarrow A = 22.62^\circ$$

$$52^2 = 63^2 + 25^2 - 2 \times 63 \times 25 \times \cos B \Rightarrow B = 53.13^\circ$$

$$\therefore V_1 = 63\angle 0^\circ \text{ V}, V_2 = 52\angle -157.38^\circ \text{ V and } V_3 = 25\angle 126.87^\circ \text{ V.}$$

$$\therefore v_1(t) = 63 \cos(100\pi t) \text{ V}, v_2(t) = 52 \cos(100\pi t - 157.4^\circ) \text{ V and } v_3(t) = 25 \cos(100\pi t + 126.9^\circ) \text{ V.}$$

8.9 APPARENT POWER, ACTIVE POWER, REACTIVE POWER AND POWER FACTOR

Consider a sinusoidal voltage source $v(t) = V_m \cos \omega t$ delivering power to a resistive load R . The current in the resistor is $i(t) = I_m \cos \omega t$, where $I_m = V_m/R$.

The instantaneous power is $p(t) = V_m I_m \cos^2 \omega t = 0.5 V_m I_m + 0.5 V_m I_m \cos 2\omega t$ W. The first term is a constant and the second term produces an average of zero over a cycle. Therefore, the average power delivered to the resistor is $0.5 V_m I_m = 0.5 V_m^2/R = 0.5 I_m^2 R$. The average power can be expressed as $V_{\text{rms}} I_{\text{rms}}$ in terms of rms values of voltage and current. Thus, a sinusoidal voltage/current is only as effective as a DC voltage/current of magnitude that is only 70.7% of the amplitude of the sinusoid. The presence of the second term – the term that has as much strength as the average power; but is oscillating at twice the supply frequency – indicates this relative inefficiency of sinusoids compared to DC quantities in carrying power to a load. This is the inevitable price that we have to pay for having opted for sinusoidal waveforms. Hence, we do not complain about the inevitable double-frequency power pulsation that has as much amplitude as the average power that is being delivered to the load.

Consider the same voltage source delivering power to the same resistor, but the resistor is in *parallel* with an inductive reactance X at ω rad/s as shown in Fig. 8.9-1.

Such a load is called a *reactive load*. The load impedance now is given by

$$Z = R // jX = \frac{jRX}{R + jX} = \left[\frac{X^2}{R^2 + X^2} \right] R + j \left[\frac{R^2}{R^2 + X^2} \right] X = \frac{RX}{\sqrt{R^2 + X^2}} \angle \tan^{-1} \frac{R}{X}$$

The current delivered by the voltage source will be

$$i(t) = \frac{V_m}{R} \cos \omega t + \frac{V_m}{X} \sin \omega t = I_m \cos(\omega t - \theta),$$

$$\text{where } I_m = \frac{V_m \sqrt{R^2 + X^2}}{RX} \text{ and } \theta = \tan^{-1} \frac{R}{X}$$

The instantaneous power is $p(t) = V_m I_m \cos \omega t \cos(\omega t - \theta)$

$$p(t) = V_m (I_m \cos \theta) \cos^2 \omega t + V_m (I_m \sin \theta) \sin \omega t \cos \omega t$$

$$p(t) = \{ [0.5 V_m (I_m \cos \theta)] + [0.5 V_m (I_m \cos \theta)] \cos 2\omega t \} + [0.5 V_m (I_m \sin \theta)] \sin 2\omega t$$

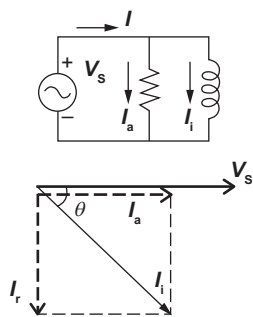


Fig. 8.9-1 A Parallel RL Load and its Phasor Diagram

$$\begin{aligned} \text{The average power is } 0.5V_m I_m \cos\theta &= 0.5V_m \times \frac{V_m \sqrt{R^2 + X^2}}{RX} \times \cos(\tan^{-1}(R/X)) = \\ 0.5V_m^2/R &\left[\text{since } \cos(\tan^{-1}(R/X)) = \frac{X}{\sqrt{R^2 + X^2}} \right]. \end{aligned}$$

Thus, $I_m \cos\theta = V_m/R$ is the same as the current drawn by the resistor alone.

Thus, the average power is due to the current drawn by the resistor and is the same as before. However, the source has to deliver a higher current to deliver the same amount of power now. The first double-frequency power pulsation (i.e., the $\cos 2\omega t$ term in $p(t)$) is the expected double-frequency pulsation when an average power is being delivered. The second double-frequency pulsating power (i.e., the $\sin 2\omega t$ term) is solely due to the inductor in parallel with the resistor – i.e., due to the *reactive* nature of load) and has an amplitude of $0.5V_m I_m \sin\theta = 0.5V_m^2/X$ (since $\sin\theta = \cos(\tan^{-1}(X/R)) = \frac{R}{\sqrt{R^2 + X^2}}$). The presence of the second pulsating power term with a non-zero amplitude is an indicator to the fact that the magnitude of current is more than the minimum magnitude of current required to pass on the average power to the load. The minimum current that is required in the circuit to deliver the average power it is delivering now is only $\cos\theta$ times the present current.

If the voltage in a DC circuit is the same as V_{rms} of this sinusoidal voltage source and the current in the DC circuit is the same as the I_{rms} in this AC circuit, then, the DC source would have delivered $V_{rms} I_{rms}$ W of average power to the load. Compared to that, the AC circuit delivers only $\cos\theta$ times this power. Thus, effectiveness of utilisation of voltage and current in a reactive circuit under the sinusoidal steady-state is compromised by the factor $\cos\theta$ compared to a DC circuit carrying a similar voltage and current. This observation leads to a definition of *apparent power* in an AC circuit.

Since the average power in an AC circuit can be different by a factor of $\cos\theta$, where θ is the angle between the voltage phasor and the current phasor, the unit of *watts* is reserved for the average power and a unit of *volt-amperes (VA)* is assigned to the apparent power. As only the average power contained in the apparent power is *active* in generating useful output from the circuit, average power is called the *active power*. The ratio between the active power and the apparent power is called the *power factor* of the circuit.

Note that the definitions of apparent power, active power and power factor are applicable to any general periodic waveform context but the expressions, $V_{rms} I_{rms} \cos\theta$ for active power and $\cos\theta$ for power factor, are applicable only under the sinusoidal steady-state condition.

8.9.1 Active and Reactive Components of Current Phasor

$I_m \cos\theta$ is the amplitude of $\cos\omega t$ term in current and $I_m \sin\theta$ is the amplitude of $\sin\omega t$ term in current. $\cos\omega t$ and $\sin\omega t$ terms are represented by phasors that have 90° between them. They are called *quadrature components* for this reason. Thus, $I_m \cos\theta$ is the *in-phase* component in the current phasor and $I_m \sin\theta$ is the *quadrature* component in the current phasor with respect to the voltage phasor. $I_m \cos\theta$, the in-phase component, carries the average power (along with an unavoidable double-frequency pulsating power of equal amplitude), and $I_m \sin\theta$, the quadrature component, produces a pure double-frequency pulsating power term with zero average content. This pulsating power term is avoidable by making $\theta = 0$ – i.e., by making the load purely resistive.

Any current phasor can be resolved into two components – one in the direction of the voltage phasor and one in a direction perpendicular to the voltage phasor. The component in the direction of voltage phasor is the *in-phase* component and this component will carry *active power*. Therefore, this component is called *active component of current* and is denoted by a phasor I_a . The component in the perpendicular direction to the voltage phasor is the *quadrature* component and this component *will not carry any average power*. This

Apparent Power

Apparent Power carried by a sinusoidal voltage of rms value V_{rms} and a sinusoidal current of rms value I_{rms} is defined as the actual power that will be carried by a DC voltage of the same effective value and a DC current of the same effective value – i.e.,
 $\text{Apparent Power} = V_{rms} \times I_{rms}$

$$\text{Apparent Power} = V_{rms} I_{rms}$$

Active Power, $P = V_{rms} I_{rms} \cos\theta$, where θ is the angle by which the voltage phasor leads the current phasor.

$$\text{Power Factor} = \frac{\text{Active Power}}{\text{Apparent Power}} = \cos\theta$$

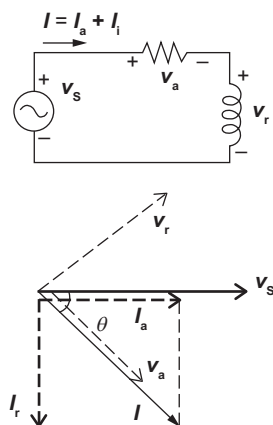


Fig. 8.9-2 A Series RL Load and its Phasor Diagram

Sign of Active and Reactive Components of Current

I_a is in phase with the voltage phasor by definition, and hence, its phase is known. Therefore, it is a common practice to consider I_a as if it is a real number with a positive or a negative sign.

In the case of I_r , it can be specified as a real number with a positive or a negative sign. Alternatively, one may add the qualifiers 'capacitive' or 'inductive' and then skip the sign.

The j in I_r is taken to be implied by the word 'reactive' in 'reactive component'. Thus, if $V_s = V_m \angle 0^\circ$ and $I = I_m \angle -\theta$, then, active component of current is $I_m \cos\theta$ ($I_{\text{rms}} \cos\theta$ A rms) and reactive component of current is $-I_m \sin\theta$ ($-I_{\text{rms}} \sin\theta$ A rms). If the negative sign is skipped, then we have to state that the inductive component of current is $I_m \sin\theta$ ($I_{\text{rms}} \sin\theta$ A rms).

For instance, a 5 A reactive current implies 5 A of capacitive current, a -5 A reactive current implies 5 A of

continued

component is decided by the reactance in the circuit and goes to zero when the circuit is purely resistive. Hence, this component is called the *reactive component of current* and is denoted by a phasor I_r . The current phasor I is the phasor sum of I_a and I_r . This is shown in the phasor diagram in Fig. 8.9-1

Note that the definition of *active and reactive components of a current phasor* is based on projecting the current phasor along and perpendicular to the voltage phasor and is applicable to any current phasor. In the circuit we considered in Fig. 8.9-1, the active component of I could be identified as the current in R and the reactive component could be identified as the current in jX . However, such an identification of active and reactive current components of a given current phasor is not a pre-condition for their definition. Consider a series RL load and its phasor diagram in Fig. 8.9-2.

The current phasor I can be resolved into in-phase component I_a and quadrature component I_r with respect to the voltage phasor as shown in the phasor diagram in Fig. 8.9-2. However, we cannot identify these components as real currents flowing in any element since there is only one current in a series circuit and that is I . Further, we observe that the voltage phasor also can be resolved into two components – one along the current phasor direction and one in a perpendicular direction. These are *active component* and *reactive component* of the voltage phasor.

We can always compare a parallel-connected resistance and reactance at a particular frequency to a series-connected resistance and reactance that has the same impedance. Then, we can identify the active component of I in a series circuit as the current that will flow in the resistor of a parallel circuit that has the same impedance as that of the series circuit. Similarly, we can identify the reactive component of I in a series circuit as the current that will flow in the reactance of a parallel circuit that has the same impedance as that of the series circuit. Hence, though resolving the voltage phasor along the current phasor and resolving the current phasor along the voltage phasor have the same effect in power equations, we choose to use the quadrature components of current phasor rather than the voltage phasor in subsequent discussions.

The side-note describes the sign convention adopted in specifying the active and reactive current components.

8.9.2 Reactive Power and the Power Triangle

We restate the concepts discussed in the last sub-section before we continue with the concept of *reactive power* since it tends to be confusing to beginners in Electrical Engineering.

- Let $v(t) = V_m \cos\omega t$ V and $i(t) = I_m \cos(\omega t - \theta)$ A be the steady-state voltage and current at a pair of load terminals as per the passive sign convention. Then 'apparent power', which is the power that a DC circuit with the same effective values of voltage and current would have delivered is, $V_{\text{rms}} I_{\text{rms}} = 0.5 V_m I_m$ VA. The average power, which is also called 'active power', is $P = V_{\text{rms}} I_{\text{rms}} \cos\theta = 0.5 V_m I_m \cos\theta$. The power factor of the circuit, which is defined as the ratio of active power to apparent power, is $\cos\theta$.
- The minimum magnitude of current required to deliver a given amount of power P is given by $I_{\text{rms}} = P/V_{\text{rms}}$ with $\cos\theta = 1$. This happens when the driving-point impedance of the load at ω rad/s is effectively a resistance. Then, the circuit has $\theta = 0$ and draws power at unity power factor with minimum magnitude of current.
- The reactive component in the driving-point impedance of the load circuit makes θ non-zero and increases the current magnitude for a given amount of active power. Thus, current is under-utilised as far as active power delivery is concerned. The power factor of the circuit will be less than unity.
- The current phasor can be resolved into 'active component' ($= I_{\text{rms}} \cos\theta$ A rms) and 'reactive component' ($= -I_{\text{rms}} \sin\theta$ A rms) by finding its projection along the

voltage phasor and along a perpendicular to the voltage phasor, respectively. The 'active current component' may be considered as the current drawn by the resistor in a parallel connected resistance-reactance combination that has the same impedance as that of the load circuit. The 'reactive component of current' is the current drawn by the reactance in that equivalent parallel circuit. The 'active current component' carries the entire 'active power'.

The utilisation of load current in its role as a vehicle to carry active power can be judged from the relative proportion of its active and reactive current components. It can be shown easily that

$$I_{\text{rms}} = \sqrt{I_{\text{a,rms}}^2 + I_{\text{r,rms}}^2} \text{ and } I_{\text{m}} = \sqrt{I_{\text{am}}^2 + I_{\text{rm}}^2},$$

where $I_{\text{a,rms}}$ and $I_{\text{r,rms}}$ indicate the rms values of the components, whereas I_{am} and I_{rm} indicate their amplitudes. The following relations also hold good between various quantities.

$$\begin{aligned} \cos \theta = \text{power factor} &= \frac{I_{\text{a,rms}}}{I_{\text{rms}}} = \frac{I_{\text{am}}}{I_{\text{m}}} \\ \sin \theta &= -\frac{I_{\text{r,rms}}}{I_{\text{rms}}} = -\frac{I_{\text{rm}}}{I_{\text{m}}} \\ \tan \theta &= -\frac{I_{\text{r,rms}}}{I_{\text{a,rms}}} = -\frac{I_{\text{rm}}}{I_{\text{am}}} \end{aligned}$$

$I_{\text{r,rms}}$ and I_{rm} contain the sign of the reactive component. Therefore, while the power factor is always positive, $\sin \theta$ and $\tan \theta$ are positive for a lagging load and negative for a leading load. Note that θ is the angle by which the voltage phasor leads the current phasor. Power factor of a load is independent of the sign of θ , and hence, a qualifier *lag* or *lead* is to be appended to the number representing the power factor to distinguish between the positive and negative values of θ . Thus, if $\theta = 45^\circ$, the power factor is '0.7 lag' and if θ is -45° , the power factor is '0.7 lead'.

Another method to describe the utilisation of load current in carrying active power will be to compare the active and reactive rms components of current after scaling the active rms current component by $+V_{\text{rms}}$ and reactive rms current component by $-V_{\text{rms}}$. But then, if the active component of current is multiplied by $+V_{\text{rms}}$, the result is *active power*. Then, it will be tempting to call the product of the reactive component of current and $-V_{\text{rms}}$, a power – but not a real average power, since this component of current produces only $V_{\text{rms}} I_{\text{r,rms}} \sin 2\omega t$ term in instantaneous power. Electrical Engineers yielded to this temptation long back and they called it *reactive power* in contrast to *active power*.

Thus, reactive power is not a power at all; it is only a power-like measure of the reactive component of current. This 'fictitious power' that is not a power at all in the normal sense of the word is, in essence, a stand-in for the reactive component of current.

It is usually denoted by Q and its unit is *Volt-Ampere-Reactive*, shortened as VAR. Thus, $Q = V_{\text{rms}} I_{\text{r,rms}} \sin \theta$ VAR, where θ is the phase angle by which the voltage phasor leads the current phasor. Therefore, the reactive power consumed by an inductive load is positive in sign and the reactive power consumed by a capacitive load is negative in sign by definition.

Notice that the Q value is the same as the amplitude of double-frequency power pulsation caused by the reactive component of current.

One may easily show that $(\text{Apparent Power})^2 = P^2 + Q^2$. Thus, a closed triangle can be constructed by treating apparent power, active power and magnitude of reactive power as its sides – the triangle will be called *power triangle*. This fact is also expressed in alternative forms as $(\text{VA})^2 = (\text{W})^2 + (\text{VAR})^2$ or $(\text{kVA})^2 = (\text{kW})^2 + (\text{kVAR})^2$.

It may also be noted that active power is alternatively called *real power* and *in-phase power*. Similarly, reactive power is also called *quadrature power*.

Many expressions are commonly employed to calculate the reactive power. The first expression is used when the load circuit is a composite circuit containing many resistive and

inductive current. A 5 A inductive current implies 5 A of inductive current and a -5 A inductive current implies 5 A of capacitive current. Similarly, a 5 A capacitive current implies 5 A of capacitive current and -5 A capacitive current implies 5 A of inductive current.

Inductive current component lags the voltage phasor by 90° and capacitive current component leads the voltage phasor by 90° .

To state that there is some reactive power flow into a load is a disguised way of stating that:

- (i) The load impedance has a reactive component.
- (ii) The load current has a reactive component that reduces the efficiency of current in carrying active power.
- (iii) Therefore, the current magnitude is more than the minimum magnitude commensurate with the actual power transfer taking place.
- (iv) Hence, the circuit is operating at a power factor less than unity.

Note carefully that the sign of the reactive component of current and the reactive power carried by that current is opposite.

Thus, an inductive load draws a 'negative reactive current' and consumes a 'positive reactive power'. A capacitive load draws a 'positive reactive current' and consumes

continued

a 'negative reactive power'. This is a matter of convention and convenience rather than of necessity.

If a circuit element is *consuming* a certain amount of reactive power, it may equivalently be considered as *delivering* negative of that amount of reactive power.

Thus, an element that *draws* positive reactive power (i.e., inductive Q) can be said to *deliver* negative reactive power (i.e., capacitive Q).

Similarly, an element that *draws* negative reactive power (i.e., capacitive Q) can be said to *deliver* positive reactive power (i.e., inductive Q).

Thus, a capacitor is a source of inductive reactive power and an inductor is a source of capacitive reactive power.

Definition of
Complex Power
under the sinusoidal
steady-state
condition.

Power
conservation under
sinusoidal steady-
state condition.

reactive elements. If $V = V_{\text{rms}} \angle \phi_v$ and $I = I_{\text{rms}} \angle \phi_i$ are the voltage phasor and current phasor at load terminals as per the passive sign convention, then the Q delivered to the load circuit is $V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i)$ VAR. The other expressions are relevant when the voltage phasor and/or current phasor across a pure reactive element is known. In this case $(\phi_v - \phi_i)$ is assured to be 90° if the element is an inductor and -90° if the element is a capacitor. Then, the reactive power delivered to that element, i.e., Q is given by $Q = V_{\text{rms}} I_{\text{rms}} = I_{\text{rms}}^2 X = \frac{V_{\text{rms}}^2}{X}$, where V_{rms} is the rms value of the voltage across the element, I_{rms} is the rms value of current through the element and X is the reactance of that element. *Note that reactance of an inductor is a positive value and that of a capacitor is a negative value.*

8.10 COMPLEX POWER UNDER SINUSOIDAL STEADY-STATE CONDITION

Can we get the P and Q values from the voltage phasor and the current phasor straightaway by multiplying them together? We will try. Let $V = V_{\text{rms}} \angle \phi_v$ V rms be the voltage phasor and $I = I_{\text{rms}} \angle \phi_i$ A rms be the current phasor; both specified as rms quantities. Then,

$$VI = V_{\text{rms}} I_{\text{rms}} \angle (\phi_v + \phi_i) = V_{\text{rms}} I_{\text{rms}} \cos(\phi_v + \phi_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\phi_v + \phi_i).$$

We are not able to identify P or Q in the real and imaginary parts of this quantity since we know that $P = V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i)$ and $Q = V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i)$, but this observation prompts us to try VI^* instead of VI .

$$\begin{aligned} VI^* &= V_{\text{rms}} I_{\text{rms}} \angle (\phi_v - \phi_i) = V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i) \\ &= P + jQ \end{aligned}$$

Thus, the quantity VI^* contains the active power as its real part and the reactive power as its imaginary part.

This quantity, VI^* , is defined as *Complex Power* and is denoted by S with unit of VA.

$$S(\text{VA}) = VI^* = P(\text{W}) + jQ(\text{VAR}) = \sqrt{P^2 + Q^2} \angle (\phi_v - \phi_i) \text{VA}.$$

Therefore, the magnitude of *complex power* is the apparent power and the angle of *complex power* is the angle by which the voltage phasor *leads* the current phasor. This angle is the same as the angle of driving-point impedance of the load circuit. S , $P + j0$ and $0 + jQ$ form a closed triangle in a complex plane.

If the reader feels uncomfortable with the way the complex power was derived, he can console himself with the fact that this was precisely how the expression for complex power was arrived at in the history of Electrical Engineering.

We had shown that both instantaneous power and average power are conserved in any circuit. Thus, the active power under the sinusoidal steady-state condition is a conserved quantity. That is, the algebraic sum of active power delivered to all the elements of an isolated circuit under the sinusoidal steady-state is zero. Does a similar conservation law hold good for reactive power? It is possible to show that it does by using the expression for instantaneous power under sinusoidal steady-state conditions. Since both the real and imaginary parts of S are conserved, S itself is a conserved quantity. That is, the algebraic sum of complex powers in all elements of an isolated circuit will be zero.

For an isolated circuit under single-frequency sinusoidal steady-state,

$$\sum_{\text{over all elements}} \text{Active power} = 0; \quad \sum_{\text{over all elements}} \text{Reactive power} = 0 \quad \text{and} \quad \sum_{\text{over all elements}} \text{Complex power} = 0$$

Let $v(t) = V_m \cos(\omega t + \phi_v) = \sqrt{2} V_{\text{rms}} \cos(\omega t + \phi_v)$ and $i(t) = I_m \cos(\omega t + \phi_i) = \sqrt{2} I_{\text{rms}} \cos(\omega t + \phi_i)$ be the voltage and the current as per the passive sign convention in a load circuit. Let the series equivalent of the load circuit be $R_s + jX_s$ and the parallel equivalent of the same circuit be $R_p + jX_p$. Then, Table 8.10-1 summarises the various phasor quantities and their inter-relations.

Table 8.10-1 Relationship Between Various Phasor Quantities in a Single-Phase System

Quantity	Relationship using peak values	Relationship using rms values	Unit
Voltage phasor $\mathbf{V}$	$V_m \angle \phi_v$	$V_{rms} \angle \phi_v$	V
Current phasor $\mathbf{I}$	$I_m \angle \phi_i$	$I_{rms} \angle \phi_i$	A
Complex power $\mathbf{S}$	$0.5 V_m I_m \cos(\phi_v - \phi_i) + j0.5 V_m I_m \sin(\phi_v - \phi_i)$	$V_{rms} I_{rms} \cos(\phi_v - \phi_i) + j V_{rms} I_{rms} \sin(\phi_v - \phi_i)$	VA
Apparent power $ \mathbf{S} $	$0.5 V_m I_m$	$V_{rms} I_{rms}$	VA
Active power P	$0.5 V_m I_m \cos(\phi_v - \phi_i)$	$V_{rms} I_{rms} \cos(\phi_v - \phi_i)$	W
Reactive power Q	$0.5 V_m I_m \sin(\phi_v - \phi_i)$	$V_{rms} I_{rms} \sin(\phi_v - \phi_i)$	Var
Power factor PF	$\cos(\phi_v - \phi_i)$	$\cos(\phi_v - \phi_i)$	—
Input impedance $\mathbf{Z}$	$(V_m/I_m) \angle (\phi_v - \phi_i)$	$(V_{rms}/I_{rms}) \angle (\phi_v - \phi_i)$	Ω
R_s	$(V_m/I_m) \cos(\phi_v - \phi_i)$; or $2P/I_m^2$	$(V_{rms}/I_{rms}) \cos(\phi_v - \phi_i)$ or P/I_{rms}^2	Ω
X_s	$(V_m/I_m) \sin(\phi_v - \phi_i)$ or $2Q/I_m^2$	$(V_{rms}/I_{rms}) \sin(\phi_v - \phi_i)$ or Q/I_{rms}^2	Ω
R_p	$V_m/[I_m \cos(\phi_v - \phi_i)]$ or $2P/[I_m \cos(\phi_v - \phi_i)]^2$	$V_{rms}/[I_{rms} \cos(\phi_v - \phi_i)]$ or $P/[I_{rms} \cos(\phi_v - \phi_i)]^2$	Ω
X_p	$V_m/[I_m \sin(\phi_v - \phi_i)]$ or $2Q/[I_m \sin(\phi_v - \phi_i)]^2$	$V_{rms}/[I_{rms} \sin(\phi_v - \phi_i)]$ or $Q/[I_{rms} \sin(\phi_v - \phi_i)]^2$	Ω

EXAMPLE: 8.10-1

Refer to Example 8.6-4. Derive expressions for complex power delivered by the first source and complex power absorbed by the second source in a synchronous link and obtain the approximate expressions for a situation when the phase difference between the sources is small and the difference in magnitude of voltages is small.

SOLUTION

The synchronous link under consideration is shown in Fig. 8.10-1.

Let the current phasor from left to right be $\mathbf{I}$.

$$\begin{aligned}
 \mathbf{I} &= \frac{V_1 \angle \delta_1 - V_2 \angle \delta_2}{jX} = \frac{V_1 \angle (\delta_1 - \pi/2)}{X} - \frac{V_2 \angle (\delta_2 - \pi/2)}{X} \\
 \mathbf{S}_1 &= \mathbf{V}_1 \mathbf{I}^* = V_1 \angle \delta_1 \left[\frac{V_1 \angle -\delta_1 - V_2 \angle -\delta_2}{-jX} \right] = \frac{V_1^2}{X} \angle \pi/2 - \frac{V_1 V_2}{X} \angle (\pi/2 + \delta_1 - \delta_2) \\
 &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) + j \frac{V_1 [V_1 - V_2 \cos(\delta_1 - \delta_2)]}{X} \\
 \therefore P_1 &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) \text{ W}
 \end{aligned}$$

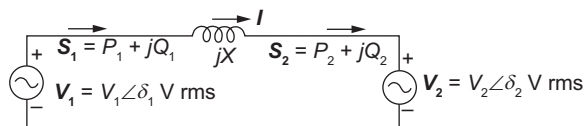


Fig. 8.10-1 A Synchronous Link

$$Q_1 = \frac{V_1 [V_1 - V_2 \cos(\delta_1 - \delta_2)]}{X} \text{ VAr}$$

Similarly,

$$\begin{aligned} \mathbf{S}_2 &= \mathbf{V}_2 \mathbf{I}^* = V_2 \angle \delta_2 \left[\frac{V_1 \angle -\delta_1 - V_2 \angle -\delta_2}{-jX} \right] = \frac{-V_2^2}{X} \angle \pi/2 + \frac{V_1 V_2}{X} \angle (\pi/2 + \delta_2 - \delta_1) \\ &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) + j \frac{V_2 [V_1 \cos(\delta_1 - \delta_2) - V_2]}{X} \end{aligned}$$

$$\therefore P_2 = \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) \text{ W}$$

$$Q_2 = \frac{V_2 [V_1 \cos(\delta_1 - \delta_2) - V_2]}{X} \text{ VAr}$$

The link is purely inductive and we do not expect any loss of active power in the link. This is borne out by the fact that $P_1 = P_2$. The link inductor has voltage across it and current through it. Therefore, this inductor will consume positive reactive power. Hence, we expect Q_2 to be less than Q_1 . Let Q_L be the reactive power absorbed by the link inductor. Then,

$$Q_L = Q_1 - Q_2 = \frac{V_1^2 + V_2^2 - 2V_1 V_2 \cos(\delta_1 - \delta_2)}{X} = \frac{(V_1 - V_2)^2 + 2V_1 V_2 [1 - \cos(\delta_1 - \delta_2)]}{X}$$

Obviously, Q_L is a positive quantity, and hence, $Q_2 < Q_1$.

Special Case – $\delta = \delta_1 - \delta_2 \ll \pi/2$ and $V_1 = V + \Delta V$, $V_2 = V$

$$P_1 \approx \frac{V^2}{X} \delta \text{ W}; Q_1 \approx \frac{V \times \Delta V + (\Delta V)^2}{X} \text{ VAr}$$

$$P_2 \approx \frac{V^2}{X} \delta \text{ W}; Q_2 \approx \frac{V \times \Delta V}{X} \text{ VAr}$$

$$Q_L \approx \frac{(\Delta V)^2}{X} \text{ VAr}$$

Thus, in a synchronous link operating with a small phase difference and small voltage magnitude difference between sources, the active power flows from the *leading* source to the *lagging* source. The amount of active power will be proportional to the phase difference (in radians) and will be relatively independent of the voltage magnitude difference. Positive reactive power flow will take place from the *source with higher voltage magnitude* to the *source with lower voltage magnitude*. Reactive power flow in the link is proportional to the voltage magnitude difference and is relatively insensitive to phase difference.

EXAMPLE: 8.10-2

A 50 Hz, 63.5 kV rms sub-station provides power to a large industrial consumer through a 20 km long high voltage line that can be modelled by a resistance of 5Ω and inductive reactance of 20Ω . The voltage magnitude at the receiving end is to be maintained at 63.5 kV (Fig. 8-10-2). This is done by adjusting the sending end voltage magnitude by tap-changing transformers or otherwise. If the consumer draws 20 MW of power at 0.707 lag power factor, find (i) the magnitude of sending end voltage and power factor, (ii) line current, (iii) sending end active and reactive power, (iv) active and reactive power absorbed by the line impedance and (v) line power efficiency.

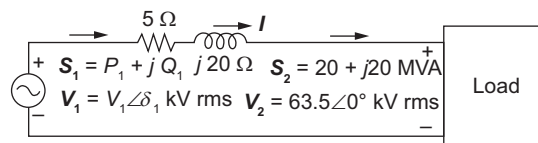


Fig. 8.10-2 Circuit for Example 8.10-2

SOLUTION

The magnitude of receiving end current = $20 \times 10^6 \div (0.707 \times 63.5 \times 10^3) = 445.5 \text{ A rms}$

The angle of current with respect to receiving end voltage phasor = $-\cos^{-1}0.707 = -45^\circ$

We take the receiving end voltage as the reference phasor.

Then, $V_R = 63.5 \angle 0^\circ \text{ kV rms}$ and $I = 445.5 \angle -45^\circ \text{ A rms}$.

Active power delivered at receiving end = 20 MW

Reactive power delivered at receiving end = $63.5 \text{ kV} \times 0.4455 \text{ kA} \times \sin 45^\circ = 20 \text{ MVAr}$

Active power consumed by the line impedance = $5 \Omega \times (0.4455 \text{ kA})^2 = 1 \text{ MW}$

Reactive power consumed by the line impedance = $20 \Omega \times (0.4455 \text{ kA})^2 = 4 \text{ MVAr}$

$\therefore$ Active power at sending end = $20 + 1 = 21 \text{ MW}$

$\therefore$ Reactive power at sending end = $20 + 4 = 24 \text{ MVAr}$

$\therefore$ Complex power at sending end, $S_1 = 21 + j24 \text{ MVA}$

Since $S_1 = V_1 I^*$, $V_1 = S_1 \div I^* = (21 + j24) \times 10^6 \div (445.5 \angle -45^\circ)^* = 71.6 \angle 3.8^\circ \text{ kV rms}$.

The angle between the sending end voltage phasor and the current phasor = $3.8^\circ - (-45^\circ) = 48.8^\circ$. Therefore, sending end power factor = $\cos 46.74^\circ = 0.66 \text{ lag}$.

- (i) Sending end voltage magnitude = 71.6 kV rms (12.75% above nominal value)
- (ii) Sending end power factor = 0.66 lag
- (iii) Sending end active power = 21 MW
- (iv) Sending end reactive power = 24 MVAr
- (v) Active power loss in line = 1 MW
- (vi) Reactive power loss in line = 4 MVAr
- (vii) Line power efficiency = 95.2%

EXAMPLE: 8.10-3

If a capacitor is connected directly across the load at customer side in the problem stated in Example 8.10-2 such that the receiving end current is at unity power factor with respect to receiving end voltage find the reactive power drawn by the capacitor and the capacitance value. Also calculate all the quantities calculated under Example 8.10-2 and comment on the differences.

Solution

Refer to the circuit in Fig. 8.10-3. The capacitor should supply all the reactive power requirement of the load, i.e., all of 20 MVAr if the current in the line at the receiving end is to be at unity power factor. Therefore, the current taken by capacitor will be $20 \times 10^6 / 63.5 \times 10^3 = 315 \text{ A}$. Therefore, capacitive reactance = $63.5 \times 10^3 / 315 = 201.6 \Omega$. This value is $1/\omega C$ and $\omega = 2\pi \times 50 = 100\pi$. Therefore, $C = 15.8 \mu\text{F}$. With this capacitor in place, the current at the receiving end of the line will have a magnitude of $20 \times 10^6 / 63.5 \times 10^3 = 315 \text{ A}$ and its angle with respect to voltage will be 0° . Then, $V_R = 63.5 \angle 0^\circ \text{ kV rms}$ and $I = 315 \angle 0^\circ \text{ A rms}$.

Active power delivered at receiving end = 20 MW

Reactive power delivered at receiving end = $63.5 \text{ kV} \times 0.315 \text{ kA} \times \sin 0^\circ = 0 \text{ MVAr}$

Active power consumed by the line impedance = $5 \Omega \times (0.315 \text{ kA})^2 = 0.5 \text{ MW}$

Reactive power consumed by the line impedance = $20 \Omega \times (0.315 \text{ kA})^2 = 2 \text{ MVAr}$

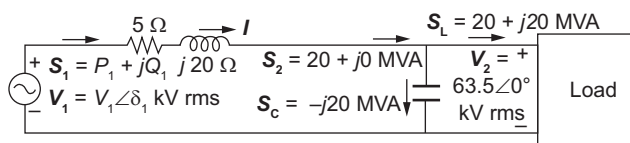


Fig. 8.10-3 Circuit for Example 8.10-3

Comments on Results Arrived at in Example 8.10-3

A load that draws power at a lag power factor results in a higher magnitude current in the line and in the source. Higher current magnitude results in higher voltage drop in the line, thereby, requiring higher value of sending end voltage to maintain a specified receiving end voltage. Higher current magnitude results in higher active power loss in the line, and thereby, reduces power efficiency of the line.

The reactive component of the load current undergoes a rotation by 90° to form the voltage drop in the line inductive reactance and results in an 'in-line' voltage drop. An 'in-line' voltage drop affects sending end voltage much more than a 'quadrature' voltage drop. See the phasor diagram in Fig. 8.10-4.

Therefore, reducing the reactive component of current drawn by a lagging load results in (i) lower current magnitude in the line and the source, (ii) lower line power loss and improved transmission efficiency and (iii) lower voltage drop in the line. This reduction is effected by making a local capacitor act as a source of lagging reactive power required by the load. The line is thereby relieved from the task of supplying this reactive power.

This is called *capacitive compensation of lagging loads*.

Capacitive compensation is routinely employed in Power Systems and Industrial Electrical Systems.

- $\therefore$ Active power at sending end = $20 + 0.5 = 20.5$ MW
- $\therefore$ Reactive power at sending end = $0 + 2 = 2$ MVAR
- $\therefore$ Complex power at sending end, $S_1 = 20.5 + j2$ MVA
- Since $S_1 = V_1 I^*$, $V_1 = S_1 \div I^* = (20.5 + j2) \times 10^6 \div (315 \angle -45^\circ)^* = 65.4 \angle 5.6^\circ$ kV rms.
- The angle between the sending end voltage phasor and the current phasor = $5.6^\circ - (0^\circ) = 5.6^\circ$. Therefore, sending end power factor = $\cos 5.6^\circ = 0.995$ lag.
- (i) Sending end voltage magnitude = 65.4 kV rms (3% above nominal value)
- (ii) Sending end power factor = 0.995 lag
- (iii) Sending end active power = 20.5 MW
- (iv) Sending end reactive power = 2 MVAR
- (v) Active power loss in line = 0.5 MW
- (vi) Reactive power loss in line = 2 MVAR
- (vii) Line power efficiency = 97.6%

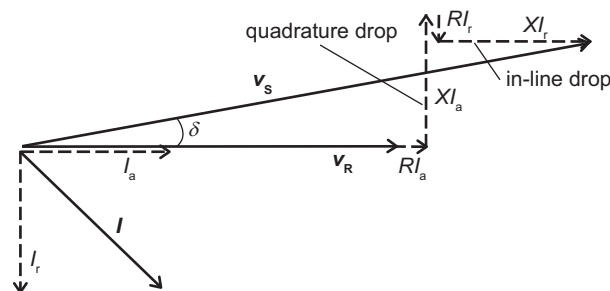


Fig. 8.10-4 Phasor Diagram of a Line Delivering Power to a Lagging Load (Not to Scale)

EXAMPLE: 8.10-4

A 230 V rms source supplies two loads in parallel. The first one draws 10 kVA at 0.8 lag power factor. The second one draws 10 kW at 0.8 lead power factor. Find the source current rms value, complex power delivered by source and source power factor.

SOLUTION

Complex power of first load = $10 \times 0.8 + j10 \times \sin(\cos^{-1}0.8) = 8 + j6$ kVA

Complex power of second load = $10 - j(10/0.8) \times \sin(\cos^{-1}0.8) = 10 - j7.5$ kVA.

Complex power is a conserved quantity.

Therefore, complex power delivered by source = total complex power delivered to loads = $18 - j1.5$ kVA.

Source voltage is $230 \angle 0^\circ$ V rms. Therefore, source current = $[(18 - j1.5) \times 10^3 / 230]^* = 78.53 \angle 4.76^\circ$ A rms.

Therefore, source current rms value is 78.53 A and source power factor is $\cos 4.76^\circ = 0.997$ lead.

8.11 SINUSOIDAL STEADY-STATE IN CIRCUITS WITH COUPLED COILS

The physical basis for mutual inductance between two coils that share a common magnetic flux component has been dealt with in Chapter 1.

Flux linkage in a coil can be produced in two ways. The first method is to send a current through the coil. The flux linkage produced in the coil will be proportional to the current through it and the proportionality constant is its self-inductance L . We now call this inductance as self-inductance because we need to distinguish it from mutual inductance. However,

the qualifier ‘self’ is often dropped in practice and the word ‘inductance’ occurring alone is taken to mean ‘self-inductance’ by default.

The flux linkage will be time-varying when the current in the coil is a time varying one. Time-varying flux linkage produces induced e.m.f. in the coil. This induced e.m.f. if viewed as a voltage drop across the element, is described by $v = L \, di/dt$, where L is the self-inductance of the coil.

The second way the flux linkage in a coil can be produced is to place it near some other coil that is carrying a current. The flux produced by the current in the second coil links the first coil too due to the coils sharing the same magnetic path. Now, two possibilities arise. The flux in the first coil due to the current in the second coil may be crossing the area of cross-section of the coil in the same direction as the flux produced in the first coil by a current flowing in it. It could also be in the opposite direction. That depends on how the coils are wound. The magnitude of flux linkage produced in first coil by the current in the second coil is proportional to that current and the proportionality constant is defined as mutual inductance M . The same value of M will decide the flux linkage in the second coil due to current in first coil.

An induced voltage $v_{12} = M di_2/dt$ will be generated in the first coil due to time-varying flux linkage produced by the time-varying current in the second coil. If the first coil is not open and is carrying a current i_1 , then $v(t) = L_1 \, di_1/dt$ is the self-induced voltage that will appear with a positive polarity at the terminal where the current i_1 enters the coil. When both coils are carrying currents, the voltage across the first coil can be $L_1 \, di_1/dt \pm M di_2/dt$ with a positive polarity at the terminal where i_1 enters the first coil. The sign connecting the two voltage terms, $+$ or $-$, will be decided by whether the flux produced in the first coil by the current in the second coil is aiding the flux produced by its own current or opposing it. Similarly, the voltage across the second coil is $L_2 di_2/dt \pm M di_1/dt$ with a positive polarity at the terminal where i_2 enters the second coil.

8.11.1 Dot Polarity Convention

One does not want to physically examine every coupled coil-set to find out the relative winding directions in order to decide the sign connecting the self-induced voltage drop and the mutually induced voltage drop in each coil. Therefore, the manufacturer marks this information in the form of two *dots* – one on each coil in a coupled set of two coils. He can choose to put the dot on any of the two terminals in the first coil. He chooses that arbitrarily. That choice will fix the terminal where the dot has to be marked in the second coil. The following rule helps us to write the voltage equation in a set of coupled coils:

- Increasing current entering a dot point in one coil will produce a mutual e.m.f. with its positive polarity at the dot point in the other coil.
- Increasing current leaving a dot point in one coil will produce a mutual e.m.f. with its negative polarity at the dot point in the other coil.
- Self-induced voltage is always positive at the current entry point for both coils.

Various possibilities for mutual e.m.f. polarity in a two-coil coupled system are shown in Fig. 8.11-1. A double-headed arrow indicates magnetic coupling with M value marked by the side.

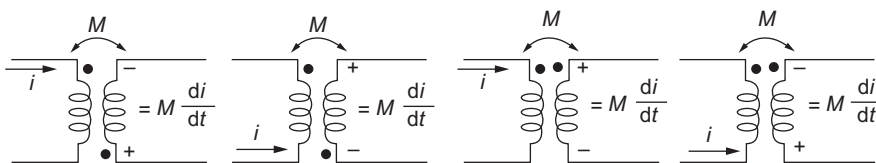


Fig. 8.11-1 Dot Polarity Convention and Sign of Mutually Induced Voltage in a Two-Coil System

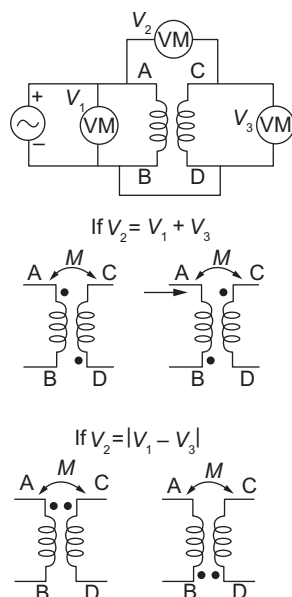


Fig. 8.11-2
Experimental
Determination of Dot
Point Assignment in a
Two-Coil System

However, markings can fade with time and people can forget to do things they are supposed to do. Therefore, the following experimental method can help us to put the *dots* if they are not known already.

Let A and B be the two terminals of the first coil and C and D be the two terminals of the second coil. Join B and D. Apply a sinusoidal source of suitable amplitude and frequency between A and B as shown in Fig. 8.11-2. Use three rms reading voltmeters to record the readings V_1 , V_2 and V_3 . Then, V_2 will be close to either $V_1 + V_3$ or $|V_1 - V_3|$. The *dot point* assignment in these two cases is shown in Fig. 8.11-2.

8.11.2 Maximum Value of Mutual Inductance and Coupling Coefficient

The maximum value of mutual inductance that a two-coil coupled system can have is the geometric mean of the self-inductances of the two coils – i.e., $\sqrt{L_1 L_2}$. This maximum value is realised when the flux produced by a current in one coil links totally with the other coil. This is impossible to achieve physically and can only be approached in practice. Winding both the coils over a common core made of high permeability material like laminated iron makes M very close to (but less than) $\sqrt{L_1 L_2}$ in practice.

The ratio between the actual mutual inductance realised to the theoretical maximum value is defined as *coupling coefficient* and is symbolised by k . Coupling coefficient is a positive number between 0 and 1. It is difficult to achieve anything more than a few tenths for k value in air-cored coils; however, coils wound on common core made of iron can have k value very close to unity.

$$M = k\sqrt{L_1 L_2} \text{ H; } 0 \leq k \leq 1.$$

8.11.3 A Two-Winding Transformer – Equivalent Models

A system of two coils with constant values of L_1 , L_2 and M with two pairs of terminals identified for application of excitation and/or measurement of response is called a *two-winding linear transformer*. Such a two-winding transformer will be referred to simply as a *transformer* in the remaining portion of this sub-section.

Figure 8.11-3 shows a two-coil system with self-inductance values of L_1 , L_2 and mutual inductance value of M . Also shown are two independent voltage sources driving the circuit at two pairs of terminals.

The mesh equations of the circuit are written assuming that the coils have zero winding resistance values.

$$\begin{aligned} L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} &= v_1(t) \\ -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} &= v_2(t) \end{aligned}$$

The current i_2 leaves the *dot* in the second coil, and therefore, the mutually induced voltage in the first coil appears with a negative polarity at the *dot*. Hence, the negative sign for the mutually induced voltage in the first mesh equation. The current i_1 enters the *dot* of the first coil, and therefore, the mutually induced voltage in the second coil appears with a positive polarity at the *dot*. When the mesh equation is written by traversing the loop in a clockwise direction, the self-induced voltage and the mutually induced voltages will enter with opposite signs. Hence, the negative sign for mutually induced voltage in the second mesh equation.

With reference to Fig. 8.11-4, the reader may easily verify that the circuit (a) and circuit (b) in this figure have the same mesh equations as those of the circuit in Fig. 8.11-3. Hence, the coupled set of coils may be replaced by two decoupled coils and two linear dependent sources as in the circuit model shown in Fig. 8.11-4(a) as far as the *v-i behaviour at the terminals is concerned*. This circuit model preserves the conductive decoupling – i.e., the galvanic isolation that exists between the two sides of the circuit.

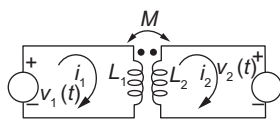


Fig. 8.11-3 A
Transformer with
Voltage Source Drive
at Both Ports

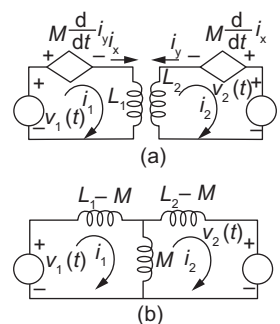


Fig. 8.11-4 Two Circuit
Models for a
Transformer (a) Model
Using Linear
Dependent Sources
(b) Conductive
Equivalent Model

A coupled set of two coils can also be replaced by a T-shaped equivalent circuit comprising three pure decoupled inductors as in the circuit model shown in Fig. 8.11-4(b) *as far as the v - i behaviour at the terminals is concerned*. This circuit model hides the galvanic isolation that is present in the transformer. Therefore, it is called the conductive equivalent circuit of coupled coils. Note that one inductance (either $L_1 - M$ or $L_2 - M$) can be negative-valued for sufficiently large values of coupling coefficient. There is no negative inductance in the physical world, but then the inductors that appear in the conductive equivalent circuit of coupled coils are *not* physical inductors – they are *mathematical inductors* that are arranged to result in the same set of circuit equations as those of the coupled coils. Therefore, they can assume negative values – we do not have to construct them!

Note that a certain *dot* polarity was assumed for the equivalent models established above. The equivalent models for the second relative *dot* polarity are shown in Fig. 8.11-5.

The circuit in Fig. 8.11-5(b) shows the dependent source based circuit model for the two-coil system in Fig. 8.11-5(a) and the circuit in Fig. 8.11-5(c) shows the conductive equivalent circuit of the two-coil system in Fig. 8.11-5(c). Note the reversal of polarity in the case of dependent sources and the change in inductance values in the case of conductive equivalent circuit.

Differentiation in time is replaced by multiplication by $j\omega$ in the phasor equivalent circuit for sinusoidal steady-state analysis in the equivalent circuit employing dependent sources. Winding resistance can be included as series resistors on both sides in both equivalent circuits.

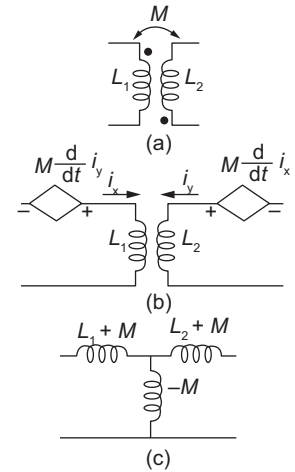


Fig. 8.11-5 Circuit Models for Coupled Coils

EXAMPLE: 8.11-1

The applied voltage $v_s(t) = 100\sqrt{2} \cos 10^3 t$ in the circuit shown in Fig. 8.11-6. (a) Find the primary (the side with source connection) and the secondary (the side with no source connection) currents and the average power delivered to the resistors for (i) $k = 0$, (ii) $k = 0.1$, (iii) $k = 0.5$ and (iv) $k = 1$. (b) Repeat calculations with $k = 1$ for $L_1 = 25$ H and $L_2 = 100$ H.

SOLUTION

(a) The conductive equivalent circuit for coupled coils is used to obtain the phasor equivalent circuits for the four values of coupling coefficients. The phasor equivalent circuits are shown in Fig. 8.11-7.

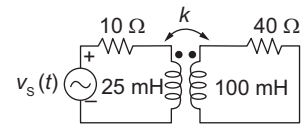


Fig. 8.11-6 Circuit for Example 8.11-1

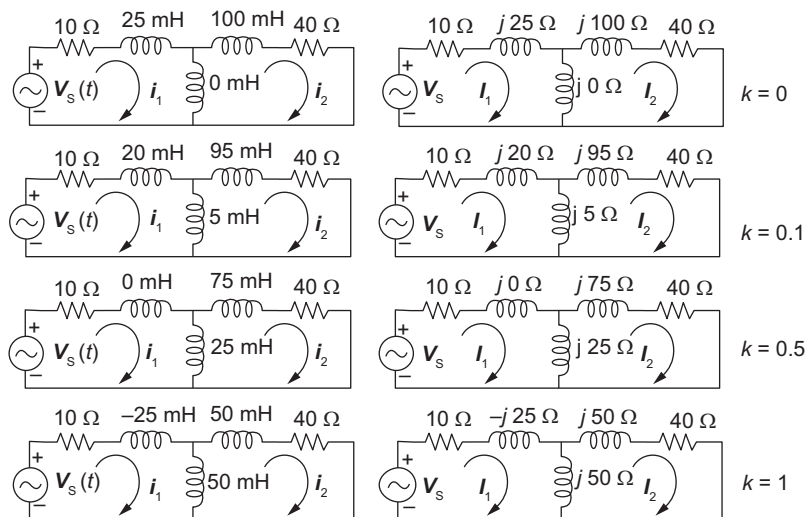


Fig. 8.11-7 Phasor Equivalent Circuits of the Circuit in Fig. 8.11-6 for Different Values of Coupling Coefficient

The solution process is illustrated for $k = 0.1$.

$\mathbf{V}_s = 100\angle 0^\circ$ V rms. The two mesh equations are expressed in matrix form as

$$\begin{bmatrix} 10 + j25 & -j5 \\ -j5 & 40 + j100 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0^\circ \\ 0 \end{bmatrix}$$

Solving these equations, we get, $I_1 = 3.737\angle -67.86^\circ$ A rms and $I_2 = 0.174\angle -46^\circ$ A rms.

Average power dissipated in $10\ \Omega$ resistor is $3.737^2 \times 10 = 139.65$ W

Average power dissipated in $40\ \Omega$ resistor is $0.174^2 \times 40 = 1.21$ W

Average power delivered by source = $100 \times 3.737 \times \cos 67.9^\circ = 140.83$ W.

Impedance seen by source = $\mathbf{Z}_{in} = 100\angle 0^\circ \div 3.737\angle -67.86^\circ = 10.09 + j24.78\ \Omega$.

The table that follows summarises the results of calculations for all four values of k .

K	I_1 (A rms)	I_2 (A rms)	$P_{10\ \Omega}$ (W)	$P_{40\ \Omega}$ (W)	PF	$\mathbf{Z}_{in}$ (Ω)
0	$3.714\angle -68.2^\circ$	0	137.94	0	0.37	$10 + j25$
0.1	$3.74\angle -67.86^\circ$	$0.174\angle -46^\circ$	139.65	1.21	0.38	$10.09 + j24.78$
0.5	$4.334\angle -58.2^\circ$	$1.006\angle -36.4^\circ$	187.83	40.48	0.53	$12.16 + j19.61$
1	$5.28\angle -10.49^\circ$	$2.45\angle 11.31^\circ$	278.89	240.29	0.98	$18.62 + j3.45$

(b) We observe from the table of results that as the coupling coefficient increases, (i) the primary and secondary currents increase in magnitude and become more active in content than reactive, (ii) the power delivered to the resistor in the secondary side increases, (iii) the power factor of the circuit increases and (iv) the source experiences a decreasing input impedance that becomes more and more resistive.

Now, we raise the self-inductance level of the coils thousand fold, but we keep the ratio between them at 4 as before. The mesh equations for $k = 1$ are

$$\begin{bmatrix} 10 + j25 \times 10^3 & -j5 \times 10^3 \\ -j5 \times 10^3 & 40 + j10^5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0^\circ \\ 0 \end{bmatrix}$$

We observe that the reactive terms in the matrix entries overpower the resistive terms. The solution is obtained as, $I_1 = 5\angle -0.01^\circ$ A rms and $I_2 = 2.5\angle 0.01^\circ$ A rms.

Average power dissipated in $10\ \Omega$ resistor is $5^2 \times 10 = 250$ W

Average power dissipated in $40\ \Omega$ resistor is $2.5^2 \times 40 = 250$ W

Average power delivered by source = $100 \times 5 \times \cos 0.01^\circ = 500$ W.

Impedance seen by source = $\mathbf{Z}_{in} = 100\angle 0^\circ \times 5\angle -0.01^\circ = 20 + j0.004\ \Omega$

The input impedance as seen by the source is almost purely resistive at $20\ \Omega$. The resistor that is in series with the source in the primary side must be contributing $10\ \Omega$ to the input impedance. Therefore, the transformer and its secondary load of $40\ \Omega$ resistor contributes $10\ \Omega$ to the input impedance. Thus, the equivalent impedance of the transformer plus the $40\ \Omega$ load seem to be an almost pure resistance of $10\ \Omega$. The ratio between $10\ \Omega$ and $40\ \Omega$ happens to coincide with the ratio $L_1:L_2$ – a coincidence that calls for further analysis.

Let us calculate the voltage appearing across the primary and secondary windings of the transformer.

Voltage across primary winding = $100\angle 0^\circ - 10 \times 5\angle -0.01^\circ \approx 50\angle 0^\circ$ V rms

Voltage across secondary winding = $40 \times 2.5\angle 0.01^\circ \approx 100\angle 0^\circ$ V rms

Thus, primary voltage:secondary voltage $\approx 1:2 = \sqrt{L_1} : \sqrt{L_2}$ and using the solution

for currents, primary current:secondary current $\approx 2:1 = \sqrt{L_2} : \sqrt{L_1}$. That makes it too many coincidences to overlook. So we analyse the transformer with $k = 1$ in greater detail now.

8.11.4 The Perfectly Coupled Transformer and The Ideal Transformer

Figure 8.11-8 shows a two-winding transformer with primary self-inductance of L_1 and secondary self-inductance of L_2 with coupling coefficient of unity. We neglect the winding resistance for the time being. The two circuits shown in this figure differ only in the *dot* polarity in the transformer.

The primary self-inductance is driven by a source $= \sqrt{2}V \cos \omega t$ V and the secondary self-inductance is *passively terminated in an impedance* $Z_L = R_L + jX_L \Omega$. Passive termination implies that there are no sources of any kind in the load circuit.

The mesh currents are I_p and I_s themselves and the mesh equations for the circuit in Fig. 8.11-8(a) in matrix form are

$$\begin{bmatrix} j\omega L_1 & -j\omega M \\ -j\omega M & Z_L + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_p \\ I_s \end{bmatrix} = \begin{bmatrix} V \angle 0^\circ \\ 0 \end{bmatrix}$$

Since $k = 1$, the value of $M = \sqrt{L_1 L_2}$. Solving for primary and secondary current phasors using this value for M , we get,

$$I_p = \frac{V \angle 0}{j\omega L_1} + \left[\frac{L_2}{L_1} \right] \frac{V \angle 0}{Z_L} = V_p \left[\frac{1}{j\omega L_1} + \frac{L_2}{L_1} \frac{1}{Z_L} \right]$$

$$I_s = V \angle 0 \left[\sqrt{\frac{L_2}{L_1}} \right] \frac{1}{Z_L} = V_p \left[\sqrt{\frac{L_2}{L_1}} \right] \frac{1}{Z_L}$$

$$V_s = I_s Z_L = V_p \left[\sqrt{\frac{L_2}{L_1}} \right]$$

$$\therefore \frac{V_s}{V_p} = \sqrt{\frac{L_2}{L_1}}.$$

We have used the fact that the source voltage phasor $V \angle 0^\circ$ itself is the primary voltage phasor V_p to make substitutions in the equation above.

We note that the secondary voltage and the primary voltage are in phase and related by a real scaling factor of $\sqrt{\frac{L_2}{L_1}}$, but this number is nothing but $\frac{n_2}{n_1}$, where n_1 and n_2 are the number of turns employed in primary and secondary windings. This is so since self-inductance of a coil is proportional to the square of number of turns in the coil. We represent this ratio as *turns ratio* ' n '.

Therefore, secondary voltage magnitude in a transformer with unity coupling coefficient is turns ratio times the primary voltage magnitude.

The equation for I_p is the sum of two terms. Each term involves a division of primary voltage phasor by the impedance. Therefore, the equation for I_p suggests that the input impedance of the circuit is a parallel combination of two impedances – an inductor of L_1 and another impedance of value $\frac{L_1}{L_2} Z_L = \frac{Z_L}{n^2}$.

Therefore, a transformer with unity coupling coefficient reflects the secondary load impedance Z_L onto the primary side as a turns-ratio transformed impedance of value Z_L/n^2 , where n = secondary turns/primary turns.

Thus, the entire transformer and load circuit can be replaced by the equivalent circuit shown in Fig. 8.11-9 as far as the input side $v-i$ behaviour under a sinusoidal steady-state condition is concerned. We will see in later chapters that this is true not only for sinusoidal steady-state analysis but also for time-domain analysis.

The primary winding resistance is included as a series resistance in the primary side, whereas, the secondary winding resistance is included as a part of R_L in the load for analysis purposes.

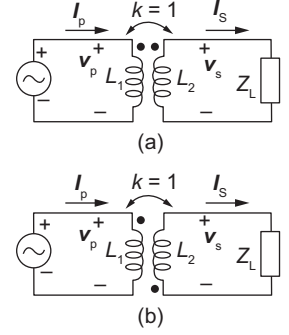


Fig. 8.11-8 A Passively Terminated Two-Winding Transformer with $k = 1$

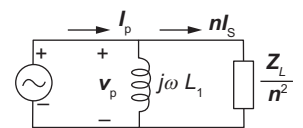


Fig. 8.11-9 Input Equivalent Circuit of Transformer with $k = 1$

Now, we consider the circuit in Fig. 8.11-8(b). The reader may easily verify that the mesh equations for this circuit will be,

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & \mathbf{Z}_L + j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_P \\ \mathbf{I}_S \end{bmatrix} = \begin{bmatrix} V \angle 0^\circ \\ 0 \end{bmatrix}$$

Both the off-diagonal terms in the mesh impedance matrix have their signs changed. The solution for primary and secondary current phasors with $k = 1$ will be

$$\begin{aligned} \mathbf{I}_P &= \frac{V \angle 0}{j\omega L_1} + \left[\frac{L_2}{L_1} \right] \frac{V \angle 0}{\mathbf{Z}_L} = V_P \left[\frac{1}{j\omega L_1} + \frac{L_2}{L_1} \frac{1}{\mathbf{Z}_L} \right] = V_P \left[\frac{1}{j\omega L_1} + \frac{n^2}{\mathbf{Z}_L} \right] \\ \mathbf{I}_S &= -V \angle 0 \left[\sqrt{\frac{L_2}{L_1}} \right] \frac{1}{\mathbf{Z}_L} = -n V_P \frac{1}{\mathbf{Z}_L} \\ \mathbf{V}_S &= \mathbf{I}_S \mathbf{Z}_L = -V_P \left[\sqrt{\frac{L_2}{L_1}} \right] = -n V_P \end{aligned}$$

There is no difference in the solution *except* for a 180° phase shift in the secondary voltage. That is anyway expected due to the change in the relative polarity of windings. The voltage magnitude ratio remains at turns ratio n .

However, the most important aspect to be noted is that the same equivalent circuit shown in Fig. 8.11-9 will describe the transformer in this case too (see the equation for primary current phasor). In fact, this is true even if $k \neq 1$.

The input impedance of a passively terminated transformer is independent of its dot point assignment.

There are many important applications of transformers with $k \approx 1$ in which we make use of the inductor L_1 that appears in the input impedance. A tuned amplifier widely employed in communication circuits is such an application example.

There are many applications of transformers with $k \approx 1$ in Electrical Power Engineering and Electronics Engineering, where the inductance in the input impedance of the transformer is not desirable. So we minimise the effect of that inductance by making L_1 (and consequently L_2 and M too) very large compared to the maximum value of $\mathbf{Z}_L$ that we may use. This is done by using a large cross-section iron core and a large number of turns in both windings keeping the turns ratio at the desired value. An *Ideal Transformer* is an idealised model for such a transformer.

Thus, the equations for an ideal transformer are:

$$\begin{aligned} \frac{V_S}{V_P} &= n; \quad \frac{\mathbf{I}_S}{\mathbf{I}_P} = \frac{1}{n} \\ \therefore V_P \mathbf{I}_P^* &= V_S \mathbf{I}_S^* \\ \mathbf{Z}_{in} &= \frac{\mathbf{Z}_L}{n^2} \end{aligned}$$

Complex power is conserved in an ideal transformer. That is, an ideal transformer absorbs zero active power, zero reactive power and zero complex power. Whatever active power and reactive power goes into the primary winding goes out of the secondary winding, into the load. An ideal transformer simply changes the voltage and current levels at which complex power transfer takes place. Moreover, an ideal transformer is an *impedance transformer*. It scales the secondary load impedance by a scaling factor of $\frac{1}{n^2}$ and presents it at the primary input. In fact, these are the major applications of transformers – efficient voltage/current level translation in Power Engineering and impedance level translation (called *impedance matching*) in Electronics Engineering.

Ideal Transformer

An ideal transformer is a two-winding transformer with perfect coupling ($k = 1$) and infinite self and mutual inductance values ($L_1 \rightarrow \infty, L_2 \rightarrow \infty, M \rightarrow \infty$).

EXAMPLE: 8.11-2

A radio frequency transformer with $k \approx 1$ and $n = 0.5$ is passively terminated by a resistor of 800Ω . It is driven by a current source $i_s(t) = I_m \cos \omega t$ A in parallel with a capacitor of 220 pF at the primary side (Fig. 8.11-10). The primary self-inductance of the transformer is $115 \mu\text{H}$. Derive an expression for the ratio of output voltage phasor to the input current phasor.

Solution

The phasor equivalent circuit of the circuit is shown in Fig. 8.11-11. The transformer is replaced by its equivalent circuit in this diagram by reflecting the 8Ω in the secondary as $800/(0.5)^2 = 3.2 \text{ k}\Omega$ in the primary.

The ratio of output voltage phasor to the input current phasor is just the impedance of the parallel combination of inductor, capacitor and resistor. Using the parameter symbols R , L and C ,

$$\frac{1}{\mathbf{Z}} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{j\omega L + R - \omega^2 LCR}{j\omega RL} = \frac{(1 - \omega^2 LC) + j\omega \frac{L}{R}}{j\omega L}$$

$$\therefore \frac{\mathbf{V}_o}{\mathbf{I}_s} = \mathbf{Z} = \frac{j\omega L}{(1 - \omega^2 LC) + j\omega \frac{L}{R}}$$

Substituting the numerical values for circuit parameters,

$$\frac{\mathbf{V}_o}{\mathbf{I}_s} = \frac{j722.6f}{(1 - f^2) + j0.226f}, \text{ where } f \text{ is the cyclic frequency of input in MHz. The magni-}$$

tude and phase of this ratio function is given by

$$\text{Magnitude of } \mathbf{Z} = \frac{722.6f}{\sqrt{(1 - f^2)^2 + (0.226f)^2}}; f \text{ in MHz}$$

$$\text{Phase of } \mathbf{Z} = \left[\frac{\pi}{2} - \tan^{-1} \frac{0.226f}{1 - f^2} \right]; f \text{ in MHz}$$

$\omega = \frac{1}{\sqrt{LC}}$ is the angular frequency at which the susceptance of the inductor and

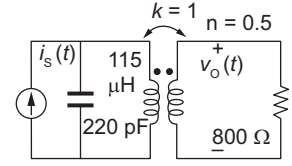
the capacitor will be equal in magnitude, but opposite in sign. Therefore, these two susceptance values will cancel each other at that frequency and leave R as the impedance value. At all other frequencies some susceptance – inductive or capacitive – will shunt R , thereby, bringing down the magnitude of impedance below R . Thus, impedance is a maximum of R at $\omega = \frac{1}{\sqrt{LC}}$. Normalising the impedance function by dividing it by value of R ,

$$\left| \frac{\mathbf{Z}}{R} \right| = \frac{722.6f}{\sqrt{(1 - f^2)^2 + (0.226f)^2}} \text{ and } \angle \text{ of } \frac{\mathbf{Z}}{R} = \theta = \left[\frac{\pi}{2} - \tan^{-1} \frac{0.226f}{1 - f^2} \right]; f \text{ in MHz}$$

Fig. 8.11-12 shows the plots for these functions.

The plots reveal that it is a highly frequency-selective circuit. The circuit produces a large magnitude output if the frequency of the sinusoidal current source is around 1 MHz. The response is small if the frequency is far away from 1 MHz on either side. Such a response characteristic is called a *band-pass characteristic* or *tuned characteristic*. A circuit with band-pass nature can extract some frequency components selectively from a mixture of sinusoidal waveforms at different frequencies presented to it at the input. Such circuits find wide application in communication and signal processing circuits. The circuit we discussed in this example illustrates the principle of *tuned amplifiers* that are used in all sorts of electronic communication circuits starting from the common transistor radio receiver. The current source is realised by a transistor circuit in a tuned amplifier.

Tuned amplifier is an example where the inductance of a transformer is intentionally designed to resonate with a chosen capacitor at a desired frequency.



8.11-10 Circuit for Example 8.11-2

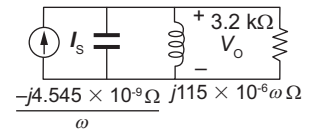


Fig. 8.11-11 Phasor Equivalent Circuit for the Circuit in Fig. 8.11-10

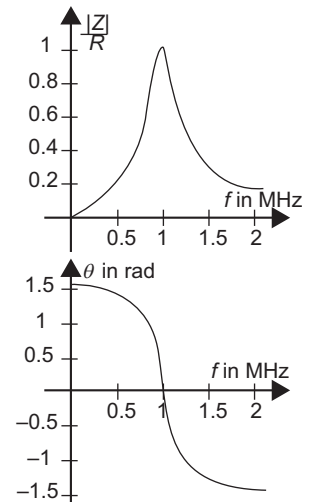


Fig. 8.11-12 Magnitude and Phase Plots of Normalised Impedance Function in Example 8.11-2

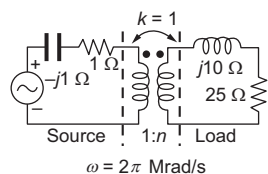


Fig. 8.11-13 Circuit for Impedance Matching Example in Example 8.11-3

EXAMPLE: 8.11-3

The sinusoidal voltage source that is driving a load of impedance $25 + j10 \Omega$ at 1 MHz has a source impedance of $1 - j1 \Omega$ at 1 MHz. An ideal transformer is introduced between the source and the load such that the power transferred to the load is a maximum. Find the turns ratio of the transformer and the value and nature of additional components needed, if any.

Solution

The circuit diagram detailing the application context is shown in Fig. 8.11-13.

The transformer should reflect the load impedance in such a way that the *conjugate impedance* criterion for maximum power transfer is met. Therefore, the 25Ω resistance in the load should appear as 1Ω resistance when reflected to the primary side. This fixes the turns ratio n as $\sqrt{25} = 5$.

However, with $n = 5$, the $j10 \Omega$ inductive reactance in the load will appear as $j0.4 \Omega$ to the primary side. As per *conjugate impedance matching* we want it to appear as $-(-j1) = j1 \Omega$ to the primary side. Turns ratio cannot be adjusted anymore. Therefore, we need to connect an extra reactance in series with the secondary winding such that the reflected total reactance becomes $j1 \Omega$. Therefore, we need an extra inductive reactance of $(j1 - j0.4) \times 5^2 = j15 \Omega$ in series with the secondary winding. The final matched network is shown in Fig. 8.11-14.

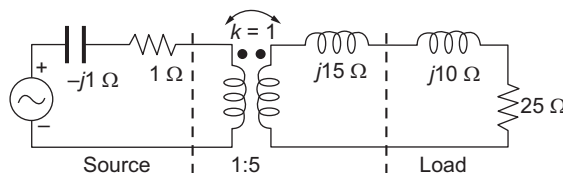


Fig. 8.11-14 Solution for Impedance Matching Problem in Example 8.11-3

8.12 SUMMARY

- *Sinusoidal steady-state* in a dynamic circuit is a state when all the response variables contain only one component with a *sinusoidal waveshape* that has the same frequency as that of the sinusoidal forcing function applied. The response will, in general, have a phase difference with respect to input. This state will be established only after a transient period that follows the application of sources. Sinusoidal steady-state response in linear circuits obeys superposition principle.
- The *real part* of the steady-state response to a complex exponential function $e^{j\omega t}$ = steady-state response to $\cos\omega t$ and the *imaginary part* of the steady-state response to $e^{j\omega t}$ = steady-state response to $\sin\omega t$.
- All element voltage variables and all element current variables in a linear dynamic circuit driven by a complex exponential function $X_m e^{j\omega t}$ will assume the form $(Y_m e^{j\phi}) e^{j\omega t}$ under the steady-state condition where $(Y_m e^{j\phi})$ represents the relevant *complex amplitude* for the variables. Y_m will be proportional to X_m .
- The amplitude of current in an inductor is $1/\omega L$ times the amplitude of voltage and the current *lags* the voltage by 90° under the sinusoidal steady-state condition.
- The amplitude of current in a capacitor is ωC times the amplitude of voltage and the current *leads* the voltage by 90° under the sinusoidal steady-state condition.
- *Phasor* is a complex number that gives the amplitude of the complex exponential function and the phase of the complex exponential function with the time-variation of the function understood as $e^{j\omega t}$. It can be used as a representation for a sinusoidal function.
- The ratio of voltage phasor to current phasor of an element is called its phasor impedance. It is $R \Omega$ for a resistor, $j\omega L \Omega$ for an inductor and $1/j\omega C \Omega$ for a capacitor. A phasor equivalent circuit is constructed for sinusoidal steady-state analysis by replacing all sources with their phasor values and all elements with their phasor impedances.

- The nodal analysis and mesh analysis techniques developed for memoryless circuits apply to the phasor equivalent circuits with no change, except that *impedance* Z takes the place of *resistance* R and *admittance* Y takes the place of *conductance* G . All circuit theorems, except the maximum power transfer theorem, apply to phasor equivalent circuits without modification.
- Maximum average power is transferred to a load circuit from a power delivery circuit under the sinusoidal steady-state when the driving-point impedance $Z_L = R_L + jX_L$ of the load is the *conjugate* of Thevenin's impedance $Z_s = R_s + jX_s$ of the power delivery circuit.
- A diagram depicting a group of coherent (i.e., of same angular frequency) complex exponential signals frozen at their initial position is called a phasor diagram. Angles measured in the counterclockwise direction in a phasor diagram are lead the angles and the angles measured in clockwise direction in a phasor diagram are lag angles.
- Apparent power carried by a sinusoidal voltage of rms value V_{rms} and a sinusoidal current of rms value I_{rms} is defined as the actual power that will be carried by a DC voltage of the same effective value and a DC current of the same effective value – i.e., Apparent Power $= V_{\text{rms}} I_{\text{rms}}$.
- Active Power, $P = V_{\text{rms}} I_{\text{rms}} \cos\theta$, where θ is the angle by which the voltage phasor leads the current phasor and Power Factor $= \cos\theta$ under the sinusoidal steady-state.
- The current phasor can be resolved into active component ($= I_{\text{rms}} \cos\theta$ A rms) and reactive component ($= -I_{\text{rms}} \sin\theta$ A rms) by finding its projection along the voltage phasor and along a perpendicular to the voltage phasor, respectively. The active current component carries the entire active power.
- Reactive Power Q under the sinusoidal steady-state condition is a quantity that stands for the reactive component of current. It is a scaled version of the reactive component of current, the scaling factor being negative of rms value of voltage.
- Complex power S in circuits under the sinusoidal steady-state condition is defined as

$$S(\text{VA}) = VI^* = P(\text{W}) + jQ(\text{Var}) = \sqrt{P^2 + Q^2} \angle(\phi_v - \phi_i) \text{VA}.$$
 Complex power and its components – P and Q – are conserved quantities.
- A system of two coils with constant values of L_1 , L_2 and M , with two pairs of terminals identified for application of excitation and/or measurement of response, is called a *two-winding linear transformer*.
- Secondary voltage magnitude in a transformer with unity coupling coefficient is turns ratio times the primary voltage magnitude. A transformer with unity coupling coefficient *reflects* the secondary load impedance Z_L on to the primary side as a *turns-ratio transformed impedance* of value Z_L/n^2 , where n = secondary turns/primary turns.
- An ideal transformer is a two-winding transformer with perfect coupling ($k = 1$) and infinite self and mutual inductance values ($L_1 \rightarrow \infty$, $L_2 \rightarrow \infty$, $M \rightarrow \infty$). The input impedance of such a transformer is Z_L/n^2 . Complex power is conserved in an ideal transformer. Practical transformers that approach ideal transformers are employed in voltage/current level conversion and impedance level conversion applications.

8.13 QUESTIONS

- Express $v(t) = 5 \cos(\omega t - 45^\circ) - 4 \sin(\omega t + 30^\circ)$ as a single complex exponential function.
- What should be the values of A , ω and ϕ if the signal $v(t) = (2 + j3)e^{j(10t + 0.5\pi)} + Ae^{j(\omega t + \phi)}$ is a real-valued signal?
- The current in a particular element in a linear circuit excited by a sinusoidal voltage source $v_s(t) = 10 \cos(100t - 60^\circ)$ V is found to be $i(t) = 0.75 \sin(100t - 15^\circ)$ A. (i) What is the complex amplitude of current if $v_s(t) = 5e^{j100t}$ V? (ii) Explain why you cannot determine the complex amplitude of current when $v_s(t) = 5e^{j200t}$ V using this data?
- Express the following signals as phasors by carrying out the addition in the phasor domain. Draw the phasor diagrams.
 - $x(t) = 10 \sin 500\pi t + 5 \cos(500\pi t - \pi/5)$
 - $x(t) = 7 \sin(120\pi t - 50^\circ) + 5 \cos(120\pi t + 30^\circ)$
 - $x(t) = 7 \sin(50\pi t) + 2 \cos(50\pi t + 30^\circ) - 3 \sin(50\pi t + \pi/4)$
- Express the following phasors as time-domain signals after carrying out the required additions in the phasor domain.
 - $15\angle -60^\circ + 5e^{j\pi/3}$
 - $20 - j10 + 10\angle -20^\circ$
 - $4e^{-j\pi/12} - 3e^{j\pi/8} + 3\angle -50^\circ$
- Let $v(t) = V_m \cos \omega t$ V and $i(t) = I_m \cos(\omega t - \phi)$ A be the voltage and the current of an element in a linear circuit under the sinusoidal steady-state. If the peak value of power pulsation is found to be twice the value of the average power, what must be the value of ϕ ?
- Express the angular frequency at which a series RLC circuit excited by a sinusoidal voltage source will have the peak

- double-frequency power pulsation equal to the average power in terms of circuit parameters.
8. The current drawn by a linear load circuit from a sinusoidal voltage source $v_s(t) = V_m \cos \omega t$ V is seen to be $I_{m1} \cos(\omega_1 t - 30^\circ)$ when the angular frequency of voltage source is ω_1 and $I_{m2} \cos(\omega_2 t + 45^\circ)$ when it is ω_2 . Explain why there should exist a value of ω between ω_1 and ω_2 such that the circuit will draw power at unity power factor at that angular frequency.
 9. A resistor R_1 in series with an inductor L is connected in parallel to another resistor R_2 that is in series with a capacitor C . (i) What will be the nature of power factor – lag or lead – for very low frequency sinusoidal input voltage? (ii) What will be the nature of power factor – lag or lead – for very high frequency sinusoidal input voltage? (iii) Will there exist a value of angular frequency at which this load will have unity power factor?
 10. The current phasor drawn by a load circuit containing two circuit elements from a voltage of $100 \angle 0^\circ$ V rms is $10 \angle -36^\circ$ A rms. Explain why the load circuit has to be an RL circuit?
 11. The power factor of a load circuit containing three circuit elements is found to be unity at $\omega = 100\pi$ rad/s. Is it necessary that all the three elements should be resistors?
 12. The power factor of a load circuit is found to be 0.7 lag at $\omega = 100\pi$ rad/s. Is it necessary that the power factor is a lag power factor for all values of ω ? If no, give an example circuit to justify the answer.
 13. The ratio of magnitudes of two load impedances is 2:1. If the first has 0.7 lag power factor, what must be the power factor of the second such that a parallel combination of these two loads draws power at unity power factor?
 14. The phasor impedance of a multi-element RLC circuit is $3 + j17 \Omega$ at 100π rad/s. Show that there must be at least one resistor and one inductor in the circuit.
 15. The phasor impedance of a multi-element passive circuit is $-j20 \Omega$ at 100π rad/s. Show that the circuit contains at least one capacitor and that it contains no resistor.
 16. The current drawn by a multi-element passive circuit is found to lag the voltage phasor by 90° . Show that the circuit contains at least one inductor and no resistor.
 17. The power factor of a multi-element passive circuit is found to be 0.3 lead at 200 rad/s. It is found to be 0.4 lag at 400 rad/s. Show that the circuit contains at least one resistor, capacitor and inductor.
 18. The sum of reactive power delivered to 5 elements in a 6-element circuit is -20 kVar. Show that the sixth element is either an independent source or an inductor.
 19. The sum of reactive power delivered to 4 elements in a 5-element circuit is $+10$ kVar and the sum of the active power delivered to the same 4 elements is 20 kW. Show that the fifth element has to be an independent source and find the power factor at which it is *delivering* power.
 20. The sum of reactive power delivered to 4 elements in a 5-element circuit is -10 kVar and the sum of active power delivered to the same 4 elements is 0 kW. Show that the fifth element has to be an inductor or a source.
 21. The sum of complex power delivered to 7 elements in an 8-element circuit is $20 - j10$ VA. Identify the nature of the eighth element.
 22. The sum of complex power delivered to 4 elements in a 6-element circuit is $20 + j10$ VA. Identify the various possibilities for the remaining two elements.
 23. Show that (i) the algebraic sum of reactive components of phasor currents entering a node has to be zero (ii) the algebraic sum of active components of phasor currents entering a node has to be zero if all the current phasors are resolved into active and reactive components with respect to the same voltage phasor.
 24. Show that (i) the algebraic sum of reactive components of phasor voltages in a circuit loop has to be zero (ii) the algebraic sum of active components of phasor voltages in a circuit loop has to be zero if all the voltage phasors are resolved into active and reactive components with respect to the same current phasor.
 25. A circuit needs at least one independent source to deliver active power to resistive elements in the circuit. Is there any such requirement with respect to reactive power in reactive elements? Discuss.
 26. A voltage source $v_s(t) = 325 \cos 100\pi t$ V delivers power to a load through an impedance $0.5 + j4 \Omega$. The load voltage peak value is seen to be 350 V. Argue with the help of a phasor diagram that the load has to be a capacitive one.
 27. What is the input impedance of a perfectly coupled 1:1 two-winding transformer with 0.5Ω primary winding resistance and 0.9Ω secondary winding resistance when the secondary side is shorted?
 28. Show that the input power factor of a perfectly coupled two-winding transformer with zero winding resistance values is 0 lag when the secondary side is open.
 29. The active power input into a perfectly coupled two-winding transformer that has zero winding resistance values is 20 kW. Explain why the load connected in the secondary side can not be purely reactive.
 30. The reactive power input into a perfectly coupled two-winding transformer that has zero winding resistance values is 20 kVar. Will the reactive power delivered by the secondary winding be 20 kVar? Explain.
 31. The complex power input into the primary side of an ideal transformer is $100 + j100$ kVA. Two loads are connected in parallel in the secondary side and the first load consumes a complex power of $25 + j125$ kVA. (i) Find the complex power consumed by the second load. (ii) Find the series equivalent circuit and parallel equivalent circuit for both loads if the secondary voltage is 230 V rms.

8.14 PROBLEMS

1. Find the equivalent impedance and admittance at $\omega = 100\pi$ rad/s for the circuits in Fig. 8.14-1.

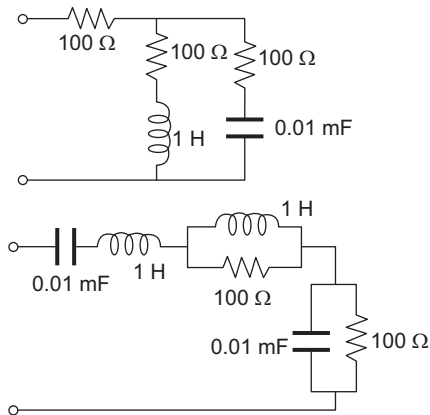


Fig. 8.14-1

2. A resistor of $200\ \Omega$ and an inductor of 0.5 H are in series. (i) Find the impedance of the combination at 60 Hz in rectangular form and polar form. (ii) Find a parallel RL circuit that has the same impedance at the same frequency. (iii) Find the impedance of the parallel equivalent in step (ii) for 50 Hz . Are the impedance values equal for the two circuits at 50 Hz ?
3. A resistor of $200\ \Omega$ and a capacitor of 0.05 mF are in parallel. (i) Find the impedance of the combination at 60 Hz in rectangular form and polar form. (ii) Find a series RC circuit that has the same impedance at the same frequency. (iii) Find the impedance of the series equivalent in step (ii) for 50 Hz . Are the impedance values equal for the two circuits at 50 Hz ?
4. The voltage across an impedance is $v(t) = 100 \sin 120\pi t$ V and the current through it is $i(t) = 5 \sin(120\pi t - 60^\circ)$ A. Find the impedance and obtain its series equivalent and parallel equivalent. Can these equivalents be used at another frequency?
5. Find the (i) current delivered by each source, (ii) the complex power delivered by each source and the power factor of each source, (iii) the voltage of LN with respect to SN, (iv) total active and reactive power delivered to the three load branches and (v) the sum of instantaneous power delivered to the three load branches in the circuit shown in Fig. 8.14-2.

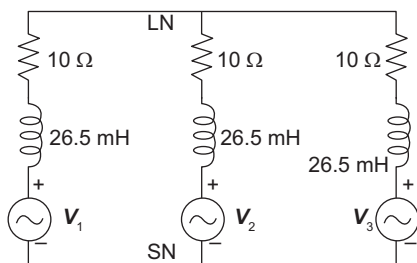


Fig. 8.14-2

$V_1 = 110\angle 0^\circ$ V rms, $V_2 = 110\angle -120^\circ$ V rms and $V_3 = 110\angle 120^\circ$ V rms. The frequency of operation is 60 Hz . Use the mesh analysis procedure. Show the phasor diagram.

6. Repeat Problem 5 by using nodal analysis procedure. The node voltages at nodes between the resistor and inductor are not required. Take SN as the reference node.
7. Find the phasor currents delivered by sources and the current I_n in the circuit in Fig. 8.14-3 by using Superposition Theorem. $V_1 = 110\angle 0^\circ$ V rms, $V_2 = 110\angle -120^\circ$ V rms and $V_3 = 110\angle 120^\circ$ V rms. The frequency of operation is 60 Hz . Draw the phasor diagram.

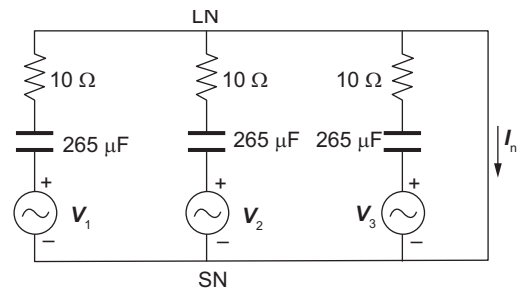


Fig. 8.14-3

8. (a) Find the phasor currents delivered by sources and the current I_n in the circuit in Fig. 8.14-4 by using the Superposition Theorem. $V_1 = 110\angle 0^\circ$ V rms, $V_2 = 110\angle -120^\circ$ V rms and $V_3 = 110\angle 120^\circ$ V rms. (b) Draw the phasor diagram showing all the voltages and currents. The frequency of operation is 60 Hz . Also find (i) the complex power delivered by each source and its power factor and (ii) the sum of instantaneous power delivered to the three load branches in the circuit.

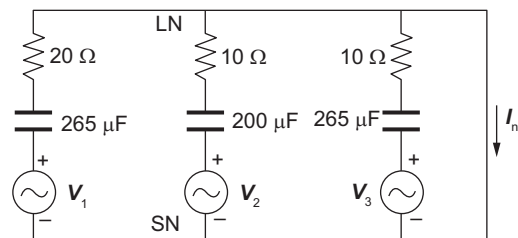


Fig. 8.14-4

9. Find the (i) current delivered by each source, (ii) S delivered by each source and the power factor of each source, (iii) the voltage of LN with respect to SN, (iv) total P and Q delivered to the three loads and (v) the sum of instantaneous power delivered to the three load branches in the circuit shown in Fig. 8.14-5. $V_1 = 230\angle 0^\circ$ V rms, $V_2 = 230\angle -120^\circ$ V rms and $V_3 = 230\angle 120^\circ$ V rms. The frequency of operation is 50 Hz . Use the node analysis procedure. Show the phasor diagram.

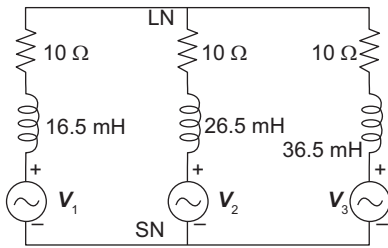


Fig. 8.14-5

10. Find the (i) current delivered by each source, (ii) S delivered by each source and power factor of each source, (iii) the voltage of LN with respect to SN and (iv) the current I_n in the circuit shown in Fig. 8.14-6. $V_1 = 100\angle 0^\circ$ V rms, $V_2 = 120\angle -140^\circ$ V rms and $V_3 = 110\angle 110^\circ$ V rms. The frequency of operation is 60 Hz. Use the node analysis procedure. Show the phasor diagram.

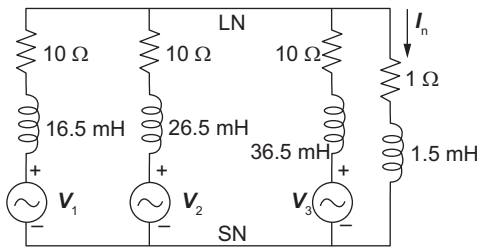


Fig. 8.14-6

11. Find the Thevenin's equivalent of the circuit in Fig. 8.14-6 with respect to terminal pair LN–SN. Thereby, find the current that would flow in a short-circuit put across the SN–LN node pair.
12. (i) What must be the value of R_L in the circuit in Fig. 8.14-7 if maximum power is to be transferred to it? What is the maximum power transferred? (ii) How will the answer differ if an adjustable reactance X_L can be put in series with R_L and independent adjustment of R_L and X_L be done for maximum power transfer is possible?

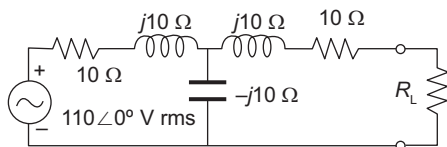


Fig. 8.14-7

13. Find the current in 1 Ω as a function of time by using Source Transformation Theorem and Current Division Principle in

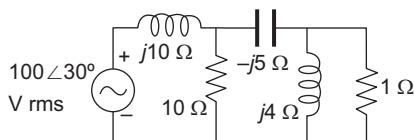


Fig. 8.14-8

the circuit shown in Fig. 8.14-8. The frequency of the source is 60 Hz.

14. Find the Norton's Equivalent with respect to a – b for the circuit in Fig. 8.14-9 for $\omega = 1000$ rad/s.

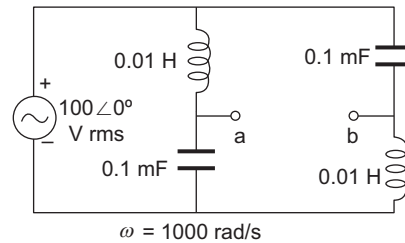


Fig. 8.14-9

15. Two impedances in parallel are supplied by $V = 100\angle 0^\circ$ V rms. The source is seen to deliver a power of 10 kW at 0.8 lag power factor. The power consumed by the first impedance is 4 kW at 0.9 lead power factor. (i) Find the currents in the impedances and impedance values, (ii) power factor and reactive power consumption of second impedance and (iii) source current, source power factor and complex power delivered to the impedances if these two impedances are connected in series across the same voltage source.
16. Find the Thevenin's equivalent of the circuit in Fig. 8.14-10 across a–b if $v_{s1}(t) = 100 \sin 120\pi t$ V and $v_{s2}(t) = 100 \sin(120\pi t - 60^\circ)$ V.

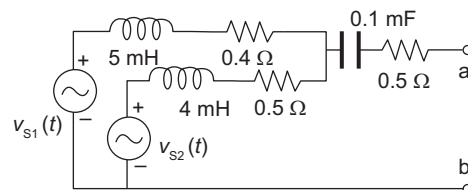


Fig. 8.14-10

17. Two impedances $Z_1 = 6 + j8 \Omega$ and $Z_2 = 8 - j6 \Omega$ are in series and the whole combination is in parallel with a third impedance $Z_3 = 5 + j5 \Omega$. The circuit is driven by a sinusoidal current source $i_s(t) = 10 \cos(120\pi t - 30^\circ)$ A. (i) Solve the circuit by phasor diagram method. (ii) Find the complex power delivered to the three impedances. (iii) Verify the conservation of active power and reactive power in the circuit.
18. The star-delta transformation equation developed for resistive circuits is applicable to impedances in phasor equivalent circuits.

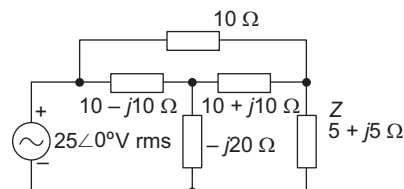


Fig. 8.14-11

Apply star-delta transformation and obtain the complex power delivered to the Z in the circuit in Fig. 8.14-11

19. A 230 V rms, 50 Hz distribution line feeds power to a load located at the far end of the line. The line can be modelled by a series combination of 0.07Ω resistance and 0.3Ω inductive reactance. The load is represented by impedance of $15 + j15 \Omega$. (i) Obtain the load voltage and line current by phasor diagram method. (ii) Calculate the source complex power, source power factor and load complex power and transmission efficiency.
20. A 110 V rms, 60 Hz distribution line feeds power to a load located at the far end of the line. The line can be modelled by a series combination of 0.05Ω resistance and 0.2Ω inductive reactance. The load is represented by impedance of $5 + j10 \Omega$. (i) Obtain the load voltage and line current. (ii) Find the capacitance value of a capacitor that should be connected across the load if the sending end and the receiving end voltage magnitudes are to be 110 V rms. (iii) Calculate the source complex power, source power factor and load complex power and transmission efficiency with and without the capacitor.
21. If the source delivers 1 kW at unity power factor in the circuit in Fig. 8.14-12, determine the impedance Z .

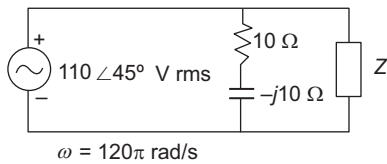


Fig. 8.14-12

22. The current delivered by the source in the circuit in Fig. 8.14-13 is $10 \cos(120\pi t - 15^\circ)$ A under the steady-state condition. The load impedance Z_1 is seen to consume 300 W of active power and the load Z_2 is found to deliver 300 VAR of reactive power. (i) Find the currents delivered to the impedances as functions of time. (ii) The impedance values.

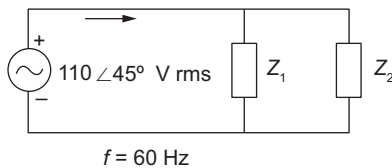


Fig. 8.14-13

23. A motor connected to a 50 Hz, 230 V rms supply line draws 9 A rms and consumes 1.6 kW. (i) Find the apparent power, reactive power and power factor. (ii) Find the capacitance of a parallel capacitor that will make the power factor at the source unity. (iii) Find the source current with this capacitor installed. (Motor is an inductive load)
24. Find the complex power delivered by the current source in the circuit in Fig. 8.14-14.

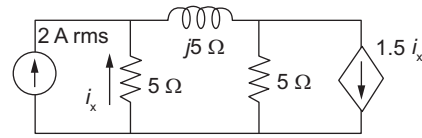


Fig. 8.14-14

25. Find the value of angular frequency such that $v_o(t)$ is out of phase with $v_s(t)$ in the circuit in Fig. 8.14-15 if $v_s(t) = V_m \cos \omega t$ V. Assume ideal opamp model for the Opamp.

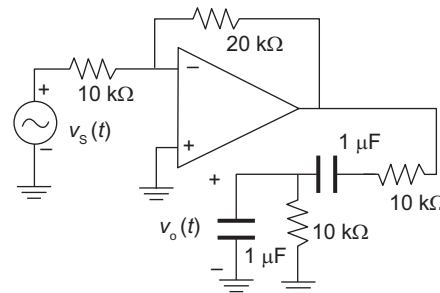


Fig. 8.14-15

26. Using the approximate power flow equations for a synchronous link, find out the complex power delivered by the two sources, currents delivered by the sources and the power factor at which they are operating in the circuit shown in Fig. 8.14-16. Also, find the voltage phasor at Node-C. (Hint: Let the voltage at C be $V \angle \delta$. Then, AC and BC are two synchronous links.)

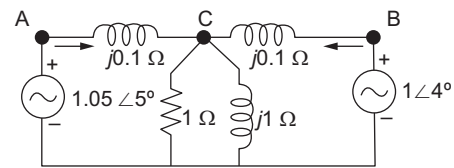


Fig. 8.14-16

27. The current source at C in Fig. 8.14-17 can deliver only Q . Q delivered by it is adjusted such that the voltage at C is 1.0 V rms (i) Find the Q that it has to deliver. (ii) Find the S delivered by the two voltage sources. (iii) Find the angle of voltage phasor at node-C. Use synchronous link equations.

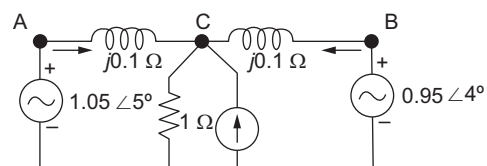


Fig. 8.14-17

28. (i) What is the turns ratio and coupling coefficient of the transformer in the circuit in Fig. 8.14-18? (ii) Find the primary and secondary currents and the voltage across the $10\ \Omega$ resistor as functions of time if the source has 110 V rms value and 60 Hz frequency.

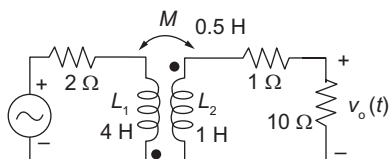


Fig. 8.14-18

29. Find all angular frequency values for which the input impedance of the circuit shown in Fig. 8.14-19 is purely resistive.

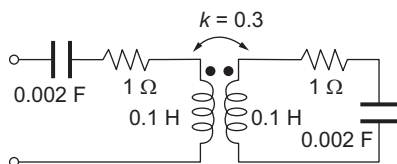


Fig. 8.14-19

30. Find the power dissipated in the resistors and the complex power delivered by the source in the circuit in Fig. 8.14-20 if the voltage source has 110 V rms value at 60 Hz.

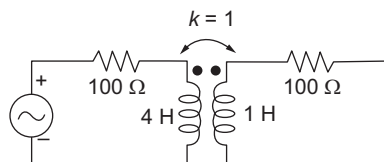


Fig. 8.14-20

31. What is the turns ratio n of the ideal transformer in the circuit in Fig. 8.14-21 if the reactive power delivered by the source (230 V rms at 50 Hz) is zero?

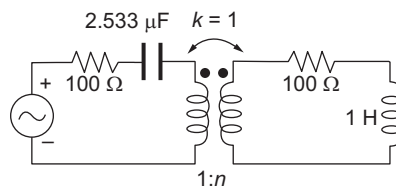


Fig. 8.14-21

32. What should the value of turns ratio n if maximum power is to be transferred to the $8\ \Omega$ load in the circuit in Fig. 8.14-22? If the source has 110 V rms value, what is the amount of power transferred to $8\ \Omega$ with this turns ratio?

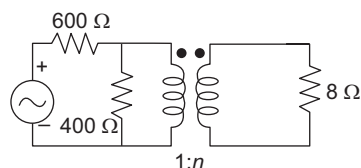


Fig. 8.14-22

33. The sinusoidal voltage source that is driving a load of impedance $16 - j10\ \Omega$ at 1 MHz has source impedance of $1 + j1\ \Omega$ at 1 MHz. An ideal transformer is introduced between the source and the load such that the power transferred to load is a maximum. Find the turns ratio of the transformer and the value and nature of additional components needed, if any.

Sinusoidal Steady-State in Three-Phase Circuits

CHAPTER OBJECTIVES

- To define and explain balanced and unbalanced three-phase systems and to extend the concepts of active power, reactive power and complex power to such systems.
- To develop and employ relations between line and phase quantities in Y-connected and Δ -connected three-phase circuits.
- To emphasise constraints on line quantities in three-phase three-wire and four-wire systems.
- To develop a systematic procedure to analyse sinusoidal steady-state in three-phase balanced circuits using Y-equivalent and single-phase equivalent circuits.
- To show how to use phasors, phasor equivalent circuits and phasor diagrams for solving three-phase circuits under sinusoidal steady-state condition.
- To introduce and illustrate the problem of neutral-shift voltage in four-wire systems.
- To illustrate circuit analysis procedures for unbalanced three-phase circuit analysis through solved examples.
- To introduce and explain symmetrical components in detail and use it in analysis of unbalanced circuits.
- To develop power relations using sequence components.

Basic familiarity with complex algebra is assumed in this chapter too. A detailed study of this chapter and the problems at the end is expected to prepare the reader for higher level courses on Electrical Machines and Electrical Power Systems.

INTRODUCTION

Electrical Power Generation, Transmission and Distribution employ a particular system of three sinusoidal voltages/currents with certain symmetry in voltage/current magnitude and relative phase. This system of three sinusoidal voltage/current waveforms is called a three-phase voltage/current and a circuit employing a three-phase voltage/current system is called a three-phase circuit. A three-phase circuit, like any other circuit, will have to go through a transient period subsequent to the application of source functions before it can settle down to a sinusoidal steady-state condition. We focus on the analysis of three-phase circuits under sinusoidal steady-state condition in this chapter.

No new concepts are required for the analysis of three-phase circuits under sinusoidal steady-state conditions. It requires only an application of all the concepts we developed in the previous two chapters. But before we take up the analysis of three-phase circuits, let us see why Electrical Engineers thought up three-phase system in the first place.

9.1 THREE-PHASE SYSTEM VERSUS SINGLE-PHASE SYSTEM

Industrial heaters are used for a variety of heating applications in various material processing industries. Typically, heater boxes that contain heating elements powered by AC supply transfer the heat developed to a gaseous (usually air) or liquid medium which conducts the heat to the material that is to be heated up. Resistor elements made of some specially constructed wire are laid out uniformly inside the heater box to develop uniform heating of the contact surface. Many such heating elements are finally connected in parallel to form the heater resistive load. Consider such a heater, which after all the parallel connection of sub-elements, is finally represented by three equal resistors of R each. These three resistors may be connected in parallel and supplied from a single-phase AC source or they may be supplied individually from different AC sources – for instance, for redundancy and reliability.

The circuit in Fig. 9.1-1(a) shows the three heater elements tied in parallel and supplied from a single-phase AC supply. The r.m.s value of source is specified as V_p . Let r be the resistance of the connecting cable and $3I$ be the phasor current in the cable. r is usually small compared to R , and hence, we can take the total circuit resistance as $R/3$, ignoring the contribution $2r$. Therefore, circuit current magnitude $\approx 3V_p/R$ A rms. Therefore, the magnitude of I is $\approx V_p/R$ A rms. The total power loss in the cable is $\approx 18r(V_p/R)^2$ W.

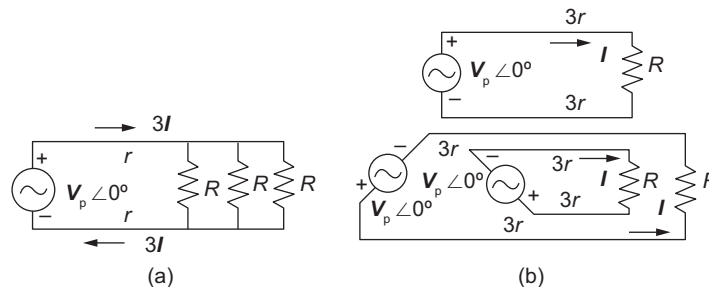


Fig. 9.1-1(a) Three Equal Resistors Powered from a Single-Phase Supply
 (b) Three Equal Resistors Supplied from Three Identical Sources

The circuit in Fig. 9.1-1(b) shows the same three heater elements supplied from the same single-phase supply – but with sources shown as three identical sources. Since the current in the connecting cable is only one-third of the earlier value we will choose a cable that has one-third the area of cross-section that the cable in the circuit in Fig. 9.1-1(a) had. This implies that the cables in the circuit in Fig. 9.1-1(b) have a resistance of $3r$ each. Thus, the total power loss in the circuit in Fig. 9.1-1(b) is $\approx 6 \times 3r \times (V_p/R)^2 = 18r(V_p/R)^2$ W.

Next, we assume that the negative polarity terminals of all the three sources are joined together to form a common node as shown in circuit of Fig. 9.1-2(a). And we imagine that a single cable connects this node to a similarly formed node on the load side. The return currents from the three load-branches flow through this cable. Then, this cable must have three times the cross-sectional area of the other cables in the circuit in Fig. 9.1-2(a). Therefore, its resistance will be r . The power loss in four cables put together will be $18r(V_p/R)^2$ W.

Non-uniform heating will result if the three branches get different powers delivered to them. Therefore, we desire to keep the power dissipated in the three resistors at the same

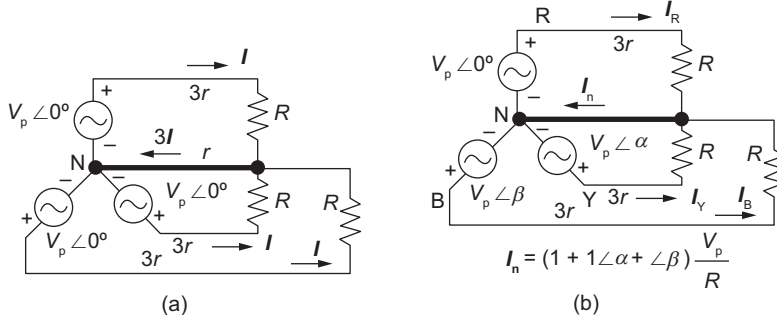


Fig. 9.1-2 From Single-Phase Circuit to Three-Phase Circuit

level. Therefore, the three sources must have the same r.m.s voltage. But that does not mean that they should be in-phase too. We are not particularly worried about the relative phase of currents flowing in the three branches of a heater resistance. But is there any advantage in making the phase angles non-zero?

We assume in the circuit as in Fig. 9.1-2(b) that the phase of the source connected at the terminal identified Y is at an angle α with respect to the source connected at the terminal marked as R. Similarly, the phase of the source connected at the terminal identified B is at an angle β with respect to the source connected at the terminal marked as R. Moreover, we assume that now we use an extra-thick cable between node-N to a similar node on the load side such that it is virtually a resistance-free connection. Now, the currents flowing out from the sources have different phases; but same magnitude of $V_p/(R + 3r)$ A rms. Note that we use 'R' as a terminal marking and 'R' as symbol and value for a resistance. The cable connecting the node-N to a similar node on the load side will have a current I_n in it.

$$I_n = (1 + 1\angle\alpha + 1\angle\beta) \frac{V_p}{R + 3r} \text{ A rms} = [(1 + \cos\alpha + \cos\beta) + j(\sin\alpha + \sin\beta)] \frac{V_p}{R + 3r} \text{ A rms.}$$

Why not we make the magnitude of this current zero so that the cable power loss will become zero in the corresponding cable even if we reduce the size of that cable? We try to find a (α, β) pair such that magnitude of I_n is zero. α and β should be such that $(\sin\alpha + \sin\beta) = 0$ and $(\cos\alpha + \cos\beta) = -1$ simultaneously. The first equation implies that $\beta = -\alpha$ and with this constraint the second equation results in $\alpha = \pm 120^\circ$. Therefore, $(\alpha, \beta) = (-120^\circ, 120^\circ)$ or $(120^\circ, -120^\circ)$ are the two choices for (α, β) pair such that the current $I_n = 0$.

If we make one of these two choices, the total power delivered to the resistors remains the same; but the total cable loss becomes $\approx 9r(V_p/R)^2$ W i.e., only 50% of cable loss in the case of single-phase supply.

If a cable (or any resistance for that matter) carries only zero current in a circuit, then, that cable can be removed without affecting the behaviour of any circuit variable in the circuit. This implies that we can remove the cable connecting the common point N of the sources to the common point in the load without affecting the load currents or load power. This is possible since the instantaneous currents in the three lines add up to zero at all t leaving only zero current to flow in the common point link. Let us verify this with $\alpha = -120^\circ$ and $\beta = 120^\circ$.

$$i_R(t) = \frac{\sqrt{2}V_p}{R + 3r} \cos\omega t \text{ A}$$

$$i_Y(t) = \frac{\sqrt{2}V_p}{R + 3r} \cos(\omega t - 120^\circ) \text{ A}$$

$$\begin{aligned}
 i_B(t) &= \frac{\sqrt{2}V_p}{R+3r} \cos(\omega t + 120^\circ) \text{ A} \\
 \therefore i_R(t) + i_Y(t) + i_B(t) &= \frac{\sqrt{2}V_p}{R+3r} [\cos \omega t + \cos(\omega t - 120^\circ) + \cos(\omega t + 120^\circ)] \\
 &= \frac{\sqrt{2}V_p}{R+3r} [\cos \omega t + \{-0.5 \cos \omega t + 0.866 \sin \omega t\} + \{-0.5 \cos \omega t - 0.866 \sin \omega t\}] = 0 \\
 &\text{for all } t.
 \end{aligned}$$

Hence, no wire is needed between the common point of sources and common point of load. Therefore, we need only three wires of $3r \Omega$ each in the circuit in Fig. 9.1-2(b) instead of two wires of $r \Omega$ each as in the circuit in Fig. 9.1-1(a). Cross-sectional area of wires in the circuit in Fig. 9.1-1(a) will be three times that of the wires in the circuit in Fig. 9.1-2(b). Therefore, the amount of copper (volume or weight) we use in wiring up the circuit in Fig. 9.1-2(b) is only 50% of copper that we use in wiring up the circuit in Fig. 9.1-1(a).

A 'balanced three-phase quantity' defined.

A set of three sinusoidal quantities, all at the same frequency, with equal peak (and hence equal r.m.s) values and shifted successively by 120° in phase is defined as a Balanced Three-Phase Quantity. Therefore, if $x_1(t)$, $x_2(t)$ and $x_3(t)$ is a three-phase set, then,

$$\begin{aligned}
 x_1(t) &= X_m \cos \omega t, x_2(t) = X_m \cos(\omega t - 120^\circ) \text{ and } x_3(t) = X_m \cos(\omega t + 120^\circ) \text{ or} \\
 x_1(t) &= X_m \cos \omega t, x_2(t) = X_m \cos(\omega t + 120^\circ) \text{ and } x_3(t) = X_m \cos(\omega t - 120^\circ) \\
 \text{and } x_1(t) + x_2(t) + x_3(t) &= 0 \text{ for all } t.
 \end{aligned}$$

The sub-elements of an industrial heater designed for three-phase operations are sometimes arranged in such a way that adjacent sub-elements belong to different phases. That is, if three sub-elements are adjacent, the first one will be a part of R that gets connected to R-phase, the second one will be a part of R that gets connected to Y-phase and the third one will be a part of R that gets connected to B-phase. Then, the cycle repeats. This is, obviously, done on purpose. We discuss the sum of instantaneous power delivered by the three phases of a three-phase system under balanced operation to appreciate the purpose involved.

Let us assume that the impedance per branch (i.e., phase) is $Z = Z \angle \theta$ and that Y-phase voltage lags R-phase by 120° and B-phase voltage lags Y-phase voltage by 120° . Then, the instantaneous currents in three lines can be expressed as:

$$i_R(t) = I_m \cos(\omega t - \theta), i_Y(t) = I_m \cos(\omega t - 120^\circ - \theta) \text{ and } i_B(t) = I_m \cos(\omega t + 120^\circ - \theta),$$

where $I_m = V_m/Z$.

$$\text{Also, } v_{RN}(t) = V_m \cos(\omega t), v_{YN}(t) = V_m \cos(\omega t - 120^\circ) \text{ and } v_{BN}(t) = V_m \cos(\omega t + 120^\circ),$$

$$\text{where } V_m = \sqrt{2}V_p.$$

Now, the instantaneous powers delivered by the sources in each phase is expressed as:

$$\begin{aligned}
 p_R(t) &= V_m I_m \cos(\omega t) \cos(\omega t - \theta) \\
 &= 0.5 V_m I_m [\cos \theta + \cos \theta \cos 2\omega t] + 0.5 V_m I_m [\sin \theta \sin 2\omega t] \text{ W.}
 \end{aligned}$$

$$\begin{aligned}
 p_Y(t) &= V_m I_m \cos(\omega t - 120^\circ) \cos(\omega t - 120^\circ - \theta) \\
 &= 0.5 V_m I_m [\cos \theta + \cos \theta \cos(2\omega t - 240^\circ)] + 0.5 V_m I_m [\sin \theta \sin(2\omega t - 240^\circ)] \text{ W.}
 \end{aligned}$$

$$\begin{aligned}
 p_B(t) &= V_m I_m \cos(\omega t + 120^\circ) \cos(\omega t + 120^\circ - \theta) \\
 &= 0.5 V_m I_m [\cos \theta + \cos \theta \cos(2\omega t + 240^\circ)] + 0.5 V_m I_m [\sin \theta \sin(2\omega t + 240^\circ)] \text{ W.}
 \end{aligned}$$

The sum of instantaneous powers can now be expressed as:

$$\begin{aligned}
 p_R(t) + p_Y(t) + p_B(t) &= 0.5 V_m I_m [3 \cos \theta \\
 &\quad + \cos \theta \{\cos 2\omega t + \cos(2\omega t - 240^\circ) + \cos(2\omega t + 240^\circ)\} \\
 &\quad + \sin \theta \{\sin 2\omega t + \sin(2\omega t - 240^\circ) + \sin(2\omega t + 240^\circ)\}] \text{ W.}
 \end{aligned}$$

Each limb or branch of a three-phase system (source or load) is termed as a phase. R, Y and B are used to designate the line terminals of a three-phase source or load.

If the peak values are unequal and/or successive phase shifts are different from 120° , the set will be called an *Unbalanced Three-Phase Quantity*.

We have seen in the discussion in the preceding paragraphs that if the load is also balanced, i.e., if the three branches of load have same impedance, it is possible to effect considerable savings in power loss incurred in transmitting power to the load while effecting a savings on the amount of copper used at the same time.

A balanced three-phase system requires only 50% of copper and incurs only 50% of power loss to transmit a given amount of power to a load compared to a single-phase system.

A phase angle of -240° is the same as $+120^\circ$ and a phase angle of $+240^\circ$ phase is the same as -120° . Therefore, the three terms in the sums within curly brackets in the preceding equation form balanced three-phase sets. Sum of a three-phase balanced set is zero for all time instants. Therefore,

$$p_R(t) + p_Y(t) + p_B(t) = 1.5V_m I_m \cos\theta = 3 \frac{V_p^2}{Z} \cos\theta \text{ W.}$$

Thus, the total power delivered to the three-phase load is seen to be a constant.

If the power dissipated in the three branches of a balanced three-phase load acts on the same physical system, then, a balanced three-phase source delivers active power to the load with zero power pulsation. The double-frequency power pulsation in the case of a single-phase AC supply can never be reduced to zero.

Some instances in which the total active power from three branches of a three-phase load acts on the same physical system are industrial heaters and three-phase motors. In a three-phase heater, power dissipated in three branches will heat up the same body. In a three-phase motor, the torque developed by the branch-windings acts on the same mechanical shaft. This leads to steady heating and steady temperature in a heater and speed without fluctuations in the case of a motor.

The next point of superiority of three-phase system over single-phase system has to be accepted as a statement of fact. It concerns matters beyond the scope of a book on circuits. It is a fact that Electrical Machines and Power Systems Components are more efficiently designed in three-phase version rather than in single-phase version in all senses of the word ‘efficiency’ – efficiency in material utilisation, efficiency in power utilisation, etc.

Advantages of Three-phase System

A balanced three-phase system comprising balanced three-phase source and balanced three-phase load is superior to a single-phase system due to the following facts:

- Power loss in transmission system is lower in three-phase system.
- Copper utilisation is superior in three-phase system.
- Power delivered to a balanced three-phase load by a balanced three-phase supply is free of pulsation.
- Electrical equipment designed for three-phase operation is more efficient than their single-phase counterparts.

9.2 THREE-PHASE SOURCES AND THREE-PHASE POWER

Three-phase sources and loads can take two forms – star connected or delta connected. The star connection is also referred to as Y-connection. The delta connection is referred to as Δ -connection or mesh connection.

9.2.1 The Y-connected Source

Two options to construct a three-phase Y-connected source are shown in Fig. 9.2-1.

In Fig. 9.2-1(a), the terminal identified by Y has a voltage phasor that lags the voltage phasor of the terminal identified as R by 120° . And the terminal identified as B has a voltage phasor that leads the R-line by 120° – i.e., B-line voltage lags R-line voltage by 240° . Thus, similarly located points on the waveform will appear first in R-line, then in Y-line after one-third of a cycle period and finally in B-line after another one-third of a cycle period. Waveforms can be seen in Fig. 9.2-1(a). Thus, the sequence of appearance of peaks will be ...-RYB-RYB-... A three-phase source that has this sequence of appearance of similarly located waveform points is called a *positive sequence three-phase source*. Note that the sequence can also be written as ...-YBR-YBR-... or BRY-BRY-...

The source in Fig. 9.2-1(b) illustrates a *negative phase sequence source*. In this case the voltage of B-line lags that of R-line by 120° and voltage of Y-line lags that of R-line by 240° . Thus, the order of appearance of similarly located waveform points in time will be ...-RBY-RBY-... (equivalently, ...-BYR-BYR-... or ...-YRB-YRB-...) in this case.

There are loads that *cannot* distinguish between the two sequences – e.g., three-phase industrial heater. There are loads that *can* distinguish between the sequences – e.g., three-phase induction motor. If it rotates in clockwise direction for positive sequence applied voltage, it will rotate in counter-clockwise direction for negative sequence applied voltage;

Phase sequence.

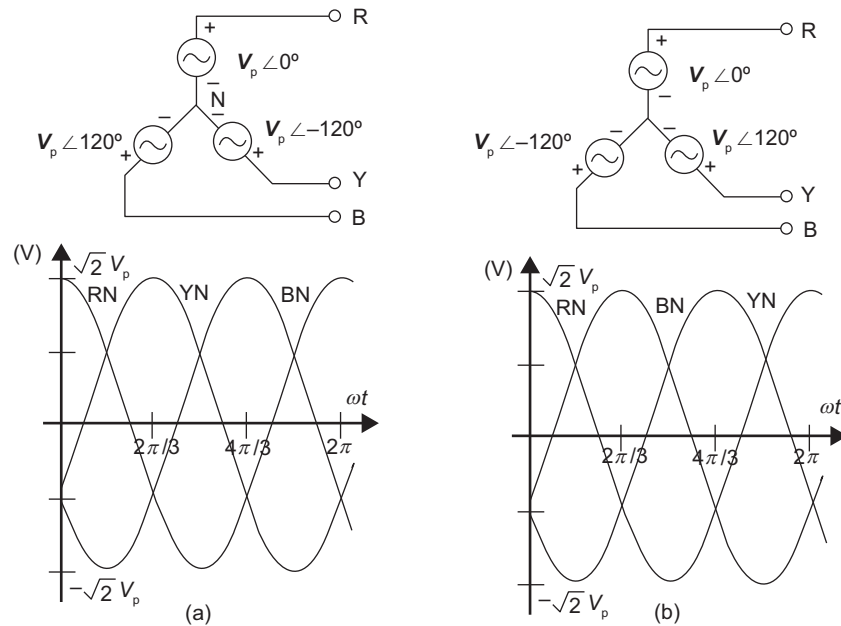


Fig. 9.2-1 (a) A Positive Sequence Y-connected Three-Phase Source
(b) A Negative Sequence Y-connected Three-Phase Source

so will a three-phase synchronous motor. The three-phase sources we consider from now on will be positive sequence sources by default.

The voltage V_p used in Fig. 9.2-1 is an r.m.s value. We will specify three-phase source in r.m.s values in this chapter.

The Y-connected source has a common point at which the three-phase sources join. This point is called the ‘neutral’ point of the source. *Phases* in the context of three-phase source or load refer to the three branches that constitute the three-phase system. *Lines* refer to the three terminals – R, Y and B – that come out of the three-phase source or load and are available for external connections. Thus, in the case of Y-connected source, the *phase-source* is connected between a *line* and the *neutral*. Therefore, the *phase-source voltage* is same as *line-to-neutral voltage* in the case of Y-connected source. As a matter of common practice, *phase-source voltage* is shortened to *phase voltage*. Beginners often get confused between *phase voltage* (i.e., phase-source voltage or phase-load voltage) and *line-to-neutral voltage* and conclude that they are the same. But they are not the same always. *Phase voltage* is the voltage across phase-source or phase-load. Phase voltages exist for any three-phase source or load. Line-to-neutral voltage will exist only if there is a neutral point – i.e., only for Y-connected source or load.

Though line-to-neutral voltage does not exist unless it is Y-connected system, *line-to-line voltages* exist for all the three-phase systems. V_{RY} is the phasor voltage of the R-line with respect to the Y-line. V_{YB} and V_{BR} are similarly defined. *Line-to-line voltage* is usually shortened to *line voltage*.

Similarly, *phase current* is the *phase-source current* or *phase-load current* flowing in one of the branches that constitute the three-phase source or load as per passive sign convention. *Line current* is the current that flows out of or into one of the three terminals R, Y or B. Line current is usually taken to be *flowing out of a source line terminal* and *flowing into a load line terminal*.

Figure 9.2-2 shows all the phase and line quantities in a Y-connected source. V_{RN} , V_{YN} and V_{BN} are the *phase voltages* and I_{RN} , I_{YN} and I_{BN} are the *phase currents*. There is a neutral

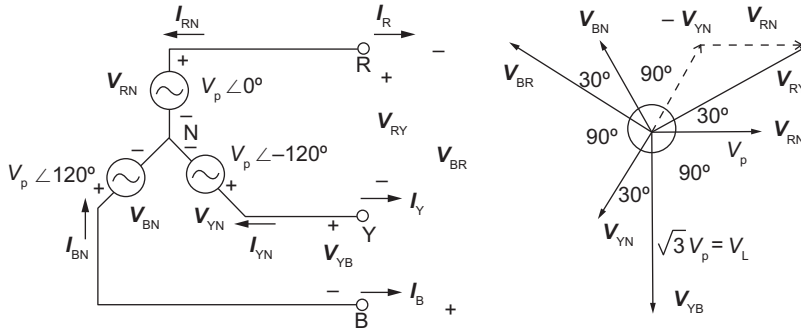


Fig. 9.2-2 A Y-connected Source with all Phase and Line Quantities Identified

The line voltages in a balanced Y-connected source form a balanced three-phase set of voltages that are $\sqrt{3}$ times in magnitude and 30° ahead in phase with respect to phase voltages.

point in a Y-connected source. Hence, *line-to-neutral voltages* exist and they are the same as *phase voltages*. *Line currents* are I_R , I_Y and I_B and are assumed to flow out of the source towards the load. In a Y-connected source, the *line currents* defined in this manner are equal to negative of *phase currents* defined as per passive sign convention.

Since the R-line is same as positive terminal of the source that feeds the R-line in a Y-connected source, this source may be referred to as the R-phase source. Similarly for other two sources also.

We start at the Y-line terminal and traverse through the two sources, reach R-line terminal and move to Y-terminal by falling through V_{RY} . The KVL equation we get is $V_{RY} = V_{RN} - V_{YN}$. Refer to the phasor diagram shown in Fig. 9.2-2. V_{RN} is taken as reference and a horizontal line of length V_p (the r.m.s value of phase-voltages) to suitable scale is drawn to represent it. The remaining two phase voltages $-V_{YN}$ and V_{BN} are at -120° and -240° positions as shown. Now, the equation $V_{RY} = V_{RN} - V_{YN}$ is implemented by rotating V_{YN} by 180° , moving a copy of V_{RN} to the tip of $-V_{YN}$ and completing the triangle. The result is the first *line voltage* V_{RY} .

We observe by projecting the lengths of V_{RN} and $-V_{YN}$ on to V_{RY} that magnitude of V_{RY} is $2 \cos 30^\circ V_p = \sqrt{3} V_p$. We represent the magnitude of line voltage by V_L . Hence, $V_L = \sqrt{3} V_p$. We observe further that the first line voltage, V_{RY} , *leads* the first phase voltage V_{RN} by 30° .

The remaining two line voltages are also obtained by using the equations $V_{YB} = V_{YN} - V_{BN}$ and $V_{BR} = V_{BN} - V_{RN}$. Obviously, the three line voltages form a three-phase balanced set of voltages.

The line currents in Y-connected source need not be co-phasal with the phase voltages. That depends on the load. Assuming that the load is a balanced one, the line currents themselves will form a three-phase balanced set of phasors that are displaced in phase by a phase-lag angle θ with respect to respective phase voltages.

Thus, $I_R = I_L \angle -\theta$, $I_Y = I_L \angle -\theta - 120^\circ$ and $I_B = I_L \angle -\theta + 120^\circ$, where I_L is the r.m.s value of line current. These current phasors are shown in relation to phase voltage phasors in Fig. 9.2-3(b). Also shown are the phase positions of line voltages with respect to phase voltages in Fig. 9.2-3(a). We make the following observations from these phasor diagrams.

1. The line current lags the phase-source voltage by θ and it lags behind the corresponding line voltage by $(30^\circ + \theta)$.
2. Each source contributes an active power of $V_p I_L \cos \theta$ W. Hence, the total three-phase active power delivered by the three-phase source $= 3 V_p I_L \cos \theta = \sqrt{3} V_L I_L \cos \theta$ W, where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RN}$ in the case of R-line) *lags* behind the phase voltage.

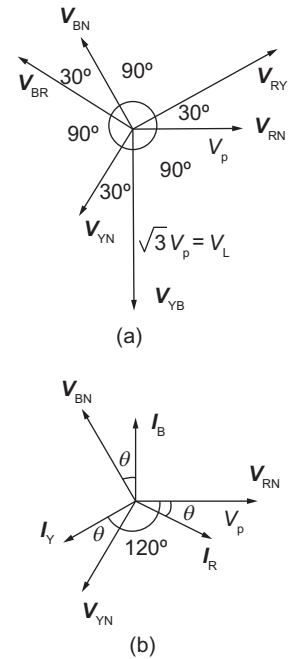


Fig. 9.2-3 (a) Phasor Diagram of Phase and Line Voltages of a Y-connected Source (ii) Phasor Diagram of Line Currents and Phase Voltages in a Y-connected Source

- Each source contributes a reactive power of $V_p I_L \sin \theta$ W. Hence, the total three-phase reactive power delivered by the three-phase source $= 3 V_p I_L \sin \theta = \sqrt{3} V_L I_L \sin \theta$ VAr, where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RN}$ in the case of R-line) lags behind the phase voltage. Equivalently, θ is the angle by which the current delivered by a phase-source lags behind the phase voltage.
- The complex power delivered by the three-phase source $S = \sqrt{3} V_L I_L \angle \theta$ VA.

9.2.2 The Δ -connected Source

The generators in a power system are usually Y-connected. However, power systems use Y- Δ transformers and a Δ -connected three-phase source can be used to model the secondary side of such a transformer. A Δ -connected three-phase source and the phasor diagram of its voltages and currents are shown in Fig. 9.2-4.

We assume that the load on the source is balanced. V_{RY} is taken as the reference phasor for phasor diagrams.

We note that *phase voltage* and *line voltage* are the same for this source and that there is no neutral point and hence no *line-to-neutral voltage* can be defined for this source. Applying KCL at the three corners of delta, we get the following equations.

$$I_{BR} - I_{RY} = I_R, \quad I_{RY} - I_{YB} = I_Y \quad \text{and} \quad I_{YB} - I_{BR} = I_B.$$

This system of equations has no unique solution since we can add a constant to I_{RY} , I_{YB} and I_{BR} without affecting these equations. Such a constant added to the three currents will represent a circulating current within the Δ -loop. Such a circulating current is possible only because the three ideal sources are connected in a loop. However, in practice, all the sources in the delta will have some small impedance which will force the condition that $(I_{RY} + I_{YB} + I_{BR})Z = 0$, where Z is the small source impedance of the sources. Assuming that the circulating current component in delta is zero, let the current $-I_{RY}$ be $I_p \angle -\theta$. Then, the remaining two phase currents will be $-I_{YB} = I_p \angle -\theta - 120^\circ$ and $-I_{BR} = I_p \angle -\theta + 120^\circ$, where I_p is the magnitude of phase currents.

Now, I_R can be constructed by $I_R = -I_{RY} - (-I_{BR})$. This is shown in the phasor diagram in Fig. 9.2-4. The resulting $I_R = \sqrt{3} I_p \angle (30^\circ + \theta)$ A rms.

Hence, the line current magnitude is $I_L = \sqrt{3} I_p$ in a Δ -connected three-phase source.

The line currents delivered by a balanced Δ -connected source form a balanced three-phase set of currents that are $\sqrt{3}$ times in magnitude and 30° behind in phase with respect to phase currents delivered by the phase-sources.

We make the following observations from these phasor diagrams.

- The phase current lags the phase-source voltage by θ and line current lags behind the corresponding line voltage by $(30^\circ + \theta)$.
- Each source contributes an active power of $V_L I_p \cos \theta$ W. Hence, the total three-phase active power delivered by the three-phase source $= 3 V_L I_p \cos \theta = \sqrt{3} V_L I_L \cos \theta$ W, where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RY}$ in the case of RY-phase source) lags behind the phase voltage.
- Each source contributes a reactive power of $V_L I_p \sin \theta$ W. Hence, the total three-phase reactive power delivered by the three-phase source $= 3 V_L I_p \sin \theta = \sqrt{3} V_L I_L \sin \theta$ VAr, where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RY}$ in the case of RY-phase source) lags behind the phase voltage. Equivalently, θ is the angle by which the current delivered by a phase-source lags behind the phase voltage.
- Therefore, the complex power delivered by the three-phase source $S = \sqrt{3} V_L I_L \angle \theta$ VA.

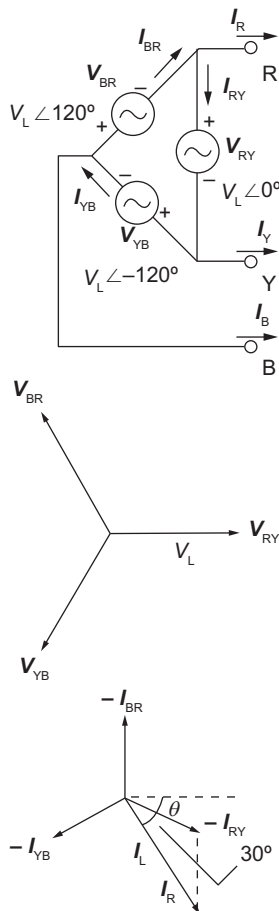


Fig. 9.2-4 A Δ -connected Source and its Phasor Diagrams

Therefore, we state the following conclusion.

The three-phase complex power delivered by a three-phase source is given by

$$S = \sqrt{3}V_L I_L \angle \theta = \left[\sqrt{3}V_L I_L \cos \theta + j\sqrt{3}V_L I_L \sin \theta \right] \text{ VA},$$

where θ is the angle by which the current phasor delivered by a phase-source lags behind its voltage phasor. This relationship is independent of whether the source is Y-connected or Δ -connected. It should be noted that θ is **not** the angle between line voltage phasor and line current phasor.

Electrical Power Engineering adopts a convention to specify a three-phase source by specifying its *line-to-line voltage in r.m.s value*. We follow this convention unless stated otherwise in the remaining portions of this chapter. Thus a 400 V, 50 Hz source means a source that has a line-to-line voltage magnitude of 400 V rms at 50 Hz. If it is Y-connected its phase voltage will be $400/\sqrt{3}$ V rms and peak of its phase voltage will be $400\sqrt{2}/\sqrt{3}$ V.

Three-phase complex power

$$S = \sqrt{3}V_L I_L \cos \theta \text{ VA.}$$

A three-phase source voltage is specified in line-line r.m.s value.

9.3 ANALYSIS OF BALANCED THREE-PHASE CIRCUITS

Balanced three-phase circuits comprising a balanced three-phase source and a balanced three-phase load can be of four varieties, viz., Y-Y, Y- Δ , Δ -Y or Δ - Δ . We develop a common analysis procedure for all the possible configurations.

9.3.1 Equivalence Between a Y-connected Source and a Δ -connected Source

Let $V_{RY} = V_L \angle 0^\circ$, $V_{YB} = V_L \angle -120^\circ$ and $V_{BR} = V_L \angle 120^\circ$ be the three line voltages observed in a three-phase circuit and let $I_R = I_L \angle -(\theta + 30^\circ)$, $I_Y = I_L \angle -(\theta + 150^\circ)$ and $I_B = I_L \angle -(\theta - 90^\circ)$ be the observed line currents where the observed phase lag of first line current is expressed as 30° plus some angle designated by θ . If the source is within a black box and we are to guess whether it is a Y-connected source or a Δ -connected source, we find that we are not able to resolve the matter using the observed line voltage phasors and line current phasors. This is so because both configurations shown in Fig. 9.3-1 will result in these observed line voltages and line currents.

A balanced Δ -connected three-phase source can be replaced by an equivalent Y-connected source for purposes of analysis.

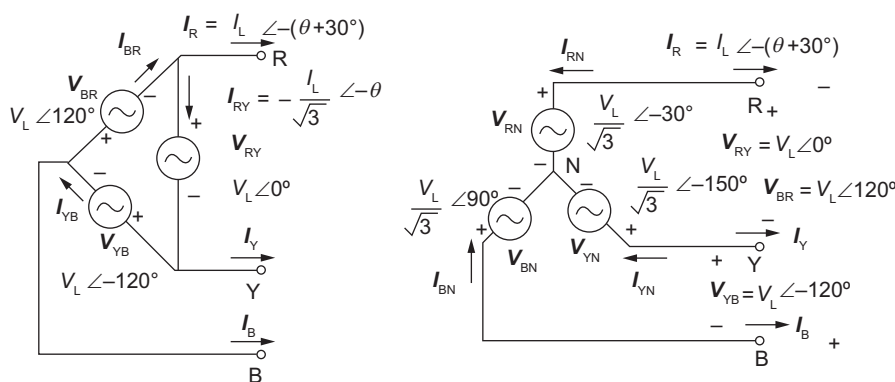


Fig. 9.3-1 Illustrating Equivalence Between a Y-connected Source and a Δ -connected Source

A three-phase balanced Δ -connected impedance can be converted into balanced Y-connected impedance without affecting the terminal voltage and current behaviour.

Hence, we conclude that, for every set of line voltages and line currents observed in a three-phase circuit there exists a Δ -connected source and a star-connected source which will justify the observed line quantities and which are indistinguishable from outside.

Therefore, if a balanced three-phase source is actually Δ -connected, it may be replaced by an equivalent Y-connected source for purposes of analysis. The details of phase-source currents in the actual Δ -connected source can be obtained after the circuit problem is solved using the star equivalent source.

9.3.2 Equivalence Between a Y-connected Load and a Δ -connected Load

But there is nothing new in this equivalence – we already know that any Y-connected set of impedances can be transformed into a Δ -connected impedance and vice-versa. Therefore, any given three-phase balanced Δ -connected impedance can be converted into a three-phase balanced Y-connected impedance such that the terminal voltage and current behaviour remain unaffected.

The individual branch currents in the branches of delta in the actual Δ -connected load can be obtained after line quantities have been obtained by using its equivalent Y-connected model. Three-phase symmetry and $1/\sqrt{3}$ factor connecting line currents and phase currents in a Δ -connected system can be used for this purpose. The impedance to be used in Y-connected load is $1/3$ times the impedance present in Δ -connected load (Fig. 9.3-2).

The Neutral Point?

The difference between a real Y-connected source/load and Y-connected equivalent of Δ -connected source/load is that the neutral point of the first is accessible for connection whereas the neutral point of the second is an inaccessible node.

However, the neutral connection in a balanced three-phase circuit does not carry any current (even if the neutral link has non-zero impedance) and hence the voltage difference between source neutral and load neutral will remain zero whether we connect the neutral points together or not.

Therefore, the circuit solution in a balanced Y-connected circuit is independent of neutral connection.

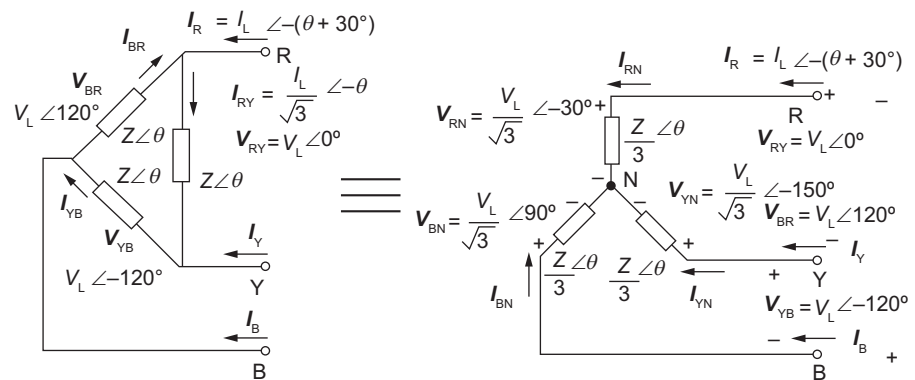


Fig. 9.3-2 Equivalent Y-connected Load for a Δ -connected Load

9.3.3 The Single-Phase Equivalent Circuit for a Balanced Three-Phase Circuit

Any balanced three-phase circuit can be equivalenced to a Y–Y balanced three-phase circuit by replacing Δ -connected sources and loads (if any) by their Y-connected equivalents developed in Sects. 9.3.1–9.3.2. The equivalence is valid as far as the line voltages, line currents and complex power flow are concerned (see side-note).

Thus, the analysis of balanced three-phase circuits reduces to a three-step procedure. The Δ -connected sources and loads are replaced by their Y-connected equivalents in the first step. The resulting balanced Y-connected circuit is solved for line voltages and line currents everywhere in the second step. The relevant line voltage values and line current values are used to determine the phase-source voltages and currents in the case of those

Δ -connected sources that were replaced by star equivalents earlier and phase-load voltages and currents in the case of those Δ -connected loads that were replaced by star equivalents earlier. This will constitute the third step.

If all the neutral points in a balanced three-phase circuit are at the same potential, then, each phase-source appears directly across the net impedance connected from the respective source line terminal (R or Y or B) to the load neutral point. Thus, the circuit may be solved phase by phase, independent of each other. But then, why should we solve all the three phases at all? We expect the circuit solution to be a balanced set of three-phase phasors in any case. Hence, we can determine the solution for Y-line and B-line if we know the solution for phasors in R-line by using three-phase symmetry. Therefore, we need to determine only the R-line phasors. The equivalent circuit that we employ to solve for R-line phasors is called the *single-phase equivalent circuit for a balanced three-phase circuit*. This equivalent circuit will contain the R-line phase-source in series with all impedances that come between R-line terminal of the source and neutral point of load.

The neutral points of source and load are shorted to form the reference node in a single-phase equivalent circuit even if the neutral points are connected through impedance in the actual circuit. This is so because the neutral connection in a balanced circuit will not carry any current.

R-line or R-phase?

Note that if a three-phase circuit is star connected everywhere, the use of 'R-phase' instead of 'R-line' does not create any confusion.

Such confusion can arise in the case of Δ -connection. A Δ -connected source has only R-line and no R-phase. But, we have now decided that all our circuits will be converted to star equivalents. Thus, use of 'R-phase' instead of R-line is permitted.

EXAMPLE: 9.3-1

A balanced Y-connected load with phase impedance of $7.9 + j5.7 \, \Omega$ at 50 Hz is supplied from a 400 V, 50 Hz balanced Y-connected source through connection impedance of $0.1 + j0.3 \, \Omega$ in each line. Find (i) the line current and load voltage and (ii) the active and reactive power delivered by the source and delivered to the load.

SOLUTION

This is a Y-Y connection. The single-phase equivalent circuit required for solving the problem is shown in Fig. 9.3-3.

A three-phase source is specified by specifying line-to-line voltage r.m.s value. Thus, the source voltage in single-phase equivalent must be $400/\sqrt{3} = 230.9 \, \text{V rms}$ in magnitude. The R-phase source voltage is taken as the reference phasor with an angle of 0° . Solving the circuit in Fig. 9.3-3, we get,

$$I_R = \frac{230.9 \angle 0^\circ}{(0.1 + j0.3) + (7.9 + j5.7)} = \frac{230.9 \angle 0^\circ}{8 + j6} = \frac{230.9 \angle 0^\circ}{10 \angle 36.9^\circ} = 23.09 \angle -36.9^\circ \, \text{A rms.}$$

$$V_L = 23.09 \angle -36.9^\circ \times (7.9 + j5.7) = 23.09 \angle -36.9^\circ \times 9.74 \angle 35.8^\circ = 224.9 \angle -1.1^\circ \, \text{V rms.}$$

Active power delivered by source =

$$3 \times 230 \times 23.09 \times \cos(0 - (-36.9^\circ)) = 12.75 \, \text{kW.}$$

Reactive power delivered by source = $3 \times 230 \times 23.09 \times \sin(36.9^\circ) = 9.56 \, \text{kVAR}$

Active power delivered to load =

$$3 \times 224.9 \times 23.09 \times \cos(-1.1^\circ - (-36.93^\circ)) = 12.64 \, \text{kW.}$$

Reactive power delivered to load =

$$3 \times 224.9 \times 23.09 \times \sin(-1.1^\circ - (-36.93^\circ)) = 9.11 \, \text{kVAR.}$$

Magnitude of line voltage across load = $\sqrt{3} \times 224.9 = 389.56 \, \text{V rms.}$

Complete Phasor Solution

Phase voltages at source: $230 \angle 0^\circ$, $230 \angle -120^\circ$, $230 \angle 120^\circ$ (V rms)

The first line voltage, i.e., V_{RY} , is known to lead the first phase voltage, i.e., V_{RN} , by 30° .

Therefore,

Line voltages at source terminals:

$$400 \angle 30^\circ, 400 \angle -90^\circ, 400 \angle 150^\circ \, (\text{V rms})$$

Phase voltages at load:

$$224.9 \angle -1.1^\circ, 230 \angle -121.1^\circ, 230 \angle 118.9^\circ \, (\text{V rms})$$

Line voltages at load terminals:

$$389.56 \angle 28.9^\circ, 389.56 \angle -91.1^\circ, 389.56 \angle 148.9^\circ \, (\text{V rms})$$

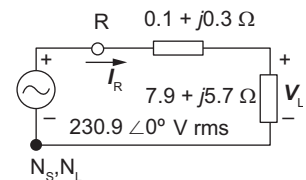


Fig. 9.3-3 Single-Phase Equivalent for Example 9.3-1

Line currents:
 $23.09\angle-36.9^\circ$, $23.09\angle-156.9^\circ$, $23.09\angle83.1^\circ$ (A rms)
 Complex power delivered by source =
 $\sqrt{3} \times 400 \times 23.09\angle(0 - (-36.9^\circ)) = 16\angle36.9^\circ$ kVA.
 Complex power delivered to load =
 $\sqrt{3} \times 389.56 \times 23.09\angle(-1.1 - (-36.9^\circ)) = 15.54\angle35.8^\circ$ kVA.

EXAMPLE: 9.3-2

A balanced Δ -connected load with phase impedance of $23.7 + j17.1 \Omega$ at 50 Hz is supplied from a 400 V, 50 Hz balanced Y-connected source through connection impedance of $0.1 + j0.3 \Omega$ in each line. Find (i) the line current and load voltage, (ii) the load-branch currents and (iii) the active and reactive power delivered by the source and delivered to the load.

SOLUTION

The circuit with all the relevant phasors identified is shown in Fig. 9.3-4.

The Δ -connected impedance is transformed into a Y-connected one by star-delta transformation equations. Since the delta impedances are equal, the impedance in star connection will be one-third of delta branch impedance, i.e., $7.9 + j5.7 \Omega$. The circuit after this transformation is shown in Fig. 9.3-5.

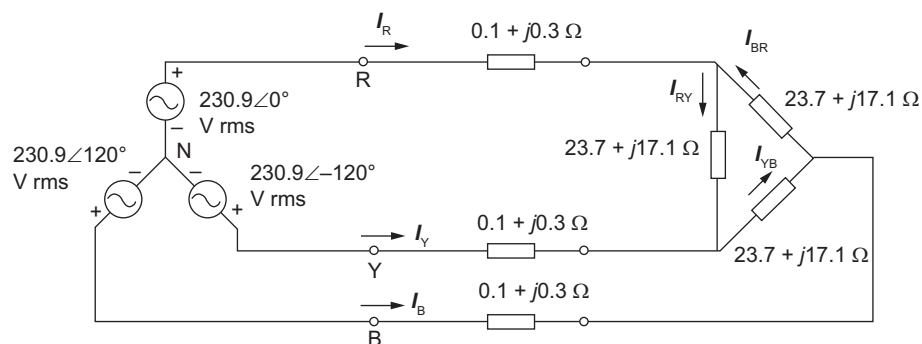


Fig. 9.3-4 A Y-connected Source Delivering Power to a Δ -connected Load Through an Impedance

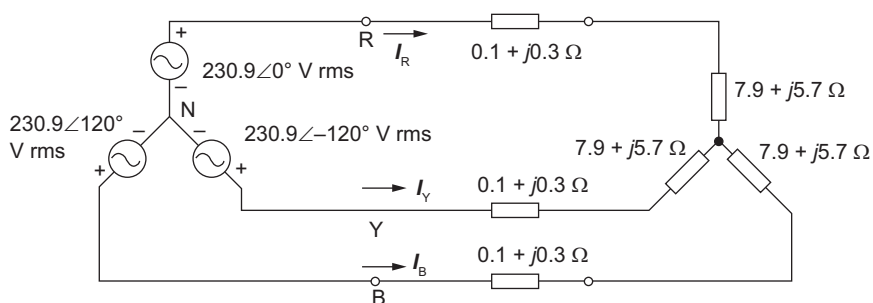


Fig. 9.3-5 The Y-connected Equivalent of Circuit in Fig. 9.3-4

Now, the single-phase equivalent of this circuit can be identified as the same as in Example 9.3-1, and hence the solution is the same as in that example.

Therefore, line current (R) = $23.09\angle-36.9^\circ$ A rms, Load phase voltage (R) = $224.9\angle-1.1^\circ$ V rms and Load line voltage (RY) = $389.56\angle28.9^\circ$ V rms.

Load active power can be found as $3 \times 224.9 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$ or as $\sqrt{3} \times 389.56 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$. The value is 12.64 kW. Similarly, the load reactive power is $3 \times 224.9 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$ or $\sqrt{3} \times 389.56 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$. The value is 9.11 kVAR.

Branch Currents in Δ -connected Load

Applying KCL at the corners of delta in R-line and Y-line in Fig. 9.3-4, we get,

$$I_{RY} - I_{BR} = I_R \text{ and } I_{YB} - I_{RY} = I_Y.$$

Multiplying the second equation by -1 and adding the result to the first equation, we get,

$$2I_{RY} - (I_{BR} + I_{YB}) = I_R - I_Y.$$

But, $(I_{RY} + I_{BR} + I_{YB}) = 0$ since we expect these three currents to be a three-phase balanced set. Therefore, $-(I_{BR} + I_{YB}) = I_{RY}$.

$$\text{Therefore, } 2I_{RY} + I_{RY} = I_R - I_Y \Rightarrow I_{RY} = (I_R - I_Y)/3$$

We know that $I_R = 23.09\angle-36.9^\circ$ A rms. I_R , I_Y and I_B form a balanced three-phase set of phasors. Therefore, $I_Y = 23.09\angle-156.9^\circ$ A rms.

$$\therefore 3I_{RY} = (23.09\angle-36.9^\circ - 23.09\angle-156.9^\circ) \text{ A rms.}$$

This is the difference between two phasors with 120° between them. We have determined the result of this kind of difference operation earlier. The result will have a magnitude that is $\sqrt{3} \times 23.09$ and will lead the first quantity by 30° .

$$\therefore 3I_{RY} = 40\angle-6.9^\circ \text{ and } I_{RY} = 13.33\angle-6.9^\circ \text{ A rms.}$$

This can also be obtained by observing that RY-line voltage at load is $389.56\angle28.9^\circ$ V rms and I_{RY} is the current drawn by $23.7 + j17.1 \Omega$ ($29.225\angle35.8^\circ \Omega$) from this voltage.

$$\text{Therefore, } I_{RY} = 389.56\angle28.9^\circ \div 29.225\angle35.8^\circ = 13.33\angle-6.9^\circ \text{ A rms.}$$

Now, the remaining two branch currents can be obtained by using three-phase symmetry as $I_{YB} = 13.33\angle-126.9^\circ$ A rms and $I_{BR} = 13.33\angle-113.1^\circ$ A rms.

EXAMPLE: 9.3-3

A Δ -connected source with an open-circuit voltage of 200 V rms at 50 Hz is used to deliver power to a star-connected load with phase impedance of $5.9 + j7.7 \Omega$. Each phase-source in the Δ -connected source has internal impedance of $0.3 + j0.9 \Omega$ in series with it. Find (i) the phase voltage and line voltage across the load, (ii) the line currents and delta branch currents and (iii) the active power and reactive power delivered to the load and consumed by the internal impedance of the source.

SOLUTION

The three-phase circuit to be analysed is shown in Fig. 9.3-6.

The open-circuit line voltage across source terminals is given to be 200 V rms. The balanced delta connection shown in Fig. 9.3-6 will not have any current in delta when the terminals are left open since the loop voltage inside delta is zero. Therefore, the source voltages must themselves be 200 V rms. A straightforward determination of Y-equivalent of Δ -connected source is not possible in this case. We transform the 'voltage source in series with impedance combinations' into 'current source in parallel with impedance combinations' by using Source Transformation Theorem. This is shown in Fig. 9.3-7.

We take the RY-line quantity as the reference phasor in this example. Now, the Δ -connected current source has a star-equivalent and the Δ -connected source impedance also has a star equivalent. Combining the current source in star-equivalent and the impedance in star-equivalent into a voltage source in series with impedance, we get $(200/\sqrt{3})\angle-30^\circ = 115.5\angle-30^\circ$ V rms in series with $(0.3 + j0.9)/3 = 0.1 + j0.3 \Omega$ in R-phase of the final star-equivalent. The complete star-equivalent of the source along with the load is shown in Fig. 9.3-8.

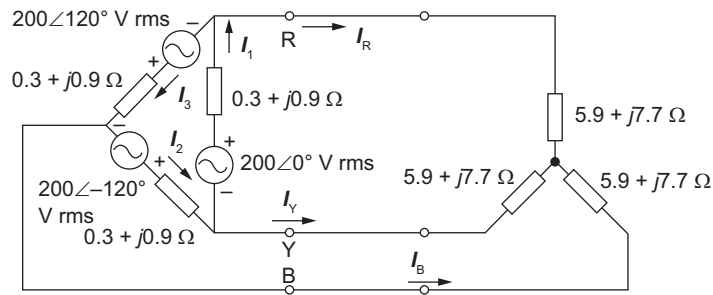


Fig. 9.3-6 A Δ -connected Source with Source Impedance Supplying a Star-connected Load

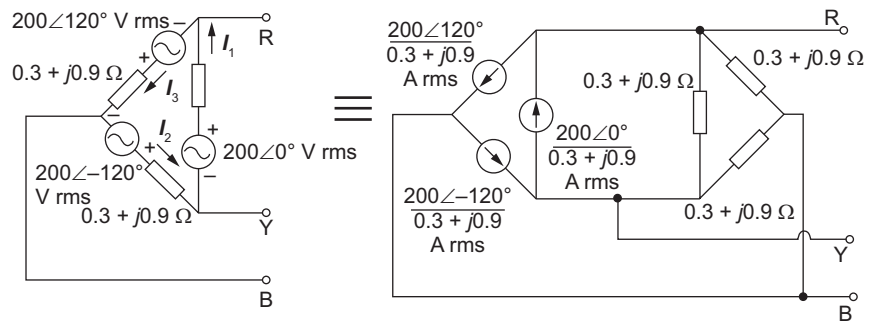


Fig. 9.3-7 Applying Source Transformation Theorem on the Δ -connected Source in Fig. 9.3-6

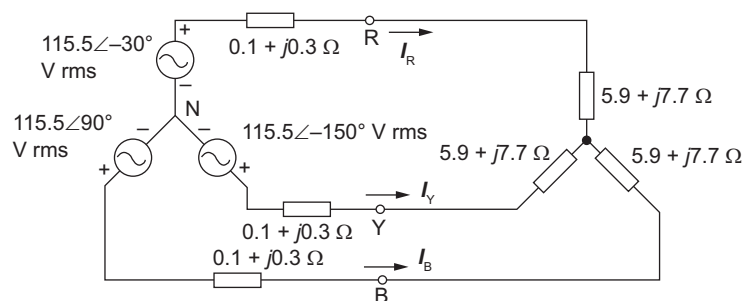


Fig. 9.3-8 The Star Equivalent of the Circuit in Fig. 9.3-6

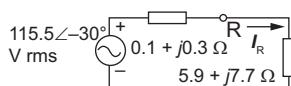


Fig. 9.3-9 Single-Phase Equivalent of Circuit in Fig. 9.3-6

The single-phase equivalent of this circuit is shown in Fig. 9.3-9.

(i) The line current (I_R) is $115.5\angle-30^\circ \text{ V rms} \div (6 + j8 \Omega) = 11.55\angle-83.13^\circ \text{ A rms}$ and the phase voltage (R_N) across load is $11.55\angle-83.13^\circ \text{ A rms} \times (5.9 + j7.7 \Omega) = 11.55\angle-83.13^\circ \times 9.7\angle52.54^\circ = 112\angle-30.6^\circ \text{ V rms}$.

Therefore, line voltage (R_Y) across the load $= 112\sqrt{3}\angle(-30.6^\circ + 30^\circ) = 194\angle-0.6^\circ \text{ V rms}$.

- (ii) The line current (R) is $115.5\angle-30^\circ \div (6 + j8) = 11.55\angle-83.13^\circ$ A rms. Refer to Fig. 9.3-6. The current marked as $I_1 = [200\angle 0^\circ - \text{line voltage (RY) across load}] \div (0.3 + j0.9) = [200\angle 0^\circ - 194\angle-0.6^\circ] \div 0.9487\angle 71.57^\circ = (6 + j2) \div 0.9487\angle 71.57^\circ = 6.68\angle-51.13^\circ$ A rms. Then I_2 will be $6.68\angle-171.13^\circ$ A rms and I_3 will be $6.68\angle 68.87^\circ$ A rms.
- (iii) The active power delivered to the load $= \sqrt{3} \times 194 \times 11.55 \times \cos(-30^\circ - (-83.13^\circ)) = 2.33$ kW. The reactive power delivered to load $= \sqrt{3} \times 194 \times 11.55 \times \sin(-30^\circ - (-83.13^\circ)) = 3.11$ kVar.

The internal impedance of the source is $0.3 + j0.9 \Omega$ and it carries an r.m.s current of 6.68 A. Therefore, the active power consumed by the internal impedance of the Δ -connected source $= 3 \times 6.68^2 \times 0.3 = 40.2$ W. The reactive power consumed by this impedance $= 3 \times 6.68^2 \times 0.9 = 120.6$ VAr.

9.4 ANALYSIS OF UNBALANCED THREE-PHASE CIRCUITS

An unbalanced three-phase circuit is one that contains at least one source or load that does not possess three-phase symmetry. A source with the three source-function magnitudes unequal and/or the successive phase displacements different from 120° can make a circuit unbalanced. Similarly, a three-phase load with unequal phase impedance values can make a circuit unbalanced.

The single-phase equivalent circuit technique of analysis *does not* work for unbalanced three-phase circuits. General circuit analysis techniques like mesh analysis or nodal analysis will have to be employed for analysing such circuits.

9.4.1 Unbalanced Y–Y Circuit

Y–Y connection is typically used in the last mile in a power system, i.e., in the low-tension (LT) distribution system.

The primary distribution system usually runs at 11 kV line voltage level. Distribution transformers that are rated for 11 kV/400 V and are Δ -connected in the primary side and Y-connected in the secondary side are used to step down the primary distribution voltage to the LT distribution level. Such transformers are located at the load centre location of the area that a particular transformer is expected to serve. *The neutral point of secondary side of a distribution transformer is usually earthed solidly.*

Power is distributed by running many four-wire LT-lines emanating from the secondary of distribution transformer to various sub-areas of the service area. Individual consumers are provided service by means of service drops tapped from these four-wire lines. Both single-phase loads and three-phase loads are provided power from these LT-lines at various points in the run of the lines. Thus, the load on the LT-line will be unbalanced in general. Even if the single phase loads are connected in such a manner that they get distributed equally among three lines (R, Y and B), only the transformer secondary will perceive a balanced load and various sections of the LT-line will see varying degree of unbalance in load. This is due to the spatially dispersed nature of loads.

Thus, the need to supply single-phase loads makes it necessary to run a neutral wire with sufficient current carrying capacity along with the three phase-lines in LT distribution. That is why LT distribution cannot be done from a Δ -connected source. The neutral wire in an LT distribution line will invariably carry current – that too, differing currents in different locations – even if the aggregate load is balanced.

The system in which the neutral wire is available for connecting single-phase loads and three-phase loads that require a neutral tie-up is called a *four-wire system*.

The system in which neutral wire is *not available* for load connection is called a *three-wire system*. This could be Y–Y system with neutrals isolated, Y– Δ , Δ –Y or Δ – Δ system.

The 'Neutral-shift' Problem

The exposed metal parts of all electrical equipments at the consumer location are tied together and earthed locally. This earth point is made available as the third pin of all three-pin power points installed in the consumer electrical system.

If the earthing resistances at the distribution transformer neutral and at the consumer location are negligibly small, then, the earth-pin at all consumer locations will be at the same potential as the transformer secondary neutral. However, the neutral-pin at the consumer locations will differ in potential from the transformer neutral potential due to the neutral wire drop.

This difference will be more if there is unbalance in the load and neutral is carrying heavy currents. Moreover, this potential difference between the earth-pin and neutral-pin will be different for different consumers due to spatially distributed nature of line load. This problem of neutral-earth voltage difference is called the *neutral-shift* problem.

Electronic equipment, in general and, computing equipment in particular, are sensitive to this neutral-shift voltage and often malfunction when it exceeds certain pre-specified level. Damage to the components can also take place due to the differences in neutral-shift potential that exist among various power points when different components of an inter-connected computing system are powered from different power points.

EXAMPLE: 9.4-1

A balanced three-phase source of 400 V, 50 Hz supplies an unbalanced three-phase load through an LT four-wire line as in Fig. 9.4-1. Each wire in the line has impedance of $0.1 + j0.3 \Omega$. (a) Find (i) phase and line voltages at load location and neutral-shift voltage, (ii) wire currents, (iii) power and reactive power delivered to the load and (iv) three-phase apparent power and power factor. (b) Repeat (a) assuming that the neutral wire is a thick conductor with zero impedance. (c) Repeat (a) assuming that the neutral wire is open.

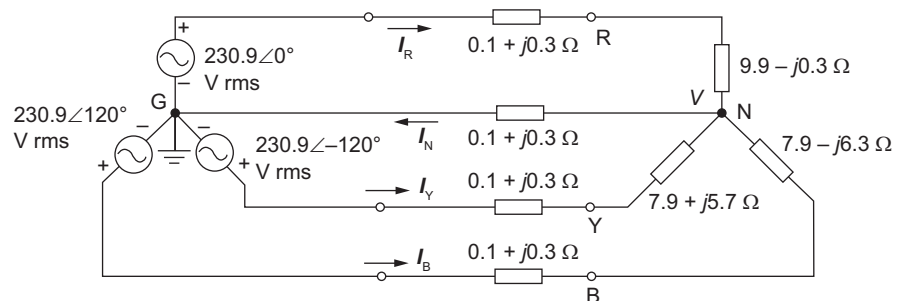


Fig. 9.4-1 Unbalanced Three-phase Circuit in Example 9.4-1

SOLUTION

(a) We solve the problem by finding the phasor voltage at load neutral with respect to the ground (earth) node-G by using the KCL equation at load neutral. The sum of currents flowing away from that node is set to zero.

$$\frac{V - 230.9\angle 0^\circ}{10} + \frac{V}{0.1 + j0.3} + \frac{V - 230.9\angle -120^\circ}{8 + j6} + \frac{V - 230.9\angle 120^\circ}{8 - j6} = 0,$$

$$\text{i.e., } V \left[\frac{1}{10} + \frac{1}{0.1 + j0.3} + \frac{1}{8 + j6} + \frac{1}{8 - j6} \right] = \frac{230.9\angle 0^\circ}{10} + \frac{230.9\angle -120^\circ}{8 + j6} + \frac{230.9\angle 120^\circ}{8 - j6}.$$

Solving for V , we get,

$$V = -2.31 - j5.49 = 5.96\angle -112.8^\circ \text{ V rms.}$$

Now, we can find wire currents as,

$$I_R = \frac{230.9\angle 0^\circ - 5.96\angle -112.8^\circ}{10} = 23.33\angle 1.35^\circ \text{ A rms.}$$

$$I_Y = \frac{230.9\angle -120^\circ - 5.96\angle -112.8^\circ}{8 + j6} = 22.5\angle -157.06^\circ \text{ A rms.}$$

$$I_B = \frac{230.9\angle 120^\circ - 5.96\angle -112.8^\circ}{8 - j6} = 23.45\angle 155.71^\circ \text{ A rms.}$$

$$I_N = I_R + I_Y + I_B = 18.83\angle 175.65^\circ \text{ A rms.}$$

The load phase voltages can be determined by multiplying line current phasors by phasor impedance of each phase.

$$V_{RN} = 23.33\angle 1.35^\circ \times (9.9 - j0.3) = 231.04\angle -0.39^\circ \text{ V rms.}$$

$$V_{YN} = 22.5\angle -157.06^\circ \times (7.9 + j5.7) = 219.18\angle -121.25^\circ \text{ V rms.}$$

$$V_{BN} = 23.45\angle 155.71^\circ \times (7.9 - j6.3) = 237\angle 123.65^\circ \text{ V rms.}$$

Now, the line voltages across the load can be determined in terms of the phase voltage phasors.

$$V_{RY} = V_{RN} - V_{YN} = 231.04\angle -0.39^\circ - 219.18\angle -121.25^\circ = 391.63\angle 28.33^\circ \text{ V rms.}$$

$$V_{YB} = V_{YN} - V_{BN} = 219.18\angle -121.25^\circ - 237\angle 123.65^\circ = 398.32\angle -90.8^\circ \text{ V rms.}$$

$$V_{BR} = V_{BN} - V_{RN} = 237\angle 123.65^\circ - 231.04\angle -0.39^\circ = 400.2\angle 147.94^\circ \text{ V rms.}$$

The active power and reactive power delivered to the load is determined by adding up the corresponding power delivered to each phase of the load.

$$\begin{aligned} P &= P_R + P_Y + P_B = 231.04 \times 23.33 \times \cos(-0.39^\circ - 1.35^\circ) \\ &\quad + 231.04 \times 22.5 \times \cos(-121.25^\circ - (-157.06^\circ)) \\ &\quad + 237 \times 23.45 \times \cos(123.65^\circ - 155.71^\circ) \\ &= 5387.1 \text{ W} + 3999 \text{ W} + 4345.9 \text{ W} \\ &= 13.73 \text{ kW.} \end{aligned}$$

$$\begin{aligned} Q &= Q_R + Q_Y + Q_B = 231.04 \times 23.33 \times \sin(-0.39^\circ - 1.35^\circ) \\ &\quad + 231.04 \times 22.5 \times \sin(-121.25^\circ - (-157.06^\circ)) \\ &\quad + 237 \times 23.45 \times \sin(123.65^\circ - 155.71^\circ) \\ &= -163.25 \text{ VAR} + 2885.3 \text{ VAR} - 3465.7 \text{ VAR} \\ &= -0.744 \text{ kVAR.} \end{aligned}$$

Two different definitions of apparent power and power factor are possible in the case of unbalanced three-phase circuits. The first one is using the total complex power, $\mathbf{S}$, delivered to the load. In this definition, the magnitude of $\mathbf{S}$ is taken as apparent power and ratio of active power to the magnitude of $\mathbf{S}$ is taken as average power factor of the circuit. The apparent power in the load in the circuit in this example then is $(13.73^2 + 0.744^2)^{0.5} = 13.75 \text{ kVA}$ and power factor is 0.9985 lead.

The second definition accepts the sum of apparent powers of load phases as the three-phase apparent power. Power factor is taken as the ratio of active power to apparent power. The apparent power in the load in the circuit in this example then is 15.88 kVA and power factor is 0.865. We cannot specify it as a lead or lag power factor in this case (see side-note).

(b) The neutral-shift voltage in this case is zero since the source neutral and load neutral are tied together by a zero impedance link. The circuit becomes a collection of three single-phase circuits sharing a common point at neutral. Therefore, the line currents can be obtained as

$$\begin{aligned} I_R &= \frac{230.9 \angle 0^\circ}{10} = 23.09 \angle 0^\circ \text{ A rms.} \\ I_Y &= \frac{230.9 \angle -120^\circ}{8 + j6} = 23.09 \angle -156.87^\circ \text{ A rms.} \\ I_B &= \frac{230.9 \angle 120^\circ}{8 - j6} = 23.09 \angle 156.87^\circ \text{ A rms.} \\ I_N &= I_R + I_Y + I_B = -19.38 \text{ A rms.} \end{aligned}$$

The load phase voltages can be determined by multiplying line current phasors by phasor impedance of each phase.

$$\begin{aligned} V_{RN} &= 23.09 \angle 0^\circ \times (9.9 - j0.3) = 228.7 \angle -1.74^\circ \text{ V rms.} \\ V_{YN} &= 23.09 \angle -156.87^\circ \times (7.9 + j5.7) = 224.93 \angle -121.06^\circ \text{ V rms.} \\ V_{BN} &= 23.09 \angle 156.87^\circ \times (7.9 - j6.3) = 223.31 \angle 118.3^\circ \text{ V rms.} \end{aligned}$$

Now, the line voltages across the load can be determined in terms of the phase voltage phasors.

$$\begin{aligned} V_{RY} &= V_{RN} - V_{YN} = 228.7 \angle -1.74^\circ - 224.93 \angle -121.06^\circ = 391.51 \angle 28.32^\circ \text{ V rms.} \\ V_{YB} &= V_{YN} - V_{BN} = 224.93 \angle -121.06^\circ - 223.31 \angle 118.3^\circ = 398.15 \angle -90.8^\circ \text{ V rms.} \\ V_{BR} &= V_{BN} - V_{RN} = 223.31 \angle 118.3^\circ - 228.7 \angle -1.74^\circ = 400.2 \angle 147.95^\circ \text{ V rms.} \\ V_{RY} + V_{YB} + V_{BR} &= 391.51 \angle 28.32^\circ + 398.15 \angle -90.8^\circ + 400.2 \angle 147.95^\circ \\ &= 344.65 + j185.73 - 5.56 - j398.11 - 339.204 + j212.37 \\ &\approx 0 \text{ within round-off error.} \end{aligned}$$

The active power and reactive power delivered to the load are determined by adding up the corresponding power delivered to each phase of the load.

$$\begin{aligned} P &= P_R + P_Y + P_B = 231.04 \times 23.33 \times \cos(-1.74^\circ - 0) \\ &\quad + 231.04 \times 22.5 \times \cos(-121.06^\circ - (-156.87^\circ)) \\ &\quad + 237 \times 23.45 \times \cos(118.3^\circ - 156.87^\circ) \\ &= 5278.2 \text{ W} + 4211.9 \text{ W} + 4211.9 \text{ W} \\ &= 13.7 \text{ kW.} \\ Q &= Q_R + Q_Y + Q_B = 231.04 \times 23.33 \times \sin(-1.74^\circ - 0) \\ &\quad + 231.04 \times 22.5 \times \sin(-121.06^\circ - (-156.87^\circ)) \\ &\quad + 237 \times 23.45 \times \sin(118.3^\circ - 156.87^\circ) \\ &= -159.94 \text{ VAR} + 3038.9 \text{ VAR} - 3358.8 \text{ VAR} \\ &= -0.48 \text{ kVAR.} \end{aligned}$$

Example 9.4-1 illustrates calculation of neutral-shift voltage.

Power Factor Under Unbalanced Conditions

The second definition based on sum of apparent powers in three phases is more appropriate in the sense that power factor and apparent power are expected to give us an indication as to the utilisation of the current in carrying active power.

The first method of calculating apparent power and power factor will hide the fact that under-utilisation of the current is taking place when one phase takes leading reactive power and the other takes lagging reactive power. The phases may consume large reactive power and yet the complex power may come out with only real part or with small imaginary part.

Therefore, the definition of apparent power and power factor based on apparent power of individual phases of load is a more meaningful one.

Need for a Neutral Tie

Observe the unacceptably large neutral-shift voltage when the source neutral and load neutral are isolated.

Further, note that the R-phase voltage is higher than nominal value. However, the line voltages show no sign of the prominent over-voltage problem in R-phase.

This example shows that a strong neutral tie is required in unbalanced four-wire circuits to maintain the phase voltages at load terminal at near-nominal values.

The three-phase apparent power is 15.86 kVA and average power factor is 0.864.
(c) The neutral points are unconnected and hence the system is a three-wire system in this case. The neutral-shift voltage $\mathbf{V}$ is found from the node equation at the load neutral.

$$\frac{\mathbf{V} - 230.9\angle 0^\circ}{10} + \frac{\mathbf{V} - 230.9\angle -120^\circ}{8 + j6} + \frac{\mathbf{V} - 230.9\angle 120^\circ}{8 - j6} = 0$$

$$\text{i.e., } \mathbf{V} \left[\frac{1}{10} + \frac{1}{8 + j6} + \frac{1}{8 - j6} \right] = \frac{230.9\angle 0^\circ}{10} + \frac{230.9\angle -120^\circ}{8 + j6} + \frac{230.9\angle 120^\circ}{8 - j6}$$

Solving for $\mathbf{V}$, we get, $\mathbf{V} = -74.55 \text{ V rms}$ (see side-note).

Now, we can find wire currents as,

$$\mathbf{I}_R = \frac{230.9\angle 0^\circ + 74.55}{10} = 30.545\angle 0^\circ \text{ A rms.}$$

$$\mathbf{I}_Y = \frac{230.9\angle -120^\circ + 74.55}{10} = 20.414\angle -138.43^\circ \text{ A rms.}$$

$$\mathbf{I}_B = \frac{230.9\angle 120^\circ + 74.55}{8 - j6} = 20.414\angle 138.43^\circ \text{ A rms.}$$

$$\mathbf{I}_N = \mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 0 \text{ A rms.}$$

The load phase voltages can be determined by multiplying line current phasors by phasor impedance of each phase.

$$\mathbf{V}_{RN} = 30.545\angle 0^\circ \times (9.9 - j0.3) = 302.53\angle -1.74^\circ \text{ V rms.}$$

$$\mathbf{V}_{YN} = 20.414\angle -138.43^\circ \times (7.9 + j5.7) = 198.87\angle -102.62^\circ \text{ V rms.}$$

$$\mathbf{V}_{BN} = 20.414\angle 138.43^\circ \times (7.9 - j6.3) = 206.27\angle 99.86^\circ \text{ V rms.}$$

Observe the over-voltage in R-phase. The line voltages across the load can be determined in terms of the phase voltage phasors now.

$$\mathbf{V}_{RY} = \mathbf{V}_{RN} - \mathbf{V}_{YN} = 302.53\angle -1.74^\circ - 198.87\angle -102.62^\circ = 392.16\angle 28.13^\circ \text{ V rms.}$$

$$\mathbf{V}_{YB} = \mathbf{V}_{YN} - \mathbf{V}_{BN} = 198.87\angle -102.62^\circ - 206.27\angle 99.86^\circ = 397.37\angle -91.2^\circ \text{ V rms.}$$

$$\mathbf{V}_{BR} = \mathbf{V}_{BN} - \mathbf{V}_{RN} = 206.27\angle 99.86^\circ - 302.53\angle -1.74^\circ = 398.94\angle 147.83^\circ \text{ V rms.}$$

The active power and reactive power delivered to the load are determined by adding up the corresponding power delivered to each phase of the load.

$$P = P_R + P_Y + P_B = 15.82 \text{ kW.}$$

$$Q = Q_R + Q_Y + Q_B = -0.53 \text{ kVAR.}$$

Apparent power is 17.11 kVA and average power factor is 0.925.

We note that, in all the cases, the three line voltages add up to zero. The sum of line voltages in any three-phase system is zero since these three voltages are the voltages in a loop formed by R to Y, Y to B and B to R.

This example demonstrated the need for a strong neutral tie in unbalanced four-wire circuits to maintain the phase voltages at load terminal at near-nominal values.

It is rarely that we encounter a Δ -connected source. One situation where it can appear is when a transformer secondary is Δ -connected. We can use a Δ -connected voltage source, with each arm of delta containing a source in series with the transformer series impedance, as a circuit model for the secondary of the transformer.

9.4.2 Circulating Current in Unbalanced Δ -connected Sources

A Δ -connected source involves connecting three sinusoidal sources in a closed loop. If the sources are *ideal independent voltage sources*, they *have to* add up to zero at all instants. Otherwise, there will be infinitely large circulating current in the closed loop. The sources will have some non-zero impedance in practice and this source impedance will limit the circulating current arising out of residual voltage in the loop. However, if the source impedance is small (and, it should be small for a good source), the circulating current will consume the current carrying capacity of the source and leave only a portion of its capacity to deliver useful power to the external load. Moreover, the large circulating current in a Δ -loop produces large dissipation within the sources.

Therefore, the phase-source voltage phasors in a Δ -connected source should add up to zero preferably. This happens naturally in the case of a balanced source. But it need not happen always in the case of unbalanced source though it can for certain combinations of magnitudes and phase angles. Essentially, the magnitudes and phase angles of the three phase-

sources must be such that the phasors form a closed triangle in complex plane. This is difficult to arrange in a synchronous generator. That is why generators are always Y-connected.

The following example illustrates the problem of circulating current in unbalanced Δ -connected sources as well as the application of mesh analysis and node analysis in solving unbalanced three-phase circuits.

EXAMPLE: 9.4-2

A Δ -connected unbalanced source delivers power to a balanced Y-connected resistive load as shown in Fig. 9.4-2. Solve the circuit completely by using mesh analysis.

SOLUTION

The mesh currents are assigned as shown in Fig. 9.4-2. The mesh equations are written in matrix form by inspection.

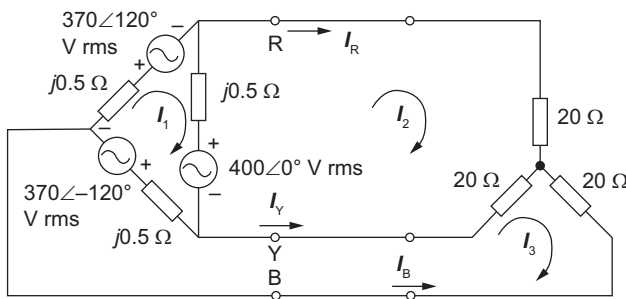


Fig. 9.4-2 Unbalanced Δ -connected Source Supplying a Y-connected Resistive Load in Example 9.4-2

$$\begin{bmatrix} j1.5 & -j0.5 & -j0.5 \\ -j0.5 & 40 + j0.5 & -20 \\ -j0.5 & -20 & 40 + j0.5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -(400\angle 0^\circ + 370\angle -120^\circ + 370\angle 120^\circ) \\ 400\angle 0^\circ \\ 370\angle -120^\circ \end{bmatrix}$$

$$= \begin{bmatrix} -30 \\ 400\angle 0^\circ \\ 370\angle -120^\circ \end{bmatrix}$$

Solving for I_1 , I_2 and I_3 ,

$I_1 = 14.98\angle 77.64^\circ$ A rms, $I_2 = 11.12\angle -29.2^\circ$ A rms and $I_3 = 10.68\angle -90.48^\circ$ A rms.

Now, the line currents are $I_R = I_2 = 11.12\angle -29.2^\circ$ A rms, $I_Y = I_3 - I_2 = 11.116\angle -151.77^\circ$ A rms and $I_B = -I_3 = 10.68\angle 89.52^\circ$ A rms.

Current delivered by $400\angle 0^\circ$ V rms source $= I_2 - I_1 = 21.08\angle -72.04^\circ$ A rms.

Current delivered by $370\angle -120^\circ$ V rms source $= I_3 - I_1 = 25.53\angle -97.42^\circ$ A rms.

Current delivered by $370\angle 120^\circ$ V rms source $= -I_1 = 14.98\angle -102.36^\circ$ A rms.

Power delivered to load $= [11.12^2 + 11.12^2 + 10.68^2] \times 20 = 7227.4$ W.

Power delivered by $400\angle 0^\circ$ V rms source $= 400 \times 21.08 \times \cos(72.04^\circ) = 2600.3$ W.

Power delivered by $370\angle -120^\circ$ V rms source $= 370 \times 25.53 \times \cos(-120^\circ + 97.42^\circ) = 8772$ W.

Power delivered by $370\angle 120^\circ$ V rms source $= 370 \times 14.98 \times \cos(120^\circ + 102.36^\circ) = -4095.57$ W.

Why Circulating Current?

Exact symmetry in a three-phase equipment is hard to achieve in practice. For instance, there can be small differences in the number of turns employed in the three phase-windings of the transformer. This may produce unbalanced voltages in the three secondary windings. And those voltage phasors may not 'close' into a triangle. Circulating current in the secondary winding of the transformer is the result.

We note the marked deviation in phase-source currents from the levels we expect from the observed line currents. We would have expected about 6–7 A rms in the sources considering the fact that the line currents are around 11 A rms. The circulating current due to the residual Δ -loop voltage of 30 V rms makes the delta currents much larger than what they need to be in view of load power. This results in excess power being delivered by some branches forcing the remaining branch to absorb power rather than delivering it. This is why the $370\angle 120^\circ$ V rms source had to absorb a large quantity of power.

Of course, the Δ -connected source used in this example can be of no practical use. A Δ -connected source is of practical utility only if the residual voltage within the Δ -loop can be kept at zero or a low value compared to source voltage.

The line voltages appearing across the load can be calculated by subtracting the voltage drop in the source impedance of each phase-source from its source voltage value. The result of this calculation is $\mathbf{V}_{RY} = 390\angle -0.48^\circ$ V rms, $\mathbf{V}_{YB} = 375.3\angle -121.8^\circ$ V rms and $\mathbf{V}_{BR} = 375.3\angle -120.83^\circ$ V rms.

9.5 SYMMETRICAL COMPONENTS

Analysis of balanced three-phase circuits is considerably simpler than analysis of unbalanced three-phase circuits. Three-phase symmetry exhibited by a balanced circuit makes it possible to solve the circuit employing a single-phase equivalent circuit. This prompts us to ask the question – is it possible to bring back the symmetry enjoyed by a balanced circuit into an unbalanced three-phase circuit by some means?

We realise that an unbalanced three-phase circuit can result from unbalanced three-phase sources acting on balanced loads or balanced sources acting on unbalanced loads or unbalanced three-phase sources acting on unbalanced loads. Let us tackle the first case – unbalanced three-phase sources acting on balanced loads.

9.5.1 Three-Phase Circuits with Unbalanced Sources and Balanced Loads

In this case, maybe we can bring three-phase symmetry back into the circuit if we can somehow express the unbalanced three-phase source voltages/currents as a *superposition of balanced sets of three-phase voltages/currents*. Is that possible?

Specialised version of a general theorem called *Fortesque's Theorem* assures us that it is possible. If a *set* of unbalanced three-phase source functions can be expressed as a sum of balanced *sets* of three-phase source functions, then, the solution of unbalanced three-phase circuit can be obtained as a *superposition* of solution of the circuit for various balanced sets of three-phase source functions. If all the loads are balanced, each of the circuit problems that needs to be solved for applying superposition principle, will be a balanced circuit problem. Thus, symmetry can be restored to unbalanced three-phase circuits this way, *provided the unbalance is only due to sources and not due to loads*. The sets of three-phase balanced source components and possible single-phase components of an unbalanced source are called its *Symmetrical Components*.

There are three symmetrical components for an unbalanced three-phase source function. *Each symmetrical component is a set of three source functions*. The first set – called the *positive sequence component* – is a balanced three-phase set of source functions that has positive phase sequence. The second set – called the *negative sequence component* – is a balanced three-phase set of source functions that has negative phase sequence. The third set – called the *zero sequence component* – is a set of three co-phasal (i.e., of same phase) single-phase source functions. *It is not a three-phase set at all*.

Symmetrical Components

The sets of three-phase balanced source components and possible single-phase components that add up to form an unbalanced source are called its *Symmetrical Components*.

Each symmetrical component is a set of three source functions.

Symmetrical components are denoted by the first phasor element in each set.

Symmetrical components are denoted by the first phasor element in each set. Thus, the positive sequence component is denoted by R-phase quantity or R-line quantity of the balanced three-phase source function of positive phase sequence in phasor form. Similarly, negative sequence component is denoted by R-phase quantity or R-line quantity of the balanced three-phase source function of negative phase sequence in phasor form. And, zero sequence component is denoted by one of the three co-phasal single-phase source functions.

For instance, let $200\angle-10^\circ$, $100\angle-50^\circ$ and $25\angle-30^\circ$ be the r.m.s values of positive, negative and zero sequence components of some unbalanced phase voltage set. Then, $200\angle-10^\circ$ stands for a three-phase balanced voltage set ($200\angle-10^\circ$, $200\angle-130^\circ$ and $200\angle110^\circ$) in RN-, YN- and BN-phase voltages, respectively. The $100\angle-50^\circ$ negative sequence component stands for ($100\angle-50^\circ$, $100\angle70^\circ$ and $100\angle-170^\circ$) in RN-, YN- and BN-phase voltages, respectively. We note the phase sequence of the bracketed quantity. The zero sequence component of $25\angle-30^\circ$ stands for ($25\angle-30^\circ$, $25\angle-30^\circ$ and $25\angle-30^\circ$) in RN-, YN- and BN-phase voltages, respectively. Then, by *Fortesque's Symmetrical Components Theorem*, the phase voltages are given by

$$V_{RN} = 200\angle-10^\circ + 100\angle-50^\circ + 25\angle-30^\circ = 308.8\angle-23.64^\circ \text{ V rms.}$$

$$V_{YN} = 200\angle-130^\circ + 100\angle70^\circ + 25\angle-30^\circ = 109.4\angle-131.7^\circ \text{ V rms.}$$

$$V_{BN} = 200\angle110^\circ + 100\angle-170^\circ + 25\angle-30^\circ = 214.66\angle132.6^\circ \text{ V rms.}$$

The phase voltage set is an unbalanced one and we have expressed it as a sum of three components – balanced positive sequence three-phase voltage, balanced negative sequence three-phase voltage and single-phase voltage.

We could have expressed these equations in another format as,

$$V_{RN} = 1 \times 200\angle-10^\circ + 1 \times 100\angle-50^\circ + 1 \times 25\angle-30^\circ.$$

$$V_{YN} = 1\angle-120^\circ \times 200\angle-10^\circ + 1\angle120^\circ \times 100\angle-50^\circ + 1 \times 25\angle-30^\circ.$$

$$V_{BN} = 1\angle120^\circ \times 200\angle-10^\circ + 1\angle-120^\circ \times 100\angle-50^\circ + 1 \times 25\angle-30^\circ.$$

Let us define a constant complex number a as $a = 1\angle120^\circ$. When a phasor is multiplied by this operator, its magnitude stays unaffected, but it gets rotated in counter-clockwise direction by 120° . Then, $a^2 = 1\angle240^\circ = 1\angle-120^\circ$ is the operator that can rotate a phasor by 120° in clockwise direction. Note that $1 + a + a^2 = 0$, $a^2 = a^*$, $a^3 = 1\angle0^\circ$ and $a^4 = a$. Now, the above equations can be expressed in terms of this operator a as

$$V_{RN} = 1 \times 200\angle-10^\circ + 1 \times 100\angle-50^\circ + 1 \times 25\angle-30^\circ.$$

$$V_{YN} = a^2 \times 200\angle-10^\circ + a \times 100\angle-50^\circ + 1 \times 25\angle-30^\circ.$$

$$V_{BN} = a \times 200\angle-10^\circ + a^2 \times 100\angle-50^\circ + 1 \times 25\angle-30^\circ.$$

Now, let us introduce X_+ , X_- and X_0 as the values of positive, negative and zero sequence components (note that they are phasors) of some voltage or current instead of the three numbers – $200\angle-10^\circ$, $100\angle-50^\circ$ and $25\angle-30^\circ$ – we used till now. Then, the unbalanced quantities are given in terms of symmetrical components (i.e., X_+ , X_- and X_0) by

$$X_R = 1 \times X_+ + 1 \times X_- + 1 \times X_0$$

$$X_Y = a^2 \times X_+ + a \times X_- + 1 \times X_0$$

$$X_B = a \times X_+ + a^2 \times X_- + 1 \times X_0$$

We can express this equation set in matrix form as below:

$$\begin{bmatrix} X_R \\ X_Y \\ X_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_+ \\ X_- \end{bmatrix} \quad (9.5-1)$$

X_R , X_Y and X_B can represent any phase voltage, phase current, line voltage or line current phasors in an unbalanced system. The 3×3 matrix in Eqn. 9.5-1 is called the *Symmetric Transformation Matrix*. The inverse equation required for determining sequence components from phase quantities is as below.

$$\begin{bmatrix} X_0 \\ X_+ \\ X_- \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} X_R \\ X_Y \\ X_B \end{bmatrix} \quad (9.5-2)$$

The Symmetrical Component RYB $\rightarrow$ 0+- transformation equation.

The Symmetrical Component 0+- $\rightarrow$ RYB transformation equation.

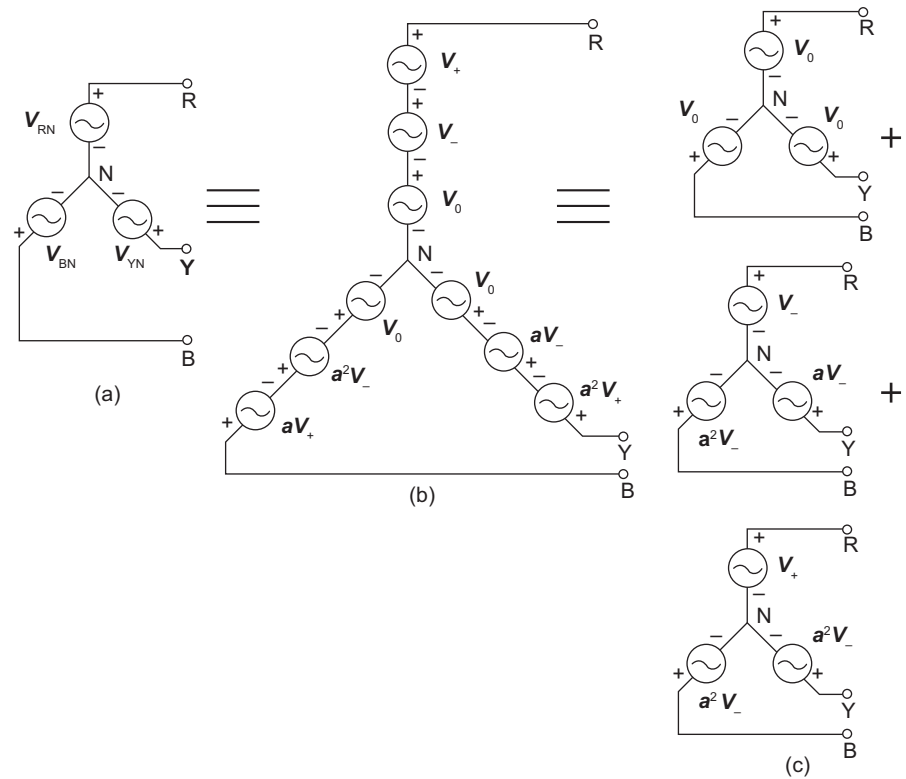


Fig. 9.5-1 Interpretation of Symmetrical Components

The reader is urged to verify the matrix inversion involved.

The circuit interpretation of symmetrical components is given in Fig. 9.5-1. Here, an unbalanced star-connected voltage source is resolved into its symmetrical components. Symmetrical Components Theorem assures us that the source in Fig. 9.5-1(a) can be viewed as the composite source in Fig. 9.5-1(b) in which each source limb contains three sources in series – one each from each sequence component set. And superposition principle assures us that applying the source in Fig. 9.5-1(b) is the same as applying the three source sets shown in Fig. 9.5-1(c) individually.

9.5.2 The Zero Sequence Component

This component needs special attention. We get an expression for zero sequence component from Eqn. 9.5-2 as

$$X_0 = \frac{1}{3}(X_R + X_Y + X_B).$$

Thus, the zero sequence component is the average of the three three-phase quantities.

Line voltages in any three-phase circuit, balanced or unbalanced, will add up to zero by KVL. *Therefore, line voltages in a three-phase system cannot have a zero sequence content anywhere in the system.*

Line currents in a three-phase three-wire system will have to add up to zero by KCL. *Therefore, line currents in a three-phase three-wire system cannot have a zero sequence content anywhere in the system.*

Phase voltages in a three-phase system *need not* add up to zero. Hence, phase voltages can have zero sequence content.

Line currents in a four-wire system *need not* add up to zero. The sum $I_R + I_Y + I_B$ can flow through the fourth wire (neutral wire) in the return direction. Therefore, three-phase four-wire systems can have zero sequence content in their line currents.

9.5.3 Active Power in Sequence Components

Let the phase voltages across a *balanced three-phase load circuit* be V_{RN} , V_{YN} and V_{BN} , where N is the neutral point in the load itself or in its Y-equivalent. Let I_R , I_Y and I_B be the line currents flowing into the load. Further, let V_0 , V_+ and V_- be the sequence components of phase voltages and I_0 , I_+ and I_- be the sequence components of line currents.

Then, the total active power that flows into the load,

$$P = \text{Re}[V_{RN} I_R^*] + \text{Re}[V_{YN} I_Y^*] + \text{Re}[V_{BN} I_B^*] \\ = \text{Re}[V_{RN} I_R^* + V_{YN} I_Y^* + V_{BN} I_B^*].$$

Let us define four column vectors as below.

$$V_{\text{ryb}} = \begin{bmatrix} V_{RN} \\ V_{YN} \\ V_{BN} \end{bmatrix}; \quad I_{\text{ryb}} = \begin{bmatrix} I_R \\ I_Y \\ I_B \end{bmatrix}; \quad V_{0+-} = \begin{bmatrix} V_0 \\ V_+ \\ V_- \end{bmatrix}; \quad I_{0+-} = \begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix}$$

Then, the power equation can be expressed as $P = \text{Re}[V_{\text{ryb}}^t I_{\text{ryb}}^*]$, where the superscripts 't' and '*' indicates matrix transpose operation and complex conjugation operation, respectively.

Equation 9.5-1 expresses the three-phase quantities in terms of its sequence components. It is reproduced below.

$$\begin{bmatrix} X_R \\ X_Y \\ X_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_+ \\ X_- \end{bmatrix}$$

Using this equation, we write the column vector V_{ryb} in terms of the column vector V_{0+-} and the column vector I_{ryb} in terms of the column vector I_{0+-} as below.

$$V_{\text{ryb}} = A V_{0+-} \quad \text{and} \quad I_{\text{ryb}} = A I_{0+-}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

Then,

$$P = \text{Re}[V_{\text{ryb}}^t I_{\text{ryb}}^*] = \text{Re}[(A V_{0+-})^t (A I_{0+-})^*] = \text{Re}[V_{0+-}^t A^t A^* I_{0+-}^*] \\ A^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & (a^2)^* & a^* \\ 1 & a^* & (a^2)^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

Therefore, $A^t A^*$ is identity matrix of 3×3 order. Hence,

$$P = \text{Re}[V_{0+-}^t I_{0+-}^*] = \text{Re}[V_0 I_0^* + V_+ I_+^* + V_- I_-^*] \\ = \text{Re}[V_0 I_0^*] + \text{Re}[V_+ I_+^*] + \text{Re}[V_- I_-^*]. \quad (9.5-3)$$

Thus, we observe that (i) active power carried by sequence components obey superposition principle and (ii) positive sequence component of voltage can send active power *only through* positive sequence component of current; negative sequence component of voltage can send active power *only through* negative sequence component of current and zero sequence component of voltage can send active power *only through* zero sequence component of current.

9.5.4 Three-Phase Circuits with Balanced Sources and Unbalanced Loads

Three-phase sources, at least at generation and transmission level, are more or less balanced. They may show considerable unbalance at LT distribution level. However, three-phase circuit unbalance is usually the result of unbalanced loads rather than unbalanced sources. Symmetrical components can be used for analysing such circuits too. However, we do not take up such analysis since it is fairly beyond the scope of an introductory text like this. Hence, we close our discussion on symmetrical components after covering two salient points in this sub-section.

Sequence Decoupling

Balanced impedance will have negative sequence current in it only if there is negative sequence voltage applied to it.

This is so because balanced impedance can produce only negative sequence voltage drop when a negative sequence current flows through it.

That is, balanced impedance is incapable of bringing about a sequence conversion involving translation of positive sequence current into negative sequence voltage and vice-versa.

This is called sequence-decoupling property of balanced three-phase impedances.

The first point is that an unbalanced load does not possess *sequence decoupling property*. Rather, it *creates sequence coupling*. Consider a balanced three-phase voltage source supplying an unbalanced three-phase load. Since the voltage source is balanced, it has no negative or zero sequence components. It is a purely positive sequence voltage source. However, the load will draw unbalanced currents since it is unbalanced. We know that unbalanced current set will contain negative sequence component. Thus, a positive sequence voltage source has produced a negative sequence current in the circuit. Moreover, the voltage drop produced across the load impedance due to the negative sequence current must be a positive sequence voltage since there is only positive sequence component in the source voltage to meet the KVL requirement. Therefore, negative sequence current succeeds in producing a positive sequence voltage drop in the case of an unbalanced load. Therefore, *an unbalanced load produces sequence coupling between positive sequence components and negative sequence components*.

The second observation is regarding active power. We observed in the previous section that a negative sequence current flowing through a positive sequence voltage could not carry active power. Hence, the negative sequence current drawn by an unbalanced three-phase load from a balanced three-phase source is a wasteful current that ties up equipment capacity and increases system losses without carrying productive power. We had come across another such current before – the reactive current component. Thus, both the reactive component of positive sequence current and the entire negative sequence current drawn by an unbalanced three-phase load from a balanced supply are unproductive in effect and detrimental to the system. *We conclude that the most energy-efficient way to draw active power from a balanced three-phase source is to draw it through balanced three-phase currents at unity power factor*. Symmetrical components afford us the required information to evaluate the effectiveness of current in carrying power in unbalanced loading of the system equipment. We can solve the unbalanced circuit problem by applying mesh or nodal analysis and obtain the symmetrical components of currents to evaluate how bad the current utilisation is.

Thus, with this introductory coverage on symmetrical components, we can now apply sequence components (i) to solve circuits with *unbalanced sources* and *balanced loads* and (ii) to evaluate the effectiveness of current flow in carrying active power in circuits with *balanced sources* and *unbalanced loads*. There is much more to symmetrical components. Power System Engineers go deep into symmetrical components in analysing power systems under faulty condition.

EXAMPLE: 9.5-1

An unbalanced four-wire system is shown in Fig. 9.5-2. (i) Find the symmetrical components of the source phase voltages. (ii) Determine the line voltages at the source and verify that line voltage does not contain zero sequence component. (iii) Determine the line currents and neutral current by using symmetrical components. (iv) Find the load

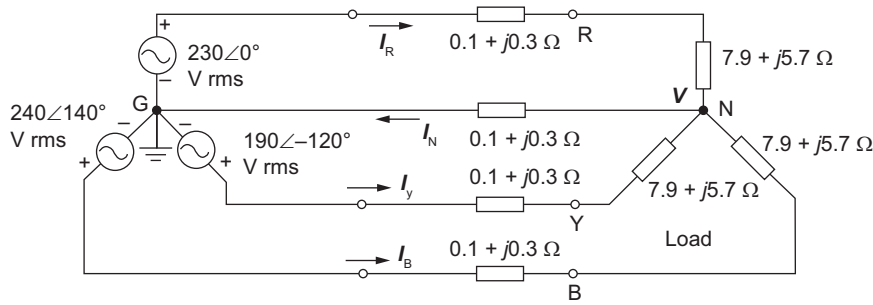


Fig. 9.5-2 Unbalanced Three-Phase Circuit for Example 9.5-1

neutral voltage with respect to earth. (v) Find the phase voltages and line voltages across the load by symmetrical components. (vi) Find the total power delivered by source and delivered to load using symmetrical components.

SOLUTION

$$(i) \begin{bmatrix} \mathbf{V}_{s0} \\ \mathbf{V}_{s+} \\ \mathbf{V}_{s-} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \mathbf{a} & \mathbf{a}^2 \\ 1 & \mathbf{a}^2 & \mathbf{a} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{sRG} \\ \mathbf{V}_{sYG} \\ \mathbf{V}_{sBG} \end{bmatrix}. \text{ The subscript 's' stands for source quantities. Substituting}$$

$$\mathbf{a} = 1 \angle 120^\circ, \mathbf{V}_{sRN} = 230 \angle 0^\circ, \mathbf{V}_{sYN} = 190 \angle -120^\circ \text{ and } \mathbf{V}_{sBN} = 240 \angle 140^\circ,$$

$$\begin{bmatrix} \mathbf{V}_{s0} \\ \mathbf{V}_{s+} \\ \mathbf{V}_{s-} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 \angle 120^\circ & 1 \angle -120^\circ \\ 1 & 1 \angle -120^\circ & 1 \angle 120^\circ \end{bmatrix} \begin{bmatrix} 230 \angle 0^\circ \\ 190 \angle -120^\circ \\ 240 \angle 140^\circ \end{bmatrix} = \begin{bmatrix} 16.64 \angle -168.12^\circ \\ 216.91 \angle 7.25^\circ \\ 39.25 \angle -37.58^\circ \end{bmatrix} \text{ V rms.}$$

(ii) Line voltages are obtained as below.

$$\mathbf{V}_{sRY} = \mathbf{V}_{sRG} - \mathbf{V}_{sYG} = 230 \angle 0^\circ - 190 \angle -120^\circ = 325 + j164.54 = 364.28 \angle 26.85^\circ \text{ V rms.}$$

$$\mathbf{V}_{sYB} = \mathbf{V}_{sYG} - \mathbf{V}_{sBG} = 190 \angle -120^\circ - 240 \angle 140^\circ = 88.85 - j318.81 = 330.96 \angle -74.43^\circ \text{ V rms.}$$

$$\mathbf{V}_{sBR} = \mathbf{V}_{sBG} - \mathbf{V}_{sRG} = 240 \angle 140^\circ - 230 \angle 0^\circ = -413.85 + j154.27 = 441.67 \angle 159.56^\circ \text{ V rms.}$$

$$\mathbf{V}_{sRY} + \mathbf{V}_{sYB} + \mathbf{V}_{sBR} = (325 + j164.54) + (88.85 - j318.81) + (-413.85 + j154.27) = 0 + j0.$$

Therefore, zero sequence component in line voltage is zero.

(iii) The circuit is solved for the three symmetrical components by applying superposition principle. We apply the positive sequence voltage source component first and obtain the positive sequence component of line currents and neutral current. We note that the connection impedances and load impedances are balanced. Therefore, we can use single-phase equivalent circuit for obtaining currents. The single-phase equivalent circuit is shown in Fig. 9.5-3. The source neutral and load neutral are at the same potential, and hence, the neutral impedance of $0.1 + j0.3 \Omega$ will not appear in the single-phase equivalent circuit for positive sequence component.

Therefore, $I_+ = 216.91 \angle 7.25^\circ \div (8 + j6) = 21.69 \angle -29.62^\circ \text{ A rms.}$

The single-phase equivalent circuit for negative sequence input is shown in Fig. 9.5-4. Passive balanced impedance cannot differentiate between positive phase sequence and negative phase sequence. However, this is not true in the case of all electrical equipment. AC motors can distinguish between the two, and hence, the equivalent circuits of motors will be different for the two phase sequences.

The load circuit is balanced and negative sequence component is a balanced three-phase component. Therefore, the neutral potential at source and load will be the same and the neutral impedance will not appear in the single-phase equivalent circuit for negative sequence input.

Therefore, $I_- = 39.25 \angle -37.58^\circ \div (8 + j6) = 3.925 \angle -74.45^\circ \text{ A rms.}$

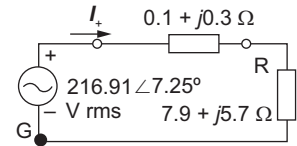


Fig. 9.5-3 Single-Phase Equivalent Circuit of the Circuit in Fig. 9.5-2 for Positive Sequence Component

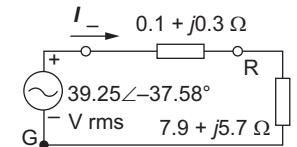


Fig. 9.5-4 Single-Phase Equivalent Circuit of the Circuit in Fig. 9.5-2 for Negative Sequence Component

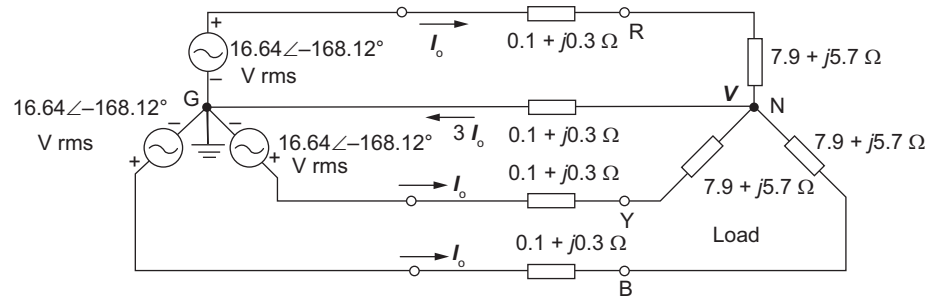


Fig. 9.5-5 Circuit with only Zero Sequence Component of Input Voltage Acting in the Circuit in Fig. 9.5-2

Note that the effective value of neutral line impedance in limiting the zero sequence current is three times its actual value due to $3I_0$ flowing in it.

The circuit to be solved with zero sequence input is shown in Fig. 9.5-5.

Note that all the three voltage sources have the same phase, and hence, all the three lines R, Y and B will carry co-phasal currents in the same direction. Therefore, the neutral return current will be $3I_0$. Applying KVL in any one mesh, we get,

$$[(8 + j6) + 3 \times (0.1 + j0.3)] \times I_0 = 16.64 \angle -168.12^\circ \Rightarrow I_0 = 1.542 \angle 152.14^\circ \text{ A rms.}$$

Now, the line currents can be obtained by applying symmetrical component transformation equation.

$$\begin{bmatrix} I_R \\ I_Y \\ I_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix}$$

The result will be, $I_R = 23.106 \angle -36.62^\circ \text{ A rms}$, $I_Y = 18.86 \angle -156.75^\circ \text{ A rms}$ and $I_B = 23.984 \angle -102.78^\circ \text{ A rms}$.

Neutral current can arise only from zero sequence component of source voltage since the remaining two components are balanced three-phase components and the load circuit is balanced. Hence, the neutral current $= 3I_0 = 4.63 \angle 152.14^\circ \text{ A rms}$.

(iv) Load neutral voltage with respect to earth = neutral current $\times$ neutral impedance, since the source neutral is earthed. The value is $= 4.63 \angle 152.14^\circ \times (0.1 + j0.3) = 1.463 \angle -136.3^\circ \text{ V rms}$.

(v) The symmetrical components of load phase voltages can be obtained as

$$\begin{bmatrix} V_{I0} \\ V_{P+} \\ V_{P-} \end{bmatrix} = \begin{bmatrix} 7.9 + j5.7 & 0 & 0 \\ 0 & 7.9 + j5.7 & 0 \\ 0 & 0 & 7.9 + j5.7 \end{bmatrix} \begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix}$$

The values are $V_{I0} = 15.018 \angle -172.05^\circ \text{ V rms}$, $V_{I+} = 211.305 \angle 6.2^\circ \text{ V rms}$ and $V_{I-} = 38.237 \angle -38.64^\circ \text{ V rms}$.

Now, the load phase voltages can be determined by symmetrical components transformation.

$$\begin{bmatrix} V_{IRN} \\ V_{IYN} \\ V_{IBN} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} V_{I0} \\ V_{I+} \\ V_{I-} \end{bmatrix}$$

The values obtained on substitution of sequence component values are, $V_{IRN} = 225.09 \angle -0.81^\circ \text{ V rms}$, $V_{IYN} = 183.73 \angle -120.93^\circ \text{ V rms}$ and $V_{IBN} = 233.65 \angle 138.54^\circ \text{ V rms}$.

The load line voltages are found as

$$V_{IRY} = V_{IRN} - V_{IYN} = 225.09 \angle -0.81^\circ - 183.73 \angle -120.93^\circ = 354.87 \angle 25.8^\circ \text{ V rms.}$$

$$V_{IYB} = V_{IYN} - V_{IBN} = 183.73 \angle -120.93^\circ - 233.65 \angle 138.54^\circ = 322.41 \angle -75.5^\circ \text{ V rms.}$$

$$V_{IBR} = V_{IBN} - V_{IRN} = 233.65 \angle 138.54^\circ - 225.09 \angle -0.81^\circ = 430.26 \angle 158.5^\circ \text{ V rms.}$$

(vi) The values of sequence components at source end are

$$V_{s0} = 16.64 \angle -168.12^\circ \text{ V rms}; V_{s+} = 216.91 \angle 7.25^\circ \text{ V rms}; V_{s-} = 39.25 \angle -37.58^\circ \text{ V rms} \\ \text{and } I_0 = 1.542 \angle 152.14^\circ \text{ A rms}; I_+ = 21.69 \angle -29.62^\circ \text{ A rms}; I_- = 3.925 \angle -74.45^\circ \text{ A rms.}$$

- $\therefore$ Power delivered through zero sequence components
 $= 16.64 \times 1.542 \times \cos(-168.12^\circ - 152.14^\circ) = 19.73 \text{ W per phase.}$
 $\therefore$ Power delivered through positive sequence components
 $= 216.91 \times 21.69 \times \cos(7.25^\circ + 29.62^\circ) = 3763.9 \text{ W per phase.}$
 $\therefore$ Power delivered through negative sequence components
 $= 39.25 \times 3.925 \times \cos(-37.58^\circ + 74.45^\circ) = 123.3 \text{ W per phase.}$
 $\therefore$ Total power delivered by the source $= 3 \times (19.73 + 3763.9 + 123.3) \text{ W} = 11.72 \text{ kW.}$
 The values of sequence components at load end are,
 $\mathbf{V}_{I_0} = 15.02 \angle -172.1^\circ \text{ V rms, } \mathbf{V}_{I_1} = 211.31 \angle 6.2^\circ \text{ V rms and } \mathbf{V}_{I_2} = 38.24 \angle -38.6^\circ \text{ V rms}$
 and $\mathbf{I}_0 = 1.54 \angle 152.14^\circ \text{ A rms, } \mathbf{I}_1 = 21.69 \angle -29.6^\circ \text{ A rms, } \mathbf{I}_2 = 3.93 \angle -74.5^\circ \text{ A rms.}$
 $\therefore$ Power delivered to the load through zero sequence components
 $= 15.018 \times 1.542 \times \cos(-172.05^\circ - 152.14^\circ) = 18.8 \text{ W per phase.}$
 $\therefore$ Power delivered to the load through positive sequence components
 $= 211.305 \times 21.69 \times \cos(6.2^\circ + 29.62^\circ) = 3716.9 \text{ W per phase.}$
 $\therefore$ Power delivered to the load through negative sequence components
 $= 38.237 \times 3.925 \times \cos(-38.64^\circ + 74.45^\circ) = 121.7 \text{ W per phase.}$
 $\therefore$ Total power delivered to the load $= 3 \times (18.8 + 3716.9 + 121.7) \text{ W} = 11.572 \text{ kW.}$

EXAMPLE: 9.5-2

Single-phasing of a three-phase load leads to an extreme case of unbalanced operation in practice. An undetected open in the wiring path in one phase or a fuse blowing in a phase leads to this kind of operation of three-phase loads in Industries. It is so common that specific measures in the form of a single-phasing prevention relay is incorporated in the control gear of high-power induction motors and drives in the Industry.

The circuit in Fig. 9.5-6 shows a balanced three-phase induction motor running from a balanced supply. The balanced operation of this circuit was studied in Example 9.3-2. Assume that the line goes open in B-phase leading to single-phased operation of the motor. Find (i) line currents and sequence components of line currents, (ii) currents in the three motor windings, (iii) active and reactive power delivered to the motor and (iv) apparent power and power factor of the source.

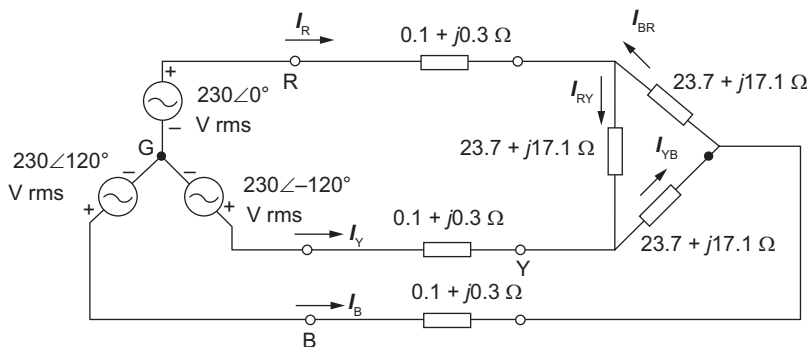


Fig. 9.5-6 Circuit for Example 9.5-2

SOLUTION

This circuit was analysed in Example 9.3-2 for balanced operation. The following results were obtained.

Line current (R) = $23.09 \angle -36.9^\circ \text{ A rms}$, Load phase voltage (R) = $224.9 \angle -1.10^\circ \text{ V rms}$ and Load line voltage (RY) = $389.56 \angle 28.9^\circ \text{ V rms}$. The delta branch currents had an r.m.s value of 13.33 A.

Load active power = 12.64 kW, load reactive power = 9.11 kVAr and source apparent power = 16 kVA.

We replace the balanced Δ -connected load impedance by a balanced star-connected impedance first to get a Y-connected equivalent circuit for the circuit in Fig. 9.5-6. This Y-connected equivalent circuit with the break in B-line is shown in Fig. 9.5-7. We note that there is only one current variable in the circuit now and that current phasor is marked as I .

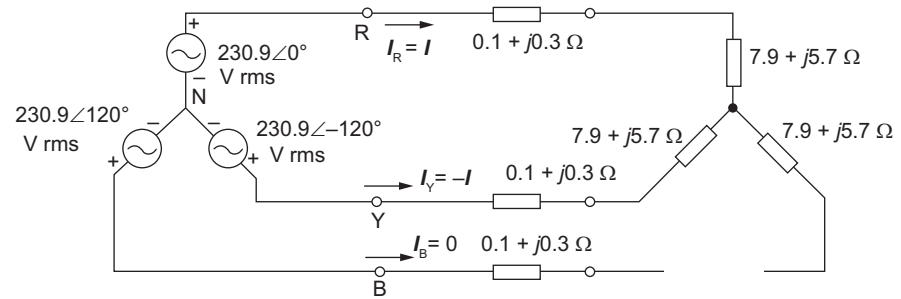


Fig. 9.5-7 The Y-connected Equivalent Circuit of Circuit in Fig. 9.5-6 with B-line Open

- (i) The current phasor I can be obtained by writing the KVL equation in the only mesh that is present in the circuit.

$$I = [230.9\angle 0^\circ - 230.9\angle -120^\circ] \div [16 + j12] = 400\angle 30^\circ \div 20\angle 36.87^\circ = 20\angle -6.87^\circ \text{ A rms.}$$

Therefore, the line currents are $I_R = 20\angle -6.87^\circ$ A rms, $I_Y = -20\angle -6.87^\circ$ A rms and $I_B = 0$.

The symmetrical components of line current are found by using the equation,

$$\begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} I_R \\ I_Y \\ I_B \end{bmatrix}.$$

The sequence components are $I_0 = 0$, $I_+ = 11.545\angle -36.87^\circ$ A rms and $I_- = 11.545\angle 23.13^\circ$ A rms. Observe that the positive sequence and the negative sequence components of current have equal magnitude.

- (ii) The current I divides in two paths in the Δ -connected windings. All the windings have equal impedances. Therefore, the current is shared in 2:1 ratio in RY winding and other two windings in series. The winding currents are 13.33 A rms, 6.67 A rms and 6.67 A rms.

- (iii) Active power delivered to motor = Active power delivered by source – Power dissipated in connection impedance = $400 \times 20 \times \cos(30^\circ - (-6.87^\circ)) - 20^2 \times 0.1 \times 2 \text{ W} = 6.32 \text{ kW}$.

Reactive power delivered to motor = Q delivered by source – Q absorbed by connection impedance = $400 \times 20 \times \sin(30^\circ - (-6.87^\circ)) - 20^2 \times 0.3 \times 2 \text{ VAr} = 4.56 \text{ kVAr}$.

- (iv) Apparent power of source = $230.9 \times 20 + 230.9 \times 20 + 230.9 \times 0 \text{ VA} = 9.236 \text{ kVA}$.
Power factor = $6.4/9.236 = 0.693$.

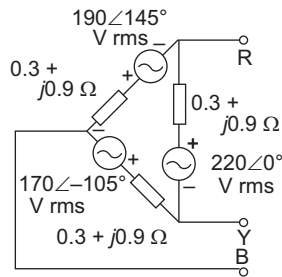


Fig. 9.5-8 Δ -connected Unbalanced Source in Example 9.5-3

EXAMPLE: 9.5-3

Convert the Δ -connected unbalanced voltage source with source impedance in Fig. 9.5-8 into its Y-connected equivalent.

SOLUTION

Let the three source voltage phasors be identified as V_1 , V_2 and V_3 . Let the source impedance be identified as Z . We convert the delta branches into current sources in parallel with impedance form as shown in Fig. 9.5-9.

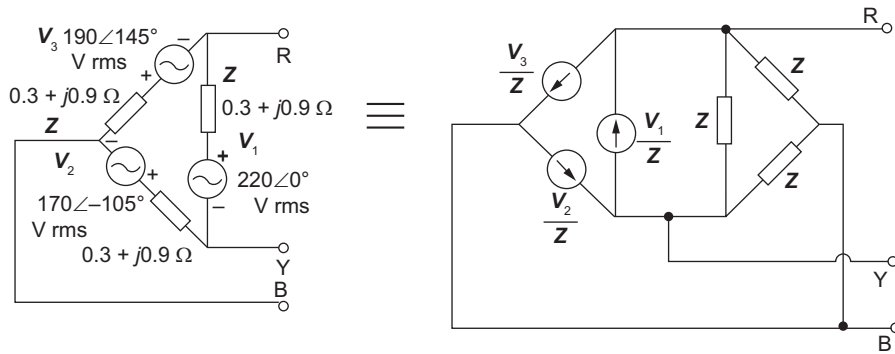


Fig. 9.5-9 Voltage Source to Current Source Transformation Applied to the Source in Fig. 9.5-8

The Δ -connected unbalanced current source is now resolved in terms of its sequence components as shown in Fig. 9.5-10, where $I_+ = V_+/Z$, $I_- = V_-/Z$ and $I_0 = V_0/Z$.

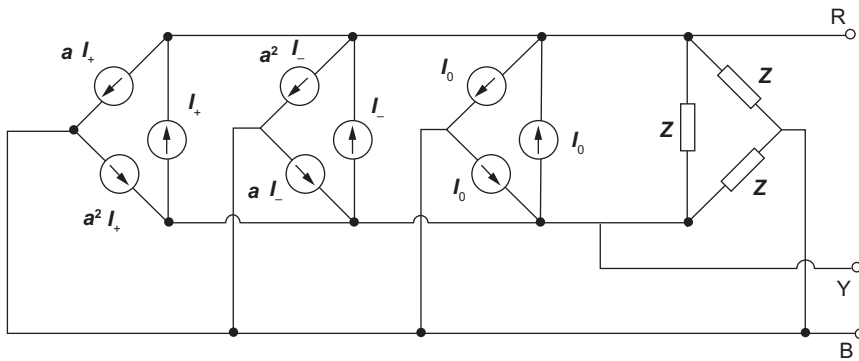


Fig. 9.5-10 Resolution of Three-Phase Current Source in Fig. 9.5-9 in Terms of its Sequence Components

The first two Δ -connected current sources are three-phase balanced sources and have Y-equivalents. Their Y-equivalent is obtained by finding out the first line current delivered by each.

The positive sequence sources deliver $I_+ - aI_+$ into R-line. The star-equivalent must have this as the source function in R-phase. $(1 - a) = \sqrt{3}\angle 30^\circ$. Therefore, the star-equivalent will have a positive sequence current source of value of $\sqrt{3}\angle 30^\circ I_+$.

The negative sequence sources deliver $I_- - \alpha^2 I_-$ into R-line. The star-equivalent must have this as the source function in R-phase. $(1 - \alpha^2) = \sqrt{3} \angle -30^\circ$. Therefore, the star-equivalent will have a negative sequence current source of value $\sqrt{3} \angle -30^\circ I_-$.

The zero sequence current inside the delta circulates within it and cannot come out. Therefore, the Y-equivalent has no zero sequence current source. This is consistent with the fact that the line currents in a three-wire system cannot have zero sequence content.

Therefore, the Y-equivalent of the circuit in Fig. 9.5-10 is as shown in the circuit in Fig. 9.5-11(a). We note that the Δ -connected impedance is also converted into equivalent star.

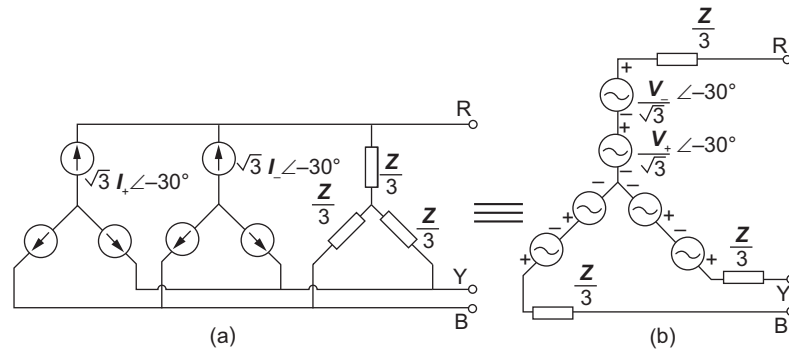


Fig. 9.5-11 Y-equivalent of Circuit in Fig. 9.5-10

Now, all the three three-phase Y-connected sub-circuits in the circuit Fig. 9.5-11(a) are balanced circuits, and hence, the neutrals will be at the same potential. Therefore, they can be joined together. Then, the resulting parallel connection of two current sources and impedance in each phase can be converted into two voltage sources in series with the same impedance. The last step is to substitute for I_+ and I_- in terms of sequence components of delta phase-source voltages. The resulting Y-connected equivalent circuit is shown in the circuit in Fig. 9.5-11(b).

Now, we have determined the positive and negative sequence components of phase voltages of the Y-connected equivalent. They are $\frac{V_+}{\sqrt{3}} \angle -30^\circ$ and $\frac{V_-}{\sqrt{3}} \angle 30^\circ$, respectively. The zero sequence component is zero. Note that there is non-zero zero sequence content in Δ -connected sources.

$$\begin{bmatrix} V_0 \\ V_+ \\ V_- \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Substituting for V_1 , V_2 , V_3 and α , we get $V_0 = 19.62 \angle -69.76^\circ$ V rms, $V_+ = 190.04 \angle 12.6^\circ$ V rms, and $V_- = 36.051 \angle -39.69^\circ$ V rms.

The star equivalent phase voltage is obtained by

$$\begin{bmatrix} V_{RN} \\ V_{YN} \\ V_{BN} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} \frac{1 \angle -30^\circ}{\sqrt{3}} V_+ \\ \frac{1 \angle 30^\circ}{\sqrt{3}} V_- \\ 0 \end{bmatrix} = \begin{bmatrix} 130.38 \angle -16.17^\circ \\ 103.63 \angle -148.11^\circ \\ 98.38 \angle 112.23^\circ \end{bmatrix} \text{ V rms.}$$

9.6 SUMMARY

- A set of three sinusoidal quantities, all at the *same frequency*, with equal peak (and hence equal r.m.s) values and shifted successively by 120° in phase is defined as a *Balanced Three-Phase Quantity*. Therefore, if $x_1(t)$, $x_2(t)$ and $x_3(t)$ is a three-phase set, then,

$$x_1(t) = X_m \cos \omega t, \quad x_2(t) = X_m \cos(\omega t - 120^\circ) \quad \text{and} \quad x_3(t) = X_m \cos(\omega t + 120^\circ) \quad \text{or}$$

$$x_1(t) = X_m \cos \omega t, \quad x_2(t) = X_m \cos(\omega t + 120^\circ) \quad \text{and} \quad x_3(t) = X_m \cos(\omega t - 120^\circ)$$

and $x_1(t) + x_2(t) + x_3(t) = 0$ for all t .

If the peak values are unequal and/or successive phase shifts are different from 120° , the set will be called an *Unbalanced Three-Phase Quantity*. A three-phase source is specified by specifying line-to-line voltage r.m.s values.

- Each limb or branch of a three-phase system (source or load) is termed as a *phase*. R, Y and B are used to designate the line terminals of a three-phase source or load.
- A balanced three-phase system comprising balanced three-phase sources and balanced three-phase loads is superior to a single-phase system due to the following facts: (i) Power loss in transmission system is lower in three-phase system. (ii) Copper utilisation is superior in three-phase system. (iii) Power delivered to a balanced three-phase load by a balanced three-phase supply is free of pulsation. (iv) Electrical equipment designed for three-phase operation is more efficient than their single-phase counter parts.
- The sum of line voltages in any three-phase system is zero since these three voltages are voltages in a loop formed by R to Y, Y to B and B to R.
- The sum of line currents in any three-phase three-wire system is zero since these three currents have to satisfy KCL.
- The line voltages in a balanced Y-connected source form a balanced three-phase set of voltages that are $\sqrt{3}$ times in magnitude and 30° ahead in phase with respect to phase voltages.
- The line currents delivered by a balanced Δ -connected source form a balanced three-phase set of currents that are $\sqrt{3}$ times in magnitude and 30° behind in phase with respect to phase-currents delivered by the phase-sources.
- The three-phase complex power delivered by a three-phase source is given by

$$S = \sqrt{3} V_L I_L \angle \theta = [\sqrt{3} V_L I_L \cos \theta + j \sqrt{3} V_L I_L \sin \theta] \text{ VA},$$
 where θ is the angle by which the current phasor *delivered* by a phase-source lags behind its voltage phasor. This relationship is independent of whether the source is Y-connected or Δ -connected. θ is *not* the angle between line voltage phasor and line current phasor.
- Any balanced three-phase circuit can be equivalenced to a Y–Y balanced three-phase circuit by replacing Δ -connected sources and loads (if any) by their Y-connected equivalents. The equivalence is valid as far as line voltages, line currents and complex power flow are concerned.
- A balanced Y–Y circuit can be solved by solving its R-phase and employing three-phase symmetry to arrive at the solution for Y-phase and B-phase. The R-phase circuit is solved by using the single-phase equivalent circuit.
- The sets of three-phase balanced source components and possible single-phase components of an unbalanced source are called its *Symmetrical Components*.
- There are three symmetrical components for an unbalanced three-phase source function. *Each symmetrical component is a set of three source functions*. The first set – called the *positive sequence component* – is a balanced three-phase set of source functions that has positive phase sequence. The second set – called the *negative sequence component* – is a balanced three-phase set of source functions that has negative phase sequence. The third set – called the *zero sequence component* – is a set of three co-phasal (i.e., of same phase) single-phase source functions. *It is not a three-phase set at all*.
- The zero sequence component is the *average* of the three three-phase quantities. Line voltages in a three-phase system cannot have a zero sequence content anywhere in the system. Line currents in a three-phase three-wire system cannot have a zero sequence content anywhere in the system.
- Active power carried by sequence components obeys superposition principle.
- Positive sequence component of voltage can send active power *only through* positive sequence component of current; negative sequence component of voltage can send active power *only through* negative sequence component of current and zero sequence component of voltage can send active power *only through* zero sequence component of current.
- Balanced impedance is incapable of bringing about a sequence conversion involving translation of positive sequence current into negative sequence voltage and vice-versa. This is called *sequence decoupling property* of balanced three-phase impedances. An unbalanced load produces sequence coupling between positive sequence components and negative sequence components
- The most energy-efficient way to draw active power from a balanced three-phase source is to draw it through balanced three-phase currents at unity power factor.

9.7 QUESTIONS

1. A balanced three-phase source of 400 V, 5 Hz rating is loaded by a balanced load. Each phase of the load is 5×100 W incandescent lamps. Will there be a visible flicker in the lamps? If yes, explain why?
2. The source in the preceding question is used to deliver power to a balanced three-phase AC motor driving an industrial fan. Will there be fluctuations in the fan speed? Assume that the supply voltage and the fan load are maintained at constant levels.
3. The double-frequency oscillation in three-phase instantaneous power is zero in balanced systems even for a reactive load. Does it mean that the three-phase reactive power is zero? If not, why?
4. Three voltage phasors $100\angle 0^\circ$ V, $100\angle -120^\circ$ V and $110\angle 120^\circ$ V cannot be the line voltages of a three-phase source. Explain why?
5. The line currents flowing into a load are found to be $10\angle -90^\circ$ A, $10\angle -200^\circ$ A and $10\angle 30^\circ$ A. Explain why the load cannot be a Δ -connected one.
6. The line currents flowing into a load are found to be $10\angle -30^\circ$ A, $10\angle -150^\circ$ A and $9\angle 90^\circ$ A. Show that the three-phase system has to be a four-wire system and that the load has to be star-connected.
7. The line currents flowing into a load are found to be $10\angle 0^\circ$ A, $5\sqrt{2}\angle -135^\circ$ A and $5\sqrt{2}\angle 135^\circ$ A. Is it possible to decide whether the load is Δ -connected or star-connected from this data? Is it possible to decide whether the load is balanced or unbalanced from this data?
8. The line currents flowing into a load from a balanced three-phase supply are found to be $100\angle -20^\circ$ A, $100\angle -140^\circ$ A and $90\angle 100^\circ$ A. Is it possible to decide whether the load is balanced or unbalanced from this data?
9. The phase-voltage phasors of a Y-connected three-phase voltage source are $100\angle 0^\circ$ V, $100\angle -90^\circ$ V and $100\angle 90^\circ$ V. Calculate the line voltages and explain why line voltage magnitude is not equal to $100\sqrt{3}$ V?
10. Three sinusoidal current sources $10\angle 0^\circ$ A, $10\angle -120^\circ$ A and $8\angle 120^\circ$ A are used to form a three-phase three-wire current source. Explain why this current source has to be a Δ -connected one.
11. A balanced three-phase voltage source of 400 V, 50 Hz delivers 10 A at unity power factor to a balanced three-phase load through a connection impedance of $1 + j0$ Ω . Is the line voltage magnitude across the load equal to $400 - 10 = 390$ V? If not, why and what is the correct value?
12. Find the symmetrical components of three currents $10\angle -35^\circ$ A, $10\angle -35^\circ$ A and $10\angle -35^\circ$ A.
13. Show that no three-phase passive impedance containing R , L and C can draw line currents with only zero sequence component from a balanced three-phase voltage source. [Hint: The magnitude of maximum phase angle possible for any RLC impedance or admittance is 90° .]

14. An Electrodynamicometer type wattmeter is a measuring instrument that measures average power. It has two coils called current coil and pressure coil. The current coil terminals are usually marked M and L and the pressure coil terminals are usually marked V and COM. The meter measures the average of the product $v(t) \times i(t)$, where $v(t)$ is the voltage of COM with respect to V and $i(t)$ is the current flowing from M to L in the meter. In the case of sinusoidal steady-state, the meter measures $V_{\text{rms}} I_{\text{rms}} \cos \phi$, where V_{rms} is the r.m.s. value of voltage applied across pressure coil, I_{rms} is the r.m.s. value of current flowing through current coil and ϕ is the angle between these two phasors. Two such wattmeters can be used to measure active power in a *three-wire system* irrespective of the nature of source and load connections (i.e., star or delta). The wattmeter connection to do this is shown in Fig. 9.7-1

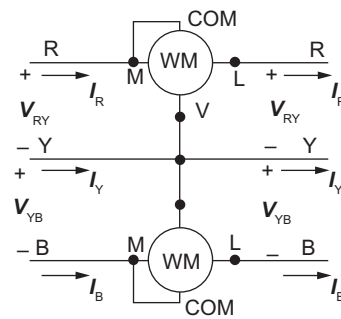


Fig. 9.7-1 Two-Wattmeter Connection

- Let the reference phasor be the R-phase phase voltage in the equivalent Y-connected system and let the current I_R be $I_L \angle -\theta$. Then, show that, under balanced operation, (i) the wattmeter in R-line reads $W_1 = \sqrt{3} V_L I_L \cos(30^\circ + \theta)$ W, (ii) the wattmeter in B-line reads $W_2 = \sqrt{3} V_L I_L \cos(30^\circ - \theta)$ W and (iii) the total active power in the three-phase system is $= W_1 + W_2$ W. Draw the required phasor diagrams.
15. What is the range of load power factor for which the first wattmeter in two-wattmeter method of power measurement in Fig. 9.7-1 will try to read negative value? What is the corrective action needed to take readings? How is the correction incorporated in the expression for total three-phase power?
 16. Wattmeters used in two-wattmeter method are often constructed with reversing switch that will reverse the voltage applied to pressure coil. What is the need for this reversing switch?

9.8 PROBLEMS

1. A 400 V rms Δ -connected source shown in Fig. 9.8-1 has $I_1 = 10\angle-30^\circ$ A rms, $I_2 = 10\angle-120^\circ$ and $I_3 = 10\sqrt{2}\angle105^\circ$ A rms. Find (i) the line currents delivered, (ii) the three-phase active power and reactive power delivered and (iii) a Δ -connected load that will result in this loading.

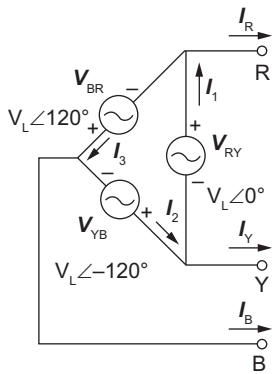


Fig. 9.8-1

2. A star-connected source with zero source impedance delivers $V_{RY} = 200\angle-20^\circ$ V rms, $V_{YB} = 210\angle-130^\circ$ V rms. The R-phase phase voltage V_{RN} is $115\angle0^\circ$ V rms. Find the third line voltage and the remaining two phase voltages. Draw the phasor diagram showing phase voltages and line voltages.
3. A Δ -connected voltage source with zero source impedance delivers $I_R = 20\angle-30^\circ$ A rms, $I_B = 15\angle130^\circ$ A rms. The source that is directly between R-line and Y-line is found to deliver $11\angle0^\circ$ A rms. Find Y-line current and the remaining two phase-source currents. Draw the phasor diagram showing all the currents.
4. A star-connected source with zero source impedance and isolated neutral delivers $V_{RY} = 400\angle-25^\circ$ V rms, $V_{YB} = 390\angle-135^\circ$ V rms. The R-phase phase voltage V_{RN} is $230\angle0^\circ$ V rms. The line currents delivered by the source in R-line and B-line are $I_R = 10\angle-30^\circ$ A rms, $I_B = 12\angle130^\circ$ A rms. Find the three-phase active power and reactive power delivered by the source. Draw the phasor diagram showing phase voltages, line voltages and line currents.
5. A Δ -connected voltage source with zero source impedance delivers $I_R = 5\angle-40^\circ$ A rms, $I_Y = 15\angle-100^\circ$ A rms. The source that is directly between R-line and Y-line is found to deliver $11\angle0^\circ$ A rms. V_{RY} and V_{YB} are found to be $200\angle-0^\circ$ V rms and $200\angle-110^\circ$ V rms, respectively. (i) Find the three-phase active power and reactive power delivered by the source. (ii) Find a Δ -connected impedance that will cause this loading in the source. (iii) Draw the phasor diagram showing phase currents, line currents and line voltages.
6. A star-connected balanced voltage source of 400 V rms, 50 Hz rating delivers $6 - j3$ kVA of complex power to a passive balanced Y-connected load. Use the R-phase phase voltage as reference phasor. (i) Find the line currents. (ii) Determine the load impedance. (iii) Draw the phasor diagrams showing all phase voltages, line voltages and line currents. (iii) What will be the readings in two wattmeters in two-wattmeter method of power measurement as in Fig. 9.7-1?
7. A star-connected balanced voltage source of 400 V rms, 50 Hz rating with a source impedance of $1 + j1 \Omega$ in series with the source in each phase delivers power to a Δ -connected balanced load impedance with a branch impedance of $20 + j20 \Omega$ at 50 Hz. Find the three-phase complex power delivered to the load.
8. A Δ -connected balanced voltage source of 200 V rms, 60 Hz rating with zero source impedance delivers power to a Δ -connected balanced load impedance with a branch impedance of $8 - j6 \Omega$ through impedance-less connection. Find the line currents, load-branch currents and three-phase complex power delivered to the load without transforming the circuit into its Y-Y equivalent. Draw the phasor diagram for the circuit taking V_{RY} as the reference phasor.
9. A Δ -connected balanced voltage source of 200 V rms, 60 Hz rating with zero source impedance delivers power to a Δ -connected balanced load impedance with a branch impedance of $8 + j6 \Omega$ through a connection impedance of $1 + j1 \Omega$ in each line. Find the line currents, load-branch currents and three-phase complex power delivered to the load by employing single-phase equivalent circuit method. Draw the phasor diagram for the circuit taking V_{RY} as the reference phasor.
10. A Δ -connected balanced voltage source of 400 V rms, 50 Hz rating with zero source impedance delivers power to a star-connected balanced load impedance with a branch impedance of $11 - j17 \Omega$ through a connection impedance of $1 + j1 \Omega$ in each line. Find the line currents, load-branch currents and three-phase complex power delivered to the load by using star-delta transformation of impedances.
11. A star-connected balanced voltage source of 400 V rms, 50 Hz rating with zero source impedance delivers power to a star-connected balanced load impedance with a branch impedance of $10 - j10 \Omega$ in parallel with a balanced Δ -connected impedance with a branch impedance of $30 + j60 \Omega$ through a connection impedance of $1 + j1 \Omega$ in each line. Find the line currents, load-branch currents in both loads, three-phase complex power delivered to each load, the complex power delivered by source and source power factor by using single-phase equivalent circuit method.
12. A star-connected balanced voltage source of 200 V rms, 60 Hz rating has an internal resistance of 1Ω in series with an internal inductance of 3 mH in each phase. It supplies power to an unbalanced Y-connected load with 10Ω , 15Ω and 8Ω in R, Y and B phases, respectively through a three-wire connection with a connection impedance of $0.8 \Omega + 4$ mH inductance in each wire. The neutral of the source is firmly earthed. Find (i) the

- neutral potential with respect to earth at the load, (ii) the line currents, line and phase voltages across the load and (iii) the active power delivered to each phase of the load and the total active power.
13. Repeat Problem 12 if it is a four-wire connection with same connection impedance in the neutral wire. Also find the neutral wire current.
 14. A star-connected unbalanced voltage source with $V_{RN} = 115\angle 0^\circ$ V rms, $V_{YN} = 115\angle -110^\circ$ V rms and $V_{BN} = 115\angle 135^\circ$ V rms delivers power to a balanced Δ -connected load with a branch-impedance of $23.4 + j17.1 \Omega$ through a three-wire line with a line impedance of $0.2 + j0.3 \Omega$. Determine (i) line currents, (ii) line voltages across the load, (iii) active power and reactive power delivered by each phase of the source and the three-phase values and (iv) three-phase active power and reactive power delivered to the load by using star-delta transformation on load impedance and mesh analysis.
 15. A Δ -connected unbalanced voltage source uses $115\angle 0^\circ$ V rms, $110\angle -110^\circ$ V rms and $122\angle 125^\circ$ V in the three branches R–Y, Y–B and B–R, respectively. Each branch source has $1 \Omega + 3$ mH inductance in series with it. The frequency of the system is 60 Hz. This source is delivering power to an unbalanced Δ -connected load that has 10Ω resistor, 10Ω resistor in series with 30 mH inductor and 10Ω resistor in series with $1000 \mu\text{F}$ capacitor between R–Y, Y–B and B–R, respectively. Determine (i) line currents, (ii) source and load branch currents, (iii) line voltages across the load, (iv) active power and reactive power delivered by each phase of the source and the three-phase values and (v) three-phase active power and reactive power delivered to the load by using star-delta transformation on load impedance and mesh analysis.
 16. The positive sequence component of line current drawn by a balanced Δ -connected load of branch impedance $8 + j6 \Omega$ is $20\angle -30^\circ$ A rms. The negative sequence component of line voltage across the load is $15\angle -90^\circ$ V rms. Find (i) the line voltages across the load and line currents, (ii) the three-phase active power and reactive power delivered to the load from three-phase quantities and (iii) the three-phase active power and reactive power delivered to the load from sequence components.
 17. An unbalanced star-connected voltage source delivers power to a balanced star-connected load with $8 - j6 \Omega$ in each phase over three-wire impedance-less connection. The load draws a line current that has $10\angle 0^\circ$ A rms positive sequence component and $2\angle -90^\circ$ A negative sequence component. The source neutral is earthed solidly and the load neutral is found to be at $10\angle -150^\circ$ V rms with respect to earth. Find (i) the phase voltages and line voltages across the source and load and (ii) the active power and reactive power delivered by the source and delivered to the load.
 18. An unbalanced star-connected voltage source delivers power to a balanced star-connected load with $7.9 + j5.7 \Omega$ in each phase over a three-wire connection with a wire-impedance of $0.1 + j0.3 \Omega$ in each wire. The load draws a line current that has $10\angle 0^\circ$ A rms positive sequence component and $2\angle -90^\circ$ A rms negative sequence component. The source neutral is earthed solidly and the load neutral is found to be at $10\angle 50^\circ$ V rms with respect to earth. Find (i) the phase voltages and line voltages across the source and load and (ii) the active power and reactive power delivered by the source and delivered to the load.
 19. An unbalanced star-connected voltage source delivers power to a balanced star-connected load with $7.9 + j5.7 \Omega$ in each phase over a four-wire connection with a wire-impedance of $0.1 + j0.3 \Omega$ in each wire. The load draws a line current that has $10\angle 0^\circ$ A rms positive sequence component and $2\angle -90^\circ$ A rms negative sequence component. The neutral return current is found to be at $1.5\angle -100^\circ$ A rms. Find (i) the phase voltages and line voltages across the source and load and (ii) the active power and reactive power delivered by the source and delivered to the load.
 20. The magnitudes of line voltages at a certain point in a three-phase system are measured as 400 V rms, 380 V rms and 410 V rms between R–Y, Y–B and B–R, respectively. Find the symmetrical components of the line voltage. [Hint: The equation $V_{RY} + V_{YB} + V_{BR} = 0$ implies that line voltage phasors in a three-phase system will have to form a closed triangle.]
 21. The line voltages of an ideal three-phase unbalanced source are measured to be 420 V rms, 390 V rms and 380 V rms between R–Y, Y–B and B–R, respectively. This source drives a balanced RLC load through a three-wire connection with wire-impedance of $1 + j3 \Omega$ in each wire. If the magnitude of positive sequence component of line voltage across load is found to be 380 V rms, find the magnitude of negative sequence component of line voltage across the load.
 22. An induction motor has different impedance values for positive sequence and negative sequence components, though it is balanced impedance for both. Let the impedance for positive sequence component be $10 + j5 \Omega$ and for negative sequence component be $0.5 + j5 \Omega$ for a motor under a particular loading condition. Find the sequence components and three-phase values of line voltage across motor if it is used as the load in Problem 21.

Part Four

Time-Domain Analysis of Dynamic Circuits

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Simple RL Circuits in Time-Domain

CHAPTER OBJECTIVES

- *Series and parallel RL circuits and their differential equations.*
- *Initial condition; its need and interpretation*
- *Solving first order differential equation by power series, integrating factor, etc.*
- *Complementary function, particular integral and their interpretation.*
- *Natural response, transient response and forced response in an RL circuit.*
- *Interpretation of various response components*
- *Nature and details of step response of RL circuit and time-domain specifications based on this.*
- *Time constant and various interpretations for it.*
- *Steady-state response versus forced response and various kinds of steady-state response.*
- *Zero-input response, zero-state response and their interpretation.*
- *Linearity and superposition principle as applied to various response components.*
- *Impulse response and its importance.*
- *Equivalence between impulse forcing function and non-zero initial condition.*
- *Zero-state response for other inputs from impulse response.*
- *Relation between $\delta(t)$, $u(t)$ and $r(t)$ and corresponding responses.*
- *Zero-state response of RL circuit for exponential and sinusoidal inputs.*
- *Frequency response of RL circuit.*
- *General analysis procedure for single time constant RL circuits.*

This is the most important chapter in the book. It introduces many basic concepts of linear dynamic system analysis in time-domain using a simple RL circuit as an illustrative system. After a study of this chapter and the problems at the end, the reader is expected to become thoroughly conversant with the topics listed above as well as with first order systems in general.

INTRODUCTION

Energy storage elements – inductance and capacitance – lend memory to electrical circuits and make possible the interesting dynamic behaviour in them. The simplest of dynamic circuits will contain one such energy storage electrical element along with passive resistors and linear dependent sources. Further simplification is possible by ruling out dependent sources

and limiting the resistors to one. We consider one such two-element circuit, containing one energy storage element, in detail in this chapter. Specifically, we study series and parallel RL circuits with voltage source and current source excitations of different waveshapes with and without initial energy storage in the inductance.

These circuits are simple in structure. However, it does not imply that their dynamic behaviour is trivial. The simple nature of these circuits does not imply that they are unimportant. In fact, simple RL and RC circuits find applications in almost all areas of Electrical and Electronics Engineering. Moreover, they display almost all the important features of dynamics displayed by the general class of ‘Linear Time-Invariant (LTI) Circuits’. They form very simple and effective circuit examples to convey important concepts in the analysis of LTI circuits.

Circuits containing a large number of inductors, capacitors and dependent sources will require simulation tools for quantitative analysis. However, a thorough understanding of the behaviour of simple circuits like the ones covered in this chapter will help an Engineer to develop qualitative insights into such circuits and arrive at approximate quantitative results. No complex or costly circuit simulation package is a satisfactory substitute for this kind of analytic capability in an Engineer.

We use a simple RL circuit to bring out many basic concepts in linear dynamic circuits in this chapter. The definitions, principles, concepts and analysis techniques explained in this chapter form a coherent set and have to appear in a single chapter. The material covered in this chapter will be used extensively in the rest of the book. That is why this is a long chapter and quite an important one. The reader will need to come back to this chapter quite often when he covers more advanced topics in the later part of the book.

10.1 THE SERIES RL CIRCUIT

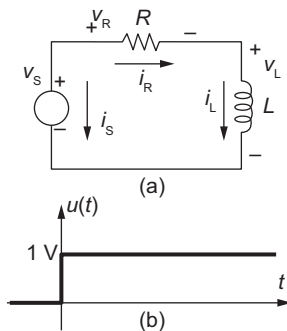


Fig. 10.1-1 (a) The Series RL Circuit (b) Unit Step Voltage Waveform

The series RL circuit with all circuit variables identified as per the passive sign convention is shown in Fig. 10.1-1. Unit step voltage applied to the circuit is also shown in the same figure. The circuit contains three two-terminal elements. Each two-terminal element is associated with two variables – a voltage variable and a current variable. We use the passive sign convention, and in passive sign convention, the positive current enters the element at the positive of voltage variable assigned to the element. Thus, v_L and i_L are the voltage and current variables of the inductor as per the passive sign convention. The corresponding variables for the remaining two elements are shown in Fig. 10.1-1 (b). All these six variables are functions of time though the dependence on time is not explicitly shown.

10.1.1 The Series RL Circuit Equations

The analysis of this circuit aims at the solution for all these six variables as functions of time. The circuit has three nodes. We know that KCL has to be satisfied at each node. Applying KCL at these three nodes will give us three equations that tie up the three current variables. However, only two of them will be independent. Thus, we have two independent equations constraining three current variables and that will result in a single current variable to be solved for. To summarise, we have three elements that are in series and they will, therefore, have a common current flowing through them, and hence, the following equation is valid at all t .

$$i_L(t) = i_R(t) = -i_s(t).$$

The circuit has one mesh and it has to satisfy KVL at all t in this mesh. Applying KVL in this mesh will give us an equation tying up the three voltage variables – v_s , v_R and v_L , and thereby, reducing the number of independent variables among the three to two. The relevant KVL equation is

$$\begin{aligned} -v_s(t) + v_R(t) + v_L(t) &= 0 \text{ for all } t \\ \therefore v_R(t) &= v_s(t) - v_L(t) \text{ for all } t. \end{aligned}$$

Thus, we have eliminated v_R by expressing it as the difference between the other two voltage variables v_s and v_L . v_s is a source function – it is the time-function representing the voltage waveform of an independent source, and hence, it is a known function. Therefore, there is only one voltage variable and we choose that to be v_L .

Now, we have two variables – v_L and i_L . We also have the element equations of the inductor and resistor to tie them up and get a single equation in i_L as follows:

$$\begin{aligned} v_R &= Ri_R = Ri_L \text{ and } v_L = L \frac{di_L}{dt} \\ v_L + v_R &= v_s \text{ by KVL in the mesh} \\ \therefore L \frac{di_L}{dt} + Ri_L &= v_s \text{ for all } t \\ \text{i.e., } \frac{di_L}{dt} + \frac{R}{L} i_L &= \frac{1}{L} v_s \text{ for all } t. \end{aligned}$$

The current in a series RL circuit is described by a first order linear differential equation with constant coefficients.

Thus, a *first order linear differential equation with constant coefficients* describes the behaviour of current in a series RL circuit for all time t . The *order* of a differential equation is the degree of the highest derivative appearing in the equation and it is one in this equation. The differential equation is linear since all derivatives of the unknown function i_L and the function i_L itself appear in the equation with an exponent of unity, the three coefficients in the three terms in the equation are not functions of i_L and the forcing function on the right-hand side of the equation is a function of only the time and is not dependent on the unknown function i_L in anyway. R and L are circuit parameters that are assumed to be constants. Therefore, the coefficients of differential equation turn out to be constants.

10.1.2 Need for Initial Condition Specification

The differential equation derived above describes the variation of the inductor current for all t in the time-domain. It is so because we have used KCL and KVL in deriving it and these two laws hold good at all time instants. Does it mean that we will be able to solve for $i_L(t)$ for all t – from infinite past to infinite future? Yes, provided we know the function $v_s(t)$ for the entire time range $-\infty < t < \infty$ and that is going to be a problem!

Do we really have to get back as far as $-\infty$ in the time axis? Not really. We need to get back to the time instant at which our inductor was manufactured. It is indeed difficult to manufacture an inductor such that it takes birth with some initial energy trapped in it! Therefore, we may safely assume that the inductor had zero initial energy, and hence, zero initial current (energy storage in an inductor is proportional to the square of the current flowing through it) when it was manufactured. In fact, when we talk of infinite past we have this time instant in mind. Therefore, we can be sure that the inductor had zero current in it in the infinite past, i.e., at $t = -\infty$. Now, if we know the time function of applied voltage $v_s(t)$ from that instant onwards up to the present instant, we will be able to solve for $i_L(t)$ from $t = -\infty$ to the present instant by integrating the governing differential equation. But did not we assume that the inductor was connected along with the resistor and the source right at the instant it was manufactured and it remained connected so from then on? We cannot assume any such thing. Therefore, we must know all that had happened to the inductor from the instant it was manufactured to the instant it was connected in the circuit we are trying to solve, if we are to solve this circuit. This is due to the fact that the function $v_s(t)$ can, at best, be known only from the instant at which this circuit came into existence and the inductor may have been subjected to various voltages in various other circuits before this circuit was wired up. In that case the inductor will be carrying a current:

$$I_0 = \frac{1}{L} \int_{-\infty}^{t_{\text{create}}} v_L dt,$$

where $-\infty$ refers to the instant of manufacturing of the inductor and t_{create} refers to the instant at which the RL circuit under discussion came into being. The voltage $v_L(t)$ in the above integral refers to all the voltage that was applied to the inductor during this time interval. Thus, the inductor carries its accumulated past in the form of an initial current I_0 given by the above integral when it enters the RL circuit which we are trying to analyse. Notice that the voltage appearing in the integrand has no relation with the source that is applied subsequently to the RL circuit.

The value of current specified for an inductor at some instant condenses all the past history of the inductor prior to that instant.

Precisely, we need to know the past of inductor. Fortunately we need not know everything about its past – we need only the value of the above integral. We will be able to solve for $i_L(t)$ for all instants after t_{create} if we know the value of I_0 along with $v_s(t)$ from the instant it was applied, i.e., from t_{create} . Obviously, we need to know everything about its past if we want to solve for $i_L(t)$ from $t = -\infty$ onwards. That is too much of a past to carry. Therefore, we would want to solve for $i_L(t)$ only for $t \geq t_{\text{create}}$. We need the value of I_0 , the initial current in the inductor at an instant just prior to t_{create} , for that. This single number condenses all the past history of the inductor as far as the effect of voltages applied to it in the past on the evolution of its current in future is concerned. This number is called the *initial condition* for the inductor.

The time instant t_{create} is to be understood as the instant from which we know the data required for solving the differential equation – the time instant at which the initial condition is specified for the inductor – and it is the time instant from which we have complete knowledge about the input source voltage. It is the time instant from which the inductor should act as an element in this circuit and only in this circuit. It is customary in Circuit Theory to set this instant as the time-zero instant, i.e., $t_{\text{create}} = 0$, unless there is some specific reason for making it otherwise.

Usually some kind of switching action takes place in the circuit at this instant. It could be a switching involving application of a specific voltage waveform at its input. It could also be a switching operation which changes the structure of the circuit – for example, one element in the circuit may have been kept shorted by closing a switch across it, and now at $t = 0$, that switch is opened. Such switching action usually brings in jump discontinuities in circuit variables. Jump discontinuities in variables involved in differential equations are difficult to handle mathematically unless singularity functions are brought in. We do not want to do that in this book. Our way of handling such discontinuities in circuit variables will be circuit-theoretic and we need to define two more time instants to facilitate our circuit-theoretic reasoning in such situations. These two time instants are $t = 0^-$ and $t = 0^+$. $t = 0^-$ is a time instant which is to the left of $t = 0$ in the time axis. However, the time interval $[0^-, 0]$ is of infinitesimal width, i.e., 0^- is arbitrarily close to 0, but always less than 0. Similarly, 0^+ is to the right of 0 and is arbitrarily close to 0. Thus, $0^- < 0 < 0^+$ while $0 - 0^- \approx 0$ and $0^+ - 0 \approx 0$.

The three important time instants $t = 0$, $t = 0^-$ and $t = 0^+$ are introduced. The current in the inductor can be different at 0^+ and 0^- under certain conditions.

10.1.3 Sufficiency of Initial Condition

We defined three important time instants – $t = 0$, 0^- and 0^+ . We also concluded that, with the initial condition for i_L specified and with the knowledge of v_s from the instant at which the initial condition is specified, we will be able to solve the differential equation for i_L . However, which among the three instants above should be used for specifying the initial condition?

The three time instants mentioned have been defined only to make it easier to handle possible discontinuities in the input function v_s . If the source function v_s is a continuous function of t , we do not need $t = 0^-$ and $t = 0^+$. In that case, the initial condition is specified at $t = 0$. However, if v_s has a discontinuity or singularity (for example, a jump discontinuity as in step function or an impulse located at $t = 0$) at $t = 0$, it is possible that either i_L or its

derivative will undergo step changes at $t = 0$. Therefore, the time instant at which the initial condition is applicable must be clearly specified. That instant has to be $t = 0^-$. Since it will not make any difference in the case where v_s is a continuous function, we will stipulate that the initial condition always be specified at $t = 0^-$.

The initial current in the inductor at $t = 0^+$ can be different from its value specified at $t = 0^-$ for certain kinds of input functions. We will illustrate how the initial condition value at 0^+ can be calculated from the initial condition value at 0^- and the nature of v_s later in this chapter. We have to address another issue at present.

We now assert that $i_L(t)$ for $t > 0^+$ can be obtained if the initial condition I_0 at $t = 0^-$ is known and $v_s(t)$ for $t \geq 0^+$ is known. This is equivalent to asserting that the net effect of all the voltage applied across the inductor in the time range $-\infty < t \leq 0^-$ on the evolution of its current in the time range $t \geq 0^+$ is encoded in a single number I_0 , the initial condition at $t = 0^-$. This, in effect, says that the past which the inductor remembers is contained in I_0 .

They are strong assertions and need to be proved. We offer a plausibility reasoning to convince ourselves that these assertions are true. We assume that the function $v_s(t)$ is continuous at $t = 0$ in the reasoning that follows.

We recast the circuit equations of the series RL circuit in Fig. 10.1-1 in the following manner.

$$v_L(t) = v_s(t) - v_R(t) \text{ for all } t \text{ (from KVL)}$$

$$\text{i.e., } v_L(t) = v_s(t) - Ri_L(t) \text{ for all } t$$

$$\text{i.e., } v_L(t) = v_s(t) - Ri_L(t) \text{ (since } i_L(t) = i_R(t))$$

$$\text{i.e., } v_L(t) = v_s(t) - \frac{R}{L} \int_{-\infty}^t v_L(t) dt \text{ for all } t \text{ (from element equation of inductor).}$$

Further, we rewrite the last equation by splitting the range of integration into two sub-ranges. Notice the change in the time-range of applicability.

$$v_L(t) = v_s(t) - \frac{R}{L} \int_{-\infty}^0 v_L(t) dt - \frac{R}{L} \int_0^t v_L(t) dt \text{ for } t \geq 0.$$

We recognise the second term on the right side as $(R/L)I_0$, where I_0 is the initial condition for the inductor. Therefore, we write

$$v_L(t) = v_s(t) - \frac{R}{L} \int_0^t v_L(t) dt - \frac{R}{L} I_0 \text{ for } t \geq 0 \quad (10.1-1)$$

It is obvious from Eqn. 10.1-1 that the solution for $v_L(t)$ for $t \geq 0$ will depend only on I_0 and the values of v_s for $t \geq 0$. A simple method to integrate the equation numerically is outlined below.

Divide the time interval $[0, t]$ into many small intervals, each of width Δt . Let N be the number of such intervals. Then,

$$v_L(0) = v_s(0) - \frac{R}{L} I_0$$

$$v_L(n\Delta t) = v_s(n\Delta t) - \frac{R}{L} v_L((n-1)\Delta t) - \frac{R}{L} I_0, n = 1 \dots N.$$

The equation gives $N + 1$ values of $v_L(t)$ in the interval $[0, t]$ at equally spaced sub-intervals. The accuracy of calculation can be improved by increasing the number of sub-intervals (i.e., by decreasing Δt). $i_L(t)$ can be found by evaluating the first derivative of $v_L(t)$ and on multiplying it by L once $v_L(t)$ is calculated with sufficient accuracy.

Initial Value of Inductor Current

The net effect of all the voltage applied across the inductor in the time range $-\infty < t \leq 0^-$ on the evolution of its current in the time range $t \geq 0^+$ is encoded in a single number I_0 , the initial condition at $t = 0^-$.

$i_L(t)$ for $t \geq 0^+$ can be obtained if the initial condition I_0 at $t = 0^-$ is known and $v_s(t)$ for $t \geq 0^+$ is known.

Thus, I_0 specified at some time instant and $v_s(t)$ from that time instant will be a sufficient set of data needed to solve for the inductor current in an RL circuit from that time instant onwards.

10.2 SERIES RL CIRCUIT WITH UNIT STEP INPUT – QUALITATIVE ANALYSIS

In this section, we study the behaviour of inductor current in a series RL circuit excited by a unit step voltage source qualitatively. The initial current in the inductor at $t = 0^-$ is assumed to be zero for the purpose of this analysis.

Unit step voltage is defined in the following manner. It has a step discontinuity at $t = 0$, and strictly speaking, the time instant $t = 0$ is excluded from the domain of the function. However, it is customary in Circuit Theory to represent this function with a solid line jumping from 0 to 1 at $t = 0$ in its plot as shown in Fig. 10.2-1

$$u(t) = \begin{cases} 0 & \text{for } t \leq 0^- \\ 1 \text{ V} & \text{for } t \geq 0^+ \end{cases}$$

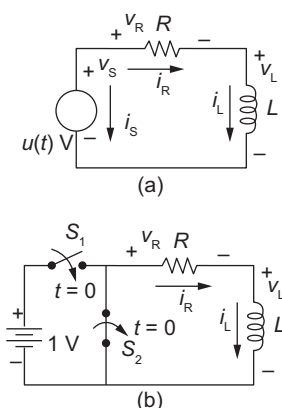


Fig. 10.2-1 (a) Series RL Circuit with Unit Step Voltage Input (b) A Practical Arrangement for Unit Step Input

The relevant circuit with the manner in which the unit step voltage is realised in practice is shown clearly in the Fig. 10.2-1.

The switch S_2 was closed at $t = -\infty$ and is opened at $t = 0$ and remains open up to $t = +\infty$. The switch S_1 is open from $t = -\infty$, is closed at $t = 0$ and remains closed up to $t = +\infty$. We assume that the switches open and close in zero time intervals. The switch S_2 applies 0 V to the circuit for all $t \leq 0^-$ and the switch S_1 applies the 1 V available from the DC Battery for all $t \geq 0^+$, thereby, effecting unit step voltage application to the circuit.

Are we justified in joining the 0 V point at $t = 0^-$ to the 1 V point at $t = 0^+$ by a vertical straight line at $t = 0$ in the unit step voltage waveform despite the fact that the point at $t = 0$ is excluded from the domain of the function $u(t)$? The question arises because we admit the discontinuity at $t = 0$, but we are asserting that the step source voltage is somewhere between 0 to 1 V at $t = 0$. Why don't we assume instead that it shoots up to, say 1000 V from 0 V and falls back to 1 V, all in zero time?

We can answer that only by going into the details of switching the switches S_1 and S_2 . Let us, for a moment, assume that the switches do not operate instantaneously; rather they take a finite time – about 1 nS – to operate. Let us assume further that the switches are equivalent to 1000 M Ω resistors when they are open and equivalent to short-circuit when they are closed. This assumption is indeed reasonable and is easily met by high-speed electronic switches. Now, we visualise the switching process as one in which a resistance value changes from a large resistance level to zero resistance level (or the other way) in 1 nS. It is possible to control the electronic switches in such a way that this change in resistance takes place linearly over the switching time. Thus, in the present case, we have S_1 resistance changing from 1000 M Ω to 0 Ω , while S_2 resistance changes from 0 Ω to 1000 M Ω over 1 nS. Forget for a moment that the series combination of R and L is connected across them and assume the junction between S_1 and S_2 to be unloaded. In that case, the potential across the switch S_2 will vary linearly from 0 V to 1 V in 1 nS under the above assumptions. Now, even if the series combination of R and L is connected, it can result only in making the voltage variation non-linear due to the current-loading effect at the junction without affecting the end point values. The starting and ending voltages will be 0 V and 1 V, respectively. Further, the voltage across S_2 will be between 0 V and 1 V during the entire switching period of 1 nS. Now, we assume that our switches have become extremely fast and they take only infinitesimally small time to switch. Within that limit, we send the switching time to zero resulting in the linear variation of voltage across S_2 becoming a vertical jump between the endpoints of 0 V and 1 V. Hence, we are indeed justified in assuming the waveshape of $u(t)$ as in Fig. 10.2-1.

10.2.1 From $t = 0^-$ to $t = 0^+$

The initial current in the inductor at $t = 0^-$ is stated to be 0 A. The voltage applied by the unit step input source has a jump discontinuity from 0 V to 1 V at $t = 0$. Can this jump discontinuity result in a change in inductor current at $t = 0^+$? We relate the possible change in inductor current over the time interval $[0^-, 0^+]$ to the voltage across the inductor during that interval by employing the element equation of inductor.

$$i_{L(0^+)} - i_{L(0^-)} = \frac{1}{L} \int_{0^-}^{0^+} v_L dt.$$

The resistor and the inductor always share the input voltage. Thus, the resistor too may absorb a part of the input voltage during the interval $[0^-, 0^+]$ if there is a current in the circuit. Therefore, the voltage available to the inductor during this interval can only be less than what is available in the source. More importantly, this voltage can assume only finite (though not determinate in view of the discontinuity in the input function) values in this interval. The interval is infinitesimal in width. The area under a finite-valued function over an infinitesimal interval is zero. Therefore, $i_{L(0^-)} = i_{L(0^+)}$, i.e., the inductor current is continuous across the interval $[0^-, 0^+]$. It takes an impulse function at $t = 0$ to generate finite area over infinitesimal time interval. Thus, we conclude that *the initial condition values at $t = 0^-$ and $t = 0^+$ for an inductor in any circuit will be the same unless the circuit can support impulse voltage across that inductor at $t = 0$.*

There is a difference between *applying impulse voltage* and *supporting impulse voltage*. If we connect an independent voltage source with its source function $= \delta(t)$ to a circuit, then we are *applying* an impulse voltage to the circuit. Imagine we are connecting an independent current source which suddenly changes its current from 0 A to 1 A at $t = 0$ to an inductor. Now, the inductor has to change its current from 0 A to 1 A instantaneously at $t = 0$ and it will produce a back emf of $L\delta(t)$ V across it in that process. The current source has to absorb that voltage in order to satisfy Kirchhoff's Voltage Law, but ideal current sources do not complain even if they are called upon to support infinite voltage. Hence, in this instance, the circuit *supports* impulse voltage, but does not *apply* it.

Inductor current remains continuous in the $[0^-, 0^+]$ time interval unless the circuit applies or supports impulse voltage across it.

10.2.2 Inductor Current Growth Process

The initial condition for inductor at $t = 0^-$ was stated to be zero in the circuit we are analysing. This implies that the initial energy storage in the inductor is zero. This state of zero initial energy is also indicated often by phrases '*initially at rest*' and '*initially relaxed*'. The input voltage is a unit step function and it was shown to remain within finite limits at all instants. Thus, the current in the inductor cannot change over the small time interval between $t = 0^-$ and $t = 0^+$. Therefore, current in the circuit at $t = 0^+$ is zero.

However, the input voltage which was at 0 V at $t = 0^-$ had become 1 V at $t = 0^+$, while the current remained at zero value. The resistor can absorb only 0 V with a zero current flowing through it. This implies that the voltage across the inductor will undergo a sudden jump at $t = 0$ from 0 V to 1 V. In fact, all sudden changes in the input voltage of a series *RL* circuit will have to appear across the inductor, i.e., the resistor will refuse to involve itself with the sudden jumps in the input voltage and will dump all such jumps on to the inductor. This is so because if the resistor absorbs even a small portion of the jump in input voltage, the circuit current too will have to have a jump discontinuity, but the inductor does not allow that unless the circuit can apply or support impulse voltage. Therefore, the inductor insists that the current through it remains continuous (unless impulse voltage can be supported) and it is willing to pay the price for that by absorbing the discontinuities in the input source voltage. Since the inductor will maintain continuous current, the voltage across the resistor too will remain continuous. In fact, this is one of the applications of *RL* circuit – to make the voltage across a load resistance smooth when the input voltage is choppy.

All sudden changes in the input voltage of a series *RL* circuit will have to appear across the inductor.

The current in an inductor is related to the voltage across it through the relation

$$v_L = L \frac{di_L}{dt}.$$

This shows that, with a voltage of 1 V at $t = 0^+$ across it, the current in the inductor cannot remain at its current value of zero forever. The current has to grow since its derivative is positive, and hence, it starts growing at the rate of $1/L$ A/s initially. This in turn implies that the slope of current versus time curve at $t = 0^+$ will be $1/L$ A/s.

However, as the current in the inductor grows under the compulsion of the voltage appearing across it, the resistor starts absorbing voltage. This results in a decrease in the inductor voltage since v_R and v_L will add up to 1 V by KVL at all $t \geq 0^+$. A decrease in v_L results in a decreasing rate of change of i_L since the rate of change of i_L is directly proportional to v_L . This means that the tangent drawn on the i_L versus t curve will start with a slope of $1/L$ A/s at $t = 0^+$ and will tilt progressively towards the time axis as t increases. The current in the inductor which starts out with a certain initial momentum loses its momentum with time and grows at a slower rate as time increases, due to the resistor robbing an increasing portion of source voltage from the inductor. Therefore, we expect the current function to be of convex shape.

Will the growth process ever end and will the inductor current reach a steady value i.e., a constant value? Let us assume that it does so. Then, what is the value of current that can remain constant in the circuit? If the current is constant, the inductor will demand only zero voltage for allowing that current. That will mean that all the source voltage, i.e., 1 V will have to be absorbed by the resistor. The resistor will do that only if the current through it is $1/R$ A. Therefore, if the circuit can reach the end of the growth process and become steady, it will do so only at this unique value of current of $1/R$ A. Thus, $i_L = 1/R$ A is a possible of steady end to the growth process we described above.

However, the inductor never allows the circuit to reach this steady state and the current will never really touch the steady value of $1/R$. To see this point clearly, let us assume that the current reaches this critical value at some instant t and then remains at that value forever as in Fig. 10.2-2(a). But then, the current function will have a kink at the time instant at which this happened. Therefore, the first derivative of current will have a jump discontinuity at that time instant. The first derivative of current multiplied by L is v_L , and therefore, v_L too will have a jump discontinuity at that instant. But the sum of v_R and v_L has to be 1 V. Therefore, v_R will also have a jump discontinuity at that instant. This will mean that i_L also must have a jump discontinuity! The only conclusion we can draw is that our assumption about i_L reaching the theoretical steady-state value of $1/R$ A and remaining at that value thereafter cannot be true.

It is possible that the current attains a steady state as in Fig. 10.2-2(b) without a kink in the waveform. Here, the current is smooth, and hence, its first derivative is continuous, but the first derivative will have a kink at that instant, and hence, the second derivative will have a jump discontinuity at that instant. But then, the first and second derivatives of current in the inductor in this circuit with step voltage input will have to be proportional to each other at all $t > 0^+$ (show this). This leads to a contradiction showing that the assumed current waveform cannot be correct.

The current in series RL circuit with a unit step voltage input shows a tendency to approach $1/R$ A as time increases, but never touches that value. It keeps growing all the time; though, at progressively decreasing rate with time. Theoretically speaking, it never gets done.

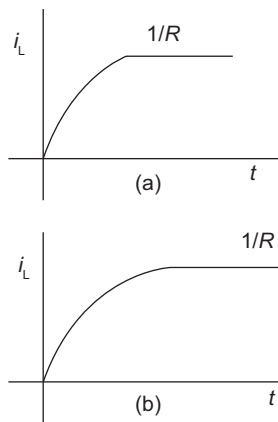


Fig. 10.2-2 Assumed Current Waveforms in RL Circuit Unit Step Response

10.3 SERIES RL CIRCUIT WITH UNIT STEP INPUT – POWER SERIES SOLUTION

The current in the RL circuit would have gone to $1/R$ A immediately on unit step voltage application had the inductance in the circuit been zero. Moreover, the current in the circuit would have tracked the applied voltage without any delay for any applied voltage waveform in that case. Thus, we see that the presence of inductance in the circuit introduces a *delay*

in the tracking performance of the circuit current with respect to applied voltage. In addition, the inductor does not ever permit the current to reach the level that it would have reached had the inductance been zero. For example, the current in the circuit with unit step voltage input does not ever reach $1/R$ A; it can reach arbitrarily close to that value as time increases, but it will never become equal to that value. Alternatively, we should say it would reach this final value as t tends to ∞ .

Infinite time is a little too much time for us to wait for a simple RL circuit to settle down. Therefore, we need to define a particular measure on the circuit current to decide whether the circuit has reached the final current value with sufficient accuracy, for *practical* purposes. Let us say we decide to call the current growth process a completed process when the current reaches 99% of its theoretical final value for the first time. That is a satisfactory measure for most of the practical applications involving RL circuits. How long do we have to wait for this to happen? How does the waiting period depend on R and L values? What is the shape of the inductor current?

We have to solve the differential equation of the circuit to answer these questions. First, we attempt to solve it in the power series form in the following sub-section.

10.3.1 Series RL Circuit Current as a Power Series

We prepare the circuit equations in the following manner for this analysis.

$$\begin{aligned} i_L(t) &= \frac{1}{L} \int_{-\infty}^t v_L(t) dt \\ &= \underbrace{\frac{1}{L} \int_{-\infty}^{0^-} v_L(t) dt + \frac{1}{L} \int_{0^-}^{0^+} v_L(t) dt}_{=I_0} + \frac{1}{L} \int_{0^+}^t v_L(t) dt. \end{aligned}$$

We use the information that unit step voltage input has no impulse content, and hence, initial condition remains unchanged across 0^- and 0^+ and that the unit step input value is simply equal to 1 for all $t \geq 0^+$, to arrive at the final step below:

$$\begin{aligned} i_L(t) &= I_0 + \frac{1}{L} \int_{0^+}^t v_L(t) dt \\ &= I_0 + \frac{1}{L} \int_{0^+}^t [1 - Ri_L(t)] dt \\ &= I_0 + \int_{0^+}^t \left[\frac{1}{L} - \frac{R}{L} i_L(t) \right] dt \\ &= \int_{0^+}^t \left[\frac{1}{L} - \frac{R}{L} i_L(t) \right] dt \quad (\because I_0 \text{ is zero in this circuit}). \end{aligned}$$

The analysis aims at an expression for i_L at a specific time instant t in the time axis. The time interval under consideration is $[0^+, t]$. Let us divide this interval into N small intervals of width Δt each, where Δt is very small and N is a large integer. We know that the current through the inductor at $t = 0^+$ is zero and the voltage across the inductor at $t = 0^+$ is 1 V. Now, we assume that the voltage across the inductor does not change significantly over the first Δt interval. In fact, this voltage does change due to the resistor absorbing a varying amount of voltage in response to the rising current during this interval. However, we can expect this change in voltage to be very small if Δt is kept small. This is the standard *assumption of continuity over small intervals* routinely employed in Calculus.

The current in the inductor rises linearly if the voltage across it is kept constant. Therefore, i_L starts increasing from 0 at a constant rate of $1/L$ A/s in the first Δt interval to reach $\Delta t/L$ A at the end of the interval. At the beginning of the second Δt interval, we update

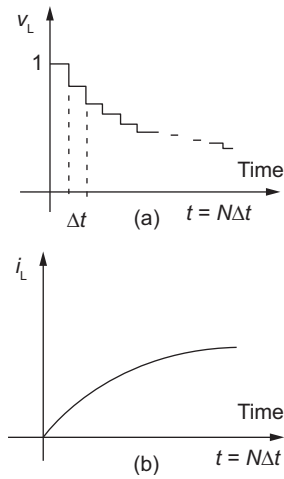


Fig. 10.3-1 (a) Step Approximation of Voltage Across Inductor (b) Corresponding Inductor Current

the value of v_L by using the i_L value at the end of first Δt interval as $(1 - Ri_L)$ and then assume that the inductor voltage remains constant at this value over the entire second Δt interval. Therefore, the inductor current at the end of second Δt interval will be $\Delta t/L + (1 - Ri_L)\Delta t/L$. We repeat this calculation process over all the N intervals to arrive at an expression for i_L at t in terms of R , L , Δt and N . This process is illustrated in Fig. 10.3-1.

The expression for current at the end of n^{th} interval for various values of n from 1 to 7 is listed below. The τ in these expressions stand for the ratio L/R and has dimensions of time.

$$\begin{aligned}
 n=1 & \quad \frac{1}{R} \left[\frac{\Delta t}{\tau} \right] \\
 n=2 & \quad \frac{1}{R} \left[\frac{2\Delta t}{\tau} - \left(\frac{\Delta t}{\tau} \right)^2 \right] \\
 n=3 & \quad \frac{1}{R} \left[\frac{3\Delta t}{\tau} - 3 \left(\frac{\Delta t}{\tau} \right)^2 + \left(\frac{\Delta t}{\tau} \right)^3 \right] \\
 n=4 & \quad \frac{1}{R} \left[\frac{4\Delta t}{\tau} - 6 \left(\frac{\Delta t}{\tau} \right)^2 + 4 \left(\frac{\Delta t}{\tau} \right)^3 - \left(\frac{\Delta t}{\tau} \right)^4 \right] \\
 n=5 & \quad \frac{1}{R} \left[\frac{5\Delta t}{\tau} - 10 \left(\frac{\Delta t}{\tau} \right)^2 + 10 \left(\frac{\Delta t}{\tau} \right)^3 - 5 \left(\frac{\Delta t}{\tau} \right)^4 + \left(\frac{\Delta t}{\tau} \right)^5 \right] \\
 n=6 & \quad \frac{1}{R} \left[\frac{6\Delta t}{\tau} - 15 \left(\frac{\Delta t}{\tau} \right)^2 + 20 \left(\frac{\Delta t}{\tau} \right)^3 - 15 \left(\frac{\Delta t}{\tau} \right)^4 + 6 \left(\frac{\Delta t}{\tau} \right)^5 - \left(\frac{\Delta t}{\tau} \right)^6 \right] \\
 n=7 & \quad \frac{1}{R} \left[\frac{7\Delta t}{\tau} - 21 \left(\frac{\Delta t}{\tau} \right)^2 + 35 \left(\frac{\Delta t}{\tau} \right)^3 - 35 \left(\frac{\Delta t}{\tau} \right)^4 + 21 \left(\frac{\Delta t}{\tau} \right)^5 - 7 \left(\frac{\Delta t}{\tau} \right)^6 + \left(\frac{\Delta t}{\tau} \right)^7 \right].
 \end{aligned}$$

The current at the end of n^{th} interval is a polynomial of degree n on the variable $\Delta t/\tau$. A close examination of the coefficients will show that the first coefficient is ${}^nC_1$, the second is ${}^nC_2$ and so on until the last term (n^{th} term with power of n) which has a coefficient of ${}^nC_n$. Therefore, we can write the expression for current at a particular time instant t as

$$i_L(t) = \frac{1}{R} \sum_{k=1}^N (-1)^{k+1} \frac{{}^NC_k}{N^k} \left(\frac{N\Delta t}{\tau} \right)^k.$$

Further, we modify the above equation as

$$\begin{aligned}
 i_L(t) &= \frac{1}{R} \sum_{k=1}^N (-1)^{k+1} \frac{{}^NC_k}{N^k} \left(\frac{N\Delta t}{\tau} \right)^k \\
 &= \frac{1}{R} \sum_{k=1}^N (-1)^{k+1} \frac{[N(N-1)\dots(N-k+1)]}{N^k \times k!} \left(\frac{N\Delta t}{\tau} \right)^k.
 \end{aligned}$$

We can make this expression more precise by increasing N while keeping the value of t the same. This amounts to decreasing Δt . In the limit, as $N \rightarrow \infty$, $N\Delta t \rightarrow t$ and $[N(N-1)\dots(N-k+1)]/N^k \rightarrow 1$. Therefore, $i_L(t)$ becomes an infinite power series as below:

$$\begin{aligned}
 i_L(t) &= \frac{1}{R} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k \\
 &= \frac{1}{R} \left[x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \dots \right], \text{ where } x = \frac{t}{\tau}
 \end{aligned}$$

This is a very familiar power series on x . In fact, the series in the square bracket is the series expansion for $(1 - e^{-x})$. Therefore,

$$i_L(t) = \frac{1}{R} (1 - e^{-t/\tau}) A \text{ for } t \geq 0^+ \quad (10.3-1)$$

Series RL circuit current for unit step voltage input arrived at by power series solution method.

is the solution for the current in the RL circuit with unit step voltage input and zero initial condition. Notice that the expression evaluated at $t = 0^+$ turns out to be 0 as it should. We will have a lot to say about exponential functions in the context of RL circuits in the remainder of this chapter. However, we will get to that after we solve the RL circuit step response problem by solving the governing differential equation in time-domain in the next sub-section.

10.4 STEP RESPONSE OF AN RL CIRCUIT BY SOLVING DIFFERENTIAL EQUATION

We have already derived the first order linear differential equation with constant coefficients that describe the behaviour of the inductor current in a series RL circuit for all t . This governing differential equation is

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} v_s(t) \text{ for all } t \quad (10.4-1)$$

where $\tau = L/R$, $i_L(t)$ is the current in the inductor (as well as the circuit current) and $v_s(t)$ is the input voltage source function (also called excitation function, forcing function, input function, etc.) which *must be defined for all t* if Eqn. 10.4-1 is to be used.

Notice that the domain of the functions appearing in the governing differential equation is the entire time axis from $-\infty$ to $+\infty$ since the differential equation is obtained by a systematic application of KCL and KVL. They are basic conservation laws in essence, and hence, must remain true at all instants of time. Therefore, we must know the input function for all t if we are to solve for $i_L(t)$ for all t . *But, neither can we know the input source function for all t in a practical circuit problem nor do we want to solve for current for all t . Also, the input function may be ill-defined at certain instants of time.* This makes a detailed discussion of the way input functions are specified in circuit problems.

10.4.1 Interpreting the Input Forcing Functions in Circuit Differential Equations

We can identify a sequence of time instants at which certain ‘switching events’ will take place in any circuit problem in general. Quite often, this sequence of time instants may contain only one entry – as in the case of an RL circuit with battery connected to it by a switch that closes at some specified time instant. However, it is quite possible that a sequence of switching events will take place in our circuit in a more complex setting. Among these various switching instants, there will be one that is the earliest, after which we have complete information about all source functions applied to the circuit – this instant is customarily (though not necessarily) marked as $t = 0$ in circuit problems. The circuit problem involves solving for all the circuit variables as functions of time from the first switching instant onwards.

A switching event in a circuit can result in (i) an applied source voltage/current changing abruptly from one time-function to another *with no change in its location in the circuit* with a possible step discontinuity in voltage/current also thrown in or (ii) change in location of source resulting in a change in structure of the circuit or (iii) application of new sources at new locations in the circuit with possible discontinuities in voltage/current or (iv) removal of sources or elements or (v) abrupt change in the value of some parameter in the circuit or (vi) introduction of new passive or active elements into the circuit, thereby, changing its structure, and hence, its describing equations or a combination of these. The circuits shown in Fig. 10.4-1 illustrate some of these.

The switch S_1 in the circuit in Fig. 10.4-1(a) is closed at $t = 0$ and the switch S_2 is opened at $t = 0$ to bring about an abrupt change in the applied voltage from 0 to V . When was S_2 closed in the infinite past? If we assume that S_2 was closed in the infinite past, the

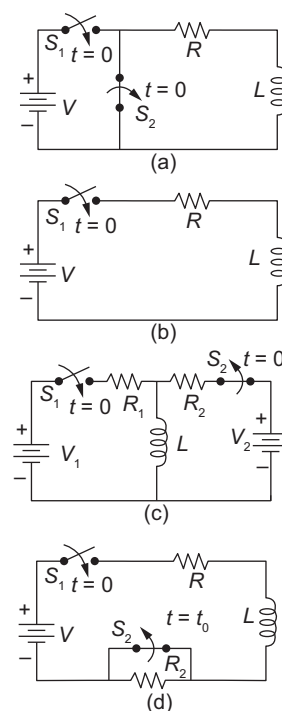


Fig. 10.4-1 Example Circuits illustrating Switching Events

current in the inductor at $t = 0^-$ can only be zero. Whatever energy that was possibly trapped in the inductor at $t = -\infty$ would have dissipated in the resistor by current flow through the closed circuit unless the inductor was a part of some other circuit simultaneously and that circuit was disconnected at $t = 0$. Hence, the answer to the question regarding the moment of closing S_2 in the past will depend on the initial condition specified and other unknown information. If a non-zero initial condition at $t = 0^-$ is specified, that will only mean that *either* S_2 was closed at some finite time point in the past and even before that the inductor was subjected to application of some voltage, resulting in some energy trapped in it when S_2 was closed *or* there were other source/s and/or component/s in the circuit that acted on the inductor and were removed at $t = 0$. Both views will be equivalent as far as circuit solution for $t > 0^+$ is concerned. They will affect the solution only for $t < 0^-$; but then we are not concerned about that.

The point of view we adopt is described now. Let us say the source function in a circuit is specified as $v_s(t) = f(t) u(t)$, where $f(t)$ is some well-behaved function of time. For example $f(t)$ could be a constant function defined for all t representing a DC source or a sinusoidal function defined for all t representing an AC source. We understand that $f(t) u(t)$ as zero voltage application from $t = -\infty$ to $t = 0^-$ and application of $f(t)$ from $t = 0^+$ to the next switching instant in the circuit or $t = \infty$, whichever is earlier. There may be a jump discontinuity at $t = 0$ in the applied source function. The transition from 0^- to 0^+ in the circuit solution has to be accomplished by using the information on discontinuities introduced by the switching operations and continuity of inductor current. If a non-zero initial condition for inductor is specified, we assume that it was created by components which were removed from the circuit at $t = 0$.

The describing equation based on these two equivalent views for the series RL circuit in Fig. 10.4-1(a) appears in Eqn. 10.4-2 below, where I_0 can be zero as a special case. The first form clearly indicates the domain of applicability of the differential equation as well as its solution. In the second form, it is somewhat hidden and we have to remember whatever we stated in the last paragraph to understand that the differential equation and its solution will be valid only for $t > 0^+$. *We employ both forms interchangeably for circuit studies from this point onwards.*

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{V}{L} \text{ for all } t \geq 0^+ \text{ with } i_L(0^-) = I_0$$

OR

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{V}{L} u(t) \text{ with } i_L(0^-) = I_0.$$

There is no ambiguity about applied voltage for $t < 0^-$ in the circuit in Fig. 10.4-1(b). We simply do not know anything about that voltage! This is so because the voltage across an open-circuit is not decided by the open-circuit but by the elements connected on the right side of the open-circuit. The initial condition at $t = 0^-$ for inductor current in this circuit can be zero or non-zero. Zero initial condition does not imply that no voltage was ever applied to the inductor in the past. Rather, it means that whatever be the voltage waveform applied to the inductor in the past, the area under that waveform from $-\infty$ to 0^- in the time axis was zero. Similarly, a non-zero initial condition will imply that, some extra circuit arrangement that is not shown was employed to apply suitable voltage to the inductor in the past in order to take its current to the specified value at $t = 0^-$. In any case, it does not matter as we are interested in the circuit solution only for $t = 0^+$, and therefore, the equation describing the circuit is the same as in Eqn. 10.4-2.

A similar situation exists in the circuit in Fig. 10.4-1(c) too. We cannot easily describe the voltage applied on the left side for $t < 0^-$, since there is an open-circuit intervening and we do not know when the switch S_2 was closed in the past. Since the specific time instant at which the switch S_2 was closed is not specified, we are expected to assume that it was

closed so far back in time that the circuit has settled down in its response by the time $t = 0^-$ point is reached. At $t = 0^-$, the circuit undergoes a structure change and becomes a simple series RL circuit with step voltage input. The components R_2 , V_2 and S_2 are shown only to indicate how the initial current is generated in the circuit and what its value is. The initial condition value is V_2/R_2 and the governing equation is the same as Eqn. 10.4-2 with $I_0 = V_2/R_2$.

In the circuit in Fig. 10.4-1(d), the switch S_1 closes at $t = 0$ and applies V to the circuit. The switch S_2 is in a closed state at that time, but opens subsequently at $t = t_0$ to introduce an additional resistance R_2 in series in the circuit. Thus, the circuit behaviour will be similar to that of the circuit in Fig. 10.4-1(b) in the time interval $[0^+, t_0^-]$. However, the circuit is described by another differential equation from $t = t_0^+$ onwards. The initial condition for that differential equation will have to be specified at $t = t_0^-$ by calculating the value from the solution of the differential equation governing the circuit in the interval $[0^+, t_0^-]$. The overall governing equation is given below:

$$\begin{aligned} \frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau_1} &= \frac{V}{L} \text{ for } 0^+ \leq t \leq t_0^- \text{ OR } \frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau_1} = \frac{V}{L} u(t) \\ \text{with } i_L(0^-) &= I_0 \text{ and } \tau_1 = \frac{L}{R_1} \\ \text{and} \\ \frac{di_L(t')}{dt'} + \frac{i_L(t')}{\tau_2} &= \frac{V}{L} \text{ for } 0^+ \leq t' \leq \infty \text{ OR } \frac{di_L(t')}{dt'} + \frac{i_L(t')}{\tau_2} = \frac{V}{L} u(t') \\ \text{with } t' = t - t_0, i_L(t' = 0^-) &= i_L(t = t_0) \text{ and } \tau_2 = \frac{L}{R_1 + R_2}. \end{aligned}$$

10.4.2 Solving the Series RL Circuit Equation by Integrating Factor Method

We have discussed the domain in which our circuit differential equation is valid and the domain in which the solution will be correct. We noted specifically that the switching instants are excluded from the domain of functions. Circuit behaviour on either side of such excluded time instants is to be worked out from our knowledge of physical elements in the circuit. It is better to be very careful about the domains of various functions and equations when we are dealing with circuit differential equations. The importance of such care will be appreciated better when we get to more complex circuits involving sequences of switching operations.

Now, we are ready to take up the task of solving the differential equation of a series RL circuit. We use the well-known integrating factor method to do so. We introduce two new symbols α defined as $\alpha = 1/\tau = R/L$ and β defined as $\beta = 1/L$ in the equations that follow. The equation governing the series RL circuit inductor current for unit step voltage input is

$$\frac{di_L}{dt} + \alpha i_L = \beta u(t)$$

or

$$di_L + \alpha i_L dt = \beta u(t) dt.$$

The integrating factor for this equation is $e^{\alpha t}$. Multiplying the above equation by the integrating factor on both sides

$$e^{\alpha t} di_L + \alpha i_L e^{\alpha t} dt = \beta e^{\alpha t} u(t) dt.$$

We identify the left side of the above equation as the exact differential of $i_L e^{\alpha t}$ since $d(i_L e^{\alpha t}) = e^{\alpha t} di_L + \alpha i_L e^{\alpha t} dt$.

Therefore,

$$d(i_L e^{\alpha t}) = \beta u(t) e^{\alpha t} dt.$$

We expect the solution for i_L to be correct only for $t \geq 0^+$. Therefore, we integrate the above equation between 0^+ and t to yield

$$(i_L e^{\alpha t})_{(t)} - (i_L e^{\alpha t})_{(0^+)} = \beta \int_{0^+}^t 1 \cdot e^{\alpha t} dt.$$

We have replaced $u(t)$ by 1 in the above equation since $u(t) = 1$ for all $t \geq 0^+$. Evaluating the integral on the right side and simplifying the left side, we get

$$i_L e^{\alpha t} - i_L(0^+) = \frac{\beta}{\alpha} (e^{\alpha t} - 1); t \geq 0^+.$$

Algebraic manipulation of the above equation and substitution of $\alpha = 1/\tau = R/L$ and $\beta = 1/L$ yield the final expression for inductor current at an arbitrary time instant $t \geq 0^+$ as

$$i_L = i_L(0^+) e^{-t/\tau} + \frac{1}{R} (1 - e^{-t/\tau}); t \geq 0^+$$

We had assumed that the initial condition specified at $t = t = 0^-$ is I_0 . The discontinuity in applied voltage at $t = 0$ in the present case is a jump from 0 to 1 V. We have also shown earlier that the value of applied voltage will remain between 0 and 1 V in the interval $[0^-, 0^+]$. Therefore, the inductor current cannot change over that infinitesimal interval, and hence, $i_L(0^+) = i_L(0^-) = I_0$ and the solution for inductor current is

$$i_L = I_0 e^{-t/\tau} + \frac{1}{R} (1 - e^{-t/\tau}); t \geq 0^+ \quad (10.4-3)$$

This time we have got the expression for inductor current in a more general form since we did not assume that initial condition is zero. The Eqn. 10.4-3 describes the inductor current in a series RL circuit with unit step voltage input with initial current present in the inductor. If $I_0 = 0$, the solution we get is

$$i_L = \frac{1}{R} (1 - e^{-t/\tau}); t \geq 0^+.$$

This is the same as the one we obtained in Eqn. 10.3-1 under power series solution.

No, we are not yet ready to explore the inductor current expression in detail. We need to arrive at the same solution by another route before we can do that. This is needed since the integrating factor method will not help us when we have to deal with circuits containing more than one memory element. Secondly, though both the power series method and the integrating factor method gave us the solution, they could not tell us why it has to be of exponential nature. We are particularly interested in that aspect of the solution. The next method of solution will lend us the required insight.

10.4.3 Complementary Function and Particular Integral

The governing equation for inductor current in a series RL circuit with zero initial condition and unit step voltage input is

$$\frac{di_L}{dt} + \alpha i_L = \beta u(t) \text{ or } \frac{di_L}{dt} + \alpha i_L = \beta \text{ for } t \geq 0^+ \quad (10.4-4)$$

where α and β have the same significance as in the previous sub-section. There is a jump discontinuity from 0 to 1 V at $t = 0$ in the applied voltage. This jump in the applied voltage travels straight to the inductor and appears as a jump in the voltage across it at $t = 0$.

Series RL circuit
current with unit step
voltage input and
non-zero initial
condition obtained
by Integrating
Factor method.

First, we ignore the switching at $t = 0$ and try to find the current in series RL circuit in which 1 V was applied at $t = -\infty$ onwards. In that case,

$$\frac{di_L}{dt} + \alpha i_L = \beta \text{ for all } t \quad (10.4-5)$$

The differential equation in Eqn. 10.4-5 is to be true for all t . This is possible only if i_L is such a function that its first derivative along with its own copy becomes a constant for all t . Almost all well-behaved functions (at least the ones we meet with in electrical circuits) can be expressed in a power series form. This motivates us to seek the required function in the form of a power series. Let us try out a general power series in t .

$$\text{Let } i_L = a_0 + a_1 t + a_2 t^2 + \dots$$

On substituting this trial solution in Eqn. 10.4-5, we get,

$$a_1 + a_2 t + \dots + \alpha (a_0 + a_1 t + a_2 t^2 + \dots) = \beta$$

Since this has to be true for all t , coefficients of each individual power of t have to be equal on the two sides of the equation. Therefore, the solution is $a_0 = \beta/\alpha$ and $a_n = 0$ for $n \neq 0$. Therefore, the solution for i_L which will satisfy the differential equation in Eqn. 10.4-5 is $i_L = \beta/\alpha$. Substituting for α and β , $i_L = 1/R$ is the solution. This solution is called the *particular integral* of the differential equation with non-zero forcing function. Obviously, the particular integral for zero input is zero. *Particular integral is the solution of a differential equation if the forcing function is applied from $t = -\infty$ onwards.*

This ‘constant’ solution conflicts with the initial condition requirement in the circuit. Such a conflict comes up in the solution because 1 V was applied only at $t = 0$ (not at $-\infty$) and the circuit has to change its response from some other particular integral to the present value of particular integral. Hence, we need another term in the solution which will force the solution to satisfy the initial condition requirement at $t = 0^+$.

This additional function has to satisfy a constraint. The solution $i_L = 1/R$ satisfies the differential equation in Eqn. 10.4-5 at all $t \geq 0^+$. Therefore, the additional function we are to going to add to this solution to enforce compliance with the initial condition requirement should not add anything to the right side of the differential equation. This implies that it has to be a function that will satisfy the following differential equation for all t .

$$\frac{di_L}{dt} + \alpha i_L = 0 \quad (10.4-6)$$

A trivial solution (in fact, its particular integral) to this equation is $i_L = 0$. We are not interested in that. We recast the above equation as

$$\frac{di_L}{dt} = -\alpha i_L \text{ for all } t$$

and note the fact that if i_L is a function of time, then both sides of this equation will be functions of time.

Two functions of time can be equal to each other over their entire domain, if, and only if, they are the same kind of functions – they must have the same shape when plotted. Hence, we look for functions which produce a copy (probably scaled versions) of themselves on differentiation.

Sinusoidal function comes to our mind first. But then, we remember that sinusoidal function can be covered by exponential function with imaginary exponent by Euler’s formula. Hence, $e^{\gamma t}$, where γ can be a complex number, is a function with the desired property. So we try out $Ae^{\gamma t}$, where A is an arbitrary constant, as a possible solution in Eqn. 10.4-6. We get,

$$\begin{aligned} A\gamma e^{\gamma t} + A\alpha e^{\gamma t} &= 0 \text{ for all } t \\ Ae^{\gamma t}(\gamma + \alpha) &= 0 \text{ for all } t \end{aligned}$$

$A = 0$ is a trivial solution. $e^{\gamma t} = 0$ cannot be a solution since the equation has to be true for all t . Therefore, γ has to be equal to $-\alpha$. Thus, $Ae^{-\alpha t}$ is the solution for the Eqn. 10.4-6.

The differential equation cannot help us in deciding the value of A . It decides only the value of exponent in the exponential function.

This solution we have arrived at, i.e., $Ae^{-\alpha t}$ is called the *complementary function* of the differential equation in Eqn. 10.4-6 and the differential equation with zero forcing function is called the *homogeneous differential equation*.

Now that we have got the function required to enforce compliance with the initial condition requirement in the circuit, let us proceed to form the *total solution* for i_L in series RL circuit with step input.

$$i_L(t) = Ae^{-\alpha t} + \frac{1}{R} \text{ for } t \geq 0^+ \text{ and } i_L(0^+) = 0 \quad (10.4-7)$$

We evaluate A to complete the solution by substituting the initial condition at $t = 0^+$ in the total solution. This gives us $A = -1/R$. Hence, the final solution is

$$i_L = \frac{1}{R}(1 - e^{-t/\tau}); t \geq 0^+,$$

where we have used $\alpha = 1/\tau$.

The expression for inductor current can be extended for the case with non-zero initial condition as shown below:

$$i_L(t) = Ae^{-\alpha t} + \frac{1}{R} \text{ for } t \geq 0^+ \text{ and } i_L(0^+) = i_L(0^-) = I_0$$

Substituting the initial condition value in this solution,

$$I_0 = A + \frac{1}{R}$$

$$\therefore A = I_0 - \frac{1}{R}$$

$$\begin{aligned} \therefore i_L(t) &= (I_0 - \frac{1}{R})e^{-\alpha t} + \frac{1}{R} \text{ for } t \geq 0^+ \\ &= I_0 e^{-\alpha t} + \frac{1}{R}(1 - e^{-\alpha t}) \text{ for } t \geq 0^+ \\ &= I_0 e^{-t/\tau} + \frac{1}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+. \end{aligned}$$

Current in a Series RL circuit excited by a unit step input obtained by solving the differential equation.

Note that the phrase ‘for $t \geq 0^+$ ’ in the expression for i_L can be replaced by a multiplication of the entire expression by the unit step function $u(t)$.

10.5 FEATURES OF RL CIRCUIT STEP RESPONSE

Step response in the electrical circuit analysis context implies the application of the unit step function, $u(t)$, as the input with a set of *zero-valued* initial conditions specified for the circuit at $t = 0^-$. The response to this unit step application can be described in terms of a chosen circuit variable, which may be a voltage variable, a current variable or a linear combination of voltage and current variables. We had chosen the current through an inductor as the response variable in the case of series RL circuit. The current waveform with zero initial condition (initially relaxed circuit) was shown to be

$$i_L = \frac{1}{R}(1 - e^{-t/\tau}); t \geq 0^+ \quad (10.5-1)$$

The primary objective of applying an input function to a circuit is to make a certain chosen output variable in the circuit behave in a desired manner. This is why the input

function is called *forcing function*. Input function is a command to the circuit to vary its response variable in a manner similar to its own time variation. The application of unit step input is equivalent to a command to the circuit to change its response variable in a step-wise manner in this sense. Similarly, when we switch on a voltage $v_s(t) = V_m \sin \omega t$ V at $t = 0$ to any circuit, we are, in effect, commanding the circuit to make the chosen response variable follow this function in shape. A purely memoryless circuit will follow the input command with no delay. However, circuits with memory elements will not do this.

Inductor constitutes electrical inertia. It does not like to change its current and resists any such current change by producing a back e.m.f. across it – the magnitude of this e.m.f. is directly proportional to the rate at which the inductor current changes. Other elements in the circuit (usually voltage sources, switches, etc.) will have to supply the voltage demanded by the inductor if the desired current change is to take place. This is the price the other elements in the circuit have to pay for demanding the lazy inductor to change its current. The price is heavier if the required change in current is to be accomplished faster.

It is still more instructive to look at the ‘inertia’ aspect of the inductor from its energy storage capability angle. An inductor stores energy in its magnetic field. The energy stored in the field is proportional to the square of the current through the inductor. Thus, if we want to change the current through the inductor, we have to supply energy to the inductor or absorb energy from the inductor. Note that we do not have to do any such thing if the current through the inductor is at a constant level. Let us assume that we want to change the inductor current from I_1 to I_2 ($I_2 > I_1$). By the time we have done it, we would have given the inductor $0.5L(I_2^2 - I_1^2)$ J of energy. We can pump energy into the inductor only by pumping power into it. Therefore, a voltage has to appear across the inductor whenever its current tries to change. Energy has to be pumped into the inductor at a fast rate if the current in the inductor is to change fast. That means that the power flow into the inductor has to be increased if the inductor current is to change fast. That is why the voltage across the inductor becomes higher when a given amount of current change is sought to be attained in shorter time intervals.

Consider a similar situation in translational mechanics. A mass M is forced to move against friction. Assume that the frictional force is proportional to the velocity of the mass and that there is no sticking friction. Now, if we apply a constant force to the mass we know that (i) the mass reaches a final speed at which the applied force is met exactly by the frictional force acting against motion and (ii) it takes some time to reach this situation. The mass M does not like to move due to its inertia – it is in the nature of objects in this world to stay put. They prefer it that way. Similarly, it is in the nature of an inductor to stay put as far as its current is concerned. However, objects in this world do yield to forces eventually. In the above case, since the mass M shows a tendency to stay put even after the force has come into action, it has to absorb the entire force initially. In that process it gets accelerated. Hence, for a brief period initially, a major portion of the applied force goes to accelerate the mass and only a minor portion goes for meeting friction. This proportion will change with time and finally no force will be spent on accelerating the mass and the entire force will be spent on countering friction. Hence, initially the ‘inertial nature’ of mass dominates the situation and puts up a stiff fight with the force that is a command to the mass to move at a constant speed. Slowly, the resistance from the mass weakens and inexorably the force subjugates the inertial nature of the mass. After sufficient time has elapsed, the applied force wins the situation. The mass yields almost completely to the force command and moves at an almost constant speed commensurate with the level of friction present in the system.

This tussle between the inherent inertial nature of systems and the compelling nature of forcing functions is a common feature in dynamic systems involving memory elements and is present in electrical circuits too. Thus, the response immediately after the application of a forcing function in a circuit will be a compromise between the inherent natural laziness of the system and the commanding nature of forcing function. The circuit expresses its dislike to *change* by spewing out a time function, which quantitatively describes its unwillingness to *change*. The forcing function wears down this natural cry from the circuit gradually

The objective of applying a forcing function to a circuit.

Inductance constitutes electrical inertia.

Inertia of inductance looked at from a stored energy point of view.

Series RL circuit compared with a mass moving against viscous friction.

and establishes its supremacy in the circuit in the long run – by forcing all circuit variables to vary as per its dictates in the long run.

The total response in the circuit is always a mixture of these two with the component from the forcing function dominating almost entirely in the long run and the natural component from the circuit's inherent inertia ruling in the beginning. It should be noted at this point that it is quite possible that neither component will succeed in overpowering the other in some circuits. Such circuits are called *marginally stable circuits*. There are circuits in which the natural component will not only refuse to yield but will grow without limit as time increases; thereby, overpowering the forcing function with time. Such circuits are called *unstable circuits*. We will take up such circuits in later chapters. At present we deal with circuits that yield to the forcing function in the long run – called *stable circuits*.

The time function that the circuit employs to protest against *change* is called the *natural response* of the circuit and the time function that the forcing function establishes in the response variable is called the *forced response*. *Natural response* means precisely that it encodes the basic nature of the circuit and has nothing to do with the nature of forcing function. Its shape and other features (except amplitude) are decided by the nature and number of energy storage elements in the circuit, the way these energy storage elements are connected with resistive elements to form the circuit etc. Thus, its shape depends only on the nature of elements and the topology of the circuit and does not depend on the particular shape and value of the forcing function – it is *natural* to the circuit – but its magnitude will depend on the initial condition and forcing function too.

The series *RL* circuit with voltage source excitation howls '*exponentially*' when the forcing function commands its current to change. In fact, all stable dynamic systems described by a 'linear first order ordinary differential equation with constant coefficients' will cry out exponentially when they are asked to change. They all have a natural response of the type $Ae^{-\alpha t}$, where α , which decides the shape of the response, is decided by system parameters (R and L in the present instance) and A is decided by the initial condition and the initial value of forced response. The forcing function along with the initial condition will decide the magnitude of natural response; but not its shape.

The shape of natural response does not depend on the forcing function, and hence, must be the same with or without the forcing function. A non-zero response with a zero forcing function can exist if the circuit starts out with initial energy at $t = 0^-$. This is similar to a mass which has been accelerated to some velocity before $t = 0$, slowing down to zero speed after $t = 0$ under the effect of friction. Thus, it follows that we can find the shape of the natural response by solving the equation describing the circuit response with the forcing function set to zero. However, that will be the homogeneous differential equation and we know that its solution is the complementary function of the equation. The complementary solution of the differential equation describing the current in the inductor in our *RL* circuit was shown to be an exponential function with negative real index earlier. Thus, we conclude that *the complementary solution of the describing differential equation of a circuit yields the natural response of the circuit, whereas, the particular integral corresponding to the applied forcing function yields the forced response.*

Natural response and forced response defined and distinguished.

The complementary solution of the describing differential equation of a circuit yields the natural response of the circuit. The particular integral corresponding to the applied forcing function yields the forced response.

10.5.1 Step Response Waveforms in Series *RL* Circuit

There are only three circuit variables in a series *RL* circuit and they are $i_L(t)$, $v_R(t)$ and $v_L(t)$ as marked in Fig. 10.1-1. The expressions for $v_R(t)$ and $v_L(t)$ may be worked out from the solution for $i_L(t)$. These expressions for zero initial condition are

$$\begin{aligned} i_L(t) &= \frac{1}{R}(1 - e^{-t/\tau}); t \geq 0^+ \\ v_R(t) &= (1 - e^{-t/\tau}); t \geq 0^+ \end{aligned} \quad (10.5-2)$$

Step response of an initially relaxed series *RL* circuit.

$$v_L(t) = L \frac{di_L(t)}{dt} = \frac{L}{R} \times \frac{e^{-t/\tau}}{\tau} = e^{-t/\tau}; t \geq 0^+.$$

We introduce normalisation in these expressions before we plot them by dividing the expressions by the final steady value or the maximum value, as applicable. Similarly, we define the normalised time variable t_n as t/τ . This results in

$$\begin{aligned} i_{Ln}(t) &= \frac{i_L(t)}{1/R} = (1 - e^{-t_n}); t_n \geq 0^+ \\ v_{Rn}(t) &= \frac{v_R(t)}{1} = (1 - e^{-t_n}); t_n \geq 0^+ \\ v_{Ln}(t) &= \frac{v_L(t)}{1} = e^{-t_n}; t_n \geq 0^+, \end{aligned} \quad (10.5-3)$$

where the second subscript ‘n’ indicates normalised variables.

These waveforms appear in Fig. 10.5-1. The inductor current rises from zero level at $t = 0^+$ and approaches the normalised final value of 1 as t_n approaches ∞ . It never touches the final value of 1 since the exponential function never becomes zero. The growth of the inductor current gradually loses momentum resulting in the convex shape of current waveform. Simultaneously, the voltage across the inductor decays exponentially and tends to go to zero as t_n approaches ∞ .

Much of the rise in i_L and fall in v_L take place within the first three units of normalised time, i.e., within 3τ s of actual time. The values of e^{-1} , e^{-2} and e^{-3} are 0.368, 0.135 and 0.05, respectively. Hence, the step response current in an *initially relaxed RL* series circuit rises to 63.2% of its final value in the first τ s. The voltage across the inductor in the same circuit falls by 63.2% from its value at $t = 0^+$ to reach 36.8% of its initial value in the first τ s. During the second τ s period, the inductor current rises by another 23.3% to reach 86.5% of its final value. Correspondingly, the voltage across the inductor falls to 13.5% of its initial value at the end of 2τ s. At the end of 3τ s, 95% of the transient phase is over and the inductor current is only 5% away from its final value.

The value of an inductor current is 99% and 99.33% of the final value at 4.6τ s and 5τ s, respectively. Hence, we can consider the natural response of an initially relaxed series *RL* circuit to be practically over within 5τ s, where $\tau = L/R$. During the first five τ periods, the response in the circuit is undergoing a *transient* phase before reaching a practically *steady* situation. This period – i.e., the period during which the natural response component is not negligible – is termed as the *transient period* and a value of 5τ is usually assigned to it in the case of first order circuits. This leads to another name for natural response – *transient response*. However, we have to be careful about this name since it gives us an impression that this response component is only transient, and hence, it will vanish with time invariably. This is not always so. There are circuits in which the natural response either persists indefinitely or increases with time. So do not expect transients to vanish with time in all cases.

In the case of an initially relaxed series *RL* circuit with unit step voltage input, the natural response (or transient response) term in the inductor current is $-e^{-t/\tau}/R$ A and the forced response is $1/R$ A.

Normalised step response of an initially relaxed Series RL circuit.

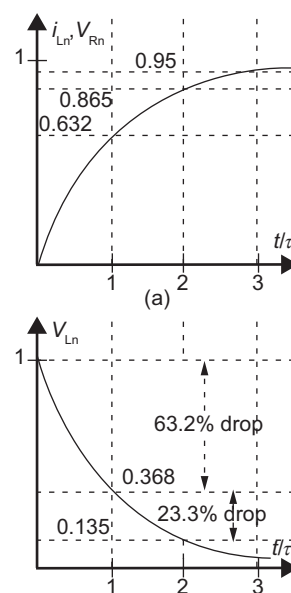


Fig. 10.5-1 Series RL Circuit Step Response – Normalised Waveforms

10.5.2 The Time Constant ‘ τ ’ of a Series *RL* Circuit

The quantity L/R symbolically represented by τ has turned out to be an important one for *RL* circuit by now. The unit of L is volt-sec/A and the unit of R is V/A resulting in a dimension of time for this quantity with seconds as its unit. Hence, this quantity is defined as the *Time Constant* of series *RL* circuit.

One of the interpretations assigned to the time constant is that it is the time taken by an initially relaxed RL circuit to reach 63.2% of its final current value. This is not very satisfactory. After all, 63.2% is not a neat round number or a particularly significant one. 50% would have been a neat measure. In fact, the time taken by a first order system step response to reach 50% of its final value is termed its *half-life* and this turns out to be $\approx 0.693\tau$ in the case of an RL circuit.

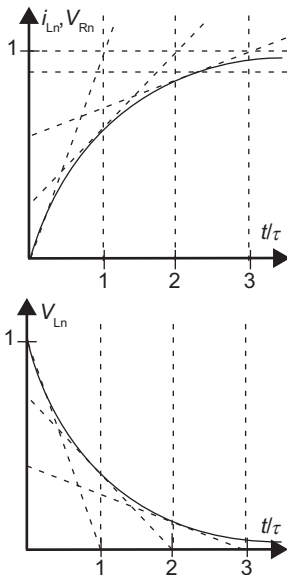


Fig. 10.5-2 Current Slope-Based Interpretation of Time Constant

Speaking qualitatively, we can appreciate the fact that τ is a measure of the duration taken by the circuit to reach the final current value. RL circuit causes delay in step response current due to the memory capability of the inductor. The inductor remembers its initial state and its memory prevents it from allowing sudden changes in current through it. But how deep is its memory in time? How persistent is its memory? Time constant provides answers to these questions.

A large τ implies deeper memory, and consequently, increased duration will be required for the forcing function to compel the inductor to go to the final current value. A larger τ implies a more persistent memory and a heightened tendency on the part of the circuit to keep its current smooth in the time-domain.

Time constant has another interesting interpretation. We had noted earlier that the current in the initially relaxed RL circuit starts rising at the rate of $1/L$ A/s at $t = 0^+$. With reference to Fig. 10.5-2, tangents to the normalised inductor current plot and the normalised inductor voltage plot are drawn at normalised time instants of 1, 2 and 3. It is clear from this figure that if the inductor current had continued to rise at the same rate of rise it had at $t = 0^+$, it would have reached the final value at $t_n = 1$ unit, i.e., at $t = \tau$. The slope of i_{Ln} at $t = 0^+$ is 1 normalised unit of current per unit of normalised time. Therefore, the normalised current would have reached 1 unit at $t_n = 1$ if the rate of rise had remained unchanged. One unit of normalised time amounts to one τ of real time. Thus, time constant is the time the current in an initially relaxed RL circuit would have taken to reach the final steady value had the initial rate of rise been maintained throughout. Equivalently, it is the time the voltage across the inductor would have taken to reach zero if the initial rate of fall could be maintained throughout.

However, time constant is even more than that. Consider the remaining two tangents at $t_n = 2$ and $t_n = 3$ in Fig. 10.5-2. Moving along either tangent line will take us to the final value of inductor current in one unit of normalised time away from the instant at which the tangent is drawn. If this is true about these two time instants, then, it must be true for all time instants since there is nothing special about these two. Let us examine the slope of an inductor current in detail.

$$i_{Ln}(t) = (1 - e^{-t_n}); t_n \geq 0^+ \\ \therefore \frac{di_{Ln}(t)}{dt_n} = e^{-t_n}; t_n \geq 0^+$$

Let Δt_n be the time required from t_n to reach the final value of 1 with rate of change of $i_{Ln}(t)$ held at this value. Then,

$$\Delta t_n = \frac{1 - (1 - e^{-t_n})}{e^{-t_n}} = 1 \\ \therefore \Delta t = \tau \text{ s.}$$

Thus, time constant of an RL circuit is the additional time required from the current instant for the step response in the initially relaxed circuit to reach the final steady value assuming that the rate of rise of response is held constant at its current value from that instant onwards.

10.5.3 Rise Time and Fall Time in First Order Circuits

‘Rise time (t_r)’ and ‘Fall time (t_f)’ are two measures of time delay defined in the context of unit step response for linear dynamic systems. These two measures are quite general in definition in order to accommodate a wide variety of systems having many terms in their transient response. However, in the case of simple first order systems, there is a direct relationship between the time constant and the rise and fall times.

Rise time is defined as the time interval between the first 10% point and the first 90% point in the rising step response of a system where the percentages are to the base of

the final step response value. Similarly, fall time is defined as the time interval between the first 90% point and the first 10% point in the step response of a system where the response variable is such that it starts at a non-zero initial value and decays to zero value in the long run. The percentages are to the base of the initial response value in this case.

These two definitions are illustrated in the case of a series RL circuit step response in Fig. 10.5-3. Normalised variables are used in this figure and the corresponding normalised time points at which the 10% and 90% crossover takes place are also marked in the figure. From the figure, it is clear that rise time and fall time of this circuit are equal to $\approx 2.2\tau$ s, where τ is the time constant of the circuit. This result is valid for any circuit described by a first order linear differential equation with constant coefficients and has nothing to do with the inductive nature of the circuit under consideration.

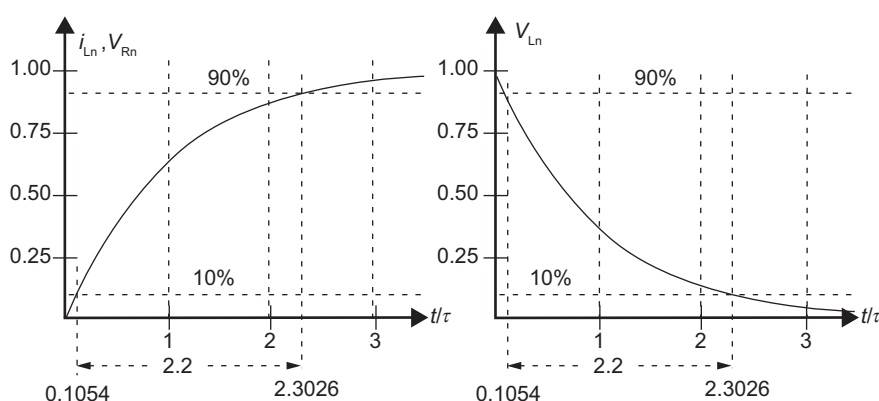


Fig. 10.5-3 Rise Time and Fall Time in Series RL Circuit

Rise time and fall time of the series RL circuit is equal to $\approx 2.2\tau$ s, where $\tau = L/R$ is the time constant of the circuit.

These two measures are defined for a general circuit of any order, and therefore, serve as measures of delay in response and depth of memory in the circuit in situations where a single time constant cannot be identified as the major delaying factor in the circuit.

10.5.4 Effect of Non-Zero Initial Condition on Step Response of RL Circuit

We have been dealing with the step response of an initially relaxed RL circuit until now. We have carefully included this constraint in our interpretation of time constant, definition of rise and fall times, etc. We will generalise our understanding in this sub-section by bringing in non-zero initial current in the inductor at $t = 0^-$.

We have already derived the expression for inductor current in this case as

$$i_L(t) = I_0 e^{-t/\tau} + \frac{1}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \quad (10.5-4)$$

Normalising this equation by using $1/R$ A as the current base and τ s as the time base, we get the normalised form of the equation with I_{0n} as the normalised initial current at $t = 0^-$.

$$i_{Ln}(t) = I_{0n} e^{-t_n} + (1 - e^{-t_n}) \text{ for } t_n \geq 0^+, \quad (10.5-5)$$

where $i_{Ln}(t) = \frac{i_L(t)}{1/R}$, $I_{0n} = \frac{I_0}{1/R}$ and $t_n = \frac{t}{\tau}$

This expression can also be written in a form that shows the natural response (transient response) and forced response components clearly separated out as below:

Normalised step response current in Series RL circuit with the forced response and natural response components identified clearly.

$$i_{L_n}(t) = \underbrace{1}_{\text{forced response}} - \underbrace{(1 - I_{0n})e^{-t/\tau}}_{\text{natural response}}; t_n \geq 0^+ \quad (10.5-6)$$

This equation is plotted in Fig. 10.5-4 with solid curves showing the total response and dotted curves showing the natural response or transient response. Curves are shown for four values of initial condition at $t = 0^-$. They are -0.5 , 0 , 0.5 and 1.5 . All values are normalised ones. A negative initial condition value indicates that the initial current in the inductor at $t = 0^-$ was in a direction opposite to that of the forced response. The forced response in all cases is represented by a horizontal line with the intercept of unity in the vertical axis.

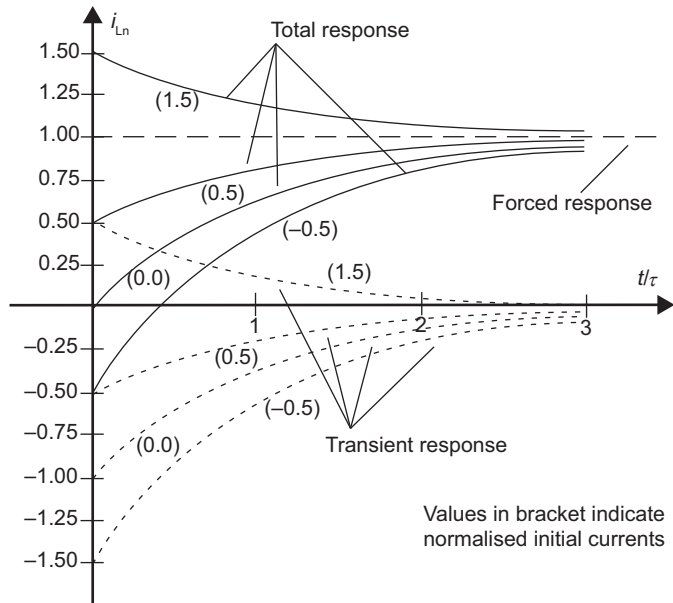


Fig. 10.5-4 Total Response and Transient Response in Series RL Circuit Step Response for Various Initial Currents

The waveforms in Fig. 10.5-4 and Eqn. 10.5-6 bring out the following aspects of RL circuit step response with non-zero initial condition.

The role of transient response is seen to be one of bridging the gap between the initial current in the inductor and the final current in the inductor.

- The transient response (natural response) of RL circuit contains two contributions – one from the initial condition specification and the other from the value of forced response at $t = 0^+$. The magnitude of the transient term is decided by these two quantities. Transient response, thus, enforces compliance with the initial condition specification in the circuit.
- The total response is a rising response if the initial current at $t = 0^-$ is less than the final current value. It is a falling response if the initial current is more than the final current.
- There will be no transient response in the circuit if the initial current specified at $t = 0^-$ is equal to the final current value in magnitude and direction.
- Consider a new current variable defined as $\Delta i_{L_n}(t) = i_{L_n}(t) - I_{0n}$, i.e., the change in inductor current from its initial value. Substituting Eqn. 10.5-6 in this definition, we get,

$$\Delta i_{L_n}(t) = (1 - I_{0n})(1 - e^{-t/\tau}); \text{ for } t_n \geq 0^+$$

- Compare this expression for the change in inductor current with the inductor current expression for initially relaxed circuit. We can see that whatever has

been said about time constant becomes applicable in relation to the change in the inductor current rather than to the total current when initial current is non-zero. The final value of this change is $1 - I_{0n}$ and the change in inductor current rises to 63.2% of its final value at one time constant, 86.5% of its final value in 2τ s, etc. Similarly, the change in the inductor current covers the 10% to 90% range in 2.2τ s where the percentages are to the base of $1 - I_{0n}$.

10.5.5 Free Response of Series RL Circuit

We consider a special case of an RL circuit with zero forcing function in this sub-section. Obviously, the solution for inductor current in this *source-free* RL circuit will contain only complementary solution. The particular integral is zero since the forcing function is zero. The complementary solution is of the form Ae^{-at} , where $\alpha = R/L$. Applying initial condition to this solution makes it clear that $A = 0$ unless the initial condition specified at $t = 0^-$ is non-zero. Thus, a source-free RL circuit can have a non-zero solution only if the inductor has some energy trapped in it at $t = 0^-$. This energy storage must have been created by some source prior to $t = 0^-$.

Consider the circuit in Fig. 10.5-5(a). The switch S_1 was closed long back and the circuit has attained the final inductor current value of I/R A by the time $t = 0^-$ is reached. At $t = 0$, the switch S_1 is opened and the switch S_2 is closed simultaneously. Thus, a source-free series RL circuit with an initial current of I_0 (which is equal to I/R in the circuit in Fig. 10.5-5) is set up at $t = 0$. The circuit in Fig. 10.5-5(b) is equivalent to the circuit in Fig. 10.5-5(a) for $t = 0^+$.

The expressions for inductor current and circuit voltages are derived as below:

$$i_L(t) = Ae^{-t/\tau}; \text{ for } t \geq 0^+$$

Since infinitely large voltage is neither applied nor supported in the circuit,

$$i_L(0^+) = i_L(0^-)$$

$$\therefore i_L(0^+) = I_0$$

$$\therefore A = I_0$$

$$\therefore i_L(t) = I_0 e^{-t/\tau}; \text{ for } t \geq 0^+$$

$$i_R(t) = i_L(t) (\because R \text{ and } L \text{ are in series})$$

$$v_R(t) = RI_0 e^{-t/\tau}, \text{ for } t \geq 0^+$$

$$v_L(t) = -v_R(t), \text{ by KVL.}$$

The current in the circuit decays exponentially from I_0 to zero with a time constant equal to L/R s. This is shown in the Fig. 10.5-5(c). The corresponding voltage across inductor is negative valued and decays with the same time constant. The circuit voltages are shown in Fig. 10.5-5(d).

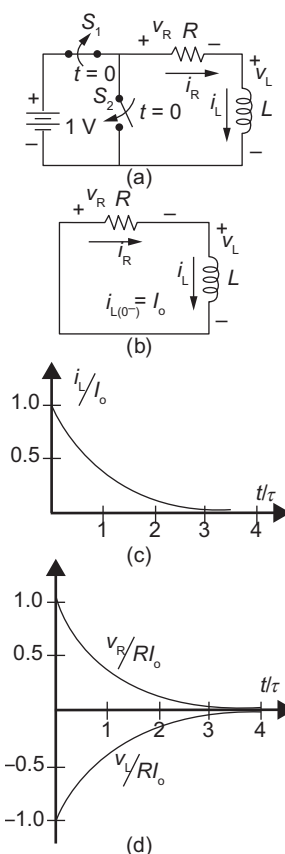


Fig. 10.5-5 Source-free RL Circuit and Waveforms

EXAMPLE: 10.5-1

Obtain an expression for voltage across the resistor in an initially relaxed series RL circuit for rectangular pulse voltage input defined as $v_s(t) = 1$ V for $0 \leq t \leq T$ and 0 V elsewhere. Use Integrating Factor Method. Plot the response for (i) $T = 0.2\tau$ (ii) $T = \tau$ and (iii) $T = 2\tau$.

Solution

The differential equation describing the circuit is

$$\frac{di_L}{dt} + \alpha i_L = \beta v_s(t), \text{ where } \alpha = \frac{1}{\tau} = \frac{R}{L} \text{ and } \beta = \frac{1}{L}$$

or

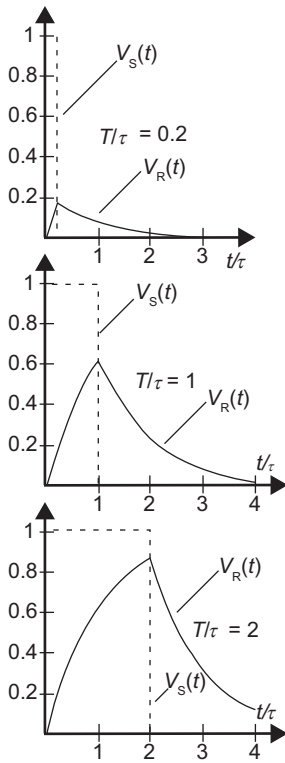


Fig. 10.5-6 Single Pulse Response of RL Circuit in Example 10.5-1

$$di_L + \alpha i_L dt = \beta v_s(t) dt$$

The integrating factor for this equation is $e^{\alpha t}$. Multiplying the above equation by the integrating factor on both sides

$$e^{\alpha t} di_L + \alpha i_L e^{\alpha t} dt = \beta e^{\alpha t} v_s(t) dt$$

We identify the left side of the above equation as the exact differential of $i_L e^{\alpha t}$, since

$$d(i_L e^{\alpha t}) = e^{\alpha t} di_L + \alpha i_L e^{\alpha t} dt.$$

Therefore,

$$d(i_L e^{\alpha t}) = \beta v_s(t) e^{\alpha t} dt.$$

Integrate the above equation between 0^+ and t to yield

$$(i_L e^{\alpha t})_{(t)} - (i_L e^{\alpha t})_{(0^+)} = \beta \int_{0^+}^t v_s(t) \cdot e^{\alpha t} dt.$$

Since the circuit is initially relaxed, the second term on the left side is zero. The input function is defined in a piece-wise manner and requires two expressions in two time ranges to define it. Therefore, the integral on the right side has to be evaluated separately for the two intervals $[0^+, T^-]$ and $[T^+, \infty)$.

$$\therefore i_L e^{\alpha t} = \beta \int_{0^+}^t 1 \cdot e^{\alpha t} dt \text{ for } 0^+ \leq t \leq T^-$$

$$\therefore i_L(t) = \frac{\beta}{\alpha} (1 - e^{-\alpha t}) = \frac{1}{R} (1 - e^{-t/\tau}) \text{ for } 0^+ \leq t \leq T^-,$$

and for $t \geq T^+$,

$$i_L e^{\alpha t} = \beta \left[\int_{0^+}^{T^-} 1 \cdot e^{\alpha t} dt + \int_{T^-}^{T^+} (\text{a bounded quantity}) \cdot e^{\alpha t} dt + \int_{T^+}^t 0 \cdot e^{\alpha t} dt \right] \text{ for } T^+ \leq t \leq \infty$$

$$\therefore i_L e^{\alpha t} = \frac{\beta}{\alpha} (e^{\alpha T} - 1) \text{ for } T^+ \leq t < \infty$$

$$= \frac{\beta}{\alpha} e^{\alpha t} (e^{-\alpha(t-T)} - e^{-\alpha t}) \text{ for } T^+ \leq t < \infty$$

$$\therefore i_L(t) = \frac{\beta}{\alpha} (e^{-\alpha(t-T)} - e^{-\alpha t}) \text{ for } T^+ \leq t < \infty$$

$$= \frac{1}{R} (1 - e^{-\alpha T}) e^{-\alpha(t-T)} \text{ for } T^+ \leq t < \infty.$$

Therefore, the expression for $v_R(t)$ is

$$v_R(t) = \begin{cases} (1 - e^{-\alpha t}) & \text{for } 0^+ \leq t \leq T^- \\ (1 - e^{-\alpha T}) e^{-\alpha(t-T)} & \text{for } T^+ \leq t < \infty \end{cases}$$

$$= \begin{cases} (1 - e^{-t/\tau}) & \text{for } 0^+ \leq t \leq T^- \\ (1 - e^{-T/\tau}) e^{-(t-T)/\tau} & \text{for } T^+ \leq t < \infty \end{cases}$$

$$= \begin{cases} (1 - e^{-t_n}) & \text{for } 0^+ \leq t \leq T_n^- \\ (1 - e^{-T_n}) e^{-(t_n - T_n)} & \text{for } T_n^+ \leq t < \infty \end{cases}$$

The subscript 'n' indicates normalisation with respect to τ . The plots of resistor voltage with normalised time for various T/τ ratios are shown in Fig. 10.5-6.

EXAMPLE: 10.5-2

Repeat the problem in Example 10.5-1 by solving for particular integral and complementary function.

SOLUTION

The differential equation for $i_L(t)$ for the interval $[0^+, T^-]$ is

$$\frac{di_L}{dt} + \alpha i_L = \beta, \text{ where } \alpha = \frac{1}{\tau} = \frac{R}{L} \text{ and } \beta = \frac{1}{L}.$$

Particular integral is the solution obtained by assuming that the forcing function was applied at infinite past. Hence, particular integral value is $\beta/\alpha = 1/R$. The complementary function is of the form $Ae^{-t/\tau}$. The circuit is initially relaxed. Applying initial condition to total solution and solving for A , we get the total solution as

$$i_L(t) = \frac{1}{R}(1 - e^{-t/\tau}) \text{ for } 0^+ \leq t \leq T^- \quad (10.5-7)$$

The inductor current would have followed this expression until there is a change in the input source function or circuit structure. There is a change in the applied voltage at $t = T$ in the present example. The voltage applied for all $t \geq T^+$ is zero. Thus, the circuit is described by the following differential equation for $[T^+ \leq t < \infty)$.

$$\frac{di_L}{dt} + \alpha i_L = 0, \text{ where } \alpha = \frac{1}{\tau} = \frac{R}{L} \text{ and } \beta = \frac{1}{L}.$$

The particular integral (i.e., the solution if 0 is applied from the infinite past) for this equation is zero. The complementary function is again $Ae^{-(t-T)/\tau}$ (but valid only for $t > T^+$) with the value of A to be decided. The value of A is found out from the value of current at $t = T^+$. Since there was no impulse voltage involved in the circuit at $t = T$, the value of current at $t = T^+$ and $t = T^-$ will be the same. This value can be obtained by substituting $t = T$ in Eqn. 10.5-7.

$$\begin{aligned} \therefore \text{Initial condition for current at } t = T^+ &= \frac{1}{R}(1 - e^{-T/\tau}) \\ \therefore i_L(t) &= \frac{1}{R}(1 - e^{-T/\tau})e^{-(t-T)/\tau} \text{ for } T^+ \leq t < \infty. \end{aligned} \quad (10.5-8)$$

We can get the expression for resistor voltage by multiplying the expressions for current by R . The solution is the same as the one worked out in Example 10.5-1 and is not repeated here.

EXAMPLE: 10.5-3

Solve for i and v as functions of time in the circuit in Fig. 10.5-7.

SOLUTION

This circuit was already in a DC steady-state at $t = 0$. At $t = 0$, the switch closes, thereby forming a source-free RL circuit on the right side and a simple resistive circuit on the left side. These two circuits do not interact after $t = 0$ except that the current through the switch will be a combination of the currents from these two circuits.

Inductor is a short for DC steady-state. Therefore, the initial current in the inductor at $t = 0^-$ was $10 \text{ V}/20 \Omega = 0.5 \text{ A}$ from top to bottom. Since the switching at $t = 0$ does not involve impulse voltage, the inductor current remains at 0.5 A at $t = 0^+$ too.

Thus, a source-free RL circuit with initial current of 0.5 A is set up at $t = 0$. The various current components in the circuit after $t = 0$ are marked in Fig. 10.5-8.

$$i_1 = \frac{10 \text{ V}}{10 \Omega} = 1 \text{ A}; i_2 = 0.5e^{-1000t} \text{ A}$$

$$\therefore \text{The current through the switch } i = i_1 - i_2 = 1 - 0.5e^{-1000t} \text{ for } t \geq 0^+$$

$$v = 0.01 \frac{di_2}{dt} = -5e^{-1000t} \text{ V for } t \geq 0^+$$

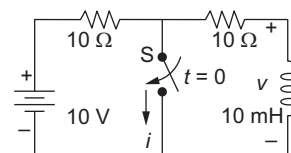


Fig. 10.5-7 Circuit for Example 10.5-3

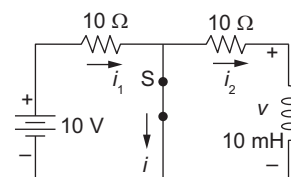


Fig. 10.5-8 Circuit for solving Example 10.5-3

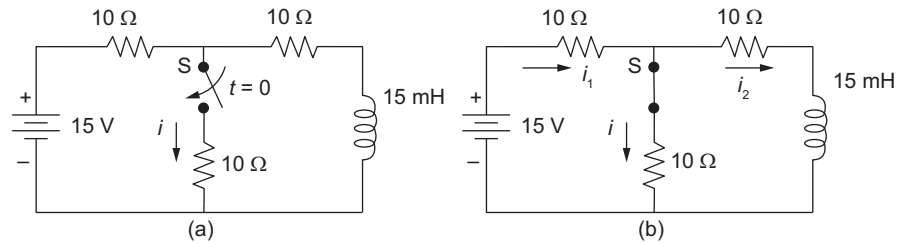


Fig. 10.5-9 Circuits for Example 10.5-4

EXAMPLE: 10.5-4

Solve the circuit in Fig. 10.5-9(a) for the current through the switch as a function of time.

SOLUTION

In this example, the two meshes in the circuit interact after $t = 0$. We can solve this circuit in many ways – branch-current method, mesh analysis, Thevenin's equivalent, etc., are some possibilities. First, we solve it by branch-current method.

Various branch currents in the circuit are identified in Fig. 10.5-9(b). We have to get a differential equation in the current variable i .

Applying KCL at the switch node gives us $i_2 = i_1 - i$.

Applying KVL in the first mesh gives us $15 = 10(i_1 + i) \Rightarrow i_1 = 1.5 - i$

$$\therefore i_2 = 1.5 - i - i = 1.5 - 2i$$

Applying KVL in the second mesh gives us $10i = 10i_2 + 0.015(di_2/dt) = 15 - 20i - 0.03(di/dt)$.

Therefore, the differential equation governing i is $di/dt + 1000i = 500$ for $t \geq 0^+$.

Initial condition for i , i.e., its value at $t = 0^+$ is needed. Initial value of i_2 is $15 \text{ V}/20 \Omega = 0.75 \text{ A}$, as the circuit was in DC steady-state prior to switching. Since $i_2 = 1.5 - 2i$, value of i at $t = 0^+$ will be $(1.5 - 0.75)/2 = 0.375 \text{ A}$. The particular integral of the differential equation for i is $500/1000 = 0.5 \text{ A}$. Time constant is $1/1000$.

Therefore, $i = Ce^{-1000t} + 0.5$. Evaluating C from initial condition for i at $t = 0^+$, we get $C = 0.375 - 0.5 = -0.125$.

Therefore, the switch current $i = 0.5 - 0.125e^{-1000t}$ for $t \geq 0^+$.

Let us solve the same problem by mesh analysis. The relevant circuit with two mesh currents – i_1 and i_2 – identified is shown in Fig. 10.5-10(a).

The two mesh equations are:

$$20i_1 - 10i_2 = 15$$

$$20i_2 - 10i_1 + 0.015 \frac{di_2}{dt} = 0.$$

Eliminating i_1 from the second equation using the first equation and simplifying, we get,

$$\frac{di_2}{dt} + 1000i_2 = 500.$$

The initial condition for i_2 at $t = 0^+$ is the same as the initial condition for inductor current at that instant. This value is 0.75 A . Particular integral is 0.5 A . Time constant is $1/1000 \text{ s}$.

$$i_2 = Ce^{-1000t} + 0.5$$

$$C = 0.25 \text{ and } i_2 = 0.5 + 0.25e^{-1000t}$$

Using this solution in the first mesh equation, we can get,

$$i_1 = 1 + 0.125e^{-1000t}$$

$$\therefore \text{Current through the switch} = i_1 - i_2 = 0.5 - 0.125e^{-1000t} \text{ A for } t \geq 0^+$$

This circuit problem can be solved by using Thevenin's theorem too.

The circuit portion to the left of the inductor may be replaced by its Thevenin's equivalent as shown in Fig. 10.5-10(b). Inductor current can be obtained from this circuit. Once inductor current is available, we will be able to get back to the switch current using KCL or KVL. This is illustrated below.

Comments on Example: 10.5-4

We derived the differential equations governing the three variables in the circuit – the branch current in the central limb, second mesh current and the inductor current – in the process of solving this circuit. The left-hand side of all the three differential equations had the same coefficients. (Why?)

We also notice that the time constant of the circuit can be easily found as L/R_{th} , where R_{th} is the Thevenin's equivalent resistance appearing across the inductor. But Thevenin's equivalent is found by deactivating all independent sources.

Therefore, the time constant of a single-inductor circuit can be found by replacing all independent voltage sources by short-circuits and all independent current sources by open-circuits and finding the equivalent resistance connected across the inductor. We illustrate this procedure further in the next example.

The initial current in the inductor is 0.75 A, again. The circuit in Fig. 10.5-10(b) is a simple series RL circuit and its particular integral is $7.5/15 = 0.5$ A. Its time constant is $15 \text{ mH}/15 \Omega = 1 \text{ ms}$. Therefore, its solution is $Ce^{-1000t} + 0.5$. Evaluating the initial condition constant C and completing the solution, we get, inductor current $= 0.5 + 0.25e^{-1000t}$ A.

$$i_L = 0.25e^{-1000t} + 0.5 \text{ A}$$

$$\begin{aligned} \text{Voltage across } 10 \Omega \text{ in the switch path} &= 10 \times (0.25e^{-1000t} + 0.5) + 0.015 \times (-250e^{-1000t}) \\ &= 5 - 1.25e^{-1000t} \text{ V} \end{aligned}$$

$$\begin{aligned} \therefore \text{Current through the switch} &= 5 - 1.25e^{-1000t} \text{ V}/10 \Omega \\ &= 0.5 - 1.25e^{-1000t} \text{ A for } t \geq 0^+. \end{aligned}$$

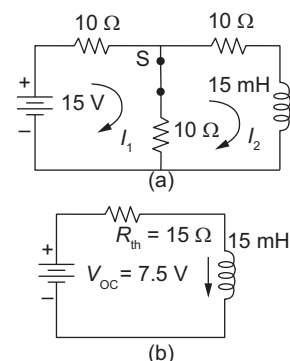


Fig. 10.5-10 Circuits for Mesh Analysis and Thevenin's Equivalent Analysis in Example 10.5-4

EXAMPLE: 10.5-5

Show that the current in 18 mH inductor in the circuit in Fig. 10.5-11(a) will go to zero as $t \rightarrow \infty$. Also, find the inductor current and currents delivered by the voltage sources as functions of time. Find how long we have to wait for the inductor current to fall below 100 mA.

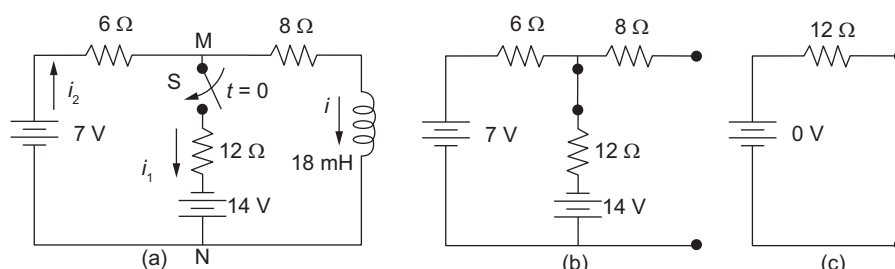


Fig. 10.5-11 Circuits for Example 10.5-5 (a) Circuit for the Problem (b) Circuit for Finding Thevenin's Equivalent (c) Thevenin's Equivalent

SOLUTION

First, we find the time constant effective after $t = 0^+$. The Thevenin's equivalent of the circuit connected across the inductor is evaluated by using the circuit in Fig. 10.5-11(b) and the resulting equivalent is shown in Fig. 10.5-11(c). Since the voltage source in Thevenin's equivalent is zero-valued, the inductor current will have a zero steady-state value. The time constant of the circuit is $16 \text{ mH}/12 \Omega = 1.5 \text{ ms}$.

We find the initial condition for inductor current next. The circuit was in steady-state prior to switching at $t = 0$. The inductor is replaced by a short-circuit for DC steady-state. Therefore, the inductor current at $t = 0^-$ must have been $7/14 = 0.5$ A and it will be 0.5 A at $t = 0^+$, since there is no impulse voltage involved in the switching.

Now, the circuit is a series RL circuit with a known initial condition and DC sources. We know the solution for such a circuit. It is of the general format $Ae^{-t/\tau} + C$, where C is the particular integral (therefore, the DC steady-state value) and A is the arbitrary constant to be found from the initial condition. *This is the general format of solution for any circuit variable in a first order circuit with DC excitation.*

$$\therefore i(t) = Ae^{-t/1.5} + 0; t \text{ in ms}$$

$$i(t) = 0.5 \text{ at } t = 0^+$$

$$\therefore i(t) = 0.5e^{-t/1.5} \text{ A}; t \text{ in ms and } t \geq 0^+ \quad (10.5-9)$$

$$\therefore \text{Voltage across inductor} = 0.018 \frac{di}{dt} = -6e^{-t/1.5} \text{ V}; t \text{ in ms and } t \geq 0^+.$$

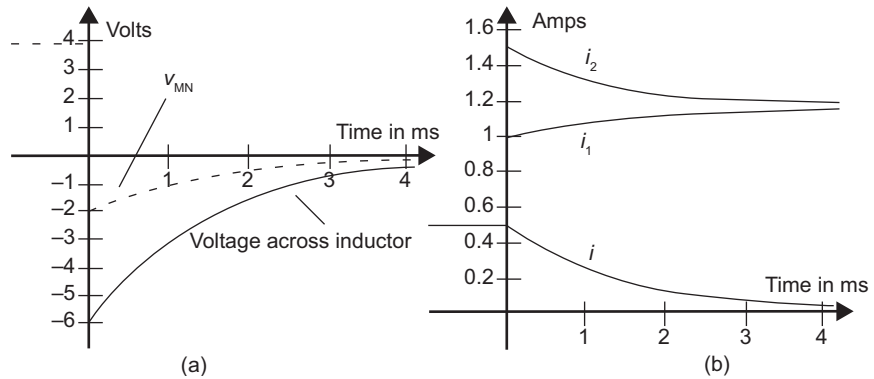


Fig. 10.5-12 Waveforms of (a) Voltages and (b) Source and Inductor Currents in Example 10.5-5

We have two ways to find the currents delivered by the voltage sources – i_1 and i_2 . In the first method, we find the voltage across M and N in the circuit in Fig. 10.5-11 (a) as $v_{MN} = 8i + 0.018 \, di/dt$ and then find i_2 as $(7 - v_{MN})/6$ and i_1 as $(v_{MN} + 14)/12$.

In the second method, we realise that all variables in this circuit will have a $Ae^{-t/\tau}$ term and a steady-state term, and that we can find the arbitrary constant A if we know the value of the particular variable at 0^+ . So, we set out to find the initial and final (i.e., steady-state) value of the source currents. The inductor current was at 0.5 A at $t = 0^+$. The voltage across the inductor at that instant is –6 V from Eqn. 10.5-9. Therefore, $v_{MN} = 8 \times 0.5 - 6 = -2$ V at $t = 0^+$. Therefore, the initial value of i_1 at $t = 0^+ = (-2 - (-14))/12 = 1$ A. Final current in the inductor is zero. Hence, final current in both the sources will be the same and equal to $(7 + 14) \text{ V} / (6 + 12) \, \Omega = 7/6 = 1.167$ A.

Therefore, the required currents are

$$\therefore i_1(t) = Ae^{-t/1.5} + 1.167; \, t \text{ in ms with } i_1 = 1 \text{ A at } t = 0^+$$

$$\therefore i_1(t) = 1.167 - 0.167e^{-t/1.5} \text{ A; } t \text{ in ms and } t \geq 0^+$$

$$\begin{aligned} i_2(t) &= i_1(t) + i(t) = 1.167 - 0.167e^{-t/1.5} + 0.5e^{-t/1.5} \\ &= 1.167 + 0.333e^{-t/1.5} \text{ A; } t \text{ in ms and } t \geq 0^+. \end{aligned}$$

These are plotted in Fig. 10.5-12. The time required for the inductor current to go below 100 mA is found as follows:

$$0.1 = 0.5e^{-t/1.5} \text{ A; } t \text{ in ms, } \therefore -\frac{t}{1.5} = \ln 0.2 = -1.6904 \text{ and } t = 2.414 \text{ ms.}$$

10.6 STEADY-STATE RESPONSE AND FORCED RESPONSE

The total response in an RL circuit to any forcing function will consist of two terms – the transient response (or natural response) and the forced response. The transition from the initial state of the circuit (which is encoded in a single number in the form of an initial inductor current specification at $t = 0^-$) to the final state (in which only forced response will be present) is accomplished with the help of the transient response. Now, we introduce a new term called the steady-state response and relate it to the response terms we are already familiar with.

Our study of the solution of differential equation describing the RL circuit has shown us that the total response will always contain two components – the transient response and the forced response. Of course, forced response will be zero if forcing function is zero, i.e., in a source-free circuit. Similarly, the transient response term may become zero under certain suitable initial condition values. But these are special situations and, in general, there will

be two terms in the total response. This is true not only for RL circuit but also for any linear circuit described by linear ordinary differential equations with constant coefficients. Such a circuit of higher order will have two groups of terms in its total solution – the first group constituting the transient response containing one or more terms and the second group constituting the forced response containing one or more terms depending on the type of forcing function. Thus, forced response is a response component which is always present in the total response of a circuit except when the forcing function itself is zero.

We have seen that the transient response of RL circuit contains exponential function of the form $e^{-\alpha t}$, where α is a positive number decided by R and L . Such an exponential function with a negative real index will taper down towards zero as t approaches ∞ . Hence, we expect the transient response in an RL circuit to vanish with time irrespective of the forced response component. Therefore, we expect that there will only be the forced response component active in the circuit in the long run, i.e., after sufficient time had been allowed for the transient response to die down. When all the transient response terms in all the circuit variables in the circuit have died down to negligible levels (they never die down to zero) and the only response component in all the circuit variables is the forced response component, we say the circuit has reached the steady-state with respect to the particular forcing function that was applied to the circuit. Notice that under the *steady-state* conditions, the transient response terms should not be present in any circuit variable at all. In other words, there cannot be a circuit which attains steady-state in some of its variables and does not attain steady-state in others.

Therefore, a circuit will reach steady-state if, and only if, all its transient response terms are of decreasing type. Moreover, the only response that will continue in the circuit after it has reached the steady-state is the forced response component. Therefore, *steady-state response* is the same as *forced response* with the condition that the steady-state will exist only if all the transient response terms are of damped nature – i.e., decreasing functions of time. Thus, steady-state response is another name for forced response when transient response is sure to die down to negligible levels. Forced response will always be present; but steady-state response need not be.

Consider the circuit in Fig. 10.6-1. The 1 V source on the left side has set up an initial current of 1 A in the inductor of 1 H at $t = 0^-$. The switch S_1 is opened and the switch S_2 is closed at $t = 0$ to apply a 1 V source right across the inductor. The current in the inductor is shown in Fig. 10.6-1(b). We note that with a bounded input (1 V DC source is a bounded input) we get a current in the inductor which is not bounded. Also, the transient response ($1e^{-0t}$) does not decrease with time. Therefore, there is no steady-state in this circuit though there is a forced response.

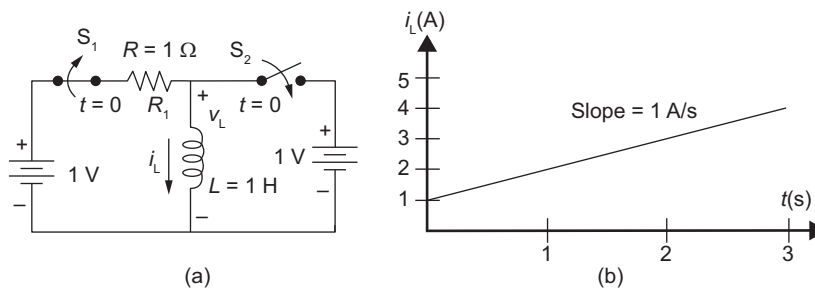


Fig. 10.6-1 A Circuit With No Steady-State and its Step Response

Steady-State

A circuit is said to have reached the steady-state with respect to a particular forcing function if all transient response terms decay down to a negligible level and the only response component that remains is the forced response component.

10.6.1 The DC Steady-State

It is to be noted that the steady-state attained by a circuit is intimately connected with the type of forcing function applied to the circuit. We cannot expect the generalisations arrived at, based on the steady-state behaviour for a particular forcing function, to hold good in the case of a steady-state behaviour for another forcing function.

For example, consider the steady-state in series RL circuit when input is a unit step voltage. The forced response in this case is a constant current of value $1/R$ A. The voltage across the inductor with a constant current through it can only be zero. But zero is indeed a constant. Thus, we see that, under the steady-state condition with step input, all the circuit variables become constants. The only constant voltage an inductor can have across it is zero if the current through it is also constrained to remain constant, since $v = L di/dt$ for an inductor. Noticing the ‘constant’ nature of the input voltage and the ‘constant’ nature of all circuit variables under steady-state, we name this kind of steady-state as the *DC steady-state*.

Under DC steady-state (provided the circuit can reach such a steady-state – transient response has to die down for that), inductors in a circuit can have any constant-valued current; but their voltages will be constrained to remain at zero. But that is similar to the definition of a short-circuit – except that the voltage across a short-circuit is zero for *any* current. Thus, we can solve for DC steady-state response in RL circuits (containing one or more inductors) by replacing all the inductors with short-circuits – provided all independent sources in the circuit are switched DC sources or step functions and a DC steady-state can exist in the circuit. Subject to the above conditions, we can state that ‘*an inductor is a short-circuit under DC steady-state*’.

While we are on the topic of steady-state, we might as well look at the remaining two kinds of steady-state we will need in the analysis of dynamic circuits.

An inductor in a circuit can be replaced by a short-circuit for analysing the circuit under DC steady-state condition, provided the circuit is capable of a DC steady-state.

10.6.2 The Sinusoidal Steady-State

Sinusoidal steady-state refers to the steady-state that gets established in a circuit when all the independent sources in the circuit are sinusoids of the same angular frequency. Like DC steady-state, this steady-state too can exist only if all the terms in transient response die down to negligible levels with time.

The word ‘*steady*’ has the literal meaning of ‘*unchanging*’. This unfortunately gives an impression that *steady-state* is that state in a circuit in which all circuit variables are *unchanging* with time. This is an error that a beginner in Circuit Analysis has to guard against. *Steady-state* does not necessarily mean that the circuit response is *unchanging* with time. It is so only in the case of DC steady-state.

To understand the meaning of the word ‘*steady*’ in the Circuit Analysis context, we have to look at the input forcing function and find out those features of the forcing function which remain unchanged with time. In the case of DC or step inputs, the input value itself is unchanging in $(0^+, \infty)$. In the case of sinusoidal input forcing function, the amplitude of the sinusoid, its angular and cyclic frequencies, its phase and its shape in one period (i.e.,

Meaning of ‘sinusoidal steady-state response’.

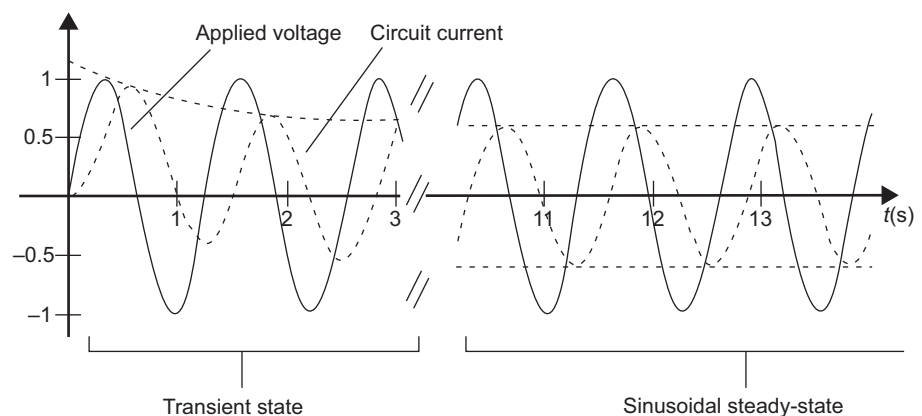


Fig. 10.6-2 Waveforms Illustrating Sinusoidal Steady-State

sinusoidal shape) remain constant in time. Therefore, if a steady-state exists in a circuit under the action of such a forcing function, we can expect all circuit variables to have sinusoidal shape, fixed amplitudes, fixed frequency *which is the same as that of the forcing function* and fixed phase with respect to the forcing function. This is what is meant by sinusoidal steady state.

Thus, a circuit excited by one or more sinusoidal forcing functions of the *same* frequency is said to have reached the *sinusoidal steady-state* if all its transient response components have died down and all its circuit variables have *sinusoidal* waveshape with the *same* frequency as that of the forcing functions and *fixed* amplitudes and phase angles.

The waveforms in Fig. 10.6-2 show the applied voltage and inductor current in an initially relaxed RL circuit with $R = 0.33 \, \Omega$ and $L = 0.33 \, \text{H}$. A sinusoidal voltage $= 1 \sin(5t) \, \text{V}$ was switched on to the circuit at $t = 0$. The current waveform shows the exponential transient response in the first few seconds clearly. After about 10 s or so, the transient response has decayed to negligible levels and the response contains only a sinusoidal waveform that is of the same frequency as that of the applied voltage. It has a fixed amplitude and a fixed phase with respect to the input sine wave. Thus, the circuit has reached the sinusoidal steady-state within a few time constants ($\tau = 1 \, \text{s}$).

Sinusoidal steady-state is also referred to as *AC steady-state* in Circuit Analysis literature.

10.6.3 The Periodic Steady-State

This is the third kind of steady-state that can come up in linear circuits. Here, the input forcing function is periodic but not sinusoidal. Therefore, a single number like amplitude in the case of a sinusoid is not available for this input. The only aspect that is *steady* about it is its frequency. Thus, we expect the circuit to reach a steady-state (if it can) in which all circuit variables will be *periodic* with the same *frequency* as that of the input. However, the waveshape of the response variables will not be the same as the waveshape of the input forcing function. The sinusoidal steady-state we discussed in the previous sub-section is indeed a periodic steady-state; but it is more than that – in sinusoidal steady-state, the response waveform is the same as that of the forcing function. In a periodic steady-state such a constraint may not be satisfied.

The waveforms shown in Fig. 10.6-3 illustrate the attainment of periodic steady-state in an RL circuit ($R = 0.33 \, \Omega$, $L = 0.33 \, \text{H}$, $I_0 = 0$, $\tau = 1 \, \text{s}$) driven by a voltage $= (1 \sin 5t + 0.4 \sin 15t) \, \text{V}$ from $t = 0^+$ onwards. Notice that the waveshape of current is not the same as that of the applied voltage. However, the current is periodic with the same

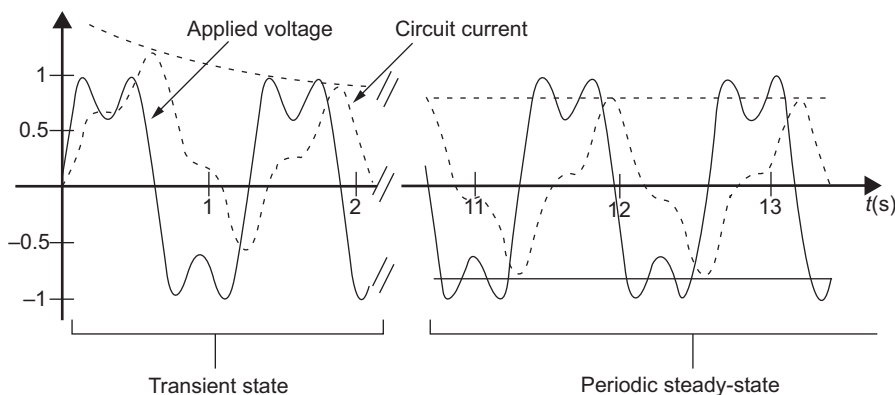


Fig. 10.6-3 Waveforms Illustrating Periodic Steady-State

period and its waveshape remains the same in successive periods after it has reached a steady-state.

We saw that the inductors can be replaced by short-circuits for DC steady-state analysis. However, we notice that the currents in the inductors in circuits are time-varying currents in sinusoidal steady-state and periodic steady-state conditions. *Hence, inductors cannot be replaced by short-circuits for all kinds of steady-state analysis – that works only for DC steady-state analysis.*

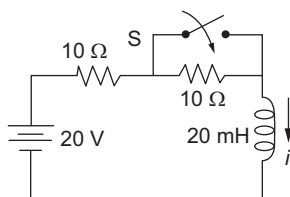


Fig. 10.6-4 Circuit for Example 10.6-1

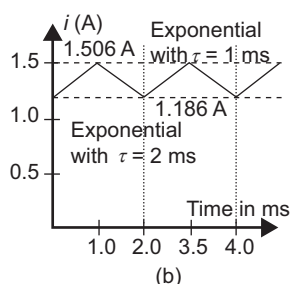
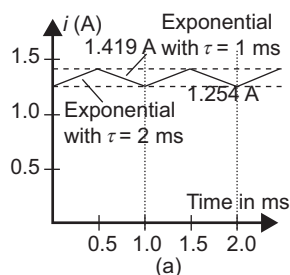


Fig. 10.6-5 Periodic steady-state in the Circuit in Example 10.6-1 for (a) 1 kHz and (b) 500 Hz Switching

EXAMPLE: 10.6-1

Periodic steady-state can come up in RL circuits when one or more sources in the circuit are periodic functions of time. But even when all sources are DC sources, there can be a periodic steady-state in the circuit if some parameter in the circuit is varying periodically with time. We consider one such circuit in this example.

The circuit for this example appears in Fig. 10.6-4. The switch S in the circuit is operated periodically at a switching frequency of 1 kHz. In every switching cycle it is kept closed for half the time and then kept open for the remaining half of cycle time, i.e., 0.5 ms. After a large number of switching cycles, a steady repeating pattern of current gets established in the circuit. We describe this steady pattern in this example.

SOLUTION

Assume that the circuit is in periodic steady-state. Let the current start with a value I_1 when the switch is closed in the beginning of a steady-state cycle. Then, the current increases along an exponential path with τ of 2 ms towards 2 A (because 2 is the particular integral under this condition). The current is not allowed to reach 2 A since the switch goes open after 0.5 ms when the current is I_2 . Now, a new transient starts, trying to take the current from this value to 1 A (because 1 is the particular integral now) exponentially with a time constant of 1 ms. But this exponential is terminated after another 0.5 ms and then the cycle starts all over again. *At the end of the second 0.5 ms, the current in the circuit will be I_1 if the circuit is in the periodic steady-state.* We convert the above reasoning into equations and solve for I_1 and I_2 .

$$I_1 e^{-0.5/2} + 2(1 - e^{-0.5/2}) = I_2 \quad \text{and} \quad I_2 e^{-0.5/1} + 1(1 - e^{-0.5/1}) = I_1$$

$$[I_1 e^{-0.5/2} + 2(1 - e^{-0.5/2})] e^{-0.5/1} + 1(1 - e^{-0.5/1}) = I_1$$

$$\therefore I_1 = 1.254 \text{ A and } I_2 = 1.419 \text{ A.}$$

The corresponding values for 500 Hz switching also are worked out as $I_1 = 1.186 \text{ A}$ and $I_2 = 1.506 \text{ A}$. The plot of inductor current for 1 kHz switching and 500 Hz switching are shown in Fig. 10.6-5(a) and (b).

The total resistance in the circuit was changing abruptly between 10 and 20 Ω , but the current in the circuit does not show any discontinuity. This once again illustrates the fact that the inductor smoothes a circuit current. Moreover, we see from Fig. 10.6-5 that the smoothing effect is more when the circuit time constants are larger than the switching cycle period. In fact, the current tends to become almost constant at 1.33 A as inductance value in this circuit is increased. (The reader is encouraged to ponder over why it should be 1.33 A and why the average value of waveforms in Fig. 10.6-5 also should be 1.33 A.)

10.7 LINEARITY AND SUPERPOSITION PRINCIPLE IN DYNAMIC CIRCUITS

We have been dealing with *unit* step response of RL circuit until now. How do we get the solution if it is not *unit* step, but a step of size V ? In short, how do we solve the series RL circuit if the input source function is $Vu(t)$? Can we just multiply the unit step response by V to get the solution for this input?

We know that memoryless circuits containing linear passive resistors, linear dependent sources and independent sources will be linear and will obey the superposition principle. We examine the issue of linearity of circuits containing one or more energy storage elements

along with resistors, linear dependent sources and independent sources in this section. We are already familiar with a particular decomposition of total response in such a circuit in terms of transient response and forced response. We will see in this section that yet another decomposition of total response into the so-called *zero-input response* and *zero-state response* is needed in view of the linearity considerations in the circuit.

An electrical element is called *linear* if its element equation obeys the superposition principle. Superposition principle involves two sub-principles – principle of additivity and principle of homogeneity. We have seen earlier that an inductor with an element relation $v = L di/dt$ and a capacitor with an element relation $i = C dv/dt$ are linear elements. But will an interconnection of such linear elements (and independent sources) into a circuit result in a *linear system*? A linear system is one in which *every* response variable in the system obeys the superposition principle.

Intuitively, we expect an interconnection of linear elements and independent sources to yield a linear circuit; but mathematically it is not that simple. It requires to be proved. The proof involves slightly advanced concepts from a mathematical topic called *linear vector spaces* and we do not take up that here. *We accept the result that a circuit formed by interconnecting linear passive elements, linear dependent sources and independent sources will be a linear circuit. Such a linear circuit has to obey the superposition principle.*

Therefore, we must be able to get $i_L(t)$ in a series RL circuit with $Vu(t)$ V as its input source function by scaling the unit step response by V . Assuming an initial condition of I_0 at $t = 0^-$, this scaling results in Eqn. 10.5-4 getting multiplied by a dimensionless scalar V to yield

$$i_L(t) = VI_0 e^{-t/\tau} + \frac{V}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+$$

as the solution. But this solution is incorrect because the current at $t = 0^+$ is VI_0 according to this equation rather than the correct value of I_0 . It looks as if the principle of homogeneity is not valid here. Let us try to get the solution without resorting to linearity.

The complementary solution is $Ae^{-\alpha t}$ with $\alpha = 1/\tau = R/L$ and the particular integral is V/R . Therefore, the total solution is $i_L(t) = Ae^{-\alpha t} + V/R$. Substituting the initial condition at $t = 0^+$ and solving for A , we get the final solution as $i_L(t) = (I_0 - V/R)e^{-\alpha t} + V/R = I_0 e^{-\alpha t} + V/R(1 - e^{-\alpha t})$ for $t \geq 0^+$. Thus, the correct solution is,

$$i_L(t) = I_0 e^{-t/\tau} + \frac{V}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \quad (10.7-1)$$

The first term in Eqn. 10.5-4 does not get multiplied by V when the step magnitude is scaled by V . The second term in the same equation gets scaled by V . This means that the forced response (the constant term in the solution) obeys the principle of homogeneity.

Now, let us solve the circuit for three situations (a) with $V_1 u(t)$, (b) with $V_2 u(t)$ and (c) with $V_1 u(t) + V_2 u(t)$ as the input source functions with the same initial condition for all three cases. The expression for $i_L(t)$ in the three cases can be derived as

$$\begin{aligned} \text{Case (a) } i_L(t) &= I_0 e^{-t/\tau} + \frac{V_1}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \\ \text{Case (b) } i_L(t) &= I_0 e^{-t/\tau} + \frac{V_2}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \\ \text{Case (c) } i_L(t) &= I_0 e^{-t/\tau} + \frac{V_1 + V_2}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+. \end{aligned} \quad (10.7-2)$$

These expressions show that the forced response part obeys the principle of additivity also. In all the cases, we see that neither the total response nor the transient response (or natural response) obeys the superposition principle.

We have arranged the terms in the expression for inductor current in Eqn. 10.7-1 and Eqn. 10.7-2 in a special manner – the dependence on the initial condition is contained in the first term and the dependence on step magnitude is contained in the second term. We notice that it is not only the forced response component which satisfies the superposition

Zero-state Response and Zero-input Response

A well-known principle in Linear System Theory states that the total response in a linear time-invariant system containing energy storage elements can be found by adding the zero-input response and zero-state response together.

Zero-input response will depend only on the initial state of the circuit as encoded in its initial condition specifications.

Zero-state response will depend only on forcing functions.

Zero-input response obeys the superposition principle with respect to the initial condition values and zero-state response obeys the superposition principle with respect to the input source functions.

principle, but the entire second term which depends on forcing function satisfies the superposition principle. However, both terms contribute to transient response and transient response does not satisfy the superposition principle.

We notice further that the first term depends only on the initial condition and will be the total response if there is no forcing function, i.e., the first term is the response in a source-free circuit. Similarly, the second term depends on the forcing function and does not depend on the initial condition. The second term will be the total response if the circuit is initially relaxed. These observations will remain valid for any forcing function. The functional nature of the second term will change with the nature of the forcing function. However, the resolution of total response into two components – one which depends entirely on the initial condition and the other which depends entirely on forcing function – will be possible for any forcing function in a linear circuit. The response component that depends only on the initial condition values is called the *zero-input response*. The response component that depends only on the forcing function is called the *zero-state response*.

Now, we focus on the zero-input response of *RL* circuit. This is the response in a source-free circuit due to its initial energy alone. It is $I_0 e^{-\alpha t}$ A with the usual interpretations for all the symbols. It must be obvious that the zero-input response will scale with I_0 , i.e., when the initial condition value is multiplied by a real constant, the zero-input response also gets multiplied by the same constant. Similarly, when two different values of initial condition I_{01} and I_{02} result in two different zero-input responses, the zero-input response with the initial condition value at $I_{01} + I_{02}$ will be the sum of the two zero-input responses observed in the first two cases. Thus, zero-input response of *RL* circuit (and all linear time-invariant circuits) obeys the superposition principle with respect to the initial condition values.

Thus, we see that both zero-state response and zero-input response obey the superposition principle individually. *Zero-input response follows the superposition principle with respect to initial condition values and zero-state response obeys superposition principle with respect to input source functions.* Therefore, the total response will not follow the superposition principle with respect to the forcing function or initial condition – only its zero-state and zero-input response components will obey the superposition principle.

Figure 10.7-1 shows the zero-input response and zero-state response components along with the total current response in an *RL* circuit excited by a unit step input for various

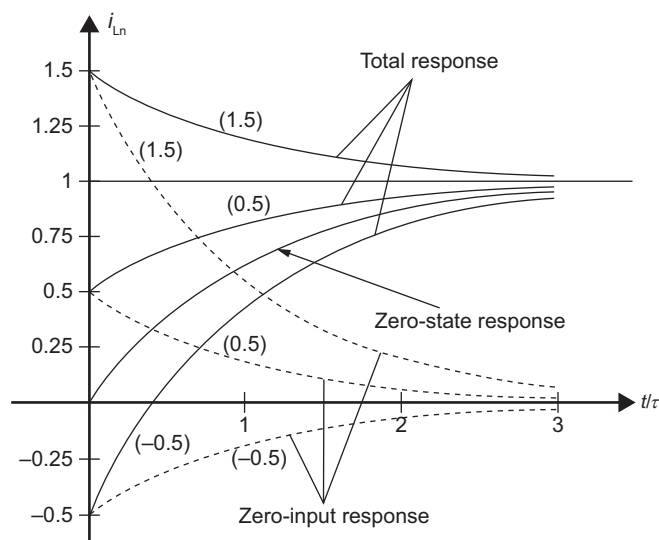


Fig. 10.7-1 Decomposition of Total Response into Zero-Input Response and Zero-State Response. Values in Brackets give the Normalised Initial Current Values.

values of normalised initial condition values. Decomposition of total response into transient response and forced response for the same circuit was shown in Fig. 10.5-4. Compare these two decompositions. One can see that (i) both zero-input response and zero-state response will contain natural response terms, (ii) however, the natural response component in the zero-state response has an amplitude which depends on the forcing function value and *does not* depend on the initial condition value, (iii) zero-input response and a part of the zero-state response together will form the transient response and (iv) the remaining part of the zero-state response will become the forced response.

We would like to understand the apparently ‘qualified’ adherence to the superposition principle exhibited by dynamic circuits a little further. We consider a simple T-circuit formed by three equal resistors with two independent voltage sources driving the circuit as shown in Fig. 10.7-2. The voltage across the shunt arm of T is accepted as the circuit response variable.

By a correct application of superposition principle to the circuit in Fig. 10.7-2(a), we get the following equation as response.

$$v = \frac{V_x}{3} + \frac{V_y}{3}. \quad (10.7-3)$$

The correct response for the remaining five cases can be worked out by substituting whatever function that takes the place of V_x and V_y . Of course, we can also apply the superposition principle afresh in each circuit to arrive at the correct response. In either case, the solution will be as below:

$$\begin{aligned} \text{Case (b)} \quad v &= \frac{kV_x}{3} + \frac{V_y}{3} \\ \text{Case (c)} \quad v &= \frac{V_x}{3} + \frac{kV_y}{3} \\ \text{Case (d)} \quad v &= \frac{V_{x1}}{3} + \frac{V_y}{3} \\ \text{Case (e)} \quad v &= \frac{V_{x2}}{3} + \frac{V_y}{3} \\ \text{Case (f)} \quad v &= \frac{V_{x1} + V_{x2}}{3} + \frac{V_y}{3} \end{aligned} \quad (10.7-4)$$

We note that (i) the solution for (b) is not the same as the solution for (a) multiplied by k , (ii) the solution for (c) is not the same as the solution for (a) multiplied by k and (iii) the solution for (f) is not the same as the sum of solutions for (d) and (e). But did we expect the above to turn out true? If we did, we would have been misapplying the superposition principle. In a two-source situation, whatever is done to one of them can affect only that component of response contributed by this particular source. That must be obvious. Hence, we did not expect any solution other than the ones which appear in Eqn. 10.7-4.

But then, how did we expect the wrong thing in the context of RL circuit? We made an error in expectation as we saw two terms in response – one depending on the initial condition and the second depending on the forcing function – and it did not occur to us that the initial condition may be a kind of source. But is it really? If the initial condition can be thought of as yet another independent source, then, there need not be any qualification for applying the superposition principle in dynamic circuits. The decomposition of response into zero-state response and zero-input response will attain a new meaning then – they will be contributions to response from two independent sources.

So, we are going to show that the initial condition can be equivalenced to an independent source. But we cannot get there if we insist on sticking on to step input. We have had enough of unit step function. Let us consider another interesting input – the unit impulse function $\delta(t)$.

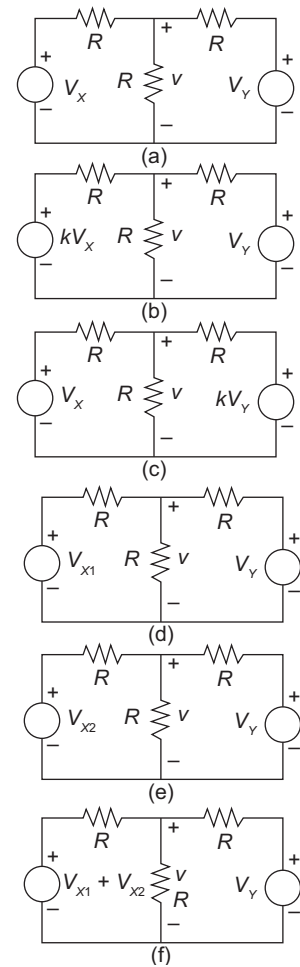


Fig. 10.7-2 Resistive Circuits for Discussion on Superposition Principle

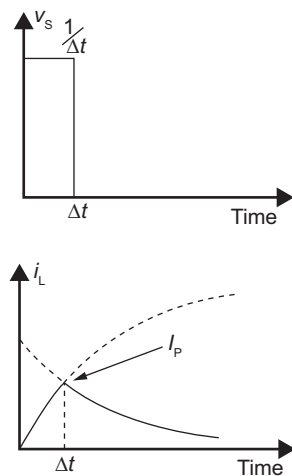


Fig. 10.8-1 Pulse Response of Series RL Circuit

10.8 UNIT IMPULSE RESPONSE OF SERIES RL CIRCUIT

We look at the response of series RL circuit in Fig. 10.1-1 with $v_s(t) = \delta(t)$, the unit impulse voltage. We have discussed the unit impulse function in detail in an earlier chapter. We saw that we can view unit impulse function as a limiting case of a rectangular pulse waveform of amplitude $1/\Delta t$ V located between $t = 0$ and $t = \Delta t$ as $\Delta t \geq 0$. The pulse always maintains unit area under it by increasing the amplitude as its duration decreases. We use this interpretation of unit impulse function to analyse impulse response of RL circuit first. The circuit is assumed to be initially relaxed.

The applied pulse and circuit current response are shown in Fig. 10.8-1. The response in the interval $[0^+, \Delta t]$ will be the same as the unit step response in initially relaxed RL circuit scaled by $1/\Delta t$. At Δt , the applied voltage at input goes to zero, i.e., the input gets shorted and a source-free RL circuit starts executing its zero-input response from that point onwards. The amplitude of this response has to be the initial current value just prior to Δt . This value is marked as I_p in the Fig. 10.8-1. An expression for I_p can be obtained by evaluating the first part of the response at Δt as shown in Eqn. 10.8-1.

$$i_L(t) = \begin{cases} \frac{1}{R\Delta t}(1 - e^{-t/\tau}) & \text{for } 0^+ \leq t \leq \Delta t \\ I_p e^{-(t-\Delta t)/\tau} & \text{for } \Delta t < t < \infty \end{cases} \quad (10.8-1)$$

$$I_p = \frac{1}{R\Delta t}(1 - e^{-\Delta t/\tau}).$$

As Δt is decreased, the value of I_p increases and moves towards left in time axis. Eventually, it attains a limit as $\Delta t \rightarrow 0$. This limiting value can be found by using series expansion for exponential function as in Eqn. 10.8-2.

$$\begin{aligned} I_p &= \frac{1}{R\Delta t}(1 - e^{-\Delta t/\tau}) \\ &= \frac{1}{R\Delta t} \left[1 - \left\{ 1 - \frac{\Delta t}{\tau} + \frac{1}{2!} \left[\frac{\Delta t}{\tau} \right]^2 - \frac{1}{3!} \left[\frac{\Delta t}{\tau} \right]^3 + \dots \right\} \right] \\ &= \frac{1}{R\tau} \text{ for small } \frac{\Delta t}{\tau} \\ &= \frac{1}{L} (\because \tau = L/R). \end{aligned} \quad (10.8-2)$$

Therefore, as $\Delta t \rightarrow 0$, the first part of the response tends to become a jump by $1/L$ A and the second part becomes an exponential from $t = 0^+$ onwards with $1/L$ as the starting value. In the limit, the impulse response becomes

$$I_L(t) = \frac{1}{L} e^{-t/\tau} \text{ A for } t \geq 0^+. \quad (10.8-3)$$

The only *steady* feature the unit impulse function possesses after $t = 0^+$ is *constancy at zero value*. Therefore, we expect the forced response to be zero. The only response component that can be there in the impulse response is the transient response component. Since the initial condition at $t = 0^-$ was stated to be zero, it follows that zero-input response is zero. Therefore, the impulse response appearing in Eqn. 10.8-3 is the zero-state response for unit impulse application.

Now, consider the series RL circuit with an initial condition of $1/L$ A at $t = 0^-$ and $v_s(t) = 0$ for $t \geq 0^+$. This is a source-free RL circuit with non-zero initial energy. It will have only zero-input response component in its response and its total response will be

$$i_L(t) = \frac{1}{L} e^{-t/\tau} \text{ A for } t \geq 0^+,$$

Zero-state unit impulse response of series RL circuit.

which is exactly the same as the zero-state unit impulse response. Thus, the effect of applying $\delta(t)$ is a change in the initial condition of the inductor between $t = 0^-$ and $t = 0^+$. Remember, we have always pointed out that the initial condition at $t = 0^-$ and $t = 0^+$ will be the same only if no impulse voltage is applied in the circuit or supported in the circuit. Applying $\delta(t)$ amounts to keeping the circuit shorted for $t \leq 0^-$ and $t \geq 0^+$ and applying an undefined voltage at $t = 0$ such that a finite volt-second area of 1 volt-sec is dumped into the circuit at $t = 0$. This results in changing the inductor current by 1 volt-sec/L H = 1/L A between $t = 0^-$ and $t = 0^+$.

The equivalence between non-zero initial condition at $t = 0^-$ and application of suitably sized impulse voltage at $t = 0$ is further clarified by the relations in Eqn. 10.8-4, where v_L and i_L are the voltages across an inductor and current through the inductor, respectively.

$$\begin{aligned} i_L(t) &= \frac{1}{L} \int_{-\infty}^t v_L(t) dt \\ &= \underbrace{\frac{1}{L} \int_{-\infty}^{0^-} v_L(t) dt + \frac{1}{L} \int_{0^-}^{0^+} v_L(t) dt}_{=i_L(0^+)} + \frac{1}{L} \int_{0^+}^t v_L(t) dt \end{aligned} \quad (10.8-4)$$

The circuit solution for $t \geq 0^+$ requires only the value of the initial current at $t = 0^+$. This value is the sum of the first two definite integrals in the Eqn. 10.8-4. It does not matter which integral contributes how much as long as the sum of their contributions remain constant as far as $i_L(t)$ after $t = 0^+$ is concerned. The first integral gives the initial current in the inductor due to all the voltages applied to it in the past. The second integral will be non-zero only if the impulse voltage is applied to the inductor at $t = 0$. As far as $i_L(t)$ after $t = 0^+$ is concerned, these two terms are interchangeable. Therefore, an initial current of I_0 in the inductor at $t = 0^-$ may be replaced by zero initial current at $t = 0^-$ and an impulse voltage $LI_0\delta(t)$ with correct polarity in series with the inductor.

However, the voltage variable that appeared in Eqn. 10.8-4 was the voltage across the inductor. If we want to replace a non-zero initial current at $t = 0^-$ by an impulse voltage source, we must ensure that the impulse source appears fully across the inductor and does not lose itself across other elements in the circuit. So we have to argue that the $\delta(t)$ we applied at the input travels through R and appears fully across L . The pulse approximation for impulse does not help us here. We have to grapple with the $\delta(t)$ itself.

The definition of unit impulse function avoids defining it at $t = 0$ and makes up for that by providing its area content in an infinitesimally small interval around $t = 0$.

$$\delta(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined} & \text{at } t = 0 \\ 0 & \text{for } 0^+ \leq t < \infty \end{cases}$$

$$\int_{0^-}^{0^+} \delta(t) dt = 1.$$

Now, consider applying this to an initially relaxed RL circuit. With reference to Fig. 10.1-1, applying KVL in the mesh, we get $v_R(t) + v_L(t) = v_s(t)$ for all t . Since this relation is true for all t , we can integrate both sides of the equation between the same two limits to get the following relation.

Area under $v_R(t)$ between two instants t_1 and t_2 + Area under $v_L(t)$ between two instants t_1 and t_2 = Area under $v_s(t)$ between two instants t_1 and t_2 . Taking $t_1 = 0^-$ and $t_2 = 0^+$ this relation results in

$$\int_{0^-}^{0^+} v_R(t) dt + \int_{0^-}^{0^+} v_L(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1.$$

Equivalence between non-zero initial current and impulse excitation.

Argument leading to the conclusion that $\delta(t)$ applied to an initially relaxed series RL circuit has to appear across the inductor.

The correct way to apply the superposition principle in dynamic circuits.

We need to show that the first integral on the left side is zero. Let us assume that it is zero. Then second integral is 1. The second integral value divided by L gives the *change* in inductor current over the interval $[0^-, 0^+]$. Therefore, the inductor current change is $1/L$ A and the inductor current is $1/L$ A since the circuit was initially relaxed. This is the maximum value that the current can have at $t = 0^+$ as we had assumed that the first integral value is zero. Thus, the current through the circuit during $[0^-, 0^+]$ is confined between 0 and a maximum of $1/L$ A. This implies that though the current during this infinitesimal interval is indeterminate in value (because there is a jump discontinuity in it), it remains upper-bounded and lower-bounded, and hence, is finite-valued in the interval. *A finite-valued function integrated over infinitesimal interval results in zero integral value.* After all, there is no area under a rectangle if the rectangle is of finite length in one direction and of infinitesimal length in the other direction. Therefore, the first integral, i.e., the portion of area content of $\delta(t)$ that gets lost across resistor, is zero and our assumption to that effect is correct. Now, we may similarly assume that this integral is non-zero and prove that the assumption leads to a contradiction. Therefore, all the area content of $\delta(t)$ appears across the inductor itself.

Therefore, now we can assert that *a non-zero initial current of I_0 in an inductor L at $t = 0^-$ in a circuit can be replaced by zero initial current at $t = 0^-$ along with an impulse voltage source $LI_0 \delta(t)$ in series with the inductor for solving the circuit in the domain $[0^+, \infty)$.*

Moreover, we have resolved the small problem that we had with linearity and the superposition principle in dynamic circuits. In fact, there was no problem. The superposition principle is fully obeyed by linear dynamic circuits, only that we have to apply it carefully when there are non-zero initial condition values specified. In that case, we have to remember that each initial condition represents a source and that it becomes a multi-source problem. When changes are effected in a source, the superposition principle has to be applied to that component of the total response contributed by the particular source.

10.8.1 Unit Impulse Response of RL Circuit with Non-Zero Initial Current

Section 10.8 dealt with the impulse response of series RL circuit with zero initial energy. We found that the current contains a zero-state response of $(1/L)e^{-t/\tau}$ and no zero-input response. Now, what if the initial condition at $t = 0^-$ is not zero?

We can reach the correct output response value by three methods. In the first method, we find the zero-input response and the zero-state response and add them up. The zero-input response is found by realising that it refers to the total response in a source-free circuit with some initial energy. Hence, it is $I_0 e^{-t/\tau}$, where I_0 is the value of the current specified at $t = 0^-$.

The value of current at $t = 0^-$ and $t = 0^+$ in a source-free circuit remains the same since we are shorting the input at $t = 0$ and there is no question of an impulse getting applied to the circuit when you are shorting it. Keep in mind that the circuit we are thinking of in order to find out the zero-input response is not the one in which we are applying the input source function. In this method, we split up our original circuit into two – one which has some non-zero initial current at $t = 0^-$ and gets shorted at input at $t = 0$ and the second with zero initial energy and gets the input source function applied across its input from $t = 0$. We solve the first one to get the zero-input response and the second one to get zero-state response and put them together in order to get the total response in our original circuit in which both, the initial energy and the input source function, are acting simultaneously. Therefore, there is no change in current over $[0^-, 0^+]$ in the circuit we are using to find the zero-input response though there is an impulse applied at $t = 0$ in the original circuit. Therefore, the zero-input response is $I_0 e^{-t/\tau}$.

The zero-state unit impulse response of a series RL circuit has been worked out already in this section. It is $(1/L) e^{-t/\tau}$.

Therefore, the total unit impulse response of a series RL circuit with an initial current of I_0 at $t = 0^-$ is the sum of these two and is $[I_0 + (1/L)]e^{-t/\tau}$ A for $t \geq 0^+$.

In the second method to solve the same problem, we employ the principle that the net effect of applying a unit impulse voltage at $t = 0$ to a series RL circuit is a *change* in its current by $1/L$ A between $t = 0^-$ and $t = 0^+$. Therefore, the inductor current at $t = 0^+$ is given by the sum of the current at $t = 0^-$ and $1/L$. Thereafter, it is the zero-input response of a source-free circuit (because $\delta(t) = 0$ for $t \geq 0^+$). Therefore, the inductor current will be $[I_0 + (1/L)]e^{-t/\tau}$ A for $t \geq 0^+$, as before.

In the third method, we use the principle that a non-zero initial condition of I_0 at $t = 0^-$ can be replaced by a zero initial condition and an impulse voltage source of magnitude LI_0 and *suitable polarity* in series with the inductor. It may be verified that this impulse source will add with the externally applied unit impulse function to make the net impulse applied to the circuit $(1 + LI_0)\delta(t)$. After this, we have a zero-state impulse response problem and the solution obviously will be $[I_0 + (1/L)]e^{-t/\tau}$ A for $t \geq 0^+$, as before.

10.8.2 Zero-State Response for Other Inputs from Zero-State Impulse Response

The *zero-state unit impulse response* (hereafter this will be referred to as ‘the impulse response’ – that it is a zero-state response, and that a unit impulse is applied will be implicit) is the most important response for a linear circuit. We will see in a later chapter that the zero-state response for any other well-behaved forcing function can be found from it. In this subsection we will show that it is easy to get the zero-state response for step input, pulse input and ramp input functions from the impulse response by integrating it.

First, we establish the relation between the unit impulse function and the unit step function. It is a simple one. Unit impulse function is defined as

$$\delta(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined at } t = 0 & \text{and } \int_{0^-}^{0^+} \delta(t) dt = 1. \\ 0 & \text{for } 0^+ \leq t < \infty \end{cases}$$

Consider the following integral $f(t) = \int_{-\infty}^t \delta(t) dt$.

Obviously, the value of $f(t)$ is zero for $-\infty < t \leq 0^-$ since $\delta(t)$ is zero in that range. But the area under the impulse function undergoes a rapid change by 1 when t goes from 0^- to 0^+ . Therefore, $f(0^-) = 0$, whereas $f(0^+) = 1$. Therefore, $t = 0$ is a point of discontinuity in $f(t)$ and has to be excluded from its domain. $f(t)$ remains at 1 for $t \geq 0^+$ since $\delta(t)$ is zero for $t \geq 0^+$. Therefore, the description of $f(t)$ coincides with the definition of $u(t)$,

$$u(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined at } t = 0. \\ 1 & \text{for } 0^+ \leq t < \infty \end{cases}$$

Now, we turn our attention to the differential equation describing a series RL circuit for a zero-state response with some input function $v_s(t)u(t)$. Here, $v_s(t)$ is a function which is defined for all t and may not be zero for $t < 0$. We make the applied voltage equal to zero in the negative time axis and equal to this function in the positive time axis by multiplying it with $u(t)$. Since we are specifically dealing only with zero-state response, it is permissible to write ‘for all t ’ at the end of the differential equation. This is so because applying zero volts in the entire past will result only in zero initial current at $t = 0^-$ and in the case of a zero-state response analysis that is the initial condition we want.

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} v_s(t)u(t) \text{ for all } t$$

Let $v_{su}(t) = v_s(t)u(t)$, then,

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} v_{su}(t) \text{ for all } t.$$

Relation Between Standard Test Functions

Unit step function is related to unit impulse function by

$$u(t) = \int_{-\infty}^t \delta(t) dt.$$

Unit ramp function is related to unit step function by

$$r(t) = \int_{-\infty}^t u(t) dt.$$

Unit step function is related to unit ramp function by

$$u(t) = \frac{dr(t)}{dt}.$$

Unit impulse function is related to unit step function by

$$\delta(t) = \frac{du(t)}{dt}.$$

Proof for the assertion that 'the zero-state response in a linear circuit for an integrated input function is equal to the integral of the zero-state response for the input function.'

Since this equation is true for all t (it is a KVL equation), we can integrate both sides from $-\infty$ to t and write

$$\int_{-\infty}^t \left(\frac{di_L(t)}{dt} \right) dt + \frac{1}{\tau} \int_{-\infty}^t i_L(t) dt = \frac{1}{L} \int_{-\infty}^t v_{SU}(t) dt \text{ for all } t.$$

Changing the order of integration and differentiation,

The last equation is the differential equation describing the zero-state response for an integrated input function. Therefore, we can state that *the zero-state response in a linear circuit for an integrated input function is equal to the integral of the zero-state response for the input function*. Strictly speaking, we showed this only for a first order differential equation; but there is nothing in the proof which limits it to the first order differential equation alone. This result is valid for any linear lumped circuit described by the linear differential equation of any order. The integration has to be performed from $-\infty$ in theory. However, since we know that the zero-state response is zero from $-\infty$ to $t = 0^-$, we need to integrate from $t = 0^-$ to t only.

The integral of $\delta(t)$ gives $u(t)$. Therefore, the integral of *impulse response* should give *zero-state unit step response* (usually referred to as *the step response*). This is verified by carrying out the integration below:

Zero-state unit impulse response of RL circuit = $\frac{1}{L} e^{-t/\tau}$ for $t \geq 0^+$ and 0 for $t \leq 0^-$

$$\begin{aligned} \text{Integrating for } t > 0^+, \quad & \int_{-\infty}^{0^-} 0 dt + \int_{0^-}^{0^+} (\text{a bounded number}) dt + \int_{0^+}^t \frac{1}{L} e^{-t/\tau} dt \\ &= \frac{\tau}{L} (1 - e^{-t/\tau}) = \frac{1}{R} (1 - e^{-t/\tau}) \text{ for } t \geq 0^+. \end{aligned}$$

If there is a non-zero initial condition, then the total solution can be found by adding the zero-input response to the zero-state response found by this integration method.

Unit Ramp Input Function is defined as $r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$.

It can be easily verified that unit ramp function is the integral of unit step function. These three basic input functions and their relations are shown in Fig. 10.8-2.

Therefore, the zero-state response for unit ramp input (called *the ramp response*) can be found by integrating the step response as below:

$$\begin{aligned} \text{Ramp response} &= \int_{0^+}^t \frac{1}{R} (1 - e^{-t/\tau}) dt \\ &= \frac{1}{R} \left[t - \tau (1 - e^{-t/\tau}) \right] \text{ for } t \geq 0^+. \end{aligned}$$

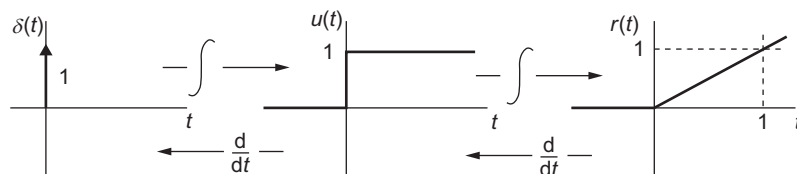


Fig. 10.8-2 Impulse, Step and Ramp Functions and Their Relations

Ramp response and its components are shown in Fig. 10.8-3. The voltage across the resistor is plotted instead of the inductor current.

The unit ramp function has a kink at $t = 0$, and hence, it is not differentiable at $t = 0$. However, it is differentiable at all other time instants. Hence, the first derivative of $r(t)$ will be a function defined as 0 for $t \leq 0^-$, 1 for $t \geq 0^+$ and *undefined* at $t = 0$. But that is the unit step function. Therefore, unit step function is the first derivative of the unit ramp function.

Unit step function is not even continuous at $t = 0$. Obviously, it cannot be differentiated there. However, we raise the question – which function when integrated will yield the unit step function? The answer is that it is the impulse function. Therefore, we can consider $\delta(t)$ to be the first derivative of $u(t)$ in the ‘anti-derivative’ sense.

KVL equations are true for all t . Both sides of an equation which holds good for all t can be differentiated with respect to time. From this observation, it is easy to see that the following is true.

The zero-state response in a linear circuit for differentiated input function is equal to the derivative of the zero-state response for the input function.

Therefore, we can get to the zero-state unit step response and the zero-state unit impulse response in any linear circuit by successive differentiation of its zero-state unit ramp response.

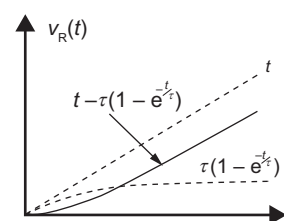


Fig. 10.8-3 Zero-State Unit Ramp Response of RL Circuit

EXAMPLE: 10.8-1

An inductor, a resistor and a unit impulse current source are connected in parallel. Find the zero-state impulse response for the resistor current and the inductor current in this circuit.

SOLUTION

The impulse current cannot flow through L , and hence, it flows through R producing an impulse voltage $R\delta(t)$ across the combination. This impulse voltage has an area content of R volt-sec. R volt-sec dumped into an inductor of L H will produce a change in its current by R/L A. Since the initial condition for zero-state response is zero, the current of an inductor at $t = 0^+$ will be this change amount itself – i.e., R/L A. Thereafter, it is a source-free circuit since the impulse current source is an open circuit after $t = 0^+$. Therefore, the current in both R and L will be $(1/\tau)e^{-t/\tau}$, where $\tau = L/R$.

EXAMPLE: 10.8-2

An inductor, a resistor and a unit step current source are connected in parallel. The initial condition for inductor is specified as I_0 . Find the time-domain expressions for current through L and R and the voltage across the combination. Also, find the total energy delivered by the current source, the energy consumed by the resistor and the energy stored in the inductor for a special case where the initial condition is zero.

SOLUTION

We use the result from the previous example in solving this. The total response for the inductor current is the sum of the zero-input response and the zero-state response. A series RL circuit with some non-zero initial condition gets formed when the current source at input is set to zero. Therefore, the zero-input response of this circuit is the same as the zero-input response of RL circuit we have discussed so far.

The zero-state response for a step current input can be found by integrating the zero-state response for impulse current input.

$$\text{Zero-input response } i_{L(z)} = I_0 e^{-t/\tau} \text{ and Zero-state impulse response } = (1/\tau) e^{-t/\tau}$$

$$\therefore \text{Zero-state step response } i_{L(zs)} = \int_0^t (1/\tau) e^{-t'/\tau} dt' = 1 - e^{-t/\tau}$$

$$\therefore \text{Total response } i_L = I_0 e^{-t/\tau} + (1 - e^{-t/\tau}) = 1 - (I_0 - 1) e^{-t/\tau}$$

$$\text{Since } i_L + i_R = 1 \text{ for } t \geq 0^+, i_R = (1 - I_0) e^{-t/\tau} \text{ and } v_R = R(1 - I_0) e^{-t/\tau}.$$

The total energy delivered by the source is found by integrating the power delivered from 0 to ∞ . The power delivered is given by the voltage across the

combination of the source current value. Similarly, the energy dissipated in R and stored in L can be found by integrating $v_R i_R$ and $v_L i_L$ from 0 to ∞ . The integral must return $L/2$ in the case of an inductor because that is the stored energy in an inductor carrying 1 A. Let us call these three energy values E_S , E_R and E_L , respectively. Then,

$$i_L = (1 - e^{-t/\tau}) \text{ (with zero initial condition)}$$

$$i_R = e^{-t/\tau} \text{ and } v_R = R e^{-t/\tau}$$

$$\therefore E_S = \int_0^\infty R e^{-t/\tau} dt = R\tau = \frac{1}{L} \text{ J}; E_R = \int_0^\infty R e^{-2t/\tau} dt = R\tau / 2 = \frac{1}{2L} \text{ J}$$

$$E_L = \int_0^\infty R e^{-t/\tau} (1 - e^{-t/\tau}) dt = R\tau / 2 = \frac{1}{L} - \frac{1}{2L} = \frac{1}{2L} \text{ J}.$$

If the current source had a magnitude of I instead of 1, all the three energy terms are to be multiplied by I^2 . Hence, we see that (i) it takes $LI^2/2$ J of energy dissipation in the parallel resistance whenever we charge an inductor to I A and this dissipated energy is as much as the energy stored in the inductor and (ii) this energy dissipation is independent of R value. The value of R will decide the time taken to dissipate this amount of energy but not the amount of energy.

EXAMPLE: 10.8-3

Find the unit impulse inductor current response with the specified initial condition in the circuit in Fig. 10.8-4(a).

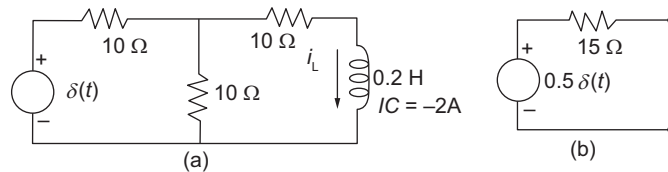


Fig. 10.8-4 Circuit for Example 10.8-3

SOLUTION

The time constant of the circuit is found by deactivating the sources and finding out the equivalent resistance connected across the inductor. The equivalent resistance is 15Ω and the time constant is 13.33 ms. The Thevenin's equivalent of the circuit connected across the inductor is shown in Fig. 10.8-4(b).

The $0.5\delta(t)$ voltage source appears across the inductor forcing its current to change by $0.5/0.2 = 2.5$ A. The inductor current at $t = 0^-$ is specified as 2 A in the opposite direction. Hence, the inductor current at $t = 0^+ = -2 + 2.5 = 0.5$ A. After $t = 0^+$, the circuit is a source-free series RL circuit and the inductor current will be $0.5e^{-75t}$ A.

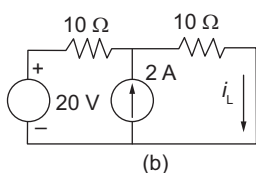
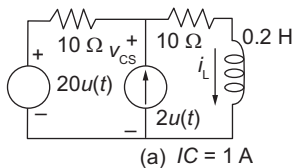


Fig. 10.8-5 Circuits for Example 10.8-4

EXAMPLE: 10.8-4

Find the inductor current and the voltage across the current source as a function of time in the circuit in Fig. 10.8-5(a). Also, find what must be the initial current in the inductor so that the inductor current will be transient-free for $t \geq 0^+$

SOLUTION

Time constant of the circuit is 0.01 s. The zero-input i_L response is $1e^{-100t}$. Zero-state response when the voltage source is acting alone is $i_L = (20/20)(1 - e^{-100t})$. The zero-state response when the current source is acting alone is $i_L = 1(1 - e^{-100t})$.

$\therefore$ Total response for $i_L = 2 - e^{-100t}$ A for $t \geq 0^+$

Zero-input v_{cs} response = voltage across the first $10\ \Omega$ resistance due to zero-input response in $i_L = -10e^{-100t}$.

Zero-state v_{cs} response when the voltage source is acting alone is $20 - 10 \times$ zero-state i_L for the same condition $= 20 - 10(1 - e^{-100t}) = 10(1 + e^{-100t})$

Zero-state v_{cs} response when the current source is acting alone is $10 \times (2 - \text{zero-state } i_L \text{ for the same condition}) = 10 \times (2 - (1 - e^{-100t})) = 10(1 + e^{-100t})$

$\therefore$ Total response for $v_{cs} = 20 - 10e^{-100t}$ V for $t \geq 0^+$

We could have obtained v_{cs} as $(10 \times \text{total response in } i_L + 0.2 \times \text{first derivative of total response in } i_L)$ too.

The inductor current will be transient-free if the natural response terms in the zero-input response cancel out the natural response terms in the *total* zero-state response. The natural response terms in the *total* zero-state response in this case is $-2e^{-100t}$, and hence, the initial current must be 2 A for transient-free inductor current in this circuit.

Another point of view would be that the inductor current will be transient-free if the initial current and the final current (i.e., steady-state current for DC steady-state) are the same. In general, the inductor current will be transient-free if the initial current at $t = 0^+$ and the value of the forced response component (this need not be a DC component) at $t = 0^+$ are equal. The DC steady-state may be obtained from the circuit in Fig. 10.8-5(b). Applying the superposition principle, we get the steady-state value of i_L as $(20\text{ V}/20\ \Omega) + 2\text{ A} \times (10\ \Omega/20\ \Omega) = 2\text{ A}$. Thus, the required initial current in the inductor for transient-free response will be 2 A again.

A single-inductor circuit can be described by a first order differential equation on the inductor current. It will be possible to work out the other circuit variables from the inductor current alone. Therefore, it follows that if the inductor current is transient-free, so will all the other circuit variables be.

10.9 SERIES RL CIRCUIT WITH EXPONENTIAL INPUTS

We take up the study of a zero-state response of series *RL* circuit for exponential inputs and sinusoidal inputs in this section. We do not worry about the zero-input response anymore since we know that it is $I_0 e^{-t/\tau}$, where I_0 is the initial current specified at $t = 0^-$. The total response is found by adding zero-input response and zero-state response together.

We permit exponential inputs of the form $e^{st}u(t)$ in this section, where s can be a complex number $s = -\sigma + j\omega$, where σ and ω are two real numbers. The reason for generalising the exponential input in this manner will be clear soon. We put a negative sign behind σ since we want the real part of s to be negative when σ is positive. This signal is a complex signal with a real part function and an imaginary part function as shown in Eqn. 10.9-1.

$$v_s(t) = e^{st}u(t) = e^{(-\sigma + j\omega)t}u(t) = (e^{-\sigma t} \cos \omega t)u(t) + j(e^{-\sigma t} \sin \omega t)u(t) \quad (10.9-1)$$

We have used Euler's Identity in expressing the complex exponential function in the rectangular form involving trigonometric functions. No signal like this can be actually generated by a physical system since there can be nothing 'imaginary' about a physical signal. Therefore, this signal does not represent a physical signal, and hence, it cannot really be applied to any circuit, but it can be the forcing function in a differential equation – the mathematics of the differential equation will not complain. Moreover, its real part and the imaginary part are real functions and can be created physically.

We know that the zero-state response of a linear circuit obeys the superposition principle. Let us apply a signal $x(t)$ to the circuit and let the zero-state response be $i_x(t)$. Similarly, let the zero-state response be $i_y(t)$ when a signal $y(t)$ is applied to the same circuit. Now, what is the response when $jy(t)$ is applied, where $j = \sqrt{-1}$? By the superposition principle, it will be $ji_y(t)$. When we apply $x(t) + jy(t)$ as the input, we must get $i_x(t) + ji_y(t)$ as the response by the superposition principle, again. *Therefore, we see that the real part of the zero-state response for a complex signal input is the zero-state response for the real part of the input*

and the imaginary part of the zero-state response for a complex signal input is the zero-state response for the imaginary part of the input. Notice that this is a direct result of the linearity of the circuit and will not be true in the case of a non-linear circuit.

Now, we can see why we opted for this complex signal. We can obtain the zero-state response of circuits to sinusoidal inputs by taking the real or the imaginary part of the zero-state response for complex exponential inputs. It is considerably easier to deal with complex exponential function than sinusoids when it comes to solving differential equations. Let us solve for the zero-state response of series RL circuit with a complex exponential input.

$$\frac{di_L(t)}{d(t)} + \frac{i_L(t)}{\tau} = \frac{1}{L} e^{st} u(t) \quad (10.9-2)$$

This equation has to be true for all t and in particular for all $t \geq 0^+$. This is possible only if the shape of the functions on both sides of the equation is the same. This will imply that the trial solution for the particular integral has to be Ae^{st} . Substituting this trial solution in the differential equation in Eqn. 10.9-2,

$$\begin{aligned} Ae^{st}(s + 1/\tau) &= (1/L)e^{st} \text{ for } t \geq 0^+ \\ \therefore A &= \frac{1}{sL + L/\tau} = \frac{1}{sL + R} (\because \tau = L/R) \\ \therefore i_L(t) &= \frac{1}{sL + R} e^{st} \text{ for } t \geq 0^+. \end{aligned} \quad (10.9-3)$$

We have to form the total solution by adding this particular integral to the complementary solution. Notice that finding the zero-state response involves finding a particular integral and a suitable complementary function also – because zero-state response involves both forced response and natural response terms.

$$\therefore i_L(t) = Be^{-t/\tau} + \frac{1}{sL + R} e^{st} \text{ for } t \geq 0^+$$

Substituting the initial current value at 0^+ ,

$$\begin{aligned} 0 &= B + \frac{1}{sL + R} \\ \therefore B &= -\frac{1}{sL + R} \text{ and} \\ i_L(t) &= \frac{1}{sL + R} (e^{st} - e^{-t/\tau}) \text{ for } t \geq 0^+. \end{aligned} \quad (10.9-4)$$

Total response of series RL circuit for complex exponential function input.

10.9.1 Zero-State Response for a Real Exponential Input

We consider a special case of complex exponential with $\omega = 0$. The input source function is then of the form $e^{-\sigma t} u(t)$. With a positive value of σ , this input function has the same format as that of the impulse response of a series RL circuit (impulse response of any first order circuit for that matter). Hence, the problem we are trying to solve may be thought of as a situation where the impulse response of one circuit is applied to a second circuit as the input. Such situations arise when we cascade different circuits. The zero-state response is obtained by putting $s = -\sigma$ in Eqn. 10.9-4 and is expressed as in Eqn. 10.9-5.

$$\begin{aligned} i_L(t) &= \frac{1}{R} \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ for } t \geq 0^+, \\ \text{where } \alpha &= 1/\tau = R/L. \end{aligned} \quad (10.9-5)$$

Total response of series RL circuit for real exponential function input.

We notice the $\alpha - \sigma$ in the denominator and raise the question – what is the response when the real exponential input has the same index as that of the impulse response of the

circuit? It is not *infinite* as Eqn. 10.9-5 would suggest. The correct answer is that this is one situation under which our assumed trial particular solution is incorrect. We have to find the particular integral afresh. There are well-established methods to arrive at the particular integral under this kind of a situation in Mathematics. But we do not take up this case here. We will deal with this in a later chapter and solve it without having to learn a new method. Therefore, we qualify the solution in Eqn. 10.9-5 by specifying that the solution is valid only if $\alpha \neq \sigma$.

$$i_L(t) = \frac{1}{R} \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ for } t \geq 0^+, \quad (10.9-6)$$

where $\alpha = 1/\tau = R/L$ and $\alpha \neq \sigma$.

The input waveform is exponential and we can think of a time constant for this waveform too. Let us denote this time constant as τ_e with the subscript reminding us that this is the time constant of an excitation waveform. Obviously $\tau_e = 1/\sigma$. Further, we normalise the current in Eqn. 10.9-6 using a normalisation base of $1/R$ A and the time by a normalisation base of τ s to put the solution in terms of the normalised variables as in Eqn. 10.9-7, where the subscript n indicates normalised variables.

$$i_{Ln}(t) = \left(\frac{1}{1 - \tau/\tau_e} \right) (e^{-(\tau/\tau_e)t_n} - e^{-t_n}) \text{ for } t \geq 0^+, \quad (10.9-7)$$

where $\tau_e = 1/\sigma$, $\tau = R/L$ and $\tau_e \neq \tau$.

The input functions and i_L waveforms are shown for two cases in Fig. 10.9-1.

The time instants at which the response reaches the maximum point and the maximum response are marked. General expressions for these may be derived by differentiating the function in Eqn. 10.9-7 and setting the derivative to zero. Note that sending τ_e to ∞ amounts to applying a unit step input and the zero-state response indeed approaches the unit step zero-state response as expected.

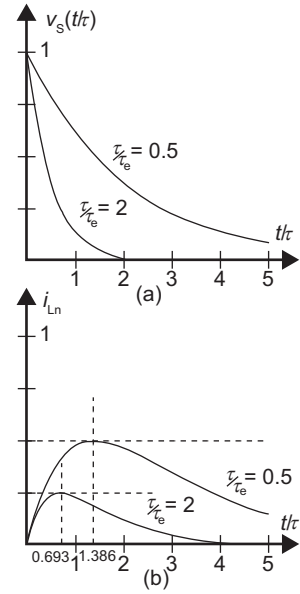


Fig. 10.9-1 Zero State Response for Exponential Input (a) Input Wave (b) Normalised Inductor Current

10.9.2 Zero-State Response for Sinusoidal Input

The input function in this case is $v_s(t) = \sin \omega t u(t)$. We can get the sinusoidal zero-state response for sinusoidal input by two methods. In the first method, we use the result in Eqn. 10.9-4 with $s = j\omega t$ and take the imaginary part of the result as our desired solution.

$$\begin{aligned} \therefore i_L(t) &= \text{Im} \left(\frac{1}{j\omega L + R} (e^{j\omega t} - e^{-t/\tau}) \right) \text{ for } t \geq 0^+ \\ &= \text{Im} \left(\frac{R - j\omega L}{(\omega L)^2 + R^2} \{ (\cos \omega t - e^{-t/\tau}) + j \sin \omega t \} \right) \text{ for } t \geq 0^+ \\ &= \frac{(R \sin \omega t - \omega L \cos \omega t) + \omega L e^{-t/\tau}}{(\omega L)^2 + R^2} \text{ for } t \geq 0^+ \end{aligned}$$

Let $\phi = \tan^{-1} \frac{\omega L}{R}$, then, $\cos \phi = \frac{R}{\sqrt{R^2 + (\omega L)^2}}$ and $\sin \phi = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}$ (10.9-8)

$$\begin{aligned} \therefore i_L(t) &= \frac{(\sin \omega t \cos \phi - \cos \omega t \sin \phi) + \sin \phi \cdot e^{-t/\tau}}{\sqrt{R^2 + (\omega L)^2}} \text{ for } t \geq 0^+ \\ &= \frac{1}{\sqrt{R^2 + (\omega L)^2}} ((\sin(\omega t - \phi) + \sin \phi \cdot e^{-t/\tau})) \text{ for } t \geq 0^+. \end{aligned}$$

In the second method, we represent the input sine wave by combining two complex exponential functions using Euler's Identity as shown.

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

Then, we get the zero-state response for the two exponential inputs separately and use the superposition principle to arrive at the solution for sinusoidal input.

$$\begin{aligned} \therefore i_L(t) &= \frac{1}{2j} \left(\frac{1}{R + j\omega L} (e^{j\omega t} - e^{-t/\tau}) - \frac{1}{R - j\omega L} (e^{-j\omega t} - e^{-t/\tau}) \right) \text{ for } t \geq 0^+ \\ &= \frac{1}{2j} \left\{ \left(\frac{1}{R + j\omega L} + \frac{1}{R - j\omega L} \right) e^{-t/\tau} + \left(\frac{e^{j\omega t}}{R + j\omega L} - \frac{e^{-j\omega t}}{R - j\omega L} \right) \right\} \text{ for } t \geq 0^+ \\ &= \frac{(R \sin \omega t - \omega L \cos \omega t) + \omega L e^{-t/\tau}}{R^2 + (\omega L)^2} \text{ for } t \geq 0^+ \\ \therefore i_L(t) &= \frac{(\sin \omega t \cos \phi - \cos \omega t \sin \phi) + \sin \phi \cdot e^{-t/\tau}}{\sqrt{R^2 + (\omega L)^2}} \text{ for } t \geq 0 \\ &= \frac{1}{\sqrt{R^2 + (\omega L)^2}} \left((\sin(\omega t - \phi) + \sin \phi \cdot e^{-t/\tau}) \right) \text{ for } t \geq 0 \end{aligned} \quad (10.9-9)$$

The angle ϕ in the Eqn. 10.9-9 is defined the same way as in Eqn. 10.9-8. Both methods lead to the same expression for the final response, as they should. The final expression may be recast in the following form, where $k = \omega\tau$ and the current and the time are normalised with respect to $1/R$ and τ , respectively.

$$i_{Ln}(t) = \frac{1}{\sqrt{1+k^2}} \left(\sin(kt_n - \tan^{-1} k) + \frac{k}{\sqrt{1+k^2}} e^{-t_n} \right); k = \omega\tau \quad (10.9-10)$$

This waveform for a case with $k = 4$ is shown in Fig. 10.9-2. The number k can be interpreted as a comparison between the characteristic time, i.e., the period of the applied voltage and the characteristic time of the circuit, i.e., its time constant. k can be expressed as $2\pi(\tau/T)$, where T is the period of input. Sinusoids undergo a full cycle of variation in one T , and hence, the value of T is indicative of the rate of change involved in the waveform, i.e., the *speed* of the waveform. Time-constant is a measure of inertia in the system. Therefore, an input sinusoid is too fast for a circuit to follow if its T is smaller than the time constant τ of the circuit. Similarly, if the input sinusoid has a T value much larger than the time constant of the circuit, the circuit will perceive it as a very slow waveform and will respond almost the same way it does to a DC input. These aspects are clearly brought out in the expression in Eqn. 10.9-10.

Total response of an initially relaxed series RL circuit for sinusoidal input.

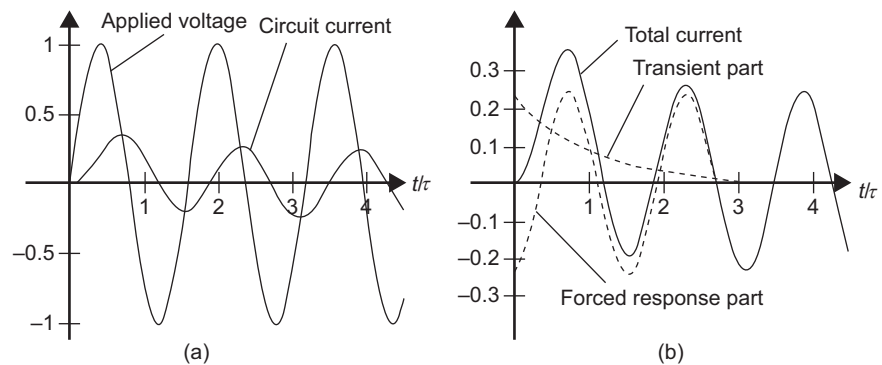


Fig. 10.9-2 Unit Sinusoidal Response of RL Circuit with $k = 4$

The amplitude of forced response component, or equivalently, the amplitude of the sinusoidal steady-state response, is a strong function of k . The amplitude decreases with increasing ω or increasing τ . Also, the steady-state current lags the applied voltage by a phase angle that increases with $\omega\tau$.

Let us imagine that we conduct an experiment. We apply a sinusoidal voltage of 1 V amplitude to a series RL circuit and wait for enough time for the transient response to die down. After the steady-state is satisfactorily established in the circuit, we measure the amplitude of current and its phase with respect to the input sine wave. We repeat this process for various values of frequency of input, keeping its amplitude at 1 V always. We ensure that the circuit is in steady-state before we measure the output every time.

The data so obtained can be plotted to show the variation of ratio of output amplitude to input amplitude and the phase of steady-state current against $k (= \omega\tau)$. Such a pair of plots will constitute what is called the *AC steady-state frequency response plots* for this RL circuit. The ratio of output amplitude to input amplitude is called the *gain* of the circuit. Its dimension will depend on the nature of input and output quantities.

Such an experiment can be performed on any circuit to get its frequency response data. However, if the differential equation of the circuit is known, we need not do the experiment. The frequency response plots can be obtained analytically in that case. The frequency response of the series RL circuit is shown in the Fig. 10.9-3 as an example.

We make the following observations on the sinusoidal steady-state response of a series RL circuit from Eqn. 10.9-10 and Fig. 10.9-3.

- The circuit current under the sinusoidal steady-state response is a sinusoid at the same angular frequency ω rad/s as that of the input sinusoid.
- The circuit current initially is a mixture of an exponentially decaying unidirectional transient component along with the steady-state sinusoidal component. This unidirectional transient imparts an offset to the circuit current during the initial period.
- The circuit current at its first peak can go close to twice its steady-state amplitude in the case of circuits with $\omega\tau \gg 1$ due to this offset.
- The amplitude of sinusoidal steady-state response is always less than the corresponding amplitude when a DC input is applied. This is due to the inductive inertia of the circuit. The amplitude depends on the product $\omega\tau$ and decreases monotonically with the $\omega\tau$ product for fixed input amplitude.
- The response sinusoid lags behind the input sinusoid under the steady-state conditions by a phase angle that increases monotonically with the product $\omega\tau$.
- The frequency at which the circuit gain becomes $1/\sqrt{2}$ times that of a DC gain is termed as a *cut-off frequency* and since this takes place as we go up in frequency, it is called *upper cut-off frequency*. Upper cut-off frequency of a series RL circuit is seen to be at $\omega = 1/\tau$ rad/s. The phase at this frequency will be -45° .
- The circuit current amplitude becomes very small at high frequencies ($\omega\tau \gg 1$) and the current lags the input voltage by $\approx 90^\circ$ at such frequencies.

We assumed that the applied voltage is $v_s(t) = \sin\omega t u(t)$ throughout this analysis. This means that the sinusoidal voltage happened to be crossing the time-axis exactly at the instant at which we closed the switch to apply it to the circuit. Though, technically it is possible to do such a switching (it is done in some applications that way), that is not the way it takes place in many practical applications. The sinusoid may be at any value between its maximum and minimum when we throw the switch. Therefore, we must analyse the response with $v_s(t) = \sin(\omega t + \theta)u(t)$ for an arbitrary θ .

The function $\sin(\omega t + \theta)$ is the imaginary part of $e^{j(\omega t + \theta)} = \cos(\omega t + \theta) + j\sin(\omega t + \theta)$. The zero-state response when e^{st} is applied to the circuit is given by Eqn. 10.9-4. So shall we substitute $s = j(\omega + \theta)$ in Eqn. 10.9-4 to get the required output? No, that will be wrong since it is only ω that gets multiplied by t , not θ . We should (i) interpret $e^{j(\omega t + \theta)}$ as

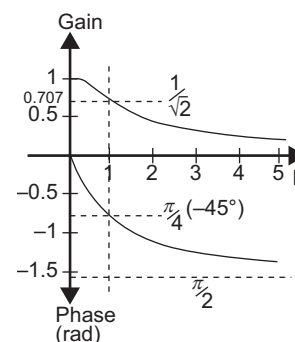


Fig. 10.9-3 Frequency Response Plots for Series RL Circuit

$e^{j\theta}e^{j\omega t}$, (ii) solve for the zero-state response for $e^{j\omega t}$, (iii) multiply the response by $e^{j\theta}$ (we apply the principle of homogeneity there) and (iv) take the imaginary part. We skip all that basic algebra and give the result below:

$$i_L(t) = \frac{1}{\sqrt{R^2 + (\omega L)^2}} (\sin(\omega t + \theta - \phi) + \sin(\phi - \theta).e^{-t/\tau}) \text{ for } t \geq 0^+ \quad (10.9-11)$$

By substituting $\theta = 90^\circ$, we get the solution for $v_s(t) = \cos\omega t u(t)$ as

$$i_L(t) = \frac{1}{\sqrt{R^2 + (\omega L)^2}} (\cos(\omega t - \phi) - \cos\phi.e^{-t/\tau}) \text{ for } t \geq 0^+ \quad (10.9-12)$$

Equation 10.9-11 indicates that it is possible to switch on an AC voltage to an initially relaxed series RL circuit in such a way that there is no transient response and the circuit immediately goes to a steady-state – the switching instant must be such that $\theta = \phi$. This principle is used sometimes in switching of heavily inductive power equipment. In such cases, the angle ϕ is close to 90° and transient-free switching is possible if the voltage is switched on to the equipment at positive or negative peak.

Note that the sinusoidal steady-state response part of the zero-state response in all the above cases could have been readily obtained by employing the phasor method that we studied in earlier chapters.

10.10 GENERAL ANALYSIS PROCEDURE FOR SINGLE TIME CONSTANT RL CIRCUITS

We have analysed the simple series RL circuit exhaustively. We would like to arrive at a generalised procedure for the analysis of RL circuits that may contain more than one resistor. The circuit may also contain more than one inductor; but in that case, we assume that the inductors will enter in series or parallel combinations and can be finally equivalenced by a single inductor. If that is not possible, the circuit will have more than one time constant and we do not handle it by the method we develop here.

A circuit with a single energy storage element can have only first order dynamics and only one time constant. A circuit with only one time constant can have only one term in its natural response for any circuit variable and that is of the form $Ae^{-t/\tau}$, where A is to be adjusted for compliance with *initial condition at $t = 0^+$ for that particular circuit variable*.

The single time constant that describes the natural response of the circuit is a property of the circuit alone and does not depend on the source function values. Therefore, it can be found from the *dead circuit* obtained by *deactivating all independent sources*. Deactivating an independent voltage source is done by setting its voltage to zero and that results in a short-circuit. Deactivating an independent current source is done by setting its current to zero and that results in an open-circuit. Thus, we get the *dead circuit* by replacing all independent voltage sources by short-circuits and all independent current sources by open-circuits. Now that the circuit will contain only one inductor and possibly many resistors (and dependent sources, if they were present in the original circuit) we can find the equivalent resistance connected across the inductor by series/parallel combinations and by star-delta transformation if necessary. If the circuit contains linear dependent sources, we may have to employ *unit current injection method* or *unit voltage application method* detailed in an earlier chapter to find the equivalent resistance across the inductor. Once we get this resistance, we can find time constant by $\tau = L/R_{eq}$.

The next step is to check whether there are impulse sources in the circuit. If there are, the amount of volt-sec dumped on the inductor at $t = 0$ has to be evaluated and the *change* in the inductor current at $t = 0$ has to be found out. This change in current, added to the

initial condition specified at $t = 0^-$, will give us the value of inductor current at $t = 0^+$. But we need the initial condition for the particular circuit variable we are solving for. Therefore, we have to carry out a DC circuit analysis at $t = 0^+$ in which the inductor is replaced by a DC current source of value equal to its current at $t = 0^+$ and all the independent sources are replaced by DC sources (if they are not DC sources already) of value equal to their values at $t = 0^+$. Solving the resulting circuit will give us the initial condition for the particular circuit variable of interest.

In the third step, we work out the forced response for all independent sources in the circuit. Since the circuit has a steady-state for DC and AC, we use the steady-state analysis for this. In particular, if the sources are DC sources, we replace the inductor by short-circuit and solve the resulting resistive circuit for the variable of interest. We may use the superposition principle along with the mesh or node analysis for this purpose. We employ the phasor analysis to solve for steady-state if there are AC sources. The steady-state solutions for all the independent sources are added up.

We add the total steady-state solution to the natural response, apply the initial condition for the relevant variable and evaluate the arbitrary constant in the natural response term in the last step.

We can also use the *zero-input response plus the zero-state response method* instead of *transient response plus forced response method*.

These methods are illustrated through a set of examples that follow.

EXAMPLE: 10.10-1

Find i_x in the circuit in Fig. 10.10-1(a) as a function of time. The initial condition in the 0.2 H inductor is 1 A in the direction shown at $t = 0^-$.

SOLUTION

The circuit contains two switched DC sources – a 2 V voltage source and a 0.5 A current source. There is only one time constant.

Step-1: Find the time constant

The independent voltage source is replaced by a short-circuit and the independent current source is replaced by an open-circuit to get the circuit in Fig. 10.10-1(b). The time constant is found by evaluating the equivalent resistance across the inductor.

$$R_{eq} = 1 + 2 // 2 = 2 \Omega \therefore \text{Time constant } \tau = \frac{0.2 \text{ H}}{2 \Omega} = 0.1 \text{ s.}$$

Step-2: Find the initial value for i_x at $t = 0^+$

The value of voltage source at $t = 0^+$ is 2 V. The value of current source at $t = 0^+$ is 0.5 A. The inductor current at $t = 0^+$ is the same as at $t = 0^-$, since there is no impulse source in the circuit. Hence, the inductor is replaced by a current source of 1 A for $t = 0^+$. The resulting circuit appears in Fig. 10.10-2(a).

This circuit contains three sources and may be solved by employing superposition principle. The three circuits in which the sources act one by one are shown in Fig. 10.10-2(b–d). Three contributions to the current i_x may be found from these single-source circuits by simple circuit analysis.

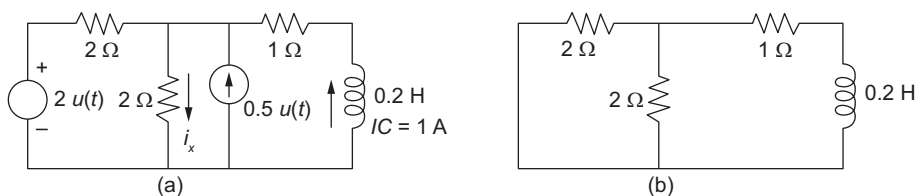


Fig. 10.10-1 Circuit for Example 10.10-1

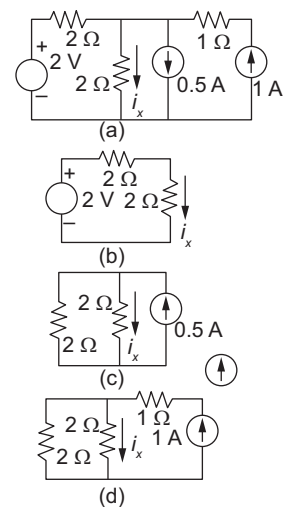


Fig. 10.10-2 Circuits for Step-2 of Example 10.10-1

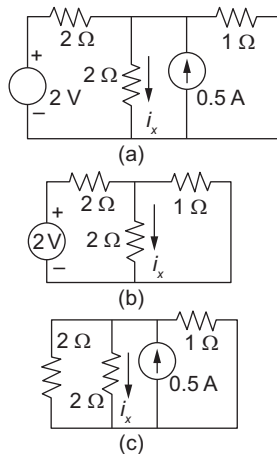


Fig. 10.10-3 Circuits for Step-3 of Example 10.10-1

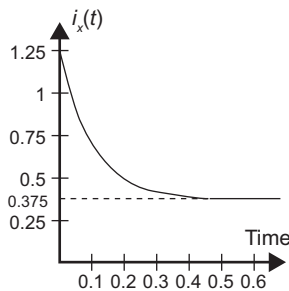


Fig. 10.10-4 The Plot of Current Waveform for Example 10.10-1

$$i_x \text{ at } t = 0^+ = \frac{2}{2+2} + \frac{0.5 \times 2}{2+2} + \frac{1 \times 2}{2+2} = 1.25 \text{ A.}$$

Step-3: Find the steady-state value of i_x

Both sources in the circuit are DC sources, and hence, they can be handled together for steady-state calculation. The inductor behaves as a short-circuit for DC steady-state. Replacing the inductor by a short-circuit and using the known values of source functions, we get the circuit in Fig. 10.10-3(a).

This circuit involves two sources and can be solved by using the superposition principle. The sub-circuits needed for that is shown in Fig. 10.10-3(b) and (c).

$$\text{Steady-state value of } i_x = \frac{2}{2+2//1} \times \frac{1}{1+2} + \frac{0.5 \times 1//2//2}{2} = 0.25 + 0.125 = 0.375 \text{ A.}$$

Step-4: Form the total solution and adjust initial condition

The transient response is of the form Ae^{-10t} , since time constant is 0.1 s

$$i_x(t) = Ae^{-10t} + 0.375 \text{ for } t \geq 0^+; \text{ initial value of this current at } 0^+ \text{ is } 1.25 \text{ A}$$

$$\therefore A + 0.375 = 1.25 \Rightarrow A = 0.875$$

$$\therefore i_x(t) = 0.375 + 0.875e^{-10t} \text{ A for } t \geq 0^+.$$

The current waveform is shown in Fig. 10.10-4.

We have calculated the required current. We obtained the solution in the form of transient response + forced response (or steady-state response). However, this form of solution is not the appropriate form if we are required to answer the supplementary questions on the problem. For example, can we work out the current if the initial condition of the inductor is changed to 2 A? Or, if the voltage source value is changed to 5 V? We will have to rework the problem almost entirely to get the answer for that. Note that it is not possible to decompose a solution into zero-input response and zero-state response components if the solution is available in the *transient response + forced response* format. However, transient response and forced response can be obtained from the solution in *zero-input response + zero-state response* format. This is where the superiority of *zero-input response + zero-state response* approach becomes evident. We work out the same example to see these two components in the solution in the next example.

EXAMPLE: 10.10-2

Obtain the solution in *zero-input response + zero-state response* form for the problem stated in Example 10.10-1 and modify the solution for (i) initial condition changed to 2 A, (ii) voltage source magnitude changed to 5 V and (iii) voltage source magnitude changed to 4 V and current source magnitude changed to 1 A along with initial condition changed to 2 A.

SOLUTION

When we want the solution in the *zero-input response + zero-state response* format, we have to split the given circuit into two sub-circuits right at the outset – one containing all independent sources and with zero initial condition for inductor current and the second with all independent sources deactivated and initial condition for inductor current at the specified value.

These two circuits are shown as Fig. 10.10-5(a) and (b). The solution of Fig. 10.10-5(b) gives the zero-input response. The solution of Fig. 10.10-5(a) gives the zero-state response. Zero-state response obeys the superposition principle. Hence, the zero-state response of the circuit in Fig. 10.10-5(a) with the two sources acting simultaneously can be obtained by summing the zero-state responses when they are acting alone. The circuits required for finding these individual zero-state responses are given as Fig. 10.10-5(c) and (d).

Step-1: Find the time constant

This step is the same as in Example 10.10-1 and the value of $\tau = 0.1$ s.

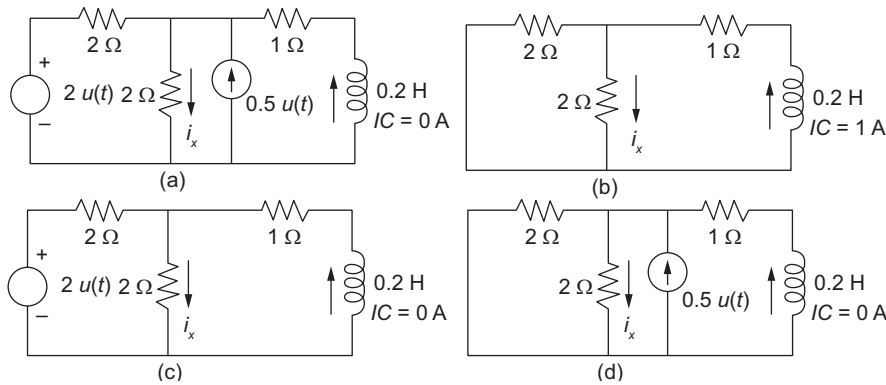


Fig. 10.10-5 Circuits for Solving Zero-Input and Zero-State Responses in Example 10.10-2

Step-2: Find the zero-input response

The initial value of i_x at $t = 0^+$ is 0.5 A from circuit Fig. 10.10-5(b). Since there is no source, the particular integral will be zero. Therefore, the zero-input response is $0.5e^{-10t}$.

Step-3: Find the zero-state response in the circuit in Fig. 10.10-5(c)

The initial value of i_x at $t = 0^+$ is 0.5 A from the circuit in Fig. 10.10-5(c). The final steady-state value is obtained by replacing the inductor with a short-circuit and solving the resulting resistive circuit. $2/(2 + 2/1)$ is the current from the voltage source. This gets divided by 2Ω and 1Ω . The value is 0.25 A. Therefore, the solution = $0.25(1 + e^{-10t})$.

Step-4: Find the zero-state response for the circuit in Fig. 10.10-5(d)

The initial value of i_x at $t = 0^+$ is 0.25 A from the circuit in Fig. 10.10-5(d). The final steady-state value is obtained by replacing the inductor with a short-circuit and solving the resulting resistive circuit. Applying the current division principle, we get the value as 0.125 A. Therefore, the solution = $0.125(1 + e^{-10t})$.

Step-5: Find the total zero-state response for the circuit in Fig. 10.10-5(a)

Total zero-state response is obtained by adding the two zero-state responses obtained in the last two steps. It is $0.25(1 + e^{-10t}) + 0.125(1 + e^{-10t}) = 0.375(1 + e^{-10t})$.

Step-6: Find the total response in the original circuit

This is obtained by adding the zero-input response obtained in *Step-1* and the total zero-state response obtained in *Step-5*. It is $0.5e^{-10t} + 0.375(1 + e^{-10t}) = 0.375 + 0.875e^{-10t}$. It is the same expression that we obtained in Example 10.10-1.

(i) If initial condition is changed to 2 A

This change will affect only the zero-input response. Zero-input response obeys the superposition principle. Therefore, the zero-input response becomes $(0.5e^{-10t}) \times 2/1 = e^{-10t}$.

Therefore, $i_x(t) = e^{-10t} + 0.375(1 + e^{-10t}) = 0.375 + 1.375e^{-10t}$ for $t \geq 0^+$.

(ii) If voltage source value is changed to 5 V

This change will affect the zero-state response contribution from the voltage source only. It was $0.25(1 + e^{-10t})$ when the voltage source value was 2 V. Applying the superposition principle, it will be 2.5 times this function with a voltage source value of 5 V.

$\therefore i_x(t) = 0.5e^{-10t} + 2.5 \times 0.25(1 + e^{-10t}) + 0.125(1 + e^{-10t}) = 0.75 + 1.25e^{-10t}$ A for $t \geq 0^+$.

(iii) If the voltage source value is changed to 4 V, the current source value is changed to 1 A and the initial condition changed to 2 A.

The solution will get affected in all the three components. Zero-input response gets scaled by 2/1, zero-state response from voltage source gets multiplied by 4/2 and zero-state response from the current source gets multiplied by 1/0.5.

$\therefore i_x(t) = 2 \times 0.5e^{-10t} + 2 \times 0.25(1 + e^{-10t}) + 2 \times 0.125(1 + e^{-10t}) = 0.75 + 1.75e^{-10t}$ A for $t \geq 0^+$.

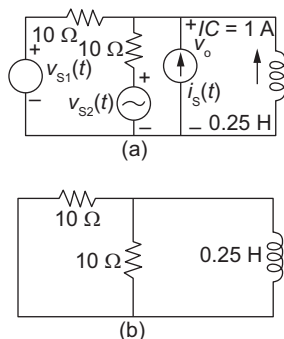


Fig. 10.10-6 Circuit for Example 10.10-3

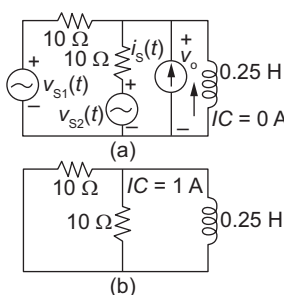


Fig. 10.10-7 Circuits for Determining Various Response Components in Example 10.10-3

EXAMPLE: 10.10-3

Solve for $v_o(t)$ as a function of time in the circuit in Fig. 10.10-6(a). Also identify (i) the transient response and the steady-state response components, (ii) zero-input response and zero-state response components in the total response and (iii) contributions to zero-state response from the individual sources. The source functions are

$$v_{s1}(t) = 10 \sin(10t + \pi/4) \text{ V}$$

$$v_{s2}(t) = 10 \sin(20t - \pi/6) \text{ V}$$

$$i_s(t) = 2 \cos(10t - \pi/3) \text{ A.}$$

SOLUTION

There are three switched sinusoidal sources in the circuit – two independent voltage sources and one independent current source. One voltage source $v_{s1}(t)$ and the current source $i_s(t)$ are of the same angular frequency of 10 rad/s. Since we are required to identify the zero-input response and the zero-state response components, we have to solve two circuits – one with all sources and zero initial condition for the inductor and another with all independent sources deactivated and specified initial condition for inductor current.

Step-1: Find the time constant

The deactivated circuit for finding the time constant is shown in Fig. 10.10-6(b).

The time constant is $0.25/(10//10) = 0.05$ s.

Step-2: Identify the circuits for zero-input response and zero-state response and solve for zero-input response.

These circuits are shown in Fig. 10.10-7. The circuit in Fig. 10.10-7(a) has to be used for determining zero-state response and the circuit in Fig. 10.10-7(b) has to be used for zero-input response.

The circuit does not apply any impulse to the inductor. Therefore, the inductor current at $t = 0^+$ is the same as at $t = 0^-$ and will be 1 A in the marked direction. What we need is the initial condition at $t = 0^+$ for our response variable v_o . The 1 A initial current in the inductor at $t = 0^+$ gets divided equally between the two equal resistors and develops a potential of 5 V across them at that instant. Hence, the required initial condition for $v_o = 5$ V.

This is a single time constant circuit, and hence, the zero-input response in any circuit variable will be of the form $Ae^{-t/\tau}$, where A has to be adjusted for compliance with initial condition on the chosen output variable. Therefore, the zero-input response component in v_o is $5e^{-20t}$ V, since the initial condition is 5 V and the time constant is 0.05 s.

Step-3: Identify circuits for obtaining zero-state response components and solve them.

Three sub-circuits derived from the circuit in Fig. 10.10-7(a) for evaluating the zero-state response contributions from individual sources are shown in Fig. 10.10-8.

We employ the phasor method to solve for the sinusoidal steady-state response in each circuit, add a transient response term of $Ae^{-t/\tau}$ format and evaluate A by substituting suitable initial condition.

Step-3a: Zero-state response in Fig. 10.10-8(a)

The circuits for evaluation of initial condition and the sinusoidal steady-state response for this circuit are shown in Fig. 10.10-9. The circuit in Fig. 10.10-9(a) shows the circuit for initial condition evaluation. The voltage source is replaced by a DC source of the same value as the value of $v_{s1}(t)$ at $t = 0^+$ i.e., $10 \sin \pi/4 = 7.07$ V. The inductor is replaced by a DC current source of value equal to its initial condition. But in a circuit for evaluating zero-state response, the initial condition for inductor current will be zero.

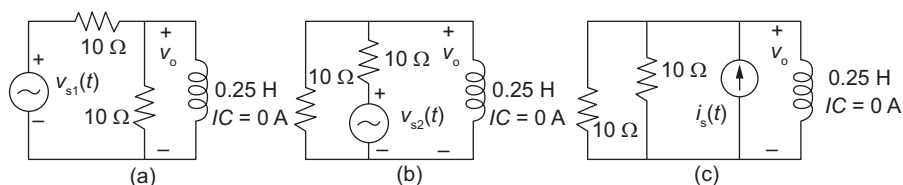


Fig. 10.10-8 Circuits for Obtaining Zero-State Response Components in Example 10.10-3

Therefore, it is a current source of zero value, i.e., it is an open-circuit. Therefore, v_o at $t = 0^+$ is $7.07/2 = 3.535$ V.

The angular frequency of $v_{s1}(t)$ is 10 rad/s. Therefore, the inductor will have an inductive reactance of $10 \times 0.25 = 2.5 \Omega$. The amplitude of $v_{s1}(t)$ is 10 V. In phasor equivalent circuit, we usually specify r.m.s value of sources. Hence, $10/\sqrt{2}$ at 45° is the voltage source value in the circuit used for sinusoidal steady-state response evaluation. The sinusoidal steady-state response is evaluated as below:

$$V_o = \frac{10 / \angle 45^\circ}{10 + 10 / \angle 45^\circ} \times \frac{10}{\sqrt{2}} \angle 45^\circ = \frac{2.24}{\sqrt{2}} \angle 108.44^\circ$$

$$v_o(t) = 2.24 \sin(10t + 108.44^\circ) \text{ V (} \because \text{Input was sine function).}$$

The zero-state response due to first voltage source is obtained by adding this solution to transient response term and evaluating the arbitrary constant in the transient response term by using the initial condition for $v_o(t)$ we have already calculated.

$$v_o(t) = Ae^{-20t} + 2.24 \sin(10t + 108.44^\circ)$$

$$v_o(0^+) = 3.535 \text{ V} \Rightarrow A + 2.24 \sin 108.44^\circ = 3.535 \text{ V} \Rightarrow A = 1.41$$

$$\therefore v_o(t) = 1.41e^{-20t} + 2.24 \sin(10t + 108.44^\circ) \text{ for } t \geq 0^+.$$

Step-3b: Zero-state response in Fig. 10.10-8(b)

The circuits for evaluation of initial condition and sinusoidal steady-state response for this circuit is shown in Fig. 10.10-10. The circuit in Fig. 10.10-10(a) shows the circuit for initial condition evaluation. The voltage source is replaced by a DC source of the same value as the value of $v_{s2}(t)$ at $t = 0^+$ i.e., $-10 \sin \pi/6 = -5$ V. The inductor is replaced by a DC current source of value zero. Therefore, v_o at $t = 0^+$ is $-5/2 = -2.5$ V.

The angular frequency of $v_{s2}(t)$ is 20 rad/s. Therefore, inductor will have an inductive reactance of $20 \times 0.25 = 5 \Omega$. The amplitude of $v_{s1}(t)$ is 10 V. Hence, $10/\sqrt{2}$ at -30° is the voltage source value in the circuit used for sinusoidal steady-state response evaluation. The sinusoidal steady-state response is evaluated as below:

$$V_o = \frac{10 / \angle 5^\circ}{10 + 10 / \angle 5^\circ} \times \frac{10}{\sqrt{2}} \angle -30^\circ = \frac{3.54}{\sqrt{2}} \angle 15^\circ$$

$$v_o(t) = 3.54 \sin(20t + 15^\circ) \text{ V (} \because \text{Input was sine function).}$$

The zero-state response due to first voltage source is obtained by adding this solution to the transient response term and evaluating the arbitrary constant in the transient response term by using the initial condition for $v_o(t)$ we have already calculated.

$$v_o(t) = Ae^{-20t} + 3.54 \sin(20t + 15^\circ)$$

$$v_o(0^+) = -2.5 \text{ V} \Rightarrow A + 3.54 \sin 15^\circ = -2.5 \text{ V} \Rightarrow A = -3.42$$

$$\therefore v_o(t) = -3.42e^{-20t} + 3.54 \sin(20t + 15^\circ) \text{ for } t \geq 0^+.$$

Step-3c: Zero-state response in Fig. 10.10-8(c)

The circuits for evaluation of initial condition and the sinusoidal steady-state response for this circuit are shown in Fig. 10.10-11. Figure 10.10-11(a) shows the circuit for initial condition evaluation. The current source is replaced by a DC current source of the same value as that of $i_s(t)$ at $t = 0^+$, i.e., $2 \cos(-\pi/3) = 1$ A. The inductor is replaced by a DC current source of value zero. Therefore, v_o at $t = 0^+$ is 5 V.

The angular frequency of $i_s(t)$ is 10 rad/s. Therefore, the inductor will have an inductive reactance of $10 \times 0.25 = 2.5 \Omega$. The amplitude of $i_s(t)$ is 2 A. Hence, $2/\sqrt{2}$ at -60° is the current source value in the circuit used for sinusoidal steady-state response evaluation. The sinusoidal steady-state response is evaluated as below:

$$V_o = 10 / 10 / \angle 2.5^\circ \times \frac{2}{\sqrt{2}} \angle -60^\circ = \frac{4.47}{\sqrt{2}} \angle 3.44^\circ$$

$$v_o(t) = 4.47 \cos(10t + 3.44^\circ) \text{ V (} \because \text{Input was a cosine function).}$$

The zero-state response due to first voltage source is obtained by adding this solution to the transient response term and evaluating the arbitrary constant in the transient response term by using the initial condition for $v_o(t)$ that we have already calculated.

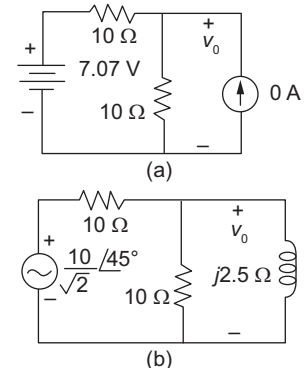


Fig. 10.10-9 Circuits for Zero-State Response due to first Voltage Source in Example 10.10-3

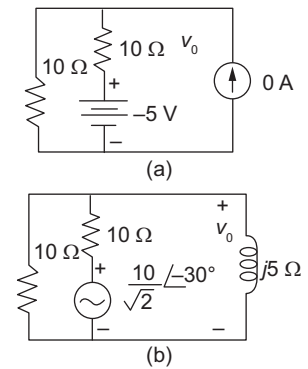


Fig. 10.10-10 Circuits for Zero-State Response due to Second Voltage Source in Example 10.10-3

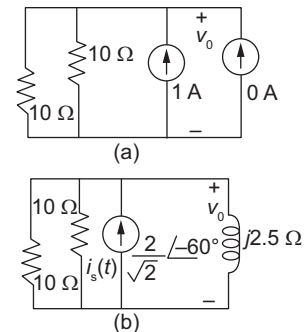


Fig. 10.10-11 Circuits for Zero-State Response due to Current Source in Example 10.10-3

$$v_o(t) = Ae^{-20t} + 4.47 \cos(10t + 3.44^\circ)$$

$$v_o(0^+) = 5 \text{ V} \Rightarrow A + 4.47 \cos 3.44^\circ = 5 \Rightarrow A = 0.54$$

$$\therefore v_o(t) = 0.54e^{-20t} + 4.47 \cos(10t + 3.44^\circ) \text{ for } t \geq 0^+.$$

Step-4: Get total response and identify various components

The zero-input response = $5e^{-20t}$ V

The zero-state response due to $v_{s1}(t) = 1.41e^{-20t} + 2.24\sin(10t + 108.44^\circ)$ V

The zero-state response due to $v_{s2}(t) = -3.42e^{-20t} + 3.54\sin(20t + 15^\circ)$ V

The zero-state response due to $i_s(t) = 0.54e^{-20t} + 4.47\sin(10t + 3.44^\circ)$ V

$\therefore$ The total response and its components are:

$$v_o(t) = 3.53e^{-20t} + 2.24\sin(10t + 108.44^\circ) + 4.47\cos(10t + 3.44^\circ) + 3.54\sin(20t + 15^\circ) \\ = 3.53e^{-20t} + 4.45\cos(10t + 32.52^\circ) + 3.54\sin(20t + 15^\circ) \text{ V for } t \geq 0^+$$

Transient component = $3.53e^{-20t}$ V

Steady-state component = $4.45\cos(10t + 32.52^\circ) + 3.54\sin(20t + 15^\circ)$ V

Zero-input response = $5e^{-20t}$ V

Zero-state response = $1.47e^{-20t} + 4.45\cos(10t + 32.52^\circ) + 3.54\sin(20t + 15^\circ)$ V.

EXAMPLE: 10.10-4

Find expressions for the inductor current and the voltage in the circuit in Fig. 10.10-12(a) for $t \geq 0^+$ and plot them in the range $0^+ \leq t \leq 1$ s.

SOLUTION

The voltage source in this circuit is specified as $15u(0.5 - t)$. $u(x)$ is 0 for $x \leq 0^-$, 1 for $x \geq 0^+$ and undefined at $x = 0$. Therefore, $u(0.5 - t)$ is a time function which is 0 for $t \geq 0.5^+$, 1 for $t \leq 0.5^-$ and is undefined at $t = 0.5$. This waveform is shown in Fig. 10.10-12(b). Physically, it implies that a 15 V DC source was connected to the circuit in the infinite past and it is removed, and instead, a short-circuit is introduced across the circuit at $t = 0.5$ s.

Since the 15 V source was connected long back, the circuit had enough time to reach the DC steady-state by the time t became zero. Therefore, the initial inductor current at $t = 0^-$ can be obtained by solving the resistive circuit with the inductor replaced by a short-circuit. This will be 0.5 A. The relevant circuit appears in Fig. 10.10-13(a).

The switch closes and puts a 15Ω resistance into the circuit at $t = 0$. A new steady-state will be established in the circuit if there is no further change in the circuit. However, the circuit cannot anticipate that a structural change or a change in source function is going to take place in future and modify its response in the present based on such anticipation. Therefore, the evolution of circuit variables will be towards the expected steady-state commensurate with the current nature and values of source functions. In the present example, we know that the circuit will not be allowed to reach the steady-state since the input is going to go down to zero at 0.5 s and the circuit will start on a new transient at that instant. But that does not prevent us from asking the question – what would have been the final value had the circuit been allowed to reach there? – Because the circuit is going to move only along that waveform up to 0.5 s. This expected final value, had the circuit been allowed to proceed to steady-state, can

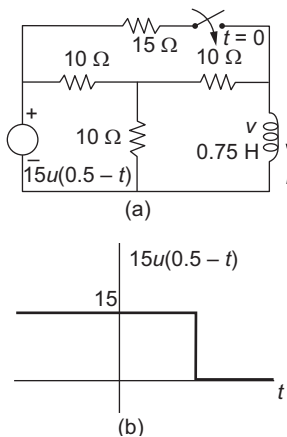


Fig. 10.10-12 Circuit for Example 10.10-4

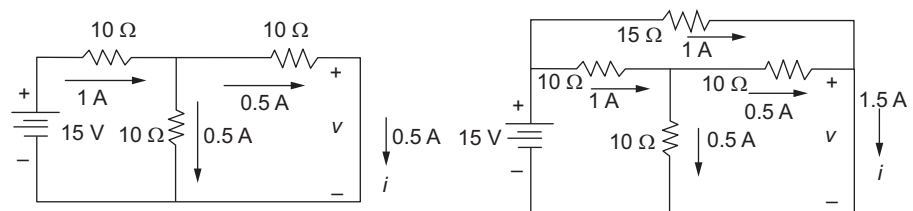


Fig. 10.10-13 Initial and Final Value Evaluation for the Circuit in Example 10.10-4

be worked out by replacing the inductor with a short-circuit. This circuit, along with the solution, is shown in Fig. 10.10-13(b).

The time constant relevant for $t \geq 0^+$ can be found by setting the voltage source value to zero with the $15\ \Omega$ already connected in place in Fig. 10.10-12(a). The equivalent resistance across the inductor will be $7.5\ \Omega$ and τ will be $0.1\ \text{s}$.

Now, the expression for i in the time range $0^+ \leq t \leq 0.5^-$ can be found as below:

$$i(t) = Ae^{-10t} + 1.5 \text{ with } i(0^+) = 0.5\ \text{A}$$

$$\therefore A + 1.5 = 0.5 \text{ and } A = -1$$

$$\therefore i(t) = -e^{-10t} + 1.5 \text{ for } 0^+ \leq t \leq 0.5^- \quad (10.10-1)$$

$$\therefore v(t) = 0.75 \frac{di}{dt} = 7.5e^{-10t} \text{ for } 0^+ \leq t \leq 0.5^-$$

At $t = 0.5\ \text{s}$, the input voltage goes to zero. A new transient in a source-free circuit starts at $t = 0.5^+$. Since there is no impulse voltage involved in the circuit, the inductor current will remain continuous between 0.5^- and 0.5^+ . The value at 0.5^- can be found by evaluating $i(t)$ at that instant using the expression for $i(t)$ in Eqn. 10.10-1. In a source-free circuit, there is no forced response term.

$$i(0.5) = -e^{-10 \times 0.5} + 1.5 \approx 1.5$$

$$\therefore i(t) = 1.5e^{-10(t-0.5)} \text{ for } t \geq 0.5^+ \quad (10.10-2)$$

$$\therefore v(t) = 0.75 \frac{di}{dt} = -11.25e^{-10(t-0.5)} \text{ for } t \geq 0.5^+$$

The plots of inductor current and voltage across inductor are given in Fig. 10.10-14.

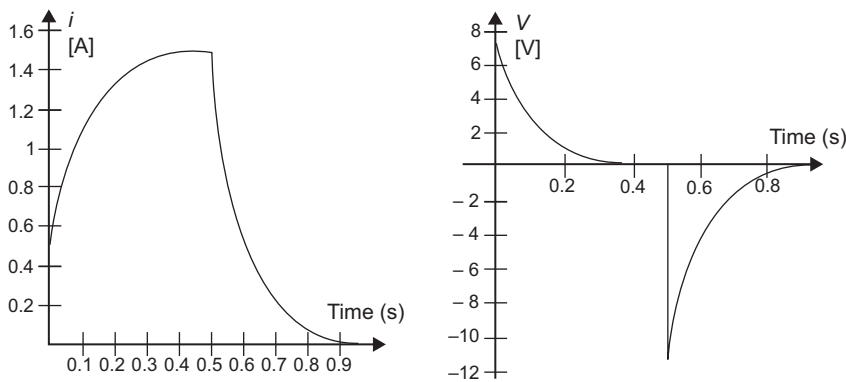


Fig. 10.10-14 Inductor Current and Voltage in Example 10.10-4

10.11 SUMMARY

- Circuits containing energy storage elements have memory in time-domain. They are described by linear ordinary differential equations with constant coefficients. We need to know the forcing function beginning from some time instant along with the initial conditions in the circuit specified at that instant to solve such differential equations.
- We focussed on series RL circuit in this chapter. RL circuit is described by a first order linear differential equation. The past history of an inductor is contained in a single initial condition specification in an RL circuit.
- The solution of the differential equation describing the inductor current in an RL circuit contains two terms – the complementary function and the particular integral. Complementary function is the solution of the differential equation with zero forcing function. Particular integral is the solution of the differential equation *with the assumption that the forcing function was applied from infinite past onwards*. The total solution is obtained by adding these two. The complementary function has an arbitrary amplitude that should be fixed by ensuring that the *total* solution complies with the specified initial condition.

- The circuit variables in the RL circuit will contain two response components – *transient response* (also called *natural response*) and *forced response*. *Natural response* is the way in which the inertia in the circuit reacts to the forcing function's command to change. Complementary solution gives the *natural response* and particular integral gives the *forced response* in a circuit.
- The nature of *natural response* of a linear time-invariant circuit is independent of the magnitude of the forcing function and depends only on circuit parameters and nature of interconnections. *Natural response* in RL circuit is an exponential of the form $Ae^{-t/\tau}$, where $\tau = L/R$ is defined as the time constant of the circuit. A is to be fixed by complying with initial condition.
- The current in an RL circuit at $t = 0^-$ and $t = 0^+$ will be the same if the circuit does not contain impulse sources and it cannot support impulse voltages.
- Step response* of a circuit is its response when a unit step input is applied, with the circuit initially at rest. In the case of RL circuit, *step response* is a rising exponential, approaching a steady-state value asymptotically as $t \rightarrow \infty$. The *step response* never gets done. However, it may be considered to be over within five time constants for practical purposes.
- Time constant* can be understood as the additional time required from the current instant for the *step response* to reach the final value, assuming that the rate of rise of response is held constant at its current value from that instant onwards.
- Free response* of an RL circuit is its response when the input is zero and there is some initial energy trapped in the inductor. It will contain only *natural response* terms. The inductor current in this case falls exponentially towards zero.
- If all the *transient response* terms are of vanishing nature, the only remaining response in the long run will be the *forced response* component. Then, the *forced response* component is termed as the *steady-state response*, provided there are constant features describing the forcing function. Three kinds of steady-states are usually studied in circuits – DC steady-state, AC steady-state and periodic steady-state. Inductors can be replaced with short-circuits for DC steady-state analysis. AC steady-state analysis can be carried out using phasor analysis.
- Impulse response* of a circuit is its response when a unit impulse input is applied to it when it is in an initially relaxed condition. *Impulse response* of circuits will contain only *natural response* terms.
- The response due to initial energy and the application of impulse are indistinguishable in an RL circuit, and hence, they can be substituted for each other. An initial current of I_0 in an inductor of value L can be replaced with zero initial condition with a voltage source $LI_0\delta(t)$ connected in series with the inductor.
- Step response* and *ramp response* in an RL circuit can be obtained by integrating its *impulse response* successively.
- Zero-input response* of a circuit is its response when there is no input but there is initial energy. It will contain only natural response terms. *Zero-state response* is the response when the circuit is initially at rest (zero initial conditions) and input is applied. It will contain both *natural response* terms and *forced response* terms. The *total response* is given by sum of zero-input response and zero-state response.
- Forced response* (and hence, the *steady-state response*) obeys the superposition principle with respect to input source functions. However, *transient response* and *total response* do not obey the superposition principle – neither with respect to initial conditions nor with respect to input source functions. However, *zero-input response* obeys the superposition principle with respect to initial conditions and *zero-state response* obeys the superposition principle with respect to input source functions.
- Total response* in single-inductor, multi-resistor circuits can be found with the help of superposition principle and Thevenin's theorem by evaluating *zero-input response* for the entire circuit and *zero-state response* for each source separately.

10.12 QUESTIONS

- What is the differential equation describing i_L for $t \geq 0^+$ in the circuit in Fig. 10.12-1?

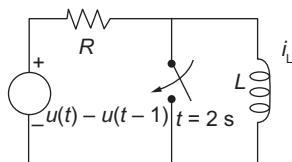


Fig. 10.12-1

- A series RL circuit with non-zero initial energy is driven by a unit step voltage source. The circuit current is found to reach 75% of its steady-state value in one time constant. Express the initial current in the inductor as a percentage of its steady-state value.
- A series RL circuit is driven by a unit step voltage source. i_L is found to be 50% at $t = \tau$. Was there any initial current in the inductor? If so, what is its magnitude and direction?
- Find the ratio E/V such that i_L in Fig. 10.12-2 becomes V/R at $t = 0.5 + \tau$, where $\tau = L/R$.

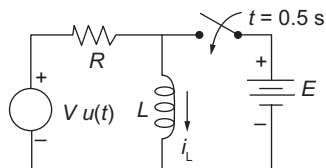


Fig. 10.12-2

5. Derive an expression for i_L in an initially relaxed series RL circuit when it is driven by $v_s(t) = t$ for $0^+ \leq t \leq \tau$ and 0 for all other t , where τ is the time constant of the circuit.
6. What is the time required for the energy stored in the inductor in a series RL circuit to reach 90% of its steady-state value in step response?
7. i_L in the circuit in Fig. 10.12-3 is -2 A at $t = 0^-$. It is found that $i_L(t) = 0$ for $t \geq 0^+$. What is the value of L ?

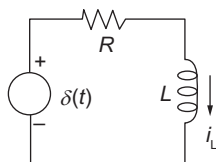


Fig. 10.12-3

8. i_L in the circuit in Fig. 10.12-4 at $t = 0^-$ is -2 A. It is found that $i_L(t) = 0$ for $t \geq 0^+$. What is the value of time constant of the circuit?

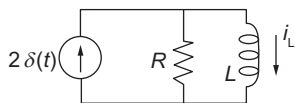


Fig. 10.12-4

9. A series RL circuit with non-zero initial energy is driven by a unit step voltage source. The inductor current is found to be $(2 - e^{-t})$ A. What is the value of inductance, resistance and initial current?
10. A series RL circuit with non-zero initial energy is driven by a unit step voltage source. The inductor current is found to be 3 A for $t \geq 0^+$. What is the value of initial current and what is its direction relative to the 3 A current?
11. An AC voltage source $= V \sin(\omega t)u(t)$ is applied to a series RL circuit and its current is found to be $= 0.7 \sin(\omega t - \pi/3)u(t)$. Was there any initial current in the inductor? If yes, what is its magnitude and relative direction?
12. A series RL circuit with non-zero initial energy is driven by a unit step voltage source. The inductor current is found to be 2 A. What is the new i_L in the circuit if (i) the initial condition is doubled and (ii) if $2u(t)$ is applied with no change in initial condition?
13. The inductor current $= 0$ for $t \geq 0^+$ in the circuit in Fig. 10.12-5. Find the values of I_0 and I_S .
14. The switch S in the Fig. 10.12-6 is operated cyclically from $t = 0$. In every switching cycle it remains open for half the

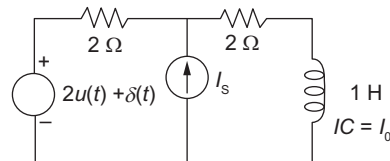


Fig. 10.12-5

cycle time and then remains in closed condition for half the cycle time. What is the time taken for the inductor current to cross 99% of its steady-state value if the frequency of switching is (i) 1 Hz and (ii) 1 kHz?

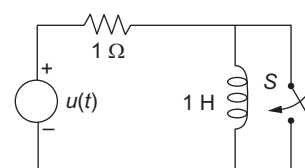


Fig. 10.12-6

15. A series RL circuit with zero initial current is driven by $v_s(t) = \delta(t) - \delta(t - 1)$. Its time constant is 1 s. (i) Starting from impulse response find the voltage across the resistor in the circuit when driven by the input $v_s(t)$. (ii) Using the result, derive an expression for voltage across the resistor when the circuit is driven by a rectangular pulse of unit amplitude and 1 s duration.
16. Derive expressions for maximum voltage across a resistor in an initially relaxed series RL circuit when it is driven by $e^{-\alpha t}u(t)$ with $\alpha \neq 1/\tau$, where τ is the time constant of the circuit. Also, find an expression for the time instant at which this maximum voltage occurs.
17. An exponential input of $k\delta(t) + 2e^{-2t/\tau}u(t)$ V is applied to an initially relaxed series RL circuit with time constant of τ s. The output for $t \geq 0^+$ is observed to contain only $e^{-2t/\tau}$ waveshape. What is the value of k ?
18. The steady-state voltage across a resistor (v_R) in a series RL circuit has an amplitude of 7 V when the circuit is driven by an AC voltage of amplitude 10 V and an angular frequency ω rad/s. (i) Find the phase angle of v_R with respect to the input sinusoid. (ii) If another sinusoidal voltage of 15 V amplitude and 3ω rad/s frequency is applied to the circuit, find the amplitude and phase of v_R under the steady-state condition.
19. The desired current in the inductor in a series RL circuit with an initial condition as shown in the Fig. 10.12-7 is given by

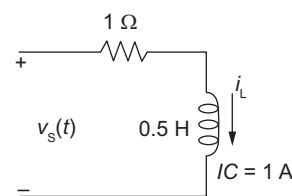


Fig. 10.12-7

$$i_L(t) = \begin{cases} 2t & \text{for } t \geq 0^+ \\ 0 & \text{for } t \leq 0. \end{cases}$$

Find the $v_s(t)$ to be applied to the circuit if the initial condition is 1 A as marked in the Fig. 10.12-7. Sketch the required voltage.

20. What is the rise time of the inductor current in the step response of the circuit in Fig. 10.12-8?

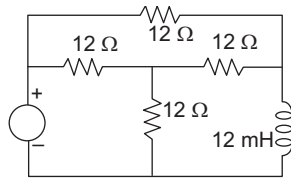


Fig. 10.12-8

21. Two equal valued inductors and one resistor are used to form the circuit in Fig. 10.12-9 at $t = 0$. The value of i_1 and i_2 at $t = 0^-$ are I_1 and I_2 , respectively. (i) Derive expressions for $i_1(t)$, $i_2(t)$, $i_R(t)$ and $v(t)$ as functions of time. (ii) What are the final values of i_1 and i_2 when $I_1 = I_2$? (iii) What are the final values of i_1 and i_2 when $I_1 = 2I_2$? (iv) What are the final values of i_1 and i_2 when $I_1 = 0.5I_2$? (v) Why do the inductors fail to discharge completely in (iii) and (iv)?

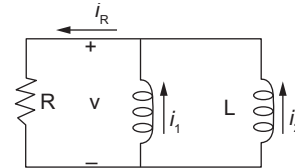


Fig. 10.12-9

10.13 PROBLEMS

(Initial current in inductor is zero unless stated otherwise.)

1. The switch S_1 in the circuit in Fig. 10.13-1 is closed at $t = 0$ and the switch S_2 is closed at $t = 0.12$ s. Both switches are ideal. (i) Find the current in the inductance as a function of time and plot it. (ii) Find the voltage across the inductance and plot it. (iii) Find the voltage across the first resistor and plot it. (iv) What is the time required to attain 90% of the final steady state value of current?

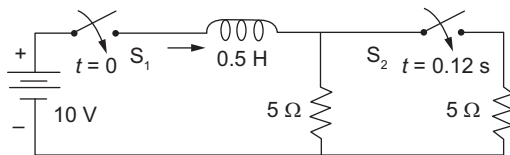


Fig. 10.13-1

2. Initial current at $t = -\infty$ in the inductor in the circuit in Fig. 10.13-2 was zero. Find $i_L(t)$ and $v_L(t)$ for $t \geq 0^+$ and plot them.

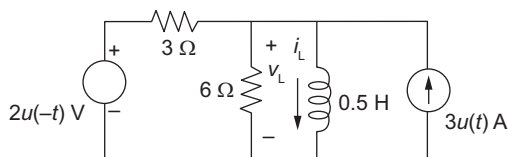


Fig. 10.13-2

3. S in Fig. 10.13-3 is a two-position switch and starts at Position 1 at $t = 0$. It is kept in that position for 10 ms and then thrown to Position 2. It is kept at the second position until the current in

the inductor goes to zero. At that instant, it is thrown back to Position 1. Then, the whole cycle repeats. (i) Calculate and plot two cycles of $i_L(t)$, $i_{S1}(t)$ and $i_{S2}(t)$. (ii) What is the frequency of switching in the circuit? (iii) What are the average currents in the two sources and inductance? (iv) What is the average power delivered to the 24 V source and the average power supplied by 12 V source. (v) If the idea was to charge the 24 V battery from 12 V battery, what is the efficiency of this charger? (vi) Suggest a method to control the average charging current in the 24 V battery.

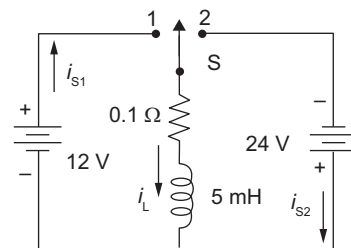


Fig. 10.13-3

4. Repeat Problem 3 with the circuit in Fig. 10.13-4.

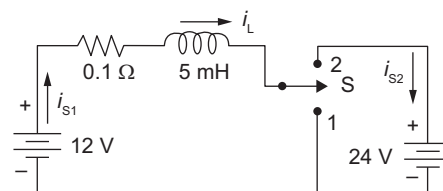


Fig. 10.13-4

5. The switch S in Fig. 10.13-5 starts with the switch thrown to Position 1 at $t = 0$. It is kept in that position for 10 ms and then thrown to Position 2. It is kept at the second position until the current in the inductor goes to zero. At that instant, it is thrown back to Position 1. Then, the whole cycle repeats.

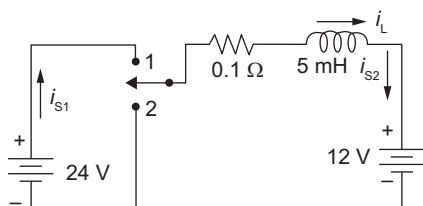


Fig. 10.13-5

- (i) Calculate and plot two cycles of $i_{S1}(t)$ and $i_{S2}(t)$. (ii) What is the frequency of switching in the circuit? (iii) What are the average currents in the two sources? (iv) What is the average power delivered to the 12 V source and the average power supplied by 24 V source. (v) If the idea was to charge the 12 V battery from the 24 V battery, what is the efficiency of this charger? (vi) Suggest a method to control the average charging current in the 12 V battery.
6. The switch in the circuit in Fig. 10.13-6 was closed for a long time and is opened at $t = 0$. Find and plot the current in the inductance and the voltage across inductance as functions of time.

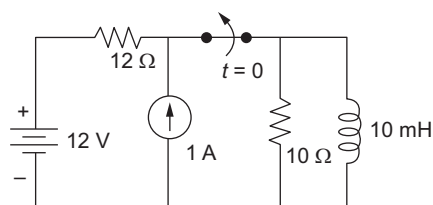


Fig. 10.13-6

7. Normalised Energy Content of an electrical waveform $f(t)u(t)$ is defined as

$$E_n = \int_0^{\infty} [f(t)]^2 dt.$$

It can be interpreted as the energy dissipated in a 1Ω resistance if this voltage (or current) waveform is applied across it.

- (i) Evaluate the E_n of a rectangular pulse of width T s and height V V. (ii) This pulse is applied to an initially relaxed series RL circuit and the output voltage is taken across the resistor. Derive the relationship between T and τ (the time constant of the circuit) such that the output voltage will have an energy content that is more than 90% of the input pulse E_n .
8. The input voltage applied to an initially relaxed series RL circuit is a single pulse defined as $v_s(t) = 3t$ for $0 < t \leq k\tau$ and zero elsewhere, where τ is the time constant of the circuit and k is a positive real number. Express the normalised energy of v_R as a fraction of E_n of $v_s(t)$. What is the value of k if 90% of the input energy content is to be passed on to the output voltage i.e., v_R ?

9. The input voltage applied to an initially relaxed series RL circuit is $v_s(t) = e^{-t/k\tau}u(t)$, where τ is the time constant of the circuit and k is a positive real number. Express the normalised energy of v_R as a fraction of E_n of $v_s(t)$. What is the value of k if 90% of the input energy content is to be passed on to the output voltage, i.e., v_R ?

10. Find the impulse response of $v(t)$ in Fig. 10.13-7.

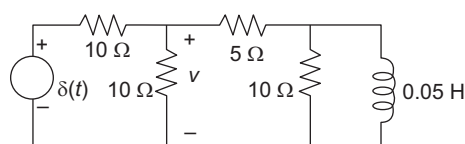


Fig. 10.13-7

11. (i) Find the zero-input response, zero-state response and total response for $i_L(t)$ and $v(t)$ in the circuit in Fig. 10.13-8. (ii) Obtain $i_L(t)$ and $v(t)$ if the current source on the right side is made $2u(t)$.

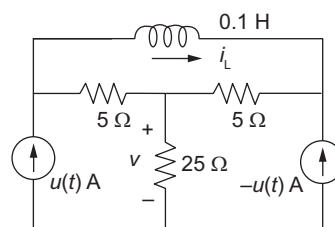


Fig. 10.13-8

12. What must be the value of k in the circuit in Fig. 10.13-9 if $v(t) = 0$ for $t \geq 0^+$?

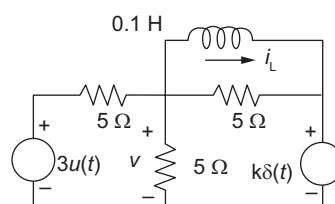


Fig. 10.13-9

13. The impulse response of $i_L(t)$ in the circuit in Fig. 10.13-10 is seen to be $= 18.75e^{-1000t}$. The steady-state value of step response is 25 mA in the inductor. (i) Find the values of R_1 , R_2 and L . (ii) Find the step response of current through R_2 .

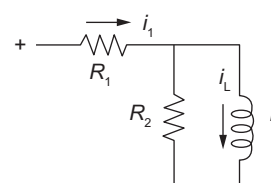


Fig. 10.13-10

14. S in the circuit in Fig. 10.13-11 operates cyclically at 10 kHz, spending equal time at both positions in a cycle. Estimate the average power delivered by the current source and the average power dissipated in the resistor.

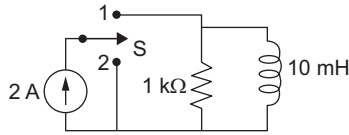


Fig. 10.13-11

15. In Fig. 10.13-12 $V_{ac} = 10 \sin(314t)$ V. (i) Find the *steady-state* output voltage waveform for $v_o(t)$ and plot it. (ii) What is the percentage peak-peak ripple with respect to average value in the output voltage? (iii) Repeat (ii) if inductance is changed to 2H. (iv) Justify the following statement – ‘The steady-state output voltage across resistance in an RL circuit will be more or less constant at the average value of driving voltage, even if the driving voltage has an AC component, if the product $\omega L \gg R$, where ω is the angular frequency of the AC component.’

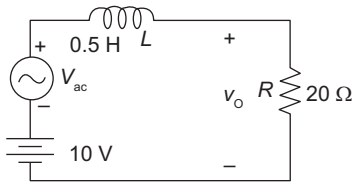


Fig. 10.13-12

16. Let $V_s(t)$ be an arbitrary time-varying periodic voltage source with a cycle average value of V_{dc} . This means that $V_s(t)$ can be written as $V_{dc} + V_{ac}(t)$, where $V_{ac}(t)$ is a time-varying periodic

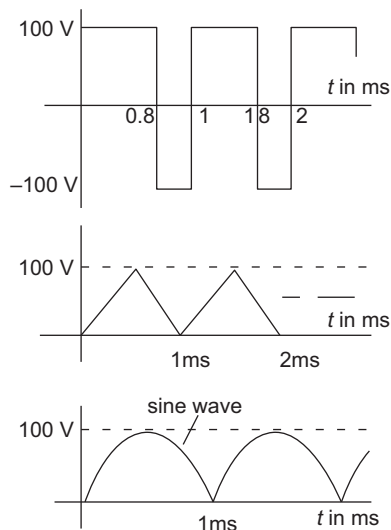


Fig. 10.13-13

component with equal positive half-cycle and negative half-cycle areas. Let that area be A volt-sec. This waveform $V_s(t)$ is applied to a series RL circuit and the output voltage is taken across R . Assume that $L/R \gg T$, where T is the period of $V_s(t)$. Show that, under periodic steady-state, the (i) average value of output voltage is V_{dc} , (ii) the peak-to-peak ripple in output voltage, A/τ V, where $\tau = L/R$ and (iii) calculate the quantities in (i) and (ii) for the three inputs given in Fig. 10.13-13, if τ is 25 ms.

17. Two series RL circuits are connected in cascade using a unity gain buffer amplifier as in Fig. 10.13-14. A buffer amplifier is an electronic amplifier that presents infinite resistance at its input and behaves like an ideal voltage source at its output. With a unity gain, its output voltage is the same as the input voltage. Buffer amplifiers are used to interconnect circuits that will interact with each other otherwise. The initial current in the inductor of the first stage circuit is 0.5 A and that of the second stage is 2 A. Find $v_o(t)$ for $t \geq 0^+$.

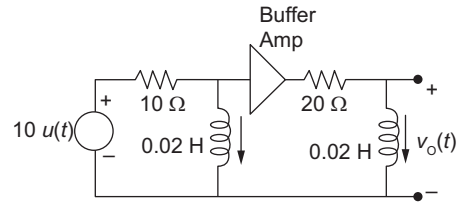


Fig. 10.13-14

18. Inductance is electrical inertia and resists change in current. Hence, an inductor in series with a resistor will tend to keep the current in the resistor and voltage across the resistor constant. This tendency will be more if the value of inductance is raised. This idea is used in producing a variable DC voltage across a load resistance from a fixed DC voltage source in the circuit in Fig. 10.13-15.

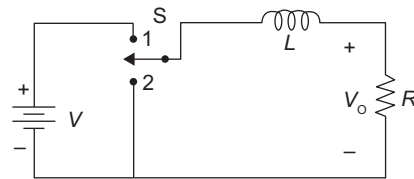


Fig. 10.13-15

The switch S spends T_1 seconds in Position 1 and $(T - T_1)$ s in Position 2 and the whole cycle is repeated. T will be kept constant and T_1 will be varied in the range 0 to T in order to control the voltage across R .

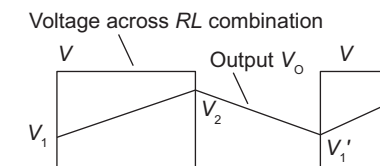


Fig. 10.13-16

A periodic steady-state behaviour of V_O will come up after a few switching cycles for any value of T_1 . This periodic steady-state is shown in Fig. 10.13-16.

The waveforms between V_1 and V_2 will be a piece of exponential and so will be the waveform between V_2 and V_1' . Under periodic steady-state, V_1' has to be equal to V_1 .

- (i) Using the above condition for periodic steady state, derive equations for V_1 and V_2 in terms of V , T_1/T and τ/T , where $\tau = L/R$. Simplify the expressions for $\tau/T \gg 1$.
- (ii) From the above expressions, find the average voltage across R in terms of V , T_1/T and τ/T . Simplify the expressions for $\tau/T \gg 1$.
- (iii) Let $d = T_1/T$. Calculate and plot the ratio of peak-peak ripple to average value for V_O for various values of ' d ' (calculate for $d = 0.1, 0.2 \dots 1$) with (a) $L/R = 10T$ and (b) $L/R = 50T$
- (iv) The above step should have shown that a very clean DC voltage can be produced across R by using a large L such

that $L/R > T$. It is possible to vary this clean DC voltage by changing ' d '. Let the system be working steadily at $d = 0.25$ with a DC output of $V/4$. The ' d ' (the so-called 'duty ratio') is changed suddenly to 0.5. Plot the growth of output DC component and calculate rise time when (a) $L/R = 10T$ and (b) $L/R = 50T$.

- (v) With a large $\tau/T \gg 1$, the current through inductance is nearly DC. Does it mean that the flux linkage in the inductance is also DC? Plot the voltage across the inductance for $d = 0.25$ and plot the corresponding flux linkage variation also. (Do not forget to include the flux due to DC current flow).

Why can we not filter the DC voltage output to near-zero ripple level? (This circuit is a simplified version of what Power Electronics Specialists call 'Buck Converter')

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RC and RLC Circuits in Time-Domain

CHAPTER OBJECTIVES

- *Impulse, step and ramp response of first-order RC circuits.*
- *Series RC circuit with real exponential input.*
- *Zero-state response of parallel RC circuit for sinusoidal input.*
- *The use of frequency response, frequency response and linear distortion.*
- *First-order RC circuits as averaging circuits.*
- *Capacitor as a signal coupling and signal bypassing element.*
- *Source-free response of series RLC circuit.*
- *The series LC circuit – A special case.*
- *The series LC circuit with small damping – another special case.*
- *Standard formats for second-order circuit zero-input response.*
- *Sinusoidal forced-response of RLC circuits from differential equation.*
- *Frequency response of RLC circuits from phasor equivalent circuit.*
- *Qualitative discussion on frequency response of series RLC circuit.*
- *A more detailed look at the band-pass output of series RLC circuit.*
- *Quality factor of practical inductors and capacitors.*
- *Zero-input response and zero-state response of parallel RLC circuit.*
- *Sinusoidal steady-state frequency response of parallel RLC circuit.*

This chapter takes up time-domain analysis of second-order circuits in detail. The frequency response of RC and RLC circuits is employed to introduce concepts on filtering, waveform distortion etc. Further, the chapter reinforces many basic concepts in Linear System Analysis through a detailed study of RC and RLC circuits.

INTRODUCTION

A circuit containing a single linear time-invariant capacitor and a linear time-invariant resistor, excited by a voltage source in series or a current source in parallel, will constitute a first-order LTI circuit. We studied a first-order circuit – namely, the *RL* Circuit – in detail and used it to bring out all the important concepts in the time-domain analysis of electrical circuits in the previous chapter. Hence, we will be able to move through simple *RC* circuits

Series and Parallel RLC circuits contain two energy storage elements. They are described by second-order differential equations.

This chapter brings out salient features of second-order system behaviour through a detailed analysis of RLC circuits.

rather quickly in this chapter. That does not mean that *RC* circuits are less important in any sense compared to *RL* circuits. *RL*, *RC* and *RLC* circuits are the basic building blocks of electrical and electronic circuits and are equally important. It is only that we chose to deal with *RL* circuits first and that in no way makes other dynamic circuits less important.

We deal with the Series *RLC* and Parallel *RLC* circuits in the second part of this chapter. We meet with a linear second-order differential equation in our attempt to describe these circuits and we go into responses of such a second-order system to various standard input signals in detail subsequently.

11.1 RC CIRCUIT EQUATIONS

First-order series *RC* circuit and parallel *RC* circuit are shown in Fig. 11.1-1 with all element variables identified. We choose the voltage across capacitor, $v_C(t)$, as the describing variable in both cases. If $v_C(t)$ is known in the series circuit, the current through capacitor can be obtained by multiplying the first derivative of $v_C(t)$ by C . Then the current through the resistor and voltage across it may be found. Similarly, all other element variables in the parallel *RC* circuit can be found if $v_C(t)$ in that circuit is known.

The differential equation governing the Series *RC* circuit is obtained by applying KCL at the positive terminal of the capacitor along with element equations of resistor and capacitor.

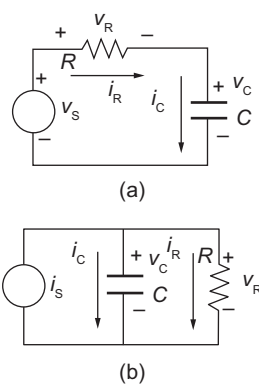


Fig. 11.1-1 (a) The Series RC Circuit (b) The Parallel RC Circuit

The time constant of a first-order RC circuit is $\tau = RC$ s.

$$i_R(t) = i_C(t) \text{ by KCL}$$

$$\text{But, } i_R(t) = \frac{v_s(t) - v_C(t)}{R} \text{ and } i_C(t) = C \frac{dv_C(t)}{dt}$$

$$\therefore C \frac{dv_C(t)}{dt} = \frac{v_s(t) - v_C(t)}{R} \text{ for all } t$$

$$\text{i.e., } \frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = \frac{1}{RC} v_s(t) \text{ for all } t.$$

The differential equation for the parallel *RC* circuit is obtained by applying KCL at the positive terminal of the capacitor.

$$i_s(t) = i_R(t) + i_C(t) \text{ by KCL}$$

$$\text{But, } i_R(t) = \frac{v_C(t)}{R} \text{ and } i_C(t) = C \frac{dv_C(t)}{dt}$$

$$\therefore C \frac{dv_C(t)}{dt} = \frac{v_C(t)}{R} = i_s(t) \text{ for all } t$$

$$\text{i.e., } \frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = \frac{i_s(t)}{C} \text{ for all } t.$$

The describing differential equation is a linear first-order equation with constant coefficients in both cases. We expect the solution to contain a natural response term of $e^{-\alpha t}$ type with $\alpha = 1/RC$. Comparing with the *RL* circuit equations, we can identify the time constant of *RC* circuit as RC s. The reader may easily verify that the product RC has dimensions of time.

11.2 ZERO-INPUT RESPONSE OF RC CIRCUIT

The differential equations derived above can be used to solve for $v_C(t)$ for all t provided the input source function is known for all t . However, we do not know the input source function for all t . The input source function is usually known only for $t > 0$. It may also contain a discontinuity at $t = 0$. The input source function is generally unknown for $t < 0$.

Therefore, the effect of all the unknown currents which went through the capacitor from infinite past to $t = 0^-$ is given in a condensed manner in the form of an initial value specification for $v_C(t)$ at $t = 0^-$. Let us denote this value as V_0 . Now, we can obtain the total response of the circuit by adding the two response components – zero-input response and zero-state response. Zero-input response is the response when *the input is held at zero from $t = 0$ onwards*. Zero-state response is the response when *the input is held zero from infinite past to $t = 0^-$ and then a specified input source function is applied from $t = 0$ onwards*.

Zero-input response is the same as the so-called *source-free response*. The describing differential equation in this situation is

$$\frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = 0 \text{ for all } t \geq 0^+ \text{ with } v_C(0^-) = V_0 \quad (11.2-1)$$

Physically this amounts to connecting a charged capacitor to a resistor to form a closed loop at $t = 0$. The initial voltage across the capacitor appears across the resistor from $t = 0$ onwards. The resistor demands a current of $v_C(t)/R$ and this current flow is of suitable polarity to discharge the capacitor. As the capacitor discharges more and more, its voltage comes down, resulting in the discharge current also going down. Thus, the capacitor keeps discharging; but at slower and slower rates as time increases. The discharge current divided by capacitance value gives the rate of decrease of voltage across the capacitor. Hence, the initial rate of decrease of voltage is V_0/τ V/s, where τ is the time constant ($= RC$) of the circuit. If the initial rate of decrease were maintained throughout, the voltage across capacitor would have gone to zero in one time constant.

There was no impulse current source in this circuit and hence the voltage across the capacitor does not change instantaneously at $t = 0$. Hence, $v_C(t)$ at $t = 0^+$ is same as $v_C(t)$ at $t = 0^-$, i.e., V_0 . Therefore, the expression for $v_C(t)$ is given by the familiar exponential function.

$$v_C(t) = V_0 e^{-t/\tau} \text{ V for } t \geq 0^+ \quad (11.2-2)$$

The graph of this function is well known and hence not repeated here.

EXAMPLE: 11.2-1

Practical dielectrics employed in capacitors have non-zero volume conductivity and surface conductivity. This results in a *leakage current* that tends to discharge an initially charged capacitor even when it is left open. This phenomenon is called *self-discharge* of a capacitor. A 1000 μ F Electrolytic capacitor is charged to 400 V and is left open from $t = 0$ onwards. The voltage across capacitor is found to be 40 V at $t = 10$ min. What will be the power dissipated in the capacitor if it is connected across a 400 V DC source for a long time?

SOLUTION

The capacitor discharges due to its internal leakage current. This leakage current may be modelled approximately by a resistor in parallel with the capacitor. Hence, the charged capacitor undergoes the zero-input response of a RC circuit with a time constant of RC , where R is its shunt resistance equivalent to its self-discharge. The value of R is to be found. Using Eqn. 11.2-2 with $V_0 = 400$ V and $v_C(t) = 40$ V,

$$\begin{aligned} 40 &= 400e^{-600/\tau} \\ \therefore \tau &= \frac{600}{\ln 10} \text{ s} = 260.6 \text{ s} \\ \therefore R &= \frac{260.6 \text{ s}}{1000 \mu\text{F}} = 0.26 \text{ M}\Omega. \end{aligned}$$

Now, if this capacitor is connected across a 400 V DC source this resistor will draw a current of 1.54 mA of current from the source. Hence, the power dissipated by the capacitor when it is connected across a 400 V source will be $400 \text{ V} \times 1.54 \text{ mA} = 0.616 \text{ W}$.

Source-free Response of RC Circuit

The source-free response (also called zero-input response) of a RC circuit is a decreasing exponential with a time constant of RC s.

The initial energy stored in the electric field in the capacitor gets dissipated in the resistor. The circuit reaches a zero-energy state practically in about 5 time constants.

An ideal capacitor connected across a DC voltage source does not draw any current after the initial charging pulse.

But a practical capacitor draws a small amount of current continuously from a DC voltage source and gets heated up (see Example 11.2-1 and Example 11.2-2).

EXAMPLE: 11.2-2

A capacitor C is initially charged to 500 V and is left open for 120 s. The voltage across capacitor at the end of this time interval is observed to be 400 V. A resistor of 100 k Ω is connected across this capacitor at 120 s. The voltage across capacitor is found to reach 100 V at 216.4 s after this connection has been made. Find the value of C and its leakage resistance.

SOLUTION

Let R be the leakage resistance of the capacitor. Then, using Eqn. 11.2-2 with suitable values for V_0 and $v_C(t)$,

$$400 = 500e^{-120/\tau_1} \Rightarrow \tau_1 = \frac{120}{\ln 1.25} = 537.8 \text{ s,}$$

and

$$100 = 400e^{-216.4/\tau_2} \Rightarrow \tau_2 = \frac{216.4}{\ln 4} = 156.1 \text{ s,}$$

where $\tau_1 = RC$ and $\tau_2 = [R//100 \text{ k}\Omega] C$

$$\therefore \frac{\tau_1}{\tau_2} = \frac{537.8 \text{ s}}{156.1 \text{ s}} = 3.445 = \frac{R(R+100)}{R \times 100} = \frac{(R+100)}{100} = 0.01R + 1; \text{ with } R \text{ in k}\Omega.$$

$$\therefore R = 244.5 \text{ k}\Omega \text{ and since } RC = 537.8 \text{ s, } C = \frac{537.8 \text{ s}}{244.5 \text{ k}\Omega} = 2.2 \text{ mF} = 2200 \mu\text{F}.$$

11.3 ZERO-STATE RESPONSE OF RC CIRCUITS FOR VARIOUS INPUTS

We consider the response of series RC circuit and parallel RC circuit for various input source functions in this section. We know that the total response of any circuit to application of input function is obtained by adding the zero-input response and zero-state response together. Hence, we consider only the zero-state response part for various input source functions in this section. We begin with impulse response (i.e., zero-state impulse response).

11.3.1 Impulse Response of First-Order RC Circuits

The series RC circuit in Fig. 11.3-1(a) has zero initial condition and is excited by a unit impulse voltage source. The capacitor cannot absorb the impulse voltage. Hence, the resistor absorbs the impulse voltage and as a result an impulse current of $1/R C$ flows through the circuit. This impulse current flow results in sudden dumping of $1/R C$ of charge on the capacitor plates; thereby changing the capacitor voltage from 0 at $t = 0^-$ to $1/RC$ V at $t = 0^+$. The unit impulse voltage source is a short circuit for $t \geq 0^+$. Therefore, the only effect of impulse voltage application is to change the initial condition of the capacitor instantaneously. The circuit effectively becomes a *source-free* circuit with initial energy for $t \geq 0^+$ and executes its zero-input response. The relevant circuit is shown in Fig. 11.3-1(b).

Initial voltage across capacitor is $1/RC$ V and all the voltages and currents in the circuit decay exponentially to zero with a time constant of $\tau = RC$ s.

$$v_C(t) = -v_R(t) = \frac{1}{RC} e^{-t/\tau} \text{ V for } t \geq 0^+$$

$$-i_C(t) = -i_R(t) = \frac{1}{R^2 C} e^{-t/\tau} \text{ A for } t \geq 0^+.$$

We had noticed the equivalence between non-zero initial condition at $t = 0^-$ and application of impulse at $t = 0$ in our analysis of RL circuits. We note that it is true in the

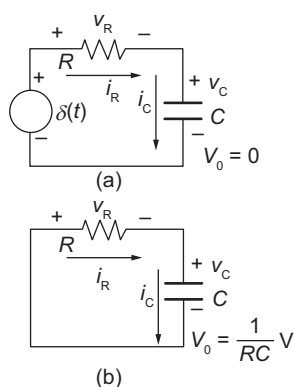


Fig. 11.3-1 Pertaining to Impulse Response of Series RC Circuit

case of RC circuits also. Specifically, a capacitor with an initial voltage of V_0 V across it at $t = 0^-$ may be replaced by a capacitor with zero initial voltage and an impulse current source of suitable magnitude (CV_0 C) and suitable polarity connected across it. This equivalence is shown in Fig. 11.3-2.

Figure 11.3-3 shows the application of a unit impulse current to a parallel RC circuit with zero initial energy. The resistor cannot support the impulse current. If it were to do so, it would have called for an impulse voltage across it and that will be resisted by the capacitor in parallel. Therefore, all the impulse content goes through the capacitor, changing its voltage by $1/C$ V instantaneously from 0 at $t = 0^-$ to $1/C$ V at $t = 0^+$. The unit impulse current source is effectively an open-circuit after $t = 0^+$. Therefore, the circuit becomes a *source-free* circuit for $t \geq 0^+$ and executes its zero-input response (Fig. 11.3-3(b)).

Therefore,

$$v_C(t) = v_R(t) = \frac{1}{C} e^{-t/\tau} \text{ V for } t \geq 0^+$$

$$i_R(t) = -i_C(t) = \frac{1}{RC} e^{-t/\tau} \text{ A for } t \geq 0^+.$$

EXAMPLE: 11.3-1

The value of I in the circuit in Fig. 11.3-4 is 3×10^{-6} C. Find the zero-state response for the current through the 10 k resistor in the direction marked in the figure.

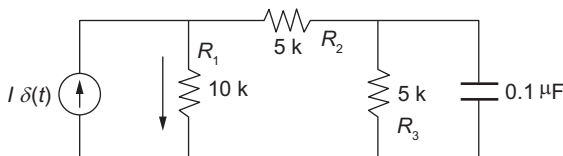


Fig. 11.3-4 Circuit for Example 11.3-1

SOLUTION

The voltage across the capacitor can, at best, change by a *finite* amount as a result of impulse current flow. This implies that the current through the 5 k resistor (R_3) across the capacitor cannot be an impulse. Therefore, the capacitor effectively shorts the 5 k resistor across it as far as the impulse current flow is concerned. Hence the $3 \times 10^{-6} \delta(t)$ gets shared by 10 k and the other 5 k (R_2) as per the current division principle in parallel resistors. Thus, $2 \times 10^{-6} \delta(t)$ goes through the other 5 k resistor (R_2) and the 0.1 μ F capacitor. This results in sudden dumping of 2 μ C of charge across the capacitor, raising its voltage to 20 V at $t = 0^+$. The current source goes open for $t \geq 0^+$ and the circuit executes its zero-input response.

Time constant of the circuit can be found by obtaining the equivalent resistance connected across the capacitor. The equivalent resistance is $5 \text{ k} // 15 \text{ k} = 3.75 \text{ k}$. Therefore, $\tau = 0.375 \text{ ms}$. The discharge current from the capacitor at 0^+ is $20 \text{ V} / 3.75 \text{ k}\Omega = 5.333 \text{ mA}$ and $20 \text{ V} / 15 \text{ k}\Omega = 1.333 \text{ mA}$ of which goes through the 10 k in the direction marked at that instant. Therefore, the zero-state response of this current for $I\delta(t)$ current excitation at input of the circuit = $1.333 e^{-2666.67t} \text{ mA}$.

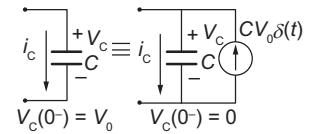


Fig. 11.3-2
Equivalence Between
Non-Zero Initial
Voltage and Impulse
Current Application in
 RC Circuits

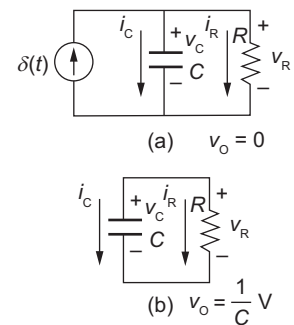


Fig. 11.3-3 Pertaining
to Unit Impulse
Response of Parallel
 RC Circuit

Time constant of a circuit containing single capacitor and multiple resistors can be determined by evaluating $R_{eq}C$, where R_{eq} is the equivalent resistance appearing across C after all independent sources have been deactivated.

Impulse Response and Step Response

'Impulse Response' is the short name for 'Zero-State Response with Unit Impulse Input'.

'Step Response' is the short name for 'Zero-State Response with Unit Step Input'.

If impulse response of a linear time-invariant circuit is known, its step response can be obtained by integrating its impulse response between the limits 0^+ and t .

This works since both impulse response and step response are zero-state responses. Note that only zero-state response gets integrated when input function gets integrated. Total response does not!

11.3.2 Step Response of First-Order RC Circuits

Step Response is understood to be the zero-state response of the circuit when unit step input is applied to it. Hence, the initial voltage across the capacitor is zero. It takes an impulse current flow through a capacitor to change its voltage by a non-zero finite amount instantaneously. Since there is no such impulse current flow in the present instance, the voltage across the capacitor remains zero at $t = 0^+$ too.

Let us consider the series *RC* circuit first. The applied voltage at $t = 0^+$ is 1 V and the voltage across the capacitor is constrained to remain at zero at that instant. And the circuit has to obey Kirchhoff's Voltage Law at that instant. This implies that the voltage across the resistor at that instant has to be 1 V and that a current of 1/R A has to flow through the circuit at that instant. Since the rate of growth of a capacitor voltage is given by the charging current divided by capacitance value, the rate of change of $v_C(t)$ at $t = 0^+$ will be $1/RC$ V/s. As the capacitor voltage increases, the voltage available across the resistor decreases, thereby bringing down the charging current in the circuit. Hence, the capacitor keeps charging up with progressively decreasing rate. This is a typical first-order process. The capacitor voltage tends to reach 1 V as $t \rightarrow \infty$ and correspondingly the current through the circuit tends to go to zero.

The detailed solution may be worked out by either '*complementary solution plus particular integral*' format or '*zero-input response plus zero-state response*' format. But we can do better than that. We have already worked out the impulse response of the series *RC* circuit in this section. And, we recollect that for a *lumped linear time-invariant* circuit, the zero-state response gets integrated when the input source function gets integrated. Unit step function is the integral of Unit impulse function. Therefore, the *step response* must be the integral of the *impulse response*.

Therefore,

$$\text{Impulse response, } v_C(t) = \frac{1}{RC} e^{-t/\tau} \text{ V for } t \geq 0^+, \text{ where } \tau = RC$$

$$\therefore \text{ Step response, } v_C(t) = \int_{0^+}^t \frac{1}{RC} e^{-t'/\tau} dt' = (1 - e^{-t/\tau}) \text{ V for } t \geq 0^+$$

$$\text{and } v_R(t) = 1 - v_C(t) = e^{-t/\tau} \text{ V for } t \geq 0^+$$

$$i_R(t) = i_C(t) = \frac{v_R(t)}{R} = \frac{1}{R} e^{-t/\tau} \text{ A for } t \geq 0^+.$$

Now consider the parallel *RC* circuit excited by a unit step current source. The voltage across the capacitor at $t = 0^+$ remains at zero. Therefore, the entire source current, i.e., 1 A has to flow through the capacitor. This results in charging up of capacitor with an initial charging rate of $1/C$ V/s. As the capacitor gets charged, the resistor takes its share of current and consequently the rate of rise of voltage comes down. We may now write down the circuit solution straightaway – $v_C(t)$ must be a rising exponential that tends towards R V, $i_C(t)$ must be a decreasing exponential starting at 1 A and $i_R(t)$ must be a rising exponential moving to 1 A. All of them will have the same time constant of $\tau = RC$ s.

$$\therefore \text{ Step response, } v_C(t) = R(1 - e^{-t/\tau}) \text{ V for } t \geq 0^+$$

$$i_C(t) = e^{-t/\tau} \text{ A and } i_R(t) = (1 - e^{-t/\tau}) \text{ A for } t \geq 0^+.$$

These step response waveforms are plotted in Fig. 11.3-5.

The zero-state responses in both cases contain a transient term (exponential in nature) and a steady-state response term (constant in nature). We had termed the steady-state response term as the *DC steady-state term* earlier. Since the current in a capacitor is proportional to the rate of change of its voltage, the only value of current such that both the voltage and the current in a capacitor remain constant in time is *zero*. It can have any constant voltage across it but its current is constrained to be zero under DC steady-state. Therefore, a capacitor may be replaced by an open-circuit for DC steady-state analysis.

A capacitor may be replaced by an open-circuit for the analysis of DC steady-state response.

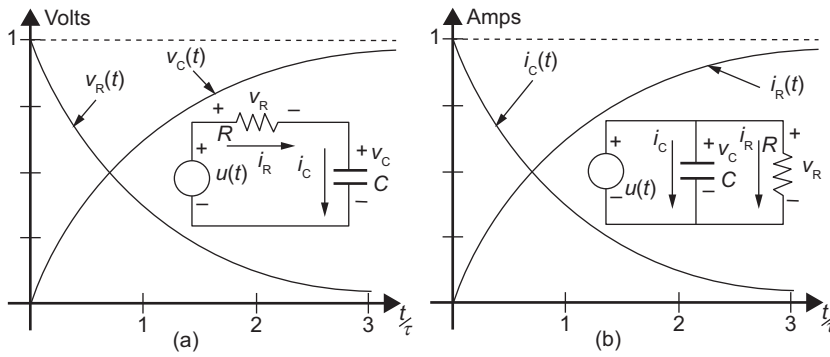


Fig. 11.3-5 Unit Step Response of RC Circuits (a) Series RC Circuit (b) Parallel RC Circuit

The DC steady-state current in a series RC circuit is zero. This implies that there is energy flow from the DC source only during the charging process. After the capacitor has charged up fully there is no energy drain from the source. Consider the charging of a capacitor to V V in a series RC circuit using a DC source of V V.

$$\text{Total energy delivered by the source} = \int_{0^+}^{\infty} V \times \frac{V}{R} e^{-t/\tau} dt = CV^2 \text{ J}$$

$$\begin{aligned} \text{Total dissipated by the resistor} &= \int_{0^+}^{\infty} R [i_R(t)]^2 dt \text{ J} \\ &= \int_{0^+}^{\infty} R \left[\frac{V}{R} e^{-t/\tau} \right]^2 dt \text{ J} \\ &= \frac{CV^2}{2} \text{ J.} \end{aligned}$$

Thus, the energy spent in charging up a capacitor to V V is CV^2 J, half of which appears in the capacitor as electrostatic energy storage. The remaining half gets dissipated in the charging resistor. *This conclusion is independent of the value of resistance of the resistor.* However, we should not stretch it to the case where $R = 0$. That is when all those *parasitic elements* that we neglected in modelling a real physical electrical device as a mathematical capacitance will start having their say in the matter.

11.3.3 Ramp Response of Series RC Circuit

The integration method is employed to arrive at the zero-state response to unit ramp input from the zero-state response to unit step input since unit ramp waveform is the integral of unit step waveform.

$$\text{Ramp response, } v_C(t) = \int_{0^+}^t (1 - e^{-t'/\tau}) dt' = t + \tau e^{-t'/\tau} \Big|_{0^+}^t = t - \tau(1 - e^{-t/\tau}) \text{ V for } t \geq 0^+.$$

Ramp waveform finds application in many contexts in electronics, instrumentation and signal processing. It is used as the internal time-base waveform in oscilloscopes. It is used in timing and counting applications too. Some dedicated electronic circuitry generates this ramp waveform and the generated waveform is conducted to the application circuit by means of a two-wire connection. This two-wire connection can often be modelled approximately by a series RC circuit where the resistor is contributed by the output

Power Loss in Repetitive Charging and Discharging of a Capacitor

Many practical applications in Electrical and Electronics Engineering involve periodic charging and discharging of capacitors in systems operating from DC power supplies. The capacitors may be present by design, or they may be parasitic.

$0.5CV^2$ J of energy is lost in the charging path every time a capacitor is charged to V V from zero volts. $0.5CV^2$ J of energy is lost every time a capacitor charged to V V is discharged to zero volts through a resistor or a passive path. Thus, every charge-discharge cycle results in CV^2 J of energy loss. If f is the frequency of operation, the resultant power loss will be CV^2f W.

This is a major factor that decides the heating in digital electronic gate circuits. This power loss along with the limited heat dissipation capability of the IC package place limitations on the frequency of operation of such devices. Especially in the case of a family of digital electronic gates called CMOS devices (see Problem 10 at the end of this chapter).

resistance of the generator circuit (i.e., the resistor that appears in the Thevenin's equivalent of the generator circuit output) and the capacitor is contributed by the input capacitance of the application circuit. The ramp waveform gets modified in this process of transmission from the output of generator circuit to the input of the application circuit. We observe from the above expression for ramp response of series RC circuit that the ramp waveform suffers two modifications – in the initial portion, i.e., for $t \ll \tau$, the output fails to follow the linear variation of the input and is rather prominently an exponential waveform. For $t \gg \tau$ the output follows the input; but with a constant difference of τ V. Hence, we may conclude that a series RC circuit will be able to transmit a ramp waveform more or less faithfully if the waveform duration is significantly larger than the time constant of the circuit. The waveforms in Fig. 11.3-6 show this clearly.

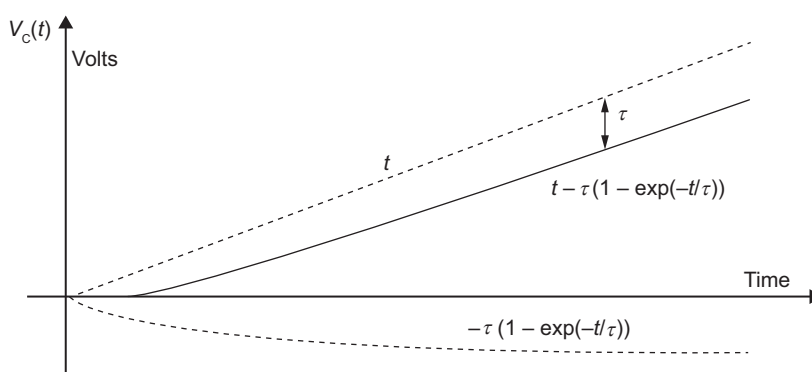


Fig. 11.3-6 Unit Ramp Response of Series RC Circuit

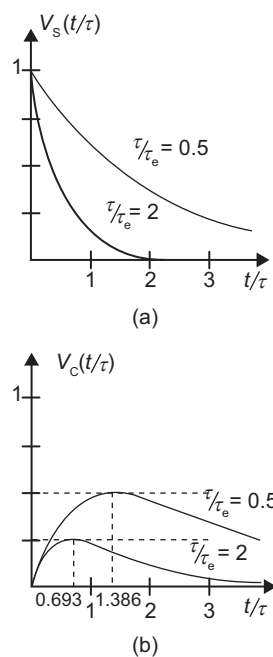


Fig. 11.3-7 Zero-State Response of RC Circuit for Exponential Input (a) Input Wave (b) Capacitor Voltage

11.3.4 Series RC Circuit with Real Exponential Input

Consider a series RC circuit excited at the input by a real exponential voltage source of the form $v_s(t) = e^{-\sigma t} u(t)$ V. The zero-state response under real exponential excitation in the case of a series RL circuit was described in Sect. 10.9 in Chap. 10. The reader is referred to Eqn. 10.9-5 in Chap. 10. The zero-state response of the capacitor voltage in the present case is written down by analogy from that equation as

$$v_C(t) = \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ V for } t \geq 0^+, \text{ where } \alpha = 1/RC \text{ and } \alpha \neq \sigma \quad (11.3-1)$$

This may also be expressed in terms of circuit time constant and excitation time constant as

$$v_C(t) = \left(\frac{1}{1 - \tau/\tau_e} \right) (e^{-(\tau/\tau_e)t_n} - e^{-t_n}) \text{ V for } t \geq 0^+,$$

where $\tau = RC$, $\tau_e = 1/\sigma$, $t_n = t/\tau$ and $\tau \neq \tau_e$.

The input source functions and the corresponding capacitor voltage waveforms are shown in Fig. 11.3-7. Time is normalised to the base of circuit time constant in this figure.

The case with $\alpha = \sigma$ was avoided in Chap. 10 with a promise that it will be taken up later in a different context. We do not avoid it any longer. The case with $\alpha = \sigma$ is taken up now as a limiting case of Eqn. 11.3-1 as $\alpha \rightarrow \sigma$.

$$\begin{aligned}
 v_C(t) &= \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ for } t \geq 0^+ \\
 &= \left(\frac{\alpha}{\alpha - \sigma} \right) \left[\left(1 - \sigma t + \frac{(\sigma t)^2}{2!} - \frac{(\sigma t)^3}{3!} + \frac{(\sigma t)^4}{4!} \dots \right) \right. \\
 &\quad \left. - \left(1 - \alpha t + \frac{(\alpha t)^2}{2!} - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} \dots \right) \right] \\
 &= \left(\frac{\alpha}{\alpha - \sigma} \right) \left[(\alpha - \sigma)t - \frac{(\alpha^2 - \sigma^2)}{2!}t^2 + \frac{(\alpha^3 - \sigma^3)}{3!}t^3 - \frac{(\alpha^4 - \sigma^4)}{4!}t^4 + \dots \right] \\
 &= (\alpha t) \left[1 + \sum_{i=2}^{\infty} (-1)^{i-1} \frac{(\alpha^i - \sigma^i)}{(\alpha - \sigma)i!} t^{i-1} \right] \\
 (\alpha^i - \sigma^i) &= (\alpha - \sigma) \underbrace{(\alpha^{i-1} + \alpha^{i-2}\sigma + \alpha^{i-3}\sigma^2 + \dots + \alpha\sigma^{i-2} + \sigma^{i-1})}_{i \text{ terms}} \\
 \therefore \text{As } \alpha \rightarrow \sigma, \frac{(\alpha^i - \sigma^i)}{(\alpha - \sigma)} &\rightarrow i \times \alpha^{i-1} \text{ and } v_C(t) \rightarrow (\alpha t) \left[1 + \sum_{i=2}^{\infty} (-1)^{i-1} \frac{i \times \alpha^{i-1}}{i!} t^{i-1} \right] \\
 \text{i.e., } v_C(t) &\rightarrow (\alpha t) \left[1 - \alpha t + \frac{(\alpha t)^2}{2!} - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} + \dots \right] \\
 \therefore v_C(t) &= \alpha t e^{-\alpha t} = \left(\frac{t}{\tau} \right) e^{-\left(\frac{t}{\tau} \right)} \text{ V for } t \geq 0^+.
 \end{aligned}$$

The wave-shape of the output voltage will be same as in Fig. 11.3-7. The output in this case will reach a maximum of $1/e = 0.3678$ V at $t = \tau$ s.

EXAMPLE: 11.3-2

Two first-order series RC circuits are cascaded using a unity gain buffer amplifier as shown in Fig. 11.3-8. Find the output $v_o(t)$ as a function of time if the circuit is initially relaxed.

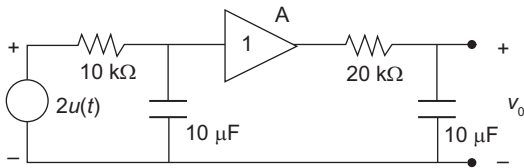


Fig. 11.3-8 Circuit for Example 11.3-2

SOLUTION

The unity gain buffer amplifier in between prevents any loading-interaction between the two RC circuits. This implies that the response of the first RC stage is independent of the presence of the second stage. The first stage produces a voltage across its capacitor that is accepted by second stage as its input source function as if it is coming from an ideal independent voltage source.

Let $v_1(t)$ be the response voltage at the terminals of the first capacitor. Then $v_1(t)$ is twice the zero-state response to unit step input (i.e., step response).

The time constant of first stage is $10 \text{ k}\Omega \times 10 \text{ }\mu\text{F} = 100 \text{ ms} = 0.1 \text{ s}$.

$$\therefore v_1(t) = 2(1 - e^{-10t}) \text{ V for } t \geq 0^+.$$

This voltage is the input to second stage since the gain of the buffer amplifier is unity. This input may be treated as the sum of two inputs: $2u(t)$ and $-2e^{-10t}u(t)$.

Zero-state response of a lumped linear time-invariant circuit obeys superposition principle and hence the responses to these inputs may be found individually and be superposed to get the desired response.

The time constant of second stage is $20 \text{ k}\Omega \times 10 \text{ }\mu\text{F} = 200 \text{ ms} = 0.2 \text{ s}$.

The component contributed to $v_o(t)$ by $2u(t)$ is $2(1 - e^{-5t}) \text{ V}$.

The contribution to $v_o(t)$ by $-2e^{-10t}u(t)$ is obtained by using Eqn. 11.3-1 with $\alpha = 5$ and $\sigma = 10$. This contribution is $-2 \times (e^{-10t} - e^{-5t}) \text{ V for } t \geq 0^+$.

Therefore,

$$\begin{aligned} v_o(t) &= 2(1 - e^{-5t}) + 2(e^{-10t} - e^{-5t}) \text{ V for } t \geq 0^+ \\ &= 2 - 4e^{-5t} + 2e^{-10t} \text{ V for } t \geq 0^+. \end{aligned}$$

The variation of capacitor voltages $v_1(t)$ and $v_2(t)$ as functions of time is shown in Fig. 11.3-9.

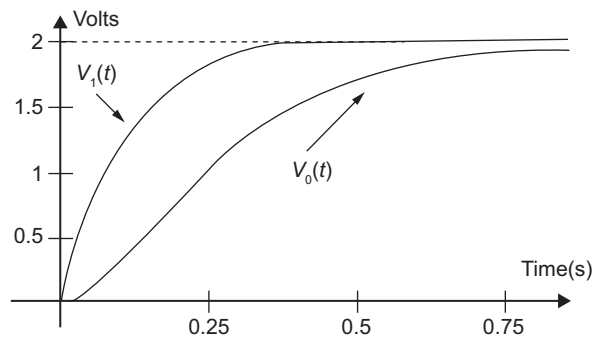


Fig. 11.3-9 Output Waveforms Across RC Stage Capacitors in Example 11.3-2

EXAMPLE: 11.3-3

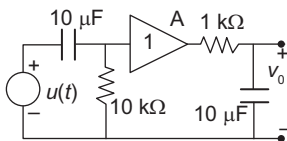


Fig. 11.3-10 Circuit for Example 11.3-3

Two first-order series RC circuits are cascaded non-interactively by employing a unity gain buffer amplifier as shown in Fig. 11.3-10. The voltage across the resistor of the first RC stage is the input to the second stage and the voltage across the capacitor of the second stage is the desired output. Find the step response of the system.

SOLUTION

Let $v_1(t)$ be the voltage across the $10 \text{ k}\Omega$ resistor in the first stage. We know that the zero-state step response of capacitor voltage in a series RC circuit is $(1 - e^{-t/\tau})$, where τ is the time constant of the circuit given by RC product. The voltage across capacitor and the voltage across resistor will have to add up to 1 V for all $t \geq 0^+$ in step response of a series RC circuit. Therefore, the step response of voltage across the $10 \text{ k}\Omega$ resistor is $[1 - (1 - e^{-t/\tau})] = e^{-t/\tau} \text{ V}$. The time constant of the first stage is 0.1 s.

Therefore, $v_1(t) = e^{-10t} \text{ V for } t \geq 0^+$.

This voltage is the input to the second stage. Its time constant is 0.01 s. We use Eqn. 11.3-1 to obtain $v_o(t)$ with $\alpha = 100$ and $\sigma = 10$.

$$\therefore v_o(t) = \left(\frac{100}{100 - 10} \right) (e^{-10t} - e^{-100t}) = 1.111(e^{-10t} - e^{-100t}) \text{ V for } t \geq 0^+$$

Note that the steady-state value of step response is zero. This is so because the first capacitor effectively opens the circuit for DC under steady-state. All the DC content of the source voltage will be found across the first capacitor in the steady-state.

EXAMPLE: 11.3-4

Show that the two circuits shown in Fig. 11.3-11 (a) and (b) have the same step response except for a sign change. The operational amplifier may be treated as an ideal one. Compare the currents drawn from the voltage source by the circuits.

SOLUTION

The time constant of the circuit in Fig. 11.3-11(a) is $100 \text{ ms} = 0.1 \text{ s}$ and its step response is $(1 - e^{-10t}) \text{ V}$ for $t \geq 0^+$.

Consider the circuit in Fig. 11.3-11(b). The Opamp is connected with negative feedback. Further, we assume that the input voltage applied is of such magnitude that the Opamp does not enter voltage saturation at its output. Moreover, we assume that the Opamp has sufficiently large slew rate capability such that it never enters rate limited operation. With these assumptions, we can analyse the Opamp using its ideal model.

The non-inverting terminal of Opamp is grounded and by *virtual short* principle the inverting input terminal is also *virtually grounded*. Therefore, the current that flows through R_1 is $u(t)/R_1$. Since the current into the input terminals of an ideal Opamp is zero, this current flows into the R_2/C combination connected in the feedback path of the Opamp. The voltage developed across this parallel combination is nothing but a scaled version of step response of a parallel RC circuit with step current excitation. The scaling factor is $1/R_1$. This step response is $R_2(1 - e^{-10t})$ since the time constant involved is 0.1 s . Therefore, the voltage developed across the parallel combination in the feedback path is $(R_2/R_1)(1 - e^{-10t}) \text{ V}$ with its positive polarity at the inverting input of Operational amplifier. Since the inverting input is at *virtual ground*, the voltage of output terminal with respect to ground (reference point) is the negative of this voltage (by KVL).

$$\therefore v_o(t) = -\frac{R_2}{R_1}(1 - e^{-10t}) = -(1 - e^{-10t}) \text{ V for } t \geq 0^+ \text{ since } R_1 = R_2 = 10 \text{ k}\Omega.$$

Therefore, the two circuits in Fig. 11.3-11 have the same step response (and hence the same dynamic behaviour) except for a change in sign.

The voltage across R in the circuit in Fig. 11.3-11(a) is $[1 - (1 - e^{-10t})] = e^{-10t} \text{ V}$ and therefore the current drawn from the unit step voltage source by this circuit is $0.1 e^{-10t} \text{ mA}$ for $t \geq 0^+$. But the current drawn by the second circuit is $u(t)/R_1 = 0.1 \text{ mA}$ for $t \geq 0^+$. Thus, the second circuit presents a constant input resistance level to the applied voltage source whereas the first circuit presents a time-varying input resistance level to the source. If the voltage source is not an ideal one, i.e., if it has a non-zero internal resistance, the time constant of circuit in Fig. 11.3-11(a) will change and hence the shape of its step response will change. However, the shape of step response will not change in the case of circuit in Fig. 11.3-11(b); but the initial magnitude will change due to change in the ratio (R_2/R_1) .

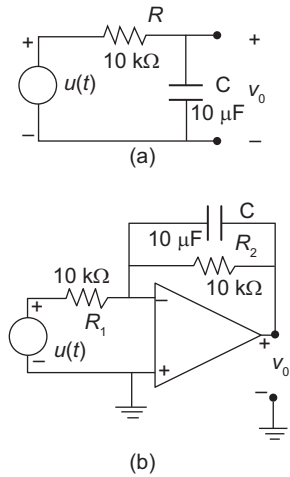


Fig. 11.3-11 Circuits for Example 11.3-4

11.3.5 Zero-State Response of Parallel RC Circuit for Sinusoidal Input

The zero-state response for sinusoidal input for any linear time-invariant circuit can be obtained in three ways.

- (1) Let the input be $\sin(\omega t) u(t)$. Obtain the zero-state response of the circuit for a complex exponential input function e^{st} , where s is a complex number. Substitute $s = j\omega$ in the solution and accept the *imaginary part of the solution* as the zero-state response for $\sin(\omega t) u(t)$.
- (2) Express $\sin(\omega t)$ as $[(e^{j\omega t} - e^{-j\omega t})/2j]$ by using Euler's formula, get the zero-state responses for the two exponential functions separately and use superposition principle.
- (3) Use phasor method to obtain the sinusoidal steady-state response and add a transient response term such that the total response satisfies initial conditions. Initial conditions will be zero-valued since we are dealing with zero-state response.

The first two methods were already illustrated in the context of sinusoidal response of RL circuits in Chap. 10. We use the third method here to obtain the zero-state response

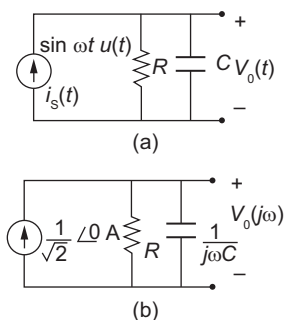


Fig. 11.3-12 (a) Parallel RC Circuit with Sinusoidal Excitation and (b) its Phasor Equivalent

for the voltage appearing across a parallel RC circuit excited by a sinusoidal current source with source function of $\sin(\omega t) u(t)$ A.

The circuit in time-domain and phasor-domain are shown in Fig. 11.3-12(a) and (b), respectively.

$$\begin{aligned} V_0(j\omega) &= R // \left(\frac{1}{j\omega C} \right) \times \frac{1}{\sqrt{2}} \angle 0 = \frac{R}{1 + j\omega RC} \times \frac{1}{\sqrt{2}} \angle 0 \\ &= \frac{R}{1 + jk} \times \frac{1}{\sqrt{2}} \angle 0; k = \omega RC = \omega\tau \\ &= \frac{R}{\sqrt{1+k^2}} \angle -\phi \times \frac{1}{\sqrt{2}} \angle 0; \phi = \tan^{-1} k. \end{aligned}$$

Going back to time-domain, we get the steady-state component of voltage across the parallel RC circuit as

$$\frac{R}{\sqrt{1+k^2}} \sin(\omega t - \tan^{-1} k) \text{ with } k = \omega\tau.$$

Now, we add a transient component of known form $Ae^{-t/\tau}$ and evaluate A such that the total solution is zero at $t = 0^+$. We get,

$$A = \frac{R}{\sqrt{1+k^2}} \sin(\tan^{-1} k)$$

$$\text{Since } \tan^{-1} k \text{ lies in the first quadrant, } \sin(\tan^{-1} k) = \frac{k}{\sqrt{1+k^2}}.$$

$$\therefore A = \frac{R}{\sqrt{1+k^2}} \frac{k}{\sqrt{1+k^2}} \text{ and}$$

$$v_0(t) = \frac{R}{\sqrt{1+k^2}} \left(\sin(\omega t - \tan^{-1} k) + \frac{k}{\sqrt{1+k^2}} e^{-t/\tau} \right) \text{ for } t \geq 0^+.$$

We normalise the time variable using the circuit time constant as the base and the output voltage by using the value of R as the base value and obtain the following expression for normalised voltage $v_{on}(t)$ as a function of normalised time t_n .

$$v_{on}(t) = \frac{1}{\sqrt{1+k^2}} \left(\sin(kt_n - \tan^{-1} k) + \frac{k}{\sqrt{1+k^2}} e^{-t_n} \right) \text{ for } t_n \geq 0^+, \quad (11.3-2)$$

$$\text{where } v_{on}(t) = \frac{v_0(t)}{R} \text{ and } t_n = \frac{t}{\tau}.$$

This waveform for a case with $k = 4$ is shown in Fig. 11.3-13.

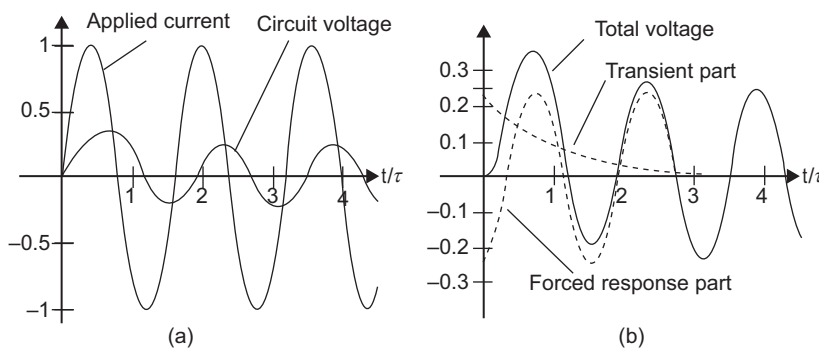


Fig. 11.3-13 Unit Sinusoidal Response (Normalised) of a Parallel RC Circuit with $k = 4$

We had noted under a similar context (Sect. 10.9 in Chap. 10 on *RL* circuits) that the number k can be interpreted as a comparison between the characteristic time – i.e., the period of the applied input and the characteristic time of the circuit – i.e., its time constant. k can be expressed as $2\pi(\tau/T)$, where T is the period of input. The value of T is indicative of the rate of change involved in the waveform, i.e., the *speed* of the waveform. Time-constant is a measure of inertia in the system. Therefore, an input sinusoid is too fast for a circuit to follow if its T is smaller than the time constant τ of the circuit. Similarly, if input sinusoid has a T value much larger than time constant of the circuit, the circuit will perceive it as a very slow waveform and will respond almost the same way it does to DC input. These aspects are clearly brought out in the expression in Eqn. 11.3-2.

We make the following observations on the sinusoidal steady-state response of parallel *RC* circuit with *current source excitation* from Eqn. 11.3-2.

- The circuit voltage under sinusoidal steady-state response is a sinusoid at the same angular frequency ω rad/s as that of input current sinusoid.
- The circuit voltage initially is a mixture of an exponentially decaying unidirectional transient component along with the steady-state sinusoidal component. This unidirectional transient imparts an offset to the voltage during the initial period.
- The circuit voltage at its first peak can go close to twice its steady-state amplitude in the case of circuits with $\omega\tau \gg 1$ due to this offset.
- The amplitude of sinusoidal steady-state response is always less than corresponding amplitude when a DC input is applied. This is due to the capacitive inertia of the circuit. When input is a current, capacitance in a circuit behaves as electrical inertia, and, when input is a voltage, inductance in a circuit behaves as electrical inertia. The amplitude depends on the product $\omega\tau$ and decreases monotonically with the $\omega\tau$ product for fixed input amplitude.
- The response sinusoid (voltage) lags behind the input sinusoid (current) under steady-state conditions by a phase angle that increases monotonically with $\omega\tau$.
- The frequency at which the circuit gain becomes $1/\sqrt{2}$ times that of DC gain is termed as a *cut-off frequency* and since this takes place as we go up in frequency it is called *upper cut-off frequency*. Upper cut-off frequency of parallel *RC* circuit is at $\omega = 1/\tau$ rad/s. The phase at this frequency will be -45° .
- Circuit voltage amplitude becomes very small at high frequencies ($\omega\tau \gg 1$) and the voltage lags the input current by $\approx 90^\circ$ at such frequencies.

Time-constant of a circuit is a measure of inertia in the circuit.

If the typical time of variation of input is small compared to the time constant of the circuit the circuit perceives the input as fast and the response will be small-valued.

If the time of variation of input is large compared to time constant, the circuit perceives such input as slow and responds almost the same way as it responds to DC input.

Definition of upper cut-off frequency

11.4 PERIODIC STEADY-STATE IN A SERIES RC CIRCUIT

We address the issue of zero-state response to *repetitive input* in *RC* circuits in this section and consider a specific example for doing so (Fig. 11.4-1).

A symmetric square wave voltage with 1 V amplitude is applied from $t = 0$ to an initially relaxed series *RC* circuit. The period of the square wave is assumed to be equal to the time constant and the time scale is marked in terms of t/τ .

A step response starts at $t = 0$ and takes the output to $(1 - e^{-0.5}) = 0.3935$ V at $t/\tau = 0.5$. Another zero-state step response starts in the negative direction at that point along with a zero-input response corresponding to an initial voltage of 0.3935 V on the capacitor. Thus, the output waveform can be expressed as $= 0.3935 e^{-(t-0.5\tau)/\tau} - (1 - e^{-(t-0.5\tau)/\tau})$ for the time range $0.5 \leq t/\tau \leq 1$. This expression may be evaluated at $t/\tau = 1$ to get the initial condition at that point and a new expression valid for $1 \leq t/\tau \leq 1.5$ may be obtained as a superposition of zero-input response and a new zero-state step response. This way the solution may be taken forward.

Periodic Steady-State

When a *repetitive input* is applied to a linear time-invariant circuit, the output is a mixture of the circuit natural response terms and forced response terms during the initial few cycles of operation.

However, after a few cycles, the circuit settles down to a mode of operation in which the output starts with a value at the beginning of the cycle and returns to the same value at the end of that cycle only to repeat that process again in the next cycle.

When the circuit reaches this kind of *repetitive operation* under the influence of any *repetitive input*, it is said to have reached a *periodic steady-state* with respect to that input.

Sinusoidal steady-state is a special case of periodic steady-state.

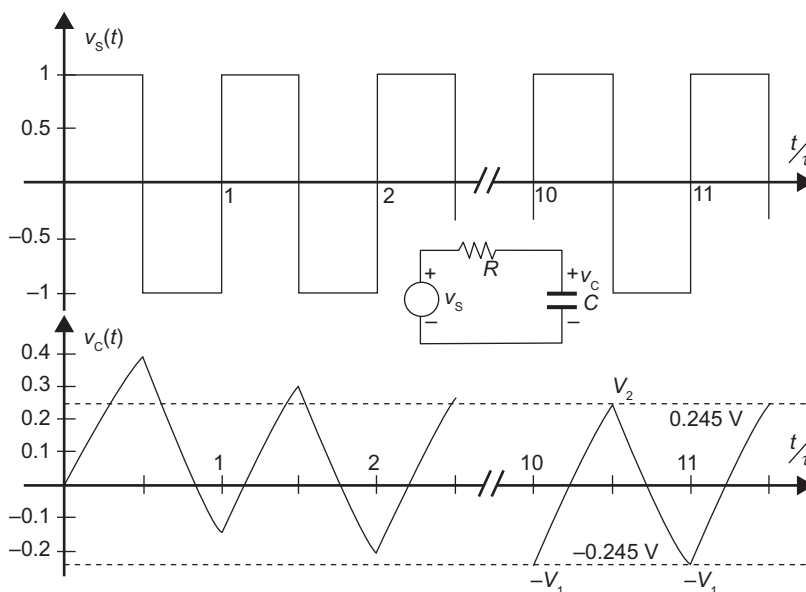


Fig. 11.4-1 Series RC Circuit with Repetitive Square Wave Input

It may be observed that the values of capacitor voltage at the beginning and at the end of a cycle are not the same in the first few cycles of operation. However, after a few cycles, the circuit settles down to a mode of operation in which the output starts with a particular value at the beginning of the cycle and returns to the same value at the end of that cycle only to repeat that process again in the next cycle. This means that the output also reaches a *repetitive pattern* after a few initial cycles. When the circuit reaches this kind of *repetitive operation* under the influence of any *repetitive input*, it is said to have reached a *periodic steady-state* with respect to that input. Figure 11.4-1 shows that the capacitor voltage oscillates between 0.245 V and -0.245 V under periodic steady-state in the present instance where T – the period of input – has been taken to be equal to τ , the time constant of the circuit.

It is not necessary to start at the beginning and proceed further till the circuit reaches the periodic steady-state in order to find out the amplitude under steady-state condition. We can proceed in the following manner to develop an expression for this amplitude.

Let $-V_1$ and $+V_2$ be the negative and positive amplitudes in a cycle as shown in Fig. 11.4-1. Then the circuit would have reached steady-state when the output value at the end of the cycle turns out to be exactly $-V_1$.

$$v_c(t) = -V_1 e^{-t/\tau} + \left(1 - e^{-t/\tau}\right) \text{ for } 0 \leq t \leq T/2 \text{ with } t \text{ measured}$$

from the beginning of cycle after the circuit has reached periodic steady-state.

$$\therefore V_2 = -V_1 e^{0.5T/\tau} + \left(1 - e^{0.5T/\tau}\right)$$

$$v_c(t) = V_2 e^{-(t-0.5T)/\tau} - \left(1 - e^{-(t-0.5T)/\tau}\right) \text{ for } T/2 \leq t \leq T.$$

This expression evaluated with $t = T$ should be equal to $-V_1$ under periodic steady-state.

If one period of the repetitive input is piecewise linear, it is possible to calculate the periodic steady-state by employing the method illustrated here for a square wave input.

$$\therefore -V_1 = V_2 e^{-0.5T/\tau} - \left(1 - e^{-0.5T/\tau}\right)$$

Substituting for V_2 in terms of V_1 and solving for V_1 , we get,

$$V_1 = \frac{1 - e^{-T/2\tau}}{1 + e^{-T/2\tau}} \text{ and } V_1 = V_2.$$

This expression evaluated with $T/\tau = 1$ gives $V_1 = V_2 = 0.245$ V for a 1 V amplitude square wave input.

The key to the above derivation was our knowledge of step response of series RC circuit. The input could be thought of as a sequence of unit steps and hence the output could be strung together employing step response and zero-input response segments. Square waves and more generalised versions of it (the so-called *rectangular pulse waveforms* that are ‘squirish’ waveforms with unequal half-cycle duration and unequal positive and negative amplitudes) appear very frequently in Pulse Electronics Applications and Digital Electronics. And, series RC circuits are routinely used to model the transmission channel that takes such signals from one location to another location in the electronic system. Hence, the periodic steady-state of series RC circuit under rectangular pulse waveforms is of crucial significance in Analog and Digital Electronics. This is the motivation behind this section on periodic steady-state.

The method described above for finding steady-state amplitudes under repetitive excitation will work only if we can identify the repetitive waveform as a sequence of some well-known shape like a step or ramp or sinusoid. However, in practice, we will be called upon to solve for periodic steady-state even when the period of input is of a complex shape. How do we proceed? The answer lies in *frequency response* of the circuit.

11.5 SINUSOIDAL STEADY-STATE FREQUENCY RESPONSE OF FIRST-ORDER RC CIRCUITS

The concept of *sinusoidal steady-state frequency response* was already introduced in Chap. 10 in the context of RL circuits. Essentially, we apply a sinusoidal input of suitable amplitude to a circuit and wait for enough time for the transient response to die down. After steady-state is satisfactorily established in the circuit, we measure the amplitude of output and its phase with respect to the input sine wave. We repeat this process for various values of frequency of input. We ensure that the circuit is in steady-state before we measure the output every time. The data so obtained is plotted to show the variation of ratio of output amplitude to input amplitude and phase of steady-state voltage against $k (= \omega\tau)$. Such a pair of plots will constitute what is called the AC *steady-state frequency response plots* for this circuit. The ratio of output amplitude to input amplitude is called the *gain* of the circuit. Its dimension will depend on the nature of input and output quantities.

The same data can be obtained from the analytical model of the circuit if such a model exists. Consider the series RC circuit and its phasor model shown in Fig. 11.5-1.

Frequency response function,

$$\begin{aligned} H(j\omega) &= \frac{V_o(j\omega)}{V_s(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} \\ &= \frac{1}{1 + j\omega\tau} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \angle -\tan^{-1}(\omega\tau). \end{aligned} \quad (11.5-1)$$

The Gain and Phase plots for the circuit are shown in Fig. 11.5-2. The gain goes to 70.7% level at $\omega = 1/\tau$ rad/s and phase delay at that frequency is 45° .

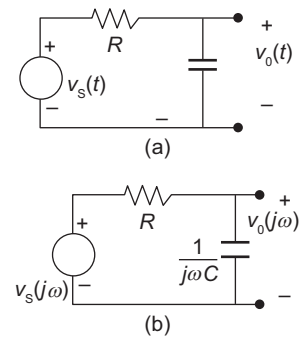


Fig. 11.5-1 Series RC Circuit and its Phasor Model

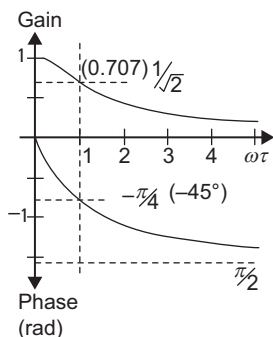


Fig. 11.5-2 Frequency Response Plots for Series RC Circuit

Steady-state response of a linear time-invariant circuit for a mixture of sinusoidal inputs at different frequencies can be obtained by applying superposition principle.

11.5.1 The Use of Frequency Response

Frequency Response information helps us to find the *steady-state output* when the input is a *mixture of sinusoids of different frequencies*. A sinusoid with a particular angular frequency is a periodic waveform. But a sum of many sinusoids with arbitrary frequencies need not be periodic. A special case where the sum of many sinusoids results in a periodic waveform is the one in which the sinusoidal components are of frequencies which are related *harmonically* – i.e., when all the frequencies are integer multiples of some basic frequency value. This is an important practical case.

Before we proceed to employ frequency response to solve circuits excited by sum of sinusoids, let us settle an issue regarding superposition principle. We know that zero-state responses due to multiple sources acting simultaneously can be obtained by superposition. But does it work for steady-state response component too?

Zero-state response contains two components – the transient response part and the steady-state response part. The transient response components, whether from zero-state response or zero-input response, will vanish with time if the circuit is passive and stable. Therefore, superposition principle can be applied on steady-state response components.

Now, if the input contains many sinusoids with different frequencies, the steady-state response component due to each sinusoid may be obtained from frequency response plots and these components may be added up so as to obtain the complete steady-state response. This procedure is illustrated in the case of a series RC circuit with a time constant of 1 s and with an input of $v_s(t) = (\sin t + 0.33 \sin 3t + 0.2 \sin 5t) u(t)$ V. The output is taken across the capacitor.

Consider the first sinusoid that has an angular frequency of 1 rad/s. The gain of the circuit at this frequency is 0.707 and phase delay is 45° (0.79 rad) (either from Eqn. 11.5-1 or from Fig. 11.5-2 with $\tau = 1$ s). Therefore, the steady-state component due to this sinusoid is $0.707 \sin(t - 0.79)$ V. The second sinusoid of $0.33 \sin 3t$ with an angular frequency of 3 rad/s meets with a gain of 0.3162 and phase delay of 71.57° (1.25 rad). Therefore, the steady-state component due to this sinusoid is $0.104 \sin(3t - 1.25)$ V. Similarly, the third sinusoid of $0.2 \sin 5t$ with an angular frequency of 5 rad/s meets with a gain of 0.1961 and phase delay of 78.7° (1.37 rad). Therefore, the steady-state component due to this sinusoid is $0.04 \sin(5t - 1.37)$ V. Therefore, $v_o(t) = 0.707 \sin(t - 0.79) + 0.104 \sin(3t - 1.25) + 0.04 \sin(5t - 1.37)$ V. The input and the output waveforms are shown in Fig. 11.5-3.

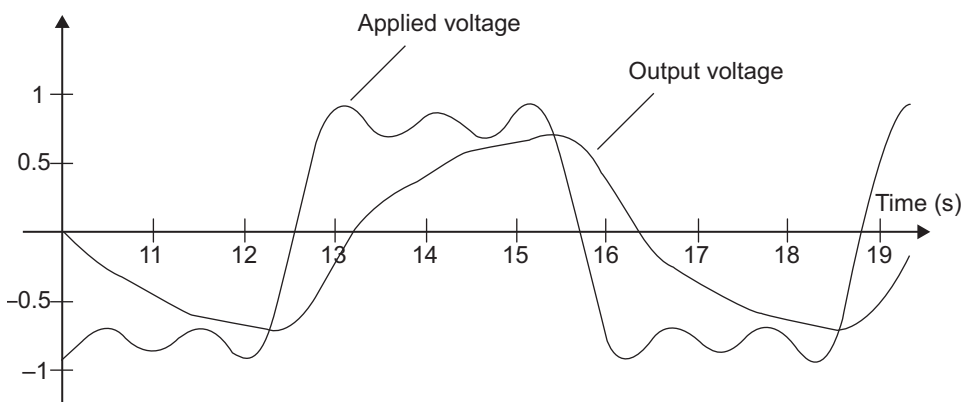


Fig. 11.5-3 Steady-State Response of Series RC Circuit for Mixed Sinusoidal Input

11.5.2 Frequency Response and Linear Distortion

We observe that our series RC circuit has meted out differential treatment to various sinusoids in the input mixture. It has shown a clear preference to the sinusoid with lowest frequency (with 1 rad/s) and passed it on with only about 30% loss of amplitude whereas the remaining two sinusoids with 3 rad/s and 5 rad/s frequencies suffered 68.5% and 80% loss of amplitude, respectively. Circuits that preferentially pass low frequency sinusoids to the output and curtail high frequency sinusoids are called *low-pass filters*. Low-pass filters will have a frequency response with a gain function that tapers down to zero as frequency goes up. Thus, a series RC circuit with output taken across the capacitor is a first-order low-pass filter. It shows a tendency to remove high frequency components in the input.

Further, we observe from Fig. 11.5-3 that the wave-shape of output voltage is considerably different from that of input. This is inevitable in a filtering context. After all, some frequency components get removed or attenuated considerably in a filtering process and therefore the output cannot but look different compared to input. When the wave-shape of output under steady-state in a circuit is different from the wave-shape of input, the circuit is said to have *distorted* the signal. Thus, *distortion* invariably follows *filtering*. When the change in wave-shape is the desired outcome we call it *filtering*; when the change in wave-shape is the undesired outcome we call it *distortion*.

This distortion of wave-shape arises out of two reasons. Sinusoids at different frequencies meet with different gains in the circuit and therefore the mix of amplitudes, i.e., the relative ratio of amplitudes of various sinusoids, will be different at output and input. In the example we considered, the ratio was 1:0.33:0.2 at input and 1:0.147:0.057 at the output. Wave-shape changes due to this change in amplitude mix. Distortion arising out of this mechanism is called *amplitude distortion* and it is due to the gain response part of the frequency response.

The second cause of distortion comes from phase response. Each sinusoid suffers a time delay when it goes through the circuit – the time delay is measured between zero crossing of that sinusoid in the input and in the output. Phase delay is equal to time delay multiplied by angular frequency. Thus, the 1 rad/s component in the previous example underwent a delay of 0.79 s, the 3 rad/s component suffered a delay of $1.25 \text{ rad}/3 \text{ rad/s} = 0.42 \text{ s}$ and the 5 rad/s component was subjected to a delay of $1.79 \text{ rad}/5 \text{ rad/s} = 0.36 \text{ s}$.

All the three cross the time-axis simultaneously at the input. But at the output they do not cross the time-axis simultaneously – the 5 rad/s crosses first followed by the 3 rad/s component and the 1 rad/s component is the last one to cross time-axis. Thus, they get *dispersed*. This *dispersion* results in change in wave-shape. The three components in the input are shown in Fig. 11.5-4(a) and the corresponding components in the output are shown in Fig. 11.5-4(b).

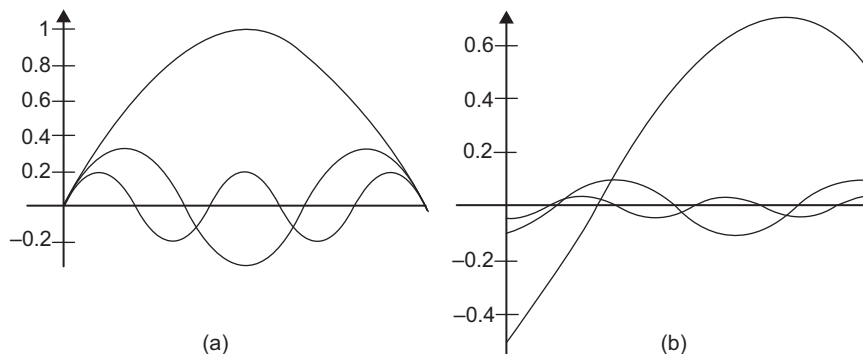


Fig. 11.5-4 Illustrating Phase Distortion Due to Dispersion

Distortion or Filtering?

Dynamic circuits provide differential treatment to various sinusoids when input is a mixture of sinusoids at different frequencies.

Such differential treatment makes the output wave-shape different from input wave-shape.

Such a change in the wave-shape is called *distortion* or *filtering* depending upon the application context.

Amplitude distortion due to frequency response.

Phase distortion from waveform dispersion due to unequal phase delays in frequency response.

The dispersion in zero-crossing instants is clearly brought out as in Fig. 11.5-4(b). The distortion resulting from dispersion of components brought about by *unequal time delays* suffered by the components in going through the circuit is termed as *phase distortion*. Of course, in any distortion context these two – amplitude distortion and phase distortion – are mixed up and cannot be separated.

Phase distortion arises essentially due to phase response part of frequency response. If all the sinusoids are delayed by *same time delay* there will be no change in wave-shape (assuming there is no amplitude distortion). The entire input wave-shape will get bodily shifted in time-axis by a definite delay and will appear as output in that case. Therefore, either zero time delay for all frequencies or constant time delay for all frequencies will prevent phase distortion. A constant time delay implies that the phase delay must be a *linear function of ω* .

Conditions for no waveform distortion.

The conditions to be satisfied by a circuit such that there is no wave-shape distortion when a signal passes through it must be evident now – *its frequency response must have a gain that is flat with ω and a phase which is either zero or linear on ω , i.e., of the form $\phi = -k\omega$, where k is a real number*. Obviously only a memory-less circuit can satisfy this. Hence, a circuit that contains at least one inductor or capacitor will cause wave-shape distortion in general. Similarly, we conclude that a memory-less circuit cannot function as a *filter*; we will need inductors and capacitors to make filters.

We observe that, in the example we analysed in this section, the input contained three sinusoids of 1 rad/s, 3 rad/s and 5 rad/s and the output contained exactly three sinusoidal components with the same frequencies as in the input. In short, the circuit did not change the frequency of sinusoids. Neither did it generate a sinusoid with a frequency that was not there in the input. This, in fact, is a property of any lumped linear time-invariant (LLTI) system. They can only scale, differentiate or integrate signals. And these three mathematical operations cannot produce a sinusoid with a frequency that is different from that of input. *Therefore, a single frequency sinusoid cannot suffer wave-shape distortion in passing through a linear time-invariant circuit.*

Linear distortion versus non-linear distortion in Circuits.

However, a non-linear circuit can change the wave-shape of a single frequency sinusoid. Apply about 10 mV of 1 kHz sinusoidal voltage to a 741 Operational Amplifier non-inverting pin after grounding the inverting pin. The Operational Amplifier is in the open loop and its large gain results in output getting saturated. We will observe a waveform that is almost a square wave at the output. That is *non-linear distortion*. The wave-shape distortion we observed in the example in this section was not due to non-linearity. It occurred due to differential treatment experienced by various sinusoidal components in a mixture of sinusoids when they went through the circuit. The distortion that occurs due to frequency response of a linear circuit is termed as *linear distortion* in order to distinguish this kind of distortion from distortion due to non-linearity. *Amplitude Distortion* and *Phase Distortion* are the two inseparable components of *Linear Distortion*.

Different filtering functions may be available from the same circuit.

One should not be under the impression that the series *RC* circuit can function only as a low-pass filter. In fact, the kind of filter realised by a given circuit will strongly depend on where exactly is the input applied and where exactly is the output taken. A series *RC* circuit excited by a voltage source at the input of the series combination with output taken across the capacitor is a low-pass filter. The same circuit with same excitation but with output taken across the resistor is a high-pass filter that passes the high frequency sinusoids to the output and curtails the low frequency components including DC.

11.5.3 Jean Baptiste Joseph Fourier and Frequency Response

We used $v_s(t) = (\sin t + 0.33 \sin 3t + 0.2 \sin 5t) u(t)$ as an input to a series *RC* circuit in Sects. 11.5.1 and 11.5.2 which dealt with the use of frequency response to solve for steady-state response when input is a mixture of sinusoids of different frequencies. We observe

from Fig. 11.5-3 that $v_s(t)$ is a periodic wave. This is so because all the three sinusoids involved are periodic with a period of 2π s. True, the sinusoid with 3 rad/s frequency is periodic with a period of $2\pi/3$ s. But if it is periodic with a period of $2\pi/3$ s, it is periodic with a period of 2π s too since there will be three full cycles of it in that duration. Therefore, in general, the sum of sinusoids with frequencies which are integer multiples of some basic frequency (called fundamental frequency) will be a periodic waveform with period corresponding to the fundamental frequency.

i.e., $v_s(t) = \sum_{n=0}^N a_n \cos(n\omega_0 t) + \sum_{n=1}^N b_n \sin(n\omega_0 t)$ will be a periodic wave with frequency

of ω_0 rad/s for any finite value of N , a_n and b_n .

For a particular value of ω_0 and for each choice of N , we can vary the $2N + 1$ numbers – a_n and b_n – to synthesise infinite number of distinct periodic waveforms of periodicity $2\pi/\omega_0$ s. Each combination of these $2N + 1$ numbers over real number field will result in a unique periodic waveform. And, we get yet another set of infinite unique periodic waveforms for another choice of N .

That prompts a question – given a non-sinusoidal periodic waveform $v_s(t)$ with angular frequency ω_0 rad/s, can we find some N and a set of $2N + 1$ real numbers for a_n and b_n such that $v_s(t) = \sum_{n=0}^N a_n \cos(n\omega_0 t) + \sum_{n=1}^N b_n \sin(n\omega_0 t)$ at all instants?

Jean Baptiste Joseph Fourier answered this question in the beginning of 19th century. Part of the answer is that N has to go to infinity. As long as N is finite, we are assured of convergence of this series. But with $N \rightarrow \infty$, the series may not converge to a finite value for all values of t for a given choice of a_n and b_n values. Fourier's Theorem assures us that a broad class of periodic waveforms satisfying certain conditions can be expanded in the form $\sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$ without any convergence problem. This series is called the Fourier Series and the reader surely must have come across this in Mathematics courses. The conditions to be satisfied by a periodic waveform for a Fourier Series to exist are fortunately not very stringent and almost all the practical waveforms used in circuits satisfy them.

We will look at Fourier Series in detail in a later chapter. For the present purpose, it suffices to understand that a periodic non-sinusoidal waveform can be thought of as a sum of a DC component (which may be zero as a special case) and infinitely many sinusoids (may be finite as a special case) with harmonically related frequencies. Therefore, the periodic steady-state solution in a circuit excited by a non-sinusoidal periodic input can be obtained by using the Fourier Series of the waveform along with the frequency response data for the circuit. That makes frequency response an extremely important mathematical description of a linear circuit.

11.5.4 First-Order RC Circuits as Averaging Circuits

A series RC circuit with voltage excitation and output taken across the capacitor is a low-pass filter. Similarly, a parallel RC circuit with current excitation and output taken across the parallel combination is also a low-pass filter.

Averaging is a signal processing application that appears often in analog and digital signal processing. The waveform to be averaged is typically a rectangular pulse waveform with a slowly varying DC content. An *averaging circuit* produces an output that is the DC content of the input signal.

Consider the rectangular pulse waveform shown in Fig. 11.5-5(a). It has amplitude of 1 V, a frequency of 1 kHz and a duty ratio of 0.2. This waveform can be thought of as the sum of a DC voltage of value 0.2 V and a pure alternating waveform with zero full-cycle area (i.e., zero DC content). These two components are shown in Fig. 11.5-5(b).

The periodic steady-state response of a linear time-invariant circuit to most of the commonly used periodic (but non-sinusoidal) inputs can be obtained by using frequency response data and Fourier series expansion for the input waveform.

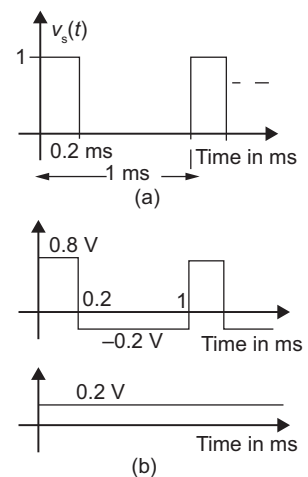


Fig. 11.5-5 A Rectangular Pulse Waveform and its Components

Assume that this waveform is applied to a series RC circuit with $R = 10\text{ k}\Omega$ and $C = 1\text{ }\mu\text{F}$ from $t = 0$. The time constant is 10 ms . Output is taken across the capacitor. The total zero-state response can be obtained by using superposition principle. Therefore, we expect the standard step response scaled by 0.2 to be present along with other components in the output. This step response component will reach steady-state in about 5τ , i.e., in 50 ms and then contribute 0.2 V steady component to the output.

The pure alternating component of the applied voltage will also reach a periodic steady-state at the output after about 50 ms . We remember that this alternating component can be thought of as the sum of infinitely many sinusoids of frequencies that are integer multiples of 1 kHz . The lowest frequency component will be 1 kHz . The phasor impedance of capacitor and the resistor share the sinusoidal voltage under steady-state and we are interested in the voltage absorbed by the capacitor. Let us calculate the phasor impedance of capacitor at 1 kHz . It is $-j159.2\text{ }\Omega$. We see that this is only about 1.6% of the resistor value ($10\text{ k}\Omega$) and hence we expect the capacitor to absorb only a very small percentage of sinusoidal voltage at 1 kHz – most of it will appear across the resistor. This will be higher for other sinusoidal components with frequencies higher than 1 kHz since phasor impedance of capacitor goes down with frequency.

As a first approximation, we assume that the alternating component that appears across capacitor is negligible when we calculate the alternating component of current in the circuit. Thus, the alternating component of applied voltage is assumed to appear across $10\text{ k}\Omega$ almost entirely, thereby resulting in a current whose wave-shape will be same as that of alternating component of voltage. This current will vary between 0.08 mA and -0.02 mA . Now we work out the small voltage that appears across capacitor due to this alternating current flow.

The half-cycle area of this current is $0.08\text{ mA} \times 0.2\text{ ms}$ (or $0.02\text{ mA} \times 0.8\text{ ms}$) = $0.016\text{ }\mu\text{C}$. Hence, the peak-to-peak voltage across the capacitor due to the alternating current flow will be $0.016\text{ }\mu\text{C}/1\text{ }\mu\text{F} = 0.016\text{ V}$. Hence, the capacitor voltage will vary in the range $0.2 \pm 0.008\text{ V}$. The variation is $\pm 4\%$ of the desired average value of 0.2 V .

This approximate solution is confirmed by the accurate solution worked out using the method to solve for periodic steady-state explained earlier in this section. This is shown in Fig. 11.5-6. The output waveform segments are actually exponential; but they appear nearly straight-line segments confirming the validity of assumption employed in our approximate reasoning.

We should never forget that a circuit reaches steady-state only after covering the transient period. This reasonably clean average value appears only after 50 ms of applying the input. We can increase the time constant of the circuit to higher levels in order to make the output appear cleaner; but there is a price to pay. The cleaner output will take longer to establish. The average value of input is not likely to remain constant forever in a practical application of averaging circuit. In fact, this value may be used to code some

RC Averaging Circuit

First-order RC circuits can be used as averaging circuits. However, there is a conflict between quality of averaged output and speed of response of the circuit.

Increasing the time constant of the circuit will result in better averaging. However, a large time constant will make the circuit very slow in following changes in the DC content of input

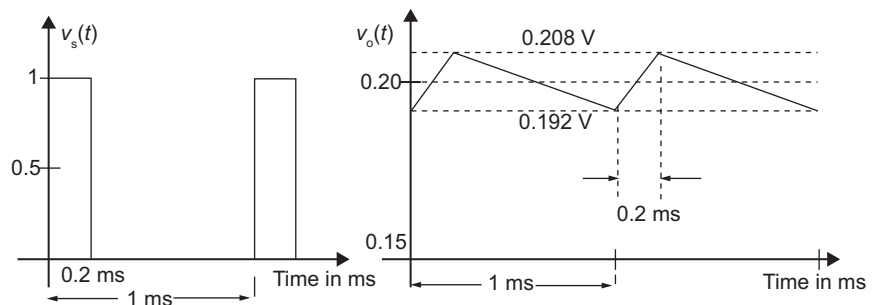


Fig. 11.5-6 Input and Output Waveforms of Series RC Averaging Circuit

information and will consequently change slowly. Typically, the duty ratio of the input wave changes slowly while its frequency is kept constant. And the averaging circuit is expected to track the change in average value faithfully. Obviously, there is a conflict between the requirement of a clean output and requirement of fast response to changing DC content at input. The time constant of the circuit must be selected in such a way that it has enough speed to catch up with the average value variation. And, if the ripple in the output is excessive with that value of time constant, we better look for some other better technique to do averaging.

Parallel RC circuit can be employed for averaging current signals subject to similar constraints. Essentially, *averaging* is only a special case of low-pass filtering. Good averaging performance requires that $\tau \gg T$, where T is the period of the input signal or its characteristic time of variation if a regular period cannot be identified.

11.5.5 Capacitor as a Signal-Coupling Element

Another application context involving the series RC circuit is the ‘*signal-coupling problem*’ in electronic amplifiers. We abstract the problem as follows.

At a certain input point in the electronic amplifier a certain value of DC voltage has to be established using a DC source and resistors. This DC voltage is needed to fix the operating point of transistors in the amplifier at suitable levels. Yet, we want to connect an AC signal source to that input point without upsetting the DC potential there. Interposing a suitably sized capacitor between the signal source and input point achieves this objective. The capacitor used for this purpose is called *coupling capacitor* and amplifiers employing this form of signal-coupling are called *RC-Coupled Amplifiers* in the study of Electronic Circuits. Note that the RC circuit is used as a high-pass filter (or *average absorber*) in this application.

This application is explained further in the following example.

The signal-coupling problem in electronic amplifiers.

EXAMPLE: 11.5-1

Consider the signal-coupling problem presented in Fig. 11.5-7. The signal $v_s(t)$ is a mixture of sinusoids and may contain sinusoids of frequency from 20 Hz to 20 kHz (the standard audio range). It is desired that 95% of signal amplitude appear at the point A for the entire frequency range without affecting the DC content of voltage at that point. (i) Calculate the value of C needed for this purpose. (ii) Assume $v_s(t)$ is a 500 mV amplitude 20 Hz sine wave and plot the potential at A and the voltage across the coupling capacitor with the value of C calculated.

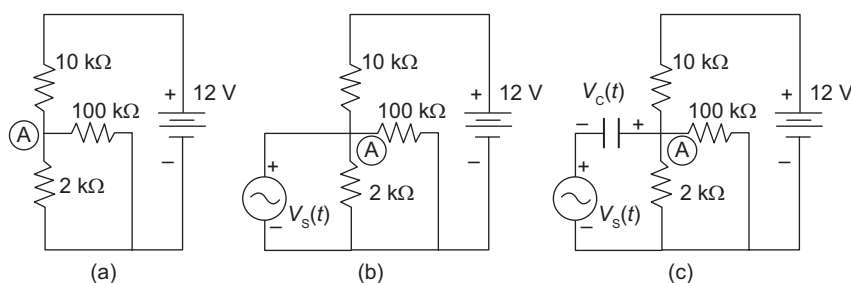


Fig. 11.5-7 Circuits for RC Signal-Coupling Example

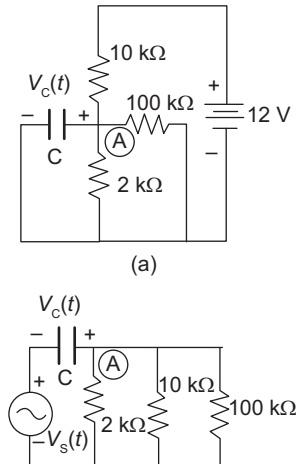


Fig. 11.5-8 Circuits for Applying Superposition in Example 11.5-1

SOLUTION

Consider the circuit Fig. 11.5-7(a). The 100 kΩ resistor produces only negligible loading at the point A and hence the current drawn by it may be ignored. Then, the DC voltage at that point is $12 \text{ V} \times (2/10 + 2) = 2 \text{ V}$ with respect to the negative of DC source.

Now, if the AC signal source is connected straight at point A as in Fig. 11.5-7(b) the DC voltage at that point gets affected. In fact it goes to zero if $v_s(t)$ has no DC content. This is a two-source problem and can be solved by applying superposition principle. When the DC source alone is considered the AC source has to be shorted (assuming it is an ideal independent voltage source). This short will reduce the DC potential at point A to zero once the signal source is connected. Hence, the direct coupling will not work.

The circuit in Fig. 11.5-7(c) is also excited by two sources. We know that the steady-state responses due to many sources obey superposition principle in a linear time-invariant circuit. The two equivalent circuits needed for calculating the steady-state response are shown in Fig. 11.5-8.

The DC component of steady-state response at point A is 2 V (neglecting the effect of 100 kΩ resistor) since a capacitor behaves as an open-circuit under DC steady-state. And the DC content in $v_c(t)$ is also 2 V.

As far as steady-state component of AC signal is concerned, we want 95% of input amplitude to appear at point A even at 20 Hz. The parallel combination of the three resistors is 1.64 kΩ. The signal voltage gets divided between the phasor impedance of capacitor, $1/(j2\pi \times 20C)$, and 1.64 kΩ.

Percentage of signal amplitude reaching point A at

$$20\text{Hz} = \left| \frac{1640}{1640 + \frac{1}{j40\pi C}} \right| \times 100\%$$

We desire this to be $\geq 95\%$.

$$\therefore 0.95 = \frac{1640 \times 40\pi C}{\sqrt{(1640 \times 40\pi C)^2}} \Rightarrow C = 14.8 \mu\text{F}.$$

Therefore, the capacitance value required is 14.8 μF.

The ratio between the voltage at A to voltage at input at 20 Hz has a gain value of 0.95.

$$\text{Its phase will be angle of } \frac{1640}{1640 + \frac{1}{j40\pi C}}.$$

Substituting $C = 14.8 \mu\text{F}$, we get the phase angle as 18.15° (0.317 rad).

Therefore, with 500 mV amplitude sine wave at input there will be a steady-state component of $(500 \times 0.95 \text{ mV}) \sin(40\pi t + 0.317)$ at point A.

$$\frac{V_c(j\omega)}{V_s(j\omega)} = -\frac{\frac{1}{j40\pi C}}{1640 + \frac{1}{j40\pi C}} = -\frac{-j537.7}{1640 - j537.7} = -0.312 \angle -1.254 \text{ rad}.$$

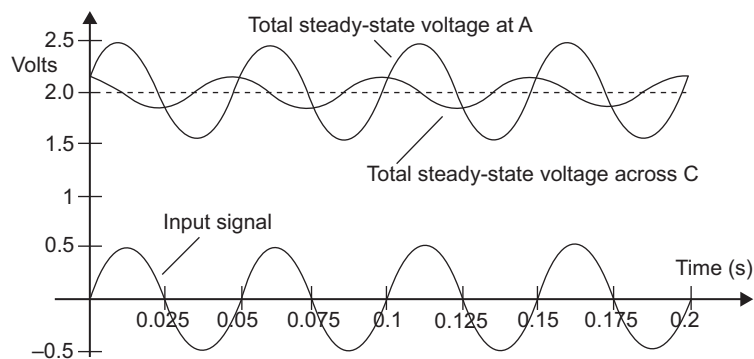


Fig. 11.5-9 Waveform Plots for Example 11.5-1

Therefore, the AC component of steady-state voltage across the capacitor
 $= -0.312 \times 0.5 \sin(40\pi t - 1.254) = -0.156 \sin(40\pi t - 1.254)$.

Therefore, total steady-state voltage at A $= 2 + 0.475 \sin(40\pi t + 0.317)$ V.

Therefore, total steady-state voltage across capacitor, $v_c(t) = 2 - 0.156 \sin(40\pi t - 1.254)$ V (see Fig. 11.5-9).

11.5.6 Parallel RC Circuit for Signal Bypassing

The need for making a resistor offer its full resistance to DC current flow and near-zero resistance to signal (i.e., alternating component) current flow arises often in Electronic Circuit applications. Constructing a parallel RC circuit by adding a capacitor in parallel to the concerned resistor is the standard solution to this ‘*signal bypassing*’ problem. The capacitor is sized suitably such that it offers a phasor impedance with magnitude very small compared to the resistance value of the resistor to be bypassed even for the lowest frequency sinusoidal component that is present in the signal current. Parallel RC circuit is used as a low-pass filter with its cut-off frequency much lower than the lowest frequency present in the signal in this application. The DC component of the total current flowing into the parallel combination goes through the resistor under steady state. But all the AC components flow almost entirely through the capacitor since it offers much lower impedance than the parallel resistor at the frequencies concerned. The voltage across the combination remains nearly DC. Example 11.5-2 illustrates this application of a parallel RC circuit.

Signal Bypassing application explained.

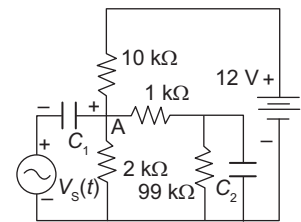


Fig. 11.5-10 Circuit for Signal Bypassing Problem in Example 11.5-2

EXAMPLE: 11.5-2

Consider the ‘*signal bypassing*’ context in Fig. 11.5-10. The signal $v_s(t)$ is a mixture of sinusoids and may contain sinusoids of frequency from 20 Hz to 20 kHz (the standard audio range). The coupling capacitor C_1 may be assumed to be so large that it is effectively a short-circuit at all frequencies except at DC. It is desired that more than 95% of the applied signal voltage appear across the $1\text{ k}\Omega$ resistor for the entire frequency range without affecting the DC content of current through that resistor. (i) Calculate the value of C needed for this purpose. (ii) Assume $v_s(t)$ is a 10 mV amplitude 20 Hz sine wave and plot the waveforms of total current through $1\text{ k}\Omega$, $99\text{ k}\Omega$ and the capacitor with the value of C calculated above.

SOLUTION

The equivalent circuits for calculating the steady-state response contributions from the DC and signal sources are shown in Fig. 11.5-11 (a) and (b), respectively.

Solving the circuit in Fig. 11.5-11 (a), we get the DC potential at the point A as 1.97 V, the DC current through $1\text{ k}\Omega$ and $99\text{ k}\Omega$ resistors as 0.0197 mA and the DC potential across $99\text{ k}\Omega$ resistor (and hence across the capacitor C_2) as 1.95 V.

Consider the circuit in Fig. 11.5-11 (b). We desire the amplitude of signal voltage across $1\text{ k}\Omega$ to be 95% of amplitude of applied signal voltage even when the signal is a 20 Hz sinusoid.

$$\frac{\text{Phasor voltage across } 1\text{ k}\Omega \text{ resistor}}{\text{Applied phasor voltage}} = \frac{1000}{1000 + 99000 // \frac{1}{j40\pi C_2}}.$$

We want the magnitude of this ratio to be 0.95. Solving for C_2 we get $C_2 = 24.5\text{ }\mu\text{F}$. Substituting this value of C_2 in the phasor ratio above we get the ratio as $0.95 \angle 0.32\text{ rad}$.

Applied signal voltage $= 0.01 \sin(40\pi t)$ V.

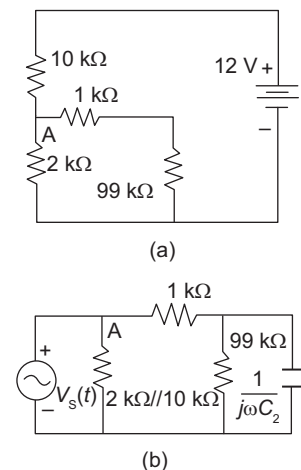


Fig. 11.5-11 Steady-State Equivalent Circuits in Example 11.5-2

$\therefore$ Signal voltage across $1\text{ k}\Omega = 0.0095 \sin(40\pi t + 0.32)\text{ V}$.
 $\therefore$ Signal current through $1\text{ k}\Omega = 0.0095 \sin(40\pi t + 0.32)\text{ mA}$.
 $\therefore$ Total current through $1\text{ k}\Omega = 0.0197 + 0.0095 \sin(40\pi t + 0.32)\text{ mA}$.
 Phasor impedance of $24.5\text{ }\mu\text{F}$ capacitor at $20\text{ Hz} = -j325\text{ }\Omega$.
 Parallel combination of $24.5\text{ }\mu\text{F}$ capacitor at 20 Hz and $99\text{ k}\Omega = 325\angle -1.567\text{ rad}\text{ }\Omega$.
 $\therefore$ Signal voltage across parallel combination
 $= 0.0095 \times 0.325 \sin(40\pi t + 0.32 - 1.567)$
 $= 0.00309 \sin(40\pi t - 1.247)\text{ V}$.
 $\therefore$ Signal current through $99\text{ k}\Omega = 0.00003 \sin(40\pi t - 1.247)\text{ mA}$.
 $\therefore$ Total current through $99\text{ k}\Omega = 0.0197 + 0.00003 \sin(40\pi t - 1.247)\text{ mA}$.
 Signal current through $24.5\text{ }\mu\text{F}$ capacitor
 $= \frac{0.00309\text{ V}}{0.325\text{ k}\Omega} \sin(40\pi t - 1.247 + \pi/2)$
 $= 0.0095 \sin(40\pi t + 0.323)\text{ mA}$.

Total voltage across parallel combination $= 1.95 + 0.00309 \sin(40\pi t - 1.247)\text{ V}$.

Almost the entire signal current flowing through $1\text{ k}\Omega$ resistor goes into the capacitor. Thus, the $99\text{ k}\Omega$ resistor has been bypassed very effectively by a $24.5\text{ }\mu\text{F}$ even at 20 Hz . The relevant waveforms are shown in Fig. 11.5-12. Voltages shown are with respect to the negative terminal of the DC source.

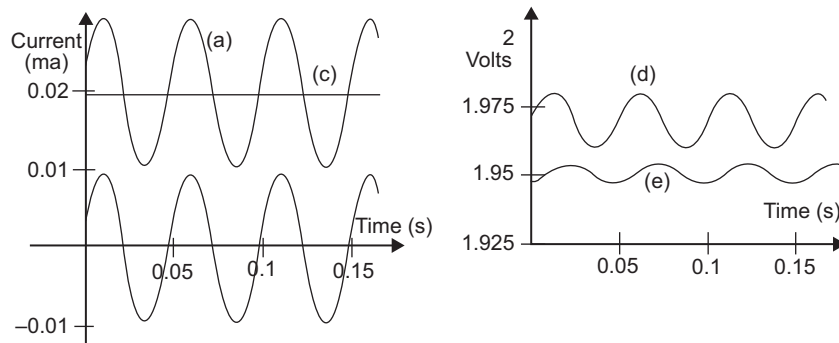


Fig. 11.5-12 Waveforms for Example 11.5-2 (a) Current in $1\text{ k}\Omega$ Resistor (b) Current in the Capacitor (c) Current in $99\text{ k}\Omega$ Resistor (d) Voltage at A (e) Voltage Across Capacitor

11.6 THE SERIES RLC CIRCUIT – ZERO-INPUT RESPONSE

Let us begin the study of RLC circuits with series RLC circuit in this section. Figure 11.6-1 shows a series RLC circuit with voltage excitation with all instantaneous variables identified.

We choose to develop the differential equation governing the circuit in terms of the voltage across the capacitor $v_C(t)$, i.e., we select $v_C(t)$ as the describing variable for the circuit. We obtain all other variables in terms of this variable once the solution is obtained.

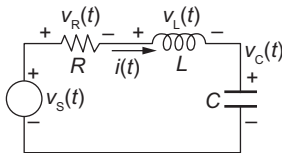


Fig. 11.6-1 Series RLC Circuit with Voltage Source Excitation

$$i(t) = C \frac{dv_C(t)}{dt} \quad (\text{By element equation of a capacitor})$$

$$-v_s(t) + Ri(t) + L \frac{di(t)}{dt} + v_C(t) = 0 \quad (\text{By KVL})$$

Substituting for $i(t)$ in terms of $v_C(t)$ and rearranging terms,

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{1}{LC} v_s(t).$$

All voltages and currents in a dynamic circuit are, in general, functions of time. Even if the variables do not include (t) explicitly we understand them to be functions of time. Hence, we write the describing differential equation of series RLC circuit in the following manner.

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{LC} v_s. \quad (11.6-1)$$

This equation is true for all t . However, we know the input source function $v_s(t)$ only from $t = 0^+$. There may be a discontinuity in $v_s(t)$ at $t = 0$ too. Therefore, we need to have additional information summarising all that happened to the inductor and capacitor from $t = -\infty$ to $t = 0^-$ in the form of initial condition specification for inductor current and capacitor voltage in order to solve this differential equation for $t \geq 0^+$. The two initial conditions needed to solve a second-order differential equation are the values of its zeroth and first derivatives at $t = 0^-$. But, if we know the initial condition for inductor current to be I_0 at $t = 0^-$, we can obtain the initial value of first derivative of $v_C(t)$ as I_0/C since $i(t) = C \frac{dv_C}{dt}$. Thus, the differential equation that we attempt to solve is restated as

$$\begin{aligned} \frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C &= \frac{1}{LC} v_s \text{ for } t \geq 0^+ \\ \text{with } v_C(0^-) &= V_0 \text{ V and } \left. \frac{dv_C}{dt} \right|_{(0^-)} = \frac{I_0}{C} \text{ V/s.} \end{aligned} \quad (11.6-2)$$

11.6.1 Source-free Response of Series RLC Circuit

We focus on the zero-input response (i.e., source-free response) of RLC circuit first. $v_s(t) = 0$ for $t \geq 0^+$ for this analysis. Initial energy storage in inductor (evidenced by a non-zero I_0) and/or initial energy storage in capacitor (evidenced by a non-zero V_0) is responsible for non-zero response under this condition. The differential equation describing zero-input response is

$$\begin{aligned} \frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C &= 0 \text{ for } t \geq 0^+ \\ \text{with } v_C(0^-) &= V_0 \text{ V and } \left. \frac{dv_C}{dt} \right|_{(0^-)} = \frac{I_0}{C} \text{ V/s.} \end{aligned} \quad (11.6-3)$$

The complementary function of homogeneous differential equations with constant coefficients is $Ae^{\gamma t}$. Substituting this trial solution in the differential equation in Eqn. 11.6-3,

$$\left(\gamma^2 + \frac{R}{L} \gamma + \frac{1}{LC} \right) A e^{\gamma t} = 0 \Rightarrow \gamma^2 + \frac{R}{L} \gamma + \frac{1}{LC} = 0$$

Let the two solutions of this algebraic equation be called α_1 and α_2 . Then,

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \text{ and } \alpha_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\therefore v_C(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t},$$

where A_1 and A_2 are two arbitrary constants to be fixed by initial conditions.

$$\text{The initial conditions are } v_C(0^-) = V_0 \text{ and } \left. \frac{dv_C}{dt} \right|_{(0^-)} = \frac{I_0}{C}.$$

But we need the initial conditions at $t = 0^+$ since the differential equation being solved is valid only for $t \geq 0^+$. There is no impulse current flowing into the capacitor in this circuit. And there is no impulse voltage appearing across inductor in this circuit. Hence, the initial conditions at $t = 0^+$ are the same as at $t = 0^-$.

Differential equation describing a series RLC circuit with voltage excitation.

The development of expressions for various responses in this chapter as well as elsewhere in this book is done with only one objective in mind – to further our understanding of various aspects of circuit behaviour by scrutinising the expressions arrived at.

The reader is neither expected nor encouraged to memorise such expressions. The reader is encouraged to solve numerical problems by building up the solution from basic principles rather than plugging in numbers in memorised expressions. This approach is illustrated in all the examples included in this book.

Consider Eqn. 11.6-4. This equation describes the variation of capacitor voltage in a RLC circuit under source-free condition. We note that two separate terms

continued

contribute to this voltage function – one term is proportional to the initial value of capacitor voltage and the other is proportional to the initial value of current in the inductor. Further, we are aware of the division of I_0 by the capacitance value in the second term and note that a quantity that is a current divided by capacitance will have V/s as its unit. That makes us think about the dimensions of denominator quantity. We are satisfied when we note that the α 's have T^{-1} dimensions.

But that draws our attention to the first term, as to its dimensions. We note that there are α 's in numerator and denominator leaving 'volt' as the unit of the entire term.

We note that both initial values contribute both exponential functions to the capacitor voltage. Thus, we realise that the initial values are not responsible for the exponential functions – rather, they decide the amplitude of exponentials. However, we note that with a proper choice of initial condition values it may be possible to reduce the amplitude of one of the exponential contributions to zero. But the only choice of initial condition values that will make the amplitudes of both exponential contributions zero is that of an initially relaxed circuit.

Equation 11.6-4 was derived for the elucidation it contributes. Further, it is needed for further elucidation in special cases of RLC circuits. But the reader is under no obligation to memorise it. In fact, he/she should not.

$$\therefore v_C(0^+) = V_0 \text{ and } \left. \frac{dv_C}{dt} \right|_{(0^+)} = \frac{I_0}{C}.$$

Applying these two initial conditions on the solution, we get,

$$A_1 + A_2 = V_0$$

$$\alpha_1 A_1 + \alpha_2 A_2 = \frac{I_0}{C}.$$

Solving for A_1 and A_2 ,

$$A_1 = \frac{\alpha_2 V_0 - I_0/C}{\alpha_2 - \alpha_1} \text{ and } A_2 = \frac{I_0/C - \alpha_1 V_0}{\alpha_2 - \alpha_1}$$

Substituting for A_1 and A_2 in the equation for $v_C(t)$ and collecting terms we get,

$$v_C(t) = V_0 \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_0}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+, \quad (11.6-4)$$

$$\text{where } \alpha_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}.$$

We observe that there are two natural response terms with exponential format in the solution. We observe further that there are two contributions in the zero-input response for $v_C(t)$ – one from initial energy storage in the capacitor and the second from initial energy storage in the inductor.

When we take a square root, we need to be concerned about the sign of quantity under the radical. Therefore, we identify three situations based on the sign of quantity under the radical in the expressions for α_1 and α_2 .

Case-1 α_1 and α_2 Real, Negative and Distinct – 'Over-Damped'

This case is a straightforward one and occurs when $R > 2\sqrt{L/C}$. The two distinct roots of characteristic equation of the homogeneous differential equation are real and negative and are arranged on either side of $R/2L$ magnitude-wise. The equation for $v_C(t)$ is as given in Eqn. 11.6-4. The capacitor voltage in this case is an additive mixture of two decaying exponential functions – one with a time constant that is less than $L/2R$ s and another with a time constant that is more than $L/2R$ s. Not only the capacitor voltage but also all the variables will have these two exponential functions in them. The following example illustrates this case. A 0.5 F capacitor 1 H inductor are assumed in the example. 1 H inductance is a practical value. However, 0.5 F capacitor is hard to come by in practice. Capacitors used in practical engineering usually range from pF to mF. However, a 0.5 F capacitor is perfectly legitimate in a numerical example aimed at illustrating theoretical concepts – at least it keeps the numbers simple.

EXAMPLE: 11.6-1

A series RLC circuit has $R = 3 \Omega$, $L = 1 \text{ H}$ and $C = 0.5 \text{ F}$. The capacitor is initially charged to 2 V and the initial current in the inductor is 1 A at $t = 0^-$. Find the zero-input response of capacitor voltage and circuit current.

SOLUTION

The differential equation governing the capacitor voltage $v_C(t)$ is

$$\frac{d^2 v_C}{dt^2} + 3 \frac{dv_C}{dt} + 2v_C = 0 \text{ for } t \geq 0^+.$$

The characteristic equation is $\gamma^2 + 3\gamma + 2 = 0$ and its roots $\alpha_{1,2} = -1 \text{ s}^{-1}$ and -2 s^{-1} .

Therefore, the general solution for $v_C(t) = A_1 e^{-t} + A_2 e^{-2t} \text{ V}$. Applying initial conditions at $t = 0^+$,

$$A_1 + A_2 = 2 \text{ V and } -1 \times A_1 + -2 \times A_2 = 1 \text{ A}/0.5 \text{ F} = 2 \text{ V/s}$$

$$\therefore A_1 = 6 \text{ and } A_2 = -4$$

$$\therefore v_C(t) = 6 e^{-t} - 4 e^{-2t} \text{ V for } t \geq 0^+.$$

$$\text{And, } i(t) = C dv_C/dt = -3 e^{-t} + 4 e^{-2t} \text{ A for } t \geq 0^+.$$

The capacitor voltage and circuit current contain two decaying exponential transients with time constants of 1 s and 0.5 s. Both exponential terms decay down to zero thereby taking the circuit to zero-energy condition in about 5–6 s.

The circuit contained total initial energy storage of 1.5 J (1 J in the capacitor and 0.5 J in the inductor). Both the voltage across capacitor and current through inductor approach zero as $t \rightarrow \infty$. Therefore, the total energy storage in the circuit goes to zero with time. Then the total energy dissipated in the resistor from $t = 0^+$ to ∞ must be 1.5 J. This is verified as under.

$$\begin{aligned} \text{Total energy dissipation in } 3\Omega \text{ resistor} &= \int_0^{\infty} 3 \times (-3e^{-t} + 4e^{-2t})^2 dt \\ &= \int_0^{\infty} 3 \times (9e^{-2t} + 16e^{-4t} - 24e^{-3t}) dt = -\frac{27}{2} e^{-2t} \Big|_0^{\infty} - 12e^{-4t} \Big|_0^{\infty} + 24e^{-3t} \Big|_0^{\infty} = 1.5 \text{ J} \end{aligned}$$

The time-variation of all the circuit variables under zero-input response conditions is shown in Fig. 11.6-2.

Both the natural response terms are decaying exponential functions. But, that, by no means, implies that all the circuit variables will decay monotonically from $t = 0^+$ onwards. Note that the voltage across capacitor increases from its initial value of 2 V to 2.25 V first before it starts decaying. This is so because the initial current in inductor charges up the capacitor further.

The difference between two decaying exponential functions can exhibit a maximum or minimum. In the present example, $v_C(t)$ reaches a maximum and $i(t)$ reaches a minimum before they settle down to zero as $t \rightarrow \infty$.

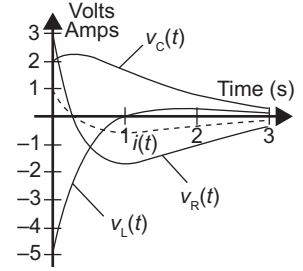


Fig. 11.6-2 Zero-Input Response of Series RLC Circuit in Example 11.6-1

Case-2 α_1 and α_2 Real, Negative and Equal – ‘Critically-Damped’

This case occurs when $R = 2\sqrt{L/C}$ and then $\alpha_1 = \alpha_2 = -R/2L$. The limiting form of expression for $v_C(t)$ given in Eqn. 11.6-4 has to be found in this case. We had found out such a limiting form in Sect. 11.3 in the context of zero-state response of a RC circuit excited by an exponential signal with a time constant equal to the time constant of the circuit itself. We had employed the series expansion of exponential function for that purpose. Here, we try a different approach. We denote α_2 as $\alpha_1 + \Delta\alpha$ and determine the limit of $v_C(t)$ as $\Delta\alpha \rightarrow 0$.

The expression for $v_C(t)$ in a general case is repeated below.

$$v_C(t) = V_0 \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_0}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+. \quad (11.6-5)$$

This equation is first recast in the following equivalent form.

$$\begin{aligned} v_C(t) &= V_0 \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_1 t} + \alpha_1 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_0}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+ \\ &= V_0 \left[e^{\alpha_1 t} + \alpha_1 \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \right] - \frac{I_0}{C} \left[\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \right] \text{ V for } t \geq 0^+ \\ &= V_0 \left[e^{\alpha_1 t} - \alpha_1 \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_1 - \alpha_2} \right] + \frac{I_0}{C} \left[\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_1 - \alpha_2} \right] \text{ V for } t \geq 0^+ \end{aligned}$$

Voltage across capacitor in a source-free critically-damped RLC circuit.

Now, let $\alpha_1 = \alpha$ and $\alpha_2 = \alpha + \Delta\alpha$. Then,

$$v_C(t) = V_0 \left[e^{\alpha t} - \alpha_1 \frac{e^{\alpha t} - e^{(\alpha + \Delta\alpha)t}}{-\Delta\alpha} \right] + \frac{I_0}{C} \left[\frac{e^{\alpha t} - e^{(\alpha + \Delta\alpha)t}}{-\Delta\alpha} \right] \text{ V for } t \geq 0^+$$

$$e^{\alpha t} - e^{(\alpha + \Delta\alpha)t} = e^{\alpha t} [1 - e^{\Delta\alpha t}] \approx -e^{\alpha t} \times \Delta\alpha t \text{ for small } \Delta\alpha$$

$$\lim_{\Delta\alpha \rightarrow 0} v_C(t) = V_0 e^{\alpha t} + \left(\frac{I_0}{C} - \alpha V_0 \right) t e^{\alpha t} \text{ V for } t \geq 0^+, \text{ where } \alpha = -\frac{R}{2L} \text{ s}^{-1}$$

Therefore, the circuit solution in this case is

$$v_C(t) = V_0 e^{-(R/2L)t} + \left(\frac{I_0}{C} + \frac{R}{2L} V_0 \right) t e^{-(R/2L)t} \text{ V for } t \geq 0^+.$$

The other circuit variables may be readily obtained now, which is illustrated in Example 11.6-2.

EXAMPLE: 11.6-2

A series RLC circuit has $R = 2 \Omega$, $L = 1 \text{ H}$ and $C = 1 \text{ F}$. The capacitor is initially charged to 2 V and the initial current in the inductor is 2 A at $t = 0^-$. Find the zero-input response of capacitor voltage and circuit current.

SOLUTION

The differential equation governing the capacitor voltage $v_C(t)$ is

$$\frac{d^2 v_C}{dt^2} + 2 \frac{dv_C}{dt} + v_C = 0 \text{ for } t \geq 0^+.$$

The characteristic equation is $\gamma^2 + 2\gamma + 1 = 0$ and its roots are $\alpha_{1,2} = -1 \text{ s}^{-1}$ and -1 s^{-1} .

From the above discussion it must be clear that the trial solution to be attempted is of the form $v_C(t) = A_1 e^{-t} + A_2 t e^{-t} \text{ V}$. Applying initial conditions at $t = 0^+$, we get two equations in A_1 and A_2 . They are,

$$v_C(0^+) = A_1 = 2 \text{ V}$$

$$\left. \frac{dv_C}{dt} \right|_{0^+} = \frac{I_0}{C} = 2 \text{ V/s} = [-A_1 e^{-t} - A_2 t e^{-t} + A_2 e^{-t}]_{0^+} = -A_1 + A_2$$

$$\therefore A_1 = 2 \text{ V and } A_2 = 4 \text{ V s}^{-1}$$

$$\therefore v_C(t) = 2e^{-t} + 4te^{-t} \text{ V for } t \geq 0^+$$

$$\text{and } i(t) = C \frac{dv_C}{dt} = 1 \frac{dv_C}{dt} = -2e^{-t} - 4te^{-t} + 4e^{-t} = 2e^{-t} - 4te^{-t} \text{ A for } t \geq 0$$

$$v_L(t) = L \frac{di}{dt} = 1 \frac{di}{dt} = -2e^{-t} + 4te^{-t} - 4e^{-t} = -6e^{-t} + 4te^{-t} \text{ V for } t \geq 0^+$$

$$v_R(t) = Ri = 2i = 4e^{-t} - 8te^{-t} \text{ V for } t \geq 0^+$$

These waveforms are shown in Fig. 11.6-3.

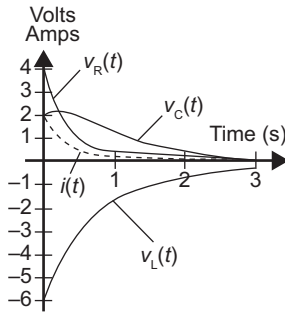


Fig. 11.6-3 Zero-Input Response Waveforms of a Series RLC Circuit in Example 11.6-2

Case-3 α_1 and α_2 Complex Conjugates with Negative Real Parts – ‘Under-Damped’

Now, we come up against the most interesting case of all. This occurs when $R < 2\sqrt{L/C}$.

The quantity $\left(\frac{R^2}{4L^2} - \frac{1}{LC} \right)$ is negative under this condition and the roots of characteristic

equation become $\alpha_{1,2} = \frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$. The roots are complex conjugate numbers.

The circuit response in this case will contain exponentially-damped sinusoidal oscillations. That is, the wave-shape of the natural response terms will be oscillatory with the oscillation losing amplitude exponentially with time. The relevant expressions are derived as below.

We define three new symbols – mainly for convenience at present. But they will turn out to be important parameters in circuit studies soon. They are defined as below.

$$\text{Let } \xi = \frac{R}{2\sqrt{L/C}}, \quad \omega_n = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{1-\xi^2} \omega_n. \quad (11.6-6)$$

The roots of characteristic equation can be expressed in terms of these new symbols as $\alpha_1 = -\xi\omega_n + j\sqrt{1-\xi^2}\omega_n$ and $\alpha_2 = -\xi\omega_n - j\sqrt{1-\xi^2}\omega_n$.

Now we use Eqn. 11.6-4 for the zero-input response for $v_C(t)$ and substitute these expressions for α_1 and α_2 and employ Euler's formula for algebraic simplification of the resulting expression.

$$\begin{aligned} v_C(t) &= V_0 \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_0}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \quad \text{V for } t \geq 0^+. \\ \alpha_{1,2} &= -\xi\omega_n \pm j\sqrt{1-\xi^2}\omega_n = -\xi\omega_n \pm j\omega_d \Rightarrow \alpha_2 - \alpha_1 = -2j\omega_d. \\ \text{Simplifying } \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1}, \\ &= \frac{-1}{2j\sqrt{1-\xi^2}\omega_n} \left[(-\xi\omega_n - j\omega_d) e^{(-\xi\omega_n + j\omega_d)t} - (-\xi\omega_n + j\omega_d) e^{(-\xi\omega_n - j\omega_d)t} \right] \\ &= \frac{-e^{-\xi\omega_n t}}{2j\sqrt{1-\xi^2}\omega_n} \left[(-\xi\omega_n - j\omega_d) e^{(j\omega_d)t} - (-\xi\omega_n + j\omega_d) e^{(-j\omega_d)t} \right] \\ &= \frac{-e^{-\xi\omega_n t}}{2j\sqrt{1-\xi^2}\omega_n} \left[-\xi\omega_n (e^{(j\omega_d)t} - e^{(-j\omega_d)t}) - j\omega_d (e^{(j\omega_d)t} + e^{(-j\omega_d)t}) \right] \\ &= \frac{-e^{-\xi\omega_n t}}{2j\sqrt{1-\xi^2}\omega_n} \left[-\xi\omega_n (2j \sin \omega_d t) - j\omega_d (2 \cos \omega_d t) \right] \quad (\text{By Euler's Formula}) \\ &= e^{-\xi\omega_n t} \left[\frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t + \cos \omega_d t \right] \quad \left(\because \omega_d = \sqrt{1-\xi^2} \omega_n \right) \end{aligned}$$

$$\begin{aligned} \text{Similarly, simplifying } \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1}, \\ \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} &= -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}\omega_n} \sin \omega_d t \\ \therefore v_C(t) &= V_0 e^{-\xi\omega_n t} \left[\frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t + \cos \omega_d t \right] + \frac{I_0}{C} \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}\omega_n} \sin \omega_d t. \end{aligned}$$

Substituting $\omega_n = \frac{1}{\sqrt{LC}}$ in the second term,

$$v_C(t) = V_0 e^{-\xi\omega_n t} \left[\frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t + \cos \omega_d t \right] + I_0 \sqrt{\frac{L}{C}} e^{-\xi\omega_n t} \frac{1}{\sqrt{1-\xi^2}} \sin \omega_d t.$$

Therefore, the zero-input response for capacitor voltage in this case with $0 \leq \xi < 1$ is given by the following expression in terms of ξ and ω_n .

Voltage across the capacitor in a source-free under-damped RLC circuit.

$$v_C(t) = e^{-\xi\omega_n t} \left[\left(\frac{V_0\xi + I_0\sqrt{\frac{L}{C}}}{\sqrt{1-\xi^2}} \right) \sin \sqrt{1-\xi^2}\omega_n t + V_0 \cos \sqrt{1-\xi^2}\omega_n t \right] \text{ for } t \geq 0^+, \quad (11.6-7)$$

$$\text{where } \omega_n = \frac{1}{\sqrt{LC}}, \xi = \frac{R}{2\sqrt{L/C}}.$$

That, indeed, looks slightly complicated. We will take it up through special cases soon. But before that, here is how it looks for a series RLC circuit with $L = 1$ H, $C = 1$ F, $R = 0.5 \Omega$, $V_0 = 2$ V and $I_0 = 2$ A (Fig. 11.6-4). The value of ξ is 0.25 and ω_n is 1.

The zero-input response of an under-damped RLC circuit has an *exponentially-damped sinusoidal* shape. All the circuit variables oscillate with the amplitude of oscillation decreasing with time.

An exponential function governs the decrease in amplitude with time. The numbers ξ and ω_n decide the decay rate of this exponential function. They also decide the interval between successive zero-crossings of the circuit variables.

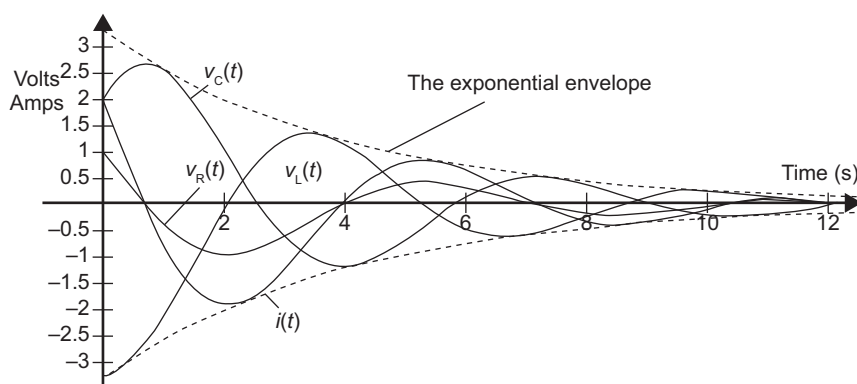


Fig. 11.6-4 Zero-Input Response of a Series RLC Circuit with $R = 0.5 \Omega$, $L = 1$ H, $C = 1$ F, $V_0 = 2$ V and $I_0 = 2$ A

11.6.2 The Series LC Circuit – A Special Case

We consider a special case of series RLC circuit – the one with resistance equal to zero. Hence, $\xi = 0$, $\omega_n = 1/\sqrt{LC}$ and $\omega_d = \omega_n$. Hence, Eqn. 11.6-7 reduces to

$$v_C(t) = V_0 \cos \omega_n t + I_0 \sqrt{\frac{L}{C}} \sin \omega_n t \text{ V for } t \geq 0^+. \quad (11.6-8)$$

Further,

$$\begin{aligned} i(t) &= C \frac{dv_C(t)}{dt} = I_0 \cos \omega_n t - V_0 \sqrt{\frac{C}{L}} \sin \omega_n t \text{ A for } t \geq 0^+ \\ v_L(t) &= -v_C(t) = -V_0 \cos \omega_n t - I_0 \sqrt{\frac{L}{C}} \sin \omega_n t \text{ V for } t \geq 0^+ \end{aligned} \quad (11.6-9)$$

Equations 11.6-8 and 11.6-9 show that the voltage variables and current variable in a pure LC circuit are sinusoidal in nature. The amplitudes of sinusoids are decided by the initial voltage across the capacitor, initial current through the inductor and the circuit parameters. But the initial voltage across capacitor and initial current through the inductor in turn decide the total energy storage in the LC circuit at $t = 0^-$. Since there are no impulse voltages and currents in the LC circuit under source-free condition, the initial condition values at $t = 0^-$ and $t = 0^+$ are the same. Therefore, the energy storage in the elements also will be the same at $t = 0^-$ and at $t = 0^+$.

Voltage across the capacitor in a source-free LC circuit.

We may recast the expressions that involve sum of the two sinusoidal functions in Eqn. 11.6-8 and 11.6-9 as single sinusoidal functions by employing trigonometric identities in the following manner.

$$\begin{aligned}
 v_C(t) &= \sqrt{V_0^2 + \frac{LI_0^2}{C}} \cos(\omega_n t - \phi) \text{ V for } t \geq 0^+ \\
 i(t) &= -\sqrt{I_0^2 + \frac{CI_0^2}{L}} \sin(\omega_n t - \phi) \text{ A for } t \geq 0^+, \\
 \text{where } \phi &= \tan^{-1} \left(\frac{I_0 \sqrt{L/C}}{V_0} \right).
 \end{aligned} \tag{11.6-10}$$

These waveforms are shown in Fig. 11.6-5 for $L = 1 \text{ H}$, $C = 1 \text{ F}$, $V_0 = 2 \text{ V}$ and $I_0 = 1 \text{ A}$.

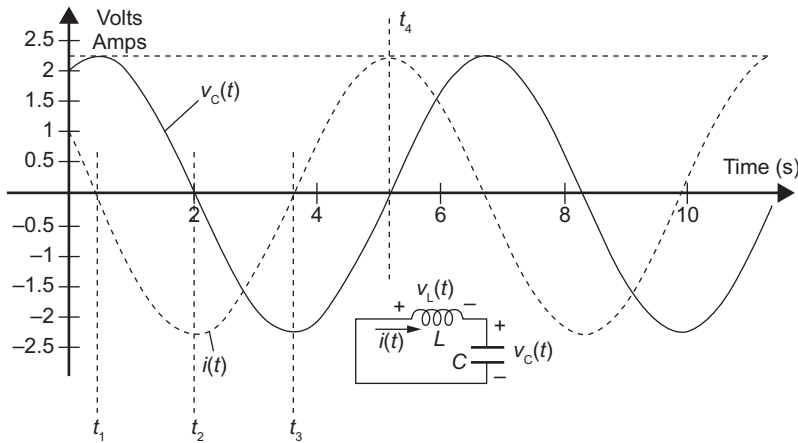


Fig. 11.6-5 Zero-Input Response of a LC Circuit ($L = 1 \text{ H}$, $C = 1 \text{ F}$, $V_0 = 2 \text{ V}$ and $I_0 = 1 \text{ A}$)

The initial voltage of 2 V across the capacitor appears across the inductor at $t = 0^+$ with a polarity such that the inductor current starts decreasing at the rate of $2 \text{ V}/1 \text{ H} = 2 \text{ A/s}$ from its initial value of 1 A. However, the circuit current is in a direction suitable for increasing the capacitor voltage. Hence, the capacitor voltage increases while the inductor current decreases. Under the action of increasing reverse voltage, the inductor current decreases more rapidly to reach zero at the instant t_1 . At that instant, the current and hence the energy storage in inductor are zero. The inductor had an initial energy of 0.5 J and the capacitor had an initial energy of 2 J. There was no dissipation in the circuit. Therefore, when the circuit current reaches zero, the capacitor must hold the total initial energy of 2.5 J in it. It will require $\sqrt{5} \text{ V}$ across it (since $C = 1 \text{ F}$ and energy = $0.5CV^2$). Equation 11.6-10 predicts exactly this value as the amplitude of $v_C(t)$. When circuit current goes through zero, capacitor voltage must go through a positive or negative peak due to two reasons – firstly, the current through a capacitor is proportional to the rate of change of voltage across it and secondly that is the instant at which it will contain the maximum possible energy equal to the total initial energy. Therefore, $v_C(t)$ reaches a positive peak at t_1 .

With such a large reverse voltage across it, the inductor has to continue its current build up in the negative direction. But, with the current changing its direction, the capacitor

Source-free response equations for a pure LC circuit.

The source-free response (equivalently, the zero-input response) of a pure LC circuit will contain undying sinusoids with steady amplitudes. The amplitude of sinusoidal waveforms is decided by the total initial energy storage in the circuit and the circuit parameters.

Circuit parameters, i.e., L and C decide the angular frequency of oscillations too – it is $(LC)^{-0.5} \text{ rad/s}$.

A Pure LC Circuit?

Strictly speaking, a pure LC circuit cannot exist in practice. The wire used to construct the inductor, the metal foil used in the capacitor and the connecting wires have non-zero resistance. The dielectric used in the capacitor will have non-zero conductivity. Thus, there will be some non-zero resistance left in any LC circuit.

continued

Even if we choose to ignore these small resistance values another mechanism that can contribute to damping exists in any circuit carrying time-varying currents. Any circuit with time-varying voltage and current in it will radiate energy into surrounding space – that is, any circuit with time-varying currents will act as a transmitting antenna (as a receiving antenna as well). The energy radiated out may be extremely small if the frequency of time-varying current is low. But this mechanism of energy loss acts in all circuits at all frequencies except at zero frequency.

Thus, theoretically, a LC circuit will be a damped circuit even if it contains no resistance.

enters discharge mode and its voltage decreases. Hence, during (t_1, t_2) the circuit current builds up in the negative direction with progressively decreasing rate and voltage across capacitor decreases to zero at t_2 . When capacitor voltage reaches zero, its energy storage also reaches zero. Therefore, the inductor must contain all the circuit energy in it at that instant. Hence the circuit current has to be $-\sqrt{5}$ A at that instant (since $L = 1$ H and energy = $0.5LI^2$). Equation 11.6-10 predicts exactly this value as the amplitude of $i(t)$. When voltage across capacitor (and hence voltage across inductor) goes through zero, circuit current must go through a positive or negative peak due to two reasons – firstly, the voltage across an inductor is proportional to the rate of change of current through it and secondly that is the instant at which it will contain the maximum possible energy equal to the total initial energy. Therefore, $i(t)$ reaches a negative peak at t_2 .

Now the large discharge current flowing out of the capacitor will result in build-up of capacitor voltage in the negative direction during the interval (t_2, t_3) . The consequent change in polarity of inductor voltage will result in the circuit current turning back from its negative peak. The circuit current decreases in magnitude and capacitor voltage increases in magnitude. At t_3 the circuit current reaches zero and capacitor voltage reaches negative peak of $-\sqrt{5}$ V.

The total initial stored energy gets shunted out from inductor to capacitor and back in this manner periodically. This initial energy cannot be dissipated since there is no resistance in the circuit (see the side-note). The circuit tries to get rid of its initial energy but fails to do so. That results in voltage and current oscillations.

In that case, the sum of energy storage in capacitor and inductor must be equal to the total initial energy storage in the circuit. Calculating the sum as $\frac{Li(t)^2}{2} + \frac{Cv_c(t)^2}{2}$ by substituting the expressions for $v_c(t)$ and $i(t)$ from Eqn. 11.6-10 will indeed show that the sum is equal to $\frac{LI_0^2}{2} + \frac{CV_0^2}{2}$, which is the total initial energy.

The frequency of voltage and current oscillations in a pure LC circuit is $\frac{1}{\sqrt{LC}}$ rad/s or $\frac{1}{2\pi\sqrt{LC}}$ Hz and period of oscillations is $2\pi\sqrt{LC}$ s. The reader may convince himself of the dimensional consistency of these expressions. We had defined ω_n to be equal to $\frac{1}{\sqrt{LC}}$ earlier. Thus, ω_n of a RLC circuit is the angular frequency of pure

sinusoidal oscillations that will take place in source-free conditions if the dissipation in the circuit is reduced to zero – in the case of a series RLC circuit this amount to reducing the resistance to zero. Energy loss mechanisms cause *damping* of oscillations in systems. Thus, resistance is a damping element in an electrical circuit. ω_n is the angular frequency of oscillations in an oscillating electrical circuit when *damping* is zero in the circuit – hence it is called the ‘undamped natural frequency of oscillations’.

Since the energy storage in the capacitor is proportional to the square of its voltage, the frequency of oscillation of stored energy will be double that of voltage waveform. This follows from the fact that $\sin^2 \theta = 0.5(1 - \cos 2\theta)$ and $\cos^2 \theta = 0.5(1 + \cos 2\theta)$. Moreover, the stored energy function will always be positive-valued and hence will have a positive average value. This is so because $(1 - \cos 2\theta)$ and $(1 + \cos 2\theta)$ cannot have negative value for any θ . Therefore, the stored energy in the capacitor and inductor will oscillate about a positive average value with a frequency of $\frac{1}{\pi\sqrt{LC}}$ Hz. These aspects

are shown clearly in Fig. 11.6-6 that shows the time-variation of stored energy in the capacitor and inductor along with the time-variation of total stored energy in the circuit for the circuit referred in Fig. 11.6-5.

Energy storage in both elements varies between 0 and 2.5 J sinusoidally. The average energy storage in both elements is 1.25 J. The total stored energy in the circuit is a constant function of time as expected.

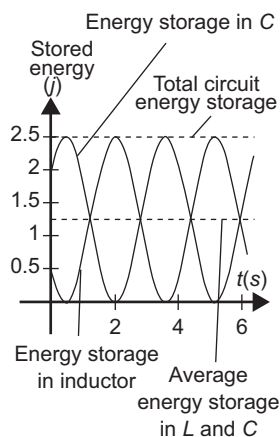


Fig. 11.6-6 Time-variation of Stored Energy Under Zero-Input Response Condition in an LC Circuit

EXAMPLE: 11.6-3

A pure LC series circuit has $L = 0.2 \text{ H}$, $C = 10,000 \mu\text{F}$, $V_0 = 100 \text{ V}$ and $I_0 = 10 \text{ A}$. (i) Find the natural frequency of oscillation and its period. (ii) Predict the amplitude of voltage and current oscillations in the source-free response of the circuit employing energy arguments. (iii) Obtain the time-domain expressions for $v_C(t)$ and $i(t)$ in the circuit.

SOLUTION

Natural frequency of oscillation $= (0.2 \times 10,000 \times 10^{-6})^{-0.5} = 22.36 \text{ rad/s}$ (3.56 Hz)

Period of oscillations $= (1/3.56 \text{ Hz}) = 0.281 \text{ s}$.

Total initial stored energy $= 0.5 \times 0.2 \times 10 \times 10 + 0.5 \times 10^{-2} \times 100 \times 100 = 60 \text{ J}$.

The capacitor must hold this much energy when its voltage reaches positive or negative peak. Therefore, the amplitude of $v_C(t) = (60 \times 2/10^{-2})^{0.5} = 109.54 \text{ V}$. Similarly, the inductor must hold 60 J of energy when its current reaches positive or negative peak. Hence, amplitude of $i(t) = (60 \times 2/0.2)^{0.5} = 24.5 \text{ A}$.

Therefore, $v_C(t) = 109.54 \cos(22.36t + \phi)$, where ϕ is to be found using initial conditions. Substituting initial condition for $v_C(t)$, $100 = 109.54 \cos(\phi) \Rightarrow \phi = \pm 0.42 \text{ rad}$. We use the initial condition for inductor to fix the sign of this phase angle.

$$i(t) = C \frac{dv_C(t)}{dt} = -10^{-2} \times 109.54 \times 22.36 \times \sin(22.36t + \phi) = -24.5 \sin(22.36t + \phi) \text{ A}$$

Applying initial condition $\Rightarrow -24.5 \sin \phi = 10 \Rightarrow \phi = -0.42 \text{ rad}$.

Therefore, $v_C(t) = 109.54 \cos(22.36t - 0.42) \text{ V}$ for $t \geq 0^+$

$$i(t) = -24.5 \sin(22.36t - 0.42) \text{ A for } t \geq 0^+$$

Of course, we could have used Eqn. 11.6-10 directly to obtain the solution. But then, that will require committing that equation to memory. And, committing too many equations to memory does not take an engineer very far in his professional career.

EXAMPLE: 11.6-4

Show that the ratio of amplitude of voltage oscillation to the amplitude of current oscillation in a LC circuit zero-input response will be $\sqrt{L/C}$.

SOLUTION

Let V_0 and I_0 be the initial values for capacitor voltage and inductor current in the LC circuit. And let V_m and I_m be the amplitudes of voltage oscillation and current oscillation, respectively.

$$\text{Total initial stored energy} = \frac{L I_0^2}{2} + \frac{C V_0^2}{2}.$$

$$\text{Maximum stored energy in the capacitor} = \frac{C V_m^2}{2}$$

$$\text{These two have to be equal, } \therefore V_m = \sqrt{V_0^2 + \frac{L}{C} I_0^2}$$

$$\text{Similarly, the maximum stored energy in the inductor} = \frac{L I_m^2}{2}$$

$$\text{This has to be equal to the total initial energy storage, } \therefore I_m = \sqrt{I_0^2 + \frac{C}{L} V_0^2}.$$

There is yet another way to arrive at I_m .

$$i(t) = C \frac{dv_C(t)}{dt}$$

$\therefore$ Amplitude of current oscillation $= \omega_n C \times$ Amplitude of voltage oscillation

$$\therefore I_m = C \times \sqrt{\frac{1}{LC}} \times \sqrt{V_0^2 + \frac{L}{C} I_0^2} = \sqrt{I_0^2 + \frac{C}{L} V_0^2}.$$

Both approaches lead to the final conclusion that $\frac{V_m}{I_m} = \sqrt{\frac{L}{C}}$. The reader may verify that this ratio has dimension of resistance. This resistance is at times referred to as the *characteristic resistance* of LC circuit.

Note that the amplitude of voltage and current oscillations in a source-free LC circuit is decided by the total initial energy storage. Stored energy is a square function of initial conditions.

Hence, the amplitude of oscillation in a source-free LC circuit depends only on the *magnitude* of the initial conditions and not on their *polarity*. The *polarity* of initial conditions decides the phase angle of oscillations.

Absence of any dissipation mechanism dooms a pure LC circuit with non-zero energy storage to oscillate forever. But suppose there is a little resistance in series – so little that there is no marked change in the shape of $v_c(t)$ and $i(t)$ in the first cycle. However, in that case, the current which flows through the resistance must dissipate some energy cycle after cycle (however minute that may be) leading to reduction of the total available stored energy in the circuit. And, that must surely lead to gradual loss of amplitude of voltage and current oscillation. Sometime or other the circuit must settle down to a zero-energy state.

Therefore, we throw in a little damping in a LC circuit in Sect. 11.6.3.

11.6.3 The Series LC Circuit with Small Damping – Another Special Case

Now we need the full expression for $v_c(t)$ given in Eqn. 11.6-7 to proceed with our analysis. This expression is reproduced below.

$$v_c(t) = e^{-\xi\omega_n t} \left[\frac{V_0\xi + I_0\sqrt{\frac{L}{C}}}{\sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\omega_n t + V_0 \cos \sqrt{1-\xi^2}\omega_n t \right] \text{ for } t \geq 0^+, \quad (11.6-11)$$

$$\text{where } \omega_n = \frac{1}{\sqrt{LC}}, \xi = \frac{R}{2\sqrt{L/C}}.$$

The resistance is small since damping is taken to be small. We note from the equation above that, for given L and C values the number ξ decides the rate of loss of amplitude of oscillations. Therefore, ξ has to be small in a case in which damping is small. Therefore, the assumption of small damping is equivalent to the assumption that $R \ll 2\sqrt{L/C}$, i.e., small resistance compared to $2\sqrt{L/C}$. This resistance, $2\sqrt{L/C}$, which is a characteristic

value for a given series RLC circuit is termed as its *critical resistance*. Since amplitude loss is basically due to *damping* and since the factor ξ governs the rate of loss of amplitude, this factor ξ is aptly termed as '*damping factor*' in circuit studies. Thus, we are dealing with series RLC circuit with $\xi \ll 1$ in this sub-section. We make use of this fact to simplify the expression in Eqn. 11.6-11 considerably.

We employ the following approximations in this context.

$$\frac{V_0\xi + I_0\sqrt{\frac{L}{C}}}{\sqrt{1-\xi^2}} \approx I_0\sqrt{\frac{L}{C}} \text{ for } \xi \ll 1$$

$$\omega_d = \sqrt{1-\xi^2}\omega_n \approx \omega_n \text{ for } \xi \ll 1.$$

Damping factor ξ
of a series RLC circuit.

$$\therefore v_C(t) \approx e^{-\xi\omega_n t} \left[I_0 \sqrt{\frac{L}{C}} \sin \omega_n t + V_0 \cos \omega_n t \right] \quad (11.6-12)$$

$$\text{and } i(t) = C \frac{dv_C(t)}{dt} = e^{-\xi\omega_n t} \left[I_0 \cos \omega_n t - V_0 \sqrt{\frac{C}{L}} \sin \omega_n t \right]$$

The algebra involved in the second step in Eqn. 11.6-12 has been skipped. Reader is urged to verify the derivation of $i(t)$. Expressing the expressions for $v_C(t)$ and $i(t)$ in Eqn. 11.6-12 as single sinusoids, we have,

$$\begin{aligned} v_C(t) &\approx \sqrt{V_0^2 + \frac{LI_0^2}{C}} e^{-\xi\omega_n t} \cos(\omega_n t - \phi) \text{ V for } t \geq 0^+ \\ i(t) &\approx -\sqrt{I_0^2 + \frac{CV_0^2}{L}} e^{-\xi\omega_n t} \sin(\omega_n t - \phi) \text{ A for } t \geq 0^+, \\ \text{where } \phi &= \tan^{-1} \left(\frac{I_0 \sqrt{L/C}}{V_0} \right). \end{aligned} \quad (11.6-13)$$

We see that the expressions for $v_C(t)$ and $i(t)$ in the case of series RLC circuit with small damping is different from corresponding functions in the case of an undamped LC circuit only by the factor $e^{-\xi\omega_n t}$ appearing as a multiplier. Compare Eqn. 11.6-10 and Eqn. 11.6-13.

Let us examine how the total stored energy in the circuit evolves in time in this case.

Total stored energy = $0.5Li(t)^2 + 0.5Cv_C(t)^2$

$$\begin{aligned} &= e^{-2\xi\omega_n t} \left[0.5L \left(I_0^2 + \frac{CV_0^2}{L} \right) \cos^2(\omega_n t - \phi) + 0.5C \left(V_0^2 + \frac{LI_0^2}{C} \right) \sin^2(\omega_n t - \phi) \right] \\ &= (0.5LI_0^2 + 0.5CV_0^2) e^{-2\xi\omega_n t}. \end{aligned}$$

It is not a constant anymore. Starting at the initial value (equal to the total initial energy storage in L and C), it decreases exponentially with time and goes to zero as $t \rightarrow \infty$. Once again, the damping factor ξ governs the rate at which the total stored energy decreases with time.

It is instructive to look at the amount of energy lost in one oscillation period. But is there a *period* of oscillation in this case?

The product of exponential function and sinusoidal function is not a sinusoid. *It will not be a periodic waveform at all.* However, the time interval between the successive zero-crossings of the waveform will remain constant since this is decided by the sinusoid in the product. Therefore, we cannot talk about a *period* in the case of damped oscillations; but we can talk about the time interval between two successive zero-crossings in the same direction. Often this interval is called ‘period’ of damped oscillation in loose manner.

Let us find the energy loss incurred over one such interval and express it as a fraction of the total energy storage at the beginning of that interval.

Let the variable $\omega_n t$ be $k\pi$ at the beginning of one such interval (i.e., at a zero-crossing), where k is an integer. The value of the total energy storage at this point is $(0.5LI_0^2 + 0.5CV_0^2) e^{-2k\pi\xi}$ J. The next similar zero crossing will take place at $\omega_n t + 2\pi$ rad.

The value of total energy storage at that time point will be $(0.5LI_0^2 + 0.5CV_0^2) e^{-2(k+2)\pi\xi}$ J.

$$\begin{aligned} \therefore \text{Fractional loss of stored energy over one oscillation} &= \frac{e^{-2k\pi\xi} - e^{-2(k+2)\pi\xi}}{e^{-2k\pi\xi}} \\ &= 1 - e^{-4\pi\xi} \\ &= 1 - \left(1 - 4\pi\xi + \frac{(4\pi\xi)^2}{2!} + \dots \right) \\ &= 4\pi\xi \text{ for } \xi \ll 1. \end{aligned}$$

Expressions for source-free response of weakly-damped RLC circuit.

Total stored energy in a source-free, weakly-damped RLC circuit decreases exponentially with time.

Voltage and Current oscillations in a pure LC circuit are periodic functions of time.

Voltage and Current oscillations in a damped RLC circuit are *not* periodic functions of time.

Loss of the stored energy in one cycle of oscillation in a weakly-damped RLC circuit is proportional to its damping factor.

By a similar reasoning, one can see that the fractional loss of amplitude of voltage and current oscillations in the circuit over one oscillation period is $\approx 2\pi\xi$, i.e., the amplitude of oscillation in the $(n + 1)$ th cycle of the oscillation will be $\approx (1 - 2\pi\xi)$ times the amplitude of n th oscillation.

All the voltage variables and the circuit current variable during the zero-input response of a weakly-damped series RLC circuit with $L = 1$ H, $C = 1$ F, $R = 0.08$ Ω , $V_0 = 2$ V and $I_0 = 1$ A are plotted in Fig. 11.6-7. The damping factor ξ in this case is 0.04 and the undamped natural frequency ω_n is 1 rad/s.

The successive positive peaks of $v_C(t)$ are 2.22 V and 1.71 V and negative peaks are 1.95 V and 1.51 V. The fractional loss of amplitude over first cycle of oscillation calculated from these numbers are 0.23 and 0.226 and are almost equal to $2\pi\xi = 0.25$. The exponential envelope function has a time constant of $1/0.04 = 25$ s. One full oscillation takes about 6.3 s and hence there will be roughly about 9 oscillations before the amplitude of oscillation get damped down to less than 10% of their starting values.

The time-variation of various energy storage functions for zero-input response of this circuit is plotted in Fig. 11.6-8.

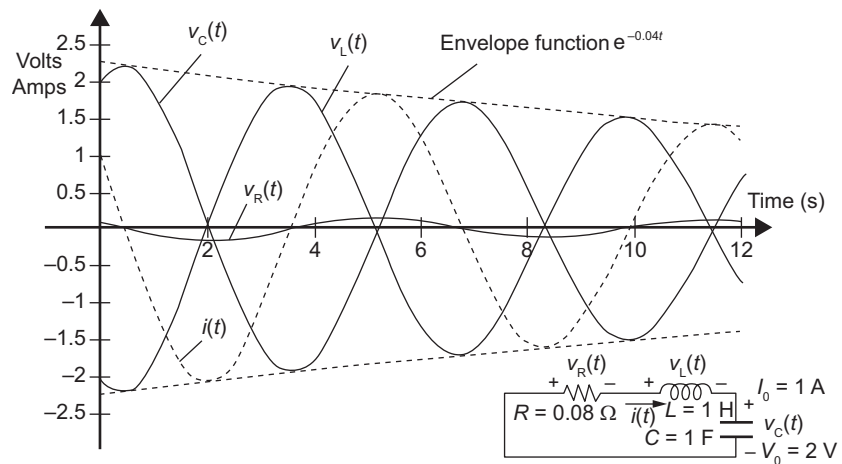


Fig. 11.6-7 Zero-Input Response of a Weakly-Damped RLC Circuit

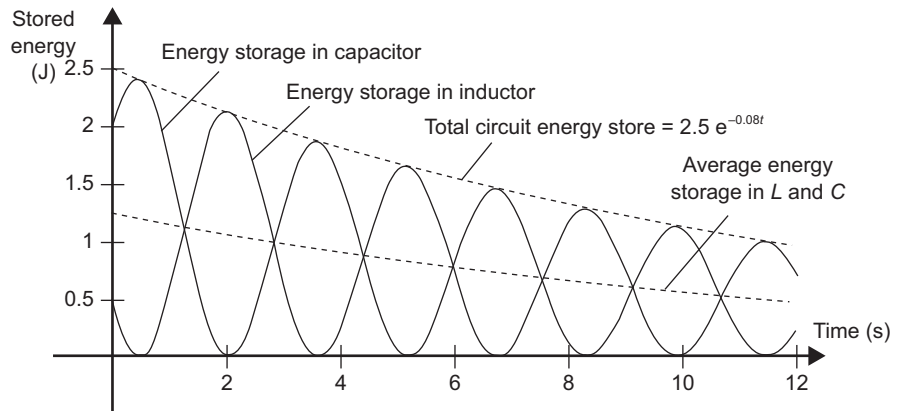


Fig. 11.6-8 Energy Storage Functions of a Weakly-Damped RLC Circuit ($L = 1$ H, $C = 1$ F, $R = 0.08$, $V_0 = 2$ V and $I_0 = 1$ A)

This *exponential decay* of total energy storage in a series *RLC* circuit is applicable only when $\xi \ll 1$. Total energy storage will come down with time for other values of ξ too; but not as per a clean exponential law. This is illustrated in Fig. 11.6-9 for a circuit with $\xi = 0.25$. $v_C(t)$ and $i(t)$ for the circuit referred in this figure was already plotted in Fig. 11.6-4.

The nature of zero-input response when ξ is not small is qualitatively similar to that of the weakly-damped case. However, the time-interval in the zero crossings (*period* of oscillation in a loose sense) is not $2\pi/\omega_n$ – it is rather $2\pi/\omega_d$, where $\omega_d = \sqrt{1-\xi^2}\omega_n$. This number is termed as ‘*damped natural frequency*’ in circuit studies.

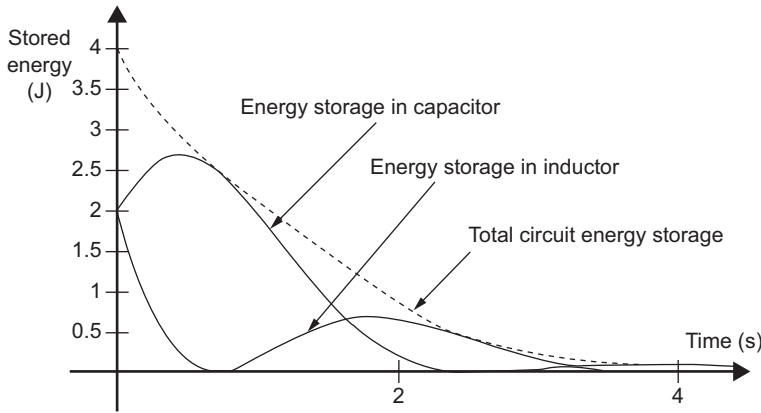


Fig. 11.6-9 Energy Storage Functions of a Moderately-damped ($\xi = 0.25$) RLC Circuit ($L = 1$ H, $C = 1$ F, $R = 0.5 \Omega$, $V_0 = 2$ V and $I_0 = 2$ A)

11.6.4 Standard Formats for Second-order Circuit Zero-input Response

Second-order circuits embody all the three kinds of responses possible in any linear system. It is a widely studied and applied response. Hence, certain standard formats have emerged in dealing with second-order system response in the areas of Circuit Theory, Linear Control Systems, Linear Electronic Circuits, Linear System Analysis, etc. We have already introduced many terms associated with this standardised format and we complete the rest in this sub-section.

A second-order circuit will have a homogeneous differential equation of the following type where $x(t)$ is some circuit variable chosen as the describing variable for the circuit.

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x = 0 \text{ for } t \geq 0^+ \text{ with } x(0^-) = x_0 \text{ and } \left. \frac{dx}{dt} \right|_{(0^-)} = x'_0 \quad (11.6-14)$$

The coefficients of this differential equation, a_1 and a_2 , will be decided by the circuit parameters like R , L and C . *The same differential equation will be obtained quite independent of which particular circuit variable happens to be chosen as the describing variable $x(t)$.*

The first standard format for this equation is

$$\frac{d^2x}{dt^2} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = 0 \text{ for } t \geq 0^+ \text{ with } x(0^-) = x_0 \text{ and } \left. \frac{dx}{dt} \right|_{(0^-)} = x'_0 \quad (11.6-15)$$

where, $\omega_n = \sqrt{a_2}$ and $\xi = \frac{a_1}{2\sqrt{a_2}}$.

The second standard format is

$$\frac{d^2x}{dt^2} + \frac{\omega_n}{Q} \frac{dx}{dt} + \omega_n^2 x = 0 \text{ for } t \geq 0^+ \text{ with } x(0^-) = x_0 \text{ and } \left. \frac{dx}{dt} \right|_{(0^-)} = x'_0 \quad (11.6-16)$$

$$\text{where, } \omega_n = \sqrt{a_2} \text{ and } Q = \frac{\sqrt{a_2}}{a_1} = \frac{1}{2\xi}.$$

The ξ -factor was seen to be responsible for damping of oscillations in series *RLC* circuit and was called *damping factor*. ω_n was seen to be the angular frequency of voltage and current sinusoidal oscillations in an *undamped LC* circuit and was called *undamped natural frequency*. These interpretations for ξ and ω_n are applicable to any second-order circuit or linear system.

The new factor that has appeared in Eqn. 11.6-16 – the *Q*-factor – is called the *Quality Factor* of the second-order circuit or system.

Four cases are identified depending on the value of ξ .

Case (i) $\xi > 1$. This is called the *over-damped circuit*. The roots of characteristic equation for such a circuit will be two distinct negative real numbers. Its zero-input response will consist of two decaying real exponential functions of time.

Case (ii) $\xi = 1$. This is called the *critically-damped circuit*. The roots of characteristic equation for such a circuit will be a negative real number with multiplicity of two. In general, its zero-input response will consist of one decaying real exponential function and another function which is the product of time and this real exponential function.

Case (iii) $0 < \xi < 1$. This is called the *under-damped circuit*. The roots of characteristic equation of such a circuit are complex conjugate numbers with negative real part. The zero-input response of such a circuit will contain an *exponentially-damped sinusoidal oscillation*. The oscillations will not be periodic in the strict sense of periodicity. However, the zero crossings of oscillations will take place at regular intervals and periodicity is understood in this sense. Then the angular frequency will be given by

$$\omega_d = \sqrt{1 - \xi^2} \omega_n = \sqrt{1 - \left(\frac{1}{2Q}\right)^2} \omega_n \text{ rad/s. } \omega_d \text{ is called the } \textit{damped natural frequency} \text{ in circuit}$$

studies.

Case (iv) $\xi = 0$. This is called the *undamped circuit*. The roots of characteristic equation are pure imaginary and conjugates. The zero-input response consists of steady sinusoidal oscillation at angular frequency of ω_n .

The response in the over-damped and critically-damped cases is not oscillations and the concept of frequency (if it is understood as the number of times something repeats in one second) is not relevant in those cases. However, linear system theorists decided to call the index of the exponential functions describing the zero-input response of a linear system as its *natural frequencies* long back.

Thus, the roots of characteristic equation of a circuit are called its *natural frequencies*. We should not get confused here trying to count the number of oscillations in a decaying real exponential function. The word ‘frequency’ in the term *natural frequency* has absolutely nothing to do with the idea of *repetitiveness*. Those numbers give us the index of exponential functions that describe the *natural* source-free response of our circuit – nothing more or nothing less. The emphasis is on ‘*natural*’ and not on ‘*frequency*’.

Since roots of characteristic equation of a circuit can be complex, natural frequencies also can be complex.

However, $e^{st} = e^{\sigma t} (\cos \omega t + j \sin \omega t)$ is a complex function of time. A physical electrical circuit cannot have a non-physical complex time function in its zero-input response. That is why, if $\sigma + j\omega$ is one of its natural frequencies, then, $\sigma - j\omega$ will also be one of its natural frequencies. Natural frequencies of a linear circuit, if complex, appear in conjugate pairs. The complex exponential response components due to conjugate pairs will always add up to yield a real physical waveform.

Quality Factor, Q

Quality factor *Q* of a second-order system is inversely proportional to its *damping factor* ξ .

Damping factor and Quality factor are used interchangeably in Linear System related studies with Electrical Filter Designers preferring Quality Factor to Damping Factor.

Reduced damping leads higher values for *Q*-factor. An undamped system has a quality factor that tends to infinity.

Natural frequency defined.

Natural Frequencies

A natural frequency value $s = \sigma + j\omega$ stands for a complex exponential signal e^{st} in time-domain. This signal will represent one of the many components present in the zero-input response of the circuit that has s as one of its natural frequencies. Thus, *natural frequency* is a stand-in for e^{st} .

The unit for real part of natural frequency is *neper/s* and for imaginary part is *rad/s*.

We complete this subsection by listing the solution of Eqn. 11.6-14 for the under-damped case in both formats.

$$x(t) = e^{-\xi\omega_n t} \left[\left(x_0 \xi + \frac{x'_0}{\omega_n} \right) \frac{\sin \omega_d t + x_0 \cos \omega_d t}{\sqrt{1-\xi^2}} \right]$$

$$\text{or } e^{-0.5 \frac{\omega_n t}{Q}} \left[\left(x_0 + \frac{2Qx'_0}{\omega_n} \right) \frac{\sin \omega_d t + x_0 \cos \omega_d t}{\sqrt{4Q^2 - 1}} \right] \text{ for } t \geq 0^+,$$

$$\text{where } \omega_d = \sqrt{1-\xi^2} \omega_n = \frac{\sqrt{4Q^2 - 1}}{2Q} \omega_n.$$

11.7 IMPULSE RESPONSE OF SERIES RLC CIRCUIT

Impulse response of a circuit is the zero-state response with unit impulse input. The relevant circuit is shown in Fig. 11.7-1(a).

The capacitor cannot absorb the impulse voltage. It cannot absorb even a finite voltage change without infinite current flow, let alone impulse voltage. Resistor cannot absorb impulse voltage, for, in that case the circuit current will become infinite and the inductor is not going to permit that. Hence, the entire impulse voltage appears across the inductor. This results in 1 Wb-T of flux linkage getting dumped into the inductor instantaneously; resulting in a sudden change in its current from zero to $1/L$ A. Therefore, $i(t)$ at $t = 0^+$ becomes $1/L$ A and $v_C(t)$ at $t = 0^+$ remains at 0. And the input source becomes short-circuit for all $t \geq 0^+$. Thus, the circuit effectively becomes a source-free circuit shown in Fig. 11.7-1(b), with initial energy of $1/2L$ J in inductor and 0 J in the capacitor. Thereafter the response is similar to zero-input response.

$$v_C(t) = \left(\frac{\omega_n}{\sqrt{1-\xi^2}} \right) e^{-\xi\omega_n t} \sin \sqrt{1-\xi^2} \omega_n t \text{ V for } t \geq 0^+$$

$$i(t) = C \frac{dv_C(t)}{dt} \quad (11.7-1)$$

$$= \frac{1}{L} \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos \left(\sqrt{1-\xi^2} \omega_n t + \tan^{-1} \frac{\xi}{\sqrt{1-\xi^2}} \right) \text{ A for } t \geq 0^+.$$

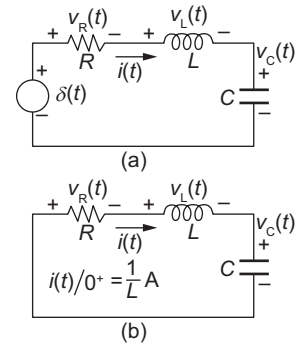


Fig. 11.7-1 Series RLC Circuit with Unit Impulse Voltage Input

Unit impulse response of under-damped series RLC circuit.

11.8 STEP RESPONSE OF SERIES RLC CIRCUIT

Step response of any linear electrical circuit is the zero-state response with unit step input. This can be found by integrating the impulse response of the same circuit. Only the under-damped case is considered here.

Impulse response of a under-damped series RLC circuit is,

$$v_C(t) = \left(\frac{\omega_n}{\sqrt{1-\xi^2}} \right) e^{-\xi\omega_n t} \sin \sqrt{1-\xi^2} \omega_n t \text{ V for } t \geq 0^+$$

Unit step response
of voltage across the
capacitor in an
under-damped series
RLC circuit.

$$= \left(\frac{\omega_n}{\sqrt{1-\xi^2}} \right) \frac{e^{(-\xi\omega_n + j\sqrt{1-\xi^2}\omega_n)t} - e^{(-\xi\omega_n - j\sqrt{1-\xi^2}\omega_n)t}}{2j} V \text{ for } t \geq 0^+.$$

Therefore, the step response of the same circuit is,

$$v_C(t) = \left(\frac{\omega_n}{2j\sqrt{1-\xi^2}} \right) \int_0^t \left(e^{(-\xi\omega_n + j\sqrt{1-\xi^2}\omega_n)t} - e^{(-\xi\omega_n - j\sqrt{1-\xi^2}\omega_n)t} \right) dt.$$

Carrying out the required integration and simplifying the result, we get,

$$v_C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos \left(\sqrt{1-\xi^2} \omega_n t - \tan^{-1} \frac{\xi}{\sqrt{1-\xi^2}} \right) V \text{ for } t \geq 0^+. \quad (11.8-1)$$

The step response waveforms of a series RLC circuit with $L = 1$ H and $C = 1$ F are shown in Fig. 11.8-1 for various values of resistance resulting in damping factors of 0.5, 0.2, 0.05 and 0. Solid curve shows $v_C(t)$ and dotted curve shows $i(t)$ in all cases.

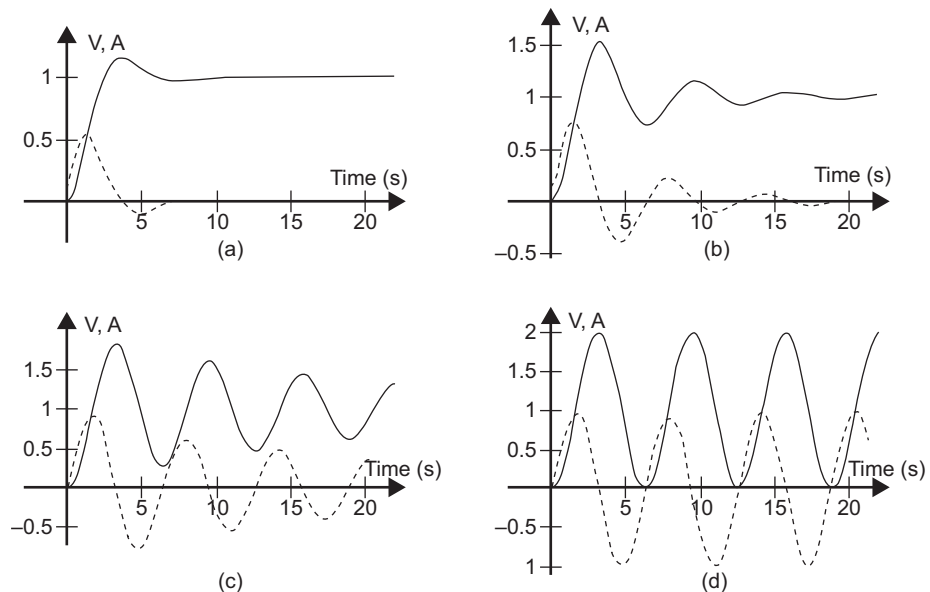


Fig. 11.8-1 Step Response of Series RLC Circuit (1 H, 1 F) (a) $\xi = 0.5$ (b) $\xi = 0.2$ (c) $\xi = 0.05$ (d) $\xi = 0$

11.9 STANDARD TIME-DOMAIN SPECIFICATIONS FOR SECOND-ORDER CIRCUITS

Certain measures of the speed and nature of response of a second-order system, defined with respect to step response, are in wide use in various areas of Electrical and Electronic Engineering. These measures are:

1. Delay time, t_d – is the time required by step response to cross 50% of final value for the first time.
2. Rise time, t_r – is the time required for step response to reach 100% of final value for the first time starting from 0% in the case of under-damped response. In the case of over-damped response and critically-damped response it is the time required for step response to reach 90% of final value from 10% of final value.

3. Peak time, t_p – is the time required for step response to reach its first peak.
4. Settling time, t_s – is the time required for the step response to reach a $\pm 5\%$ band around the final value and remain within that band thereafter.
5. Maximum (percentage) overshoot, M_p – is the difference between the maximum peak value attained by the step response and the final value of step response expressed as a fraction or percentage of the final value of step response.

These measures are illustrated on a typical step response plot of a series RLC circuit (capacitor voltage is plotted) in the Fig. 11.9-1.

It is possible to derive expressions for these measures starting from Eqn. 11.8-1. The algebra involved is skipped here and the final expressions are given in Eqn. 11.9-1.

$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}}{\sqrt{1-\xi^2}\omega_n}, \quad t_p = \frac{\pi}{\sqrt{1-\xi^2}\omega_n}, \quad t_s = \frac{3}{\xi\omega_n}, \quad M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}. \quad (11.9-1)$$

Time-domain specifications for a second-order circuit.

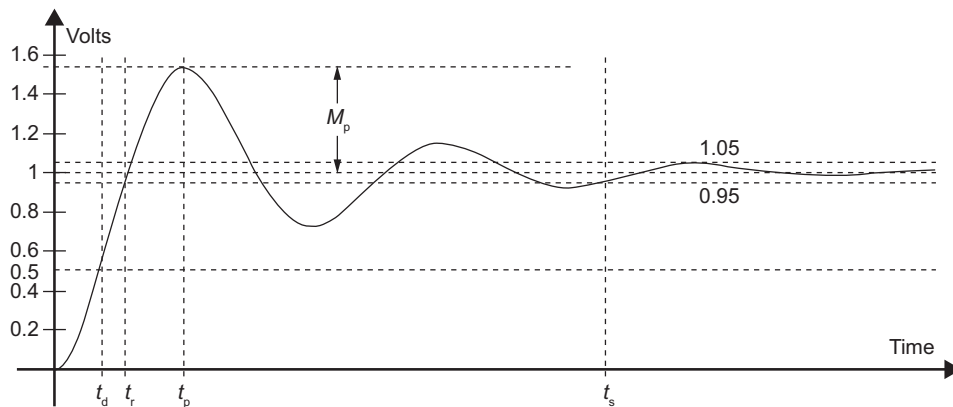


Fig. 11.9-1 Time-domain Specifications for a Second-Order Circuit

The relationship between various time-domain specifications and the damping factor is highly non-linear. Figure 11.9-2 shows the variation of rise time (the lower curve), peak time (the upper curve) and maximum overshoot in step response of capacitor voltage in a series RLC circuit with an undamped natural frequency of 1 rad/s. $\xi \approx 0.7$ is a critical value. When ξ increases above 0.7, the response becomes *sluggish*. When ξ decreases below 0.7, the maximum overshoot increases rapidly without much change in the rise time. Thus, $\xi = 0.7$ is a value of damping factor that results in a step response which has optimal speed without incurring too heavy a penalty in the form of excessive overshoot.

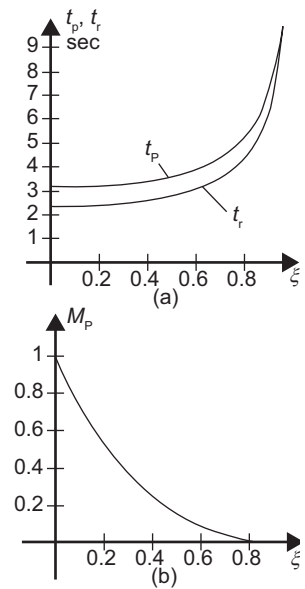


Fig. 11.9-2 (a) Rise Time, Peak Time and (b) Maximum Overshoot in a Second-Order Circuit versus ξ ($\omega_n = 1$ rad/s)

11.10 EXAMPLES ON IMPULSE AND STEP RESPONSE OF SERIES RLC CIRCUITS

The circuit variables and polarity conventions for specifying initial conditions in the following examples are the same as in the series RLC circuit in Fig. 11.6-1.

EXAMPLE: 11.10-1

A series RLC circuit has $L = 10 \text{ mH}$, $C = 1000 \text{ }\mu\text{F}$ and $R = 10 \text{ }\Omega$. The capacitor is initially charged with 10 V . Find an expression for the voltage across the capacitor when the circuit is excited by $10 u(t)$.

SOLUTION

This is a problem that involves both zero-input response and zero-state response. The zero-input response component arises due to the initial voltage across the capacitor. The zero-state response component is nothing but step response scaled by 10.

Zero-Input Response Component:

Characteristic equation is $\gamma^2 + 10^3 \gamma + 10^5 = 0$. Natural frequencies of the circuit are -112.7 and -887.3 . This is an over-damped circuit.

$$\therefore v_C(t) = A_1 e^{-112.7t} + A_2 e^{-887.3t}$$

$$\text{and } i(t) = 10^{-3} \times (-112.7 A_1 e^{-112.7t} - 887.3 A_2 e^{-887.3t})$$

Applying initial conditions of 10 V and 0 A at 0^+ ,

$$A_1 + A_2 = 10 \text{ and } -0.1127 A_1 - 0.8873 A_2 = 0 \Rightarrow A_1 = 11.455 \text{ and } A_2 = -1.455.$$

$$v_C(t) = 11.455 e^{-112.7t} - 1.455 e^{-887.3t} \text{ V for } t \geq 0^+ \text{ is the zero-input response.}$$

Zero-State Response Component:

The initial conditions at $t = 0^-$ are set to zero for determining zero-state response. The final value of step response in a series RLC circuit is 1 V across the capacitor. Hence, the final value of response is 10 V across the capacitor in this example. The format of transient response is known from the values of natural frequencies. Therefore, $v_C(t) = 10 + A_1 e^{-112.7t} + A_2 e^{-887.3t}$ is the format of zero-state response for $10 u(t)$ input.

The applied voltage suddenly becomes 10 V at $t = 0^+$. Since the inductor current cannot change instantaneously unless impulse voltage comes across it and capacitor voltage cannot change instantaneously unless impulse current flows through it, the capacitor voltage and inductor current remain at zero values at $t = 0^+$. The 10 V step appears across the inductor first, thereby changing the initial value of derivative of current. But, we need initial values only for capacitor voltage and inductor current to solve for A_1 and A_2 in the zero-state response above. Applying these initial values on

$$v_C(t) = 10 + A_1 e^{-112.7t} + A_2 e^{-887.3t} \text{ and } i(t) = -0.1127 A_1 e^{-112.7t} - 0.8873 A_2 e^{-887.3t},$$

$$\text{we get } A_1 + A_2 = -10 \text{ and } -0.1127 A_1 - 0.8873 A_2 = 0 \Rightarrow A_1 = -11.455 \text{ and } A_2 = 1.455.$$

$$\therefore v_C(t) = 10 - 11.455 e^{-112.7t} + 1.455 e^{-887.3t} \text{ V for } t \geq 0^+ \text{ is the zero-state response.}$$

$$\therefore \text{Total response of } v_C(t)$$

$$= \text{zero-input response} + \text{zero-state response}$$

$$= 11.455 e^{-112.7t} - 1.455 e^{-887.3t} + 10 - 11.455 e^{-112.7t} + 1.455 e^{-887.3t}$$

$$= 10 \text{ V for } t \geq 0^+.$$

We need not have to go through all this to see that $v_C(t) = 10$. If the expected final state of a circuit is same as the given initial state, then, there is no transient response in the circuit.

Note the method employed in solving this problem. We could have obtained the zero-input response by direct substitution in Eqn. 11.6-5. But such direct substitutions in equations should be avoided as far as possible and problems should be worked out employing basic principles governing the circuit behaviour.

EXAMPLE: 11.10-2

A series RLC circuit has $L = 10 \text{ mH}$, $C = 100 \text{ }\mu\text{F}$ and $R = 10 \text{ }\Omega$. The capacitor is initially charged with -10 V and the inductor has 1 A flowing in it at $t = 0^-$. Find an expression for the voltage across the capacitor when the circuit is excited by $10 u(t)$.

SOLUTION

This is a problem that involves both zero-input response and zero-state response. The zero-input response component arises due to the initial voltage across the capacitor and initial current through inductor. The zero-state response component is nothing but step response scaled by 10.

Zero-Input Response Component:

Characteristic equation is $\gamma^2 + 10^3 \gamma + 10^6 = 0$. Natural frequencies of the circuit are -500 and -500 . This is a critically-damped circuit.

$$\therefore v_C(t) = A_1 e^{-500t} + A_2 t e^{-500t}$$

$$\text{and } i(t) = 10^{-4} (-500 A_1 e^{-500t} - 500 A_2 t e^{-500t} + A_2 e^{-500t})$$

Applying initial conditions of -10 V and 1 A at 0^+ ,

$$A_1 = -10 \text{ and } -0.05 A_1 + 0.0001 A_2 = 1 \Rightarrow A_1 = -10 \text{ and } A_2 = 5000$$

$$\therefore v_C(t) = -10 e^{-500t} + 5000t e^{-500t} \text{ V for } t \geq 0^+ \text{ is the zero-input response.}$$

Zero-State Response Component:

The initial conditions at $t = 0^-$ are set to zero values for determining zero-state response. The final value of step response in a series RLC circuit is 1 V across the capacitor. Hence, the final value of response here is 10 V across the capacitor in this example. The format of transient response is known from the values of natural frequencies. Therefore, $v_C(t) = 10 + A_1 e^{-500t} + A_2 t e^{-500t}$ is the format of zero-state response for $10 u(t)$ input.

The initial values of $v_C(t)$ and $i(t)$ remain at zero at $t = 0^+$ as explained in the previous example. Applying these initial values on

$$v_C(t) = 10 + A_1 e^{-500t} + A_2 t e^{-500t} \text{ and}$$

$$i(t) = -0.05 A_1 e^{-500t} - 0.05 A_2 t e^{-500t} + 0.0001 A_2 e^{-500t},$$

$$\text{we get } A_1 = -10 \text{ and } -0.05 A_1 + 0.0001 A_2 = 0 \Rightarrow A_1 = -10 \text{ and } A_2 = -5000$$

$$\therefore v_C(t) = 10 - 10 e^{-500t} - 5000t e^{-500t} \text{ V for } t \geq 0^+ \text{ is the zero-state response.}$$

$$\begin{aligned} \therefore \text{Total response of } v_C(t) &= \text{zero-input response} + \text{zero-state response} \\ &= -10 e^{-500t} + 5000t e^{-500t} + 10 - 10 e^{-500t} - 5000t e^{-500t} \\ &= 10 - 20 e^{-500t} \text{ V for } t \geq 0^+. \end{aligned}$$

EXAMPLE: 11.10-3

Develop the differential equation governing $v_C(t)$ in the circuit in Fig. 11.10-1 and find its step response for $L = 16$ mH, $C = 10$ μ F and $R = 100$ Ω . Obtain the rise time, peak time, settling time and maximum percentage overshoot for this response.

SOLUTION

Applying KCL at the junction between R , L and C along with the element equation of a capacitor yields $i(t) = C \frac{dv_C(t)}{dt} + \frac{v_C(t)}{R}$.

$$\text{Applying KVL in the first mesh yields } L \frac{di(t)}{dt} + v_C(t) = u(t).$$

Substituting the first equation in the second one results in,

$$LC \frac{d^2 v_C(t)}{dt^2} + \frac{L}{R} \frac{dv_C(t)}{dt} + v_C(t) = u(t)$$

$$\text{i.e., } \frac{d^2 v_C(t)}{dt^2} + \frac{1}{RC} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{1}{LC} u(t).$$

$$\text{The critical resistance of this circuit is } \frac{1}{2} \sqrt{\frac{L}{C}}$$

Substituting the parameter values,

$$\frac{d^2 v_C(t)}{dt^2} + 10^3 \frac{dv_C(t)}{dt} + 2500^2 v_C(t) = 2500^2 \text{ for } t \geq 0^+$$

$$\omega_n = 2500 \text{ rad/s, } 2\xi \omega_n = 10^3 \Rightarrow \xi = 0.2.$$

$$\therefore \omega_d = \sqrt{1 - \xi^2} \omega_n = 2449.5 \text{ rad/s.}$$

The final value of step response is obtained by replacing the capacitor by open-circuit and the inductor by short-circuit. This shows that the final value is 1 V.

The initial conditions required for solving the differential equation are the values of the capacitor voltage at $t = 0^+$ and of its first derivative at $t = 0^+$.

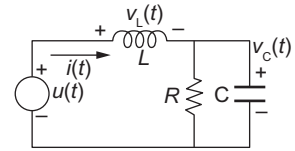


Fig. 11.10-1 Circuit for Example 11.10-3

$$v_c(0^+) = v_c(0^-) = 0$$

$$\frac{dv_c(t)}{dt} = \frac{i_c(t)}{C} = \frac{i(t) - \frac{v_c(t)}{R}}{C}$$

$$i(0^+) = i(0^-) = 0$$

$$\therefore \left. \frac{dv_c(t)}{dt} \right|_{t=0^+} = 0.$$

The step response for $v_c(t)$ will have the following format with A_1 and A_2 decided by initial conditions.

$$v_c(t) = 1 + e^{-500t} (A_1 \cos 2449.5t + A_2 \sin 2449.5t).$$

Then,

$$v_c(t) = -500 e^{-500t} (A_1 \cos 2449.5t + A_2 \sin 2449.5t) + e^{-500t} (-2449.5 A_1 \sin 2449.5t + 2449.5 A_2 \cos 2449.5t)$$

Substituting initial conditions,

$$A_1 = -1 \text{ and } -500 A_1 + 2449.5 A_2 = 0$$

Solving for A_1 and A_2 , we get, $A_1 = -1$ and $A_2 = -0.2041$

$$\therefore \text{Step response } v_c(t) = 1 - e^{-500t} (\cos 2449.5t + 0.2041 \sin 2449.5t) = 1 - 1.0206 e^{-500t} \cos (2449.5t - 0.2013 \text{ rad}) \text{ V for } t \geq 0^+.$$

$$\text{Rise time, } t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}}{\sqrt{1-\xi^2} \omega_n} = \frac{\pi - 1.37 \text{ rad}}{0.978 \times 2500} = 0.725 \text{ ms}.$$

$$\text{Peak time, } t_p = \frac{\pi}{\sqrt{1-\xi^2} \omega_n} = \frac{\pi}{0.978 \times 2500} = 1.29 \text{ ms}.$$

$$\text{Settling time, } t_s = \frac{3}{\xi \omega_n} = \frac{3}{0.2 \times 2500} = 6 \text{ ms}.$$

$$\text{Percentage overshoot, } M_p = 100 \times e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}} = 100 \times e^{-0.6425} = 52.6\%.$$

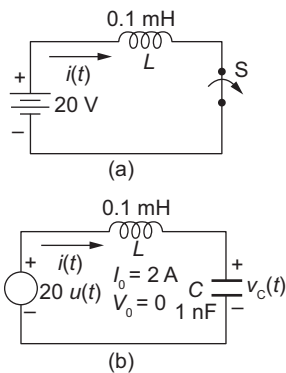


Fig. 11.10-2 Circuit for Example 11.10-4

EXAMPLE: 11.10-4

An inductor in a circuit tends to keep the current in that circuit smooth. This makes it somewhat difficult to open a switch in such a circuit. The inductor demands an infinitely large voltage across it for its current to go to zero instantaneously and this voltage appears across the switch.

In practice, as the switch opens, the voltage across the inductor and the voltage across the switch rise rapidly to large values and soon the air (or whatever that is there between the switch contacts) breaks down electrically and becomes ionised and conducting. And, an arc is struck through the ionised medium and the current continues to flow through the arc, dissipating the stored energy in inductor in the form of heat and light. However, the arc requires more voltage than the supply voltage to maintain itself and this excess voltage appears across the inductor with suitable polarity to effect a decrease in current. Specially designed switches called circuit breakers are employed to open abnormally large currents that result when the system suffers a short circuit somewhere in the electrical power systems. These will have mechanical systems to remove the ionised medium (air or oil) and supply fresh air or oil to the opening switch gap to help the current decrease faster. The arc gets extinguished at the next natural zero-crossing of AC cycle. After that, the switch gap can be modelled by a capacitance that includes all the circuit capacitance that comes in parallel with the open switch and the intrinsic parasitic capacitance of the open switch itself. A new transient involving the circuit inductance and switch capacitance starts at that instant. The voltage of AC supply at the instant arc gets extinguished may not be zero (in fact, it is close to the peak value of voltage since a power system under faulted conditions has a very low lag power factor). This voltage suddenly gets applied to a LC circuit

formed by circuit inductance and switch capacitance and the short-circuit. This is a circuit with low damping and will have a natural frequency that is much larger than the AC supply frequency. Hence the AC voltage will not change much during the first few cycles of oscillations in this circuit. Therefore, we may solve for step response of the circuit with the size of the step equal to the peak AC system voltage to analyse the transient voltage that appears across the switch immediately after the arc gets quenched.

This voltage transient that appears across a switch gap which has just recovered from an ionised conducting state is called *recovery voltage* in power system studies. The gap has not recovered completely and if the *rate of rise of recovery voltage* (RRRV) is excessive the switch gap will lose its blocking capability and a re-strike of arc will take place. Hence, RRRV is an important parameter in circuit breaker application.

If the switch is a semiconductor device, as soon as the device is switched off the parasitic capacitance across it comes into action. The current in the inductor continues to flow and charges up this capacitance. Soon (unless counter-measures are not employed) the voltage across the device capacitance reaches the break-down voltage of the device. The device breaks down and the current continues to flow through it. Since the device breaks down only at a voltage larger than the supply voltage usually, the voltage across inductor will be of suitable polarity to cause decrease in the current. Thus, the inductor discharges its stored energy into the semiconductor device working in the break-down mode till the circuit current reaches zero. Dissipating stored energy in the inductor in a semiconductor switch usually burns it out.

Consider the circuit in Fig. 11.10-2(a). The inductor started with zero current when the switch S was closed at $t = -10 \mu\text{s}$. The switch is kept closed for $10 \mu\text{s}$ and then opened. It is a semiconductor switch with 1 nF capacitance when it is open. (i) Find the voltage that appears across the switch and the current through the circuit after the switch goes open (a) if the switch break-down voltage is 1000 V (b) if the switch break-down voltage is 100 V (ii) Find the additional capacitance that has to be connected in parallel to the switch to prevent break-down.

SOLUTION

The current in the inductor at the end of $10 \mu\text{s} = 10 \mu\text{s} \times 20 \text{ V}/0.1 \text{ mH} = 2 \text{ A}$

The transient in the circuit after the switch goes open can be obtained by solving the circuit in Fig. 11.10-2(b) with initial conditions $V_0 = 0 \text{ V}$ and $I_0 = 2 \text{ A}$. This is a step response problem.

$$\text{Natural frequency of the circuit, } \omega_n = \sqrt{\frac{1}{0.1 \times 10^{-3} \times 1 \times 10^{-9}}} = 3.162 \times 10^6 \text{ rad/s.}$$

$$\text{Damping factor, } \xi = 0.$$

$$\therefore \omega_d = \sqrt{1 - \xi^2} \omega_n = 3.162 \times 10^6 \text{ rad/s.}$$

The circuit is an undamped one. Therefore, the format of solution for $v_C(t)$ is

$$v_C(t) = 20 + A_1 \sin 3.162 \times 10^6 t + A_2 \cos 3.162 \times 10^6 t$$

Then,

$$i(t) = C \frac{dv_C(t)}{dt} = 3.162 \times 10^{-3} A_1 \cos 3.162 \times 10^6 t - 3.162 \times 10^{-3} A_2 \sin 3.162 \times 10^6 t.$$

$$\text{Applying } v_C(0^+) = 0 \text{ V, } i(0^+) = 2 \text{ A,}$$

$$A_2 = -20$$

$$3.162 \times 10^{-3} A_1 = 2.$$

$$\therefore A_1 = 632.5 \text{ and } A_2 = -20.$$

$$\therefore v_C(t) = 20 + 632.5 \sin 3.162 \times 10^6 t - 20 \cos 3.162 \times 10^6 t \\ = 20 + 632.82 \sin (3.162 \times 10^6 t - 0.032 \text{ rad}) \text{ V for } t \geq 0^+. \quad (11.10-1)$$

$$i(t) = 2 \cos (3.162 \times 10^6 t - 0.032 \text{ rad}) \text{ A for } t \geq 0^+$$

Consider the situation when the switch break-down voltage is $\pm 1000 \text{ V}$. In this case the switch voltage is an undamped oscillation centred around 20 V and with a frequency of $\sim 1 \text{ MHz}$. The positive peak goes to 652.82 V and negative peak goes to -612.82 V . The switch voltage goes to these high voltage levels since the initial energy of inductor has to be absorbed by the switch capacitor at voltage peak points.

The current in the circuit continues as a 2 A oscillation at $\sim 1 \text{ MHz}$. The current never gets switched off unless there is some damping in the circuit.

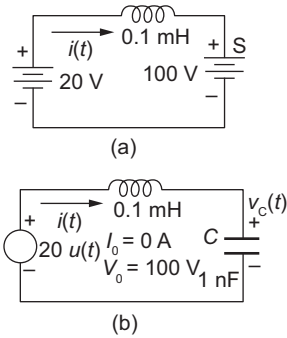


Fig. 11.10-3 Equivalent Circuits for Example 11.10-4

An inductor carrying current has certain amount of energy stored in it. It is necessary to dissipate this stored energy or reabsorb it before the current in an inductive circuit can be switched off.

The capacitance that appears across the open switch is forced to absorb this inductive energy in a switching context. Example 11.10-4 makes it clear that it may be necessary to shunt the switch with an additional capacitor in order to limit the voltage that appears across the open switch. It also points to the need to employ a dissipating element across the switch to damp the oscillation that follows a switching operation.

A series combination of judiciously chosen capacitor and resistor with the resistor shunted by a diode is commonly employed across semiconductor switches used in inductive circuits in practice.

Now, let us consider the situation if the break-down voltage of the semiconductor switch is ± 100 V. In this case too the switch voltage follows the curve given by Eqn. 11.10-1 till the voltage reaches 100 V. This happens at about $0.05\mu\text{s}$ as shown below.

$$100 = 20 + 632.82 \sin(3.162 \times 10^6 t - 0.032 \text{ rad})$$

$$\therefore t = \frac{\sin^{-1} \frac{80}{632.82} + 0.032}{3.162 \times 10^6} = 0.05 \mu\text{s}$$

$$i(t) = 2 \cos(3.162 \times 10^6 t - 0.032 \text{ rad}) \text{ A for } t \geq 0^+$$

$$\therefore i(t) \text{ at } t = 0.05 \mu\text{s} = 1.984 \text{ A.}$$

From that time, the semiconductor switch enters the break-down mode and conducts till current becomes zero. The semiconductor switch in break-down mode maintains a constant voltage of 100 V across itself and thereby acts similar to a battery of 100 V. Therefore, the circuit now is equivalent to the circuit in Fig. 11.10-3(a).

The voltage across inductance is -80 V now and hence its current decreases with a constant slope of $-0.8 \text{ A}/\mu\text{s}$ from $t = 0.05 \mu\text{s}$ to reach zero at $t = 2.53 \mu\text{s}$. The switch voltage will remain constant at 100 V till then. At $t = 2.53 \mu\text{s}$ the switch recovers and a new equivalent circuit as shown in Fig. 11.10-3(b) becomes applicable from that instant onwards. The initial conditions (at $t = 2.53 \mu\text{s}$) are $V_0 = 100$ V and $I_0 = 0$. The circuit starts a new response from that time point. Thus, $v_c(t)$ may be expressed as follows.

$$\therefore v_c(t) = \begin{cases} 20 + 632.82 \sin(3.162 \times 10^6 t - 0.032 \text{ rad}) \text{ V for } 0^+ \leq t < 0.05 \mu\text{s} \\ 100 \text{ V for } 0.05 \mu\text{s} \leq t < 2.53 \mu\text{s} \\ 20 + 80 \cos 3.162 \times 10^6 t \text{ V for } t \geq 2.53 \mu\text{s} \end{cases}$$

$$i(t) = \begin{cases} 2 \cos(3.162 \times 10^6 (t - 2.53 \times 10^{-6}) - 0.032 \text{ rad}) \text{ A for } 0^+ \leq t < 0.05 \mu\text{s} \\ 1.984 \left(1 - \frac{t - 0.05 \times 10^{-6}}{2.48 \times 10^{-6}} \right) \text{ A for } 0.05 \mu\text{s} \leq t < 2.53 \mu\text{s} \\ -0.253 \sin(3.162 \times 10^6 (t - 2.53 \times 10^{-6}) - 0.032 \text{ rad}) \text{ A for } t \geq 2.53 \mu\text{s} \end{cases}$$

The switch voltage waveform is shown in Fig. 11.10-4.

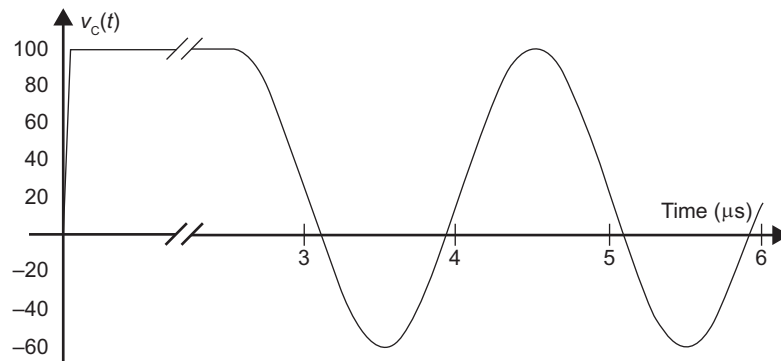


Fig. 11.10-4 Switch Voltage Waveform in Example 11.10-4

The switch voltage will oscillate with a frequency of $\sim 1 \text{ MHz}$ between 100 V and -60 V forever unless there is some damping in the circuit. There will be an accompanying current oscillation of 0.253 A amplitude. Here too, the circuit current does not get switched off unless there is some damping in the circuit.

To find the additional capacitance needed across the switch to limit the peak to 100 V

Let C_1 be the total capacitance across the switch. Then,

$$v_c(t) = 20 + A_1 \sin \omega_n t \times 10^6 t + A_2 \cos \omega_n t$$

Then,

$$i(t) = C_1 \frac{dv_c(t)}{dt}$$

$$= \omega_n C_1 A_1 \cos \omega_n t - \omega_n C_1 A_2 \sin \omega_n t$$

Applying $v_c(0^+) = 0$ V, $i(0^+) = 2$ A,

$$A_2 = -20$$

$$\omega_n C_1 A_1 = 2$$

$$\therefore A_1 = \frac{2}{\omega_n C} = 2\sqrt{\frac{L}{C}}, \text{ and } A_2 = -20.$$

$\sqrt{A_1^2 + A_2^2}$ must be ≤ 80 . Therefore, $C_1 \geq 66.7$ nF. Hence, the additional capacitance needed across the switch is ≥ 65.7 nF.

11.11 FREQUENCY RESPONSE OF SERIES *RLC* CIRCUIT

Consider a series *RLC* circuit excited by a voltage source $v_s(t) = 1 \sin \omega t u(t)$ V with a set of specified initial conditions. We continue to use $v_c(t)$, the voltage across capacitor, as the describing variable for the circuit.

The total response of the circuit is, as usual, given by the sum of zero-input response and zero-state response. Zero-input response contains only transient terms that are exponentially-damped sinusoid in nature. The zero-state response contains transient terms as well as the forced response term. The transient terms in zero-state response will be of the same nature as that of transient terms in zero-input response. In short, the total response will contain natural response terms of exponentially-damped sinusoidal nature and forced response term. We expect the natural response terms to vanish when time increases without limit. Only the forced response term acts in the long run, and of course, it is also termed steady-state response.

We are interested in the sinusoidal steady-state response of series *RLC* circuit in this section. In particular, we look at variation of the ratio of output amplitude to input amplitude and the phase angle by which output leads the input with frequency under sinusoidal steady-state – that is, we look at frequency response of series *RLC* circuit in detail.

11.11.1 Sinusoidal Forced-Response from Differential Equation

The differential equation governing $v_c(t)$, the capacitor voltage, in a series *RLC* circuit excited by a voltage source $v_s(t)$ was derived earlier in this chapter.

$$\frac{d^2 v_c(t)}{dt^2} + 2\xi \omega_n \frac{dv_c(t)}{dt} + \omega_n^2 v_c(t) = \omega_n^2 v_s(t), \quad (11.11-1)$$

$$\text{where } \omega_n = \sqrt{\frac{1}{LC}}, \quad \xi = \frac{R}{2\sqrt{\frac{L}{C}}}.$$

The steady-state component of response when $v_s(t) = 1 \sin \omega t u(t)$ is obtained by ‘method of undetermined coefficients’. We assume a trial solution of the form $v_c(t) = A \sin \omega t + B \cos \omega t$ and determine the values of A and B by substituting the assumed solution in the differential equation and equating coefficients of $\sin \omega t$ and $\cos \omega t$ on both sides of the resulting equation.

Trial solution, $v_c(t) = A \sin \omega t + B \cos \omega t$.

Sinusoidal steady-state response by method of undetermined coefficients.

Substituting in the differential equation and collecting terms,

$$[(\omega_n^2 - \omega^2)A - 2\xi\omega_n B] \sin \omega t + [(\omega_n^2 - \omega^2)B + 2\xi\omega_n A] \cos \omega t = \omega_n^2 \sin \omega t.$$

The only way this equation can remain true independent of t is by the coefficients of $\sin \omega t$ (and $\cos \omega t$) becoming equal on both sides of the equation. Therefore,

$$(\omega_n^2 - \omega^2)A - 2\xi\omega_n B = \omega_n^2.$$

$$(\omega_n^2 - \omega^2)B + 2\xi\omega_n A = 0.$$

Solving these two equations simultaneously we get,

$$A = \frac{\omega_n^2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 - 4\xi^2\omega_n^2\omega^2}, \quad B = \frac{-2\xi\omega_n^2(\omega_n\omega)}{(\omega_n^2 - \omega^2)^2 - 4\xi^2\omega_n^2\omega^2}.$$

Substituting these in the assumed solution and simplifying the solution to a 'single sinusoid with phase' form we get,

$$v_C(t) = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \sin(\omega t + \phi_C) \quad (11.11-2)$$

$$\text{where } \phi_C = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}.$$

Therefore, the magnitude part of frequency response function for $v_C(t)$ is

$$\frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \text{ and the phase part of the frequency response function is}$$

$$-\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}$$

11.11.2 Frequency Response from Phasor Equivalent Circuit

We remember at this point that phasor analysis is another way to arrive at the sinusoidal steady-state response of linear circuits. We verify the result in Eqn. 11.11-2 by employing phasor analysis. The series RLC circuit with unit amplitude sinusoidal excitation and its phasor equivalent circuit are shown in Fig. 11.11-1.

The frequency response function for any circuit variable is the ratio of output phasor to the input phasor. This ratio will be a complex ratio and its magnitude part will give the amplitude ratio and its angle part will give the phase angle by which the output sine wave *leads* the input sine wave under steady-state condition. We obtain three phasor ratios for the voltage variables in this circuit by employing voltage division principle in a series circuit.

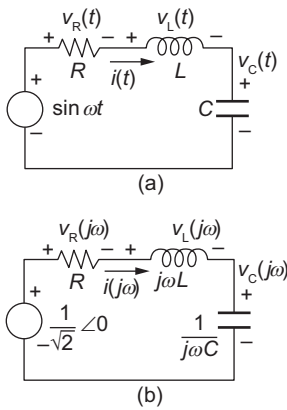


Fig. 11.11-1 (a) Series RLC Circuit and (b) its Phasor Equivalent

Frequency response of capacitor voltage in a series RLC circuit.

$$\begin{aligned} \frac{V_C(j\omega)}{V_S(j\omega)} &= \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{1 - \omega^2 LC + j\omega RC} \\ &= \frac{1/LC}{1/LC - \omega^2 + j\omega R/L} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega}. \end{aligned}$$

This ratio can be written in polar form as

$$\frac{V_C(j\omega)}{V_S(j\omega)} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_C \quad (11.11-3)$$

$$\text{where } \phi_C = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}$$

We see that the frequency response obtained by solving the differential equation is the same as the one obtained by employing phasor equivalent circuit as expected. Similar evaluation of phasor ratios leads to the frequency response functions for the remaining two voltage variables in the circuit. They are,

$$\begin{aligned}\frac{V_R(j\omega)}{V_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{\omega_n^2 - \omega^2 + j2\xi\omega\omega_n} \\ &= \frac{2\xi\omega\omega_n}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2} \angle\phi_R,\end{aligned}\quad (11.11-4)$$

$$\text{where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}$$

$$\begin{aligned}\frac{V_L(j\omega)}{V_S(j\omega)} &= \frac{(j\omega)^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega} \\ &= \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle\phi_L,\end{aligned}\quad (11.11-5)$$

$$\text{where } \phi_L = \pi - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}$$

Frequency response of resistor voltage in a series RLC circuit.

Frequency response of inductor voltage in a series RLC circuit.

The remaining variable, $i(t)$, is directly related to $v_R(t)$ and hence its frequency response need not be obtained separately.

11.11.3 Qualitative Discussion on Frequency Response of Series RLC Circuit

The ratio of voltage appearing across an element in a series circuit to the source voltage is equal to the ratio between the phasor impedance of that element to the sum of all the impedances in series. The impedance of an inductor increases linearly with angular frequency and impedance of a capacitor decreases in inverse proportion to angular frequency. We use these basic principles to discuss the shape of frequency response plots for the three possible outputs in the series RLC circuit.

The Voltage Across Resistor – The Band-pass Output

At zero frequency (that is, for DC steady-state) the inductor appears as a short and capacitor appears as open. Therefore, the magnitude part of frequency response is zero for $v_R(t)$, zero for $v_L(t)$ and 1 for $v_C(t)$ at this frequency.

The inductor appears as impedance of infinite magnitude and capacitor appears as impedance of zero magnitude as ω increases without limit. Therefore, all the high frequency voltage will appear across the inductor. Thus as $\omega \rightarrow \infty$, the magnitude part of frequency response is zero for $v_R(t)$ and $i(t)$, zero for $v_C(t)$ and 1 for $v_L(t)$.

The sign of impedance of inductor is positive and the sign of impedance of capacitor is negative for any ω . Thus, they tend to cancel each other partially in the sum at all frequencies. The cancellation is 100% at one particular frequency. The value of frequency at which this happens is when $\omega L = 1/\omega C \Rightarrow \omega = 1/\sqrt{LC}$. But this frequency was named as *undamped natural frequency* earlier. Thus, we conclude that, the reactance part of the total series impedance of a series RLC circuit goes to zero at ω_n and the circuit appears purely resistive under steady-state conditions at that frequency. Therefore, the current in the circuit at that frequency will be $v_S(t)/R$ and will be in phase with the input voltage. The power factor of the circuit will be unity at that frequency.

Resonance

In general, in a circuit excited by a single sinusoidal voltage source (current source) across a pair of terminals, *resonance* is the condition under which the current drawn at the terminals (voltage appearing across the terminals) is in phase with the source voltage (current).

Equivalently, *resonance* is the condition under which the input impedance (admittance) offered to the sinusoidal source is resistive.

The cancellation between the inductive reactance and capacitive reactance is only partial at all other frequencies. Hence, the magnitude of total series impedance of the circuit at any frequency other than ω_n will be more than R and amplitude of current will be less than $1/R$ A (assuming unit amplitude excitation) at all other frequencies. *Thus, in a series RLC circuit, the impedance is a minimum and amplitude of current (and hence amplitude of voltage across the resistor) is a maximum at ω_n .* Moreover, the current in the circuit will be at unity power factor at that frequency. This condition in the series RLC circuit is called the *resonance* condition and the frequency at which this happens is called the *resonant frequency*. Obviously, in a series RLC circuit, the *resonant frequency* and the *undamped natural frequency* are the same.

The amplitude of voltage appearing across the resistor in a series RLC circuit under resonance condition is same as the amplitude of input. Therefore, the magnitude of frequency response for $v_R(t)$ begins with zero at zero frequency, goes to unity at ω_n and tapers down to zero as $\omega \rightarrow \infty$. The total reactance in a series RLC circuit is capacitive for $\omega < \omega_n$ and it is inductive for $\omega > \omega_n$. Therefore, the voltage across resistor leads the input voltage for frequencies lower than resonant frequency and lags the input voltage for frequencies higher than resonant frequency. Thus, the phase of frequency response of $v_R(t)$ starts at 90° at $\omega = 0$, becomes zero at $\omega = \omega_n$ and decreases to -90° as $\omega \rightarrow \infty$.

Equation 11.11-4 confirms all these conclusions. The shape of magnitude response and phase response for the voltage across resistor is plotted against ω/ω_n ratio for various damping factors in Fig. 11.11-2.

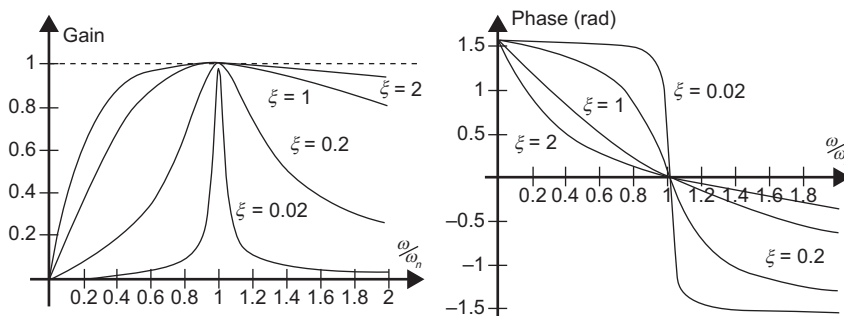


Fig. 11.11-2 Magnitude and Phase Plots for Frequency Response of Resistor Voltage in a Series RLC Circuit

Series RLC circuit with output taken across R is a *narrow band-pass filter* for low values of ξ ($\xi < 0.1$ or $Q > 5$) and it is a *wide band-pass filter* for high value of ξ ($\xi > 0.25$ or $Q < 2$).

The magnitude curve is found to become narrower as the damping ratio is reduced. The resistor voltage in a series RLC circuit exhibits the so-called *band-pass* characteristic. A frequency response is said to be of *band-pass* nature when it attenuates low frequency sinusoids and high frequency sinusoids considerably and passes on mid-frequency sinusoids preferentially. Resistor voltage in a series RLC circuit is a band-pass output for all values of damping factor – that is, even an over-damped series RLC circuit behaves as a band-pass filter if the output is taken across the resistor. However, the *band-pass* characteristic becomes sharper and sharper when the damping in the circuit is reduced. That is, the circuit becomes highly *frequency-selective* as ξ approaches zero.

Another point of great significance is that the output in this band-pass filter is *in-phase* with the input at a frequency that is at the centre of the band – i.e., at ω_n . Output signal undergoes a phase change by about 180° when its frequency varies in a small band around ω_n if ξ is very small. The phase curve for $\xi = 0.02$ is shown in Fig. 11.11-2. This kind of rapid variation of phase of output over a small frequency range has considerable negative implications in designing control systems for systems that involve RLC circuits.

The Voltage Across Capacitor – The Low-pass Output

The magnitude response for this output starts at unity at zero frequency and goes to zero as $\omega \rightarrow \infty$. In between it may be a monotonically decreasing function or it may attain a maximum depending on the damping present in the circuit. The frequency response function shown in Eqn. 11.11-3 is plotted in Fig. 11.11-3 for the various values of ξ .

This output is essentially a *low-pass* output. However, it is a bad low-pass filter if it is too under-damped or over-damped. This is so because we want the magnitude response of a low-pass filter to be reasonably flat till a particular value of ω and then fall to zero more or less rapidly. Only curves as in Fig. 11.11-3(c) and (d) look good from this point of view. In fact the value of ξ used in filter design is 0.7 typically and this corresponds to curve as in Fig. 11.11-3(d).

A series RLC circuit with voltage input and output taken across the capacitor can be designed to function as a good low-pass filter.

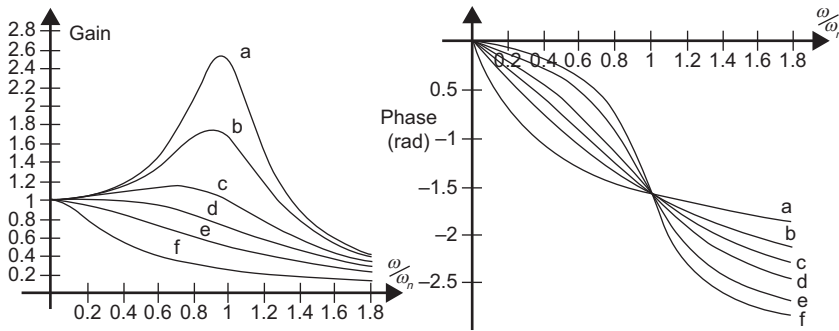


Fig. 11.11-3 Magnitude and Phase Plots for Capacitor Voltage Frequency Response in a Series RLC Circuit (a) $\xi = 0.2$ (b) $\xi = 0.3$ (c) $\xi = 0.5$ (d) $\xi = 0.7$ (e) $\xi = 1$ (f) $\xi = 2$

Observe that for $\xi < 0.7$ the magnitude response exhibits a peak. The ratio of this peak gain value to DC gain value is called *resonant peak factor*. The frequency at which the resonant peak in gain occurs *is not* the frequency at which resonance occurs in the circuit. Resonance occurs in the circuit at ω_n and the voltage across the resistor goes to a maximum value at that frequency. But the frequency at which the capacitor voltage goes to a maximum for fixed input amplitude is different from ω_n and it depends on ξ too. This frequency is less than ω_n and shifts more to the left with increase in ξ . The expression for magnitude response Eqn. 11.11-3 may be differentiated with respect to ω and set to zero to find the frequency at which maximum takes place (if at all there is a maximum) and the value of the maximum. The results will be

$$\omega_{cp} = \omega_n \sqrt{1 - 2\xi^2} \quad \text{and} \quad R_{cp} = \frac{1}{2\xi \sqrt{1 - \xi^2}} = \frac{Q}{\sqrt{1 - 1/4Q^2}},$$

where ω_{cp} is the frequency at which gain maximum takes place and R_{cp} is the resonant peak factor. The expressions reveal that there is a resonant peak only for $\xi < 1/\sqrt{2} \approx 0.7$.

The gain for capacitor voltage at ω_n is $1/(2\xi)$ ($=Q$) as shown in Eqn. 11.11-3 with $\omega = \omega_n$. This predicts large amplitude voltage across the capacitor when the damping factor is small. For example, the amplitude of voltage across the capacitor is 50 V when a 1 V sinusoid is applied to the circuit at resonant frequency if the ξ -factor is 0.01 (equivalently, Q -factor is 50). How does this voltage amplification take place?

We have seen that the series circuit impedance is resistive and a minimum at ω_n . The reactance of L and C cancel each other at that frequency. Hence, R decides the amplitude of current and the reactance of C multiplies this current amplitude to convert it into voltage

Resonant peak frequency and resonant peak factor in the low-pass output in a series RLC circuit.

The voltage gain available across the capacitor in a series RLC circuit under sinusoidal steady-state exhibits a peak for all $\xi > 0.707$.

The gain available at resonant frequency is equal to the quality factor Q of the circuit. However, this is not the peak gain.

The peak gain is $>Q$ and is available at a frequency less than resonance frequency.

amplitude. Therefore, the ratio of amplitude of capacitor voltage to amplitude of input voltage at ω_n must be $1/\omega_n RC$. Similarly, the ratio of amplitude of inductor voltage to amplitude of input voltage at ω_n must be $\omega_n L/R$. But, both these ratios are equal to the Q -factor of the circuit as given below.

$$\frac{1}{\omega_n RC} = \frac{\sqrt{LC}}{RC} = \frac{\sqrt{L/C}}{R} = \frac{1}{2\xi} = Q$$

$$\frac{\omega_n L}{R} = \frac{L}{\sqrt{LCR}} = \frac{\sqrt{L/C}}{R} = \frac{1}{2\xi} = Q.$$

Thus, the voltage amplification factor for capacitor voltage and inductor voltage in a series RLC circuit at resonant frequency is its Q -factor.

Resonant amplification of voltage in a series RLC circuit is one method used in High Voltage Engineering to generate high AC voltages from low voltage sources. Such amplification becomes possible because at resonant frequency the voltage across L will be of same amplitude and opposite phase as that of the voltage across C and therefore they cancel each other without absorbing any portion of input voltage to sustain them.

The Voltage Across Inductor – The High-pass Output

The magnitude response for this output starts at zero at zero frequency and goes to unity as $\omega \rightarrow \infty$. In between it may be a monotonically increasing function or it may attain a maximum depending on the damping present in the circuit. The frequency response function shown in Eqn. 11.11-5 is plotted in Fig. 11.11-4 for various values of ξ .

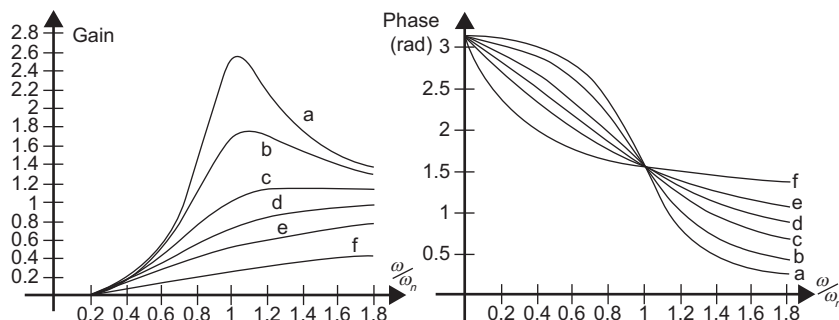


Fig. 11.11-4 Magnitude and Phase Plots for Inductor Voltage Frequency Response in a Series RLC Circuit (a) $\xi = 0.2$ (b) $\xi = 0.3$ (c) $\xi = 0.5$ (d) $\xi = 0.7$ (e) $\xi = 1$ (f) $\xi = 2$

The plots as well as our qualitative discussion reveal that the inductor voltage in a series RLC circuit has *high-pass nature* under sinusoidal steady-state. But, it is a bad high-pass filter if the series RLC circuit is only lightly-damped. A good high-pass filter is expected to have near-zero gain at low frequencies till a particular frequency and a gain that rapidly rises to unity and remains there after that frequency. The prominent resonant peaks in the gain evident in magnitude response plot in Fig. 11.11-4 for low ξ values are not acceptable in a good high-pass filter. Such resonant peaks will lead to considerable distortion even for signals that do not have any low frequency sinusoids in them.

The voltage across inductor in series RLC circuit *leads* the input voltage by a phase angle that varies from 180° to 0° with ω . The phase angle is 90° at ω_n .

The angular frequency at which the voltage across the inductor peaks for given input voltage amplitude is *not* ω_n . It is always greater than ω_n and moves further to the right of ω_n as ξ increases. Exact expression for the angular frequency at which the magnitude response peaks and the value of the peak (if such resonant peaking occurs at all) can be obtained by differentiating the magnitude expression in Eqn. 11.11-5 with respect to ω and setting the derivative to zero. The result will be

$$\omega_{lp} = \frac{\omega_n}{\sqrt{1-2\xi^2}} \text{ and } R_{lp} = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{Q}{\sqrt{1-1/4Q^2}},$$

where ω_{lp} is the frequency at which gain maximum takes place and R_{lp} is the resonant peak factor. The expressions reveal that there is a resonant peak only for $\xi < 1/\sqrt{2} \approx 0.7$.

We observe that $\omega_{cp}\omega_{lp} = \omega_n^2$ and $R_{cp} = R_{lp}$. Thus, the geometric mean of resonant peak frequencies at high-pass and low-pass outputs is equal to the resonant peak frequency at the band-pass output.

The magnitude response for all the three outputs are shown in Fig. 11.11-5 for $\xi = 0.3$. The points brought out in the discussion on resonant peaks and the frequencies at which resonant peaks occur are illustrated in this plot.

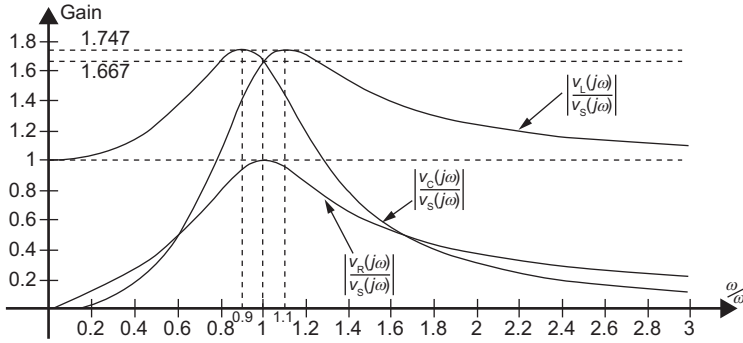


Fig. 11.11-5 Magnitude Response for Low-Pass, Band-Pass and High-Pass Outputs in a Series RLC Circuit with Damping Factor = 0.3

The capacitor voltage attains a peak amplitude of 1.747 V at $\sim 0.9\omega_n$ and the inductor voltage attains a peak amplitude of 1.747 V at $\sim 1.1\omega_n$ for an input amplitude of 1 V. Both attain amplitude of 1.667 V at resonant frequency ω_n . The Q -factor of the circuit is 1.67.

11.11.4 A More Detailed Look at the Band-pass Output of Series RLC Circuit

The frequency response at the band-pass output of series RLC circuit (i.e., the voltage across the resistor) was obtained as

$$\begin{aligned} \frac{V_R(j\omega)}{V_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \\ &= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R \end{aligned} \quad (11.11-6)$$

$$\text{where, } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}$$

Frequency response function for band-pass output in a series RLC circuit.

'Half-power frequencies' defined and explained as a measure of frequency selectivity of band-pass circuits.

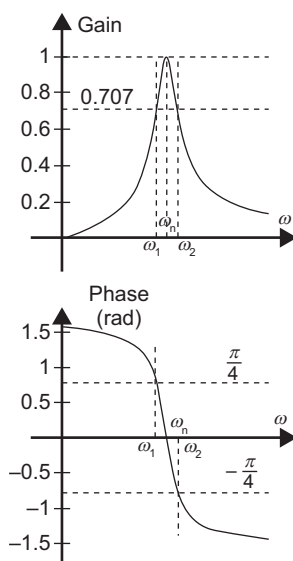


Fig. 11.11-6 A Typical Band-pass Circuit Frequency Response

The ratio of centre frequency to bandwidth of a narrow band-pass filter is equal to its quality factor. The half-power frequencies of such a filter are at approximately equal distance on either side of the centre frequency.

Different interpretations for Q -factor of a RLC circuit.

It was shown that this output has a maximum gain of unity and that occurs at $\omega = \omega_n$. We wish to develop a measure for the *frequency selectivity* exhibited by this output in this sub-section.

The resistor gets all of the input voltage if input frequency is resonant frequency. Therefore, the resistor receives maximum power for fixed amplitude input if the input frequency is resonant frequency. The power dissipated at all other frequencies will be less than this value. There will be one or more frequencies at which the power dissipated in the resistor is exactly 50% of the power that will be dissipated if input is at resonant frequency (assuming the input amplitude is kept fixed). These frequencies have been used traditionally to define measures of frequency selectivity in band-pass circuits. They are called *half-power frequencies* for obvious reason.

Power dissipated in a resistor becomes 50% when voltage developed across it becomes $\frac{100\%}{\sqrt{2}} \approx 70.7\%$. Therefore, *half-power frequencies* are the angular frequencies at which the magnitude response of the circuit output becomes 70.7% of some reference gain value. The reference gain value in the case of low-pass circuit is the DC gain, in the case of band-pass circuit it is the maximum gain and in the case of high-pass circuit it is the gain as $\omega \rightarrow \infty$.

The frequency at which the gain of a band-pass circuit reaches maximum is termed *centre frequency* in filter studies. A typical band-pass response is shown in Fig. 11.11-6. The half power frequencies ω_1 and ω_2 are marked in the magnitude plot.

The difference between the two half-power frequencies is called the *bandwidth* of the band-pass circuit. We develop an expression for bandwidth of a *narrow band-pass* circuit below and develop interesting insight into the relation between the bandwidth and quality factor of the circuit.

$$\frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{(2\xi\omega\omega_n)^2}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2} = \frac{1}{2}$$

$$\text{Let } x = \frac{\omega}{\omega_n}. \text{ Then, } \frac{4\xi^2x^2}{(1-x^2)^2 + 4\xi^2x^2} = \frac{1}{2}$$

$$\therefore (1-x^2)^2 - 4\xi^2x^2 = 0 \Rightarrow (1-x^2) = \pm 2\xi x.$$

$$\therefore x^2 \pm 2\xi x - 1 = 0 \Rightarrow x = \mp \xi \pm \sqrt{1+\xi^2}.$$

$$\text{Taking only positive values for } \omega, \omega_{2,1} = (\sqrt{1+\xi^2} \pm \xi)\omega_n \Rightarrow bw = 2\xi\omega_n.$$

$$\therefore \frac{\text{Centre frequency}}{\text{Bandwidth}} = \frac{1}{2\xi} = Q \quad (11.11-7)$$

This Q -factor (equivalently, the damping factor) has indeed turned out to be an important parameter for series RLC circuit. Equation 11.11-7 is the third interpretation for Q .

We had seen earlier that, in a weakly-damped series RLC circuit, the fractional loss of total stored energy in the circuit over one cycle of oscillation is given by $4\pi\xi$. Since $Q = 1/2\xi$,

$$Q = 2\pi \frac{\text{Total stored energy in the source-free circuit}}{\text{Energy lost in one cycle of free response}}.$$

Second interpretation is based on the same energy ratio under sinusoidal steady-state conditions at resonant frequency. Let the circuit be at resonance with 1 V amplitude input. Then,

$$v_s(t) = 1 \sin \omega_n t, \therefore i(t) = \frac{1}{R} \sin \omega_n t \text{ and } v_C(t) = \frac{-1}{\omega_n RC} \cos \omega_n t$$

$$\begin{aligned}\text{Total stored energy} &= \frac{Li(t)^2}{2} + \frac{Cv_C(t)^2}{2} = \frac{L}{2R^2} \sin^2 \omega_n t + \frac{C}{2\omega_n^2 R^2 C^2} \cos^2 \omega_n t \\ &= \frac{L}{2R^2} \left(\sin^2 \omega_n t + \cos^2 \omega_n t \right) = \frac{L}{2R^2} \left(1 \right) = \frac{L}{2R^2}\end{aligned}$$

$$\text{Energy dissipated in one cycle} = \frac{1}{\omega_n} \int_0^{2\pi} R \times \left(\frac{1}{R} \sin \omega_n t \right)^2 d(\omega_n t) = \frac{\pi}{\omega_n R}$$

$$\therefore \frac{\text{Total stored energy}}{\text{Energy dissipated in one cycle}} = \frac{1}{2\pi} \frac{\omega_n L}{R} = \frac{1}{2\pi} \frac{\sqrt{L/C}}{R} = \frac{1}{2\pi} \frac{1}{2\xi} = \frac{Q}{2\pi}$$

$$\therefore Q = 2\pi \frac{\text{Total stored energy under resonance condition}}{\text{Energy dissipated in one cycle under resonance condition}}$$

11.11.5 Quality Factor of Inductor and Capacitor

We are already familiar with the concept of Q -factor for a series RLC circuit. We discuss a similar factor for the elements themselves in this section.

We had assumed till now that the inductor and capacitor used in the series RLC circuit are ideal elements and that they have no parasitic elements associated with them. This is not true in practice. The non-zero parasitic elements associated with inductor and capacitor will affect the performance of RLC circuits considerably in narrow band-pass circuit applications.

An inductor has a non-zero wire resistance that goes along with its inductance in series. Further, if the inductor uses iron core, there will be hysteresis and eddy current losses in the iron core due to time-varying magnetic fields in the core. These losses are strongly dependent on frequency of operation and flux level in the core. Core loss is usually modelled by a resistance in parallel to the inductance. However, due to its complex dependence on frequency of operation, it cannot be satisfactorily modelled by a single value of resistance at all frequencies.

In addition, a physical inductor will have distributed capacitance of winding shunting its inductance value. Thus, a practical inductor is more like the equivalent circuit shown in Fig. 11.11-7(a).

However, in frequency response studies, the distributed capacitance C_p is usually ignored since its value is generally too small to affect the circuit performance. Circuit equivalent in Fig. 11.11-7(b) is usually employed in studying the effect of losses in the inductor on the Q -factor of a circuit employing this inductor. R_p can, at the best, represent the core losses in the inductor only for a small band of frequencies around a frequency value at which it was measured.

The circuit in Fig. 11.11-7(b) cannot be reduced to the circuit in Fig. 11.11-7(c) such that the circuit in Fig. 11.11-7(c) remains equivalent to the circuit in Fig. 11.11-7(b) at all frequencies. However, the circuit in Fig. 11.11-7(b) can be reduced to the circuit as in Fig. 11.11-7(c) – that is, a value of L' and R_s' can be found such that the circuit in Fig. 11.11-7(c) will have same phasor impedance as that of the circuit in Fig. 11.11-7(b) – at some particular frequency. Usually the value of L' will be close to L and it is approximated that way in practice. R_s' will include the effects of R_p and R_s together. Since, in any case, a specific value of R_p is valid only over a small band around a specific frequency, the circuit in Fig. 11.11-7(b) can be equivalenced to circuit in Fig. 11.11-7(c) subject to the condition that it can be expected to give reasonably accurate results only over a small band of frequencies around the specific frequency at which R_p and R_s' are measured or calculated. This is satisfactory in the case of resonance studies in under-damped circuits since the frequency range of interest is a small band of frequencies around ω_n . The value of R_s' is usually indicated in an indirect manner by specifying it through a ratio. That ratio is the Q -factor of Inductor.

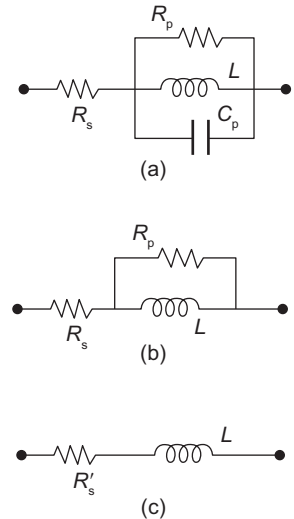


Fig. 11.11-7 Equivalent Circuits for a Practical Inductor

The Q -factor of a reactive element is defined as the ratio between maximum energy storage in the reactance to the energy lost in one cycle under steady-state operation at the frequency at which Q is calculated.

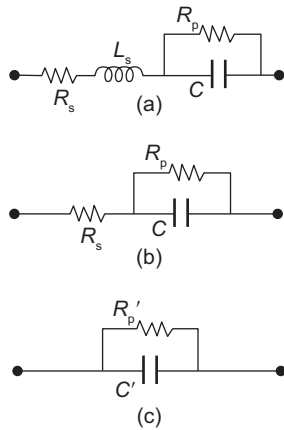


Fig. 11.11-8 Equivalent Circuits for a Practical Capacitor

From the definition of Q -factor of a reactive element (see the side-box) it follows that it is the ratio of reactance of the inductor at ω to the resistance value R_s' relevant to that frequency. Therefore,

$$Q\text{-factor of an inductor at } \omega = \frac{\omega L}{R_s' \text{ measured at } \omega}. \quad Q\text{-factor of an inductor will}$$

change with frequency.

A commercial Q -meter that is available in any well-equipped laboratory will have features that permit measurement of Q of inductors at various frequencies.

A practical capacitor also has three parasitic elements associated with it. The foil resistance and lead inductance come in series with the capacitance. The leakage current that flows through the imperfect dielectric employed in the capacitor is modelled by a resistance in parallel to the capacitor. The loss mechanisms in the capacitor are also frequency-dependent and hence an equivalent circuit for a practical capacitor will be valid only for a small band of frequencies around a specific frequency at which the measurement is carried out.

The series inductance L_s in the detailed equivalent circuit in Fig. 11.11-8(a) is usually ignored (or absorbed along with the inductor in series) in studies on under-damped resonant circuits. And, the equivalent circuit in Fig. 11.11-8(b) is approximated by the circuit in Fig. 11.11-8(c) with $C' \approx C$ and R'_p as measured by a Q -meter. The equivalent circuit in Fig. 11.11-8(c) is understood to be valid only for a small band ($\approx 20\%$ max) around the frequency at which R'_p was measured. The Q -meter measures it indirectly and displays the *Quality factor of capacitor*. Q -factor for a capacitor is defined as the ratio of resistance value R'_p at ω to the reactance of the capacitor at ω . Q -factor of a capacitor will change with frequency. Therefore,

$$Q\text{-factor of a capacitor at } \omega \text{ rad/s} = \omega C \times R'_p \text{ measured at } \omega.$$

EXAMPLE: 11.11-1

Series RLC circuit is widely employed as *averaging filters* in Switched Mode Power Supplies, DC–DC Converters, DC–AC Inverters, etc. The load appears across the capacitor of the averaging filter. There will be no external resistor in series with the inductor. However, the winding resistance and core losses of inductor along with other series dissipating mechanisms present in the power electronic circuitry behind the filter stage will provide damping for the otherwise undamped LC Filter circuit. We look at an example of use of LC circuit as an averaging filter in a DC–DC converter stage.

The power electronics in the converter converts a fixed DC voltage of 24 V into a switched wave as shown in Fig. 11.11-9(b).

The average value of the input is 12 V and it has a frequency of 10 kHz. The circuit used to extract the average value from this input and deliver it to a load

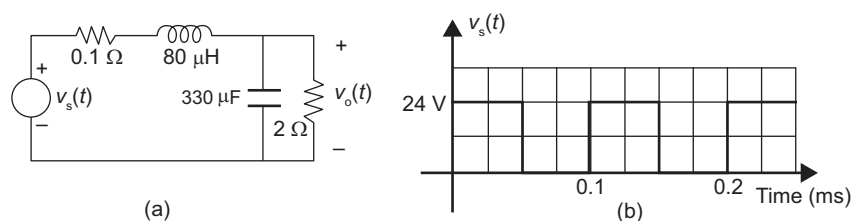


Fig. 11.11-9 Circuit and Input waveform for Example 11.11-1

resistance of $2\ \Omega$ is shown in Fig. 11.11-9(a). Calculate and plot the output voltage under steady-state.

[Hint: A symmetric square wave of ± 1 V amplitude and f Hz frequency can be approximated by the truncated series $\frac{4}{\pi} \left(\sin 2\pi ft + \frac{1}{3} \sin 6\pi ft + \frac{1}{5} \sin 10\pi ft + \frac{1}{7} \sin 14\pi ft \right)$.]

SOLUTION

When the rectangular waveform is switched on to the filter circuit, the system undergoes the usual transient response and settles down to a steady-state in the long run. We look at only the steady-state output in this example.

The waveform shown in Fig. 11.11-9(b) can be resolved into two components – the first component is a DC of 12 V (that is the cyclic average value of $v_s(t)$) and the second component is a symmetric square wave of ± 12 V amplitude and 10 kHz frequency.

Steady-state responses due to simultaneous application of two sources obey superposition principle. Therefore, $v_o(t)$ can be found as superposition of DC steady-state response and steady-state response to square wave input. The DC steady-state response is $12 \times 2/2.1 = 11.43$ V since inductor is short and capacitor is open under DC steady-state.

The ± 12 V square-wave can be approximated by sum of four sinusoids as given in the hint. Therefore, we have frequency response problem at hand. We use phasor equivalent circuit for this purpose.

The lowest frequency we are interested in is 10 kHz. The other frequencies of interest are 30 kHz, 50 kHz and 70 kHz. The resonant frequency, ω_r , of the circuit is

$$\frac{1}{2\pi} \frac{1}{\sqrt{80 \times 10^{-6} \times 330 \times 10^{-6}}} \text{ Hz} = 979.5 \text{ Hz}.$$

The phasor impedances at 10 kHz are $j5.026\ \Omega$ for $80\ \mu\text{H}$ inductor and $-j0.0482\ \Omega$ for $330\ \mu\text{F}$ capacitor.

$$\therefore \text{Gain at 10 kHz} = \frac{2 / -j0.0482}{0.1 + j5.026 + 2 / -j0.0482} = 0.00968 \angle -177.5^\circ.$$

Amplitude of the input at this frequency = $12 \times 4/\pi = 15.28$ V.

$\therefore$ 10 kHz component in steady-state output = $15.28 \times 0.00968 \sin(2\pi \times 10^4 t - 177.5^\circ)$ V.

The next input component to be considered is $5.09 \sin(6\pi \times 10^4 t)$ V. The gain at 30 kHz is obtained by noting the inductor impedance as $j5.026 \times 3 = j15.08\ \Omega$ and the capacitor impedance as $-j0.0482/3 = -j0.016\ \Omega$. Compared to the $2\ \Omega$ in parallel, the impedance of capacitor is so small in magnitude that the parallel combination is practically $-j0.016\ \Omega$ itself. Therefore, the required gain is

$$\frac{-j0.016}{0.1 + j15.08 - j0.016} \approx \frac{-j0.016}{j15.064} \approx -0.00106.$$

Therefore, 30 kHz component in a steady-state output = $5.09 \times 0.00106 \sin(6\pi \times 10^4 t)$ V.

Similar calculations show that the gain at 50 kHz is -0.0004 and the gain at 70 kHz is -0.0002 . Thus, the total steady-state output due to the square wave input component is constructed as below.

$$\begin{aligned} &= 15.28 \times 0.00968 \sin(2\pi \times 10^4 t - 177.5^\circ) - 5.09 \times 0.00106 \sin 6\pi \times 10^4 t \\ &\quad - 3.062 \times 0.0004 \sin 10\pi \times 10^4 t - 2.22 \times 0.0002 \sin 14\pi \times 10^4 t \\ &= 0.15 \sin(2\pi \times 10^4 t - 177.5^\circ) - 0.005 \sin 6\pi \times 10^4 t \\ &\quad - 0.001 \sin 10\pi \times 10^4 t - 0.0004 \sin 14\pi \times 10^4 t. \end{aligned}$$

All the components, except the first component, are really negligible and we need not even have calculated them. We will accept the first two components and neglect the last two. Thus, the total steady-state voltage at the output = $11.43 + 0.15 \sin(2\pi \times 10^4 t - 177.5^\circ) - 0.005 \sin 6\pi \times 10^4 t$ V. The AC component of output is plotted in Fig. 11.11-10 and it is clear that it is heavily dominated by the 10 kHz component.

The peak-to-peak ripple in DC output is 0.3 V and represents 2.6% of the DC value of 11.43 V.

Let us see how the low-pass filter has achieved this extent of attenuation of AC component. We noted that the resonant frequency of the circuit is about 0.98 kHz. Thus,

Example 11.11-1 illustrates the use of frequency response calculations in analysing the performance of a LC circuit employed as an averaging filter.

It explains why an LC averaging filter is superior to a RC averaging filter in terms of ripple content in output and power dissipation in the filtering process.

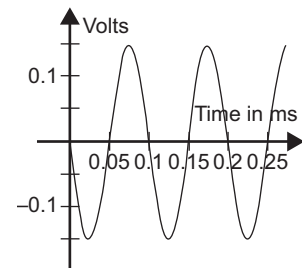


Fig. 11.11-10 The Rippling Component of Output Voltage in Example 11.11-1

The attenuation for AC components in a well-designed LC averaging filter.

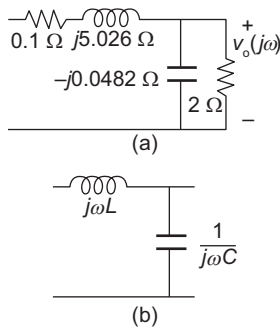


Fig. 11.11-11 Phasor Equivalent Circuits for Example 11.11-1

LC filter is superior to RC filter in 'averaging' applications.

the ratio between the lowest frequency present in the input, that is 10 kHz, and the resonant frequency of the circuit is about 10. We now show that the gain of a LC low-pass filter for a sinusoidal input at f Hz is $\propto (f_n/f)^2$, where f_n is the resonant frequency in Hz unit and $f \gg f_n$. We show it in two ways.

Equation 11.11-3 gives the frequency response function for capacitor voltage in a standard series RLC circuit. It is reproduced below.

$$\frac{V_C(j\omega)}{V_S(j\omega)} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle\phi_C$$

$$\text{where } \phi_C = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}$$

$$\frac{|V_C(j\omega)|}{|V_S(j\omega)|} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}}$$

$$= \frac{1}{\sqrt{(1-x^2)^2 + 4\xi^2x^2}}, \text{ where } x = \frac{\omega}{\omega_n} = \frac{f}{f_n}$$

$$\approx \frac{1}{\sqrt{(1-x^2)^2 + 4\xi^2x^2}} \approx \frac{1}{\sqrt{(1-x^2)^2}} = \frac{1}{x^2} = \left(\frac{\omega_n}{\omega}\right)^2 = \left(\frac{f_n}{f}\right)^2 \text{ for } x \gg 1.$$

The second way to appreciate the second-order gain characteristic of LC low-pass filter is by considering the following qualitative argument based on phasor impedances. The purpose of the averaging filter is to eliminate AC components in output. Some intervening element has to absorb the AC component in the input if it is not to appear at output. An element which can take large AC voltage across it while keeping current low is inductance (since its impedance increases with frequency). And, a capacitor across the output will shunt out whatever AC current that tries to get into the load resistance (because capacitor has low impedance at high frequency). Thus, the inductor *chokes* the high frequency current while absorbing almost all the input AC voltage content and the capacitor located across the output *absorbs* whatever AC current that appears even after the inductor *chokes* it. This is how a LC filter does averaging. There is a two-fold action – a series element that makes it more and more difficult for AC current to flow as frequency increases and a shunt element which makes it more and more difficult for AC voltage to develop across it as frequency increases. This explains the inverse square dependence of gain on frequency at high frequency values.

At a sufficiently high frequency, any resistance or inductor that is connected in parallel with a capacitor may be ignored for approximate calculation. Similarly, at a sufficiently high frequency, any resistance or capacitor connected in series with an inductor may be ignored for approximate calculations. For example, see the phasor equivalent of circuit in this example at 10 kHz in Fig. 11.11-11(a).

Obviously, the 2Ω and 0.1Ω can be safely ignored. Therefore, the circuit can be approximated by the circuit in Fig. 11.11-11(b) for high frequencies. Then, the gain

$$\text{function is } = \left| \frac{1/j\omega C}{j\omega L + 1/j\omega C} \right| = \left| \frac{1}{1 - \omega^2 LC} \right| = \frac{1}{|1 - \omega^2 / \omega_n^2|} \approx \left(\frac{\omega_n}{\omega} \right)^2 = \left(\frac{f_n}{f} \right)^2 \text{ for } f \gg f_n. \text{ Current in}$$

the circuit in Fig. 11.11-11(b) lags voltage by 90° at high frequency because the net reactance is inductive at $\omega > \omega_n$. And, the capacitance voltage lags behind circuit current by 90° . Therefore, the output voltage will be 180° out of phase with respect to input at high frequencies.

Note that a LC filter will offer superior performance in averaging applications compared to RC circuit. This is due to the fact that the gain of an RC averaging circuit falls off in inverse proportion to ω for large values of ω (large compared to $1/\tau$) whereas the gain of an LC filter falls in inverse proportion to ω^2 for large values of ω (large compared to ω_n). Moreover, averaging by a LC filter is more efficient since an inductor does not dissipate power. Hence, averaging at high power levels (few 10's of watts and higher) is usually done by LC filters. RC Averaging is commonly used in low-power signal processing applications.

EXAMPLE: 11.11-2

Standard test signals such as square wave, triangular wave, sinusoidal wave, etc. are routinely used in Electronics, Communication and allied areas for a variety of purposes. It is rather easy to generate high quality square waves in electronic circuits. It is not so easy to make pure sinusoids. One of the commonly used methods to generate low power sinusoidal signals of high waveform quality is to generate a good square wave and pass it through a narrow band-pass filter with a large Q-factor. The centre frequency of filter is adjusted to be equal to the frequency of square wave.

This example deals with applying a series RLC circuit for this purpose. Consider the circuit in Fig. 11.11-12.

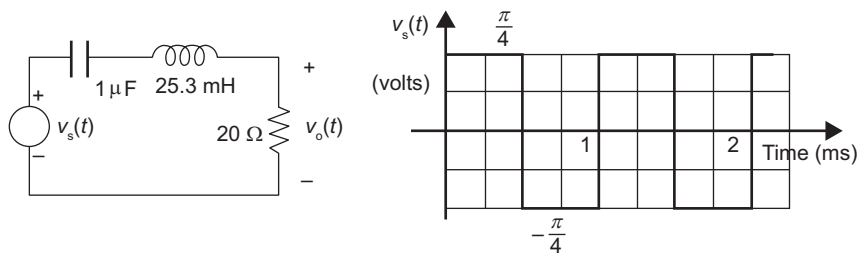


Fig. 11.11-12 Circuit and Input waveform for Example 11.11-2

This circuit is used to filter the square wave shown and deliver a sine wave to the $20\ \Omega$ load resistance.

- Find and plot the steady-state output waveform expected from the circuit.
- If the quality factor of the inductor used was measured to be 50 at 1 kHz and the test square wave was obtained from a function generator which has an output resistance of $50\ \Omega$ find and plot the steady-state output voltage waveform that will be observed.

[Hint: The square wave shown in Fig. 11.11-12 can be expressed in the form of a

Fourier series $v_s(t) = \sum_{n=1, \text{ odd}}^{\infty} \frac{1}{n} \sin 2\pi \times 10^3 t$ V]

SOLUTION

- The circuit is the familiar series RLC circuit and the frequency response function for resistor voltage in such a circuit has been shown to be

$$\begin{aligned} \frac{V_R(j\omega)}{V_s(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \\ &= \frac{j2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R, \text{ where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.} \end{aligned}$$

We define $x = \omega/\omega_n$ and express the above function as

$$\frac{V_R(j\omega)}{V_s(j\omega)} = \frac{j2\xi x}{(1-x^2) + j2\xi x}.$$

The input signal is a sum of sinusoids with odd multiples of 1 kHz as their frequencies. Thus, the separation between frequencies is large and it is quite possible that x may turn out to be quite large compared to 1 for many of the sinusoidal components. We calculate ω_n to verify this.

$$f_n = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{25.3 \times 10^{-3} \times 1 \times 10^{-6}}} = 1 \text{ kHz.}$$

$$\text{Critical resistance} = 2\sqrt{\frac{L}{C}} = 318.2\ \Omega.$$

Example 11.11-2 introduces a standard application of narrow band-pass filter – it is used to convert a square wave into a high quality sine wave. The filter circuit must have a high Q (>10 preferred) in this application.

This example illustrates the use of frequency response calculations in evaluating the performance of a RLC circuit in this application.

It also brings out the need to pay special attention to component Q-factors and source resistance level in evaluating the performance of narrow band-pass filters.

This expression shows that the gain of a band-pass filter circuit is inversely proportional to the frequency for frequencies much higher than the centre frequency of the filter.

$$\therefore \text{Damping factor, } \xi = \frac{20}{318.2} = 0.063 \text{ and Quality factor } Q = \frac{1}{2\xi} = 7.95.$$

Therefore, the values of x of interest to us are the odd integers. For $x \geq 3$, we can approximate the gain and phase expressions as shown below.

$$\frac{V_R(j\omega)}{V_S(j\omega)} = \frac{j2\xi x}{(1-x)^2 + j2\xi x}.$$

$$\left| \frac{V_R(j\omega)}{V_S(j\omega)} \right| = \frac{2\xi x}{\sqrt{(1-x^2)^2 + 4\xi^2 x^2}} \approx \frac{2\xi}{x} \text{ for } x \gg 1.$$

$$\text{Phase} = \frac{\pi}{2} - \left(\pi - \tan^{-1} \frac{2\xi x}{x^2 - 1} \right) \text{ for } x > 1$$

$$\approx -\frac{\pi}{2} + \tan^{-1} \frac{2\xi}{x} = -\frac{\pi}{2} + \frac{2\xi}{x} \text{ for } x \gg 1.$$

The following table shows gain and phase of output for x values from 1 to 11.

x	1	3	5	7	9	11
Gain	1	0.042	0.025	0.018	0.014	0.012
Phase(rad)	0	-1.53	-1.545	-1.553	-1.557	-1.56

Now, the first six terms of output may be constructed using the amplitude information.

$$v_o(t) = 1 \sin(2\pi \times 10^3 t) + 0.014 \sin(6\pi \times 10^3 t - 1.53) + 0.005 \sin(10\pi \times 10^3 t - 1.545) + 0.0026 \sin(14\pi \times 10^3 t - 1.553) + 0.0013 \sin(18\pi \times 10^3 t - 1.557) + 0.0011 \sin(22\pi \times 10^3 t - 1.56).$$

Obviously, the first three terms are significant. All the others can easily be ignored (including the higher frequency components that we did not evaluate).

$$\therefore v_o(t) \approx 1 \sin(2\pi \times 10^3 t) + 0.014 \sin(6\pi \times 10^3 t - 1.53) + 0.005 \sin(10\pi \times 10^3 t - 1.545) \text{ V.}$$

This waveform has only about 1.5% of other frequency content. It is almost pure sinusoid at 1 kHz.

(b) The 25.3 mH inductor has a reactance of $2\pi \times 10^3 \times 25.3 \times 10^{-3} = 159 \Omega$ at 1 kHz and since its Q at that frequency is given as 50, it has a series resistance of $159/50 = 3.18 \Omega$. The function generator that provides the square wave contributes 50Ω . Thus, the total series resistance in the series RLC circuit is 73.18Ω . The value of f_n of the circuit remains unchanged at 1 kHz. But its ξ -factor becomes 0.23 now. Also, the total voltage that develops across 73.18Ω has to be multiplied by $20/73.18 = 0.273$ to calculate the voltage across the load resistance. Thus, the frequency response expression now is

$$\frac{V_R(j\omega)}{V_S(j\omega)} = \frac{j0.547\xi x}{(1-x^2) + j2\xi x}$$

We continue to use the same approximations for $x \gg 1$.

The following table shows gain and phase of output for x values from 1 to 11.

x	1	3	5	7	9	11
Gain	0.273	0.042	0.025	0.018	0.014	0.011
Phase(rad)	0	-1.418	-1.479	-1.505	-1.52	-1.53

Now, the first six terms of output may be constructed using the amplitude information.

$$v_o(t) = 0.273 \sin(2\pi \times 10^3 t) + 0.014 \sin(6\pi \times 10^3 t - 1.418) + 0.005 \sin(10\pi \times 10^3 t - 1.479) + 0.0026 \sin(14\pi \times 10^3 t - 1.505) + 0.0015 \sin(18\pi \times 10^3 t - 1.52) + 0.001 \sin(22\pi \times 10^3 t - 1.53).$$

Higher frequency terms are neglected. The output waveforms for the two cases are plotted in Fig. 11.11-13.

The output in case (b) shows distortion clearly. This example illustrates the importance of high quality factor in filtering a square wave to a high quality sine wave. It also tells us that an otherwise high quality factor circuit may appear to be a low Q circuit if we forget about the output resistance of the signal generator that we use to test the circuit. The test signal should be passed on to the test circuit through a unity gain buffer amplifier with negligible output resistance.

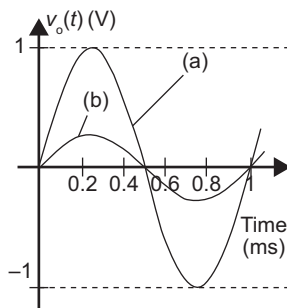


Fig. 11.11-13 Output Waveforms for case (a) and case (b) in Example 11.11-2

11.12 THE PARALLEL RLC CIRCUIT

We have dealt with the series RLC circuit in great detail and have developed considerable insight into the time-domain behaviour of second-order circuits in general. Moreover, we have also studied the frequency response of second-order circuits using series RLC circuit as an example. This will help us to draw parallels between the behaviour of series RLC circuit with another equally important, if not more important, circuit – the parallel RLC circuit.

Parallel RLC circuit finds application in almost all communications equipment (starting from radio receiver), sinusoidal oscillators, low-power and high-power filters, and electrical power systems. In fact, one can even state that analog communications will be impossible without using parallel RLC circuit. Of the two important and commonly appearing circuits – the series and parallel RLC circuits – we chose the series RLC circuit as the vehicle for carrying all the important concepts of second-order resonant circuits. And now we use what we have learnt from series RLC circuit to understand parallel RLC circuit.

11.12.1 Zero-Input Response and Zero-State Response of Parallel RLC Circuit

Figure 11.12-1 shows a parallel RLC circuit excited by a current source $i_s(t)$. There are four other circuit variables apart from $i_s(t)$ – they are the three current variables and one common voltage variable.

We choose $i_L(t)$ as the variable for deriving the differential equation. However, the variable that is commonly used as output variable in practice is $v(t)$. All the three possible output voltage variables are used in practical applications in the case of series RLC circuit. But in the case of parallel RLC circuit, it is $v(t)$ almost invariably. However, $v(t)$ can be obtained easily once we solve for $i_L(t)$.

$$v(t) = L \frac{di_L(t)}{dt} \text{ (By element equation of inductance).}$$

$$\therefore i_R(t) = \frac{L}{R} \frac{di_L(t)}{dt} \text{ and } i_C(t) = LC \frac{d^2i_L(t)}{dt^2} \text{ (By element equations of } R \text{ and } C \text{).}$$

Now we apply KCL at the positive node of current source and make use of expressions for $i_C(t)$ and $i_R(t)$ in terms of $i_L(t)$.

$$i_C(t) + i_R(t) + i_L(t) = i_s(t) \text{ for } t \geq 0^+$$

$$\therefore LC \frac{d^2i_L(t)}{dt^2} + \frac{L}{R} \frac{di_L(t)}{dt} + i_L(t) = i_s(t) \text{ for } t \geq 0^+$$

$$\therefore \frac{d^2i_L(t)}{dt^2} + \frac{1}{RC} \frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = \frac{1}{LC} i_s(t) \text{ for } t \geq 0^+.$$

Comparing with the standard format using ξ and ω_n ,

$$\omega_n = \frac{1}{\sqrt{LC}} \text{ and } 2\xi\omega_n = \frac{1}{RC} \Rightarrow \xi = \frac{1}{2} \sqrt{\frac{L}{C}}.$$

We see that if we write the differential equations in the standard format using ξ and ω_n , the differential equations for $v_C(t)$ in series RLC circuit and $i_L(t)$ in parallel RLC circuit are identical except that $v_C(t)$ gets replaced by $i_L(t)$ and $v_S(t)$ gets replaced by $i_S(t)$. The only

point to be remembered is that the damping factor is $R/2\sqrt{L/C}$ in the series RLC circuit

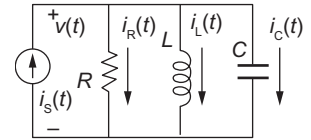


Fig. 11.12-1 The Parallel RLC Circuit

whereas it is $\frac{1}{2}\sqrt{\frac{L}{C}}/R$ for parallel RLC circuit. Thus, increasing resistance will increase damping in series RLC circuit whereas it will decrease damping in parallel RLC circuit. The initial conditions needed for solving the differential equation remain the same – inductor current I_0 at $t = 0^-$ and capacitor voltage V_0 at $t = 0^-$.

Similar to series RLC circuit, when $\xi > 1$ the parallel RLC circuit is over-damped, when $\xi = 1$ it is critically-damped and when $\xi < 1$ it is under-damped.

Zero-input response and zero-state response of parallel RLC circuit are illustrated by a series of examples that follow. The polarity convention for specifying initial conditions is as per Fig. 11.12-1.

EXAMPLE: 11.12-1

A parallel RLC circuit with $L = 10$ mH, $C = 100$ μ F, $R = 2.5$ Ω , $V_0 = 10$ V and $I_0 = -1$ A is allowed to execute its free-response from $t = 0$. Obtain and plot all the circuit variables as functions of time.

SOLUTION

Undamped natural frequency, $\omega_n = \frac{1}{\sqrt{LC}} = 1000$ rad/s.

Critical resistance $= \frac{1}{2}\sqrt{\frac{L}{C}} = 5$ Ω .

$\therefore$ Damping factor, $\xi = 5/2.5 = 2$

$\therefore$ Natural frequencies $= (-\xi \pm \sqrt{\xi^2 - 1})\omega_n = -0.268 \times 10^3$ s⁻¹ and -3.732×10^3 s⁻¹.

Hence, this circuit is over-damped with two real negative natural frequencies corresponding to two time constants of 3.732 ms and 0.268 ms. There is no forced response component. Therefore, the total response for all variables will contain sum of two decaying exponential functions with time constants calculated above.

$$\therefore i_L(t) = A_1 e^{-0.268 \times 10^3 t} + A_2 e^{-3.732 \times 10^3 t} \quad (11.12-1)$$

A_1 and A_2 have to be evaluated from initial conditions. One of the initial conditions – inductor current – can be directly applied on this equation. The initial current given is specified at $t = 0^-$. But since there is no impulse voltage in this circuit, the inductor current cannot change instantaneously. Therefore, the value of inductor current at $t = 0^+$ and at $t = 0^-$ will be the same. Applying this initial condition at $t = 0^+$ on the assumed solution gives one equation on A_1 and A_2 .

$$A_1 + A_2 = I_0 = -1.$$

The second initial condition, the value of capacitor voltage at $t = 0^-$, can be employed to obtain the second equation needed to solve for A_1 and A_2 . We recognise that (i) in a parallel RLC circuit, voltage is the common variable and hence the initial value of voltage across inductor is same as V_0 (ii) the voltage across an inductor is proportional to first derivative of current.

$$\therefore \left. \frac{di_L(t)}{dt} \right|_{0^+} = \frac{v_L(t)}{L} \Big|_{0^+} = \frac{v_C(t)}{L} \Big|_{0^+}.$$

$$\text{But } \frac{v_C(t)}{L} \Big|_{0^+} = \frac{v_C(t)}{L} \Big|_{0^-} \text{ if there is no impulse current flow in the capacitor at } t = 0.$$

$$\therefore \left. \frac{di_L(t)}{dt} \right|_{0^+} = \frac{V_0}{L} \text{ A/s.}$$

Differentiate the assumed response in Eqn. 11.12-1 and apply this initial condition to get the second equation on A_1 and A_2 .

$$-268 A_1 - 3732 A_2 = 10/0.01 = 1000 \text{ A/s.}$$

Solving for A_1 and A_2 ,

$$A_1 = -0.789 \text{ and } A_2 = -0.211$$

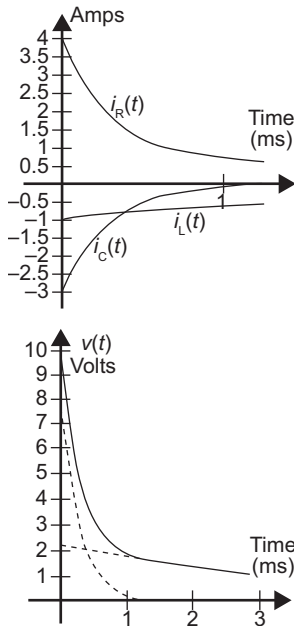


Fig. 11.12-2 Current and Voltage Waveforms for Example 11.12-1

$$\therefore i_L(t) = -0.789e^{-268t} - 0.211e^{-3732t} \text{ A for } t \geq 0^+$$

$$v(t) = L \frac{di_L(t)}{dt} = 2.11e^{-268t} + 7.89e^{-3732t} \text{ V for } t \geq 0^+$$

$$i_C(t) = C \frac{dv(t)}{dt} = -0.0565e^{-268t} - 2.945e^{-3732t} \text{ A for } t \geq 0$$

$$i_R(t) = \frac{v(t)}{R} = 0.846e^{-268t} + 3.156e^{-3732t} \text{ A for } t \geq 0^+.$$

The sum of the three currents should be zero and it is verified.

Figure 11.12-2 shows the three currents and the common voltage waveforms. The two exponential functions that make the voltage waveform are also shown. In all cases the exponential with lower time constant (0.268 ms) dominates the initial behaviour whereas the exponential with the higher time constant (3.732 ms) affects the behaviour after 1 ms.

EXAMPLE: 11.12-2

A parallel RLC circuit with $L = 1 \text{ mH}$, $C = 1000 \mu\text{F}$, $R = 2.5 \Omega$, $V_0 = 0 \text{ V}$ and $I_0 = 0 \text{ A}$ is driven by a single pulse of current of amplitude 100 A lasting for $10 \mu\text{s}$. Obtain and plot all the circuit variables as functions of time.

SOLUTION

A single rectangular pulse can be expressed as a sum of two step functions. Let the pulse height be I , pulse duration be t_0 and let it start at $t = 0$. Then, this pulse can be expressed as $I[u(t) - u(t - t_0)]$, where $u(t - t_0)$ is a unit step which is delayed by t_0 s in the time-axis. This looks like a step response problem.

$$\text{Undamped natural frequency, } \omega_n = \frac{1}{\sqrt{LC}} = 1000 \text{ rad/s.}$$

$$\text{Critical resistance} = \frac{1}{2} \sqrt{\frac{L}{C}} = 0.5 \Omega.$$

$$\therefore \text{Damping factor, } \xi = 0.5/2.5 = 0.2$$

$$\therefore \text{Natural frequencies, } = \left(-\xi \pm j\sqrt{1-\xi^2} \right) \omega_n = -200 + j979.8 \text{ and } -200 - j979.8.$$

Hence, the circuit is under-damped and its natural response terms will be exponentially-damped sinusoidal functions of time.

We note that the time constant of the exponential which damps the sinusoid is $1/200 = 5 \text{ ms}$. The period of the sinusoid term will be $2\pi/979.8 = 6.413 \text{ ms}$.

Observe that the duration of current pulse applied to the circuit is only 0.01 ms . Thus, the duration of pulse applied is very small compared to 5 ms and 6.413 ms . An approximate solution to the circuit problem can usually be obtained under such situations by approximating the driving pulse as an impulse. The area content of impulse – that is, the magnitude of impulse – must be the same as the area content of the pulse being approximated. The pulse can be arbitrary in shape – the only constraint is that the pulse duration has to be much less than the characteristic times involved in the circuit natural response.

Hence, this problem can be solved by obtaining the impulse response (that is, zero-state response to the unit impulse input) and scaling it by the magnitude to the impulse used to approximate the rectangular pulse, i.e., $100 \text{ A} \times 10 \mu\text{s} = 1000 \mu\text{C}$.

No part of an impulse current can flow through the resistor in a parallel RLC circuit since that will result in impulse voltage across the combination. Capacitor will not let that happen. No part of impulse current can flow through the inductor since inductor does not even permit a finite change in current over infinitesimal time duration. Hence, all of the input impulse current has to flow through the capacitor at $t = 0$. The magnitude of impulse is $1000 \mu\text{C}$ and this charge will be deposited on the capacitor

An approximate solution to a circuit analysis problem with pulse input can be obtained by approximating the driving pulse by an impulse.

This is permissible only if the duration of pulse is very small compared to the lowest time constant or period appearing in the natural response terms.

The pulse can have any shape. But the magnitude of impulse used to replace it must be equal to the area content of the pulse.

Example 11.12-2 illustrates this technique.

instantaneously at $t = 0$. Therefore, the capacitor voltage changes instantaneously to $1000 \mu\text{C}/1000 \mu\text{F} = 1 \text{ V}$ from 0. The impulse current source is open (because a current source which is zero-valued is an open-circuit) for all $t \geq 0^+$. Thus, once again, the only effect of impulse excitation is to change initial condition abruptly. The subsequent response is the free-response of the circuit.

Now the problem has reduced to finding the zero-input response (that is, free-response) of the circuit with $V_0 = 1 \text{ V}$ and $I_0 = 0$.

$$i_L(t) = e^{-200t} (A_1 \sin 979.8t + A_2 \cos 979.8t)$$

$$i_L(0^+) = 0 \text{ and } \left. \frac{di_L(t)}{dt} \right|_{t=0^+} = \frac{V_0}{L} = 1000 \text{ A/s.}$$

Differentiating $i_L(t)$ and applying initial conditions,

$$A_2 = 0 \text{ and } 979.8 A_1 - 200 A_2 = 1000.$$

Solving for A_1 and A_2 ,

$$A_1 = 1.0206 \text{ and } A_2 = 0$$

$$\therefore i_L(t) = 1.0206 e^{-200t} \sin 979.8t \text{ A for } t \geq 0^+$$

$$v(t) = L(i_L(t))' = e^{-200t} (-0.204 \sin 979.8t + \cos 979.8t)$$

$$= 1.0206 e^{-200t} \cos(979.8t + 0.2 \text{ rad}) \text{ V for } t \geq 0^+$$

$$i_C(t) = C(v(t))' = e^{-200t} [-0.204 \cos(979.8t + 0.2 \text{ rad}) - \sin(979.8t + 0.2 \text{ rad})]$$

$$= -e^{-200t} \sqrt{0.204^2 + 1^2} \sin \left[(979.8t + 0.2 \text{ rad}) + \tan^{-1} \frac{0.204}{1} \right]$$

$$= -1.0206 e^{-200t} \sin(979.8t + 0.4 \text{ rad}) \text{ A for } t \geq 0^+$$

$$i_R(t) = \frac{v(t)}{R} = 0.408 e^{-200t} \cos(979.8t + 0.2 \text{ rad}) \text{ A for } t \geq 0^+.$$

Figure 11.12-3 shows the current and voltage waveforms.

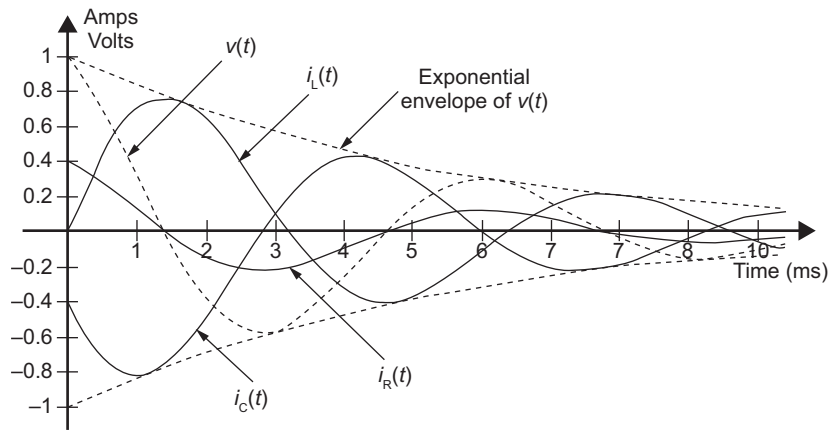


Fig. 11.12-3 Current and Voltage Waveforms for Example 11.12-2

EXAMPLE: 11.12-3

A parallel RLC circuit with $R \rightarrow \infty$ has an initial voltage of $V_0 \text{ V}$ across the capacitor and $I_0 \text{ A}$ in the inductor at $t = 0^-$. Find the expressions for all the variables under free-response conditions.

SOLUTION

A series RLC circuit with $R = 0$ and a parallel RLC circuit with $R \rightarrow \infty$ will be the same circuit and hence the free-response in this case is same as the free-response in the case of undamped series RLC circuit. It is given in Eqn. 11.6-10 and is reproduced below.

$$v(t) = \sqrt{V_0^2 + \frac{L I_0^2}{C}} \cos(\omega_n t - \phi) \text{ V for } t \geq 0^+$$

$$i_C(t) = -i_L(t) = -\sqrt{I_0^2 + \frac{C V_0^2}{L}} \sin(\omega_n t - \phi) \text{ A for } t \geq 0^+,$$

$$\text{where } \phi = \tan^{-1} \left(\frac{I_0 \sqrt{L/C}}{V_0} \right)$$

The ratio of amplitude of voltage to amplitude of current is $\sqrt{\frac{L}{C}}$. Total energy storage in the circuit will be $\frac{L I_0^2}{2} + \frac{C V_0^2}{2}$ J and it will remain constant at that value.

EXAMPLE: 11.12-4

Find the unit step response of a current excited parallel RLC Circuit with $L = 1$ mH, $C = 1000$ μ F, $R = 2.5$ Ω .

SOLUTION

Undamped natural frequency, $\omega_n = \frac{1}{\sqrt{LC}} = 1000$ rad/s

$$\text{Critical resistance} = \frac{1}{2} \sqrt{\frac{L}{C}} = 0.5 \Omega$$

$$\therefore \text{Damping factor, } \xi = 0.5/2.5 = 0.2$$

$$\therefore \text{Natural frequencies, } (-\xi \pm j\sqrt{1-\xi^2})\omega_n = -200 + j979.8 \text{ and } -200 - j979.8.$$

Capacitor behaves as a short-circuit and inductor behaves as an open-circuit under DC steady-state condition. Therefore, the 1 A current in the unit step will go through the inductor under steady-state. Hence, the solution for $i_L(t)$ can be assumed as

$$i_L(t) = 1 + e^{-200t} (A_1 \sin 979.8t + A_2 \cos 979.8t)$$

$$\text{with } i_L(0^+) = 0 \text{ and } i_L'(0^+) = V_0/L = 0.$$

The initial conditions are zero-valued since it is a step response problem. Step response means zero-state response to unit step input by default. We get the two equations needed to solve for A_1 and A_2 by applying initial conditions. They are $A_2 = -1$ and $979.8A_1 - 200A_2 = 0$.

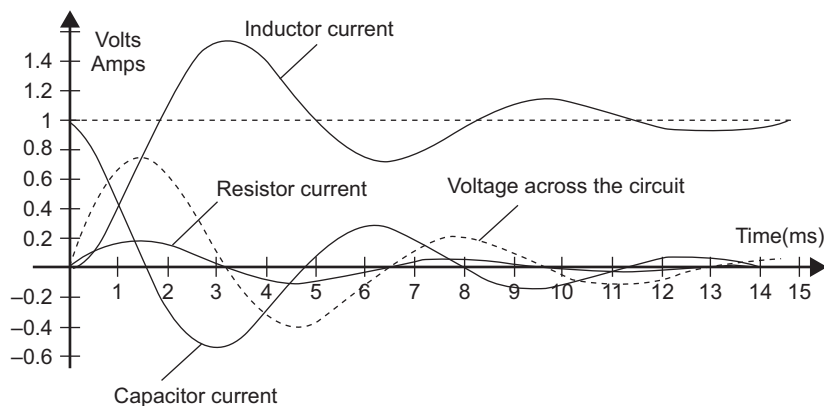


Fig. 11.12-4 Voltage and Current Waveforms in Example 11.12-4

Solving for A_1 and A_2 , $A_1 = -0.204$ and $A_2 = -1$.

$$\therefore i_L(t) = 1 - e^{-200t} (0.204 \sin 979.8t + \cos 979.8t) \text{ A for } t \geq 0^+$$

$$= 1 - 1.0206 e^{-200t} \cos (979.8t - 0.2 \text{ rad}) \text{ A for } t \geq 0^+$$

$$v(t) = L(i_L(t))' = e^{-200t} [0.204 \cos (979.8t - 0.2 \text{ rad}) + \sin (979.8t - 0.2 \text{ rad})]$$

$$= 1.0206 e^{-200t} \sin 979.8t \text{ V for } t \geq 0^+$$

$$i_R(t) = v(t)/2.5 = 0.255 e^{-200t} \sin 979.8t \text{ A for } t \geq 0^+$$

$$i_C(t) = 1 - i_R(t) - i_L(t) = 1.0206 e^{-200t} \cos (979.8t - 0.2 \text{ rad}) - 0.255 e^{-200t} \sin 979.8t$$

$$= e^{-200t} [-0.051 \sin 979.8t + \cos 979.8t]$$

$$= e^{-200t} \cos (979.8t + 0.05 \text{ rad}) \text{ A for } t \geq 0^+$$

The variation of resistor current, inductor current, capacitor current and the voltage across the parallel combination with time is shown in Fig. 11.12-4.

All the input current goes through the capacitor at $t = 0^+$ since neither the voltage across the circuit nor the current through inductor can become non-zero at that instant. Inductor current shows 52.7% overshoot which is the value predicted by Eqn. 11.9-1 for $\xi = 0.2$.

11.12.2 Sinusoidal Steady-State Frequency Response of Parallel RLC Circuit

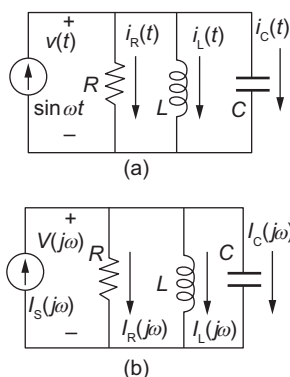


Fig. 11.12-5 (a) Parallel RLC Circuit and (b) its Phasor Equivalent Circuit

Frequency response of inductor current in a current-excited parallel RLC circuit.

Frequency response of resistor current in a current-excited parallel RLC circuit.

The parallel RLC circuit and its phasor equivalent circuit are shown in Fig. 11.12-5.

Almost the entire source current flows through the inductor at low frequency since the inductor is short-circuit at DC and low impedance for low frequency AC. Similarly, almost the entire source current passes through the capacitor at high frequencies since the capacitor impedance approaches zero as frequency increases without limit. Thus, the magnitude response (i.e., gain) of inductor current must be a low-pass function. Magnitude response of the capacitor current must be a high-pass function and that of resistor current (and hence that of circuit voltage $v(t)$) must be a band-pass function. These frequency response functions are obtained by applying current division principle to the parallel RLC circuit phasor equivalent circuit.

$$\begin{aligned} \frac{I_L(j\omega)}{I_S(j\omega)} &= \frac{1/j\omega L}{\frac{1}{R} + j\omega C + 1/j\omega L} = \frac{1}{1 - \omega^2 LC + j\omega L/R} \\ &= \frac{1/LC}{1/LC - \omega^2 + j\omega 1/RC} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega} \end{aligned}$$

This ratio can be written in polar form as

$$\frac{I_L(j\omega)}{I_S(j\omega)} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_L, \text{ where } \phi_L = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad}$$

Thus, the frequency response function for $i_L(t)$ in a parallel RLC circuit is found to be the same as the frequency response function for $v_C(t)$ in series RLC circuit. It is a low-pass output.

Similarly, the frequency response of $i_C(t)$ and $i_R(t)$ are also obtained.

$$\begin{aligned} \frac{I_R(j\omega)}{I_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2)^2 + j2\xi\omega_n\omega} \\ &= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R, \text{ where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.} \end{aligned}$$

$$\begin{aligned}\frac{I_C(j\omega)}{I_S(j\omega)} &= \frac{(j\omega)^2}{(\omega_n^2 - \omega^2)^2 + j2\xi\omega_n\omega} \\ &= \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_L, \text{ where } \phi_L = \pi - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad.}\end{aligned}$$

The frequency response of voltage developed across the circuit is the frequency response function for $i_R(t)$ multiplied by R . It will be a band-pass function.

These frequency response functions have been dealt in detail in the context of series RLC circuit and nothing further need be added. Whatever that has been stated with respect to the capacitor voltage in the series circuit can be applied directly to the inductor current in the parallel circuit and so on.

Resonance in parallel RLC circuit takes place when input frequency is ω_n . Under resonant condition the input admittance (and impedance) of parallel RLC circuit becomes purely resistive and equal to $1/R$ S. This is so because at that frequency susceptance of inductor and capacitor are exactly equal in magnitude and opposite in sign and they cancel each other when added. They do not cancel completely at any other frequency and hence the admittance of a parallel RLC circuit is a *minimum* of $1/R$ at resonant frequency.

All the current from the source flows through R under resonance conditions. Thus, the amplitude of voltage across the parallel combination is a maximum of R V (assuming unit amplitude for source current) at ω_n . The amplitude of current through the capacitor at that frequency will then be $\omega_n RC$. The amplitude of the current through inductor at resonant frequency will then be $R/\omega_n L$. Thus, the current amplification factor at resonance in a parallel RLC circuit, defined as the ratio of amplitude of current in capacitor or inductor to the amplitude of source current is

$$= \omega_n RC = \frac{RC}{\sqrt{LC}} = \frac{1}{\sqrt{L/C}} \frac{1}{R} = \frac{1}{2\xi} = Q.$$

Thus, a high Q circuit will carry very high amplitude currents in L and C even when the source current amplitude is small if the source frequency is equal to or near about the circuit resonant frequency. These currents cancel themselves due to their phase opposition and they do not take any portion of the source current. Entire source current flows through the resistance under resonant condition.

Frequency response of the capacitor current in a current-excited parallel RLC circuit.

Parallel RLC circuit with high Q -factor (that is, low damping factor) will work as a narrow band-pass filter if it is excited by a current signal and the voltage across the circuit is accepted as the output.

And, that is the most frequently used application of a parallel RLC circuit.

The admittance of a parallel RLC circuit is a *minimum* of $1/R$ at resonant frequency ω_n .

The current amplification factor at resonant frequency is equal to quality factor.

EXAMPLE: 11.12-5

A parallel LC circuit used in a tuned amplifier circuit has $L = 25.3 \mu\text{H}$ and $C = 1 \text{ nF}$. There is no resistance load across it. A frequency response test on the circuit revealed that the band-pass output (that is, the voltage across the circuit when the excitation is a sinusoidal current source) has a centre frequency that is approximately the expected value. But its bandwidth was found to be 6 kHz. (i) Find the Q -factor of the inductor at the centre frequency. Losses in the capacitor may be ignored. (ii) What is the resistance that has to be connected across the LC combination such that the band-pass filter has a bandwidth of 10 kHz?

SOLUTION

A pure parallel LC circuit excited by a current source should have a bandwidth that goes to zero. This is due to the fact that for a narrow band-pass filter, the bandwidth and centre frequency are related by $\text{bw} = \frac{\omega_n}{Q}$. The Q of a pure LC parallel circuit is ∞ since its damping factor is zero.

The fact that the experiment conducted revealed a bandwidth of 6 kHz implies that there is damping in the circuit. Winding and core losses in the inductor produce

Parallel RLC circuits are employed in tuned amplifiers for band-pass filtering. The load resistance connected at the output of amplifier decides the value of R . In addition, the losses in the inductor (or load-coupling transformer) also affect the value of R - and Q -factor.

This example illustrates the effect of coil losses on the frequency selectivity of a tuned circuit.

damping in the circuit. So does the dielectric losses in the capacitor. But, the effect of capacitor losses is usually small compared to the effect of inductor losses. Hence, we assume that, the limiting of bandwidth observed is essentially due to losses in the inductor. We can obtain the effective resistance that has come across the inductor at and around resonant frequency using the data provided.

$$\text{Undamped natural frequency of the circuit} = \frac{1}{2\pi\sqrt{LC}} = 1 \text{ MHz.}$$

$\therefore$ Centre frequency observed = 1 MHz (because that is the expected value and the experiment confirmed it).

Observed bandwidth = 6 kHz.

$\therefore$ The Q -factor that is effective in the circuit = $1000/6 = 166.7$.

$\therefore$ The ξ -factor that is effective in the circuit = $1/2Q = 0.003$.

$$\therefore \text{The parallel resistance that is effective in the circuit} = \frac{1}{2\xi} \sqrt{\frac{L}{C}} = 26.5 \text{ k}\Omega.$$

This resistance represents the losses at 1 MHz in the inductor in the form of a resistance in parallel with inductance. The Q -factor of a reactive element is defined as the ratio between maximum energy storage in the reactance to the energy lost in one cycle under steady-state operation at the frequency at which Q is calculated. From

this definition it follows that Q of an inductor = $\frac{\omega L}{R_s} = \frac{R_p}{\omega L}$, where R_s is the resistance that

comes in series to represent the losses in the inductor and R_p is the resistance in parallel to the inductor if the losses are to be represented by such a parallel resistance instead of a series resistance.

The reactance of $25.3 \mu\text{H}$ inductor at 1 MHz is 159Ω .

Therefore, Q of the inductor = $26,500/159 = 166.7$.

If the bandwidth is to be raised to 10 kHz, the Q of the circuit has to be lowered to 100. Then the damping factor has to be 0.005 and the effective parallel resistance

has to be $= \frac{1}{2\xi} \sqrt{\frac{L}{C}} = 15.9 \text{ k}\Omega$. There is already $26.5 \text{ k}\Omega$ effective parallel resistance from

the losses in the inductor. Let R_1 be the extra resistance that has to be connected in parallel. Then $R_1/26.5 \text{ k}\Omega$ has to be $15.9 \text{ k}\Omega$. Therefore, R_1 is $39.75 \text{ k}\Omega$.

11.13 SUMMARY

- RC circuits are described by first-order linear differential equations. The past history of the circuit is contained in a single initial condition specification for capacitor voltage in RC circuits.
- The solution of the differential equation describing the capacitor voltage in a RC circuit contains two terms – the complementary function and particular integral. Complementary function is the solution of differential equation with zero forcing function. Particular integral is the solution of the differential equation *with the assumption that the forcing function was applied from infinite past onwards*. The total solution is obtained by adding these two. The complementary function has arbitrary amplitude that should be fixed by ensuring that the *total* solution complies with the specified initial condition.
- The circuit variables in the RC circuit will contain two response components – *transient response* (also called *natural response*) and *forced response*. *Natural response* is the way in which the inertia in the circuit reacts to the forcing function's command to change. Complementary solution gives the *natural response* and particular integral gives the *forced response* in a circuit.
- The *natural response* of a circuit is independent of the magnitude of forcing function and depends only on circuit parameters and nature of the interconnections. *Natural response* in RC circuit is exponential of the form $A e^{-t/\tau}$, where $\tau = RC$ is defined as time constant of the circuit. A is to be fixed by complying with initial condition.
- The initial capacitor voltage in a RC circuit at $t = 0^-$ and $t = 0^+$ are the same if the circuit does not contain impulse sources.
- In the case of RC circuit, *step response* is a rising exponential, approaching a steady-state value asymptotically as $t \rightarrow \infty$.

The *step response* never gets done. But for practical purposes it may be considered to be over within 5 time constants.

- *Free-response* of a RC circuit is its response when input is zero and there is some initial energy trapped in the capacitor. It will contain only *natural response* terms. The capacitor voltage in this case falls exponentially towards zero.
- The response due to initial energy and application of impulse are indistinguishable in a RC circuit and hence they can be replaced for each other. An initial voltage of V_0 in a capacitor of value C can be replaced by zero initial condition with a current source $CV_0 \delta(t)$ connected in *parallel* with the capacitor.
- *Step response* and *ramp response* in a RC circuit can be obtained by integrating its *impulse response* successively.
- Wave-shape distortion occurs in linear circuits due to differential treatment experienced by various sinusoidal components in a mixture of sinusoids when they go through the circuit. The conditions to be satisfied by a circuit such that there is no wave-shape distortion when a signal passes through it, is that its frequency response must have a gain that is flat with ω and a phase which is either zero or linear on ω , i.e., of the form $\phi = -k\omega$, where k is a real number.
- A series RC circuit excited by a voltage source at the input of the series combination with output taken across the capacitor is a low-pass filter with a cut-off frequency of $1/\tau$ rad/s and a monotonically decreasing gain. The same circuit with same excitation but with output taken across the resistor is a high-pass filter with a cut-off frequency of $1/\tau$ rad/s and a monotonically increasing gain.
- Series RC circuit with output taken across capacitor can be used as averaging filter for voltage signals and parallel RC circuit with output taken across the combination can be used as averaging filter for current signals. Good averaging performance requires that $\tau \gg T$, where T is the period of the input signal or its characteristic time of variation if a regular period cannot be identified.
- Series and parallel RLC circuits are described by second-order linear differential equation with constant coefficients. The natural response of such circuits contains two exponential functions. Three different types of natural response terms are possible in such circuits depending on the size of elements.
- A pure LC circuit with initial energy oscillates sinusoidally forever with a frequency of $\omega_n = 1/LC$ rad/s. $v_C(t)$ will

have an amplitude of $\sqrt{2E_0/C}$, where E_0 is the total initial

energy storage in the circuit, i.e., $E_0 = \frac{LI_0^2}{2} + \frac{CV_0^2}{2}$, where

V_0 and I_0 are the initial capacitor voltage and inductor current,

respectively. $i(t)$ will have an amplitude of $\sqrt{2E_0/L}$ and the ratio between voltage amplitude and current amplitude is $\sqrt{L/C}$.

- The parameter called damping factor, ξ , decides the nature of natural response in RLC circuits. $\xi = \frac{R}{2\sqrt{L/C}}$ for series RLC

circuit and $\xi = \frac{1}{2}\sqrt{L/C}$ for parallel RLC circuit. If $\xi > 1$ the

circuit is over-damped and its natural response will contain two decaying real exponential functions. If $\xi = 1$ the circuit is critically-damped and its natural response will contain an exponential function and a product of time with same exponential function. If $\xi < 1$, the circuit will be under-damped and its natural response will contain exponentially-damped sine function and cosine function.

- The natural frequency $s = \sigma + j\omega$ stands for a complex exponential signal e^{st} in time-domain. This signal will represent one of the many components present in the zero-input response of the circuit that has 's' as one of its natural frequencies. Thus, *natural frequency* is a stand-in for e^{st} .

- Natural frequencies for an RLC circuit are:

$(-\xi \pm \sqrt{\xi^2 - 1})\omega_n$ if the circuit is over-damped, $-\xi\omega_n$ with multiplicity of 2 if the circuit is critically-damped and $(-\xi \pm j\sqrt{1 - \xi^2})\omega_n$ if the circuit is under-damped.

- Quality factor Q of a RLC circuit is another parameter that quantifies damping in the circuit. It is related to ξ through the relationship $Q = 1/2\xi$. In lightly-damped RLC circuits,

the fractional loss of energy in one oscillation in zero-input response is given by $2\pi/Q$ or $4\pi\xi$.

- The capacitor voltage in a series RLC circuit is a low-pass output. This leads to application of series RLC circuit as a good averaging filter.

- The resistor voltage in a series RLC circuit is a band-pass output with centre frequency at ω_n and a bandwidth of ω_n/Q or $2\xi\omega_n$. The two half-power frequencies are asymmetrically located around centre frequency in general. However, in a narrow band-pass case (that is, $Q > 5$ or $\xi < 0.1$), they are more or less symmetric about ω_n .

- In a circuit excited by a single sinusoidal voltage source (current source) across a pair of terminals, *resonance* is the *sinusoidal steady-state condition* under which the current drawn at the terminals (voltage appearing across the terminals) is in phase with the source voltage (current). Equivalently,

resonance is the condition under which the input impedance (admittance) offered to the sinusoidal source is resistive.

- Series *RLC* circuit is resonant at ω_n . At that frequency, the impedance of the circuit is a *minimum* at R and is resistive. Circuit draws current at unity power factor. Voltage across capacitor and inductor will be of equal amplitude; but opposite in phase. The voltage amplification factor at resonance will be Q or $1/2\xi$.
- Parallel *RLC* circuit is resonant at ω_n . At that frequency, the impedance of the circuit is a *maximum* at R and is resistive. Circuit develops voltage at unity power factor. Current through the capacitor and the inductor will be of equal amplitude; but opposite in phase. The current amplification factor at resonance will be Q or $1/2\xi$.
- The voltage across a parallel *RLC* circuit is a band-pass output with centre frequency at ω_n and a bandwidth of ω_n/Q or $2\xi\omega_n$.

The two half power frequencies are asymmetrically located around centre frequency in general. However, in a narrow band-pass case (that is, $Q > 5$ or $\xi < 0.1$), they are more or less symmetric about ω_n .

- Quality factor Q of a *RLC* circuit is 2π times the ratio between total energy storage in the beginning of an oscillation cycle to the energy lost during that cycle. Equivalently, it is 2π times the ratio of the total energy stored in the circuit to the energy dissipated in one cycle when the circuit is in sinusoidal steady-state at resonant frequency.
- However, Quality factor of an element like L or C is 2π times the ratio between maximum energy stored in reactive part of that element to the energy lost in one cycle in that element under steady-state operation at a particular ω . It will be a frequency dependent number.

11.14 QUESTIONS

1. What is the differential equation describing $v_C(t)$ for $t \geq 0^+$ in the circuit in Fig. 11.14-1?

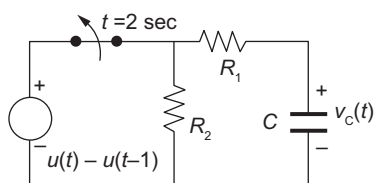


Fig. 11.14-1

2. A parallel *RC* circuit with non-zero initial energy is driven by $u(t)$ A. $v_C(t)$ is found to reach 75% of its steady-state value in one time constant. Express the initial voltage across the capacitor as a percentage of its steady-state value.
3. A parallel *RC* circuit is driven by $u(t)$ A. $v_C(t)$ is found to be 50% at $t = \tau$. Was there any initial voltage across C ? If yes, what is its magnitude and polarity?
4. Find and plot the $v_C(t)$ and voltage across the current source as functions of time in Fig. 11.14-2. Is there any exponential function in the expression for $v_C(t)$? If not, why?

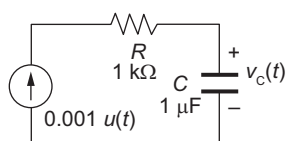


Fig. 11.14-2

5. Derive an expression for v_C in an initially relaxed series *RC* circuit when it is driven by $v_s(t) = t$ for $0^+ \leq t \leq \tau$ and 0 for all other t , where $\tau = RC$ s.
6. What is the time required for the energy stored in C in a series *RC* circuit to reach 99% of its steady-state value in step response?
7. v_C in the circuit in Fig. 11.14-3 is -10 V at $t = 0^-$. If $v_C(t) = 0$ for $t \geq 0^+$, what is the value of C ?

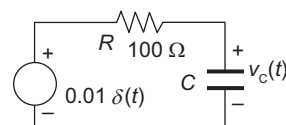


Fig. 11.14-3

8. A parallel *RC* circuit is driven by $u(t)$ A. The resistor voltage is found to be $(10 - 5e^{-t})$ V. What are the values of C , R and initial voltage?
9. v_C in the circuit in Fig. 11.14-4 at $t = 0^-$ is -12 V. It is found that $v_C(t) = 0$ for $t \geq 0^+$. What is the rise time of step response of the circuit?

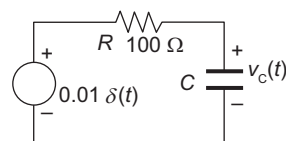


Fig. 11.14-4

10. A parallel RC circuit with non-zero initial energy is driven by a unit step current source. The capacitor voltage is found to be 15 V for $t \geq 0^+$. What is the value of initial voltage and what is its polarity relative to the observed voltage? What is the value of R in the circuit?
11. An AC voltage source $= V \sin(\omega t) u(t)$ is applied to a series RC circuit and its current is found to be $= 0.7 \sin(\omega t - \pi/3) u(t)$ A. Was there any initial voltage across the capacitor? If so, what is its magnitude and relative polarity?
12. A parallel RC circuit with non-zero initial energy is driven by a unit step current source. The resistor current is found to be 12 mA under steady-state. What is the new steady-state value of this current if (i) the initial condition is doubled (ii) if $2u(t)$ is applied with no change in initial condition?
13. The capacitor voltage $v_c(t) = 0$ for $t \geq 0^+$ in the circuit in Fig. 11.14-5. Find the values of V_0 and I_s .

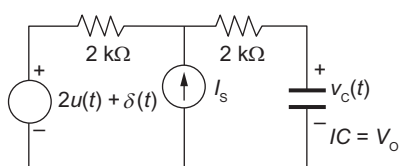


Fig. 11.14-5

14. A 50 Hz symmetric triangular voltage of 10 V peak value is passed through a non-linear circuit that outputs the absolute value of the input applied to it. Design a series RC circuit to extract the average value of the output waveform. The output stage of absolute value circuit cannot deliver more than 10 mA.
15. A series RC circuit with zero initial current is driven by $v_s(t) = \delta(t) - \delta(t-1)$. Its time constant is 1 s. (i) Starting from impulse response, find the voltage across the resistor in the circuit when driven by the input $v_s(t)$. (ii) Using the result derive an expression for voltage across the resistor when the circuit is driven by a rectangular pulse of unit amplitude and 1 s duration.
16. Derive expressions for maximum voltage across resistor in an initially relaxed series RC circuit when it is driven by $e^{\alpha t} u(t)$ A with $\alpha \neq -1/\tau$, where τ is the time constant of the circuit. Also, find an expression for the time instant at which this maximum voltage occurs.
17. An input of $k\delta(t) + 2e^{-2t/\tau}$ V is applied to an initially relaxed series RC circuit with time constant of τ s. The output across the capacitor for $t \geq 0^+$ is observed to contain only $e^{-2t/\tau}$ wave-shape. What is the value of k ?
18. The steady-state voltage across resistor (v_R) in a series RC circuit has an amplitude of 7 V when the circuit is driven by an AC voltage of amplitude 10 V and angular frequency ω rad/s. (i) Find the phase angle of v_R with respect to the input sinusoid. (ii) If another sinusoidal voltage of 15 V amplitude and 3ω rad/s frequency is applied to the circuit, find the amplitude and phase of v_R under steady-state condition.
19. The desired voltage across a parallel RC with initial condition as shown in the Fig. 11.14-6 is given by $v(t) = 2t$ V for $t \geq 0$ and 0 for $t < 0$. Find the $i_s(t)$ to be applied to the circuit if the initial condition is 5 V. Sketch the required $i_s(t)$.

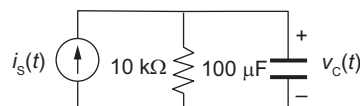


Fig. 11.14-6

20. The fall time of a series RC circuit is 6.6 ms. If the same components are used to make a parallel RC circuit excited by a sinusoidal current source $i_s(t) = 0.5 \sin 300 \pi t$ A find the steady-state current in the resistor in the circuit.
21. A pure LC circuit with $L = 100$ mH and $C = 10 \mu\text{F}$ has 0.05 J in the capacitor and 0.2 J in the inductor at $t = 0^-$. The circuit is allowed to carry out its free-response from $t = 0$. (i) Find the frequency of oscillations and amplitude of voltage and current oscillations. (ii) If the voltage across the capacitor is found to be 150 V at a particular instant, what is the current in the circuit at that instant?
22. A pure LC circuit in free-response is found to have 15 V amplitude voltage and 1.5 A current sinusoidal oscillations with a frequency of $2\pi \times 10^3$ rad/s. (i) Find L and C . (ii) Find the total initial energy storage. (iii) Can the initial voltage across capacitor and initial current through inductor be found out from this data? Explain.
23. A lightly-damped series RLC circuit uses 100 mH inductor and $1 \mu\text{F}$ capacitor and starts its free-response at $t = 0$ with an initial energy of 0.1 J. The voltage across the capacitor is 200 V and the current through the circuit is 0.4 A after 4 ms. Find ξ , Q , ω_d and initial conditions for the circuit.
24. The Q -factor of a 1 mH inductor at 10 kHz is measured to be 25. (i) What is the value of capacitor needed to make a series RLC circuit resonant at 10 kHz using this inductor? (ii) What will be the maximum percentage overshoot in the step response of the series circuit made with this inductor and the calculated value of capacitance?
25. Show that the amplitude of steady-state voltage across capacitor in a series RLC circuit is $1/\omega^2$ for $\omega \gg \omega_n$, where ω is the angular frequency of sinusoidal input voltage and ω_n is the undamped natural frequency of the circuit.
26. Show that the amplitude of steady-state voltage across inductor in a series RLC circuit is $\propto \omega^2$ for $\omega \ll \omega_n$, where ω is the angular frequency of sinusoidal input voltage and ω_n is the undamped natural frequency of the circuit.
27. Show that the amplitude of steady-state voltage across resistor in a series RLC circuit is $\propto 1/\omega$ for $\omega \gg \omega_n$, where ω is the angular frequency of sinusoidal input voltage and ω_n is the undamped natural frequency of the circuit.
28. The total energy storage in a parallel RLC circuit in free-response mode is found to be 70% of total initial energy storage after three full oscillations. The input admittance of the circuit is found to be resistive with a value of 0.01 S at 100 kHz. If this circuit is used as a band-pass filter, find the centre frequency, half-power frequencies and bandwidth of the filter.
29. The values of ω_n and ξ for a series RLC circuit with unknown parameters are to be found with the help of a square wave generator and oscilloscope. Suggest a suitable experimental procedure and explain how the values may be calculated from the observations.

30. The values of ω_n and ξ for a parallel RLC circuit with unknown parameters are to be found with the help of a sine wave generator and oscilloscope. Suggest a suitable experimental procedure and explain how the values may be calculated from the observations.
31. A series RLC circuit with zero damping (i.e., with $R = 0$) is driven by a sinusoidal voltage source of unit amplitude. The angular frequency of the source coincides with the value of ω_n

of the circuit. Show that the zero-state response for capacitor voltage in this case is $= 0.5 \sin \omega_n t - 0.5 (\omega_n t) \cos \omega_n t$ if the input is unit amplitude sine function. Does the circuit reach a steady-state?

[Hint: Solve for a general value of ω and then use the

following limit, $\lim_{x \rightarrow y} \frac{\sin x - \sin y}{x - y} = \cos x$.]

11.15 PROBLEMS

(Initial condition values are zero unless specified otherwise.)

1. The switches in Fig. 11.15-1 are ideal. (i) Find the voltage across C and plot it. (ii) Find the current through C and plot it. (iii) Find the voltage across the first resistor and plot it.

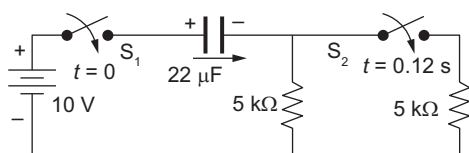


Fig. 11.15-1

2. Initial voltage at $t = -\infty$ in the capacitor in the circuit in Fig. 11.15-2 was zero. Find the voltage across capacitor and current through it for $t \geq 0^+$ and plot them.

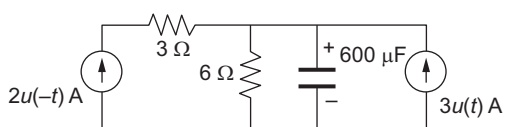


Fig. 11.15-2

3. The switch in the circuit in Fig. 11.15-3 was closed for a long time and is opened at $t = 0$. Find and plot the current in C and voltage across C as functions of time.

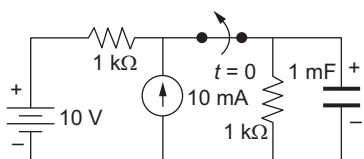


Fig. 11.15-3

4. Find the response of the voltage variable $v(t)$ in the circuit in Fig. 11.15-4.

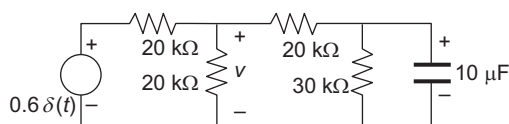


Fig. 11.15-4

5. $v_C(t)$ at 0^- in the circuit in Fig. 11.15-5 is $-V$. The switch S is closed at $t = 0$. Show that $v_C(t)$ will cross zero at $t = 0.69 RC$ s.

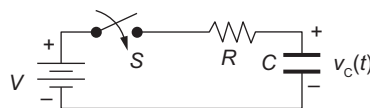


Fig. 11.15-5

6. (i) Plot the output for (a) $RC = T$ (b) $RC = 10T$ and (c) $RC = 0.1T$ (ii) Find the relation between RC and T if the area of output pulse after T s is to be less than 10% of the area of output pulse from 0 to T s. (Fig. 11.15-6)

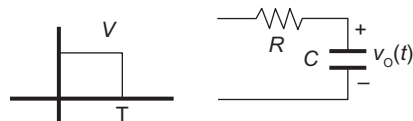


Fig. 11.15-6

7. What must be the value of k in the circuit in Fig. 11.15-7 if $v(t) = 0$ for $t \geq 0^+$?

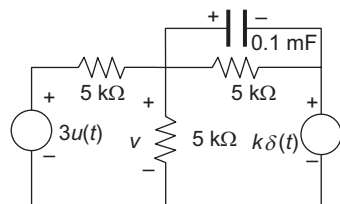


Fig. 11.15-7

8. A single pulse defined as $v_s(t) = 10t$ for $0 \leq t \leq 1$ and 0 otherwise is applied to a series RC circuit with a time constant of 0.3 s. Find and plot the capacitor voltage.
9. Find the time constants for each circuit in Fig. 11.15-8 for voltage excitation and current excitation. All resistors are 1 kΩ and all capacitors are 1 μF.

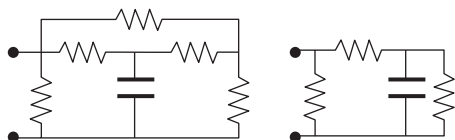


Fig. 11.15-8

10. The inverter in the circuit in Fig. 11.15-9 is a digital electronic gate circuit and its input and output behaviour is as shown in the waveforms. Each inverter gate has 15 pF input capacitance. The gate circuit will draw only zero current from +V supply if its input is held at V or 0 V steadily. The output of one such gate is connected as input to four such gates. Calculate (i) the power dissipation in the driving gate and (ii) average power supply current drawn by the driving gate when the input to driving gate is a square wave varying between +V and 0 with a frequency f for (a) $V = 5$ V, $f = 100$ kHz (b) $V = 5$ V, $f = 10$ MHz (c) $V = 15$ V, $f = 100$ kHz (d) $V = 15$ V, $f = 10$ MHz.

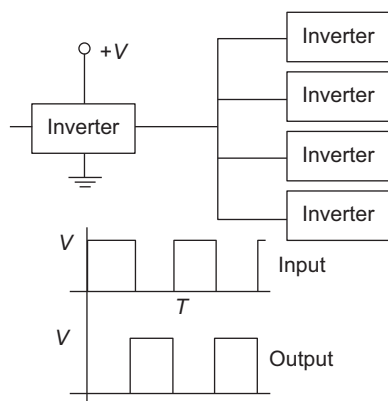


Fig. 11.15-9

11. Capacitor resists change in voltage. Hence a capacitor in parallel with a resistor will tend to keep the voltage across resistor constant. This will be more if the value of capacitance is raised. This idea is used in producing a variable DC voltage across a load resistance from a fixed DC current source in the circuit in Fig. 11.15-10.

The switch S spends T_1 s in position-1 and $(T - T_1)$ s in position-2 and the whole cycle is repeated. T will be kept constant and T_1 will be varied in the range 0 to T in order to control the voltage across R .

For any value of T_1 , a periodic steady-state behaviour of I_R will come up after a few switching cycles. This periodic steady-state is also shown in Fig. 11.15-10.

The waveforms between I_1 and I_2 will be a piece of an exponential and so will be the waveform between I_2 and I_1 . Under periodic steady-state I_1 has to be equal to I_1' .

- Using the above condition for periodic steady state derive equations for I_1 and I_2 in terms of I_0 , T_1/T , τ/T , where $\tau = RC$. Simplify the expressions for $\tau/T \gg 1$.
- From the above expressions find the average voltage across R in terms of I_0 , T_1/T , τ/T . Simplify the expressions for $\tau/T \gg 1$.
- Let $d = T_1/T$. Calculate and plot the ratio of peak-peak ripple to average value for V_0 for various values of ' d ' (calculate for $d = 0.1, 0.2 \dots 1$) with (a) $RC = 10T$ and (b) $RC = 50T$.
- The above step should have shown that a very clean DC voltage can be produced across R by using a large C such that $RC \gg T$. And it is possible to vary this clean DC voltage by changing ' d '. Let the system be working steadily at $d = 0.25$ with a DC output of $I_0 R/4$. The ' d ' (so called 'duty ratio') is changed suddenly to 0.5. Plot the growth of output DC component and calculate rise time when (i) $RC = 10T$ and (ii) $RC = 50T$.

Why can we not filter the DC voltage output to near zero ripple level?

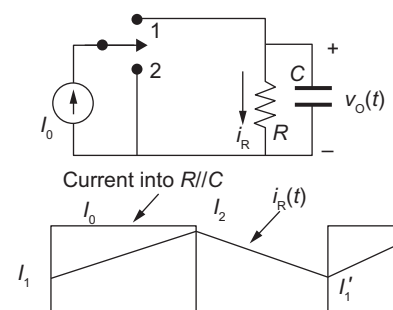


Fig. 11.15-10

12. A symmetric square wave of $\pm V$ amplitude and period of T s is applied to a high pass RC circuit as shown in Fig. 11.15-11. After a few cycles of initial transient the output waveform settles to a periodic steady-state as shown, where V_1' and V_2'

will be equal to V_1 and V_2 , respectively under periodic steady-state. The slanting portion of output wave will be exponential with time constant $= RC$. However, if $RC \gg T$ (equivalently, if the cut-off frequency of the circuit is much less than the square wave frequency) we can approximate the exponential by straight-line segments. Use this approximation and find expressions for V_1 and V_2 . Also find an expression for the so-called 'percentage tilt' defined as $100 \cdot (V_1 - V_2)/V$.

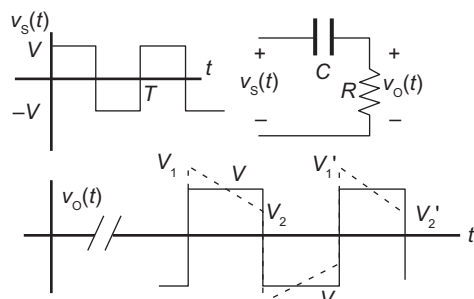


Fig. 11.15-11

13. Let $V_s(t)$ an arbitrary time varying periodic voltage source with a cycle average value of V_{DC} . This means that $V_s(t)$ can be written as $V_{DC} + V_{AC}(t)$, where $V_{AC}(t)$ is a time varying periodic component with equal positive half cycle and negative half cycle areas. Let that area be A volt-sec.

This waveform $V_s(t)$ is applied to a series RC circuit and output voltage is taken across C . Assume that $RC \gg T$, where T is the period of $V_s(t)$. Show that under periodic steady-state (i) the average value of output voltage is V_{DC} (ii) the peak-to-peak ripple in output voltage $\approx A/\tau$ V, where $\tau = RC$. (iii) Calculate the quantities in (i) and (ii) for the three inputs given in Fig. 11.15-12 if τ is 20 ms. (The series RC circuit can be used to extract average value of the input. The basic issue involved in this application is the trade-off between ripple in the average value versus response time).

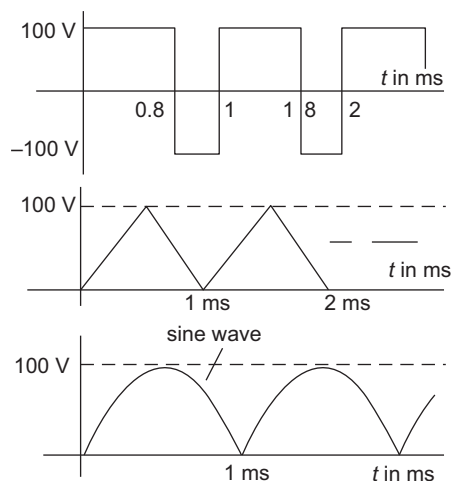


Fig. 11.15-12

14. The switch S_1 in Fig. 11.15-13 is closed at $t = 0$ with zero initial condition in the capacitor. The switch S_2 is kept open for T_1 s and closed for $(T - T_1)$ s periodically. Obtain the voltage across capacitor and plot it assuming the following data. $V = 12$ V, $R = 12$ k Ω , $C = 1$ μ F, $T_1 = 10$ ms, $T = 11$ ms and resistance of S_2 when it is ON $= 100$ Ω .

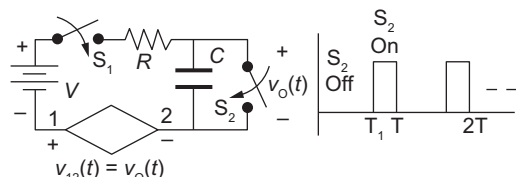


Fig. 11.15-13

15. A current signal $i_s(t) = (2 \sin \omega t - 1.2 \sin 3\omega t + 0.8 \sin 5\omega t)$ mA is applied to a parallel RC circuit of bandwidth 0.8ω rad/sec. Find and plot the steady-state waveform of voltage across the circuit ($R = 1$ k Ω).
16. A series RLC circuit with $L = 16$ mH, $C = 16$ μ F and $R = 100$ Ω has initial energy storage of 0.0016 J in the inductor and 0.0032 J in the capacitor. It is driven by a $10u(t)$ V source. Find the total response for the resistor voltage in the circuit.
17. A series RLC circuit with $L = 16$ mH, $C = 16$ μ F and $R = 10$ Ω has initial energy storage of 0.0016 J in the inductor and 0.0032 J in the capacitor. It is driven by a $10u(t)$ V source. Find the total response for the resistor voltage in the circuit.
18. A series LC circuit uses $L = 4$ mH, $C = 64$ μ F. The Q -factor of inductor is measured to be 80 at 1975 rad/s and the Q -factor of capacitor at that frequency is found to be 300. If the circuit starts its free-response with total initial energy storage at 0.3 J what is the time at which the total stored energy in the circuit is 0.1 J?
19. (i) Derive the differential equation governing the output voltage $v_o(t)$ in the circuit in Fig. 11.15-14. (ii) Obtain expressions for ω_n and ξ of the circuit and calculate them for $L = 1$ mH, $C = 100$ μ F, $R_1 = 0.5$ Ω and $R_2 = 10$ Ω . (iii) Find the zero-input response of the output voltage if the initial value of capacitor voltage is 10 V.

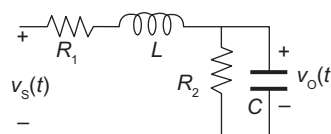


Fig. 11.15-14

20. A DC power supply of 12 V is connected to an electronic circuit by means of a two-wire connection that has an inductance of 10 μ H and resistance of 0.5 Ω . The circuit draws 0.2 A from the power supply may be modelled as a resistor. A 0.22 μ F ceramic capacitor is connected directly across the electronic circuit to hold its power supply constant. The circuit can withstand only

± 2.5 V variation on supply and will get damaged if the power supply exceeds 15 V. Examine whether the circuit will get damaged when the 12 V supply is switched on. If so, what solution will you suggest for the problem? [Hint: Consider the maximum percentage overshoot in the step response of RLC circuit model.]

21. Series LC filters are frequently employed at the output stage of DC power supplies to reduce the ripple content in the DC output. One such filter in a 12 V DC supply (Fig. 11.15-15) uses $L = 4$ mH and $C = 220$ μ F. The inductor has a resistance of 1 Ω . Two loads (electronic circuits) which can be modelled as 5 Ω and 20 Ω resistors are connected across the capacitor of the filter. The system was working under steady-state for long time. Suddenly, the load resistance of 5 Ω is switched off. Find the power supply output voltage as function of time. The maximum voltage that the electronic circuit represented by 20 Ω can withstand is 15 V. Will it get damaged when the other load is thrown open?

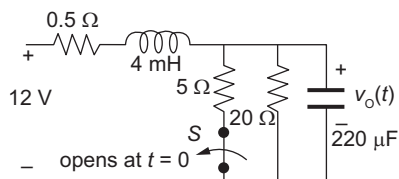


Fig. 11.15-15

22. A series RLC circuit is resonant at 1 kHz and is found to have critically-damped step response if a 10 Ω resistance is used in the circuit. (i) Find L and C . (ii) If $R = 1$ Ω is actually used, find rise time, peak time, settling time and maximum percentage overshoot in step response of capacitor voltage.
23. A series RLC circuit was found to draw 5 W power at unity power factor when driven by a 10 V, 2 kHz source. It has 60% maximum percentage overshoot in its step response. (i) Find R , L and C . (ii) Find the additional resistance that has to be included in the circuit to make it critically-damped. (iii) Find the voltage across the resistor under critical damping condition if the circuit with zero initial energy storage is excited by $10u(t)$ V.
24. A series RLC circuit was driven by a 10 V amplitude sinusoidal wave with variable frequency. The voltage across R was found to be of 10 V amplitude at 5 kHz and of 7.07 V amplitude at 4.9 kHz and 5.1 kHz. (i) Find the ω_n , ω_d , ξ and Q of the circuit. (ii) Find the amplitude of voltage across C and voltage across L at 5 kHz if input amplitude is 10 V. (iii) Find the maximum amplitude of voltage that appears across L and C and the corresponding frequencies.
25. A series RLC circuit is resonant at 1 kHz and has 50% maximum overshoot in step response. It is driven by $v_s(t) = 1 \sin \omega t$ V. Find the steady-state voltage across L and

C as functions of time approximately for (a) $f = 10$ kHz and (b) $f = 100$ Hz.

26. The switch in Fig. 11.15-16 was open for a long time and closes at $t = 0$. Find $v_C(t)$ and $i_L(t)$ for $t \geq 0^+$.

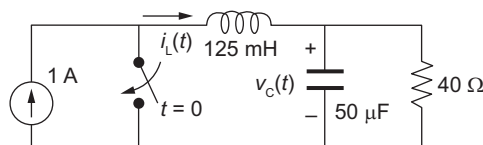


Fig. 11.15-16

27. The circuit in Fig. 11.15-17 is initially relaxed. Find $v_C(t)$ and $i_L(t)$ for $t \geq 0^+$. [Hint: Natural frequencies of a circuit will not depend on the excitation source value.]

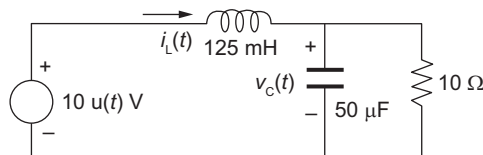


Fig. 11.15-17

28. The circuit in Fig. 11.15-18 is initially relaxed. Find $v_C(t)$ for $t \geq 0^+$. [Hint: Refer to Problem 27]

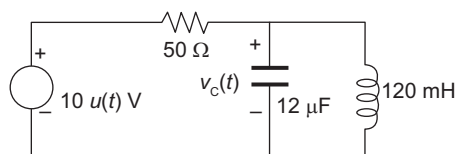


Fig. 11.15-18

29. A parallel LC circuit was driven by a sinusoidal current source of 10 mA amplitude. The frequency of the source was variable. Maximum voltage amplitude observed across the LC combination was 10 V and this happened at 20 kHz. Moreover, the voltage amplitude was seen to be 7.07 V at 20.1 kHz as well as at 19.9 kHz. (i) Find the values of L , C and Q -factor of inductor at 20 kHz. The capacitor may be assumed as loss-less. (ii) If a bandwidth of 2 kHz is desired, calculate the extra resistance that has to be put in parallel with the LC parallel combination.
30. A parallel RLC Circuit is resonant at 0.8 kHz and has 60% maximum overshoot in step response of inductor current. It is driven by $i_s(t) = 2 \cos \omega t$ A. Find the steady-state current through L and C as functions of time approximately for (a) $f = 8$ kHz and (b) $f = 80$ Hz.

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Higher Order Circuits in Time-Domain

CHAPTER OBJECTIVES

- To extend the concepts of linear circuit analysis in time-domain to higher order circuits.
- To emphasise the important properties exhibited by n^{th} order linear time-invariant circuits arising out of linearity and time-invariance properties of circuit elements in them.
- To show that any arbitrary input source function can be expanded in terms of scaled and shifted impulse functions.
- To show that the zero-state response of a linear time-invariant circuit to an arbitrary input source function can be obtained from its impulse response through the convolution integral.
- To provide a graphical interpretation of time-domain convolution and explain the meaning of Scanning Function.
- To explain how to obtain the DC steady-state response and the sinusoidal steady-state response using the convolution integral.
- To relate the sinusoidal steady-state frequency response function of a linear time-invariant circuit to its impulse response.

This chapter shows that the impulse response of a linear time-invariant circuit characterises it completely. It also shows that the frequency response function of such a circuit can be obtained from its impulse response and it ends with a question – is the sinusoidal steady-state frequency response a complete description of a linear time-invariant circuit? Thus, this chapter winds up the time-domain analysis of circuits and leads to frequency-domain analysis.

INTRODUCTION

Many of the basic principles governing the behaviour of a linear time-invariant circuit containing passive elements have been brought out in the two previous chapters. Such circuits are described in time-domain by a linear constant coefficient ordinary differential equation.

This chapter aims at generalising the principles brought out in earlier chapters for a linear time-invariant circuit of n^{th} order. The circuit may contain linear dependent sources along with passive elements (i.e., R , L , C and M).

We start with the mesh and nodal analysis of linear dynamic circuits containing two or more energy storage elements (and dependent sources as well) in the first part of this chapter. We bring out many generally applicable concepts in linear systems through a set of specific circuit examples.

Subsequently, in the second part of this chapter, we generalise the concepts evolved through these examples. We introduce the *complex signal space* and identify the natural response components in the circuit as *signal points* in this complex signal space as a part of this generalisation.

Then, we show that the impulse response of a dynamic circuit is a complete characterisation of the circuit in the sense that the zero-state response for any arbitrary input can be obtained from the input time-function and the impulse response function. The specific relationship tying up the input function, the output function and the impulse response is called the *convolution integral*. This integral is taken up for a detailed study in the third part of this chapter. As part of this study, we will observe that the sinusoidal steady-state frequency response function is intimately related to the time-domain impulse response function of the circuit. This observation will prompt us to ask the question – can the zero-state response for an arbitrary input function be determined using that function and the frequency response function? Or, in short, *is the sinusoidal steady-state frequency response function a complete characterisation of the circuit?*

This chapter ends at this point. The question raised at the end of this chapter signals the end of time-domain analysis and the entry into frequency-domain analysis. An entire part called *Frequency-domain analysis of Dynamic Circuits* is devoted for answering that question in detail.

12.1 ANALYSIS OF MULTI-MESH AND MULTI-NODE DYNAMIC CIRCUITS

Mesh analysis and nodal analysis were introduced in Chap. 4 in the context of analysis of memoryless circuits. However, the formulation of mesh and nodal analysis techniques was general in nature.

Node voltage variables are a set of minimum number of independent voltage variables defined in such a way that all the element voltages can be determined if they are known. Similarly, mesh currents are a set of minimum number of independent current variables defined in such a way that all element currents can be determined if they are known. Node voltage variables are solved by using KCL equation for all nodes except the reference node. If the elements in the circuit are memoryless elements, the resulting KCL equations will be algebraic in nature. If the elements are dynamic elements, the resulting KCL equations will be integro-differential equations.

Thus, mesh and nodal analysis formulations are applicable to dynamic circuits in time-domain without any modification, whatsoever. However, the node equations and the mesh equations in the case of a dynamic circuit will be integro-differential equations instead of algebraic equations since the *element equation* of a dynamic element is an integro-differential equation. We illustrate the mesh analysis and the nodal analysis procedure for dynamic circuits through a series of examples.

The examples that follow illustrate various techniques that are employed in solving higher order linear circuits in time-domain. Simultaneously, they bring out important general properties of linear circuits. Further, they go into the details of determination of initial conditions required to solve the describing differential equation from the initial currents in the inductors and the initial voltages across the capacitors.

EXAMPLE: 12.1-1

(i) Find the differential equations describing the node voltage variables in the circuit in Fig. 12.1-1 and determine the order of the differential equation in each case. (ii) Examine whether the natural response components in the node voltage variables are of the same natural frequencies. (iii) Obtain the impulse response of all the node voltage variables. (iv) Obtain the step response of the third node voltage variable.

SOLUTION

(i) There are only three nodes that require node voltage variable assignment. The node voltage variables assigned are shown in Fig. 12.1-1. Node equations are obtained by summing up the currents flowing away from the node and equating the sum to zero. We obtain the node equations as below:

$$\text{Node-1: } 2v_1 + \frac{dv_1}{dt} - v_2 = v_s(t) \quad (12.1-1)$$

$$\text{Node-2: } 2v_2 + \frac{dv_2}{dt} - v_1 - v_3 = 0 \quad (12.1-2)$$

$$\text{Node-3: } v_3 + \frac{dv_3}{dt} - v_2 = 0 \quad (12.1-3)$$

From Eqn. 12.1-1 we get,

$$v_2 = 2v_1 + \frac{dv_1}{dt} - v_s(t) \quad (12.1-4)$$

Substituting Eqn. 12.1-4 in Eqn. 12.1-2, we get,

$$\begin{aligned} v_3 = 2v_2 + \frac{dv_2}{dt} - v_1 &= 4v_1 + 2\frac{dv_1}{dt} - 2v_s(t) + 2\frac{dv_1}{dt} + \frac{d^2v_1}{dt^2} - \frac{dv_s(t)}{dt} - v_1 \\ &= \frac{d^2v_1}{dt^2} + 4\frac{dv_1}{dt} + 3v_1 - 2v_s(t) - \frac{dv_s(t)}{dt} \end{aligned} \quad (12.1-5)$$

Substituting Eqn. 12.1-5 and Eqn. 12.1-4 in Eqn. 12.1-3 and simplifying, we get,

$$\frac{d^3v_1}{dt^3} + 5\frac{d^2v_1}{dt^2} + 6\frac{dv_1}{dt} + v_1 = \frac{d^2v_s(t)}{dt^2} + 3\frac{dv_s(t)}{dt} + v_s(t) \quad (12.1-6)$$

From Eqn. 12.1-3 we get,

$$v_2 = v_3 + \frac{dv_3}{dt} \quad (12.1-7)$$

Substituting Eqn. 12.1-7 in Eqn. 12.1-2, we get,

$$v_1 = 2v_2 + \frac{dv_2}{dt} - v_3 = \frac{d^2v_3}{dt^2} + 3\frac{dv_3}{dt} + v_3 \quad (12.1-8)$$

Using Eqn. 12.1-8 and Eqn. 12.1-7 in Eqn. 12.1-1 and simplifying, we get,

$$\frac{d^3v_3}{dt^3} + 5\frac{d^2v_3}{dt^2} + 6\frac{dv_3}{dt} + v_3 = v_s(t) \quad (12.1-9)$$

Obtaining the differential equation for second node voltage variable is somewhat tedious. We proceed as below:

Differentiating the second node equation and adding it to itself, we get,

$$2v_2 + \frac{dv_2}{dt} - v_1 - v_3 + 2\frac{dv_2}{dt} + \frac{d^2v_2}{dt^2} - \frac{dv_1}{dt} - \frac{dv_3}{dt} = 0 \quad (12.1-10)$$

Adding Eqn. 12.1-10 to Eqn. 12.1-3 we get,

$$\frac{d^2v_2}{dt^2} + 3\frac{dv_2}{dt} + v_2 - v_1 - \frac{dv_1}{dt} = 0 \quad (12.1-11)$$

We have eliminated v_3 now. Differentiating Eqn. 12.1-11 with respect to time and adding it to two times the same equation, we get,

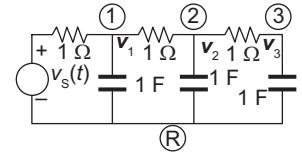


Fig. 12.1-1 Circuit for Example 12.1-1

Time-domain analysis of circuits proceeds by choosing one circuit variable as the describing variable. The process of eliminating other variables in order to arrive at a differential equation describing this chosen variable can be tedious as illustrated in this example.

Comments on**Example: 12.1-1**

We differentiated the entire set of equations many times in the elimination process leading to Eqn. 12.1-15. They were all KCL equations and that is why it was possible.

However, even KCL equations can be differentiated only if all the terms are continuous functions of time. Hence, points of discontinuities in the input function are not in the domain of the equations in Eqn. 12.1-15. We assume that if at all $v_s(t)$ has a discontinuity, it is at $t = 0$, and hence, we state that Eqn. 12.1-15 is valid for $t \geq 0^+$.

We account for input discontinuity in the evaluation of the initial conditions required for solving the differential equations.

Natural Frequencies

The natural response terms in the total response of any linear time-invariant circuit will be of the form e^{st} , where 's' is called the natural frequency and is, in general, a complex number.

Natural response terms come from the complementary solution of the differential equation, i.e., from the solution of the homogeneous differential equation with right-hand side as zero.

Substitution of the expected solution Ae^{st} in the homogeneous differential equation leads to the characteristic equation which is a polynomial equation in 's' with the degree of polynomial same as the order of the differential equation.

Roots of the characteristic equation give the natural frequencies of the circuit.

$$\frac{d^3 v_2}{dt^3} + 5 \frac{d^2 v_2}{dt^2} + 7 \frac{dv_2}{dt} + 2v_2 - \frac{d^2 v_1}{dt^2} - 3 \frac{dv_1}{dt} - 2v_1 = 0 \quad (12.1-12)$$

Differentiating Eqn. 12.1-1 with respect to time and adding the result to it, we get,

$$\frac{d^2 v_1}{dt^2} + 3 \frac{dv_1}{dt} + 2v_1 - \frac{dv_2}{dt} - v_2 = v_s(t) + \frac{dv_s(t)}{dt} \quad (12.1-13)$$

Adding Eqn. 12.1-12 and Eqn. 12.1-13, we get,

$$\frac{d^3 v_2}{dt^3} + 5 \frac{d^2 v_2}{dt^2} + 6 \frac{dv_2}{dt} + v_2 = \frac{dv_s(t)}{dt} + v_s(t) \quad (12.1-14)$$

Thus, the equations describing the node voltage variables are:

$$\begin{aligned} \frac{d^3 v_1}{dt^3} + 5 \frac{d^2 v_1}{dt^2} + 6 \frac{dv_1}{dt} + v_1 &= \frac{d^2 v_s(t)}{dt^2} + 3 \frac{dv_s(t)}{dt} + v_s(t) \\ \frac{d^3 v_2}{dt^3} + 5 \frac{d^2 v_2}{dt^2} + 6 \frac{dv_2}{dt} + v_2 &= \frac{dv_s(t)}{dt} + v_s(t) \\ \frac{d^3 v_3}{dt^3} + 5 \frac{d^2 v_3}{dt^2} + 6 \frac{dv_3}{dt} + v_3 &= v_s(t) \end{aligned} \quad (12.1-15)$$

(ii) We observe that the left-hand sides of the differential equations governing the node voltage variables are of the same format. Hence, all the three third-order differential equations in Eqn. 12.1-15 will possess the same characteristic equation. Hence, all the node voltage variables will contain natural response terms with the same natural frequencies.

Since all element voltages in a circuit can be expressed in terms of its node voltage variables, it follows that the differential equation written in terms of any circuit voltage variable will have the same format on the left-hand side as that of the equations in Eqn. 12.1-15. Similarly, all element currents can also be expressed in terms of node voltage variables, and hence, the differential equation written for any current variable in the circuit also will have the same format on the left-hand side as that of the equations in Eqn. 12.1-15. For instance, let us write the differential equation for the current through the first capacitor.

$$i_{C1} = 1 \times \frac{dv_1}{dt}.$$

The differential equation governing the node voltage variable v_1 is

$$\frac{d^3 v_1}{dt^3} + 5 \frac{d^2 v_1}{dt^2} + 6 \frac{dv_1}{dt} + v_1 = \frac{d^2 v_s(t)}{dt^2} + 3 \frac{dv_s(t)}{dt} + v_s(t).$$

Differentiating this equation with respect to time again, we get,

$$\frac{d^4 v_1}{dt^4} + 5 \frac{d^3 v_1}{dt^3} + 6 \frac{d^2 v_1}{dt^2} + \frac{dv_1}{dt} = \frac{d^3 v_s(t)}{dt^3} + 3 \frac{d^2 v_s(t)}{dt^2} + \frac{dv_s(t)}{dt}$$

Substituting $i_{C1} = 1 \times \frac{dv_1}{dt}$ in the above equation, we get,

$$\frac{d^3 i_{C1}}{dt^3} + 5 \frac{d^2 i_{C1}}{dt^2} + 6 \frac{di_{C1}}{dt} + i_{C1} = \frac{d^3 v_s(t)}{dt^3} + 3 \frac{d^2 v_s(t)}{dt^2} + \frac{dv_s(t)}{dt} \quad (12.1-16)$$

We verify that Eqn. 12.1-16 has the same left-hand side format as that of the equations in Eqn. 12.1-15. Thus, we conclude that the characteristic equation will be the same irrespective of the circuit variable that was chosen as the describing variable for preparing the circuit differential equation. The choice of circuit variable will influence only the right-hand side of the differential equation. The format of the left-hand side is decided by the nature of elements and the way they are interconnected. Therefore, a linear time-invariant circuit has a unique characteristic equation that is decided by the nature of elements and the way they are interconnected.

Thus, a linear time-invariant circuit will have a unique set of natural frequencies that are independent of the circuit variable chosen to describe the circuit. The natural

frequencies are decided by the nature of elements and the way they are interconnected.

This implies that if a particular circuit variable is found to contain an e^{st} function in the natural response with a specific value of natural frequency equal to s_1 , then, all circuit variables in that circuit will *in general* contain an e^{st} function in their natural response. If that term is missing in the natural response for a particular circuit variable, that can only be the result of a specific choice of initial conditions. For some other initial conditions, the term may reappear in that variable too.

(iii) Impulse response means 'zero-state impulse response'. Thus, the initial voltage across all capacitors is zero at $t = 0^-$. Moreover, the value of $v_s(t)$ and all its derivatives is zero for $t \geq 0^+$, when $v_s(t) = \delta(t)$. Therefore, the differential equations governing the node voltage variables under impulse response conditions are:

$$\left. \begin{aligned} \frac{d^3 v_1}{dt^3} + 5 \frac{d^2 v_1}{dt^2} + 6 \frac{dv_1}{dt} + v_1 &= 0 \\ \frac{d^3 v_2}{dt^3} + 5 \frac{d^2 v_2}{dt^2} + 6 \frac{dv_2}{dt} + v_2 &= 0 \\ \frac{d^3 v_3}{dt^3} + 5 \frac{d^2 v_3}{dt^2} + 6 \frac{dv_3}{dt} + v_3 &= 0 \end{aligned} \right\} \text{ for } t \geq 0^+ \quad (12.1-17)$$

We need the natural frequencies of the differential equation to solve for the impulse response. We also need the initial values of all node voltage variables and their first and second derivatives in order to solve for the impulse response.

The characteristic equation is $s^3 + 5s^2 + 6s + 1 = 0$. We can resort to some root-finding numerical procedure of root-finding software to factorise the left-hand side polynomial. The roots in this case are $s_1 = -0.1981$, $s_2 = -3.247$ and $s_3 = -1.555$.

Now, we need to calculate the required initial conditions. The circuit conditions at $t = 0^+$ are marked in Fig. 12.1-2.

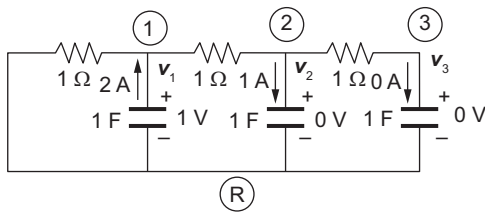


Fig. 12.1-2 The Circuit in Fig. 12.1-1 at $t = 0^+$ with Unit Impulse Input

The $\delta(t)$ V applied is converted into a $\delta(t)$ A current by a 1Ω resistor. This current deposits 1 C of charge across the first capacitor, changing its voltage from 0 V to 1 V instantaneously at $t = 0$. The impulse voltage source is a short after $t = 0$. The initial voltage across the other two capacitors remains unaffected at $t = 0^+$. Therefore, the first capacitor is delivering 2 A of current and 1 A of this current goes into the second capacitor. The third capacitor has no current in it since there is no voltage drop across the last resistor.

Therefore, $v_1(0^+) = 1$ V, $v_2(0^+) = 0$ V, $v_3(0^+) = 0$ V.

The first derivative of a voltage at $t = 0^+$ can usually be found by determining the current through a capacitor that has this voltage as its element voltage. In the case of the present circuit, we can easily identify the first derivatives of v_1 , v_2 and v_3 at $t = 0^+$ as the currents that flow through the first, second and third capacitors at that instant (since all three capacitors have 1 F value).

$$\text{Therefore, } \left. \frac{dv_1}{dt} \right|_{(0^+)} = -2 \text{ V/s, } \left. \frac{dv_2}{dt} \right|_{(0^+)} = +1 \text{ V/s and } \left. \frac{dv_3}{dt} \right|_{(0^+)} = 0 \text{ V/s.}$$

The node equations of the circuit are reproduced next.

A linear time-invariant circuit has a unique characteristic equation.

Determining the required initial conditions in time-domain analysis often turns out to be the most challenging part. The reader is urged to pay special attention to the logical reasoning involved in the determination of initial conditions in this Example.

Time-domain Analysis of Dynamic Circuits

This example illustrates the general procedure to carry out time-domain analysis of circuits.

Step-1: Decide on nodal analysis or mesh analysis and prepare the circuit suitably.

Step-2: Write the nodal equations or mesh equations and convert them into differential equations if they are integro-differential equations.

Step-3: Choose a circuit variable as the describing variable and eliminate all other variables from the set of equations obtained in Step-2. The result will be an n^{th} order linear differential equation.

Step-4: Form the characteristic equation for the differential equation and solve for natural frequencies of the circuit.

Step-5: Obtain the steady-state response (if present) for the required output variable. This may be done by solving a memoryless circuit obtained by replacing all inductors with short-circuits and all capacitors with open-circuits in the case of DC excitation. Phasor analysis may be employed if the excitation is sinusoidal. Note that the output variable need not be the same as the variable chosen for formulating the circuit differential equation.

Step-6: Let $y(t)$ be the output variable. Then, the total response for this variable is expressed as

$$y(t) = \sum_{i=1}^{i=n} A_i e^{s_i t} + y_p(t),$$

where s_i are values of natural frequencies arrived at in Step-4, A_i are arbitrary constants

continue

$$\text{Node-1: } 2v_1 + \frac{dv_1}{dt} - v_2 = v_s(t)$$

$$\text{Node-2: } 2v_2 + \frac{dv_2}{dt} - v_1 - v_3 = 0$$

$$\text{Node-3: } v_3 + \frac{dv_3}{dt} - v_2 = 0.$$

We recast these equations as below:

$$\frac{dv_1}{dt} = -2v_1 + v_2 \text{ for } t \geq 0^+$$

$$\frac{dv_2}{dt} = v_1 + v_3 - 2v_2 \text{ for } t \geq 0^+$$

$$\frac{dv_3}{dt} = v_2 - v_3 \text{ for } t \geq 0^+.$$

Differentiating these equations with respect to time, we get,

$$\frac{d^2v_1}{dt^2} = -2 \frac{dv_1}{dt} + \frac{dv_2}{dt} \text{ for } t \geq 0^+$$

$$\frac{d^2v_2}{dt^2} = \frac{dv_1}{dt} + \frac{dv_3}{dt} - 2 \frac{dv_2}{dt} \text{ for } t \geq 0^+$$

$$\frac{d^2v_3}{dt^2} = \frac{dv_2}{dt} - \frac{dv_3}{dt} \text{ for } t \geq 0^+.$$

We already know the values for first derivatives of node voltage variables. We can substitute those values in these equations to obtain the values of second derivatives of node voltage variables at $t = 0^+$. We get,

$$\left. \frac{d^2v_1}{dt^2} \right|_{(0^+)} = 5 \text{ V/s}^2, \left. \frac{d^2v_2}{dt^2} \right|_{(0^+)} = -4 \text{ V/s}^2 \text{ and } \left. \frac{d^2v_3}{dt^2} \right|_{(0^+)} = 1 \text{ V/s}^2.$$

We solve for impulse response of node voltage variables one by one now.

Let $v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}$, where s_1, s_2 and s_3 are the natural frequencies and A_1, A_2 and A_3 are the arbitrary constants representing the amplitudes of natural response terms.

Then,

$$\frac{dv_1(t)}{dt} = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t} + s_3 A_3 e^{s_3 t}$$

and

$$\frac{d^2v_1(t)}{dt^2} = s_1^2 A_1 e^{s_1 t} + s_2^2 A_2 e^{s_2 t} + s_3^2 A_3 e^{s_3 t}.$$

Applying initial conditions on these three equations at $t = 0^+$ and expressing the resulting equations on A_1, A_2 and A_3 in the form of a matrix equation, we get,

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \quad (12.1-18)$$

Solving this system of equations, we get, $A_1 = 0.1076$, $A_2 = 0.3493$ and $A_3 = 0.5431$. Therefore,

$v_1(t) = 0.1076e^{-0.1981t} + 0.3493e^{-3.47t} + 0.543e^{-1.555t}$ V is the impulse response of the first node voltage.

The values of A_1, A_2 and A_3 for $v_2(t)$ will be decided by a similar system of equations shown next.

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \quad (12.1-19)$$

Solving this system of equations, we get, $A_1 = 0.1939$, $A_2 = -0.4356$ and $A_3 = 0.2417$. Therefore, $v_2(t) = 0.1939e^{-0.1981t} - 0.4356e^{-3.47t} + 0.2417e^{-1.555t}$ V is the impulse response of the second node voltage.

The values of A_1 , A_2 and A_3 for $v_3(t)$ will be decided by a similar system of equations shown below:

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (12.1-20)$$

Solving this system of equations, we get, $A_1 = 0.02417$, $A_2 = 0.1938$ and $A_3 = -0.4356$. Therefore,

$v_3(t) = 0.2417e^{-0.1981t} + 0.1938e^{-3.47t} - 0.4356e^{-1.555t}$ V is the impulse response of the third node voltage.

(iv) The circuit conditions at $t = 0^+$ immediately after the application of a unit step voltage to the initially relaxed circuit is shown in Fig. 12.1-3.

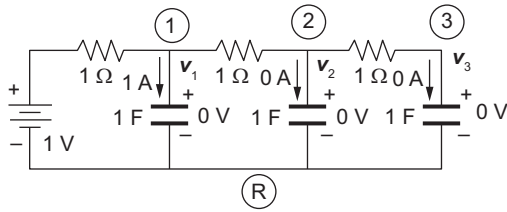


Fig. 12.1-3 Circuit Condition at $t = 0^+$ Immediately After Unit Step Input

The particular integral of the governing differential equation is obtained by solving for the DC steady-state value of the third node voltage. It is 1 V since all capacitors go open under the DC steady-state in a circuit.

Therefore, $v_3(t) = 1 + A_1e^{s_1t} + A_2e^{s_2t} + A_3e^{s_3t}$ with $s_1 = -0.1981$, $s_2 = -3.47$ and $s_3 = -1.555$. The values of A_1 , A_2 and A_3 are to be found by applying the initial conditions on this solution.

The initial conditions are $v_3(0^+) = 0$ (since there is no impulse current flow through the third capacitor at $t = 0$), $\frac{dv_3}{dt}\bigg|_{(0^+)} = 0$ (since there is no current flow through capacitor at $t = 0^+$) and $\frac{d^2v_3}{dt^2}\bigg|_{(0^+)} = 0$ (since $\frac{d^2v_3}{dt^2} = \frac{dv_2}{dt} - \frac{dv_3}{dt}$ for $t \geq 0^+$ and $\frac{dv_2}{dt}\bigg|_{(0^+)} = 0$, $\frac{dv_3}{dt}\bigg|_{(0^+)} = 0$).

Therefore, applying initial conditions on the total solution, we get,

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad (12.1-21)$$

Solving this system of equations, we get, $A_1 = -1.2205$, $A_2 = -0.0597$ and $A_3 = 0.2802$. Therefore,

$v_3(t) = 1 - 1.2205e^{-0.1981t} - 0.0597e^{-3.47t} + 0.2802e^{-1.555t}$ V is the unit step response of the third node voltage.

and $y_p(t)$ is the steady-state response obtained in Step-5.

Step-6: Obtain the initial conditions for the output variable $y(t)$ by using the initial values of inductor currents and capacitor voltages and Kirchhoff's laws. n initial conditions – the value of output variable and the values of its first $(n - 1)$ derivatives at $t = 0^+$ – are needed. Initial values of inductor currents and capacitor voltages at $t = 0^+$ are to be evaluated from corresponding values at $t = 0^-$ after accounting for impulse excitation sources (if present) and discontinuities in the input source functions at $t = 0$.

Step-7: Apply the initial values arrived at in Step-6 to the total response formulated in Step-5 to obtain n equations in n arbitrary constants $A_1 \dots A_n$. Solve for the arbitrary constants and complete the solution for total response.

EXAMPLE: 12.1-2

Three first-order RC sections are cascaded using unity gain ideal buffer amplifiers as shown in Fig. 12.1-4. Find the impulse response for v_3 .

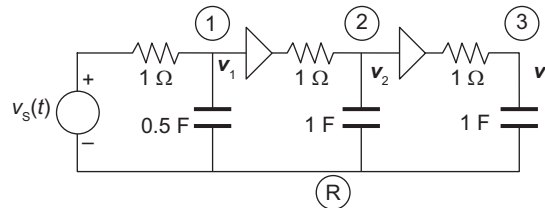


Fig. 12.1-4 Circuit for Example 12.1-2

SOLUTION

The node voltage variables are identified in Fig. 12.1-4. We write the node equations at the three nodes by applying KCL on currents flowing away from nodes. We note that the buffer amplifiers do not draw input current. The node equations are:

$$0.5 \frac{dv_1}{dt} + v_1 = v_s(t) \quad (12.1-22)$$

$$\frac{dv_2}{dt} + v_2 = v_1 \quad (12.1-23)$$

$$\frac{dv_3}{dt} + v_3 = v_2 \quad (12.1-24)$$

Using Eqn. 12.1-24 in Eqn. 12.1-23, we get, $\frac{d^2v_3}{dt^2} + 2\frac{dv_3}{dt} + v_3 = v_1$.

Using this equation in Eqn. 12.1-22, we get, $\frac{d^3v_3}{dt^3} + 4\frac{d^2v_3}{dt^2} + 5\frac{dv_3}{dt} + 2v_3 = 2v_s(t)$ as the differential equation governing $v_3(t)$. It is a third-order linear ordinary differential equation with constant coefficients.

The characteristic equation is $s^3 + 4s^2 + 5s + 2 = 0$ and its roots are $s_1 = -2$, $s_2 = -1$ and $s_3 = -1$. There is a root with a multiplicity of 2.

The impulse response can now be written as $v_3(t) = A_1 e^{-2t} + A_2 e^{-t} + A_3 t e^{-t}$ V. We had seen earlier (in Chap. 11, Sect. 11.6) that a natural frequency of multiplicity of 2 invites a contribution of the form $Ae^{st} + Bte^{st}$, where s is the natural frequency with multiplicity of 2 and A and B are the arbitrary constants. We had also observed in that section that (and in Sect. 11.3 of Chap. 11) that te^{st} is the result of a limiting operation $\lim_{\Delta \rightarrow 0} \frac{e^{(s+\Delta)t} - e^{st}}{\Delta}$. This indicates that the signal te^{st} could be viewed as a linear

combination of two closely spaced complex exponential functions scaled by the reciprocal of the difference between their natural frequencies. Thus, even te^{st} can be viewed as a mix of complex exponential functions. This is the point of view we adopt here. We write, $s_2 = -1 + \Delta$ and $s_3 = -1 - \Delta$ and send Δ to zero in the final result.

Now, the impulse response can be written as $v_3(t) = A_1 e^{-2t} + A_2 e^{(-1+\Delta)t} + A_3 e^{(-1-\Delta)t}$.

The $\delta(t)$ volts gets converted into $\delta(t)$ A current into the first capacitor. Hence its voltage changes from 0 to 2 V at $t = 0$ and is 2 V at $t = 0^+$. The voltage across second and third capacitors remains at zero. Therefore, $v_3(0^+) = 0$. The current through third capacitor at $t = 0^+$ is zero, and hence, $\left. \frac{dv_3}{dt} \right|_{(0^+)} = 0$. Differentiating the third node equation in Eqn. 12.1-24

$$\text{and evaluating the result at } t = 0^+ \text{ gives } \left. \frac{d^2v_3}{dt^2} \right|_{(0^+)} = \left. \frac{dv_2}{dt} \right|_{(0^+)} - \left. \frac{dv_3}{dt} \right|_{(0^+)} = 2 - 0 \text{ V/s}^2.$$

Comments on Example 12.1-2

The most important point in this example is that even when the natural frequencies of a circuit have a multiplicity of 2 and the impulse response contains terms of the type te^{st} , we can still view the natural response as the sum of complex exponential functions. This conclusion can be extended to $t^n e^{st}$, where n is an integer, thereby, accommodating natural frequencies with multiplicity of more than 2.

Applying these initial conditions and expressing the resulting equations in A_1 , A_2 and A_3 in matrix form, we get,

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -1+\Delta & -(1+\Delta) \\ 4 & 1-2\Delta & 1+2\Delta \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad (12.1-25)$$

We have neglected higher order terms in Δ considering the fact that it is very small compared to 1. The determinant of the square matrix in Eqn. 12.1-25 is $-\Delta$. Solving for A_1 , A_2 and A_3 , we get, $A_1 = 2$, $A_2 = -1 + 1/\Delta$ and $A_3 = -1 - 1/\Delta$. Therefore,

$$\begin{aligned} v_3(t) &= \lim_{\Delta \rightarrow 0} \left[2e^{-2t} - e^{(-1+\Delta)t} + \frac{1}{\Delta} e^{(-1+\Delta)t} - e^{(-1-\Delta)t} - \frac{1}{\Delta} e^{(-1-\Delta)t} \right] \\ &= 2e^{-2t} - 2e^{-t} + 2 \lim_{\Delta \rightarrow 0} \frac{e^{(-1+\Delta)t} - e^{(-1-\Delta)t}}{2\Delta} \\ &= 2e^{-2t} - 2e^{-t} + 2te^{-t} \text{ V.} \end{aligned}$$

The limit in the above equation is evaluated using the general limit $\lim_{a \rightarrow b} \frac{e^{at} - e^{bt}}{(a-b)} = te^{at}$ (see Sect. 11.3 in Chap. 11 for the proof).

Thus, we may state that the natural response of a linear time-invariant circuit contains only complex exponential functions.

In certain special situations, two or more complex exponential functions in the natural response of a linear time-invariant circuit may combine in such a manner that, in the limit of more than one natural frequency coinciding at the same value, the combination reduces to functions of $t^n e^{st}$ nature.

EXAMPLE: 12.1-3

Find the impulse response and step response of current in the resistor in the circuit in Fig. 12.1-5.

SOLUTION: IMPULSE RESPONSE

The two mesh equations are:

$$\begin{aligned} \frac{di_1}{dt} + \int_{-\infty}^t (i_1 - i_2) dt &= v_s(t) \\ \int_{-\infty}^t (i_2 - i_1) dt + \frac{di_2}{dt} + i_2 &= 0. \end{aligned}$$

Adding the two mesh equations results in

$$\frac{di_1}{dt} + \frac{di_2}{dt} + i_2 = v_s(t) \Rightarrow \frac{di_1}{dt} = v_s(t) - \frac{di_2}{dt} - i_2.$$

Differentiating the second mesh equation twice with respect to time and using the above equation in the result gives us,

$$\frac{d^3 i_2}{dt^3} + \frac{d^2 i_2}{dt^2} + 2 \frac{di_2}{dt} + i_2 = v_s(t).$$

The characteristic equation is $s^3 + s^2 + 2s + 1 = 0$ and its roots are $s_1 = -0.2151 + j1.307$, $s_2 = -0.2151 - j1.307$ and $s_3 = -0.5698$. There is a pair of *complex conjugate* roots.

The first capacitor voltage cannot change instantaneously, and hence, all the unit impulse voltage applied at the input appears across the first inductor. This results in an instantaneous change of its current from 0 to 1 A, from left to right. Thus, $i_1(0^+) = 1$ A. The current in second inductor is unaffected at $t = 0$, and hence, $i_2(0^+) = 0$ A.

Now, we need $\left. \frac{di_2}{dt} \right|_{0^+}$. We remember that the voltage across an inductor is proportional to the first derivative of current through it and we can identify i_2 as the current through the second inductor. Hence, we can find $\left. \frac{di_2}{dt} \right|_{0^+}$ if we can find the voltage across it at $t = 0^+$. This voltage is zero since the capacitor voltage is zero and the current in 1Ω is zero at $t = 0^+$. Hence $\left. \frac{di_2}{dt} \right|_{0^+} = 0$.

Further, we need $\left. \frac{d^2 i_2}{dt^2} \right|_{0^+}$. Differentiating the second mesh equation with respect to time, we get,

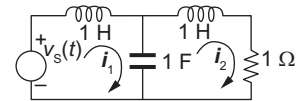


Fig. 12.1-5 Circuit for Example 12.1-3

Comments on Example 12.1-3

The coefficients of characteristic equation of a linear time-invariant circuit are decided by the circuit constants, and hence, will have to be *real-valued*. If such a characteristic polynomial with real coefficients has a complex root, the conjugate of that root also has to be a root of the polynomial. Otherwise, the coefficients will not turn out to be real.

Each natural frequency stands for a term Ae^{st} in the natural response of the circuit.

A complex natural frequency $s = \sigma + j\omega$ stands for a complex signal $e^{st}[\cos\omega t + j\sin\omega t]$ in the natural response and its conjugate companion results in $e^{st}[\cos\omega t - j\sin\omega t]$ term in the natural response.

The natural response of a physical circuit has to be a *real* function of time. Hence, the amplitudes of these two complex conjugate exponential functions will have to be complex conjugate numbers in order to get rid of the imaginary part in the combination. This is illustrated in Example 12.1-3.

$$(i_2 - i_1) + \frac{d^2 i_2}{dt^2} + \frac{di_2}{dt} = 0$$

$$\text{Therefore, } \left. \frac{d^2 i_2}{dt^2} \right|_{(0^+)} = - \left. \frac{di_2}{dt} \right|_{(0^+)} + i_1(0^+) - i_2(0^+) = -0 + 1 - 0 = 1 \text{ A/s.}$$

The impulse response can be written as $i_2(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}$ A with $s_1 = -0.2151 + j1.307$, $s_2 = -0.2151 - j1.307$ and $s_3 = -0.5698$. Substituting the initial conditions and expressing the resulting equations in matrix form, we get,

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (12.1-26)$$

Now, we show that A_1 and A_2 are complex conjugate numbers if s_1 and s_2 are complex conjugate numbers. We put $s_2 = s_1^*$ and $s_2^2 = (s_1^*)^2$ in Eqn. 12.1-26 and solve for A_1 and A_2 by Cramer's rule to show that $A_2 = A_1^*$ as below. We use the fact that s_3 is real in the proof.

$$A_1 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 0 & (s_1)^* & s_3 \\ 1 & (s_1^*)^2 & s_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ s_1 & (s_1)^* & s_3 \\ s_1^2 & (s_1^*)^2 & s_3^2 \end{vmatrix}} = \frac{s_3 - (s_1)^*}{\Delta} \quad \text{and} \quad A_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ s_1 & 0 & s_3 \\ s_1^2 & 1 & s_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ s_1 & (s_1)^* & s_3 \\ s_1^2 & (s_1^*)^2 & s_3^2 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 1 & 1 \\ -0 & s_1 & s_3 \\ 1 & s_1^2 & s_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ s_1 & (s_1)^* & s_3 \\ s_1^2 & (s_1^*)^2 & s_3^2 \end{vmatrix}} = \frac{s_1 - s_3}{-\Delta^*} = A_1^*$$

Note that the proof did not depend on the particular values of natural frequencies, and hence, is quite general. We can easily extend this proof for a case that has more than three natural frequencies. Hence, we state that, the arbitrary constants that represent the amplitudes of complex exponential functions that have complex conjugate natural frequencies, will themselves be complex conjugate numbers.

First we complete the solution for A_1 , A_2 and A_3 by solving Eqn. 12.1-26 after substituting for s_1 , s_2 and s_3 . The result is

$$A_1 = -0.2726 - j0.074 = 0.2824e^{-j2.8766}$$

$$A_2 = -0.2726 + j0.074 = 0.2824e^{-j2.8766}$$

$$A_3 = 0.5451.$$

Then,

$$\begin{aligned} \therefore i_2(t) &= 0.5451e^{-0.57t} + 0.2824e^{-j2.877} e^{-0.215t + j1.307t} + 0.2824e^{j2.877} e^{-0.215t - j1.307t} \\ &= 0.5451e^{-0.57t} + 0.2824e^{-0.215t} e^{j(1.307t - 2.877)} + 0.2824e^{-0.215t} e^{-j(1.307t - 2.877)} \\ &= 0.5451e^{-0.57t} + 0.2824e^{-0.215t} [e^{j(1.307t - 2.877)} + e^{-j(1.307t - 2.877)}] \\ &= 0.5451e^{-0.57t} + 0.2824e^{-0.215t} [2 \cos(1.307t - 2.877 \text{ Rad})] \text{ (by Euler's Formula)} \\ &= 0.5451e^{-0.57t} + 0.5648e^{-0.215t} \cos(1.307t - 164.82^\circ) \text{ A.} \end{aligned}$$

SOLUTION: STEP RESPONSE

The steady state component of a unit step response is found by replacing all capacitors with open-circuits and inductors with short-circuits. Therefore, the steady-state value of $i_2(t)$ is 1 A. Therefore, the step response of $i_2(t)$ can be written as $i_2(t) = 1 + A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}$ A.

$$i_2(0^+) = 0 \text{ (since there is no impulse voltage across the inductors), } \left. \frac{di_2}{dt} \right|_{(0^+)} = 0 \text{ (since}$$

there is no current in the second mesh at $t = 0^+$ and the voltage across the capacitor is 0 at $t = 0^+$) and $\left. \frac{d^2 i_2}{dt^2} \right|_{(0^+)} = - \left. \frac{di_2}{dt} \right|_{(0^+)} + i_1(0^+) - i_2(0^+) = -0 + 0 - 0 = 0 \text{ A/s.}$

Substituting these initial conditions and expressing the resulting equations in A_1 , A_2 and A_3 in matrix form, we get,

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system of equations by Cramer's rule, we get,

$$A_1 = -0.0217 + j0.2121 = 0.2132e^{j1.6727}$$

$$A_2 = -0.0217 - j0.2121 = 0.2132e^{-j1.6727}$$

$$A_3 = -0.9566.$$

We observe that A_1 and A_2 are complex conjugate numbers as expected.

$$\begin{aligned} \therefore i_2(t) &= 1 - 0.957e^{-0.57t} + 0.213e^{j1.673}e^{-0.215t+j1.307t} + 0.213e^{-j1.673}e^{-0.215t-j1.307t} \\ &= 1 - 0.957e^{-0.57t} + 0.213e^{-0.215t}e^{j(1.307t+1.673)} + 0.213e^{-0.215t}e^{-j(1.307t+1.673)} \\ &= 1 - 0.957e^{-0.57t} + 0.213e^{-0.215t} \left[e^{j(1.307t+1.673)} + e^{-j(1.307t+1.673)} \right] \\ &= 1 - 0.957e^{-0.57t} + 0.213e^{-0.215t} \left[2 \cos(1.307t + 1.673 \text{ rad}) \right] \text{ (By Euler's Formula)} \\ &= 1 - 0.957e^{-0.57t} + 0.426e^{-0.215t} \cos(1.307t + 95.84^\circ) \text{ A.} \end{aligned}$$

EXAMPLE: 12.1-4

(i) Determine the differential equation governing the output voltage of the active circuit shown in Fig. 12.1-6. The Opamp may be assumed to be an ideal one. (ii) Determine the natural frequencies of the circuit as functions of k and plot their variations in complex plane when k varies from 1 to $+\infty$ and $RC = 1$ s. (iii) Find the impulse response of the circuit for (a) $k = 1.586$ (b) $k = 3$ (c) $k = 4.414$ with $RC = 1$ s.

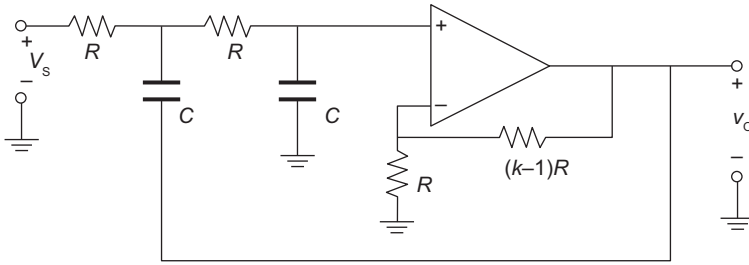


Fig. 12.1-6 Opamp Circuit for Example 12.1-4

SOLUTION (I)

The Opamp is an ideal one. Hence, it provides a gain of $1 + (k-1)R/R = k$ between the non-inverting terminal and the output terminal. It may be replaced with an ideal dependent voltage source as shown in Fig. 12.1-7.

There are only two nodes that need node voltage assignment and the assigned node voltage variables are shown in Fig. 12.1-7. The node equations are written at the nodes by equating the sum of currents leaving a node to zero.

$$\begin{aligned} \text{Node-1: } \frac{(v_1 - v_s)}{R} + \frac{(v_1 - v_2)}{R} + C \frac{d(v_1 - kv_2)}{dt} &= 0 \\ \text{Node-2: } \frac{(v_2 - v_1)}{R} + C \frac{dv_2}{dt} &= 0 \end{aligned} \quad (12.1-27)$$

We get $v_1 = v_2 + RC \frac{dv_2}{dt}$ from the second node equation. Using this elimination equation in the first node equation, we get,

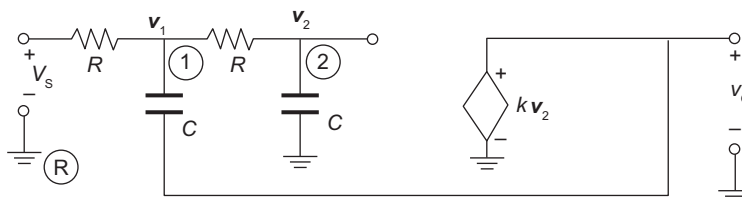


Fig. 12.1-7 The Amplifier in the Circuit in Fig. 12.1-6 Replaced with an Ideal Dependent Voltage Source

$$(RC)^2 \frac{d^2 v_2}{dt^2} + (3-k)RC \frac{dv_2}{dt} + v_2 = v_s \quad (12.1-28)$$

$$\text{i.e., } \frac{d^2 v_2}{dt^2} + (3-k) \frac{1}{RC} \frac{dv_2}{dt} + \frac{1}{(RC)^2} v_2 = \frac{1}{(RC)^2} v_s$$

The output voltage $v_o(t)$ is k times $v_2(t)$. Therefore, the differential equation governing the behaviour of the output voltage is

$$\frac{d^2 v_o}{dt^2} + (3-k) \frac{1}{RC} \frac{dv_o}{dt} + \frac{1}{(RC)^2} v_o = \frac{k}{(RC)^2} v_s \text{ for } t \geq 0^+ \quad (12.1-29)$$

SOLUTION (II)

Natural frequencies are roots of characteristic equations. The characteristic equation of the differential equation in Eqn. 12.1-29 is $s^2 + \frac{3-k}{RC}s + \frac{1}{(RC)^2} = 0$ and its roots are

$$s_{1,2} = \left(\frac{1}{RC} \right) \left(\frac{(k-3) \pm \sqrt{[(3-k)^2 - 4]}}{2} \right) = \left(\frac{1}{RC} \right) \left(\frac{(k-3) \pm \sqrt{k^2 - 6k + 5}}{2} \right).$$

The natural frequencies are coincident at -1 (with $RC = 1$ s) when k is 1. As k increases from 1, the natural frequencies become complex conjugate numbers with a negative real part. When k assumes the value of 3, the natural frequencies become $+j1$ rad/s and $-j1$ rad/s. As k increases above 3, the natural frequencies become complex conjugate numbers with a positive real part. When k reaches 5, the natural frequencies become coincident at 1. As $k \rightarrow \infty$, one of the natural frequencies approaches zero from the right and the other moves to $+\infty$.

Thus, the natural frequencies are complex conjugate numbers for the range $1 < k < 5$. The natural frequencies will be given by $s_{1,2} = \left(\frac{1}{RC} \right) \left(\frac{(k-3)}{2} \pm j \frac{\sqrt{-k^2 + 6k - 5}}{2} \right)$. If we square and add the real part and the imaginary part of natural frequencies in this range of k , we will get the $\frac{1}{(RC)^2}$ as the result. This indicates that the natural frequencies

move in a circle of radius $\frac{1}{(RC)^2}$ as k varies from 1 to 5. This variation is shown in Fig. 12.1-8 with $RC = 1$ s.

SOLUTION (III)

The entire impulse voltage appears across the first resistor since the second capacitor prevents the second resistor from absorbing any part of it. This results in an impulsive $\delta(t)/R$ current flow in the first capacitor, resulting in a sudden change of its voltage from 0 to $1/RC$ V.

Therefore, $v_1(0^+) = 1$ V, $v_2(0^+) = 0$ and hence, $v_o(0^+) = 0$. The current through the second resistor at 0^+ is $(1 - 0)/R$ and this current flows into the second capacitor since

the Opamp is ideal. Therefore, $\left. \frac{dv_2}{dt} \right|_{(0^+)} = \frac{1}{RC}$ V/s, and hence $\left. \frac{dv_o}{dt} \right|_{(0^+)} = \frac{k}{RC}$ V/s.

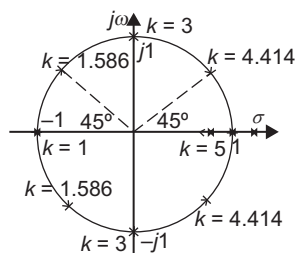


Fig. 12.1-8 Variation of Natural Frequencies of the Circuit in Fig. 12.1-6 with the Parameter k

(a) $k = 1.586$ and $RC = 1$ s. The natural frequencies in this case are $s_1 = -0.707 + j0.707$ and $s_2 = -0.707 - j0.707$. The impulse response at the output can be written as $v_o(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

Applying $v_o(0^+) = 0$ and $\left. \frac{dv_o}{dt} \right|_{(0^+)} = \frac{k}{RC} \text{ V/s} = 1.586 \text{ V/s}$, we get,

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.586 \end{bmatrix}.$$

Solving for A_1 and A_2 , $A_1 = -j1.1216$ and $A_2 = j1.1216$.

$$\begin{aligned} \therefore v_o(t) &= 1.1216 e^{-j90^\circ} e^{-0.707t + j0.707t} + 1.1216 e^{j90^\circ} e^{-0.707t - j0.707t} \\ &= 2.2432 e^{-0.707t} \cos(0.707t - 90^\circ) = 2.2432 e^{-0.707t} \sin 0.707t \text{ V.} \end{aligned}$$

(b) $k = 3$ and $RC = 1$ s. The natural frequencies in this case are $s_1 = j1$ and $s_2 = -j1$. The impulse response at the output can be written as $v_o(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

Applying $v_o(0^+) = 0$ and $\left. \frac{dv_o}{dt} \right|_{(0^+)} = \frac{k}{RC} \text{ V/s} = 3 \text{ V/s}$, we get,

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Solving for A_1 and A_2 , $A_1 = -j1.5$ and $A_2 = j1.5$.

$$\begin{aligned} \therefore v_o(t) &= 1.5 e^{-j90^\circ} e^{jt} + 1.5 e^{j90^\circ} e^{-jt} \\ &= 3 \cos(t - 90^\circ) = 3 \sin t \text{ V.} \end{aligned}$$

(c) $k = 4.414$ and $RC = 1$ s. The natural frequencies in this case are $s_1 = 0.707 + j0.707$ and $s_2 = 0.707 - j0.707$. The impulse response at the output can be written as $v_o(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

Applying $v_o(0^+) = 0$ and $\left. \frac{dv_o}{dt} \right|_{(0^+)} = \frac{k}{RC} \text{ V/s} = 4.414 \text{ V/s}$, we get,

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.414 \end{bmatrix}.$$

Solving for A_1 and A_2 , $A_1 = -j3.1216$ and $A_2 = j3.1216$.

$$\begin{aligned} \therefore v_o(t) &= 3.1216 e^{-j90^\circ} e^{0.707t + j0.707t} + 3.1216 e^{j90^\circ} e^{0.707t - j0.707t} \\ &= 6.2432 e^{0.707t} \cos(0.707t - 90^\circ) = 6.2432 e^{0.707t} \sin 0.707t \text{ V.} \end{aligned}$$

We observe that if a natural response term is represented by a point on the *left-half* of the complex signal space, i.e., if the *natural frequency* which represents this natural response term has a *negative real part*, the response term will decrease with time and go to zero as $t \rightarrow \infty$.

Further, if a natural response term is represented by a point on the *imaginary axis* of a complex signal space, i.e., if the *natural frequency* which represents this natural response term has *zero real part*, the response term will have a constant amplitude and it will not decay with time. Such a circuit behaves as a *sinusoidal oscillator* with the amplitude of oscillation decided by the initial energy storage in the circuit and/or the impulse excitation content.

If a natural response term is represented by a point on the *right-half* of complex signal space, i.e., if the *natural frequency* which represents this natural response term has a *positive real part*, the response term will be a growing transient and its amplitude will grow with time. The growing amplitude will be limited by the circuit variables reaching a certain ceiling level in practice. Obviously, such a circuit cannot have a DC, sinusoidal or periodical steady-state, since its transient response components will never die down to zero. Therefore, such circuits do not find many applications. This does not mean that circuits with natural frequencies on the *right-half* of complex signal space are totally useless. They do have applications; however, we do not deal with such applications in this text.

The circuit in this example is usually employed as a *low-pass active filter* with k between 1 and 3. With $k = 1.586$, it realises a so-called *Butterworth Second-order Low-pass Filter*. The circuit in filter application is called '*Sallen-Key Low-pass Filter*'.

Stability in Dynamic Circuits

Dynamic circuits with all their natural frequencies on the *left-half* of the s -plane are called *stable* circuits. All passive circuits, i.e., dynamic circuits containing no dependent sources, are unconditionally stable.

Dynamic circuits with one or more conjugate pairs of natural frequencies lying on the *imaginary axis* in the s -plane are called *marginally stable* circuits. Such circuits can be used to construct *sinusoidal oscillators*. A circuit will need dependent sources in it for it to be marginally stable.

Dynamic circuits with one or more natural frequencies in the *right-half* of the s -plane are called *unstable* circuits. Circuits with dependent sources can exhibit instability.

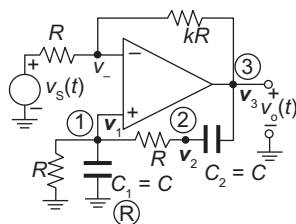


Fig. 12.1-9 Opamp Circuit in Example 12.1-5

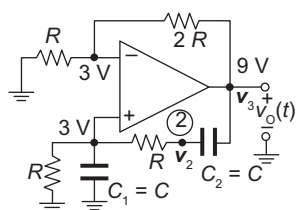


Fig. 12.1-10 Circuit Variables at 0^+ in the Circuit in Fig. 12.1-9

Non-zero output resistance and finite input resistance of a practical Opamp will introduce further damping in the circuit in Example 12.1-5 and will contribute a negative real part to the natural frequencies. Hence, the natural frequencies with $k = 2$ will not be on the $j\omega$ axis in the s -plane. Therefore, the circuit starts oscillating with $9\sqrt{2}$ V amplitude and gradually loses its amplitude. Eventually, the circuit settles down with zero output.

The various dissipation mechanisms in an Opamp or in any amplifier cannot be accurately predicted and modelled. Hence, the adjustment of k for sustained oscillation in the circuit is impossible in practice. Moreover, this circuit in the present form can oscillate only with an amplitude decided by its initial conditions. Hence, though this circuit can

(continue)

EXAMPLE: 12.1-5

Assume that the Opamp is ideal in the circuit shown in Fig. 12.1-9. (i) Find the ranges of k for stability in the circuit. (ii) What is the value of k for which the circuit will function as an oscillator? What is the frequency of oscillation? (iii) If the input is shorted and the circuit is started with v_1 at 3 V by pre-charging the capacitor C_1 , find the output voltage as a function of time with the value of k arrived at in Step (ii) and $RC = 1$ s.

SOLUTION

(i) We need to find the characteristic equation and its roots for determining the range of value of k for stability. We need the differential equation describing $v_o(t)$ for that. Three salient nodes in the circuit are identified and node voltages are assigned as shown in Fig. 12.1-9. We do not need a node voltage variable at the inverting terminal since the voltage at that point can be expressed as $v_- = \frac{k}{k+1}v_s(t) + \frac{1}{k+1}v_3(t)$ by applying the superposition principle and the zero-input current principle for an ideal Opamp.

We write the node equations at Node-1 and Node-2 and eliminate v_2 from them first.

$$\text{Node-1: } \frac{2v_1}{R} + C \frac{dv_1}{dt} - \frac{v_2}{R} = 0$$

$$\text{Node-2: } \frac{v_2}{R} + C \frac{dv_2}{dt} - C \frac{dv_3}{dt} - \frac{v_1}{R} = 0$$

$$\text{The first node equation implies that } v_2 = 2v_1 + RC \frac{dv_1}{dt}.$$

Using this in the second node equation after multiplying by R throughout,

$$(RC)^2 \frac{d^2v_1}{dt^2} + 3RC \frac{dv_1}{dt} + v_1 - RC \frac{dv_3}{dt} = 0.$$

Now, $v_1 = v_-$ by virtual-short principle for an ideal Opamp. Therefore, we substitute $v_1 = \frac{k}{k+1}v_s(t) + \frac{1}{k+1}v_3(t)$ in the last equation.

$$\begin{aligned} (RC)^2 \frac{k}{k+1} \frac{d^2v_s}{dt^2} + (RC)^2 \frac{1}{k+1} \frac{d^2v_3}{dt^2} + 3RC \frac{k}{k+1} \frac{dv_s}{dt} + \\ 3RC \frac{1}{k+1} \frac{dv_3}{dt} + \frac{k}{k+1}v_s + \frac{1}{k+1}v_3 - RC \frac{dv_3}{dt} = 0 \\ \frac{d^2v_3}{dt^2} + \frac{1}{RC}(2-k) \frac{dv_3}{dt} + \frac{1}{(RC)^2}v_3 = -k \left(\frac{d^2v_s}{dt^2} + \frac{3}{(RC)} \frac{dv_s}{dt} + \frac{1}{(RC)^2}v_s \right). \end{aligned}$$

$$\text{Therefore, the characteristic equation is } s^2 + \frac{2-k}{RC}s + \frac{1}{(RC)^2} = 0. \text{ The roots of this}$$

equation will have negative real values or complex conjugate values with negative real parts only if $k < 2$. Hence, the permitted value of k for a stable circuit will be $0 \leq k < 2$.

(ii) A circuit can function as an oscillator if it has a pair of natural frequencies that are pure imaginary conjugate numbers. This circuit will have such a pair at

$j\frac{1}{RC}$ and $-j\frac{1}{RC}$ when k assumes the value of 2. The frequency of sinusoidal oscillation will be $\frac{1}{2\pi RC}$ Hz.

(iii) The circuit is started with 3 V across C_1 at $t = 0^+$. The circuit condition at $t = 0^+$ is shown in Fig. 12.1-10.

The 3 V available across C_1 reaches the inverting terminal by the virtual-short existing across the input terminals of an ideal Opamp. The input terminals of an ideal Opamp do not draw any current. Therefore, the voltage at the right end of $R-2R$ chain must be 9 V. Therefore, $v_3(0^+) = 9$ V.

The current through C_1 at $t = 0^+$ is $3/R$ A towards ground node. Therefore, the rate of change of voltage across C_1 is $3/RC$ V/s. The voltage at the inverting terminal also changes at the same rate. The rate of change of voltage at the inverting terminal is

1/3rd of the rate of change of v_3 . Therefore, $\left| \frac{dv_3}{dt} \right|_{t=0^+} = \frac{9}{RC}$ V/s. With $RC = 1$ s, the value will

be 9 V/s. The angular frequency of operation will be $1/RC = 1$ rad/s. Therefore, the output has to be a sinusoidal waveform with $\omega = 1$ rad/s, initial value of 9 V and initial slope of 9 V/s. Let $v_3(t) = A \cos(t - \phi)$. Then $A \cos \phi = 9$ and $A \sin \phi = 9$.

$$\therefore v_3(t) = 9\sqrt{2} \cos(t - 45^\circ) \text{ V for } t \geq 0^+.$$

This circuit is called *Wien's Bridge Oscillator Circuit* and finds wide application in audio frequency sinusoidal generators (see the side-box).

oscillate, many other components are needed to ensure that it will positively start up and to control its amplitude of oscillation without affecting the frequency of oscillation.

EXAMPLE: 12.1-6

Obtain the differential equation governing the first mesh current in the circuit shown in Fig. 12.1-11. What is the order of differential equation?

SOLUTION

The mesh equations of the circuit are:

$$2i_1 + \frac{di_1}{dt} - 0.2 \frac{di_2}{dt} - i_2 = v_s \quad (12.1-30)$$

$$2i_2 + \frac{di_2}{dt} - 0.2 \frac{di_1}{dt} - i_1 = 0 \quad (12.1-31)$$

The Eqn. 12.1-30 is differentiated with respect to time and added to two times that equation itself to get,

$$\frac{d^2 i_1}{dt^2} + 4 \frac{di_1}{dt} + 4i_1 - 0.2 \frac{d^2 i_2}{dt^2} - 1.4 \frac{di_2}{dt} - 2i_2 = \frac{dv_s}{dt} + 2v_s \quad (12.1-32)$$

0.2 times the Eqn. 12.1-31 is differentiated with respect to time and added to that equation itself to get,

$$0.04 \frac{d^2 i_1}{dt^2} + 0.4 \frac{di_1}{dt} + i_1 + 0.2 \frac{d^2 i_2}{dt^2} + 1.4 \frac{di_2}{dt} + 2i_2 = 0 \quad (12.1-33)$$

Adding Eqn. 12.1-32 to Eqn. 12.1-33 yields

$$0.96 \frac{d^2 i_1}{dt^2} + 3.6 \frac{di_1}{dt} + 3i_1 = \frac{dv_s}{dt} + 2v_s$$

$$\text{i.e., } \frac{d^2 i_1}{dt^2} + 3.75 \frac{di_1}{dt} + 3.125i_1 = 1.04167 \frac{dv_s}{dt} + 2.083v_s.$$

Thus, the differential equation governing the first mesh current is of second order.

We observe that the order of differential equation describing the circuit is equal to the number of energy storage elements in the circuit in all the examples taken up in this section except this one.

The circuit in this example involves a node at which only inductors are connected. Every such node, where only inductors and current sources (in this case the current sources are not present) are connected will reduce the order of differential equation describing the circuit by one. Such a node is shown in Fig. 12.1-12(a).

In Fig. 12.1-12(a), $i_1 + i_2 + i_3 + i_s = 0$ by KCL. Therefore, if two of them are known, the third can be determined. For instance, if i_1 and i_2 are known then $i_3 = -i_1 - i_2 - i_s$. Moreover, the voltage across the inductor carrying i_3 can be obtained as

$$\begin{aligned} v_3 &= L_3 \frac{di_3}{dt} = -L_3 \left[\frac{di_1}{dt} + \frac{di_2}{dt} + \frac{di_s}{dt} \right] \\ &= -\frac{L_3}{L_1} \times L_1 \frac{di_1}{dt} - \frac{L_3}{L_2} \times L_2 \frac{di_2}{dt} - L_3 \frac{di_s}{dt} \\ &= -v_1 - v_2 - L_3 \frac{di_s}{dt}. \end{aligned}$$

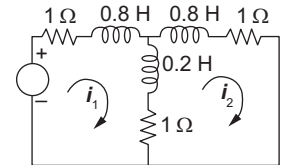
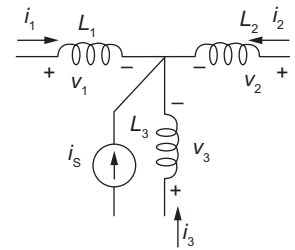
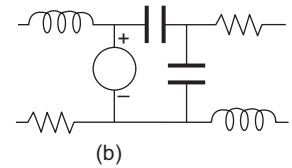


Fig. 12.1-11 Circuit for Example 12.1-6



(a)



(b)

Fig. 12.1-12 (a) Circuit with an All-Inductor-Current Source Node (b) An All-Capacitor-Voltage Source Loop

Order of a Circuit

The order of differential equation describing a linear time-invariant circuit is equal to the number of independent capacitors + the number of independent inductors – the number of all inductor nodes – the number of all capacitor loops.

'Independent inductors and capacitors' mean inductors and capacitors that remain after their number has been reduced to a minimum by series-parallel equivalents.

Differential equation describing an n^{th} order linear time-invariant circuit.

Initial value requirements for solving an n^{th} order circuit in time-domain.

Therefore, the element voltage and the element current of one inductor is completely decided by the element voltages and element currents of the remaining inductors connected at the node in the case of a node at which only inductors and independent current sources are incident. Therefore, the number of *dynamic variables* will be one less than the number of inductors incident at such a node.

A similar situation comes up in the case of a loop containing only capacitors and independent voltage sources. With a reasoning similar to the one employed in the preceding paragraph it can be shown that the number of dynamic variables is less than the total number of energy storage elements by one in this case. Such a loop is shown in Fig. 12.1-12(b).

12.2 GENERALISATIONS FOR AN n^{th} ORDER LINEAR TIME-INVARIANT CIRCUIT

We are almost through with the time-domain analysis of linear time-invariant circuits. This section summarises all the major concepts we discussed in the context of time-domain analysis in this chapter as well as in previous chapters.

Concept-1

A linear time-invariant circuit containing R, L, C, M , linear dependent sources and a single input source is described by a linear ordinary differential equation with constant coefficients. The differential equation for such a circuit can be expressed in a standard format as below:

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x \quad (12.2-1)$$

The variable y is any circuit voltage or current variable chosen as the describing variable for the circuit and x is the input source function. Standard mesh analysis or nodal analysis technique along with variable elimination will help us to arrive at Eqn. 12.2-1. However, the variable elimination involved can be considerably tricky in the case of large circuits containing many energy storage elements. This is a serious shortcoming of time-domain analysis.

Concept-2

The *order* of a circuit is equal to the *order* of the differential equation that describes it. The order of the circuit will be equal to the total number of independent inductors and capacitors – number of all-inductor nodes – and number of all-capacitor loops in the circuit.

The order of a circuit depends also on the kind and location of input. The same circuit will have a different order if a voltage source input is replaced by a current source input.

The coefficients $a_{n-1} \dots a_0$ and $b_m \dots b_0$ are decided by the circuit parameters. They are real-valued. $a_{n-1} \dots a_0$ are positive real numbers if the circuit is passive i.e., if it contains only R, L, C and M . They can be zero or negative real numbers if the circuit contains dependent sources. $b_m \dots b_0$ can be positive or negative or zero in all circuits.

Concept-3

The format of the left-side of the differential equation that describes a circuit is independent of the particular circuit variable chosen as the describing variable. That is, $a_{n-1} \dots a_0$ will remain the same even if some other circuit variable is used as the variable y . However, $b_m \dots b_0$ will depend on the variable chosen.

Concept-4

The n^{th} order differential equation in Eqn. 12.2-1 requires n initial values for solving it. The required initial values are $y(0^+), \left. \frac{dy}{dt} \right|_{(0^+)}, \left. \frac{d^2 y}{dt^2} \right|_{(0^+)}, \dots, \left. \frac{d^{n-1} y}{dt^{n-1}} \right|_{(0^+)}$. The differential equation can be solved using $x(t)$ for $t \geq 0^+$ and these initial values.

The initial values available in a circuit are the initial currents in all inductors and the initial voltage across all capacitors. It requires considerable effort to translate these values into the required initial values in the case of a circuit containing many energy storage elements. This is another serious shortcoming of time-domain analysis.

Concept-5

The solution for the describing equation in Eqn. 12.2-1 contains two parts – the complementary solution and the particular solution. The complementary solution is the solution of the homogeneous differential equation given in Eqn. 12.2-2.

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = 0 \quad (12.2-2)$$

The particular solution is the solution when the input x is active. The complementary solution of the differential equation in Eqn. 12.2-2 will have n terms. Each term will be of Ae^{st} form, where s is a complex number and A is an arbitrary complex constant. Complex exponential functions are the eigen functions of linear ordinary constant coefficient differential equations.

Concept-6

The values of s in n complementary solution terms are obtained by equating the characteristic polynomial $s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0$ to zero. Each root of $s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$ gives a complementary solution term of the form $Ae^{s_i t}$.

Thus, the complementary solution $= A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_n e^{s_n t} = \sum_{i=1}^n A_i e^{s_i t}$.

The complementary solution is called the *natural response* in the context of linear time-invariant circuits.

Concept-7

It is possible that the equation $s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$ has multiple roots. Let s_i be a root with multiplicity of r . Then, the natural response will contain $A_{i0}e^{s_i t} + A_{i1}te^{s_i t} + A_{i2}t^2e^{s_i t} + \dots + A_{i(r-1)}t^{r-1}e^{s_i t}$. However, it is possible to view terms of the kind $t^k e^{s_i t}$ as a limiting case of combination of complex exponential functions with closely spaced values in their indices. Thus, conceptually, we may state that the natural response terms of Eqn. 12.2-1 are complex exponential functions always.

Concept-8

Thus, the function format for the natural response terms for a linear time-invariant circuit is fixed as a complex exponential function. Therefore, we may represent these natural response terms by simply listing the indices of the complex exponential functions. Thus, the value of index of a complex exponential function can be used as a symbolic representation of that function. The index of a complex exponential function, with this additional role of a symbolic stand-in for the function, is called *natural frequency* of the dynamic circuit. Natural frequency has nothing to do with periodicity in time, in general. It is the index of a complex exponential time function. However, it is given in units of frequency. Its real part has a unit of *neper/sec* and its imaginary part has a unit of *radian/sec*.

These natural frequency values, being complex numbers in general, may be represented as points on a complex plane. Then, each point in that complex plane will represent a complex exponential function with the index the same as the complex number represented by that point. Therefore, we may call this complex plane the *complex signal space*. Each point in this space stands for a complex exponential signal. Real signals are only special cases of complex signals. Complex signal space is also called '*s-plane*', s standing for 'signal'. Fig. 12.2-1 shows the signal shape for various signal points on the s -plane.

The homogeneous differential equation of an n^{th} order linear time-invariant circuit.

Natural response terms of a linear time-invariant circuit are complex exponential functions.

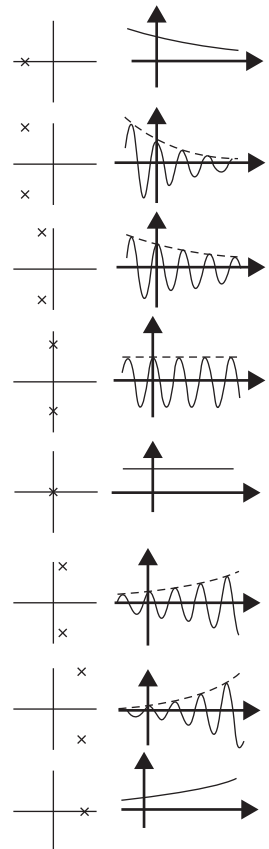


Fig. 12.2-1 Signal Point in S-Plane Versus Signal Shape

Concept-9

The roots of a polynomial with *real* coefficients will either be real or will occur as complex conjugate pairs.

Hence, a polynomial with *real* coefficients and odd order will have to have at least one real root.

Concept-10

The natural response of a circuit given by $= A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_n e^{s_n t}$ has to be added to the particular solution $y_{ps}(t)$ to form the total solution of the circuit. The total solution $y_{ts}(t)$ has to satisfy all the initial conditions on y and its first $(n - 1)$ derivatives at $t = 0^+$.

$$y_{ts}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_n e^{s_n t} + y_{ps}(t) \quad (12.2-3)$$

Substituting the initial conditions in Eqn. 12.2-3 yields a set of n algebraic equations on $A_1 \dots A_n$.

Concept-11

If the natural frequency s_i is real-valued, the corresponding A_i will also be real-valued. If s_i and s_j are a pair of complex conjugate natural frequencies, then A_i and A_j will also be a complex conjugate pair.

Concept-12

If all natural frequencies of a circuit are either negative real or complex conjugate numbers with negative real part, then, all natural response terms will be decaying real exponentials in time or damped sinusoidal waveforms in time. Therefore, all of them will go to zero as $t \rightarrow \infty$. Such a circuit is called a *stable* circuit.

Therefore, the condition for stability in a linear time-invariant circuit is that all its natural frequencies must lie on the left-half of the s -plane, excluding the $j\omega$ -axis.

If a circuit has one or more natural frequencies on $j\omega$ axis, it is called a *marginally stable* circuit. The natural response term corresponding to those natural frequencies on $j\omega$ axis in the s -plane will remain steady in amplitude with time.

If a circuit has one or more natural frequency values lying on the right-half of the s -plane, the corresponding natural response terms will grow with time. Such circuits are called *unstable* circuits.

Concept-13

If a circuit is stable, the only response that remains in the long run is the particular solution. *Forced response* is another name for particular solution. The natural response in a *stable* circuit, with the amplitudes of complex exponential functions decided by the initial conditions, is also called *transient response*. Thus, the transient response in a *stable* circuit dies down with time, leaving only the forced response component.

Now, if this forced response component in a *stable circuit* is produced by a DC input, then, it is called the *DC steady-state response*. If it is produced by a sinusoidal input, then, it is called the *sinusoidal steady-state response*. If it is produced by a non-sinusoidal periodic input, then, it is called the *periodic steady-state response*.

Steady-state response can be defined only for these three kinds of inputs. Therefore, while forced response is defined for any arbitrary input function, steady-state response is not. Forced response is present in stable and unstable circuits. But steady-state response is present only in *stable* circuits with DC/AC/periodic excitation.

Concept-14

A passive circuit that does not contain dependent sources is unconditionally stable. A *necessary, but not sufficient, condition* for all natural frequencies to be on the left-half of the s -plane (i.e., for stability) is that all a 's in Eqn. 12.2-1 are non-zero, positive values. A *sufficient, but not necessary, condition* for instability is that some a 's in Eqn. 12.2-1 are zero or negative. Thus, there may be circuits with all a 's positive and yet are unstable. Dependent sources in a circuit can make a circuit unstable; though not necessarily.

It takes all natural frequencies on the left-half of the s -plane to make a *stable* circuit. It takes only one natural frequency on the right-half of the s -plane to make it an *unstable* circuit.

Relation between forced response and steady-state response.

Concept-15

In a circuit with multiple input functions acting simultaneously, the superposition principle is applicable only for the forced response component. Transient response does not go by the superposition principle.

Concept-16

A more powerful way to partition the total response in a circuit is in terms of the zero-input response and the zero-state response. Total response is the sum of the two. Zero-input response is the response of the circuit if x is held zero for all $t \geq 0$. Zero-state response is the total response to x applied from $t = 0$ onwards, with the circuit initially relaxed (i.e., all initial conditions at zero) at $t = 0$.

Zero-state response in a linear time-invariant circuit obeys the superposition principle on inputs. Zero-input response obeys superposition principle on initial conditions.

Concept-17

The initial current through an inductor can be replaced with a suitably sized impulse voltage source in series. The initial voltage across a capacitor can be replaced with a suitably sized impulse current source across it. Thus, even the zero-input response problem can be converted into a zero-state response problem if necessary.

Concept-18

The impulse response of a linear time-invariant circuit is a short name for ‘zero-state response for unit impulse input’ and the step response of a linear time-invariant circuit is a short name for ‘zero-state response for unit step input’.

Concept-19

If the input to a linear time-invariant circuit is replaced with its derivative, the zero-state response is the derivative of the zero-state response before replacement. Similarly, if the input to a linear time-invariant circuit is replaced with its integral, the zero-state response is the integral of the zero-state response before replacement.

Concept-20

The DC steady-state response in a stable circuit can be obtained by solving a memoryless circuit, formed by replacing all capacitors with open-circuits and all inductors with short-circuits.

Concept-21

The sinusoidal steady-state response of a linear time-invariant circuit can be obtained by phasor analysis of the phasor equivalent circuit. Phasor analysis involves replacing every time-domain differentiation with multiplication by $j\omega$, where ω is the angular frequency of the applied sinusoidal input. Carrying out this operation in Eqn. 12.2-1, we get the *frequency response function* $H(j\omega)$ as:

$$H(j\omega) = \frac{\sum_{k=0}^m b_k (j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i (j\omega)^i}$$

If $x = A \cos \omega t$, then, $y = |H(j\omega)| A \cos(\omega t + \phi)$, where $\phi = \angle H(j\omega)$.

12.3 TIME-DOMAIN CONVOLUTION INTEGRAL

We show in this section that the impulse response is a complete characterisation of a linear time-invariant circuit. It is a complete characterisation in the sense that the zero-state response of a linear time-invariant circuit to the application of any arbitrary input source function can be determined from the impulse response and the input source function.

We show this in three steps. In the first step, we show that the zero-state response of a linear time-invariant circuit to a narrow rectangular pulse input of unit area content can be approximated by its impulse response. In the second step, we show that any arbitrary input source function can be resolved into a stream of weighted impulses. In the third step, we show how the zero-state response of a linear time-invariant circuit to an arbitrary input source function at any time instant t can be constructed by superposing weighted and time-shifted impulse responses corresponding to the impulse-stream resolution of the input source function arrived at in the second step.

12.3.1 Zero-State Response to Narrow Rectangular Pulse Input

Let $x(t)$ be the input source function and $y(t)$ be the response function in an n^{th} order linear time-invariant circuit. We identify two special response functions. They are unit step response and unit impulse response. Let $s(t)$ represent the unit step response and $h(t)$ represent the unit impulse response of the system. Note that these two are only special symbols for $y(t)$, when $x(t) = u(t)$ and $x(t) = \delta(t)$, respectively.

Now, we let $x(t)$ be a narrow rectangular pulse with a duration of $\Delta\tau$ s and a height of $1/\Delta\tau$ so that it contains unit area under the waveform. It is centred about $t = 0$. Thus, $x(t)$ is given by

$$x(t) = \begin{cases} 0 & \text{for } -\infty < t < -\frac{\Delta\tau}{2} \\ \frac{1}{\Delta\tau} & \text{for } -\frac{\Delta\tau}{2} < t < \frac{\Delta\tau}{2} \\ 0 & \text{for } \frac{\Delta\tau}{2} < t < \infty \end{cases}$$

This waveform can be expressed as a sum of two scaled step functions as below:

$$x(t) = \frac{1}{\Delta\tau} \left[u\left(t + \frac{\Delta\tau}{2}\right) - u\left(t - \frac{\Delta\tau}{2}\right) \right]$$

Therefore, the zero-state response of the circuit to $x(t)$ can be obtained as the sum of the zero-state response to $\frac{1}{\Delta\tau} u\left(t + \frac{\Delta\tau}{2}\right)$ and the zero-state response to $-\frac{1}{\Delta\tau} u\left(t - \frac{\Delta\tau}{2}\right)$.

The zero-state response to $u(t)$ is the step response $s(t)$. The circuit is a *time-invariant* one. Hence, when the input source function is advanced by $\Delta\tau/2$ s, the response also gets advanced by the same amount in time-axis. Similarly, when the input source function is delayed by $\Delta\tau/2$ s, the response also gets delayed by the same amount in time-axis.

Therefore, the zero-state response to $\frac{1}{\Delta\tau} u\left(t + \frac{\Delta\tau}{2}\right)$ is $\frac{1}{\Delta\tau} s\left(t + \frac{\Delta\tau}{2}\right)$ and the zero-state

response to $-\frac{1}{\Delta\tau} u\left(t - \frac{\Delta\tau}{2}\right)$ is $-\frac{1}{\Delta\tau} s\left(t - \frac{\Delta\tau}{2}\right)$. We have employed the fact that the circuit is a linear one to scale the response by $\frac{1}{\Delta\tau}$. Therefore, the zero-state response to narrow rectangular pulse input of unit area can be expressed as $y(t) = \frac{1}{\Delta\tau} \left[s\left(t + \frac{\Delta\tau}{2}\right) - s\left(t - \frac{\Delta\tau}{2}\right) \right]$.

Now, we let $\Delta\tau$ approach zero. That is, we reduce the width of the pulse and increase its height such that the area under the waveform remains unity. Then, the response can be expressed

as $y(t) = \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \left[s\left(t + \frac{\Delta\tau}{2}\right) - s\left(t - \frac{\Delta\tau}{2}\right) \right] = \frac{ds(t)}{dt}$. But this limit is nothing but the first

derivative of $s(t)$ evaluated at t . Therefore, $y(t) = \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \left[s\left(t + \frac{\Delta\tau}{2}\right) - s\left(t - \frac{\Delta\tau}{2}\right) \right] = \frac{ds(t)}{dt}$.

However, the derivative of step response of a linear time-invariant circuit is nothing but its impulse response, since $\delta(t)$ is the first derivative of $u(t)$ (in the sense that integrating $\delta(t)$ yields $u(t)$). Therefore, the response for a narrow rectangular pulse of unit area approaches the impulse response in a linear time-invariant circuit when the width of the pulse is made small arbitrarily, keeping the area at unity. Hence, the response for a sufficiently narrow pulse of area A can be approximated by A times the unit impulse response. The approximation will get better when the width of the pulse reaches very low values compared to the time constants in the circuit.

12.3.2 Expansion of an Arbitrary Input Function in Terms of Impulse Functions

Now, let $x(\tau)$ be an arbitrary input source function which is assumed to be known in the entire time-axis. We denote the time-variable by τ this time for a reason that will emerge soon.

We divide the entire τ -axis into small strips (i.e., intervals) of width $\Delta\tau$ as shown in Fig. 12.3-1.

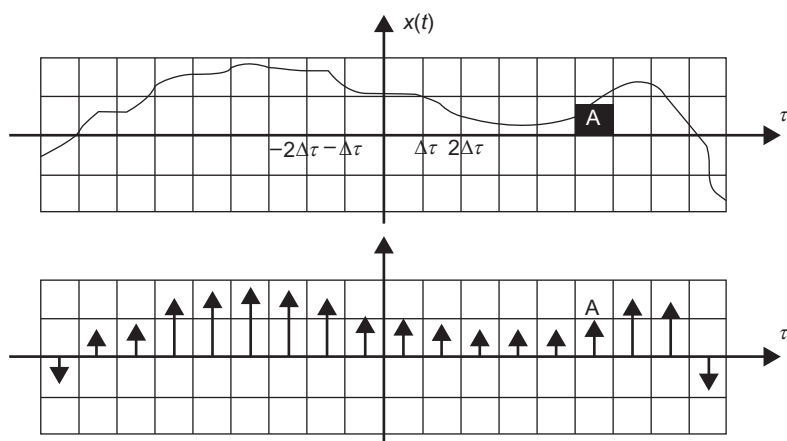


Fig. 12.3-1 An Arbitrary Input Function Approximated by an Impulse Stream

$x(\tau)$ varies within each strip. However, any curve can be treated as a straight line if the interval of observation is sufficiently small. Therefore, we may approximate the waveform by straight line segments within small intervals of width $\Delta\tau$. Then, the area under the waveform for each waveform-strip is given by the value of $x(\tau)$ at the middle of the interval multiplied by the width of the strip. This area content can be expressed as $x(n\Delta\tau - \Delta\tau/2) \times \Delta\tau$. Now, we replace this strip with a rectangular strip of the same area. Obviously, all these approximations will tend to become exact as $\Delta\tau \rightarrow 0$. We have shown in the previous sub-section that applying a rectangular pulse of area A to a linear time-invariant circuit results in a response that can be approximated by A times the impulse response of the circuit, provided the width of the rectangular pulse is very small. Therefore, each waveform strip under intervals of $\Delta\tau$ can be replaced by impulses of magnitude equal to the area under the waveform strip as far as the effect of $x(\tau)$ on a linear time-invariant circuit is concerned. The impulses are to be located at the centre of the corresponding intervals. This is shown in Fig. 12.3-1. Let us denote the resulting impulse stream by $x_i(\tau)$. Then, $x_i(\tau)$ can be expressed as

An arbitrary input function $x(t)$ expressed as a sum of scaled and shifted impulses called 'impulse resolution of $x(t)$ '.

$$x_i(\tau) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} x \left[\left(n - \frac{1}{2} \right) \Delta\tau \right] \delta \left[\tau - \left(n - \frac{1}{2} \right) \Delta\tau \right] \Delta\tau \quad (12.3-1)$$

The ' n ' in Eqn. 12.3-1 is an integer. This equation expresses an arbitrary function $x(\tau)$ as a sum of many infinitely scaled and time-shifted impulse functions. The response of a linear time-invariant circuit to $x(\tau)$ is approximately the same as the response of the same circuit to $x_i(\tau)$ as given by Eqn. 12.3-1. The approximation becomes exact as $\Delta\tau \rightarrow 0$.

12.3.3 The Convolution Integral

Now, we address the issue of determining the response of a linear time-invariant circuit from its impulse response and the input source function.

We consider the most general case of a two-sided input source function and a two-sided impulse response. The impulse response of a physical linear time-invariant circuit will invariably be one-sided, i.e., $h(t) = 0$ for $t < 0$. This is so since all physical systems are *causal systems*. A two-sided impulse response implies that the system could somehow predict that an impulse is going to hit at $t = 0$ and it started responding in anticipation from the infinite past onwards. This clearly goes against the law of causality. However, since we impose only two requirements on the system now – that it be linear and time-invariant – and do not impose the requirement of *causality* at this point in our discussion, we use a two-sided $h(t)$ in this discussion. Moreover, we assume that we have full knowledge of the input applied to the circuit from $t = -\infty$ to $+\infty$.

Let us imagine that we want to find out y at a time-instant t located on the τ -axis.

The input $x(\tau)$ can be treated as a stream of scaled and shifted impulses, as explained in the previous sub-section. Each impulse in the impulse stream starts a new scaled-impulse response in the circuit. Since the circuit is time-invariant, the response to a shifted impulse is the same as the impulse response itself shifted by the same time-shift. That is, the response to $\delta \left[\tau - \left(n - \frac{1}{2} \right) \Delta\tau \right]$ is equal to $h \left[\tau - \left(n - \frac{1}{2} \right) \Delta\tau \right]$. The impulse $\delta \left[\tau - \left(n - \frac{1}{2} \right) \Delta\tau \right]$ is located at the middle of the n th interval in τ -axis and it is scaled by the value of x at the middle of that interval $\Delta\tau$. The impulse response due to this single impulse will contribute to the value of output at t . This contribution can be obtained as the instantaneous value of the scaled and shifted impulse response given by $x \left[\left(n - \frac{1}{2} \right) \Delta\tau \right] h \left[\tau - \left(n - \frac{1}{2} \right) \Delta\tau \right] \Delta\tau$ evaluated

at $\tau = t$. Therefore, contribution from the impulse located at $\tau = \left(n - \frac{1}{2} \right) \Delta\tau$ to the output y at

$$t = x \left[\left(n - \frac{1}{2} \right) \Delta\tau \right] h \left[\tau - \left(n - \frac{1}{2} \right) \Delta\tau \right] \Delta\tau \Big|_{\tau=t} = x \left[\left(n - \frac{1}{2} \right) \Delta\tau \right] h \left[t - \left(n - \frac{1}{2} \right) \Delta\tau \right] \Delta\tau.$$

If the system is causal, its impulse response will be right-sided, and hence, only those impulses that appear *before or at* t in τ -axis will contribute to the output at t . But, we are considering a two-sided impulse response here. Therefore, even those impulses which are yet to appear will contribute to the output at t . That is, all impulses from $-\infty$ to $+\infty$ in τ -axis will contribute to y at t . The circuit is stated to be linear and it obeys the superposition principle. Thus, the *instantaneous* value of y at t can be constructed by adding all the contributions to the output from all the impulses in Eqn. 12.3-1.

$$\therefore y(t) \approx \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} x \left[\left(n - \frac{1}{2} \right) \Delta\tau \right] h \left[t - \left(n - \frac{1}{2} \right) \Delta\tau \right] \Delta\tau \quad (12.3-2)$$

The result in Eqn. 12.3-2 is only approximate since Eqn. 12.3-1 is only an approximation. We take the limit of Eqn. 12.3-2 as $\Delta\tau \rightarrow 0$ to make the result exact.

Convolution Integral

Convolution integral expresses the zero-state response of a linear time-invariant circuit in terms of its impulse response and the input source function.

Only the properties of 'linearity' and 'time-invariance' are employed in the derivation of this integral. Hence, *convolution integral* is only a restatement of properties of 'linearity' and 'time-invariance'.

$$\therefore y(t) = \lim_{\Delta\tau \rightarrow 0} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} x\left[\left(n - \frac{1}{2}\right)\Delta\tau\right] h\left[t - \left(n - \frac{1}{2}\right)\Delta\tau\right] \Delta\tau.$$

The variable $\left(n - \frac{1}{2}\right)\Delta\tau$ is a discrete variable. It represents a sequence of equidistant time-points in the τ -axis. The spacing between these equidistant time-points goes to zero as $\Delta\tau \rightarrow 0$, and in the limit, they cover every point in τ -axis. Equivalently, the variable $\left(n - \frac{1}{2}\right)\Delta\tau$ gets replaced by the variable τ . Moreover, the summation becomes an integration as $\Delta\tau \rightarrow 0$. Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (12.3-3)$$

The integral in Eqn. 12.3-3 is called the *convolution integral* and the process of evaluating this integral is called ‘*convolving $x(t)$ with $h(t)$* ’. The convolution operation is also indicated by $x(t)*h(t)$. Therefore,

$$y(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Any two time-functions can be convolved. When the first one is an input to a linear time-invariant circuit and the second one is the impulse response of that circuit, the convolution between them yields the response of the circuit.

Note that the integral is a definite integral, yielding a single value that denotes the instantaneous value of y at a particular time instant t . Therefore, the determination of one value in the output waveform requires the evaluation of an integral over the entire time-axis.

But which response does the Eqn. 12.3-3 give us? Total response, forced response or zero-state response?

We had assumed that the input source function $x(t)$ is known from $-\infty$ to ∞ in the time-axis. Hence, the output given by Eqn. 12.3-3 is the forced response which also happens to be the total response, since there is no transient response when the input starts from $-\infty$.

However, in practice, the input function is known only from $\tau = 0$. Then, we assume that $x(\tau) = 0$ for $\tau < 0$. The response obtained with that assumption is what we called the zero-state response. Therefore, the response given by the convolution integral is the zero-state response, if the input source function is one-sided. Convolution integral cannot yield the zero-input response part. That has to be found separately. Also, the limit of integration can now be changed to 0 to ∞ . Therefore,

$$y_{\text{zsr}}(t) = x(t) * y(t) = \int_0^{\infty} x(\tau)h(t-\tau)d\tau \text{ if } x(t) = 0 \text{ for } t < 0 \quad (12.3-4)$$

If the impulse response is causal, $h(t)$ will be right-sided. Therefore, the value of the integrand in Eqn. 12.3-3 will be zero for all $\tau > t$ and the upper limit on the integral can be set at t . Therefore,

$$y(t) = x(t) * y(t) = \int_{-\infty}^t x(\tau)h(t-\tau)d\tau \text{ if } h(t) = 0 \text{ for } t < 0 \quad (12.3-5)$$

If both $x(t)$ and $h(t)$ are right-sided, convolution integral returns the zero-state response of a causal linear time-invariant circuit and the integration limits can be set at 0 and t .

$$y_{\text{zsr}}(t) = x(t) * y(t) = \int_0^t x(\tau)h(t-\tau)d\tau \text{ if } x(t) \text{ and } h(t) = 0 \text{ for } t < 0 \quad (12.3-6)$$

Convolution integral for the total response of a circuit with a two-sided impulse response $h(t)$ excited by a two-sided input function $x(t)$.

EXAMPLE: 12.3-1

The impulse response of a linear time-invariant circuit is $h(t) = 2e^{-t}u(t)$ and the input to the circuit is $x(t) = e^{-2t}u(t)$. The circuit was initially relaxed. Find its output by the convolution $x(t)*h(t)$. Also, find the convolution $h(t)*x(t)$ and verify that it is the same as $x(t)*h(t)$.

SOLUTION

Since the circuit was initially relaxed, the total response is given by the zero-state response itself. The zero-state response to any input is obtained by convolving the input source function with the impulse response. Therefore, the output $y(t)$ is obtained as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t e^{-2\tau} \cdot 2e^{-(t-\tau)}d\tau.$$

The limits of integration were changed from $-\infty$ to 0 and ∞ to t since both the input source function and the impulse response are right-sided functions of time.

$$y(t) = \int_0^t e^{-2\tau} \cdot 2e^{-(t-\tau)}d\tau = 2 \int_0^t e^{-2\tau} e^{\tau} e^{-t} d\tau = 2 \int_0^t e^{-\tau} e^{-t} d\tau$$

The integration variable is τ , and not t . Therefore, the factor e^{-t} can be pulled out of the integration sign as below:

$$y(t) = 2e^{-t} \int_0^t e^{-\tau} d\tau = 2e^{-t} [-e^{-\tau}]_0^t = 2e^{-t} [1 - e^{-t}] = 2e^{-t} - 2e^{-2t}.$$

Any two time-functions can be convolved with each other. Hence, we can convolve $h(t)$ with $x(t)$. Then,

$$\begin{aligned} h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_0^t 2e^{-\tau} \cdot e^{-2(t-\tau)}d\tau = 2e^{-2t} \int_0^t e^{\tau} d\tau = 2e^{-2t} [e^t - 1] = 2e^{-t} - 2e^{-2t}. \end{aligned}$$

We see that $x(t)*h(t) = h(t)*x(t)$ in this example. But, $h(t)*x(t)$ can be interpreted as the output of a linear time-invariant circuit with $x(t)$ as its impulse response and $h(t)$ as its input. Thus, the roles of impulse response and the input source function seem to be interchangeable as far as this example is concerned. In fact, it is generally true as shown below. We use a substitution of variable $\tau' = t - \tau$. Then, $\tau = t - \tau'$, $d\tau = -d\tau'$ and

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = - \int_{\infty}^{-\infty} h(t-\tau')x(\tau')d\tau' = \int_{-\infty}^{\infty} x(\tau')h(t-\tau')d\tau' = x(t) * h(t).$$

Therefore, the roles of the impulse response and the input source function are interchangeable in a linear time-invariant circuit.

If $x_1(t)$ and $x_2(t)$ are two arbitrary functions of time, then $x_1(t)*x_2(t) = x_2(t)*x_1(t)$.

EXAMPLE: 12.3-2

The impulse response of a second order linear time-invariant circuit is seen to be $h(t) = 2e^{-t} \cos 2t u(t)$. Find its steady-state output when it is driven by $x(t) = \cos 4t u(t)$ by convolution.

SOLUTION

We find the zero-state response to $x(t) = \cos 4t$ first.

$$h(t) = 2e^{-t} \cos 2t u(t) = 2e^{-t} \frac{e^{j2t} + e^{-j2t}}{2} u(t) = [e^{(-1+j2)t} + e^{(-1-j2)t}] u(t)$$

$$x(t) = 2 \cos t u(t) = [e^{jt} + e^{-jt}] u(t)$$

We have used Euler's formula in these steps.

$$\begin{aligned}
y(t) &= x(t) * h(t) = \int_0^t (e^{j\tau} + e^{-j\tau}) \left[e^{(-1+j2)(t-\tau)} + e^{(-1-j2)(t-\tau)} \right] d\tau \\
&= \int_0^t (e^{j\tau} + e^{-j\tau}) \left[e^{(-1+j2)(t-\tau)} \right] d\tau + \int_0^t (e^{j\tau} + e^{-j\tau}) \left[e^{(-1-j2)(t-\tau)} \right] d\tau \\
&= e^{(-1+j2)t} \int_0^t (e^{j\tau} + e^{-j\tau}) \left[e^{-(-1+j2)\tau} \right] d\tau + e^{(-1-j2)t} \int_0^t (e^{j\tau} + e^{-j\tau}) \left[e^{-(-1-j2)\tau} \right] d\tau \\
&= e^{(-1+j2)t} \int_0^t \left[e^{(1-j)\tau} + e^{(1-j3)\tau} \right] d\tau + e^{(-1-j2)t} \int_0^t \left[e^{(1+j3)\tau} + e^{(1+j)\tau} \right] d\tau \\
&= e^{(-1+j2)t} \left[\frac{e^{(1-j)\tau} - 1}{1-j} + \frac{e^{(1-j3)\tau} - 1}{1-j3} \right] + e^{(-1-j2)t} \left[\frac{e^{(1+j3)\tau} - 1}{1+j} + \frac{e^{(1+j)\tau} - 1}{1+j3} \right] \\
&= \left[\frac{e^{(j)t} - e^{(-1+j2)t}}{1-j} + \frac{e^{(-j)t} - e^{(-1+j2)t}}{1-j3} \right] + \left[\frac{e^{(-j)t} - e^{(-1-j2)t}}{1+j} + \frac{e^{(j)t} - e^{(-1-j2)t}}{1+j3} \right] \\
&= \left[\frac{e^{(j)t}}{1-j} + \frac{e^{(-j)t}}{1+j} + \frac{-e^{(-1+j2)t}}{1-j} + \frac{-e^{(-1-j2)t}}{1+j} + \frac{e^{(-j)t}}{1-j3} + \frac{e^{(j)t}}{1+j3} + \frac{-e^{(-1+j2)t}}{1-j3} + \frac{-e^{(-1-j2)t}}{1+j3} \right] \\
&= \cos t - \sin t - e^{-t} \cos 2t + e^{-t} \sin 2t + \frac{1}{5} \cos t + \frac{3}{5} \sin t - \frac{1}{5} e^{-t} \cos t - \frac{3}{5} e^{-t} \sin t.
\end{aligned}$$

The sinusoidal steady-state component in this output is $= \cos t - \sin t + 0.2 \cos t + 0.6 \sin t = 1.2 \cos t - 0.4 \sin t = 1.265 \cos(t + 18.44^\circ)$.

The steady-state response to an input function (provided the notion of steady-state is applicable with that particular input) can be found in two ways. In the first method, we find the zero-state response when the particular input is applied to the circuit and accept the part of zero-state response that remains after the natural response terms damp down to zero as the steady-state response. This is the approach we followed in this example.

This does not imply that the steady-state response can be found as the limit of zero-state response as $t \rightarrow \infty$. That is a wrong interpretation of 'steady-state'. Steady-state response is the response that remains after the transient terms die down. The response that remains after the transients die down need not be a *constant* value. Therefore, a *limit* operation is not valid. It may appear to be valid in the case of DC steady-state. However, viewing the steady-state response as the limit of response as $t \rightarrow \infty$ is *not a correct interpretation*.

The second approach is to assume that the input was applied to the circuit from $t = -\infty$ onwards. In that case, we would be determining the forced response of the circuit and the forced response is the steady-state response whenever the notion of steady-state is applicable. Therefore, the convolution integral $y(t) = \int_{-\infty}^t x(\tau)h(t-\tau)d\tau$ will yield the steady-state response straightaway if $x(t)$ is assumed to have been applied from $t = -\infty$ onwards. Of course, the integral specialises to $y(t) = \int_{-\infty}^t x(\tau)h(t-\tau)d\tau$ for a causal circuit. The reader may verify the steady-state result by this method in the case of this example.

Note that the evaluation of convolution integral in this example has been rendered easy by the liberal use of Euler's formula. Integrating exponential functions is an easy job. We do not need the table of integrals or integration by parts for that.

The reader is cautioned against a common error in interpreting 'steady-state'!

EXAMPLE: 12.3-3

Show that the steady-state step response value is equal to the area under the impulse response for a stable linear time-invariant circuit.

SOLUTION

We use the second approach mentioned under the previous example to determine the steady-state. We assume that unit input was applied to the circuit from $t = -\infty$ onwards. Then,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad (\because \text{Role of } x(t) \text{ and } h(t) \text{ can be interchanged}) \\
&= \int_{-\infty}^{\infty} h(\tau)d\tau \quad (\because \text{input is unit constant}) \\
&= \text{Area under impulse response.}
\end{aligned}$$

Therefore, the forced response to unit DC input is equal to the area under impulse response. But, forced response to unit DC input is the same as the steady-state response to unit step input. Therefore, steady-state step response value is equal to the area under impulse response.

12.3.4 Graphical Interpretation of Convolution in Time-Domain

We develop a graphical interpretation for convolution integral $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ in this sub-section, and thereby, appreciate why the impulse response function is often called the *Scanning Function*.

Two functions are involved in a product in the integrand. The first one is simple – it is x laid out in τ -axis. The second one, $h(t-\tau)$, calls for some interpretation before we can visualise it.

$h(\tau)$ is the circuit impulse response laid out in τ -axis. We define a new function $h_i(\tau)$ as $h_i(\tau) = h(-\tau)$. For every τ , the value of h_i is the value that h has at $-\tau$. This means that the graph of $h_i(\tau)$ will be the mirror image of the graph of $h(\tau)$ with the mirror image taken about the vertical axis. For instance, if $h(\tau)$ is a right-sided function, then, $h_i(\tau)$ will be a left-sided function.

Now, we construct a delayed version of $h_i(\tau)$ and call it $h_{id}(\tau)$. The delay involved is taken as t s. Thus $h_{id}(\tau) = h_i(\tau - t)$. Since $h_i(\tau)$ is the mirror image of $h(\tau)$, $h_i(\tau - t)$ must be equal to $h[-(\tau - t)] = h(t - \tau)$. Therefore, $h(t - \tau)$ is a waveform obtained by mirror-reflecting $h(\tau)$ about the vertical axis and moving it forward in τ -axis by t s if t is positive or moving it backward by $|t|$ s if t is negative. t is the value of the time-instant at which the output y is to be calculated. The two waveforms $x(\tau)$ and $h(t - \tau)$ are multiplied to form the product waveform and the area under the product waveform is obtained in order to determine *one value* of y at a particular time-instant t . The process is repeated for different values of t to construct y as a function of time.

Figure 12.3-2 shows convolution graphically for $x(t) = e^{-0.5t}[u(t) - u(t - 5)]$ and $h(t) = e^{-t}u(t)$ for four values of t . The shaded curves show the integrand and the shaded area gives the output values.

The impulse response function moves in from the left side of the time-axis and sweeps through the input source function as t increases. The overlap region between the mirror-reflecting and the shifted impulse response and the input source function increases with t . Impulse response function does a kind of *selective scanning-in* on the input function as t increases. It places more emphasis on the recent values of input; nevertheless it *scans in* all the past values of input too – but scales them down depending on the time interval between the current instant and the past instant.

The impulse response of a stable linear time-invariant circuit can contain only decreasing exponential functions, and hence, it will be a function that tapers down to zero as time increases. Therefore, though the impulse response scans in the past input values from the infinite past theoretically, the significance of the scanned-in past values will be negligible after a certain point in the past. This is due to the tapering nature of the impulse response. For instance, an exponential impulse response function goes down to 1% of its initial value in about 5 time constants. Thus we may assume without much error that the input values that occurred before 5 time constants of impulse response will not influence the output significantly. This may equivalently be stated as ‘the depth of memory of the circuit is 5 time constants of the impulse response’. If the impulse response contains many

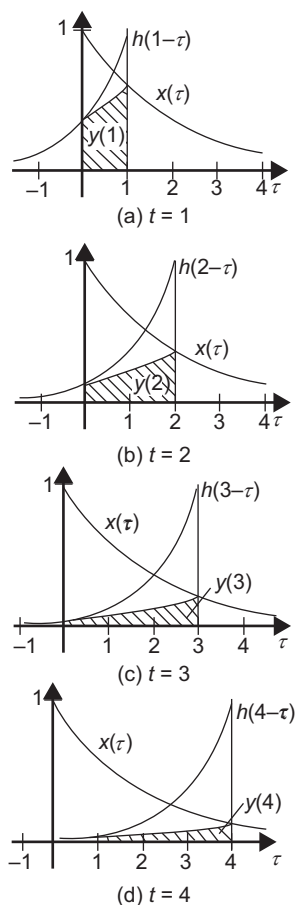


Fig. 12.3-2 Graphical Interpretation of Convolution

exponential functions, the largest time constant in the impulse response is to be used to decide the depth of circuit memory.

The impulse response of a stable linear time-invariant circuit is a tapering function of time. Therefore, after a sufficiently long interval of time after the application of input, we may assume that almost the entire significant content of the impulse response function has moved into the input function from the left-side of τ -axis. Then, if the input is a step function, the output will remain constant with further increase in t , since the region of overlap and its area remain unchanging once the impulse response has completely *moved into* the input source function. This explains the DC steady-state.

If the input is a periodic function, the output also varies periodically with the same period once the impulse response has moved into the input function completely. This gives rise to a periodic steady-state. Sinusoidal steady-state is a special case of periodic steady-state.

Scanning Function

Convolution integral indicates that the process of generation of response in a circuit may be viewed as a process in which the circuit scans the input function using its impulse response template as a scanner.

This is the reason why the impulse response function is called the *Scanning Function*.

12.3.5 Frequency Response Function from Convolution Integral

Let $x(t) = 1 \cos \omega t$ be applied to a stable and causal linear time-invariant circuit from the infinite past. Then, the response will contain only the forced response component. The forced response component is the same as the steady-state response in the case of a stable circuit. We find the steady-state response by employing the convolution integral as below:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} 1 \cos \omega \tau \times h(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) \cos \omega(t - \tau) [\because x(t) * h(t) = h(t) * x(t)] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} h(\tau) [e^{j\omega(t-\tau)} + e^{-j\omega(t-\tau)}] d\tau \quad (\text{By Euler's formula}) \end{aligned}$$

$$y(t) = \frac{e^{j\omega t}}{2} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau + \frac{e^{-j\omega t}}{2} \left[\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right]^*$$

Let $A \angle \theta = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$; then,

$$y(t) = \frac{A \angle \theta e^{j\omega t}}{2} + \frac{A \angle -\theta e^{-j\omega t}}{2} = \frac{A e^{j(\omega t + \theta)}}{2} + \frac{A e^{-j(\omega t + \theta)}}{2} = A \cos(\omega t + \theta).$$

$\therefore$ Magnitude of frequency response function = magnitude of $\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

Phase of frequency response function = phase of $\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

$\therefore$ Frequency response function $H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

We can write $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$ by a simple change of variable.

Therefore, we see that the sinusoidal steady-state frequency response function is completely decided by the impulse response of the circuit. The frequency response function is a disguised version of the impulse response function. The convolution integral has assured us that the zero-state response to any arbitrary input can be obtained from the impulse response. Then, if the impulse response is contained in the frequency response function, it must be possible to determine the zero-state response to an arbitrary input from the frequency response function too. In other words, if the impulse response function is a complete characterisation of a linear time-invariant circuit, then, the frequency response function must also be an equally complete characterisation of the same circuit. The next major part of this text shows that this is true.

Frequency Response and Impulse Response

Sinusoidal steady-state frequency response function $H(j\omega)$ of a linear time-invariant circuit is related to its impulse response $h(t)$ by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

EXAMPLE: 12.3-4

The impulse response of a first order circuit is $h(t) = 2e^{-2t}u(t)$. Find its frequency response function.

SOLUTION

The frequency response function is given by

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\ &= \int_0^{\infty} 2e^{-2t} e^{-j\omega t} dt = 2 \int_0^{\infty} e^{-(2+j\omega)t} dt = \frac{2}{2+j\omega} = \frac{2}{\sqrt{2+\omega^2}} \angle -\tan^{-1}(0.5\omega). \end{aligned}$$

EXAMPLE: 12.3-5

The impulse response of a second order circuit is $h(t) = e^{-2t} \cos 3t u(t)$. Find its frequency response function.

SOLUTION

The frequency response function, $H(j\omega)$, is given by

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-2t} \cos 3t e^{-j\omega t} dt = 0.5 \int_0^{\infty} e^{-2t} [e^{j3t} + e^{-j3t}] e^{-j\omega t} dt \\ &= 0.5 \int_0^{\infty} [e^{(-2-j(\omega-3))t} + e^{(-2-j(\omega+3))t}] dt \\ &= \frac{0.5}{2+j(\omega-3)} + \frac{0.5}{2+j(\omega+3)} = \frac{2+j\omega}{(\omega^2-5)+4j\omega}. \end{aligned}$$

12.3.6 A Circuit with Multiple Sources – Applying Convolution Integral

The convolution integral gives the response of a circuit to a single input in terms of that input source function and the impulse response function between the point of application of that input and the point at which the output is observed. It gives the forced response if input is known to have been applied from $t = -\infty$. It gives the zero-state response for a right-sided input. The forced response and the zero-state response in a linear time-invariant circuit obey the superposition principle for multiple inputs. Therefore, the response of a linear time-invariant circuit to simultaneous application of two inputs from two different locations in the circuit can be obtained by finding the components of response when each is acting alone and then adding the components. Convolution integrals can be employed to determine the two response components, provided we know the response of the circuit to unit impulses applied at the two input locations.

EXAMPLE: 12.3-6

Find the zero-state response of $v_o(t)$ in the circuit in Fig. 12.3-3 if $v_s(t)$ is a rectangular pulse of 10 V lasting for 1 s, starting from $t = 0$, and $i_s(t)$ is a rectangular pulse of 1.5 A lasting for 1 s, starting from $t = 0$.

SOLUTION

The two circuits for applying the superposition principle are shown in Fig. 12.3-4. We need to determine the impulse responses first.

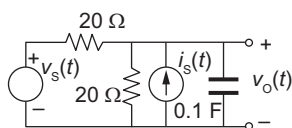


Fig. 12.3-3 Circuit for Example 12.3-6

Impulse response in the first circuit – $h_v(t)$:

The applied unit impulse voltage gets dropped entirely across the first $20\ \Omega$ resistor, since the capacitor can absorb no part of it. Therefore, $0.05\ \delta(t)$ current of magnitude $0.05\ \text{C}$ gets dumped into the capacitor at $t = 0$. This results in the capacitor voltage changing from 0 to $0.5\ \text{V}$ at $t = 0$. Therefore, $v_o(0^+) = 0.5\ \text{V}$, and after that, it is a free response of an RC circuit with time constant $= (20\ \Omega // 20\ \Omega) \times 0.1\ \text{F} = 1\ \text{s}$. Therefore, $h_v(t) = 0.5 e^{-t}\ \text{V}$.

Impulse response in the second circuit – $h_i(t)$:

The applied unit impulse current carrying $1\ \text{C}$ flows entirely through the capacitor, changing its voltage from 0 to $10\ \text{V}$ instantaneously at $t = 0$. Therefore, $v_o(0^+) = 10\ \text{V}$, and after that, it is a free response of an RC circuit with time constant $= (20\ \Omega // 20\ \Omega) \times 0.1\ \text{F} = 1\ \text{s}$. Therefore, $h_i(t) = 10 e^{-t}\ \text{V}$.

Now, the response components can be obtained by the convolution integrals.

Response component due to $v_s(t) = v_s(t) * h_v(t)$

$$\begin{aligned} & \int_0^t 10 \times 0.5 e^{-(t-\tau)} d\tau = 5e^{-t} \int_0^t e^{\tau} d\tau = 5e^{-t} [e^{\tau} - 1] = 5(1 - e^{-t})\ \text{V for } 0^+ < t < 1 \\ & = \int_0^1 10 \times 0.5 e^{-(t-\tau)} d\tau = 5e^{-t} \int_0^1 e^{\tau} d\tau = 5e^{-t} [e^1 - 1] = 8.591e^{-t}\ \text{V for } 1 < t < \infty. \end{aligned}$$

Response component due to $i_s(t) = i_s(t) * h_i(t)$

$$\begin{aligned} & \int_0^t 1.5 \times 10 e^{-(t-\tau)} d\tau = 15e^{-t} \int_0^t e^{\tau} d\tau = 15e^{-t} [e^{\tau} - 1] = 15(1 - e^{-t})\ \text{V for } 0^+ < t < 1 \\ & = \int_0^1 1.5 \times 10 e^{-(t-\tau)} d\tau = 15e^{-t} \int_0^1 e^{\tau} d\tau = 15e^{-t} [e^1 - 1] = 25.774e^{-t}\ \text{V for } 1 < t < \infty. \end{aligned}$$

The zero-state response when both sources are acting together = sum of the response components.

$$\therefore v_o(t) = \begin{cases} 20(1 - e^{-t})\ \text{V for } 0 \leq t < 1 \\ 34.365e^{-t}\ \text{V for } 1 < t \end{cases}$$

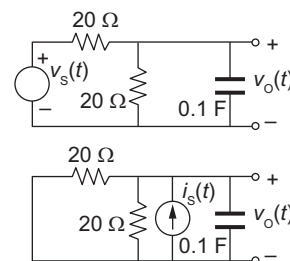


Fig. 12.3-4
Component Circuits
for Applying the
Superposition Principle
in Example 12.3-6

12.3.7 Zero-Input Response by Convolution Integral

We had observed that the convolution integral gives the zero-state response for a right-sided input. That raises the question – can we not bring the zero-input response also under the convolution integral?

The zero-input response arises from the non-zero initial energy storage in the inductors and capacitors. We remember that the inductors and capacitors with non-zero initial condition can be replaced with elements with zero initial condition along with suitably connected impulse sources to account for the initial conditions. Thus, we can reduce any circuit problem with non-zero initial energy storage into a zero-state response after replacing the initial voltage across the capacitors and the initial current through the inductors with impulse sources. The resulting circuit will have more than one source. We have seen how the response to multiple inputs can be obtained by convolution integral in the previous sub-section.

However, the zero-state response to initial condition sources does not even require convolution. These are impulse sources. Therefore, the convolution is to be done between the impulse functions and the impulse response functions. The convolution of any function with unit impulse function results in that function itself.

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau = \int_{t^-}^{t^+} x(\tau) \delta(t - \tau) d\tau = x(t).$$

Therefore, the response to initial condition sources will be scaled impulse response functions relevant to the locations at which the initial condition impulse sources are connected.

EXAMPLE: 12.3-7

The initial voltage across the capacitor in the circuit in Fig. 12.3-3 is -10 V. Find the total response of the circuit.

SOLUTION

The circuit, after replacing the -10 V initial condition on the capacitor with an impulse current source of magnitude $10 \text{ V} \times 0.1 \text{ F} = 1 \text{ C}$, is shown in the circuit in Fig. 12.3-5(a).

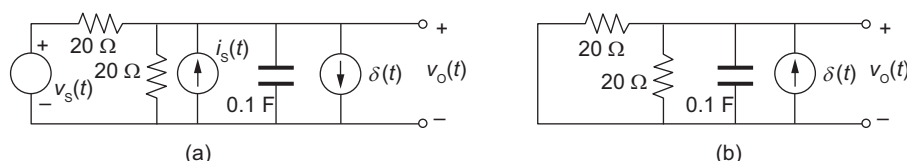


Fig. 12.3-5 (a) Circuit After Replacement of Initial Voltage by Impulse Current Source (b) Circuit for Impulse Response for Current Source Excitation Across Capacitor

This is a three-source problem. We have already solved for the zero-state response due to $v_s(t)$ and $i_s(t)$ in Example 12.3-6. We need to find the impulse response for current source excitation across the capacitor to determine the third component of output. The circuit required for this is shown in Fig. 12.3-5(b).

The unit impulse current applied across the capacitor as shown in Fig. 12.3-5(b) flows entirely through the capacitor, changing its voltage from 0 to $1 \text{ C}/0.1 \text{ F} = 10 \text{ V}$ instantaneously at $t = 0$. Thereafter, the impulse current source behaves as an open-circuit and the circuit executes its free response. Therefore, the required impulse response $h_{ic}(t) = 10 e^{-t} \text{ V}$.

Now, we need to convolve this impulse response with the actual initial condition source function. But the actual initial condition source function is $-\delta(t)$. There is no need to carry out the convolution. The result is known to be $-1 \times h_{ic}(t) = -10 e^{-t}$.

The total zero-state response due to the three sources is given by

$$v_o(t) = \begin{cases} 20(1 - e^{-t}) - 10e^{-t} = 20 - 30e^{-t} \text{ V for } 0^+ \leq t < 1 \\ 34.365e^{-t} - 10e^{-t} = 24.365e^{-t} \text{ V for } 1 < t \end{cases}$$

We have accepted the zero-state response for $v_s(t)$ and $i_s(t)$ from Example 12.3-6 for arriving at this expression.

Since all sources including the initial condition sources are accounted for in this zero-state response, this zero-state response itself is the total response.

12.4 SUMMARY

- A linear time-invariant circuit containing R , L , C , M , linear dependent sources and a single input source is described by a linear ordinary differential equation with constant coefficients. The differential equation for such a circuit can be expressed in a standard format as below:

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$$
- The *order* of a circuit is equal to the *order* of the differential equation that describes it. The order of the circuit will be equal to the total number of independent inductors and capacitors – the number of all-inductor nodes – the number of all-capacitor loops in the circuit.
- The natural response of a linear time-invariant circuit contains n complex exponential functions $= A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_n e^{s_n t}$, where $s_1, s_2, \dots, s_n$ are given by

the roots of $s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$. The index of a complex exponential function, with the additional role of a symbolic stand-in for the function, is called *natural frequency* of the dynamic circuit. Its real part has a unit of Np/s and its imaginary part has a unit of rad/s .

- Each point in the *complex signal space* stands for a complex exponential signal. Real signals are only special cases of complex signals. Natural frequencies can be represented as points in this complex signal space.
- The total solution $y_{\text{ts}}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_n e^{s_n t} + y_{\text{ps}}(t)$ obtained by adding the particular solution to the natural response *has* to satisfy all the initial conditions on y and its first $(n-1)$ derivatives at $t = 0^+$.
- Substituting the initial conditions in the total response yields a set of n algebraic equations on $A_1 \dots A_n$. If the natural frequency s_i is real-valued, the corresponding A_i will also be real-valued. If s_i and s_j are a pair of complex conjugate natural frequencies, then A_i and A_j will also be a complex conjugate pair.
- If all natural frequencies of a circuit are either negative real or complex conjugate numbers with negative real part, the circuit is a *stable* circuit.
- If a circuit has one or more natural frequencies on $j\omega$ -axis, it is called a *marginally stable* circuit.
- If a circuit has one or more natural frequency values lying on the right-half of the s -plane, the corresponding natural response terms will grow with time. Such circuits are called *unstable* circuits.
- Impulse response is a complete characterisation of a linear time-invariant circuit in the sense that the zero-state response of the circuit to the application of any arbitrary input function can be determined from the impulse response and the input function by convolution integral $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$. Convolution integral is only a restatement of properties of linearity and time-invariance.
- Graphical interpretation of convolution leads to the view that the process of generation of response in a circuit is a process in which the circuit *scans* the input function using its impulse response template as a scanner. The impulse response function is called the *Scanning Function* based on this viewpoint.
- The sinusoidal steady-state frequency response function of a linear time-invariant circuit can be obtained as $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$.

12.5 QUESTIONS

- The impulse response of a linear time-invariant circuit is seen to be $h(t) = (2 e^{-2t} + 1.7 e^{-0.3t} + 2 e^{-t} \cos(10t - 45^\circ))u(t)$. What is the order of the circuit and what are its natural frequencies? Is the circuit a stable one?
- The impulse response of a linear time-invariant circuit is seen to be $h(t) = (0.5 e^{-2.7t} + 1.7 e^{-3t} \sin(30t - 55^\circ) + 2 \cos(10t - 45^\circ))u(t)$. What is the order of the circuit and what are its natural frequencies? Is the circuit a stable one?
- The impulse response of a circuit is $h(t) = (0.7 e^{0.3t} \sin(3t - 55^\circ) + 2 e^{-0.25t} \cos(5t - 45^\circ))u(t)$. What is the order of the circuit and what are its natural frequencies? Is the circuit a stable one?
- The impulse response of a linear time-invariant circuit is $h(t) = (10t e^{-0.2t} + 2e^{-3t}) u(t)$. The circuit does not contain dependent sources. What is the minimum number of energy storage elements in the circuit?
- The step response of a circuit is $0.5 - 0.2 e^{-t} - 0.3t e^{-2t}$. What is its zero-state response for an input $2.5[u(t) - u(t-1)]$?
- The impulse response of a circuit is $h(t) = (10 e^{-0.2t} + 2e^{-3t})u(t)$. (i) What is its response for input $2\delta(t+2)$? (ii) Obtain the differential equation describing the circuit under source-free condition.
- Explain why the differential equation $y''' + 2y' + y = 2x$ cannot be the describing equation of a linear time-invariant circuit containing no dependent sources.
- Explain why the set of complex numbers $\{-1.2, -2 + j3, -2 + j5\}$ cannot be the natural frequencies of a linear time-invariant circuit?
- The impulse response of a circuit is $h(t) = (10 e^{-0.2t} + 2 e^{-3t})u(t)$. Find the value of its step response at $t = 1$ s.
- What is the order of a pure memoryless circuit?
- The impulse response of a linear time-invariant circuit is $h(t) = 0.7\delta(t)$. What is the order of the circuit?
- Show that the impulse response of a stable linear time-invariant circuit is absolutely summable, i.e., $\int_{-\infty}^{\infty} |h(t)| dt$ is finite.
- The function $h_1(t) = 2 e^{-3t}$ cannot be the impulse response of a physical circuit whereas the function $h_2(t) = 2 e^{-3t}u(t)$ can be. Explain why?
- Find the order of circuits in the Fig. 12.5-1 and explain why they are different.

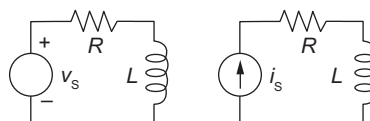


Fig. 12.5-1

- Find the order of circuits in the Fig. 12.5-2 and explain why they are different.

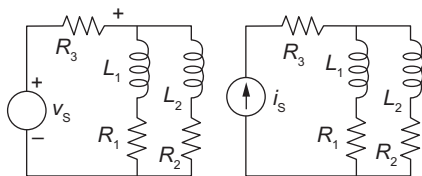


Fig. 12.5-2

16. Two terminals **a** and **b** are identified in a linear time-invariant circuit containing no independent sources. The order of the circuit when it is driven by an independent voltage source connected across **a–b** is found to be 5. Will the order of the circuit be 5 when it is driven by an independent current source connected across **a–b**? Discuss.
17. Find the order of the circuit in Fig. 12.5-3 for voltage excitation and current excitation across **a–b**.

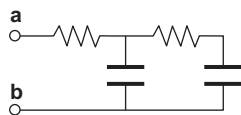


Fig. 12.5-3

18. Show that $[v_1(t) + v_2(t)] * v_3(t) = v_1(t) * v_3(t) + v_2(t) * v_3(t)$.

19. Two linear time-invariant circuits are connected in cascade using an ideal buffer amplifier of unity gain. The output of the first circuit, thus, becomes the input to the second one. If $h_1(t)$ and $h_2(t)$ are their impulse responses, show that (i) the overall cascade will have an impulse response given by $h_1(t) * h_2(t)$ and (ii) the impulse response of the overall cascade is independent of the order of cascading.
20. A finite duration waveform $v(t)$ has non-zero content in the interval $[0, 2 \text{ s}]$ and 0 elsewhere. If $v_1(t) = v(t) * v(t)$, find its duration.
21. The impulse response of a circuit is approximated as $h(t) = 2 e^{-t} \text{ V}$ in the interval $[0, 5]$ and 0 elsewhere. The input to the initially relaxed circuit is a rectangular pulse of height 2 V and duration 2 s. What is the output at 7.5 s after the start of the pulse?
22. The circuit in the previous question is operating in the sinusoidal steady-state with an input of $2 \cos 100t$. The input undergoes a disturbance in the form of a narrow spike superimposed on the cosine wave at some instant t . How much time should elapse before the sinusoidal steady-state will be re-established in the circuit?
23. The natural response terms of a linear time-invariant circuit of finite order are of the form e^{-st} . Using this fact show that the sinusoidal steady-state frequency response function of a finite order linear time-invariant circuit is a continuous function of angular frequency ω .

12.6 PROBLEMS

1. The initial currents in L_1 and L_2 in the circuit in Fig. 12.6-1 at $t = 0^-$ are 1 A and 0.5 A, respectively, in the directions shown. (i) Write the mesh equations of the circuit. (ii) Obtain the differential equation governing the variation of $v_o(t)$ for $t \geq 0^+$ and calculate the natural frequencies of the circuit. (iii) Obtain the complete response for $v_o(t)$ if $v_s(t) = \delta(t)$. (iv) Obtain the complete response for $v_o(t)$ if $v_s(t) = u(t)$. (v) Could you have obtained the step response by integrating the impulse response? Explain. (vi) Obtain the total response for the current through L_1 when $v_s(t) = u(t)$ by expressing the total solution as the sum of the natural response terms and the steady-state value and evaluating the amplitude of natural response terms by applying initial values for zeroth and first order derivatives of the current.

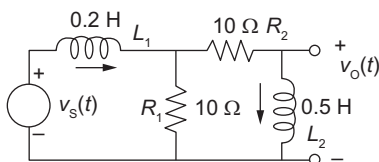


Fig. 12.6-1

2. The initial voltages across C_1 and C_2 at $t = 0^-$ are -2 V and -0.5 V in the circuit in Fig. 12.6-2. (i) Write the node equations

of the circuit. (ii) Obtain the differential equation governing the variation of $v_o(t)$ for $t \geq 0^+$ and calculate the natural frequencies of the circuit. (iii) Obtain the complete response for $v_o(t)$ if $v_s(t) = \delta(t)$. (iv) Obtain the complete response for $v_o(t)$ if $v_s(t) = u(t)$. (v) Obtain the total response for the current through C_1 when the input is a unit step function by expressing the total solution as the sum of natural response terms and the steady-state value and evaluating the amplitude of natural response terms by applying initial values for zeroth and first order derivatives of the capacitor current.

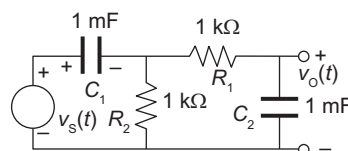


Fig. 12.6-2

3. The impulse response of the current in R_1 in the circuit in Fig. 12.6-3 is found to contain two exponential functions that decay with time constants of 20 ms and 10 ms, respectively. (i) Find the values of L_1 and L_2 . (ii) Find the steady-state step response of the circuit from its impulse response.

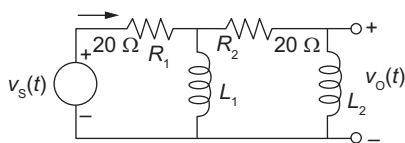


Fig. 12.6-3

4. The initial voltage across the capacitor in the circuit in Fig. 12.6-4 is -1 V with the polarity shown. The initial current in the inductor is zero. (i) Develop the circuit differential equation using the voltage across capacitor as the describing variable and identify the natural frequencies of the circuit. (ii) Find the voltage across the capacitor as a function of time for $t \geq 0^+$ if the input is a unit step voltage. (iii) Develop the circuit differential equation using the source current as the describing variable, identify the natural frequencies of the circuit and calculate the initial values needed for solving this differential equation.

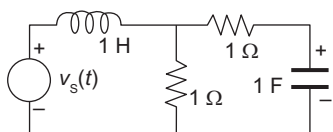


Fig. 12.6-4

5. The current source in the circuit in Fig. 12.6-5 is a unit impulse current. The values of v_{C1} and v_{C2} at $t = 0^-$ are -10 V and 0 V, respectively. Find v_{C1} , v_{C2} , dv_{C1}/dt and dv_{C2}/dt at $t = 0^+$.

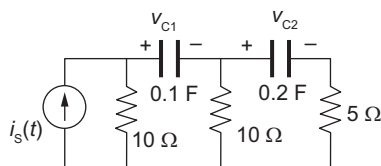


Fig. 12.6-5

6. The voltage source in the circuit in Fig. 12.6-6 is a unit impulse voltage. The values of i_1 and i_2 at $t = 0^-$ are -10 A and 5 A, respectively. Find i_1 , i_2 , di_1/dt and di_2/dt at $t = 0^+$.

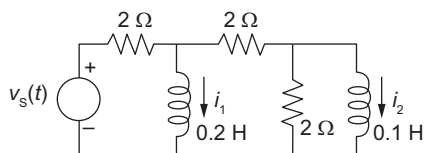


Fig. 12.6-6

7. The impulse response of i_x in the circuit in Fig. 12.6-7 is found to contain a real exponential function of time that has a time constant of 1.755 s along with other response terms. (i) Find the natural frequencies of $v_o(t)$ (ii) Find the step response of $v_o(t)$.

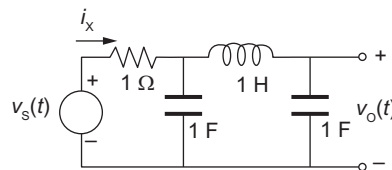


Fig. 12.6-7

8. (i) Find the natural frequencies of the circuit in Fig. 12.6-7 if the voltage source is replaced by an independent current source. (ii) Is the circuit stable? (iii) Find the impulse response of $v_o(t)$ in this case. Explain why the response is undamped despite the presence of a resistor in the circuit.
9. The impulse response of an LTI circuit is $h(t) = 1.5e^{-t} - 1.5e^{-10t}$ V. (i) Find the output at $t = 10$ s when a unit step input is applied to this circuit. (ii) A noise signal which can be approximated as an impulse of strength 0.5 V/s appears in the step input at $t = 11$ s. Find the disturbance in the output at $t = 13$ s due to this disturbance in the input at $t = 11$ s. (iii) How long will it take for the disturbance in the output to taper down to less than 1% of the steady-state output?
10. A first order series RC circuit with $R = 10$ k Ω and $C = 100$ μ F is driven by a unit ramp voltage source. The circuit was initially relaxed. Find the current in the circuit by using the convolution integral.
11. Two first order RL circuits are cascaded by an ideal unity gain buffer amplifier as shown in Fig. 12.6-8. (i) Find the impulse response of the cascaded circuit by convolution. (ii) Find the zero-state response of $v_o(t)$ when the input is a 10 V rectangular pulse of duration 2 ms starting from $t = 0$ by convolution.

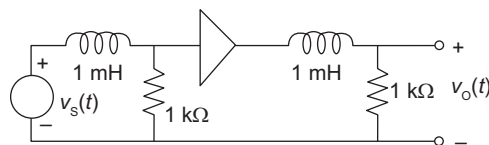


Fig. 12.6-8

12. The sinusoidal steady-state frequency response function of circuit can be found from its impulse response by the convolution integral using $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$.
- (i) The impulse response of an LTI circuit is $h(t) = 2e^{-1.5t}$ V. Find and plot its frequency response using the above result. Find the bandwidth of this circuit.
- (ii) The impulse response of a second order LTI circuit is $h(t) = 1.5e^{-0.5t} \sin(2t + \pi/4)$ V. Find and plot its frequency response.
13. Show that the two expressions for the frequency response of a linear time-invariant circuit given by

$$H(j\omega) = \frac{\sum_{k=0}^m b_k(j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i(j\omega)^i} \text{ and}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt \text{ are equivalent.}$$

14. Verify the statement in Problem 13 for the circuit shown in Fig. 12.6-7.
15. The impulse response of the circuit in Fig. 12.6-9 is found to contain an exponentially damped oscillation of $e^{-0.105t} \sin(1.5525t + \phi)$ form. (i) Obtain the differential equation describing the circuit in terms of $v_o(t)$. (ii) Obtain the natural frequencies present in $v_o(t)$. (iii) Find the impulse response of $v_o(t)$. (iv) Find the frequency response function from the impulse response.

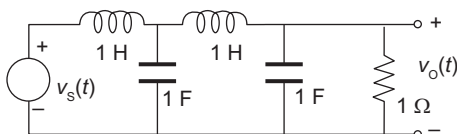


Fig. 12.6-9

16. Obtain the differential equation governing i_x in the circuit in Fig. 12.6-10 and find its natural frequencies. Decide whether the circuit is a stable one.

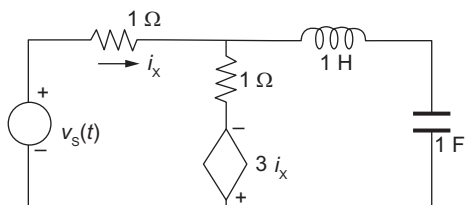


Fig. 12.6-10

17. With reference to the circuit in Fig. 12.6-11, (i) Find the impulse response for v_x from the two source locations. (ii) Find the zero-state response of v_x when $v_{s1}(t) = 2u(t)$ V and $v_{s2}(t) = 2e^{-t}u(t)$ V by convolution.

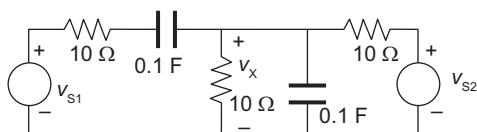


Fig. 12.6-11

18. Three identical RC circuits are cascaded using ideal unity gain buffer amplifiers as shown in Fig. 12.6-12. (i) Find the impulse response of the cascaded circuit if output is taken across the third capacitor. (ii) Obtain the frequency response function of the circuit from its impulse response and verify that it is the same as the frequency response function obtained by phasor equivalent circuit analysis.

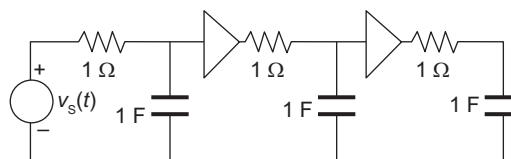


Fig. 12.6-12

19. Assume an ideal Opamp in the circuit in Fig. 12.6-13. (i) Find the natural frequencies in $v_o(t)$. (ii) Find the impulse response of the circuit. (iii) Find the frequency response function of the circuit from its impulse response. (iv) What kind of filtering does the circuit perform?

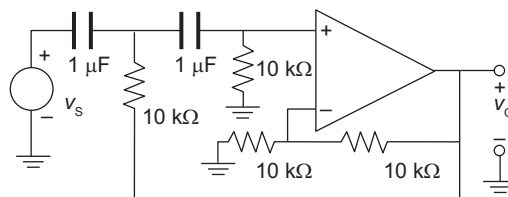


Fig. 12.6-13

Part Five

Frequency-Domain Analysis of Dynamic Circuits

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Dynamic Circuits with Periodic Inputs – Analysis by Fourier Series

CHAPTER OBJECTIVES

- This chapter (i) explains how a periodic waveform can be expanded in terms of sinusoids and why such an expansion is necessary, (ii) shows how such an expansion may be obtained for a given periodic waveform and (iii) shows how the expansion can be used to solve for the forced response of the circuit.
- The introductory sections discuss the difference between steady-state response and forced response of a circuit to set the background for application of sinusoidal expansion of periodic waveforms in Circuit Analysis.
- Important properties of Fourier Series expansion are brought out through solved examples.
- In addition, this chapter sets the background for expansion of aperiodic waveforms in terms of sinusoids that will appear in the next chapter on Fourier Transforms.

This chapter provides a partial answer to an important question in Circuit Analysis – ‘Is the Sinusoidal Steady-State Frequency Response Function a complete characterisation for an LTI Circuit?’

INTRODUCTION

This chapter deals with the determination of forced response component in the output of dynamic circuits when they are excited by periodic input waveforms. The problem of expressing a periodic waveform as a sum of pure sinusoidal components is addressed first. Subsequently, the use of frequency response data to solve for forced response when the forcing function is such a sum of sinusoidal components will be dealt with.

Fourier series expansion deals with periodic waveforms. It *resolves* a periodic waveform *into* pure sinusoidal components. Equivalently, it expands the periodic waveform as a linear combination of infinitely many *harmonically* related sinusoidal waveforms. Two sinusoids are *harmonically* related if their frequencies are integer multiples of some common frequency value. Fourier series expresses the periodic waveform as a sum of infinitely many sinusoids with frequencies which are integer multiples of the frequency of that waveform.

Periodicity of Waveforms

A waveform $v(t)$ is periodic on t with a period of T if and only if the following condition is satisfied by it for all t and n , where n is a positive integer.

$$v(t) = v(t \pm nT) \text{ for any } t \text{ and for any } n; \\ n = 1, 2, 3, \dots$$

This implies that the values of $v(t)$ at similarly

positioned time points, at an interval of T s between adjacent points, will be the same.

Obviously, if $v(t)$ is periodic, it must extend from $-\infty$ to $+\infty$ in the time-axis since the value of integer n in the definition of periodicity is not limited to any finite number. Therefore, it is equally obvious that, there is no strictly periodic waveform in nature. All waveforms in electrical circuits start at some time-instant and stop at some other time-instant. Hence, all practical circuit waveforms are necessarily aperiodic.

Complex exponential functions are eigen functions of the linear time-invariant circuits.

Hence, there is an advantage in expanding a complex wave-shape in terms of complex exponential functions.

This frequency is called the *fundamental frequency*. The sinusoidal component that is at the same frequency as that of the periodic waveform is termed as *the fundamental component* and all the other sinusoidal components with frequencies which are integral multiples of this frequency are called the *harmonic components*.

Granted that a periodic waveform can be expanded in this manner and that frequency response information for a circuit can then be employed to find the forced response of the circuit for such a periodic input, two questions arise at this point.

- Why the obsession with sinusoids? Is it because a periodic waveform can be expanded only in terms of sinusoids?
- If a periodic waveform is only a mathematical entity with no corresponding physical counterpart (see side-box), why bother to study the forced response of circuits to such a hypothetical input?

13.1 PERIODIC WAVEFORMS IN CIRCUIT ANALYSIS

Let us consider the first question.

Expansion in terms of sinusoids is not the only expansion possible for a periodic waveform. However, this expansion turns out to be of a great utility in Circuit Analysis.

Complex exponential functions of the form e^{st} , where s is a complex number, are *eigen functions* of linear time-invariant circuits. This means that the *total response* of the circuit when $v_s(t) = e^{st}$ for all t is applied, is a scaled copy of e^{st} itself. The scaling factor will be a function of s and circuit parameters and hence it will be a complex number (this complex number is called an *eigen value*). We note that e^{st} has to be applied from $t = -\infty$ for this to be true. Since the waveform is applied to the circuit from infinite past, there is no natural response component and forced response itself is the total response.

Forced response components in a linear time-invariant circuit (as well as the natural response terms in zero-state response) due to simultaneously acting forcing functions obey superposition principle. Therefore, if an input function with an arbitrary wave-shape can be expressed as a sum of many (finite or infinite) functions of a type which has a simple wave-shape, we will be able to arrive at the forced response of the circuit for this arbitrary wave-shape by superposing forced response components for the simple wave-shape. The advantage involved will be further enhanced if the forced response component for the simple wave-shape is easy to determine. Therefore, the simple function that should be used as a *basis* function for expanding the complex wave-shape must be e^{st} since that is the input function for which a simple scaling will result in the forced response component.

s in e^{st} is, in general, a complex number and e^{st} is a complex signal. A real physical waveform that we find in a physical circuit cannot have an imaginary part. Therefore, we expect that if we find e^{st} with a particular complex value for $s = \alpha + j\omega$ necessary in the expansion for a real waveform, we will find that (i) the conjugate signal of e^{st} will also be needed and (ii) the scaling factors for e^{st} and its conjugate will turn out to be conjugate numbers. Thus, these two conjugate contributions will combine to yield a component of $e^{st} \sin(\omega t + \theta)$ type. Depending on the values of α and ω , this component may be a decaying exponential, a growing exponential, an exponentially damped sinusoid, an exponentially growing sinusoid or a steady amplitude sinusoid.

But, is it possible to expand any arbitrary function of time, say $v(t)$, as a sum of scaled complex exponential functions even if we allow all possible values of s (i.e., all signals in the signal space) to contribute to the sum? This is a question for mathematicians. The short answer for the student of Circuit Analysis is that, yes, it is possible, except for some very peculiar and pathological functions which exist only in the pages of books on Mathematics and never in a physical system.

An exponentially damped sinusoid is a complex wave-shape, so is an exponentially growing sinusoid. True, they are *eigen functions* of linear circuits and do not pose any difficulty in solving for forced response in the circuits. But let us try to take the simpler ones first. The real exponential function is also not simple enough for our purpose since there is nothing *steady* about it. Therefore, we choose the simplest – the steady amplitude sinusoid which is a special case of e^{st} with the real part of s set to zero. That is, we choose all those signals of type $e^{j\omega t}$ along the $j\omega$ axis in the signal plane as our *basis* functions. Limiting ourselves to sinusoidal signals drawn from the entire imaginary axis of the signal plane, we raise the question – can any arbitrary waveform $v(t)$ be expressed as a sum of scaled functions of $e^{j\omega t}$ type?

The answer is no. But all the waveforms that can really appear in a physical circuit can be expanded this way. Some of the commonly used idealised waveforms like step, impulse, ramp, etc., do pose some problems in this regard – but the problems are not insurmountable.

We constrain $v(t)$ further and state that it is periodic with period T . Now, do we need all the signals on the entire $j\omega$ axis for its expansion? We note that the two waveforms with *same period* will result in a periodic waveform with same period when they are combined in a linear combination. Therefore, it is reasonable to expect that a periodic $v(t)$ can be expressed as a sum of many sinusoidal waveforms with same period. But how many sinusoids can be there with a particular period T ? There are infinite – because a sinusoid with a period of T/n , where n is an integer, is periodic with a period of T too; there will be n full cycles of it in T s. Therefore, the signal represented by the finite series, $\sum_{n=0}^N a_n \cos(n\omega_0 t)$

$+\sum_{n=1}^N b_n \sin(n\omega_0 t)$ will be a periodic wave with frequency of ω_0 rad/s for any finite value of N , a_n and b_n .

For a particular value of ω_0 and for each choice of N , we can vary the $2N+1$ numbers – a_n and b_n – to synthesise infinite number of distinct periodic waveforms of periodicity $2\pi/\omega_0$ s. Each combination of these $2N+1$ numbers over real number field will result in a unique periodic waveform. And we will get yet another set of infinite unique periodic waveforms for another choice of N .

That prompts a question – given a non-sinusoidal periodic waveform $v(t)$ with angular frequency ω_0 rad/s, can we find some N and a set of $2N+1$ real numbers for a_n and b_n such that $v(t) = \sum_{n=0}^N a_n \cos(n\omega_0 t) + \sum_{n=1}^N b_n \sin(n\omega_0 t)$ at all instants? Yes, provided N is allowed to become infinite if necessary and $v(t)$ satisfies certain criteria. Almost all periodic functions appearing in Circuit Analysis satisfy the required criteria.

We have answered the first question raised in the introductory section. We take up the second one now.

Forced response in a circuit is the component of total response contributed by the input source function. It is found out by solving the differential equation *assuming that it is valid for all time and that the forcing function was applied from $t = -\infty$* . But then, what is the difference between response when the forcing function is applied from $t = -\infty$ and when it is applied from $t = 0$? There is no change in the forced response component, but there will be a difference in natural response components.

We recapitulate that the total response of a circuit contains two components – they are *zero-input response* and *zero-state response*. Of these two, the zero-input response depends only on the initial conditions at the instant from which the total response is to be found out. Zero-input response does not depend on the particular nature of forcing function or the instant at which it is applied. Zero-input response contains only natural response terms. Natural response terms are of e^{st} type, where s values are the natural frequencies (i.e., roots of its characteristic equation) of the circuit. Zero-state response, on the other hand, contains two kinds of terms – natural response and forced response. The amplitude of natural

Sinusoidal function is a special case of complex exponential function. An arbitrary time-function can be expanded in terms of sinusoidal functions under certain conditions. . .

response terms in zero-state response is *independent of initial conditions* since zero-state response by definition is the response of an initially relaxed circuit. However, they depend on the nature of input and the time instant at which the input source function was applied to the circuit. The role of natural response terms in zero-state response is to force it to comply with the zero initial conditions when forced response component is added.

We also remember that though the terms – *forced response* and *steady-state response* – are used interchangeably, they are not the same. If a forcing function is applied to a circuit, there will be a forced response. But there need not be a steady-state response always. There can be a steady-state in the circuit only if three conditions are satisfied. Firstly, the circuit natural response must be damped and should approach zero as time increases without limit.

Secondly, there must exist features in the input source function waveform that can be used to define what is meant by *steady-state* at the output (that is, the meaning of ‘*steady-state*’ is decided by input waveform. If there is nothing about the waveform that can be used to define a meaning for ‘*steady-state*’, then, there is no meaning for ‘*steady-state*’ at the output too). For example, if the input is a periodic waveform with period of T , switched on to the circuit at $t = 0$, then *steady-state output* is a waveform that is periodic with same period – such a *steady-state* is called *periodic steady-state*. Sinusoidal steady-state is a special case of periodic steady-state. The difference between the two is that the output waveform need not have the same wave-shape as that of input in the case of a general periodic steady-state whereas the output waveform will be of the same wave-shape as that of input in the case of sinusoidal steady-state.

The third condition is that the input source function must remain applied to the circuit for enough time for the circuit to reach steady-state. For example, we may switch on a periodic wave such as a 50 Hz sine wave to an RC circuit with 0.2 s time constant and switch it off after 0.3 s. The circuit transient response does not get enough time to decay down to zero and there is no question of circuit reaching sinusoidal steady-state. For that matter, no linear circuit can ever reach steady-state for any input whatsoever since damped exponential functions never really touch zero at any time theoretically. But we do not mean theoretical steady-state when we refer to steady-state in circuits.

Many circuits do reach periodic steady-state after the switch-on transient and operate for long periods before they are switched off. Electrical power system and the loads that draw power from it constitute an important example of this kind of operation. Many features of such steady-state operation are of considerable practical interest to the Electrical Engineer. Therefore, one reason why we study the forced response of circuits to periodic waveforms is that we are interested in steady-state response of circuits when such waveforms are switched on to them at $t = 0$. Expanding the periodic waveform in terms of sinusoidal components and using frequency response and superposition principle will help us to get the required steady-state solution with relative ease. However, if we want to get the complete response we have to bring in the natural response terms too and arrange for compliance with initial conditions.

Consider an example. An RL circuit with $R = 1\ \Omega$ and $L = 1\ \text{H}$ is connected to a source $v(t) = \sin t + 0.3 \sin 3t + 0.2 \sin 5t\ \text{V}$ at $t = 0$. Subsequently it is removed from the supply at $t = 100\ \text{s}$ and the circuit is kept shorted after that. We wish to find the *zero-state response* of the circuit current.

Let $v_s(t)$ stand for the voltage applied to the circuit. What is $v_s(t)$?

$$v_s(t) = (\sin t + 0.3 \sin 3t + 0.2 \sin 5t)[u(t) - u(t - 100)]$$

However, this will make the applied voltage zero for $t < 0$. This does not make any difference to the problem since we are going to use this function to solve for zero-state response current and applying zero from $-\infty$ up to $t = 0$ will result in only zero initial condition in the circuit. Note that if the circuit was kept open after 100 s, we could not have used this expression for applied voltage.

The forcing function is a sum of three sinusoids. Forced response component for each sinusoid may be obtained using phasor analysis or frequency response data. Forced response

Steady-state can exist only in a stable circuit.

Steady-state can arise in a stable circuit only if the notion of ‘steady-state’ is applicable to the input currently active in the circuit.

Steady-state response – if ‘steady-state’ gets established in the circuit – is the same as forced response for the period during which ‘steady-state’ remains established in the circuit.

components for the three sinusoids may be combined to obtain the total forced response. The result will be $= 0.707 \sin(t - 0.7 \text{ rad}) + 0.0945 \sin(3t - 1.25 \text{ rad}) + 0.04 \sin(5t - 1.37 \text{ rad}) \text{ A}$.

This solution predicts that the current at $t = 0^+$ is -0.63 A . But the initial condition for current at $t = 0^+$ is zero. Therefore, the natural response component has to be $-0.836 e^{-t} \text{ A}$. Therefore, the solution for current for $t \geq 0^+$ is $i(t) = -0.63 e^{-t} + 0.707 \sin(t - 0.7 \text{ rad}) + 0.0945 \sin(3t - 1.25 \text{ rad}) + 0.04 \sin(5t - 1.37 \text{ rad}) \text{ A}$.

The circuit reaches periodic steady-state after about 5 time constants, i.e., at about 5 s. It operates under periodic steady-state till 100 s. At that point, the applied voltage in the circuit goes to zero suddenly. Therefore, the solution we determined above is valid only up to 100 s. The value of current at $t = 100^-$ can be obtained by evaluating $i(t)$ with $t = 100$. The value is 0.66 A. This current decreases exponentially with a time constant of 1 s since the circuit is kept shorted after 100 s. Therefore, the complete solution for current in the circuit is

$$i(t) = \begin{cases} -0.63e^{-t} + 0.7 \sin(t - 0.7) + 0.095 \sin(3t - 1.25) \\ + 0.04 \sin(5t - 1.37) \text{ A for } 0^+ \leq t \leq 100^- \\ 0.66e^{-(t-100)} \text{ A for } t \geq 100^+ \end{cases}$$

We could arrive at the solution only because we could solve for the forced response component using frequency response analysis. We could do that since the statement of the problem already expressed the applied source as a sum of sinusoids. Fourier series helps us get this sum for a wide class of periodic waveforms.

But, what if we cannot identify any periodicity in the applied waveform? What if we cannot visualise $v_s(t)$ as a chunk of some periodic $v(t)$?

For instance, consider the problem of finding the current in the above circuit when the applied voltage is a single rectangular pulse of 1 V height and 1 s duration, starting from $t = 0$. Whether we think of the applied waveform as a single, non-repetitive rectangular pulse of 1 s duration or as a gated version of a periodic square wave with period, $T > 1 \text{ s}$ and a gate width T_g such that $1 < T_g \leq T$ is matter of view point (Fig. 13.1-1).

Obviously, this technique of visualising an aperiodic waveform as a chunk of a periodic waveform will work for any aperiodic waveform with finite duration – i.e., $v_s(t)$ must become identically zero after some point in time for this to work.

Now, if we can express the underlying periodic square wave as a sum of harmonically related sinusoids, we can obtain the forced response component due to rectangular pulse by using frequency response data. Then, the solution for current will be,

$$i(t) = \begin{cases} -i_{fr}(0^+)e^{-t} + i_{fr}(t) \text{ A for } 0^+ \leq t \leq T_g^- \\ [-i_{fr}(0^+)e^{-T_g} + i_{fr}(T_g^-)]e^{-(t-T_g)} \text{ A for } t \geq T_g^+ \end{cases} \quad (13.1-1)$$

where $i_{fr}(t)$ is the forced response component worked out by using the sinusoidal expansion (i.e., the Fourier series) for $v(t)$ (i.e., the square wave with period T) and frequency response information for the RL circuit (i.e., admittance function $Y(j\omega) = \frac{1}{1+j\omega}$). The term $-i_{fr}(0^+)e^{-t}$

If a waveform that is applied to a circuit can be expressed as a *gated periodic waveform* – i.e., if $v_s(t) = v(t) \cdot [u(t) - u(t - t_0)]$, where $v_s(t)$ is the applied waveform, $v(t)$ is the underlying periodic waveform and t_0 is the duration for which the underlying periodic waveform is passed on to the circuit – then, the forced response component in the output can be obtained by using sinusoidal expansion of the underlying waveform along with frequency response data for the circuit.

Fourier series is needed and will be of great help in a circuit problem of this kind.

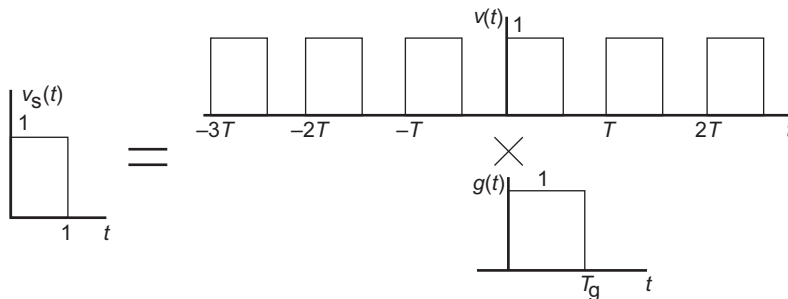


Fig. 13.1-1 A Rectangular Pulse as a Gated Periodic Waveform

Fourier series and frequency response of a circuit help us to solve for forced response component when the input function is aperiodic and of finite duration.

The input source function is thought of as a chunk of a periodic waveform in that case. We construct that periodic waveform by replicating the wave-shape of input function once in every T s in the time-axis from $-\infty$ to $+\infty$, where T is chosen to be more than the duration of signal.

Then this periodic waveform is expanded in terms of sinusoids using Fourier series techniques. Frequency response data and superposition principle help us to solve for forced response.

We complete the solution by adjusting for initial conditions at $t = 0$ and for final condition at the end of input waveform duration by introducing suitably sized natural response terms.

is the natural response term needed to meet the zero initial condition requirement at 0^+ . The term $[-i_{fr}(0^+)e^{-T_g} + i_{fr}(T_g^-)]$ is the value of current at $t = T_g$ and that value becomes the initial value for the exponentially decaying transient that starts from $t = T_g$.

But, what is the value of T that we must use and for a chosen value of T what, must be the value of T_g , the gate width? There is no constraint on T other than that it has to be more than the duration of $v_s(t)$ – 1 s in the example. And, for a chosen value of T , the value of T_g has to be between the duration of $v_s(t)$ and the value of T . Otherwise, it is arbitrary.

But the current waveform must, indeed, be independent of the particular values of T and T_g that one happened to choose when one employs this method of solution. The circuit solution cannot depend on the view point we adopt towards some of the waveforms in it. Therefore, it follows that the forced response and natural response terms in the above solution will change suitably such that the final waveform for $i(t)$ for all $t \geq 0^+$ will appear the same whatever be the values of T and T_g that one happens to choose, provided the above mentioned constraints on them are satisfied.

We note from the Eqn. 13.1-1 that the first solution for current is valid up to the duration of gating waveform. The gating waveform can have as much width as the assumed period T of underlying periodic waveform. We choose to fix T_g at T itself for subsequent discussions from this point onwards and modify the equation suitably. Of course, we realise that $i_{fr}(t)$ will differ for different choices of T and T_g .

$$i(t) = \begin{cases} -i_{fr}(0^+)e^{-t} + i_{fr}(t) \text{ A for } 0^+ \leq t \leq T^- \\ [-i_{fr}(0^+)e^{-T} + i_{fr}(T^-)e^{-(t-T)}] \text{ A for } t \geq T^+ \end{cases} \quad (13.1-2)$$

Now, the first solution term for current is valid up to T .

We use a sleight of hand (or mind) at this point. We relocate the gating waveform between $-T/2$ and $+T/2$, and constrain T to be ≥ 2 times the duration of $v_s(t)$. Nothing changes really. The waveform $v(t)$ is zero-valued between $-T/2$ and 0 with this choice of T (Fig. 13.1-2). Only that we have to replace T in Eqn. 13.1-2 by $T/2$.

$$i(t) = \begin{cases} -i_{fr}(0^+)e^{-t} + i_{fr}(t) \text{ A for } 0^+ \leq t \leq \frac{T^-}{2} \\ [-i_{fr}(0^+)e^{-\frac{T^-}{2}} + i_{fr}\left(\frac{T^-}{2}\right)e^{-\left(t-\frac{T^-}{2}\right)}] \text{ A for } t \geq \frac{T^+}{2} \end{cases} \quad (13.1-3)$$

Now, let T increase without limit. For any finite T , we express the underlying periodic waveform $v(t)$ as a sum of sinusoids and the forced response due to $v(t)$ (i.e., $i_{fr}(t)$) as the sum of sinusoidal steady-state response terms. Consider a large value of T , such as 1000 s, in the

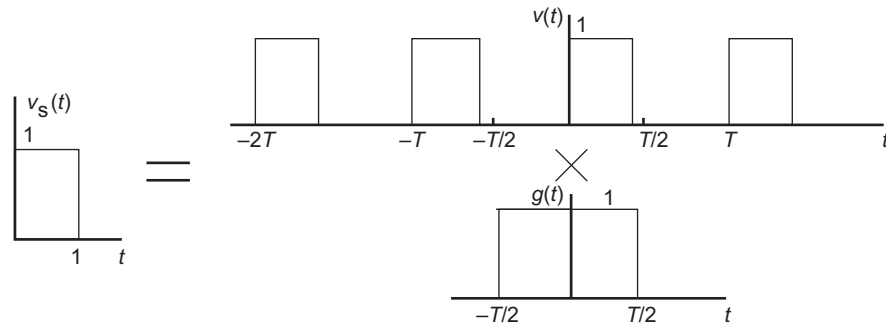


Fig. 13.1-2 A Rectangular Pulse as a Symmetrically Gated Periodic Waveform

example in this section. Thus, the last rectangular pulse just prior to $t = 0$ would have taken place 1000 s back. The circuit in the example has a time constant of 1 s. Whatever be the response value at the end of that pulse, we can expect it to have gone down to zero during the intervening 999 s before the pulse located at $t = 0$ comes up; $-e^{-999}$ is indeed a very small number. Therefore, the term $-i_{fr}(0^+)e^{-t}$ in the solution for $0^+ \leq t \leq 0.5T^-$ can be dropped because $i_{fr}(0^+)$ will be close to zero. All the more so when $T \rightarrow \infty$. Moreover, as $T \rightarrow \infty$, the second solution term in Eqn. 13.1-3 is not needed since the first solution term itself takes care of the entire right side time-axis. Therefore, we get $i(t) = i_{fr}(t)$ as $T \rightarrow \infty$.

But if T is finally going to be infinite in value, why should $v_s(t)$ be of finite duration to begin with? Hence, we bring those aperiodic waveforms that are infinite in extent (for example, $e^{-0.5t} u(t)$) into the ambit of our current discussion. The gist of this discussion is abstracted in the side-box.

But what happens to the sinusoidal expansion of $v(t)$ as $T \rightarrow \infty$? It will also reach a more general form – the Fourier series will transform itself to Fourier transform.

Thus, the expansion of periodic waveforms in terms of sinusoids – i.e., Fourier series – helps us to study the performance of circuits under periodic steady-state. Therefore, we take up the study of Fourier series and its application to periodic steady-state solution in circuits in the remaining sections of this chapter.

More importantly, Fourier series leads to Fourier transform. Fourier transform expresses any reasonably well-behaved time-function, whether of finite duration or of infinite duration, as a continuous sum of sinusoids. And, Fourier transform will translate a zero-state response problem into a sinusoidal steady-state problem. We deal with Fourier transform in Chap. 14.

The forced response of a circuit to a periodic waveform that has an aperiodic input waveform contained in its basic period, approaches the zero-state response (including both natural response terms and forced response terms) of the circuit to that aperiodic waveform, as the period of the periodic waveform goes to infinity.

Thus, the Fourier series and the Fourier transform techniques convert the problem of determining the zero-state response of a linear time-invariant circuit into one of determining sinusoidal steady-state response of the circuit.

13.2 THE EXPONENTIAL FOURIER SERIES

Having settled the ‘why’ of Fourier series we move on to the ‘how’ of Fourier series now.

Let $v(t)$ be a periodic function of time and let it satisfy the conditions required to be satisfied for its expansion in terms of sinusoids to exist. Let the period of $v(t)$ be T s and let its angular frequency be ω_0 rad/s ($\omega_0 = 2\pi/T$). Then, Fourier theorem, in effect, states that $v(t)$ may be represented by the infinite series

$$v(t) = \dots + \tilde{v}_{-3}e^{-j3\omega_0 t} + \tilde{v}_{-2}e^{-j2\omega_0 t} + \tilde{v}_{-1}e^{-j\omega_0 t} + \tilde{v}_0 + \tilde{v}_1e^{j\omega_0 t} + \tilde{v}_2e^{j2\omega_0 t} + \tilde{v}_3e^{j3\omega_0 t} + \dots$$

Using summation notation we write this series as,

$$v(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{jn\omega_0 t}, \quad n \text{ integer.} \quad (13.2-1)$$

This equation states that the periodic function $v(t)$ can be *constructed* or *synthesised* from infinitely many complex exponential functions of time drawn from $j\omega$ axis in the complex signal plane (s -plane). $\tilde{v}_n$ in Eqn. 13.2-1 are called the *coefficients* of exponential Fourier series. ω_0 is the *fundamental frequency*.

$v(t)$ is usually a real function of time since it represents some voltage or current waveform in a circuit. The only way the imaginary components from the pair of signals $e^{jn\omega_0 t}$ and $e^{-jn\omega_0 t}$ can disappear is by $\tilde{v}_{-n}$ being equal to $\tilde{v}_n^*$. Therefore, we expect that $\tilde{v}_{-n}$ will turn out to be $\tilde{v}_n^*$ for any non-zero value of n .

$$\text{Let } \tilde{v}_n = \frac{a_n}{2} - j\frac{b_n}{2}. \text{ Then, } \tilde{v}_n^* = \frac{a_n}{2} + j\frac{b_n}{2}.$$

Then, the contribution of n^{th} harmonic to $v(t)$ can be expressed as

Exponential Fourier series synthesis equation.

$$\begin{aligned}
&= \tilde{v}_{-n} e^{-jn\omega_0 t} + \tilde{v}_n e^{jn\omega_0 t} \\
&= \tilde{v}_n^* e^{-jn\omega_0 t} + \tilde{v}_n e^{jn\omega_0 t} = \left[\frac{a_n}{2} + j \frac{b_n}{2} \right] e^{-jn\omega_0 t} + \left[\frac{a_n}{2} - j \frac{b_n}{2} \right] e^{jn\omega_0 t} \\
&= \left(\frac{a_n}{2} \cos n\omega_0 t + \frac{b_n}{2} \sin n\omega_0 t \right) - j \left(\frac{a_n}{2} \sin n\omega_0 t - \frac{b_n}{2} \cos n\omega_0 t \right) \\
&\quad + \left(\frac{a_n}{2} \cos n\omega_0 t + \frac{b_n}{2} \sin n\omega_0 t \right) + j \left(\frac{a_n}{2} \sin n\omega_0 t - \frac{b_n}{2} \cos n\omega_0 t \right) \\
&= a_n \cos n\omega_0 t + b_n \sin n\omega_0 t.
\end{aligned}$$

Hence, $\tilde{v}_{-n} = \tilde{v}_n^*$ will result in the two complex exponential contributions adding up to yield a real function of time.

Equation 13.2-1 tells us how to construct the periodic waveform $v(t)$ from its harmonic components. But how do we get the exponential Fourier series coefficients given the function $v(t)$?

We proceed as follows.

First, we introduce a new index variable k in place of n in Eqn. 13.2-1 and restate that equation as below.

$$v(t) = \sum_{k=-\infty}^{k=\infty} \tilde{v}_k e^{jk\omega_0 t}$$

Then, we multiply both sides of the equation by $e^{-jn\omega_0 t}$ where n is a particular value of k .

$$\begin{aligned}
v(t) e^{-jn\omega_0 t} &= e^{-jn\omega_0 t} \sum_{k=-\infty}^{k=\infty} \tilde{v}_k e^{jk\omega_0 t} \\
&= \tilde{v}_n + \sum_{\substack{k=-\infty \\ k \neq n}}^{k=\infty} \tilde{v}_k e^{j(k-n)\omega_0 t} \\
&= \tilde{v}_n + \sum_{\substack{k=-\infty \\ k \neq n}}^{k=\infty} \tilde{v}_k [\cos(k-n)\omega_0 t + j \sin(k-n)\omega_0 t].
\end{aligned}$$

We wish to extract $\tilde{v}_n$. We remember an interesting property of sinusoids – the area under a sinusoidal curve over one period is zero. This is so because the area accumulated under the positive half-cycle is cancelled exactly by the area accumulated under the negative half-cycle. More generally, the area under a sinusoid over any time interval equal to its period or integer multiples of its period will be zero.

$k - n$ is an integer. Thus, a sinusoid with angular frequency of $(k - n)\omega_0$ will have integral number of cycles in T s since $T = 2\pi/\omega_0$. Therefore,

$$\int_t^{t+T} \cos(k-n)\omega_0 t dt = 0 \text{ and } \int_t^{t+T} \sin(k-n)\omega_0 t dt = 0 \text{ for } k \neq n.$$

We make use of this fact to extract $\tilde{v}_n$ as,

$$\begin{aligned}
\int_t^{t+T} v(t) e^{-jn\omega_0 t} dt &= \int_t^{t+T} \tilde{v}_n dt + \sum_{\substack{k=-\infty \\ k \neq n}}^{k=\infty} \int_t^{t+T} \tilde{v}_k [\cos(k-n)\omega_0 t + j \sin(k-n)\omega_0 t] dt \\
&= \tilde{v}_n T + 0. \\
\therefore \tilde{v}_n &= \frac{1}{T} \int_t^{t+T} v(t) e^{-jn\omega_0 t} dt.
\end{aligned}$$

The required integration can be carried out over any interval of width T . However, this interval is usually chosen to be $[-T/2, +T/2]$ in order to exploit certain symmetries that the waveform $v(t)$ may possess. Therefore,

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt \quad \text{— analysis equation.} \quad (13.2-2)$$

Exponential
Fourier series
analysis equation.

Equations 13.2-1 and 13.2-2 are called *synthesis equation* and *analysis equation*, respectively and the two together form the *Fourier series pair*.

We expect that $\tilde{v}_{-n}$ will turn out to be $\tilde{v}_n^*$ for any non-zero value of n . We show that it is indeed so.

$$\begin{aligned} \tilde{v}_{-n} &= \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-j(-n)\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{jn\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} v(t) (e^{-jn\omega_0 t})^* dt \\ &= \left(\frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt \right)^* \quad (\text{since } v(t) \text{ is a real function } t) \\ &= \tilde{v}_n^*. \end{aligned}$$

The value $n = 0$ is a special one. The harmonic coefficient at $n = 0$ appears alone without a conjugate companion. We examine this coefficient further.

$$\tilde{v}_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-j \cdot 0 \cdot \omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt.$$

Thus, $\tilde{v}_0$ is a *real value* representing the *cyclic average* value of $v(t)$. The area of $v(t)$ in one cycle is divided by the period to arrive at $\tilde{v}_0$. It represents the DC content in the waveform. If this DC content is removed from $v(t)$, it becomes a pure AC that has zero area under one cycle.

13.3 TRIGONOMETRIC FOURIER SERIES

The trigonometric form of Fourier series affords better insight into how sinusoids combine to produce the periodic waveform $v(t)$. This form is derived from the exponential form below.

$$\begin{aligned} \text{Let } \tilde{v}_n &= \frac{a_n}{2} - j \frac{b_n}{2}. \text{ Then } \tilde{v}_n^* = \frac{a_n}{2} + j \frac{b_n}{2}. \text{ Then,} \\ v(t) &= \tilde{v}_0 + \sum_{n=1}^{\infty} [\tilde{v}_{-n} e^{-jn\omega_0 t} + \tilde{v}_n e^{jn\omega_0 t}] \\ &= \tilde{v}_0 + \sum_{n=1}^{\infty} [\tilde{v}_n^* e^{-jn\omega_0 t} + \tilde{v}_n e^{jn\omega_0 t}] \\ &= \tilde{v}_0 + \sum_{n=1}^{\infty} \left[\left[\frac{a_n}{2} + j \frac{b_n}{2} \right] e^{-jn\omega_0 t} + \left[\frac{a_n}{2} - j \frac{b_n}{2} \right] e^{jn\omega_0 t} \right] \\ &= \tilde{v}_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \\ \therefore v(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t, \end{aligned}$$

Trigonometric
Fourier series.

where,

$$a_0 = \tilde{v}_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt, \quad (13.3-1)$$

$$a_n = \tilde{v}_n + \tilde{v}_{-n} = \tilde{v}_n + \tilde{v}_n^* = 2 \operatorname{Re}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \cos n\omega_0 t \, dt \text{ for } n = 1, 2, 3 \dots$$

$$b_n = -\tilde{v}_n + \tilde{v}_{-n} = -\tilde{v}_n + \tilde{v}_n^* = -2 \operatorname{Im}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \sin n\omega_0 t \, dt \text{ for } n = 1, 2, 3 \dots$$

This can be written in the following form by combining the cosine and sine contributions for a particular harmonic order n using trigonometric identities.

$$\therefore v(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \phi_n),$$

$$\text{where } c_0 = \tilde{v}_0,$$

$$c_n = \sqrt{a_n^2 + b_n^2} = 2\tilde{v}_n \tilde{v}_n^* = 2|\tilde{v}_n| \quad (13.3-2)$$

$$\text{and } \phi_n = \tan^{-1} \frac{b_n}{a_n} = -\angle \text{ of } \tilde{v}_n \text{ for } n = 1, 2, 3 \dots$$

13.4 CONDITIONS FOR EXISTENCE OF FOURIER SERIES

The exponential and trigonometric Fourier series exists for all $v(t)$ which satisfy a set of conditions known as Dirichlet's conditions (see side-box).

There are functions that violate one or more of Dirichlet's conditions. But they do not come up in Electrical Circuits. Hence, we can safely assert that all waveforms we encounter in physical circuits will satisfy these conditions.

If $v(t)$ satisfies all the Dirichlet's conditions, its Fourier series is guaranteed to exist and converge to the function value *except at the points of discontinuity*. At the points of discontinuity, the Fourier series will converge to the average value – i.e., to half the sum of value of $v(t)$ at the left and right of the discontinuity.

This implies that, for a $v(t)$ that satisfies all the four conditions, the partial sum of its Fourier series tends to approach the value of $v(t)$ as the number of series terms included in the partial sum approaches infinity.

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^{n=N} \tilde{v}_n e^{jn\omega_0 t_0} = v(t_0) \text{ if } v(t) \text{ is continuous at } t_0, \text{ and} \quad (13.4-1)$$

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^{n=N} \tilde{v}_n e^{jn\omega_0 t_0} = \frac{v(t_0^-) + v(t_0^+)}{2} \text{ if } v(t) \text{ is discontinuous at } t_0.$$

All the terms in a Fourier series are continuous functions of time. Equation 13.4-1 seems to suggest that the Fourier series converges to a discontinuity – i.e., the series converges to $v(t_0^-)$ at $t = t_0^-$, to $0.5v(t_0^-) + 0.5v(t_0^+)$ at $t = t_0$ and to $v(t_0^+)$ at $t = t_0^+$. How can a sum of continuous functions produce jump discontinuity?

In fact, it does not. The partial sum of Fourier series in Eqn. 13.4-1 oscillates with time around the point of discontinuity. As N is increased, these oscillations in partial sum value get crowded more and more towards a small neighbourhood of t_0 . The oscillations never really disappear. We study in detail about convergence issue in Chap. 14.

13.5 WAVEFORM SYMMETRY AND FOURIER SERIES COEFFICIENTS

Waveforms can exhibit symmetry about the vertical axis. Figure 13.5-1 shows two waveforms that exhibit the so-called *even symmetry*.

If the right side portion of the waveform – i.e., the portion of the waveform for $t > 0$ – is mirror reflected about vertical axis, the reflection coincides with the portion of waveform

Dirichlet's Conditions

(i) The function $v(t)$ must be single-valued everywhere.

(ii) The integral $\int_{t_0}^{t_0+T} |v(t)| dt$ is finite for any t_0 . This condition ensures that all Fourier series coefficients are finite-valued.

(iii) $v(t)$ has only a finite number of maxima and minima in any one period.

(iv) $v(t)$ has only a finite number of discontinuities in any one period. Moreover, each discontinuity is a finite discontinuity.

located in the left side of time-axis. This can be expressed as $v(-t) = v(t)$ for any t . A function $v(t)$ that exhibits this kind of symmetry is said to possess the property of *even symmetry* and is called an *even function*.

Examine the two waveforms shown in Fig. 13.5-2. If the right-side portion of the waveform is mirror reflected about vertical axis, the reflection is equal and opposite to the waveform located in the left-side of time-axis. This can be expressed as $v(-t) = -v(t)$ for any t . A function $v(t)$ that exhibits this kind of symmetry is said to possess the property of *odd symmetry* and is called an *odd function*.

Even and odd symmetry in waveforms is dependent on choice of the $t = 0$ point in the horizontal axis. If the vertical axis is moved to the right or left of its current position all the waveforms in Figs. 13.5-1 and 13.5-2 will lose their symmetry properties.

Consider a $v(t)$ which does not exhibit *even* symmetry or *odd* symmetry. Let us express $v(t)$ in the following equivalent form.

$$v(t) = \frac{v(t) + v(-t)}{2} + \frac{v(t) - v(-t)}{2}$$

$v(t)$ is expressed here as a sum of two functions. The first function is an even function. So we call it $v_e(t)$.

$$v_e(t) = \frac{v(t) + v(-t)}{2}. \text{ Then,}$$

$$v_e(-t) = \frac{v(-t) + v(t)}{2} = v_e(t).$$

Therefore, $v_e(t)$ is an *even* function on t for any $v(t)$.

The second function is an odd function. So we call it $v_o(t)$.

$$v_o(t) = \frac{v(t) - v(-t)}{2}. \text{ Then,}$$

$$v_o(-t) = \frac{v(-t) - v(t)}{2} = -v_o(t).$$

Therefore, $v_o(t)$ is an *odd* function on t for any $v(t)$.

Any time-function $v(t)$ can be expressed as the sum of an even function and an odd function. $v_e(t) = \frac{1}{2}[v(t) + v(-t)]$ is termed as the *even part* of $v(t)$ and $v_o(t) = \frac{1}{2}[v(t) - v(-t)]$ is called the *odd part* of $v(t)$.

How will *even* and *odd* symmetry affect the exponential Fourier series coefficients? If $v(t)$ is *even* on t ,

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_0^{T/2} v(t) [e^{-jn\omega_0 t} + e^{-jn\omega_0 (-t)}] dt = \frac{2}{T} \int_0^{T/2} v(t) \cos n\omega_0 t dt.$$

Thus, its Fourier series coefficients $\tilde{v}_n$ will be real numbers. This indicates that its trigonometric Fourier series will contain only cosine terms.

If $v(t)$ is even on t , then, $\tilde{v}_n$ is real and $b_n = 0$ for all n

$$\therefore v(t) = a_n + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_0 = \frac{2}{T} \int_0^{T/2} v(t) dt \text{ and } a_n = \frac{4}{T} \int_0^{T/2} v(t) \cos n\omega_0 t dt.$$

The coefficients of exponential Fourier series of an *even* periodic waveform on t will be *real* and its trigonometric Fourier series will contain only *cosine* terms.

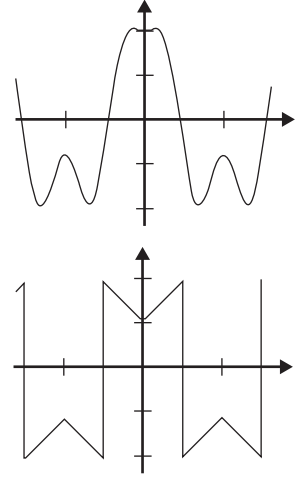


Fig. 13.5-1 Waveforms Exhibiting Even Symmetry

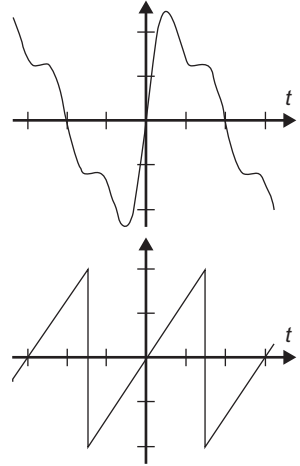


Fig. 13.5-2 Waveforms Exhibiting Odd Symmetry

This is quite reasonable since sum of *odd* functions will produce only an odd function and hence an *even* $v(t)$ cannot contain even a single sine term.

It can be shown similarly that,

If $v(t)$ is *odd* on t , then,

$$\tilde{v}_n \text{ is pure imaginary and } = -j \frac{2}{T} \int_0^{T/2} v(t) \sin n\omega_0 t \, dt, a_n = 0 \text{ for all } n$$

and $v(t)$ will contain only sine terms in its trigonometric Fourier series. The DC content in the waveform has to be zero since DC value is an *even* function of time.

If $v(t)$ is *odd* on t , then, $\tilde{v}_n$ is imaginary and $a_n = 0$ for all n

$$\therefore v(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \text{ and } b_n = \frac{4}{T} \int_0^{T/2} v(t) \sin n\omega_0 t \, dt.$$

The coefficients of exponential Fourier series of an *odd* periodic waveform on t will be *imaginary* and its trigonometric Fourier series will contain only *sine* terms.

Another kind of symmetry exhibited by waveforms is called *half-wave symmetry*. A periodic waveform $v(t)$ is *half-wave symmetric* if one half-cycle of the waveform has the same appearance of the other half-cycle inverted about time-axis. This is expressed as

$$\text{Half-wave symmetry} \Rightarrow v(t) = -v\left(t \pm \frac{1}{2}T\right) \text{ for all } t.$$

Figure 13.5-3 shows three waveforms exhibiting *half-wave symmetry*. The waveform in Fig. 13.5-3(a) shows *half-wave symmetry*; but it does not possess *even* or *odd* symmetry. The waveform represented by *solid* curve in Fig. 13.5-3(b) shows *half-wave symmetry* along with *even* symmetry and the one represented by *dotted* curve in Fig. 13.5-3(b) shows *half-wave symmetry* along with *odd* symmetry. When a periodic waveform possesses both *half-wave symmetry* and *even/odd* symmetry it is said to have *quarter-wave symmetry*. The effect of *half-wave symmetry* on Fourier series coefficients is derived below.

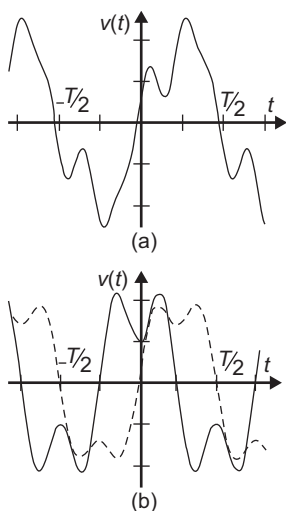


Fig. 13.5-3 Waveforms Showing Half-Wave Symmetry

$$\tilde{v}_n = \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} v(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_0^{\frac{1}{2}T} v(t) e^{-jn\omega_0 t} dt + \frac{1}{T} \int_{-\frac{1}{2}T}^0 v(t) e^{-jn\omega_0 t} dt = I_1 + I_2.$$

Substitute $t' = t + \frac{T}{2}$ in the second integral

$$\begin{aligned} I_2 &= \frac{1}{T} \int_{-\frac{1}{2}T}^0 v(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_0^{\frac{1}{2}T} v\left(t' - \frac{T}{2}\right) e^{-jn\omega_0 t'} e^{jn\omega_0 \frac{T}{2}} dt' \\ &= \frac{1}{T} \int_0^{\frac{1}{2}T} -v(t') e^{-jn\omega_0 t'} e^{jn\omega_0 \frac{T}{2}} dt' \quad (\because v(t) \text{ is half-wave symmetric}) \\ &= \frac{1}{T} \int_0^{\frac{1}{2}T} -v(t') e^{-jn\omega_0 t'} e^{jn\pi} dt' \quad \left(\because \omega_0 = \frac{2\pi}{T}\right) \\ &= (-1)^n \frac{1}{T} \int_0^{\frac{1}{2}T} -v(t') e^{-jn\omega_0 t'} dt' \quad (\because e^{jn\pi} \text{ is } 1 \text{ for even } n \text{ and } -1 \text{ for odd } n) \\ &= (-1)^{n+1} \frac{1}{T} \int_0^{\frac{1}{2}T} v(t') e^{-jn\omega_0 t'} dt' = -I_1 \text{ for even } n \text{ and } +I_1 \text{ for odd } n \end{aligned}$$

$$\therefore \tilde{v}_n = \begin{cases} 0 & \text{for even } n \\ \frac{2}{T} \int_0^{\frac{1}{2}T} v(t) e^{-jn\omega_0 t} dt & \text{for odd } n \end{cases}$$

$$a_0 = 0$$

$$a_n = \begin{cases} 0 & \text{for even } n \\ \frac{4}{T} \int_0^{\frac{1}{2}T} v(t) \cos n\omega_0 t dt & \text{for odd } n \end{cases} \quad \text{and } b_n = \begin{cases} 0 & \text{for even } n \\ \frac{4}{T} \int_0^{\frac{1}{2}T} v(t) \sin n\omega_0 t dt & \text{for odd } n \end{cases}$$

13.6 PROPERTIES OF FOURIER SERIES AND SOME EXAMPLES

Let us take some examples of Fourier series and develop the important properties of this series through them. The waveforms used in the examples that follow are very important signal waveforms that appear frequently in Electrical and Electronic Engineering applications and are not just some functions used to illustrate Fourier series. The waveforms appearing in the examples are important in their own right.

The first property of Fourier series is almost self-evident. It is the property of *linearity*. It states that if $v_1(t)$ and $v_2(t)$ are two periodic waveforms with the same period T and $v_3(t) = a_1 v_1(t) + a_2 v_2(t)$, then,

$\tilde{v}_{3n} = a_1 \tilde{v}_{1n} + a_2 \tilde{v}_{2n}$ for all n , where $\tilde{v}_{1n}$, $\tilde{v}_{2n}$ and $\tilde{v}_{3n}$ are the coefficients of exponential Fourier series of $v_1(t)$, $v_2(t)$ and $v_3(t)$, respectively. This may be proved easily.

EXAMPLE: 13.6-1

Find the exponential Fourier series of $v(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$ and derive the trigonometric Fourier series from it.

SOLUTION

This waveform is a periodic sequence of unit impulses with a period of 1 s. It is shown in Fig. 13.6-1.

$$\tilde{v}_n = \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} v(t) e^{-jn\omega_0 t} dt = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} v(t) e^{-jn2\pi t} dt = \int_0^{0^+} \delta(t) e^{-jn2\pi t} dt = \int_0^{0^+} \delta(t) dt = 1.$$

The waveform has zero value at all points in the first period $[-0.5, 0.5]$ except between 0^- and 0^+ . In that interval it is an impulse of unit magnitude. And the value of exponential in that interval is 1. Therefore, all coefficients in the exponential Fourier series are equal to 1.

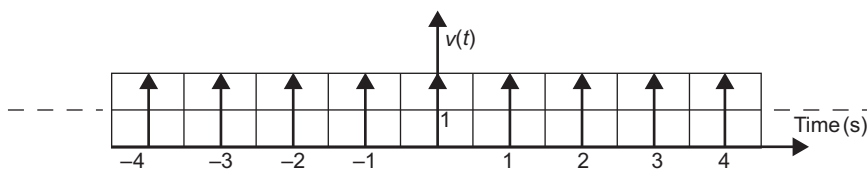


Fig. 13.6-1 Waveform $v(t)$ in Example 13.6-1

Waveform Symmetry and Fourier Series

The coefficients of exponential Fourier series of an even periodic waveform on t will be *real* and its trigonometric Fourier series will contain only cosine terms.

The coefficients of exponential Fourier series of an odd periodic waveform on t will be *imaginary* and its trigonometric Fourier series will contain only sine terms.

A periodic waveform with *half-wave symmetry* does not have any average value (or DC content) and does not contain any *even harmonics*.

If a periodic waveform is *even* on t and is *half-wave symmetric*, its Fourier series expansion will contain only cosine functions at *odd harmonic frequencies*.

If a periodic waveform is *odd* on t and is *half-wave symmetric*, its Fourier series expansion will contain only sine functions at *odd harmonic frequencies*.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt} \text{ is the exponential Fourier series of } v(t).$$

We obtain the trigonometric Fourier series from exponential Fourier series by taking two components with harmonic order $-n$ and $+n$ together.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt} = e^{j2\pi \cdot 0 \cdot t} + \sum_{n=1}^{\infty} (e^{j2\pi nt} + e^{-j2\pi nt}) = 1 + \sum_{n=1}^{\infty} 2\cos 2\pi nt \quad (13.6-1)$$

The waveform contains a DC component of 1 unit. This should be so because the total area content of the waveform in one period is the area content of unit impulse, i.e., unity. This area divided by T must be the DC content in the waveform. T in the example is 1 s.

The waveform contains only cosine terms. This too is expected since $v(t)$ is an even function of t .

The most important aspect to be noted is that periodic impulse train contains *all harmonics with equal strength*. That is, the amplitude of harmonic components does not show any let up as the frequency of harmonic component goes up. Sinusoids of all frequencies with uniform amplitude are required to synthesise the periodic impulse train.

But didn't we stretch the concept of Fourier series a bit too far? Impulse is a highly discontinuous waveform. In fact, the $v(t)$ in this example violates the Dirichlet's condition that, discontinuity, if present, must be of finite value. Hence, though we have found a Fourier series for $v(t)$, will it really converge to $v(t)$ at all t ?

The answer, strictly speaking, is no. As usual, faced with such mathematical difficulties, we just change our view point and make the Fourier series we derived in this example, a useful one. We simply view the impulse as rectangular pulse of large height and small width and unit area. Then, we argue that over the width of rectangular pulse in the first period (which is centred around $t = 0$) the exponential factor $e^{-jk\omega_0 t}$ is close to 1 and may be approximated as such. But will the approximation fail if k becomes very large though t is small? It may, but we will not let it fail; we will state that we will compress the pulse a little more while keeping its area at unity. Hence, the Fourier series in Eqn. 13.6-1 represents the Fourier series of a periodic rectangular pulse train with each pulse containing unit area as the width of the pulse is made infinitesimal and height of the pulse is made infinitely large. We keep Dirichlet happy that way.

A periodic impulse train contains *all harmonics with equal strength*.

EXAMPLE: 13.6-2

Obtain the Fourier series of $v(t) = \sum_{n=-\infty}^{\infty} \delta(t-n-0.5)$ shown in Fig. 13.6-2.

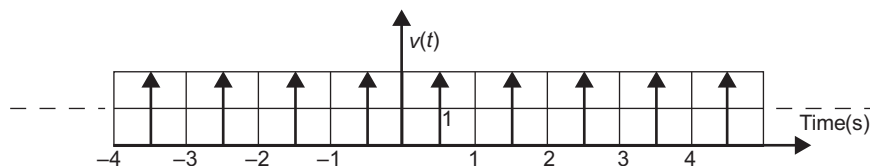


Fig. 13.6-2 Waveform $v(t)$ for Example 13.6-2

SOLUTION

We observe by comparing Figs. 13.6-1 and 13.6-2 that this waveform is only a *delayed* version in Fig. 13.6-1. The delay involved is 0.5 s. Therefore, if we delay all the sinusoidal components in the Fourier series of waveform in Fig. 13.6-1 we should get the Fourier series of waveform in Fig. 13.6-2. Delaying a sinusoid by t_d s amounts to adding a phase delay of ωt_d rad to its argument, where ω is its radian frequency. That is, ωt has to be replaced by $(\omega t - \phi)$, where $\phi = \omega t_d$.

k th harmonic component in the exponential Fourier series of any waveform is denoted by $\tilde{v}_k e^{jk\omega_0 t}$, where ω_0 is the fundamental radian frequency. Delaying this

component by t_d s results in $\tilde{v}_n e^{j(n\omega_0 t - n\omega_0 t_d)} = [\tilde{v}_n e^{-jn\omega_0 t_d}] e^{jn\omega_0 t}$. Therefore, the Fourier series of a time-shifted waveform can be expressed in terms of Fourier series of the unshifted version as follows.

If $\tilde{v}_n$ are the coefficients of exponential Fourier series of a periodic waveform $v(t)$, then, the coefficients of exponential Fourier series of the time-shifted periodic waveform $v(t - t_d)$ are given by $\tilde{v}_n e^{-jn\omega_0 t_d}$. This is called the 'time-shifting property of Fourier series expansion'.

In this example, the time delay is 0.5 s and that value is half the period. Therefore, $e^{-jn\omega_0 t_d} = e^{-j\pi n} = 1$ for even n and -1 for odd n . Therefore, the Fourier series of $v(t)$ is as shown below.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t - k - 0.5) = \sum_{n=-\infty}^{\infty} e^{-j\pi n} e^{j2\pi n t} = 1 + \sum_{n=1}^{\infty} (-1)^n 2 \cos 2\pi n t. \quad (13.6-2)$$

There is no change in the amplitude of cosine waves, but the phase of waves alternate between 0 and 180° with harmonic order n .

*Time-shifting
property of Fourier
series.*

EXAMPLE: 13.6-3

Find the exponential Fourier series and trigonometric Fourier series of waveforms in Examples 13.6-1 and 13.6-2, if the magnitude of each impulse in both cases is V units and the period is T s.

SOLUTION

We remember that there is a $1/T$ factor in the Fourier series analysis equation. It happened to be 1 when T was set to 1 s. We have to bring that factor back. Similarly, since $\omega_0 = \frac{2\pi}{T}$ we have to bring in $1/T$ in the index of exponential in exponential Fourier

series and in the argument of trigonometric functions in trigonometric Fourier series.

Hence, the required Fourier series for waveform in Example 13.6-1 is

$$\therefore \sum_{k=-\infty}^{\infty} V \delta(t - kT) = \sum_{n=-\infty}^{\infty} \frac{V}{T} e^{j\frac{2\pi}{T} n t} = \frac{V}{T} + \sum_{n=1}^{\infty} \frac{2V}{T} \cos \frac{2\pi}{T} n t$$

and the required Fourier series for waveform in Example 13.6-2 is

$$\therefore \sum_{k=-\infty}^{\infty} V \delta\left(t - kT - \frac{T}{2}\right) = \sum_{n=-\infty}^{\infty} \left(\frac{V}{T} e^{-j\pi n}\right) e^{j\frac{2\pi}{T} n t} = \frac{V}{T} + \sum_{n=1}^{\infty} \frac{2V}{T} (-1)^n \cos \frac{2\pi}{T} n t.$$

EXAMPLE: 13.6-4

Let $v_1(t)$ be a periodic waveform same as the one used in Example 13.6-1 and $v_2(t)$ be a periodic waveform same as the waveform used in Example 13.6-2. Then, find $v(t) = v_1(t) - v_2(t)$ and its Fourier series.

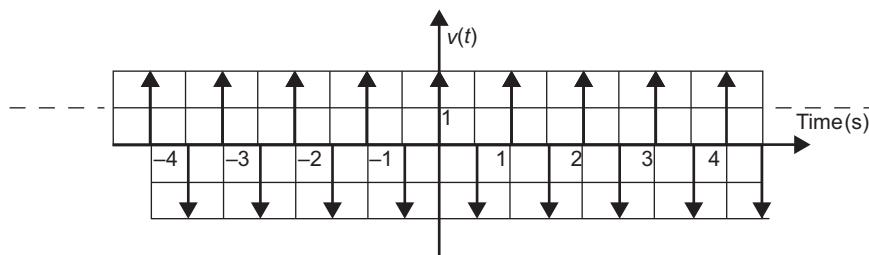
SOLUTION

The waveform $v(t)$ is constructed and shown in Fig. 13.6-3

We construct the Fourier series of this waveform by using Eqn. 13.6-1 and 13.6-2 and property of linearity of Fourier series.

$$\therefore v(t) = \sum_{n=-\infty}^{\infty} e^{j2\pi n t} - \sum_{n=-\infty}^{\infty} e^{-j\pi n} e^{j2\pi n t} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} 2e^{j2\pi n t} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} 4 \cos 2\pi n t.$$

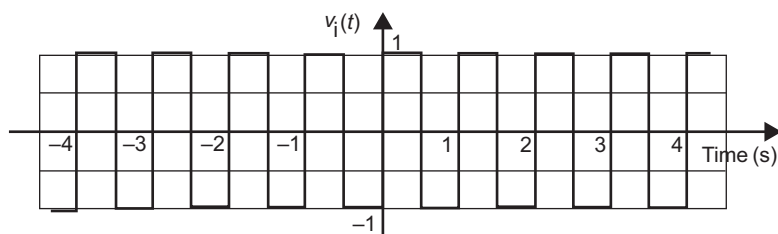
It contains only odd harmonics of cosine format. Its average value is zero. It should be so because this waveform has even symmetry and half-wave symmetry.

Fig. 13.6-3 Waveform of $v(t)$ in Example 13.6-4**EXAMPLE: 13.6-5**

$v(t)$ is a periodic waveform with same waveform as that of Fig. 13.6-3, but with 2 units of magnitude in each impulse. Find and plot $v_i(t) = \int_{-\infty}^t v(t)dt$ and obtain Fourier series of $v_i(t)$.

SOLUTION

Integration usually results in an arbitrary constant. This constant will become a DC component in $v_i(t)$. Since there is no way to find out this arbitrary constant we choose to ignore it and we qualify the Fourier series of $v_i(t)$ by adding a clause that a DC term can always come in the series without upsetting other coefficients. The waveform of $v_i(t)$, ignoring a possible DC term, is shown in Fig. 13.6-4.

Fig. 13.6-4 Waveform of $v_i(t)$ in Example 13.6-5

The DC content was ignored. Therefore, $v_i(t)$ must be pure AC. At $t = 0$, its value will change by the area content of impulse at that point. This area content is 2 units. Hence, the value of $v_i(t)$ must change by 2 units at $t = 0$. Then, it must remain constant. At $t = 0.5$ s, the negative impulse of 2 unit magnitude will again change $v_i(t)$ by -2 units. With the assumption of AC output, $v_i(t)$ must then be alternating between $+1$ and -1 with a period of 1 s. Hence, $v_i(t)$ is a symmetric square wave of amplitude 1 unit and period 1 s.

We wish to find the Fourier series of $v_i(t)$. There are two ways to do it. The first method is to employ the analysis equation of Fourier series – Eqn. 13.2-2 – and carry out the required integration to determine the Fourier series coefficients. The second method relies upon the fact the waveform $v_i(t)$ is the integral of $v(t)$ and that we already know the Fourier series of $v(t)$. That leads us to the question – what is the relationship between Fourier series coefficients for a periodic waveform and Fourier series coefficients for its integral?

$$v_i(t) = \int_{-\infty}^t v(t)dt = \int_{-\infty}^t \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{jn\omega_o t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^t \tilde{v}_n e^{jn\omega_o t} dt = \sum_{n=-\infty}^{\infty} \frac{\tilde{v}_n}{jn\omega_o} e^{jn\omega_o t} \quad (13.6-3)$$

$$\therefore \tilde{v}_{in} = \frac{\tilde{v}_n}{jn\omega_o}$$

We note that $v(t)$ should not have an average value, i.e., $\tilde{v}_0 = a_0 = 0$. If it has a non-zero average value, then its integral will have a linearly increasing term which makes it aperiodic and unbounded. Fourier series for such a waveform does not exist.

We have arrived at the *Integration in Time* property of Fourier series.

If $v(t)$ is a zero-average periodic waveform with $\tilde{v}_n$ as its exponential Fourier series coefficients, then, the Fourier series coefficients of $\int_{-\infty}^t v(t) dt$ is given by $\frac{\tilde{v}_n}{jn\omega_0}$.

The Fourier series coefficients of half of $v(t)$ in this example was derived in Example 13.6-4.

$$v(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} 4e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} 8\cos 2\pi nt$$

$$\therefore v_i(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{4}{j2\pi n} e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt} + \frac{2}{j\pi(-n)} e^{j2\pi(-n)t} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4}{\pi n} \sin 2\pi nt.$$

A unit amplitude symmetric square wave of period T will also have same coefficients, i.e., $\frac{4}{\pi n}$. The factor of $\frac{1}{T}$ gets cancelled by a similar factor that is included in ω_0 in Eqn. 13.6-3.

Integration in Time property of Fourier series.

A symmetric unit amplitude square wave contains only odd harmonics and the harmonic amplitude ($= \frac{4}{\pi n}$ in trigonometric Fourier series) decreases in inverse proportion to harmonic order n .

EXAMPLE: 13.6-6

$v(t)$ is a ± 4 unit symmetric square wave with $T = 1$ s. Find and plot $v_i(t) = \int_{-\infty}^t v(t) dt$ and find its Fourier series.

SOLUTION

$v(t)$ and $v_i(t)$ are shown in Fig. 13.6-5.

Using the result from the previous example, we get,

$$v(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{8}{j\pi n} e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{16}{\pi n} \sin 2\pi nt.$$

The exponential Fourier series coefficients get divided by $jn\omega_0$ when the waveform gets integrated in time.

$$\therefore v_i(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{16}{(j2\pi n)^2} e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} -\frac{4}{\pi^2 n^2} (e^{j2\pi nt} + e^{-j2\pi nt}) = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} -\frac{8}{\pi^2 n^2} \cos 2\pi nt.$$

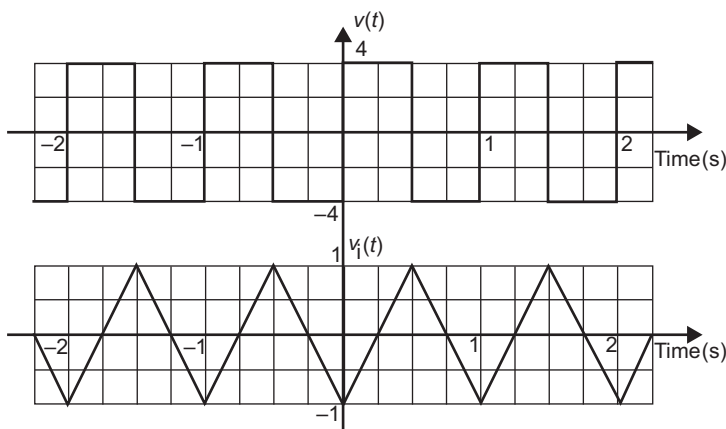


Fig. 13.6-5 Waveforms of $v(t)$ and its Integral, $v_i(t)$, for Example 13.6-6

A symmetric unit amplitude triangle wave contains only odd harmonics and the harmonic amplitude ($= \frac{8}{\pi^2 n^2}$ in trigonometric Fourier series) decreases in inverse proportion to square of harmonic order n .

The triangle wave in Fig. 13.6-5 has ± 1 unit amplitude and has even *half-wave* symmetry. Hence, it contains odd cosine harmonics.

EXAMPLE 13.6-7

Let $v(t)$ be a periodic waveform with period T and let $v_c(t)$ be defined as $v_c(t) = v(\alpha t)$, where $\alpha > 0$. Show that $v(t)$ and $v_c(t)$ have same coefficients in their Fourier series.

SOLUTION

$v_c(t)$ will be a time-compressed version of $v(t)$ for $\alpha > 1$ and time-expanded version of $v(t)$ for $\alpha < 1$. Therefore, the period of $v_c(t)$ will be $\frac{T}{\alpha}$. Let $\tilde{v}_{cn}$ and $\tilde{v}_n$ be the exponential Fourier series coefficients of $v_c(t)$ and $v(t)$, respectively.

$$\begin{aligned}\tilde{v}_{cn} &= \frac{\alpha}{T} \int_{-\frac{T}{2\alpha}}^{\frac{T}{2\alpha}} v_c(t) e^{j\frac{2\pi}{T}nt} dt = \frac{\alpha}{T} \int_{-\frac{T}{2\alpha}}^{\frac{T}{2\alpha}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} dt \\ &= \frac{\alpha}{T} \int_{-\frac{T}{2\alpha}}^{\frac{T}{2\alpha}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} \frac{1}{\alpha} d(\alpha t) \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} d(\alpha t) = \tilde{v}_n.\end{aligned}$$

Therefore, *compression or expansion of a periodic waveform in time does not change its Fourier series coefficients. But fundamental frequency and harmonic frequency values will change. This property is called Time-scaling property of Fourier series.*

We note that *time-scaling* is not the same as a simple change in T alone. Let $v(t)$, be a periodic train of rectangular pulses of unit amplitude and 0.1 s width repeating with a period of 1 s. Then, $v(0.5t)$ is a periodic train of rectangular pulses of unit amplitude and 0.2 s repeating with a period of 2 s. Its Fourier series coefficients will be the same as that of $v(t)$; but its fundamental frequency will be π rad/s, whereas that of $v(t)$ will be 2π rad/s.

However, consider another periodic train of rectangular pulses of unit height and 0.1 s width repeating with a period of 2 s. This is not the same as $v(0.5t)$. Its Fourier series coefficients will be different from that of $v(t)$. In fact, all Fourier series coefficients will get multiplied by 1/2.

Time-scaling
property of Fourier
series.

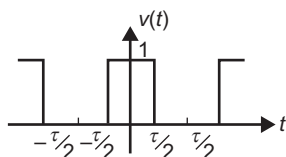


Fig. 13.6-6 Waveform for Example 13.6-8

EXAMPLE: 13.6-8

Find the Fourier series of the periodic rectangular pulse train shown in Fig. 13.6-6.

SOLUTION

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-jn\omega_0 t} dt = \frac{-1}{jn\omega_0 T} \left[e^{-jn\omega_0 \tau/2} - e^{jn\omega_0 \tau/2} \right]$$

$$e^{-jn\omega_0 \tau/2} - e^{jn\omega_0 \tau/2} = -2j \sin \frac{n\omega_0 \tau}{2} \text{ by Euler's Formula.}$$

$$\therefore \tilde{v}_n = \frac{\tau}{T} \frac{2 \sin \frac{n\omega_0 \tau}{2}}{n\omega_0 \tau} = \frac{\tau}{T} \frac{\sin \frac{n\omega_0 \tau}{2}}{n\omega_0 \tau / 2} = \left(\frac{\tau}{T} \right) \left(\frac{\sin x}{x} \right), \text{ where } x = \frac{n\omega_0 \tau}{2}.$$

The DC content of the waveform is $\frac{\tau}{T}$. The Fourier series contains only cosine terms since exponential Fourier series coefficients are real.

EXAMPLE: 13.6-9

Figure 13.6-7 shows the output of an absolute value circuit when the input is a sinusoidal wave. This is the shape of output voltage of a full-bridge diode rectifier routinely used in AC–DC conversion applications. We wish to obtain the Fourier series for this wave-shape. We bring out an important property of Fourier series first and use that property to obtain the required Fourier series in this example.

SOLUTION

The waveform $v(t)$ is visualised first as the product of two waveforms as in Fig. 13.6-8.

The product of $v_2(t)$ which is a unit amplitude square wave with $v_1(t)$ which is a pure sine wave results in $v(t)$, the full-wave rectified waveform.

That raises the question – what are the exponential Fourier series coefficients of a product of two waveforms with same period in terms of exponential Fourier series coefficients of the constituent waveforms?

$$v_1(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} e^{jn\omega_0 t} \text{ and } v_2(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_{2n} e^{jn\omega_0 t}$$

$$v(t) = v_1(t)v_2(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_k e^{jk\omega_0 t}$$

Consider k^{th} component in the exponential Fourier series of $v(t)$. The waveform contributed to $v(t)$ by this component is $\tilde{v}_k e^{jk\omega_0 t}$. Since $v(t)$ is the product of $v_1(t)$ and $v_2(t)$, this contribution can come up in $v(t)$ due to products of n^{th} contribution $\tilde{v}_{1n} e^{jn\omega_0 t}$ in $v_1(t)$ and $(k-n)^{\text{th}}$ contribution $\tilde{v}_{2(k-n)} e^{j(k-n)\omega_0 t}$ with n varying from $-\infty$ to $+\infty$.

$$\therefore \tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}.$$

This is the so-called *Multiplication in Time* property of Fourier series.

If $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period and $v(t) = v_1(t) \times v_2(t)$, then, the exponential Fourier series coefficients of $v(t)$ is given by $\tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$ for $-\infty < k < \infty$, where $\tilde{v}_{1n}$ and $\tilde{v}_{2n}$ are the exponential Fourier series coefficients of $v_1(t)$ and $v_2(t)$, respectively.

$v_1(t)$ is a sine wave in this example.

$$v_1(t) = \sin 2\pi t$$

$$= \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} \quad (\text{By Euler's Formula})$$

$$\therefore \tilde{v}_1 = \frac{1}{2j}, \tilde{v}_{-1} = -\frac{1}{2j} \text{ and } \tilde{v}_n = 0 \text{ for all other values of } n$$

$v_2(t)$ is a unit amplitude square wave. Its Fourier series was obtained in Example 13.6-5 as

$$\tilde{v}_{2n} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi n t}$$

$$\therefore \tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)} = -\frac{1}{2j} \frac{2}{j\pi(k+1)} + \frac{1}{2j} \frac{2}{j\pi(k-1)} \text{ for even } k$$

$$= \frac{-2}{\pi(k^2-1)} \text{ for even } k.$$

Since n takes only two values -1 and 1 , and square wave has only odd harmonics, $(k+1)$ and $(k-1)$ have to be odd for $\tilde{v}_k$ to be non-zero.

$$\therefore v(t) = \sum_{\substack{k=-\infty \\ \text{even } k}}^{\infty} \frac{-2}{\pi(k^2-1)} e^{j2\pi k t} = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{k=1 \\ \text{even } k}}^{\infty} \frac{1}{(k^2-1)} \cos 2\pi k t \text{ is the Fourier series of a full-}$$

wave rectified sinusoid of unit amplitude.

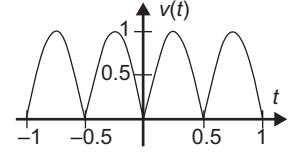


Fig. 13.6-7 Full-Wave Rectified Waveform in Example 13.6-9

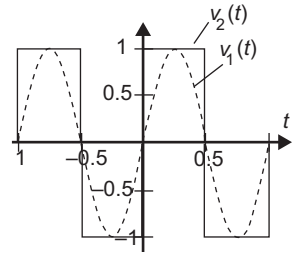


Fig. 13.6-8 Two Waveforms in a Product Results in $v(t)$ in Example 13.6-9

Multiplication in Time property of Fourier series.

The average value of rectified waveform is $\frac{2}{\pi}$ or $\frac{2V_m}{\pi}$ in general, where V_m is the peak value of sine wave undergoing rectification. It has only even cosine harmonics in it. What kind of symmetry is responsible for this? It is even function, and hence, only cosines are expected. We notice that the fundamental period of the waveform is 0.5 s and not 1 s. This 1 s period comes up only in the square waveform and sine waveform we used to form the product. This is why there is no component at odd harmonic order in the rectified output. If we treat it as a waveform with 0.5 s period, all odd and even harmonics of its fundamental frequency are present in it. Its fundamental frequency will be double the fundamental frequency of the sinusoid that underwent rectification.

13.7 DISCRETE MAGNITUDE AND PHASE SPECTRUM

Spectral plot for a time-domain waveform displays the Fourier series coefficients graphically against frequency. Since the frequencies involved in a Fourier series are discrete values (fundamental frequency and its multiples), a plot of Fourier series coefficients cannot be a continuous curve. Therefore, the spectral plot is called a *discrete spectrum*. The exponential Fourier series coefficients are complex in general and two plots will be needed – one for magnitude of the coefficients and the other for phase angle of the coefficients.

The information on coefficients is portrayed as a series of vertical lines located at harmonic frequencies. These lines will be equidistant and the length of the lines will be proportional to the magnitude of the coefficient in the case of magnitude spectrum and to the phase in the phase spectrum. The harmonic order n is also used in the abscissa instead of ω or f . The discrete spectral plots of the unit amplitude square wave we covered in Example 13.6-5 is shown in Fig. 13.7-1 for illustration. Its exponential Fourier series is $\tilde{v}_n = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt}$

The Fourier series coefficients of exponential Fourier series were plotted in the spectrum and that results in the so-called *two-sided spectrum*. It has been pointed out in earlier discussions that the two companion components at n and $-n$ always go together in exponential Fourier series. Two such components will always add up to yield a *real* sinusoid. They cannot be split.

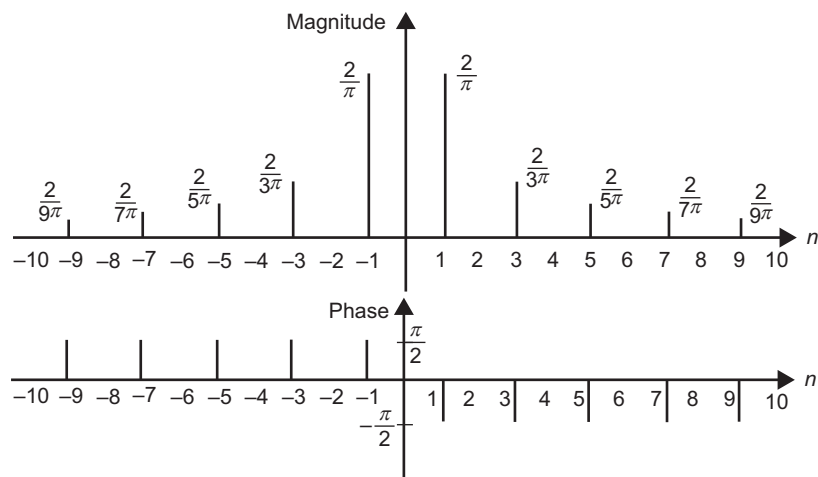


Fig. 13.7-1 Discrete Magnitude Spectrum and Phase Spectrum for a ± 1 Square Wave Against Harmonic Order

Discrete
spectrum defined.

They always stay
together

That the two components similarly placed on the left and the right of origin in a two-sided spectrum should be viewed as an integral unit rather than as two separate components is to be kept in mind especially when interpreting two-sided spectral plots drawn against ω . If we forget that, we will be tempted to ask the often repeated question – *what is the meaning of negative frequency?*

There is no negative *cyclic frequency* or negative *angular frequency*. There are only two complex exponential functions $-e^{j\omega t}$ and $e^{-j\omega t}$. These two always get scaled by complex conjugate numbers and enter into a sum. They never appear individually once the circuit problem has been solved. They always go together and produce either a $\sin\omega t$ or a $\cos\omega t$ or a mixture of the two. Whatever they produce at the end will have an angular frequency of ω rad/s and a cyclic frequency of $\omega/2\pi$ Hz.

Both of them are complex exponential functions of time. Hence, they have real and imaginary parts. Both, real and imaginary, are sinusoids. Those sinusoids have angular frequency of ω rad/s and cyclic frequency of $\omega/2\pi$ Hz whether they come from $e^{j\omega t}$ or from $e^{-j\omega t}$. Therefore, *there is no negative radian frequency or cyclic frequency*.

However, we want to represent the magnitudes of scaling factors of $e^{j\omega t}$ and $e^{-j\omega t}$ and phases of scaling factors separately in a spectral plot. Therefore, as a part of notation for presenting information efficiently, we decide to extend the ω -axis to the left and put the data on scaling factor of $e^{-j\omega t}$ there. That does not make a value on the left side of ω -axis a *negative frequency*.

We note that the magnitude spectrum of a real $v(t)$ has to be necessarily *even* on ω and its phase spectrum has to be necessarily *odd* on ω . (Why?)

Negative frequency?

No linear physical electrical circuit can ever do any processing on $e^{j\omega t}$ without carrying out the same processing on $e^{-j\omega t}$.

EXAMPLE: 13.7-1

Some desktop off-line Uninterruptible Power Supply (UPS) units used for supplying single PC units deliver the waveform shown in Fig. 13.7-2 instead of a sine wave. (a) Find α if the third harmonic content is to become zero. (b) With this value of α , find V such that the r.m.s voltage is 220 V. (c) Plot the magnitude and phase spectra with this value of α and V . (d) The purity of a sine wave is measured in terms of a quantity called 'Total Harmonic Distortion (THD)'. It is usually quoted in percentage and is defined as below.

$$\text{THD} = \frac{\sqrt{\sum_{n=2}^{\infty} |\tilde{V}_n|^2}}{|\tilde{V}_1|} \times 100\%, \text{ where } \tilde{V}_n \text{ is the exponential Fourier series coefficient.}$$

The r.m.s value of all the harmonic components together is expressed as a percentage of r.m.s value of fundamental component in the THD measure. Amplitudes may be used instead of r.m.s values since it is a ratio. Calculate the THD of the waveform in this example.

SOLUTION

The trigonometric Fourier series of the waveform with $V = 1$ and $T = 1$ is determined first. It is an *odd half-wave symmetric* waveform. Its trigonometric Fourier series will contain only odd sine harmonics. With $V = 1$ and $T = 1$,

$$\therefore v(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \text{ and } b_n = \frac{4}{T} \int_0^{T/2} v(t) \sin n\omega_0 t \, dt.$$

$$\begin{aligned} b_n &= 4 \int_{\alpha/2}^{(1-\alpha)/2} \sin 2\pi n t \, dt \\ &= \frac{4}{2\pi n} [\cos n\pi\alpha - \cos n\pi(1-\alpha)] \\ &= \frac{4}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi(1-2\alpha)}{2}. \end{aligned}$$

The trigonometric Fourier series coefficients go to zero for even n as expected. Further, the Fourier series of this waveform approach that of a unit amplitude square wave as $\alpha > 0$, as expected.

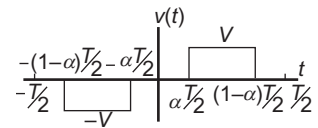


Fig. 13.7-2 Waveform for Example 13.7-1

(a) If the third harmonic content is to be zero, $\sin \frac{3\pi(1-2\alpha)}{2}$ must become zero.

Therefore, $\alpha = 1/6$. With this value of α , the waveform will be zero for one-third of a half-cycle.

(b) Therefore, r.m.s value = $\sqrt{\frac{2}{3}}V = 0.8165V$. If this is to be 220 V, V must be $\approx 270V$.

(c) The exponential Fourier series of $v(t)$ can be constructed from trigonometric Fourier series by noting that coefficients of sine terms will be twice the negative of imaginary part of exponential Fourier series coefficients. Therefore, with $V = 1$, $T = 1$ and $\alpha = 1/6$,

$$v(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} -\frac{j2}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi}{3} e^{j2\pi n t} \quad (13.7-1)$$

The two-sided spectrum is plotted in Fig. 13.7-3 with the scaling factor of 270 V incorporated.

(d) Refer to Eqn. 13.7-1. $\sin \frac{n\pi}{2}$ has a magnitude of 1 for all odd n . $\sin \frac{n\pi}{3}$ has a magnitude of 0 for all odd multiples of 3 and 0.866 for all other odd n including $n = 1$. $\frac{8}{\pi}$ is a common factor.

$$\text{Therefore, THD} = \sqrt{\frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \dots} \times 100 \approx 28.5\%$$

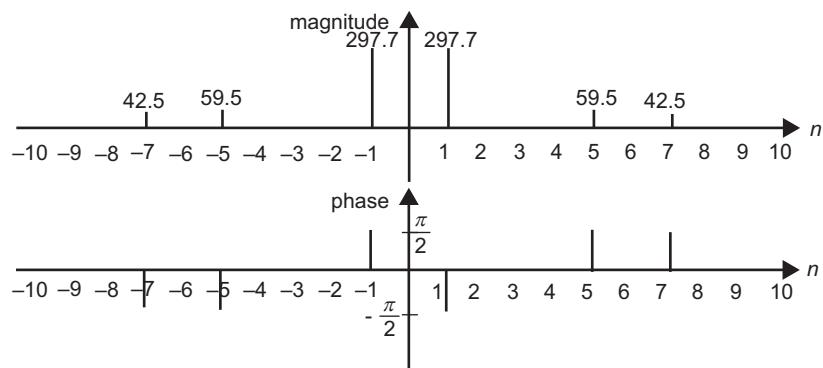


Fig. 13.7-3 Spectral Plots for $v(t)$ in Example 13.7-1

Harmonic amplitude varies with harmonic order in general. Information on rate of variation is needed in analysis and design of circuits dealing with switched waveforms.

13.8 RATE OF DECAY OF HARMONIC AMPLITUDE

Fourier series is an infinite series. It requires infinite number of sinusoids with frequencies ranging from fundamental frequency to infinitely large frequency to synthesise a non-sinusoidal periodic waveform in general. There may be special cases where the Fourier series terminates at some finite harmonic order but they are only special cases.

This indicates that we have to find the AC steady-state response of the circuit to each and every component in Fourier series of input and sum them up to get the periodic steady-state response of the circuit. That calls for infinite computation – we will not get done with it. Hence, the issue of rate of decay of harmonic amplitudes is of practical significance in deciding how many terms from the Fourier series should we carry in any analysis problem.

Circuits carrying high frequency voltages and currents cause electromagnetic interference (EMI) in them as well as in the neighbouring circuits. This EMI takes the form of induced voltages and currents due to electromagnetic coupling and electrostatic coupling between circuits as well as due to electromagnetic radiation. Every circuit carrying time-varying voltage and current acts as a transmitting antenna and receiving antenna simultaneously. The induced voltages and currents can lead to malfunction in circuits if not actual damage.

Electromagnetic interference happens at all frequencies. However, the induced voltages are usually of negligible magnitude at low frequencies. Therefore, a designer will often be forced to take out high frequency content from circuit waveforms for reducing destructive electromagnetic interference. An appreciation of how the Fourier series coefficients vary with harmonic order and the factors governing such variation helps him in such a task.

We noted in Example 13.6-1 and in subsequent examples in Sect. 13.6 that periodic impulse train waveforms of different type will have Fourier series with coefficients which *do not vary* with harmonic order n . The amplitude of harmonics is independent of harmonic order in such waveforms.

Integration brings a factor of $1/n$ in the Fourier series coefficients. Integration of an impulse train results in a waveform that will have step discontinuities in one period. Then, *integration in time* property of Fourier series shows us that such waveforms which contain step discontinuities will have Fourier series coefficients $\propto 1/n$, where n is the harmonic order.

Integrating a square or a rectangular pulse waveform results in a piece-wise linear waveform. Such waveforms are continuous; but their first derivative will have step discontinuities. Their second derivative will contain impulse train along with other possible components. Thus, we see that periodic waveforms that have impulses in their second derivative will have Fourier series with at least some coefficients $\propto 1/n^2$. There may be terms involving $1/n^3$, $1/n^4$ etc., in the Fourier series of such a waveform; but the terms involving $1/n^2$ are the ones which decide how many terms in the Fourier series are to be included in a circuit analysis context or EMI context.

By extending this reasoning we may state qualitatively that if a periodic waveform $v(t)$ requires m successive differentiation operations before impulses make their appearance, then, the harmonic amplitude in its Fourier series will decrease with $1/n^m$ at the least.

If a periodic waveform $v(t)$ requires m successive differentiation operations before impulses make their appearance, then, the harmonic amplitude in its Fourier series will decrease with $1/n^m$ at the least.

EXAMPLE: 13.8-1

Some desktop UPS units supplying single PC units deliver the waveform $v(t)$ as shown in Fig. 13.8-1. The trapezoidal shape is expected to reduce the THD of the wave compared to the square wave in Example 13.7-1. (a) Obtain an expression for r.m.s value of this voltage in terms of V and α . (b) Find the Fourier series coefficients for $v(t)$. (c) Find V and α such that the third harmonic content is zero and the r.m.s value is 220 V. (d) With these values for V and α find the THD of $v(t)$.

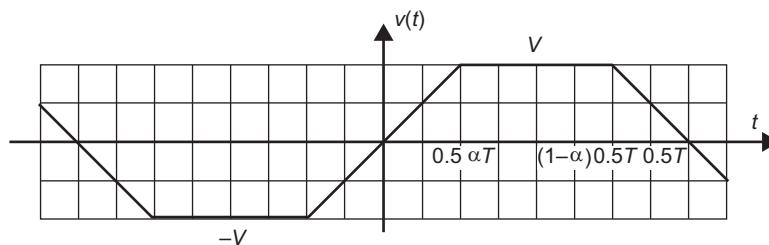
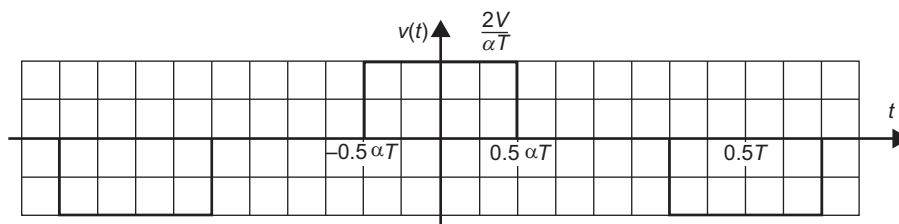


Fig. 13.8-1 Waveform for Example 13.8-1

Fig. 13.8-2 Waveform of Derivative of $v(t)$ in Example 13.8-1**SOLUTION**

The waveform $v(t)$ exhibits odd symmetry and half-wave symmetry. Its Fourier series will contain only odd harmonics of sine format.

$$(a) \text{ r.m.s value of } v(t) = \sqrt{\frac{4}{T} \int_0^{0.5\alpha T} \left(\frac{2Vt}{\alpha T}\right)^2 dt + V^2(0.25T - 0.5\alpha T)} = V\sqrt{1 - \frac{4}{3}\alpha}.$$

(b) Fourier series coefficients can be found by a straight application of the analysis equation of Fourier series Eqn. 13.2-2. However, a more elegant method will be to think of a waveform which will produce $v(t)$ on integration, find its Fourier series coefficients and obtain the Fourier series coefficients of $v(t)$ by dividing those coefficients by $j n \omega_0$. The waveform shown in Fig. 13.8-2 is the required one.

Exponential Fourier series coefficients for this waveform is found by splitting the waveform into two – the upper half-cycles alone will constitute a periodic rectangular pulse train with period T and the lower half-cycles alone will constitute the same rectangular pulse train delayed by $0.5T$ and negated. Example 13.6-8 may be referred for Fourier series of such a periodic rectangular pulse train.

$$\tilde{v}'_n = \frac{2V}{\alpha T} \frac{1}{j n \omega_0 T} (e^{j0.5n\omega_0\alpha T} - e^{-j0.5n\omega_0\alpha T}) - \frac{2V}{\alpha T} \frac{e^{-j0.5n\omega_0 T}}{j n \omega_0 T} (e^{j0.5n\omega_0\alpha T} - e^{-j0.5n\omega_0\alpha T})$$

$$e^{-j0.5n\omega_0 T} = e^{-jn\pi} = 1 \text{ for even } n \text{ and } -1 \text{ for odd } n.$$

$$\begin{aligned} \therefore \tilde{v}'_n &= \begin{cases} 0 & \text{for even } n \\ \frac{2V}{\alpha T} \frac{2}{j n \omega_0 T} (e^{j0.5n\omega_0\alpha T} - e^{-j0.5n\omega_0\alpha T}) & \text{for odd } n \end{cases} \\ &= \begin{cases} 0 & \text{for even } n \\ \frac{4V}{T} \frac{\sin \pi n \alpha}{\pi n \alpha} & \text{for odd } n \end{cases} \end{aligned}$$

Coefficients of exponential Fourier series of $v(t)$ is obtained by dividing these coefficients by $j n \omega_0$.

$$\therefore \tilde{v}_n = \begin{cases} 0 & \text{for even } n \\ -j \frac{2V}{\pi n} \frac{\sin \pi n \alpha}{\pi n \alpha} & \text{for odd } n \end{cases} \quad (13.8-1)$$

The rate of decay of harmonic amplitude must be $\propto \frac{1}{n^2}$ and it is seen to be so.

We notice that the waveform becomes a square waveform when $\alpha = 0$. The Fourier series coefficients given in Eqn. 13.8-1 become the same as that of a square wave, i.e., $\frac{2V}{\pi n}$

for odd n . When $\alpha = 0.5$, $v(t)$ becomes a triangle waveform with $\frac{4V}{\pi^2 n^2}$ for odd n as magnitude of exponential Fourier series coefficient.

(c) Third harmonic content will go to zero when $\sin 3\pi\alpha = 0$. Therefore, $\alpha = 1/3$ is the required value. The r.m.s value with this value of α is 0.7453V and for 220 V rms value V must be ≈ 295 V.

(d) The THD is evaluated as

$$\text{THD} = 100 \times \sqrt{\sum_{\substack{n=3 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} \left(\frac{\sin n\pi/3}{n\pi/3} \right)^2} \div \frac{\sin \pi/3}{\pi/3} \approx 4.6\%.$$

The faster decay of harmonic amplitude with harmonic order due to the slanting portion of this trapezoidal waveform has yielded a much better approximation to a pure sine wave than the waveform in Example 13.7-1. The next step to improve THD further will be to replace the flat portion of the waveform by another slanting portion with lesser slope than in the first section.

13.9 ANALYSIS OF PERIODIC STEADY-STATE USING FOURIER SERIES

The first step of analysis is the determination of Fourier series of the periodic input waveform in exponential format or trigonometric format. The second step is the determination of frequency response function connecting the required output variable to the input variable. The third step is the determination of steady-state response for each term in the Fourier series of input waveform. The fourth step is to combine all these steady-state response components and to obtain the instantaneous waveform of output after deciding the number of terms to be retained in the truncated Fourier series of the output waveform.

This procedure is illustrated through examples that follow.

EXAMPLE: 13.9-1

A periodic ramp voltage waveform shown in Fig. 13.9-1 is applied to an RC low-pass circuit with time constant of 0.1 s. The period of the input waveform is 1 s. Find and plot the steady-state output voltage across the capacitor.

SOLUTION

Step-1: Find the Fourier series of $v(t)$

$$\tilde{v}_n = \frac{1}{T} \int_0^T v(t) e^{-jn\omega_0 t} dt$$

($\because$ Integration can be performed over any interval that is one period wide)

$$\begin{aligned} \therefore \tilde{v}_n &= \int_0^1 t e^{-j2\pi n t} dt \quad (\because T = 1, \omega_0 = 2\pi) \\ &= -\frac{1}{(-j2\pi n)^2} e^{-j2\pi n t} \Big|_0^1 + \frac{1}{-j2\pi n} t e^{-j2\pi n t} \Big|_0^1 \\ &= 0 + \frac{1}{-j2\pi n} = \frac{j}{2\pi n}. \end{aligned}$$

This expression is valid only for $n \neq 0$

$$\tilde{v}_0 = \frac{1}{T} \int_0^T v(t) dt = \int_0^1 t dt = \frac{1}{2}.$$

$$\therefore a_0 = 0.5, a_n = 0 \text{ for } n \neq 0, b_n = -\frac{1}{\pi n}$$

$$v(t) = 0.5 - \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin 2n\pi t.$$

Step-2: Find the frequency response function

$$\frac{V_o(j\omega)}{V(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega\tau} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \angle -\tan^{-1} \omega\tau$$

Substituting $\tau = 0.1$ s

$$\frac{V_o(j\omega)}{V(j\omega)} = \frac{1}{\sqrt{1 + 0.01\omega^2}} \angle -\tan^{-1} 0.1\omega.$$

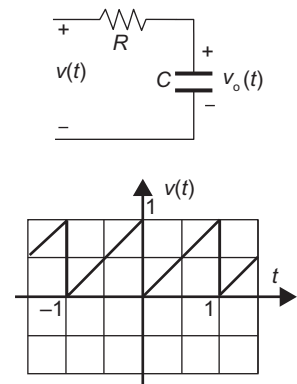


Fig. 13.9-1 The Input Waveform and the Circuit for Example 13.9-1

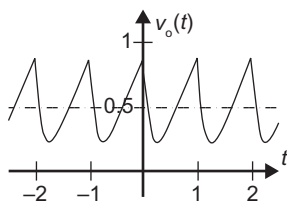


Fig. 13.9-2 Output Waveform $v_o(t)$ for Example 13.9-1

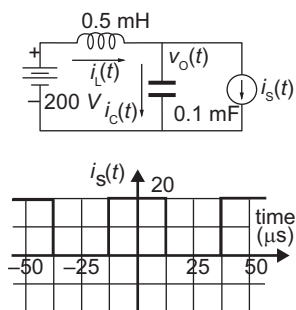


Fig. 13.9-3 Circuit and Waveform for Example 13.9-2

ESR of Capacitor

However, in practice, the 100 μF Electrolytic Capacitor used will have a series resistance along with its capacitance. It is called Effective Series Resistance (ESR). It comes up due to the lead resistance and foil resistance.

A 100 μF capacitor is likely to have an ESR of 0.3 – 0.8 Ω depending on the grade and quality of the capacitor chosen. Note that the impedance of 100 μF capacitor at 20 kHz is $-j0.08 \Omega$ and that of 0.5 mH inductor is $j62.8 \Omega$. Obviously, the ESR of capacitor rules the situation. Current division will take place between the impedance of inductor and ESR of capacitor and the filter performance will not be as good as we calculated.

Step-3: Find the steady-state response to input components

Output corresponding to DC content in input = 0.5 (since DC gain = 1)

Output phasor for n th harmonic component

$$= -\frac{1}{\pi n} \frac{1}{\sqrt{1 + 0.01 \times (2\pi n)^2}} \sin(2n\pi t - \tan^{-1} 0.2\pi n)$$

$$= -\frac{1}{\pi n} \frac{1}{\sqrt{1 + 0.395n^2}} \sin(2n\pi t - \tan^{-1} 0.63n).$$

Step-4: Form the total output and decide the number of terms needed in the truncated series.

$$v_o(t) = 0.5 - \sum_{n=1}^{\infty} \frac{1}{\pi n} \frac{1}{\sqrt{1 + 0.395n^2}} \sin(2n\pi t - \tan^{-1} 0.63n).$$

The DC content in the output is 0.5. We can choose to ignore all those harmonic terms which have amplitude less than 1% of this value.

$$\frac{1}{\pi n} \frac{1}{\sqrt{1 + 0.395n^2}} = 0.005$$

Ignore 1 under square root for an approximate solution.

$$\Rightarrow n = 10$$

$$\text{Then, } v_o(t) \approx 0.5 - \sum_{n=1}^{10} \frac{1}{\pi n} \frac{1}{\sqrt{1 + 0.395n^2}} \sin(2n\pi t - \tan^{-1} 0.63n).$$

This output is plotted in Fig. 13.9-2. More terms will have to be included to remove the fine oscillations that appear in the waveform. Note that the circuit is not able to follow the sharp fall that takes place in the input at the end of every period. Any non-zero time constant is too slow to follow an instantaneous change in input. $v_o(t)$ can almost touch 1 if time constant is reduced further, but it will not touch zero for any time constant.

EXAMPLE: 13.9-2

The current source in the circuit in Fig. 13.9-3 represents a power electronic load (called a DC-DC chopper – it is used to step down DC voltages) that is drawing a pulsed current at 20 kHz. The current drawn by the load under a particular operating condition is shown as $i_s(t)$ in Fig. 13.9-3. The LC filter is expected to hold the voltage presented to the load at a constant level and to smooth the current in the battery. Pulsed current has adverse impact on battery life. Solve for $i_L(t)$, $i_C(t)$ and $v_o(t)$ under steady-state.

SOLUTION

The Fourier series coefficients of a symmetric square wave was found out in Example

Example 13.6-5. They are $\frac{4}{\pi n}$ in the trigonometric Fourier series. The waveform of $i_s(t)$ here

contains a DC component of 10 A. Hence, $[i_s(t) - 10]$ will be a symmetric square wave that has even symmetry. The square wave in Example 13.6-5 had odd symmetry and had sine terms in its Fourier series. Here, it will be cosine terms.

$$\therefore i_s(t) = 10 + \frac{40}{\pi} \sum_{n=1, \text{ odd}}^{\infty} \frac{1}{n} \cos 40 \times 10^3 n\pi t \text{ A.}$$

Inductor behaves as an open-circuit and capacitor behaves as short-circuit under DC steady-state conditions. Hence, the DC component in $v_o(t) = 200 \text{ V}$, in $i_L(t) = 10 \text{ A}$ and in $i_C(t) = 0 \text{ A}$.

The 200 V battery is replaced by a short-circuit when the circuit solution for the AC components of $i_s(t)$ is attempted. This results in a parallel LC circuit with current excitation. Current division principle in parallel impedances gives the frequency response of inductor current as

$$\frac{I_L(j\omega)}{I_s(j\omega)} = \frac{1/j\omega C}{j\omega L + 1/j\omega C} = \frac{1}{1 - \omega^2 LC} = \frac{1}{1 - 789.6n^2} \approx -\frac{1.267 \times 10^{-3}}{n^2}.$$

Thus, the amplitude of inductor current at 20 kHz will be $1.267 \times 10^{-3} \times 40/\pi = 0.016$ A. This is 0.16% of the DC current in it. We need not calculate the contribution due to other current harmonics. They will be very negligible due to the n^2 factor in the denominator of the current division ratio and the $1/n$ factor in the Fourier series of $i_s(t)$. Therefore, the inductor current is practically DC. Similarly, it can be shown that $v_o(t)$ is practically DC.

Increasing the capacitance value decreases the Effective Series Resistance (ESR). Thus, the size of capacitor chosen in these kinds of applications is based on ESR considerations rather than the kind of calculations we carried out without taking the ESR into account (see side-box).

EXAMPLE: 13.9-3

L_s in the circuit in Fig. 13.9-4 represents the Thevenin's impedance of the power system at a point where a non-linear load is connected. The current drawn by the load is modelled as a current source $i(t)$. The waveform of the load current is also given in the same figure. L_f and C_f are the harmonic filter components. They are tuned to 250 Hz, i.e., it is a fifth harmonic current filter. R_f is the series resistance of the inductor. The component values are $L_s = 0.3$ mH, $L_f = 1.84$ mH, $C_f = 0.22$ mF and $R_f = 0.072 \Omega$. C_p is a power factor correction capacitor that is employed to draw leading current in order to cancel the lagging current drawn by another load which is not active in the current context. Its value is 0.47 mF. The source voltage is a 320 V amplitude, zero phase sinusoid at 50 Hz.

Solve for the three currents and the output voltage under steady-state conditions and plot them.

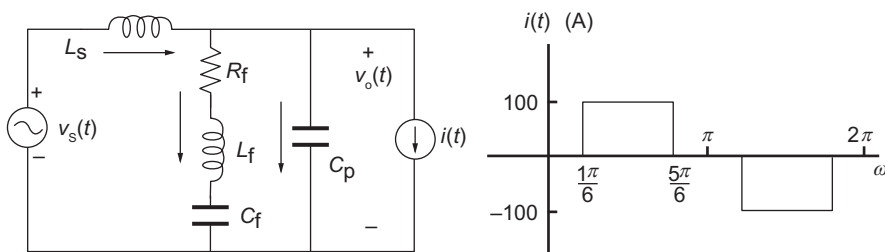


Fig. 13.9-4 Circuit Diagram for the Harmonic Filtering Context in Example 13.9-3

SOLUTION

Step-1: Fourier series of $i(t)$

Example 13.7-1 refers to the Fourier series of this waveform. It is

$$i(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{j200}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi}{3} e^{j\omega_0 n t} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \left(\frac{400}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi}{3} \right) \sin n\omega_0 t.$$

All the harmonics of 3rd order and its multiples will have amplitude of zero.

Harmonics in Power Systems and Harmonic Filtering

A linear electrical element draws only sinusoidal current from a pure sinusoidal source. However, a non-linear load can draw a non-sinusoidal current that is rich in harmonics from a voltage source of sinusoidal nature. Various industrial electronic equipment like AC-DC converters, inverter fed induction motor drives, thyristor controlled industrial heaters, converter fed DC motor drives, uninterruptible power supplies, fluorescent lamps etc., belong to this category.

Distorted voltage results in malfunctioning of electronic equipment, inefficient operation of motors, increased losses everywhere in the system etc. It can also cause damage to the power capacitors used in a power system for various purposes.

Harmonic current filters are used to prevent the harmonic currents generated by a load from getting into the power system. Both passive filters and filters using power semiconductor devices are in use. A passive harmonic filter is essentially a tuned LC series combination connected in parallel to the offending load. The LC circuit is tuned to the harmonic frequency to be eliminated. The tuned combination will have zero impedance at that frequency and the currents at that frequency will flow into the filter instead of getting into the power system. Obviously more than one such combination will have to be used in parallel if elimination of multiple harmonics is desired.

Example 13.9-3 illustrates Fourier series analysis technique in this harmonic-filtering context.

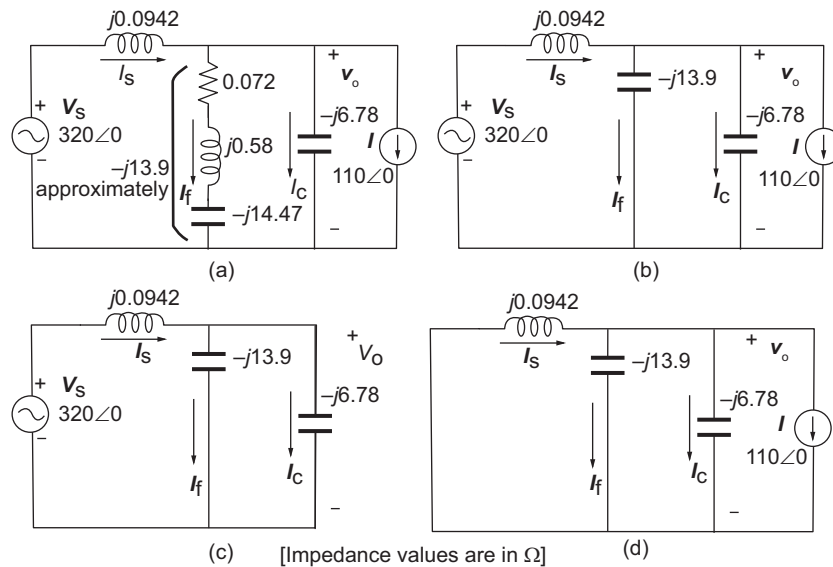


Fig. 13.9-5 (a) and (b) Phasor Equivalent Circuit (c) and (d) Circuits for Applying Superposition Principle

Step-2: Solve for steady-state response for each harmonic

The phasor equivalent circuit for fundamental component is shown in circuit in Fig. 13.9-5(a). The value of ω_0 is ≈ 314 rad/s.

Peak values of voltage and current sources are marked instead of the r.m.s value. It is a two-source problem and the solution is obtained by applying superposition principle.

Solution for (c)

Total effective impedance = $j0.0942 + (-j13.9) // (-j6.78) = j0.0942 - j4.56 = -j4.463 \Omega$

$\therefore I_s = 320 \angle 0^\circ \div -j4.463 = 71.7 \angle 90^\circ \text{ A}$

$V_o = 320 \angle 0^\circ \times (-j4.56 \div -j4.463) = 327 \angle 0^\circ \text{ V}$

$I_f = 71.7 \angle 90^\circ \times (-j6.78 \div -j20.68) = 23.5 \angle 90^\circ \text{ A}$

$I_c = 71.7 \angle 90^\circ \times (-j13.9 \div -j20.68) = 48.2 \angle 90^\circ \text{ A}$

Solution for (d)

Total effective admittance = $-j10.4 \text{ S}$

Total effective impedance = $j0.096$

$\therefore V_o = -110 \angle 0^\circ \times j0.096 = -10.6 \angle 90^\circ \text{ V}$

$I_s = 10.6 \angle 90^\circ \div j0.094 = 112.8 \angle 0^\circ \text{ A}$

$I_c = 10.6 \angle 90^\circ \div (-j6.78) = 1.54 \angle 180^\circ \text{ A}$

$I_f = 10.6 \angle 90^\circ \div (-j13.9) = 0.74 \angle 180^\circ \text{ A}$

Solution for (c) + (d)

$V_o = 327 \angle 0^\circ - 10.6 \angle 90^\circ \text{ V} = 327.2 \angle -1.9^\circ \text{ V}$

$I_s = 71.7 \angle 90^\circ + 112.8 \angle 0^\circ \text{ A} = 133.66 \angle 32.4^\circ \text{ A}$

$I_f = 23.5 \angle 90^\circ + 0.74 \angle 180^\circ \text{ A} = 23.5 \angle 91.8^\circ \text{ A}$

$I_c = 48.2 \angle 90^\circ + 1.54 \angle 180^\circ \text{ A} = 48.2 \angle 91.8^\circ \text{ A}$

Time-domain solution

$v_o(t) = 327.2 \sin(314t - 1.9^\circ) \text{ V}$

$i_s(t) = 133.7 \sin(314t + 32.4^\circ) \text{ A}$

$i_f(t) = 23.5 \sin(314t + 91.8^\circ) \text{ A}$

$i_c(t) = 48.2 \sin(314t + 91.8^\circ) \text{ A}$

The equivalent circuits for solving the harmonic currents are shown in Fig. 13.9-6.

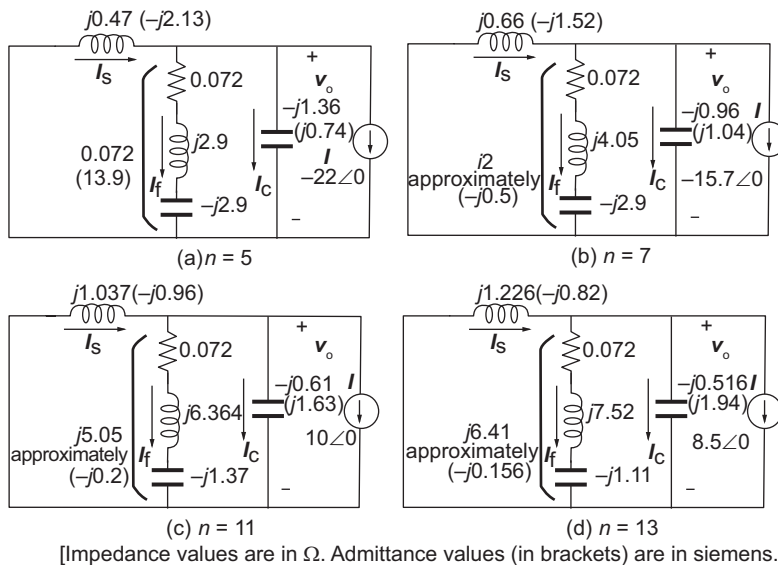


Fig. 13.9-6 Circuit Equivalents for Current Harmonics

Only the first four non-zero harmonics in load current are considered.

The circuits can be solved by current division principle – current divides among parallel elements in proportion to their admittance values. Solution is illustrated in the case of $n = 11$.

$$\begin{aligned} V_o &= 10\angle 0 \div (j1.63 - j0.2 - j0.96) = 21.3\angle -90^\circ \text{ V} \\ I_s &= 10\angle 0 \times -j0.96 \div (j1.63 - j0.2 - j0.96) = -20.4\angle 0^\circ \text{ A} \\ I_L &= 10\angle 0 \times -j0.2 \div (j1.63 - j0.2 - j0.96) = -4.26\angle 0^\circ \text{ A} \\ I_C &= 10\angle 0 \times j1.63 \div (j1.63 - j0.2 - j0.96) = 34.7\angle 0^\circ \text{ A} \end{aligned}$$

Observe that the load current had only 10 A at 550 Hz (11th harmonic). However, the source current has 21.3 A and the power factor correction capacitor C_p has 34.7 A in it. If C_p were not there, the source current would have been close to 10 A and the 11th harmonic component in voltage would have been close to 10 V instead of its present value of 21.3 V. This amplification of 11th harmonic illustrates a potential problem that can exist in harmonic context in power systems. The admittance of source inductance and the power factor capacitor cancel each other partially at 11th harmonic and this results in an increase in the impedance level presented to 11th harmonic. Higher impedance results in higher voltage and higher currents in individual parallel paths. There can be dangerous harmonic resonance if the parallel resonant frequency of the circuit coincides or comes near one of the harmonics of supply frequency. In this case, the resonant frequency is around 424 Hz (decided practically by the 0.3 mH inductor and 0.47 mF capacitor) and there can be large harmonic amplification if the load current contains 8th or 9th harmonics.

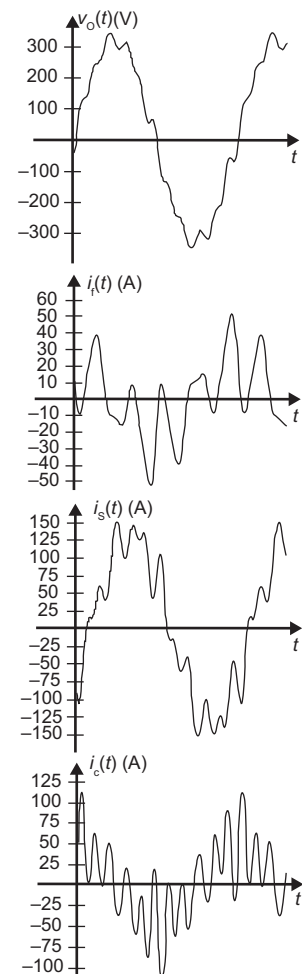
The time-domain solution is obtained by using $\omega = 11 \times 314 \text{ rad/s}$.

$$\begin{aligned} v_o(t) &= 21.3 \sin(11 \times 314t - 90^\circ) \text{ V} \\ i_s(t) &= -20.4 \sin(11 \times 314t) \text{ A} \\ i_L(t) &= -4.26 \sin(11 \times 314t) \text{ A} \\ i_C(t) &= 34.7 \sin(11 \times 314t) \text{ A} \end{aligned}$$

The equivalent circuits for other harmonics too can be solved similarly.

The time-domain solution for $n = 5$ is given below. We note that the tuned circuit diverts most of the 22 A 5th harmonic into it. The source current still contains 5th harmonic and this is due to the fact that the filter path has a resistance of 0.072 Ω .

$$\begin{aligned} v_o(t) &= -1.58 \sin(5 \times 314t + 5.7^\circ) \text{ V} \\ i_s(t) &= -3.35 \sin(5 \times 314t - 84.3^\circ) \text{ A} \\ i_L(t) &= -21.9 \sin(5 \times 314t + 5.7^\circ) \text{ A} \\ i_C(t) &= -1.17 \sin(5 \times 314t + 95.7^\circ) \text{ A} \end{aligned}$$

Fig. 13.9-7 Waveforms of $v_o(t)$, $i_s(t)$, $i_C(t)$ and $i_L(t)$ for Example 13.9-3

The time-domain solution for $n = 7$ is given below.

$$\begin{aligned}v_o(t) &= -16.1 \sin(7 \times 314t + 90^\circ) \text{ V} \\i_s(t) &= -24.4 \sin(7 \times 314t) \text{ A} \\i_i(t) &= -8.05 \sin(7 \times 314t) \text{ A} \\i_c(t) &= 16.7 \sin(7 \times 314t) \text{ A}\end{aligned}$$

The time-domain solution for $n = 13$ is given below.

$$\begin{aligned}v_o(t) &= 8.8 \sin(13 \times 314t - 90^\circ) \text{ V} \\i_s(t) &= -7.2 \sin(13 \times 314t) \text{ A} \\i_i(t) &= -1.38 \sin(13 \times 314t) \text{ A} \\i_c(t) &= 17 \sin(13 \times 314t) \text{ A}\end{aligned}$$

The total solution is obtained by combining all these solutions. We neglect the harmonics above 13^{th} in this example.

$$\begin{aligned}v_o(t) &= 327 \sin(\omega_0 t - 1.9^\circ) - 1.6 \sin(5\omega_0 t + 5.7^\circ) - 16.1 \sin(7\omega_0 t + 90^\circ) + 21.3 \sin(11\omega_0 t - 90^\circ) + 8.8 \sin(13\omega_0 t - 90^\circ) \text{ V} \\i_s(t) &= 133.7 \sin(\omega_0 t + 32.4^\circ) - 3.35 \sin(5\omega_0 t - 84.3^\circ) - 24.4 \sin(7\omega_0 t) - 20.4 \sin(11\omega_0 t) - 7.2 \sin(13\omega_0 t) \text{ A} \\i_i(t) &= 23.5 \sin(\omega_0 t + 91.8^\circ) - 21.9 \sin(5\omega_0 t + 5.7^\circ) - 8.05 \sin(7\omega_0 t) - 4.26 \sin(11\omega_0 t) - 1.38 \sin(13\omega_0 t) \text{ A} \\i_c(t) &= 48.2 \sin(\omega_0 t + 91.8^\circ) - 1.17 \sin(5\omega_0 t + 95.7^\circ) + 16.7 \sin(7\omega_0 t) + 34.7 \sin(11\omega_0 t) + 17 \sin(13\omega_0 t) \text{ A}.\end{aligned}$$

These waveforms are plotted in Fig. 13.9-7. The waveforms reveal that the presence of power factor correction capacitor has caused deterioration in the harmonic performance of the system and that the harmonic filter is not very effective in this context. Observe the high frequency currents in the capacitor.

13.10 NORMALISED POWER IN A PERIODIC WAVEFORM AND PARSEVAL'S THEOREM

Normalised power in a periodic signal defined.

Normalised power of a waveform is equal to zeroth coefficient in the exponential Fourier series of its squared version.

The concept of normalised power in a periodic waveform is often employed in Communication Engineering and allied areas as a measure of signal strength. It is defined as the *average power* that will be delivered to 1Ω resistance if the periodic waveform is thought of as a voltage waveform applied to that resistor. The averaging is done over any interval equal to the period of the waveform.

$$P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt. \quad (13.10-1)$$

If we think of $v_3(t) = [v(t)]^2$ as a new time-function, the term on the right side of Eqn. 13.10-1 can be identified as the DC component of $v_3(t)$. Hence, P_n must be equal to the exponential Fourier series coefficient of $v_3(t)$, $\tilde{v}_{3k}$ for $k = 0$.

We developed the *multiplication-in-time* property of Fourier series in Example 13.6-9. This property states that if $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period and $v_3(t) = v_1(t) \times v_2(t)$, then, the exponential Fourier series coefficients of $v_3(t)$ is given by $\tilde{v}_{3k} = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$ for $-\infty < k < \infty$, where $\tilde{v}_{1n}$ and $\tilde{v}_{2n}$ are the exponential Fourier series coefficients of $v_1(t)$ and $v_2(t)$, respectively. We use this property with $v_1(t) = v_2(t) = v(t)$ and evaluate the exponential Fourier series coefficient of $[v(t)]^2$ for $k = 0$ as $\tilde{v}_{30} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n}$.

Therefore,

$$P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2 = |\tilde{v}_0|^2 + 2 \sum_{n=1}^{\infty} |\tilde{v}_n|^2.$$

This is Parseval's Theorem on normalised power of periodic waveforms.

The trigonometric Fourier series for $v(t)$ is

$$v(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t,$$

where $a_0 = \tilde{v}_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt,$

$$a_n = \tilde{v}_n + \tilde{v}_{-n} = \tilde{v}_n + \tilde{v}_n^* = 2 \operatorname{Re}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \cos n\omega_0 t dt \text{ for } n = 1, 2, 3 \dots$$

$$a_n = -\tilde{v}_n + \tilde{v}_{-n} = -\tilde{v}_n + \tilde{v}_n^* = 2 \operatorname{Im}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \sin n\omega_0 t dt \text{ for } n = 1, 2, 3 \dots$$

Therefore, Parseval's Theorem can be expressed in terms of trigonometric Fourier

series coefficients as $P_n = a_0^2 + \sum_{n=1}^{\infty} \left(\frac{a_n^2 + b_n^2}{2} \right).$

The second form of trigonometric Fourier series is shown below.

$$\therefore v(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \phi_n)$$

where $c_0 = \tilde{v}_0, c_n = \sqrt{a_n^2 + b_n^2} = 2\tilde{v}_n \tilde{v}_n^* = 2|\tilde{v}_n|$

and $\phi_n = \tan^{-1} \frac{b_n}{a_n} = -\angle \text{ of } \tilde{v}_n \text{ for } n = 1, 2, 3 \dots$

$$\therefore P_n = c_0^2 + \sum_{n=1}^{\infty} \frac{c_n^2}{2}$$

The normalised power of a periodic waveform $v(t)$, P_n , is given by

$$\begin{aligned} P_n &= \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2 = |\tilde{v}_0|^2 + 2 \sum_{n=1}^{\infty} |\tilde{v}_n|^2 \\ &= a_0^2 + \sum_{n=1}^{\infty} \frac{(a_n^2 + b_n^2)}{2} \\ &= c_0^2 + \sum_{n=1}^{\infty} \frac{c_n^2}{2}. \end{aligned}$$

Parseval's Power Relation for a periodic waveform.

Though the *multiplication-in-time* property easily led us to Parseval's theorem, it does not help us to see the significance of this theorem. Neither does it tell us how this total normalised power is distributed among various frequency components. Hence, we use the trigonometric Fourier series $v(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \phi_n)$ for further appreciation of P_n .

Consider a simpler situation in which $v(t)$ contains just three components.

$$v(t) = c_0 + c_m \cos(m\omega_0 t - \phi_m) + c_k \cos(k\omega_0 t - \phi_k), \text{ } k \text{ and } m \text{ are integers.}$$

$$\begin{aligned} \therefore [v(t)]^2 &= c_0^2 + c_m^2 \cos^2(m\omega_0 t - \phi_m) + c_k^2 \cos^2(k\omega_0 t - \phi_k) \\ &\quad + 2c_0 c_m \cos(m\omega_0 t - \phi_m) + 2c_0 c_k \cos(k\omega_0 t - \phi_k) \\ &\quad + 2c_m c_k \cos(m\omega_0 t - \phi_m) \cos(k\omega_0 t - \phi_k). \end{aligned}$$

$$\begin{aligned} \therefore [v(t)]^2 &= c_0^2 + \frac{1}{2} c_m^2 + \frac{1}{2} c_k^2 + \frac{1}{2} c_m^2 \cos 2(m\omega_0 t - \phi_m) + \frac{1}{2} c_k^2 \cos 2(k\omega_0 t - \phi_k) \\ &\quad + 2c_0 c_m \cos(m\omega_0 t - \phi_m) + 2c_0 c_k \cos(k\omega_0 t - \phi_k) \\ &\quad + c_m c_k \cos[(m+k)\omega_0 t - (\phi_m + \phi_k)] + c_m c_k \cos[(m-k)\omega_0 t - (\phi_m - \phi_k)]. \end{aligned}$$

k and m are integers. Thus, if $k \neq m$, the frequencies $m\omega_0, k\omega_0, 2m\omega_0, 2k\omega_0, (m-k)\omega_0$ and $(m+k)\omega_0$ are integer multiples of ω_0 . Hence, all the cosine waves in $[v(t)]^2$ will have integer number of cycles in T s, where T is the period of $v(t)$. Therefore, their average over one T will be zero.

Parseval's relation interpreted in terms of DC component and r.m.s values of harmonic components.

R.M.S value of a non-sinusoidal periodic waveform.

Does Power Obey Superposition Principle?

Consider two arbitrary waveforms $v_1(t)$ and $v_2(t)$. Let average of $[v_1(t)]^2$ and $[v_2(t)]^2$ over some interval be a_1 and a_2 , respectively. Will the average of $[v_1(t) + v_2(t)]^2$ over the same interval be $a_1 + a_2$? The answer depends on whether the average of $2v_1(t)v_2(t)$ in that interval is zero or not. In general, it is not zero, and, average of $[v_1(t) + v_2(t)]^2$ is not the same as the sum of averages of $[v_1(t)]^2$ and $[v_2(t)]^2$. Thus, the power does not obey superposition principle in general.

But, if $v_1(t)$ and $v_2(t)$ are two sinusoids with different frequencies, and, if their frequencies are integer multiples of some basic frequency, the average of $2v_1(t)v_2(t)$ in an interval that is equal to the period corresponding to the basic frequency, is zero.

Hence, if $v(t)$ is a mixture of harmonically related sinusoids and DC, the normalised

$$\begin{aligned}\therefore P_n &= \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = \frac{1}{T} [c_0^2 + \frac{1}{2}c_m^2 + \frac{1}{2}c_k^2] \times T \\ &= c_0^2 + \frac{1}{2}c_m^2 + \frac{1}{2}c_k^2 = c_0^2 + \left(\frac{c_m}{\sqrt{2}}\right)^2 + \left(\frac{c_k}{\sqrt{2}}\right)^2\end{aligned}$$

Generalising the result for infinite term Fourier series,

$$\therefore P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = c_0^2 + \sum_{n=1}^{\infty} \left(\frac{c_n}{\sqrt{2}}\right)^2 \quad (13.10-2)$$

$$\text{i.e., } P_n = (\text{DC component})^2 + \sum_{n=1}^{\infty} (\text{r.m.s value of } n^{\text{th}} \text{ harmonic component})^2$$

Since the r.m.s value of a DC component is same as its value, we can express this as $P_n = \sum_{n=0}^{\infty} (\text{r.m.s value of } n^{\text{th}} \text{ harmonic component})^2$. (13.10-3)

Square root of this quantity will give the r.m.s value of $v(t)$ itself.

$$\text{R.M.S value of } v(t) = \sqrt{\sum_{n=1}^{\infty} (\text{r.m.s value of } n^{\text{th}} \text{ harmonic component})^2} \quad (13.10-4)$$

The normalised power of a particular harmonic component with amplitude c_n when acting alone will be $0.5 c_n^2$. Equation 13.10-2 shows that it contributes the same amount to the total power even when it is acting along with other harmonics.

Each harmonic component in the trigonometric Fourier series of a waveform contributes to normalised power (see side-box). We can ascribe the power contributed by a particular component to its frequency and plot the information against frequency or harmonic order as a line spectrum. This spectral plot is called the *discrete power spectrum*. However, it will be a single-sided spectrum since we derived it from trigonometric Fourier series. Spectral lines will be located at $0, \omega_0, 2\omega_0, 3\omega_0$, etc., and the length of the spectral line will be proportional to $0.5 c_n^2$. By Parseval's theorem,

$$P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2.$$

Therefore, we can draw the two-sided discrete power spectrum by plotting two lines of height proportional to $|\tilde{v}_n|^2$ at $n\omega_0$ and $-n\omega_0$. We had noted earlier that, two spectral components located at $\pm n\omega_0$ in the two-sided magnitude and phase spectra based on exponential Fourier series, have to be treated as an integral unit rather as individual components. Those two components always go together and form a *real* sinusoid. Similarly, it is understood that the power spectral components located at $\pm n\omega_0$ in the two-sided power spectrum always go together to make a total contribution of $2|\tilde{v}_n|^2$ to P_n .

EXAMPLE: 13.10-1

The output of a fully-controlled AC–DC converter operating from a sinusoidal voltage of 320 V peak and 50 Hz frequency is shown in Fig. 13.10-1. (i) Find and plot its discrete power spectrum. (ii) This waveform is applied to RL circuit with $L = 150 \text{ mH}$ and $R = 10 \Omega$. Find and plot the discrete power spectrum of voltage appearing across the resistor and find the power dissipation in it.

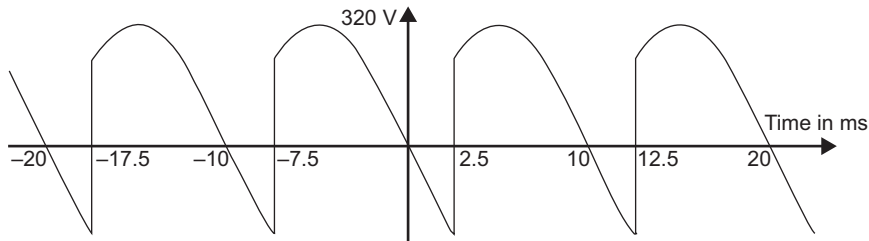


Fig. 13.10-1 Waveform for Example 13.10-1

SOLUTION

(i) This waveform, $v(t)$, can be expressed as the product of two waveforms – $v_1(t) = 320 \sin 100\pi t$ and $v_2(t)$ which is a symmetric ± 1 , 50 Hz square wave that is delayed by 2.5 ms.

$$\begin{aligned} v_1(t) &= 320 \sin 100\pi t \\ &= 320 \frac{e^{j100\pi t} - e^{-j100\pi t}}{2j} \quad (\text{By Euler's Formula}) \\ \therefore \tilde{v}_1 &= \frac{160}{j}, \tilde{v}_{1-1} = \frac{-160}{j} \text{ and } \tilde{v}_n = 0 \text{ for all other values of } n. \end{aligned}$$

$v_2(t)$ is a unit amplitude square wave. Fourier series of a unit amplitude square wave of 1s period that crosses zero at origin was obtained in Example 13.6-5 as $\sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi n t}$.

Using *time-shift property* of exponential Fourier series, we write the exponential Fourier series of the square wave in this example as $\tilde{v}_{2n} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{-j\frac{\pi}{4}n} e^{j100\pi n t}$. Using the

multiplication-in-time property of exponential Fourier series we write the exponential Fourier series coefficients of the waveform $v(t)$ as

$$\begin{aligned} \therefore \tilde{v}_k &= \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2[k-n]} = -\frac{160}{j} \frac{2e^{-j\frac{k+1}{4}\pi}}{j\pi(k+1)} + \frac{160}{j} \frac{2e^{-j\frac{k-1}{4}\pi}}{j\pi(k-1)} \text{ for even } k \\ &= \frac{-320}{\pi} \left(\frac{e^{-j\frac{k+1}{4}\pi}}{k+1} + \frac{e^{-j\frac{k-1}{4}\pi}}{k-1} \right) \text{ for even } k. \end{aligned}$$

The magnitude part of the quantity inside the brackets can be shown as $\frac{\sqrt{2(1+k^2)}}{k^2-1}$.

$$\therefore |\tilde{v}_n|^2 = 2 \times \left(\frac{320}{\pi} \right)^2 \frac{k^2+1}{(k^2-1)^2} \text{ for even } k.$$

The discrete power spectrum of $|\tilde{v}_n|^2$ is plotted against harmonic order n in Fig. 13.10-2. The common factor $2 \times \left(\frac{320}{\pi} \right)^2$ is normalised to unity.

(ii) The frequency response of resistor voltage in a series RL circuit is obtained as $\frac{V_R(j\omega)}{V(j\omega)} = \frac{R}{R+j\omega L} = \frac{1}{1+j\omega\tau}$. $\tau = 150 \text{ mH}/10 \Omega = 15 \text{ ms}$ and $\omega = 100\pi \text{ rad/s}$ in this case.

$$\frac{V_R(j\omega)}{V(j\omega)} = \frac{1}{1+j1.5\pi n} = \frac{1}{\sqrt{1+22.21n^2}} \angle \tan^{-1} 4.71n.$$

Exponential Fourier series coefficients of output = Exponential Fourier series coefficients of output $\times$ value of frequency response function at the corresponding frequency.

power contributions from each component is unaffected by the presence of other components. Hence, the normalised power of the waveform is the sum of normalised power of individual components.

Therefore, the power obeys superposition principle if the waveforms involved are periodic waveforms with the same period.

88% of P_n of the waveform in Example 13.10-1 is contributed by the DC component and the 100 Hz component ($n = 2$) together. The next harmonic component at 200 Hz ($n = 4$) adds another 6.3%. Thus, all the harmonics above 200 Hz contribute only 5.7% of power to P_n . (Fig. 13.10-2)

The contribution from a spectral component to normalised power depends on the square of amplitude of spectral component. Hence, the smaller amplitude components add only negligible power to the waveform though they affect the wave-shape of the signal considerably. For example, the sharp edge in the waveform is synthesised by small amplitude high frequency components. But they contribute almost nothing to the normalised power.

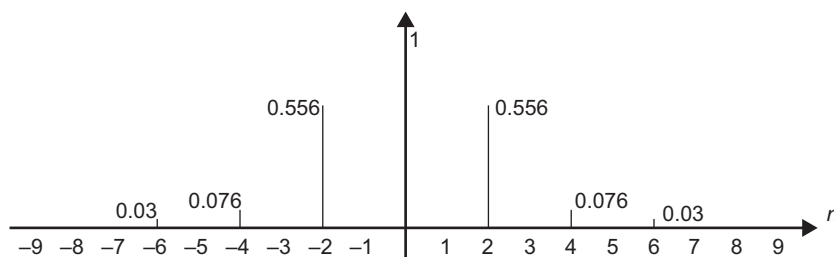


Fig. 13.10-2 Two-sided Discrete Power Spectrum of Waveform $v(t)$ for Example 13.10-1

Effect frequency
response on power
spectral
amplitudes.

Therefore, power spectral component in output = power spectral component at input $\times$ square of magnitude of frequency response function value at the corresponding frequency.

The values of square of magnitude of frequency response function is 1 at $n = 0$, 0.1055 at $n = 2$, 0.053 at $n = 4$ and 0.035 at $n = 6$. Hence, the power spectral component (normalised with respect to maximum value) in the output at $n = 0$ is 1, at $n = \pm 2$ is $0.1055 \times 0.556 = 0.059$, at $n = \pm 4$ is $0.053 \times 0.076 = 0.004$ and at $n = \pm 6$ is $0.035 \times 0.03 = 0.001$. Thus, almost the entire normalised power in the output waveform comes from its DC component. This spectrum is plotted in Fig. 13.10-3.

Power dissipated in the $10\ \Omega$ resistor = $P_n / 10$

$$\approx 2 \times \left(\frac{320}{\pi} \right)^2 \times (1 + 0.06 + 0.06 + 0.04 + 0.04 + 0.01 + 0.01 + \dots) / 10 = 2.53\ \text{kW}.$$

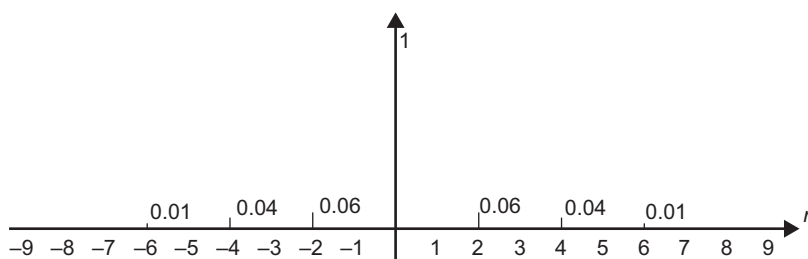


Fig. 13.10-3 Discrete Power Spectrum of Output Waveform for Example 13.10-1

13.11 POWER AND POWER FACTOR IN AC SYSTEM WITH DISTORTED WAVEFORMS

Let $v(t)$ and $i(t)$ be the voltage across an electrical element and current through that element under periodic steady-state conditions in an AC system working under distorted waveform conditions. Both $v(t)$ and $i(t)$ are expressed in the form of trigonometric Fourier series below where ω_0 is the fundamental radian frequency of the system. AC system voltages and currents usually do not contain any DC component under steady-state and hence DC components are not included in these Fourier series.

$$v(t) = \sum_{n=1}^{\infty} \sqrt{2} V_n \cos(n\omega_0 t - \phi_{vn})$$

$$i(t) = \sum_{n=1}^{\infty} \sqrt{2} I_n \cos(n\omega_0 t - \phi_{in})$$

V_n and ϕ_{vn} represent the r.m.s value of n^{th} harmonic component of voltage and its phase. Similarly I_n and ϕ_{in} represent the r.m.s value of n^{th} harmonic component of the current and its phase.

The power delivered to the element, assuming passive sign convention, is given by $v(t) \times i(t)$. The average power delivered, P , is obtained by averaging this quantity over one period of the AC system.

The product $v(t)i(t)$ contains two kinds of terms – product of cosine functions of the same harmonic order which have the general form of $2V_k I_k \cos(k\omega_0 t - \phi_{vk}) \cos(k\omega_0 t - \phi_{ik})$ is the first kind. The second kind of terms will be of the form $2V_k I_r \cos(k\omega_0 t - \phi_{vk}) \cos(r\omega_0 t - \phi_{ir})$. The first kind can be rewritten as $V_k I_k [\cos(2k\omega_0 t - \phi_{vk} - \phi_{ik}) + \cos(\phi_{ik} - \phi_{vk})]$. When integrated over a period and divided by the period, this term will result in a contribution of $V_k I_k \cos(\phi_{ik} - \phi_{vk})$ to the average delivered power. The second kind of terms can be expressed as $V_k I_r [\cos((k+r)\omega_0 t - \phi_{vk} - \phi_{ir}) + \cos((k-r)\omega_0 t + \phi_{ir} - \phi_{vk})]$; $k \neq r$. Since k and r are integers, $(k+r)$ and $(k-r)$ are also integers. Therefore, the two cosine components at the sum and difference frequencies will have zero average value over one period of the system. Hence, only terms of the first kind contribute to the average power.

$$\therefore P = \sum_{n=1}^{\infty} V_n I_n \cos(\phi_{in} - \phi_{vn}) \text{ W.}$$

Thus, the active power (i.e., the average power) delivered by each voltage and current harmonic pair with same harmonic order is independent of whether they are acting alone or acting along with other harmonic pairs. Each harmonic pair delivers an active power given by the product of r.m.s voltage, r.m.s current and power factor of that pair.

However, the VA (Volt–Ampere product) is given by the product of r.m.s values of $v(t)$ and $i(t)$. We express the r.m.s value of $v(t)$ in terms of r.m.s values of its harmonic components by using Eqn. 13.10-4. $V_{\text{rms}} = \sqrt{\sum_{n=1}^{\infty} (V_n)^2}$ and the r.m.s value of $i(t)$ as

$$I_{\text{rms}} = \sqrt{\sum_{n=1}^{\infty} (I_n)^2}. \text{ Then, VA product } S = V_{\text{rms}} I_{\text{rms}} = \sqrt{\sum_{n=1}^{\infty} (V_n)^2} \sqrt{\sum_{n=1}^{\infty} (I_n)^2} \text{ V/A.}$$

The power factor (*apparent* power factor) is the ratio between P and S . It includes the effect of reactive power of each harmonic voltage–current pair as well as the cross-power terms resulting from product of voltage and current of different harmonic order. Cross-power terms do not contribute to active power; but they contribute to VA.

The *fundamental power factor* is $\cos(\phi_{i1} - \phi_{v1})$ and is lagging if $(\phi_{i1} - \phi_{v1})$ is positive and is leading if $(\phi_{i1} - \phi_{v1})$ is negative.

Total active power in an AC system with distorted voltage and current waveforms.

EXAMPLE: 13.11-1

Fourier analysis of the AC voltage received at the substation of an industry that uses a large number of variable-speed drives shows that the received phase voltage can be represented by the following truncated Fourier series approximately.

$$v(t) = 6.35\sqrt{2} \sin(100\pi t) + 0.2\sqrt{2} \sin(500\pi t - 0.5 \text{ rad}) + 0.15\sqrt{2} \sin(700\pi t - 1 \text{ rad}) \text{ kV.}$$

The industry draws a distorted current from this supply due to the power electronics involved in variable speed drives. Fourier analysis of current drawn shows that the following Fourier series can represent it approximately.

$$i(t) = 110\sqrt{2} \sin(100\pi t - 0.6 \text{ rad}) + 22\sqrt{2} \sin(500\pi t - 1.5 \text{ rad}) + 15\sqrt{2} \sin(700\pi t - 1 \text{ rad}) + 10\sqrt{2} \sin(1100\pi t - 0.5 \text{ rad}) + 7\sqrt{2} \sin(1300\pi t - 0.5 \text{ rad}) \text{ A.}$$

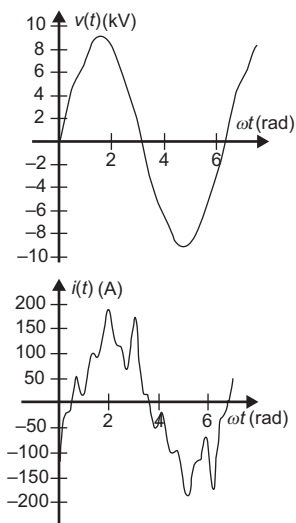


Fig. 13.11-1 Phase Voltage and Current Waveforms for Example 13.11-1

These waveforms are shown in Fig. 13.11-1.

Find the active power, VA, power factor, fundamental power factor and THD of voltage and current assuming balanced three-phase operation.

SOLUTION

$$v(t) = 6.35\sqrt{2} \sin(100\pi t) + 0.2\sqrt{2} \sin(500\pi t - 0.5 \text{ rad}) + 0.15\sqrt{2} \sin(700\pi t - 1 \text{ rad}) \text{ kV}$$

$$i(t) = 110\sqrt{2} \sin(100\pi t - 0.6 \text{ rad}) + 22\sqrt{2} \sin(500\pi t - 1.5 \text{ rad}) + 15\sqrt{2} \sin(700\pi t - 1 \text{ rad}) + 10\sqrt{2} \sin(1100\pi t - 0.5 \text{ rad}) + 7\sqrt{2} \sin(1300\pi t - 0.5 \text{ rad}) \text{ A}$$

$$\begin{aligned} \therefore P &= 6.35 \times 110 \times \cos(0.6 \text{ rad}) + 0.2 \times 22 \times \cos(1.5 - 0.5 \text{ rad}) + 0.15 \times 15 \times \cos(1 - 1 \text{ rad}) \\ &= 581.1 \text{ kW per phase} \\ &= 1.743 \text{ MW (three-phase)} \end{aligned}$$

$$\begin{aligned} V_{\text{rms}} &= \sqrt{6.35^2 + 0.2^2 + 0.15^2} = 6.355 \text{ kV (phase)} \\ &= 11 \text{ kV (line)} \end{aligned}$$

$$I_{\text{rms}} = \sqrt{110^2 + 22^2 + 15^2 + 10^2 + 7^2} = 113.8 \text{ A}$$

$$\begin{aligned} \therefore S &= V_{\text{rms}} I_{\text{rms}} = 723 \text{ kVA (per phase)} \\ &= 2.17 \text{ MW (three-phase)}. \end{aligned}$$

$$\text{Power factor} = \frac{P}{S} = \frac{1.74}{2.17} = 0.8.$$

$$\text{Fundamental power factor} = \cos(0.6 \text{ rad}) = 0.825 \text{ lag.}$$

$$\text{Total harmonic distortion in voltage} = \frac{\sqrt{0.2^2 + 0.15^2}}{6.35} \times 100 = 3.94\%.$$

$$\text{Total harmonic distortion in current} = \frac{\sqrt{22^2 + 15^2 + 10^2 + 7^2}}{110} \times 100 = 26.63\%.$$

13.12 SUMMARY

- Almost all periodic signals employed in circuits can be expressed as a sum of infinite number of sinusoids. Given a periodic waveform $v(t)$ with period T , its exponential Fourier series is given by $v(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{-jn\omega_0 t}$, where $\omega_0 = \frac{2\pi}{T}$ and $\tilde{v}_n$ are the exponential Fourier series coefficients of $v(t)$.

- The exponential Fourier series coefficients $\tilde{v}_n$ are found by

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt.$$

- The Fourier series may also be expressed in trigonometric form as below.

$$v(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t,$$

where $a_0 = \tilde{v}_0$, $a_n = 2 \operatorname{Re}(\tilde{v}_n)$ and $b_n = -2 \operatorname{Im}(\tilde{v}_n)$.

- Another form of trigonometric Fourier series is given below.

$$v(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \phi_n),$$

$$\text{where } c_0 = \tilde{v}_0, c_n = \sqrt{a_n^2 + b_n^2} = 2\tilde{v}_n \tilde{v}_n^* = 2|\tilde{v}_n|$$

$$\text{and } \phi_n = \tan^{-1} \frac{b_n}{a_n} = -\angle \text{ of } \tilde{v}_n \text{ for } n = 1, 2, 3 \dots$$

- Properties exhibited by the Fourier series for a *real* $v(t)$ are listed in Table 13.12-1.
- The magnitude of exponential Fourier series coefficients plotted against frequency in a two-sided line plot is called the discrete magnitude spectrum of the waveform. A similar plot of phase of exponential Fourier series coefficients is the discrete phase spectrum.
- The two components at $\pm n\omega_0$ in the discrete spectrum form a pair that cannot be separated in any analysis. They always go together to form a *real* sinusoidal component.
- The steady-state response of a circuit to a periodic input can be found in four steps. (i) Find the Fourier series of the input.

(ii) Find out the frequency response function between the desired output variable and the input variable by using phasor equivalent circuit. (iii) Obtain the steady-state response in time-domain for each sinusoidal component in the input. (iv) Decide how many terms in the output Fourier series are needed to represent the output waveform with the desired degree of accuracy.

- Normalised power P_n of $v(t)$ is defined as the average power that will be dissipated in a 1Ω resistance if this

waveform is applied to it as a voltage. Parseval's theorem states that the normalised power of a periodic waveform $v(t)$, P_n , is given by

$$P_n = |\tilde{v}_0|^2 + 2 \sum_{n=1}^{\infty} |\tilde{v}_n|^2$$

$$= c_0^2 + \sum_{n=1}^{\infty} \frac{c_n^2}{2}.$$

Table 13.12-1 Properties of Exponential Fourier Series for a Real $v(t)$

Signal/Property	Fourier Series
$v(t)$	$\tilde{v}_n, a_n$ and b_n, c_n and ϕ_n – Definition
$a v_1(t) + b v_2(t)$	$a \tilde{v}_{1n} + b \tilde{v}_{2n}$ – Linearity
$v(t - t_d)$	$e^{-j\omega t_d} \tilde{v}_n$ – Time-shifting
$e^{jk\omega_0 t} v(t)$	$\tilde{v}_{n-k}$ – Frequency-shifting
$v(-t)$	$\tilde{v}_{-n}$ – Time reversal
$v(\alpha t), \alpha > 0$	$\tilde{v}_n$ – Time-scaling
$v_1(t) v_2(t)$	$\sum_{k=-\infty}^{\infty} \tilde{v}_{1k} \tilde{v}_{2(n-k)}$
$\frac{dv(t)}{dt}$	$j n \omega_0 \tilde{v}_n$ – Time-domain differentiation
$\int_{-\infty}^t v(t) dt$ ($v(t)$ has zero DC content)	$\frac{\tilde{v}_n}{j n \omega_0}$ – Time-domain integration
$v(t)$ real	$\tilde{v}_{-n} = \tilde{v}_n^*$ – Conjugate symmetry $\text{Re}(\tilde{v}_n) = \text{Re}(\tilde{v}_{-n})$ and $\text{Im}(\tilde{v}_n) = -\text{Im}(\tilde{v}_{-n})$ $ \tilde{v}_n = \tilde{v}_{-n} $ and $\angle \tilde{v}_n = -\angle \tilde{v}_{-n}$
$v(t) = -v\left(t \pm \frac{1}{2}T\right)$ for all t – half-wave symmetry	$\tilde{v}_n = 0$ for even n
$v(-t) = v(t)$, even symmetry	$\text{Im}(\tilde{v}_n) = 0$, for all n . Only cosines in trigonometric Fourier series.
$v(-t) = -v(t)$, odd symmetry	$\text{Re}(\tilde{v}_n) = 0$, only sine terms in trigonometric Fourier series.
$v_e(t) = 0.5[v(t) + v(-t)]$	$\text{Re}(\tilde{v}_n)$
$v_o(t) = 0.5[v(t) - v(-t)]$	$\text{Im}(\tilde{v}_n)$
Parseval's Power Relation	$P_n = \int_{-T/2}^{T/2} [v(t)]^2 dt = \sum_{n=-\infty}^{\infty} \tilde{v}_n ^2$

13.13 QUESTIONS

1. A current $i(t) = 10 + 10 \sin 100\pi t + 5 \sin 300\pi t$ A is delivered to the positive terminal of a 12 V lead-acid battery for 1 h. Find the charge delivered to the battery.
2. Is the signal $v(t) = \sin(21t) + \sin(15t - 75^\circ)$ periodic? If it is, what is the period?
3. Is the signal $v(t) = \sin(\sqrt{2}t) + \sin(t - 75^\circ)$ periodic? If it is, what is the period?
4. Show that (i) product of two odd functions of t will be an even function of t (ii) product of two even functions of t will be an even function of t (iii) product of an odd function of t and an even function of t will be an odd function of t .
5. The waveform in Fig. 13.13-1 has $\omega = 1$ rad/s. It is time-shifted by t_0 to produce $v_1(t) = v(t - t_0)$. (i) Can a value for t_0 be found such that $v_1(t)$ has odd symmetry? (ii) Is it possible to find value for t_0 such that $v_1(t)$ is even on t ?

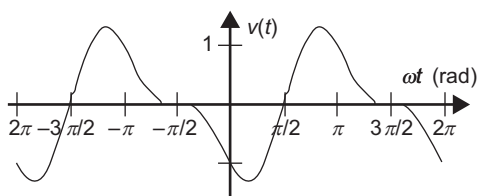


Fig. 13.13-1

6. The waveform in Fig. 13.13-2 has $\omega = 1$ rad/s. It is time-shifted by t_0 to produce $v_1(t) = v(t - t_0)$. (i) Can a value for t_0 be found such that $v_1(t)$ has odd symmetry? (ii) Is it possible to find value for t_0 such that $v_1(t)$ is even on t ?

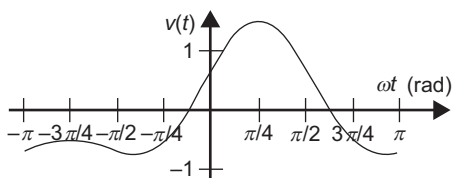


Fig. 13.13-2

7. The waveform in Fig. 13.13-3 has $T = 1$ s. (i) Does it possess odd or even symmetry? (ii) If not, will its time-shifted version,

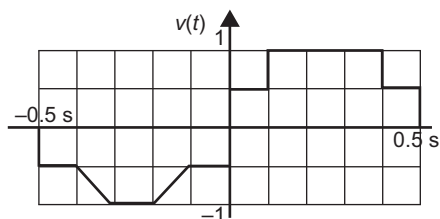


Fig. 13.13-3

8. A waveform $v(t) = 2 \sin(\omega t + \pi/7) + 0.5 \sin(5\omega t + \phi)$ is known to possess odd symmetry when it is delayed by $1/14$ s. Find ω and one possible value of ϕ .
9. A periodic voltage waveform $v(t)$ that is odd and half-wave symmetric is applied to a series RLC circuit and the output is taken across the capacitor. (a) Will the output voltage be odd on t ? (b) Will it be half-wave symmetric? (c) Are the answers dependent on the nature of circuit?
10. A periodic voltage waveform $v(t)$ that is half-wave symmetric is applied to a series RLC circuit and the output is taken across the capacitor. Explain why the DC component of capacitor voltage is zero under steady-state.
11. Show that (i) exponential Fourier series coefficients of *even part* of a waveform is the real part of exponential Fourier series coefficients of that waveform and (ii) exponential Fourier series coefficients of *odd part* of a waveform is the imaginary part of exponential Fourier series coefficients of that waveform.
12. Show that if $v(t)$ is a periodic waveform with ω_0 as its fundamental frequency, the exponential Fourier series coefficients of $\frac{dv(t)}{dt}$ are given by $j n \omega_0 \tilde{v}_n$, where $\tilde{v}_n$ are its exponential Fourier series coefficients.
13. Show that the power spectral components of the output voltage of a circuit are given by $|H(j\omega)|^2 \times$ power spectral component of input where $H(j\omega)$ is the frequency response function of the circuit.
14. If $v(t)$ is a periodic waveform with a period of T s and $v_1(t) = 2v(t - 0.5T) + 7$, find the relationship between the trigonometric Fourier series coefficients of $v(t)$ and $v_1(t)$?
15. $v(t)$ has a fundamental frequency of ω_0 rad/s and zero DC content. $v_1(t) = \frac{dv(t)}{dt}$ and $v_2(t) = \int_{-\infty}^t v(t) dt$. Will the r.m.s value of $v_1(t)$ and $v_2(t)$ be less than, equal or greater than that of $v(t)$? How does the answer depend on ω_0 ?
16. $v(t)$ is a distorted sinusoidal waveform with a fundamental frequency of ω_0 rad/s and zero DC content. $v_1(t) = \frac{dv(t)}{dt}$ and $v_2(t) = \int_{-\infty}^t v(t) dt$. Will the THD value of $v_1(t)$ and $v_2(t)$ be less than, equal or greater than that of $v(t)$? How does the answer depend on ω_0 ?
17. A battery of open circuit voltage V and internal resistance R is delivering a load current $i(t) = I \sin^2 \omega t$ A. Power is measured by connecting a voltmeter across the battery terminals and an ammeter in series with the battery and multiplying the readings. Calculate the percentage error (ignore meter errors) in measured power if meters are of (i) moving coil type (ii) moving iron type.

13.14 PROBLEMS

1. One cycle of a waveform $v(t)$ is shown in Fig. 13.14-1. It is a symmetrically clipped sinusoid. (i) Obtain a time-shifted version of this waveform such that the resulting waveform has odd symmetry. (ii) Find the trigonometric Fourier series of the shifted version and thereby obtain the discrete Fourier spectrum for $v(t)$.

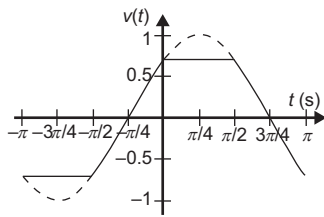


Fig. 13.14-1

2. One cycle of a periodic impulse train is shown in Fig. 13.14-2. Find its exponential Fourier series and plot the two-sided spectra.

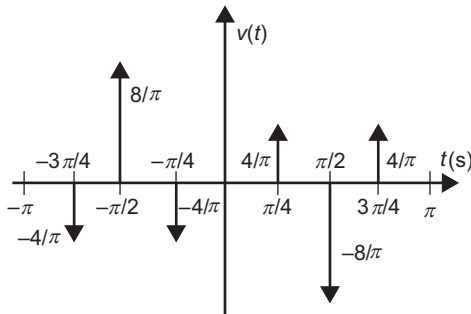


Fig. 13.14-2

3. One cycle of $v(t)$ is shown in Fig. 13.14-3. (i) What is the relationship between this waveform and the one in Fig. 13.14-2? (ii) Obtain the exponential Fourier series of this waveform by using this relationship. (iii) Obtain the trigonometric Fourier series of this waveform and plot the one-sided spectrum of this waveform.

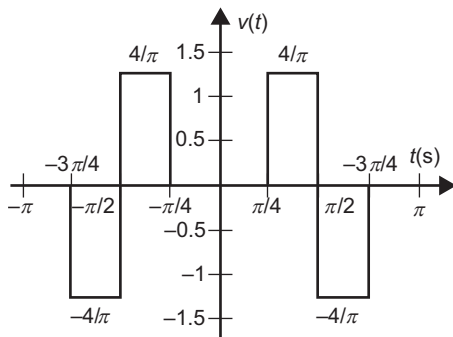


Fig. 13.14-3

4. One cycle of a periodic pulse train is shown in Fig. 13.14-4. (i) What is the relationship between this waveform and the one in Fig. 13.14-3? (ii) Obtain the exponential Fourier series of this waveform by using this relationship. (iii) Obtain the trigonometric Fourier series of this waveform and plot the one-sided spectrum of this waveform.

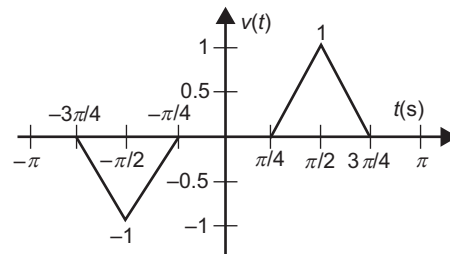


Fig. 13.14-4

5. Find the trigonometric Fourier series of the waveform $v(t)$ in Fig. 13.14-5 and plot its spectrum.

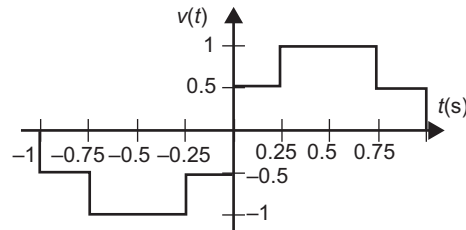


Fig. 13.14-5

6. $v(t)$ is a cosine wave and $v_1(t)$ is a square wave in Fig. 13.14-6. (i) Find $v_2(t) = v(t)v_1(t)$ and plot it. (ii) Find the trigonometric Fourier series of $v_3(t)$ from Fourier series of $v(t)$ and $v_1(t)$ and plot its spectrum.

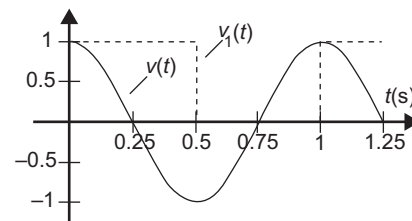


Fig. 13.14-6

7. $v(t)$ is a sine wave and $v_1(t)$ is a square wave in Fig. 13.14-7. (i) Find $v_2(t) = v(t)v_1(t)$ and plot it. (ii) Find the trigonometric Fourier series of $v_3(t)$ from Fourier series of $v(t)$ and $v_1(t)$ in terms of α . (iii) Plot its spectrum for $\alpha = \pi/6$.

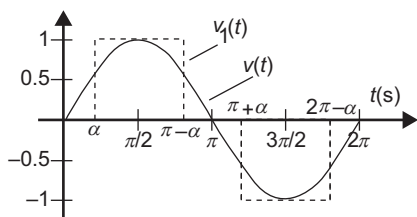


Fig. 13.14-7

8. Positive half-cycle of $v(t)$ with a period of 2 s is shown in Fig. 13.14-8. The waveform has odd symmetry. Find the exponential and trigonometric Fourier series of this waveform and plot its one-sided spectrum. If this waveform is used as an approximation to a sine wave find its THD.

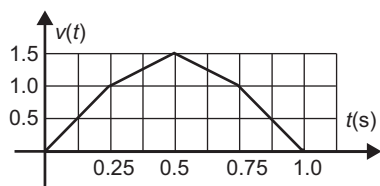


Fig. 13.14-8

9. $v_s(t) = 5|\sin \omega_0 t|$ V with $\omega_0 = 100\pi$ rad/s as in Fig. 13.14-9. Assume that the Opamp is ideal. Find the output voltage $v_o(t)$ as a function of time and draw its one-sided spectrum. What function does this circuit perform?

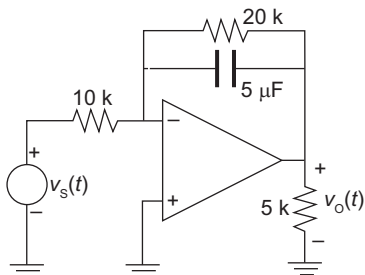


Fig. 13.14-9

10. The input voltage applied to the Opamp circuit as in Fig. 13.14-10 is a symmetric triangle periodic waveform

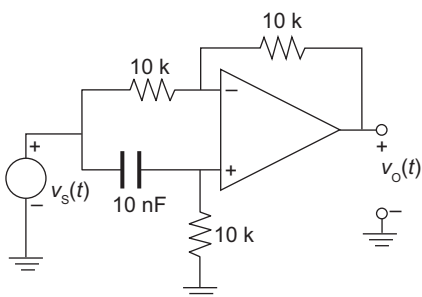


Fig. 13.14-10

moving between +5 V and -5 V with a period of 1 ms. Find the plot the output voltage as a function of time. What function does this circuit perform?

11. The circuit as in Fig. 13.14-11 is a *practical differentiator circuit* using an Opamp. The components C and R are sufficient to carry out differentiation. However, the non-ideal frequency response of the Opamp makes the circuit highly under-damped usually and the additional component, R_d , imparts damping to the circuit. But, with R_d present, the circuit is no more a differentiator at high frequencies. The input voltage applied to the practical differentiator circuit using Opamp in Fig. 13.14-11 is a ± 1 V symmetric triangular periodic waveform at 2.5 kHz. Obtain and plot the output voltage waveform. What is the expected output from a good differentiator for this input waveform? How does the calculated output compare with it?

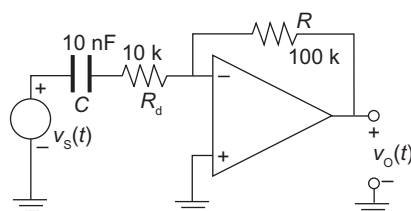


Fig. 13.14-11

12. The circuit as in Fig. 13.14-12 is a *practical integrator* using an Opamp. The resistor R_{off} is needed to control the DC offset at output terminals. However, R_{off} makes the circuit an imperfect integrator. The input to this integrator is the waveform shown in Fig. 13.14-3. Find and plot the output taking the first five non-zero harmonics of input into account.

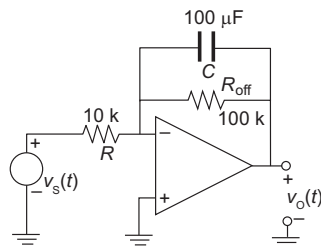


Fig. 13.14-12

13. (i) Predict the DC content in current through 6Ω and in voltage across the parallel combination without finding out Fourier series coefficients in the circuit in Fig. 13.14-13. (ii) Find the

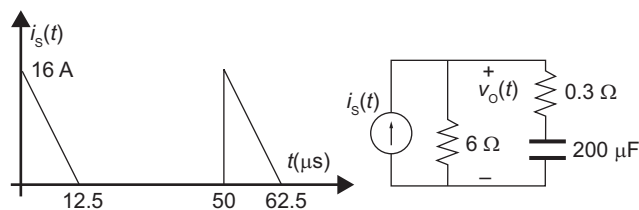


Fig. 13.14-13

output voltage $v_o(t)$ as a function of time and plot its one-sided Fourier spectrum. (iii) Find the r.m.s value of current through 0.3Ω and the power dissipated in it.

14. Find the output voltage $v_o(t)$ in the circuit in Fig. 13.14-14 considering the DC component and first two non-zero harmonics in the input current source.

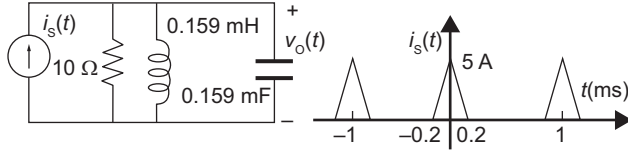


Fig. 13.14-14

15. $R = 1 \text{ k}\Omega$ and $C = 1 \mu\text{F}$ in the circuit in Fig. 13.14-15. The source voltage is a periodic impulse train given by $v_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n \times 10^{-3}) \text{ V}$. Find and plot the two-sided discrete power spectrum of $v_o(t)$.

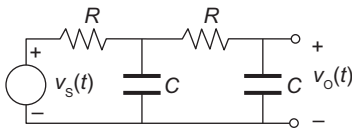


Fig. 13.14-15

16. The output voltage of a Power Electronic Inverter Circuit is related to the DC voltage used in the inverter by the equation $v_o = V_{\text{DC}} \times m \sin 100\pi t$, where m is the so-called modulation index. Assume that V_{DC} is not a pure DC source and it contains AC components. Let $V_{\text{DC}} = 400 + 20 \cos 200\pi t - 10 \cos 400\pi t \text{ V}$ and $m = 0.8$. (i) Find and plot the output $v_o(t)$ of the Inverter. (ii) Find the THD and r.m.s value of Inverter output. (iii) Plot the two-sided power spectrum of output.
17. The switch S in the circuit in Fig. 13.14-16 operates periodically with a frequency of 10 kHz, spending $27 \mu\text{s}$ in position-1 and $73 \mu\text{s}$ in position-2. (i) Find the average charging current in the 12 V battery under periodic steady-state operation. (ii) Find the exponential Fourier series of $i(t)$ under steady-state operation and plot its power spectrum. (iii) Find the r.m.s value of $i(t)$ and the power dissipated in the resistor.

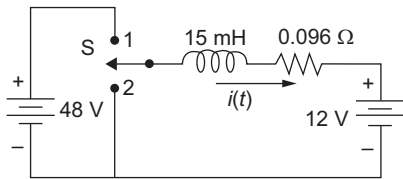


Fig. 13.14-16

18. The switch S in the circuit in Fig. 13.14-17 operates periodically with a frequency of 10 kHz, spending $77 \mu\text{s}$ in position-1 and $23 \mu\text{s}$ in position-2. (i) Find the average current delivered by the 12 V battery under periodic steady-state operation. (ii) Find the exponential Fourier series of $i(t)$ under steady-state operation and plot its power spectrum. (iii) Find the r.m.s value of $i(t)$, the power dissipated in the resistor and the power delivered by the 12 V battery. (iv) Thereby find the average charging current in 48 V battery. (v) Can this circuit be used to charge the 12 V battery from the 48 V battery? Discuss.

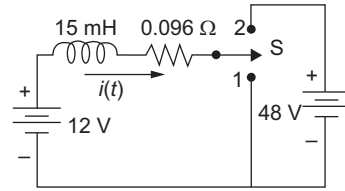


Fig. 13.14-17

19. The applied voltage $v_s(t)$ in the circuit in Fig. 13.14-18 is $320 \sin 100\pi t - 40 \sin 300\pi t - 20 \sin 500\pi t \text{ V}$. (i) Find the r.m.s value of applied voltage. (ii) Find the current delivered by the source as a function of time. (iii) Find the power delivered by the source and the VA delivered by it.

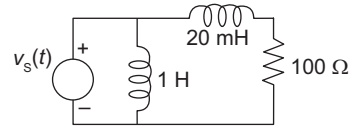


Fig. 13.14-18

20. The exponential Fourier series coefficients of $i(t)$ in the circuit in Fig. 13.14-19 are $\tilde{i}_0 = 1 \text{ A}$, $\tilde{i}_1 = 1 - j1 \text{ A}$, $\tilde{i}_{-1} = 1 + j1$, $\tilde{i}_3 = 0.3 + j0.2$ and $\tilde{i}_{-3} = 0.3 - j0.2$. The value of L is 10 mH and value of R is 100 ohm. The period of $v_s(t)$ is 50 ms. Find the exponential Fourier series of $v_s(t)$.

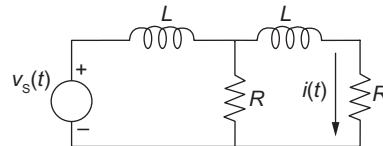


Fig. 13.14-19

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Dynamic Circuits with Aperiodic Inputs – Analysis by Fourier transforms

CHAPTER OBJECTIVES

- *Expansion of periodic and aperiodic waveforms in terms of sinusoids.*
- *Definition of Fourier transform and inverse integral.*
- *Dirichlet's conditions and Gibb's oscillations*
- *Various properties of Fourier transform of a real function of time.*
- *Time-limiting and band-limiting of signals*
- *Basic Fourier transform pairs.*
- *Fourier transform representation for periodic waveforms.*
- *Method of partial fractions.*
- *Relation between system function and Fourier transform of impulse response.*
- *Ideal filters and why they can not be made*
- *Solving circuit problems using Fourier transforms.*
- *Linear distortion in signal transmission context.*
- *Conditions on system function for distortion-free transmission of signals.*
- *Parseval's relation for finite energy signals and its application in signal analysis.*

This chapter provides a detailed answer to an important question in Circuit Analysis – 'Is the sinusoidal steady-state frequency response function a complete characterisation for a stable LTI Circuit?' – and concludes that it is.

INTRODUCTION

The impulse response of a dynamic circuit completely characterises it. The zero-state response for any arbitrary input can be obtained by convolving the input function with the impulse response in the time-domain. Non-zero initial conditions, if present, can also be handled by replacing them with impulse sources; however, the impulse response relevant to the corresponding point of application of source should be used.

We found that the sinusoidal steady-state frequency response function of a *stable* circuit is intimately related to its impulse response. Thus, the frequency response function contains the impulse response in a disguised manner. This prompted us to ask whether we can work out the zero-state response of a *stable* circuit to an arbitrary input from its

This chapter addresses the following questions:

(i) Can an aperiodic waveform be resolved into a sum of sinusoids of different amplitudes and frequencies? If yes, which are the frequencies that are present in the signal decomposition?

(ii) Can the steady-state frequency response function help us get the output when such an aperiodic waveform is applied to the circuit? If yes, how do we explain a *steady-state* term yielding a *transient response* term?

The first question will lead us to an important analytical tool in the study of signals and systems – *Fourier transforms*.

The second question will lead us to an important concept in the study of linear systems – the *System Function*.

The answers we develop for these two questions will reveal that the *sinusoidal steady-state frequency response function* is a full and complete characterisation of a linear, time-invariant (LTI) stable circuit.

We will also derive considerable insight into time-domain versus frequency-domain behaviour of signals and circuits using Fourier transforms.

frequency response function. In short, does frequency response function of a *stable* circuit characterise it completely?

The previous chapter provided a partial answer to this question. We considered the application of periodic inputs to a circuit in that chapter. We found that a well-behaved periodic input can be *resolved* into infinitely many pure sinusoidal components with their frequencies related harmonically. We were able to obtain the steady-state response to such inputs by using the superposition principle and the frequency response function. A waveform is periodic only if it exists for all t from $-\infty$ to $+\infty$. There is no natural response component in the output when such a periodic input is applied to a circuit. But that is not the way we drive circuits in practice. We switch on a periodic waveform at some time-instant (usually at $t = 0$) to the circuit. Then, the response will contain a transient part and a forced response (or steady-state response) part. The method of analysis we discussed in the previous chapter gives the steady-state response component in this case. It is so because the forced response is the particular integral component of the solution of the differential equation governing the circuit and the particular integral is the solution obtained by assuming that the input was applied from $-\infty$ onwards.

We notice that when we switch on a periodic input to a circuit, the resulting applied input is *not* a periodic function. For example $v_s(t) = \sin(\omega t)$ is periodic; but $v_s(t) = \sin(\omega t) \times u(t)$ is not periodic. That is why there is a transient response component.

An aperiodic input is an input for which no periodicity can be identified. It may be of a finite duration in time – i.e., it may be identically zero in the entire time-domain except in a finite interval or it may be semi-infinite, i.e., it may be non-zero in $[0^+, \infty)$. It may have finite *normalised* energy or infinite *normalised* energy. We take up the issue of obtaining the complete response of a circuit to such aperiodic inputs in this chapter.

14.1 APERIODIC WAVEFORMS

All inputs get switched on to the circuit at some time-instant and get switched off or removed from the circuit at some other time-instant in practice. Hence, all waveforms applied to circuits are *aperiodic* in practice. However, in steady-state studies, we may assume that the input is applied at $t = 0$, but not removed later. In that case, we have an aperiodic waveform that is *semi-infinite* in time. A periodic waveform switched on to the circuit at $t = 0$ is an example of an aperiodic input that is semi-infinite in time-domain. Such an aperiodic input has finite average power over a cycle, infinite duration and infinite energy – power and energy in the context of waveforms are to be understood as *normalised power* and *normalised energy*.

A periodic waveform switched on to a circuit at $t = 0$ and switched off later at $t = t_0$ is an example of an aperiodic input with finite duration and finite energy. The underlying waveform in these two examples is assumed to be periodic, but this assumption is not necessary. The time-function describing a semi-infinite or finite duration signal can be any well-behaved function.

Aperiodic waveforms are classified in terms of their duration, energy and power. They can be of *semi-infinite* or of *finite duration* in time. They may have *finite energy* or *finite power*.

An aperiodic waveform that is finite in value and is of finite duration will have finite energy too. An aperiodic waveform of semi-infinite duration can have finite energy or finite power. Some examples follow.

An aperiodic waveform described by $v_s(t) = [u(t - \tau/2) - u(t + \tau/2)]$ is a *finite-duration, finite-energy* waveform. It is a rectangular pulse of unit height located in the interval $(-\tau/2, \tau/2)$. Its duration is τ s and its normalised energy is τ J. The unit of normalised energy is Joules if it is understood as the energy that will be dissipated in a 1Ω resistance if this signal is either the voltage across it or the current through it. However, if

normalised energy is understood as the value of the integral of $[v_s(t)]^2$ over the entire time-domain, then its unit will be volt²-sec. Both units are used in practice. We use *Joules* (J).

An aperiodic waveform described by $v_s(t) = e^{-\alpha t}u(t)$ with a positive real value for α is a *semi-infinite duration, finite-energy* signal. Its energy content is $(0.5/\alpha)$ J.

An aperiodic waveform described by $v_s(t) = u(t)$, i.e., the unit step function, is a *semi-infinite duration, infinite-energy* signal. However, it has a *finite-power* of 1 W for $t > 0$.

An aperiodic waveform described by $v_s(t) = \sin(\omega t)u(t)$ is a *semi-infinite duration, infinite-energy* signal. Its average power content is 0.5 W for $t \geq (2\pi/\omega)$. The average power content is calculated over one period.

An aperiodic waveform described by $v_s(t) = \sin(\omega t)[u(t) - u(t - t_0)]$ is a *finite-duration, finite-energy* signal. Its energy content is $0.5\omega t_0[1 - (\sin 2\omega t_0/2\omega t_0)]$ J.

Examples for
different kinds of
aperiodic signals.

14.1.1 Finite-Duration Aperiodic Signal As One Period of a Periodic Waveform

It is not that aperiodic waveforms can not be expressed as a Fourier series. At least, finite-duration aperiodic waveforms can be expressed as one period of a periodic waveform that has a Fourier series.

Let $v(t)$ be an aperiodic waveform with finite duration and let its duration be τ s. Then, we construct a periodic waveform $v_-(t)$ with a period of T s ($T \geq \tau$) by placing the waveshape of $v(t)$ in every interval of width T s from $-\infty$ to $+\infty$. Clearly, every period of $v_-(t)$ will be identical to $v(t)$. This periodic $v_-(t)$ will have a Fourier series, i.e.,

$$v_-(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{-j\omega t}$$

where $\tilde{v}_n$ represents the n^{th} coefficient of the exponential Fourier series of $v_-(t)$. Now, we may write the aperiodic waveform as $v(t) = v_-(t) \times g(t)$, where $g(t)$ is a gate function defined as 1 for $0 < t < T$ and zero elsewhere. Figure 14.1-1 shows this operation for an example triangular pulse aperiodic waveform. Here, $\tau = 4$ s and $T = 5$ s.

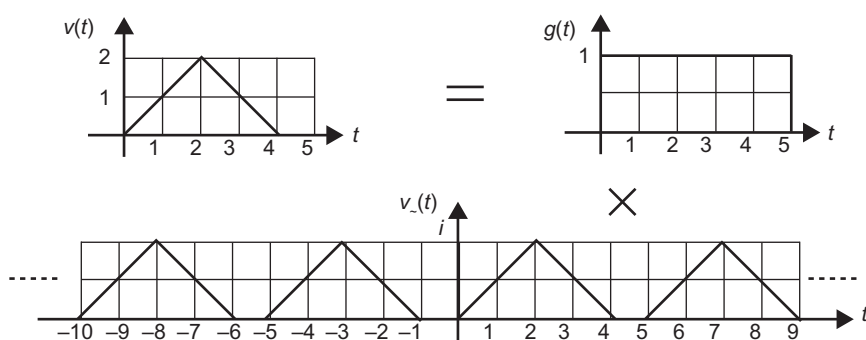


Fig. 14.1-1 A Finite-Duration Aperiodic Waveform As One Period of a Periodic Waveform

But, can we obtain the *total response* of a circuit when $v(t)$ is applied as the input from the output of the same circuit when $v_-(t)$ is applied to it? Essentially, we are asking another equivalent question here – if we know the output $r_1(t)$ of an LTI circuit for an input $v_1(t)$, can we find the output of the same circuit when $v_2(t) v_1(t)$ is applied to it by multiplying $r_1(t)$ by $v_2(t)$? The answer, obviously, is *no* since this is not *scaling* an input by a constant, and hence, the principle of homogeneity does not apply. Similarly, if $r_1(t)$ is the output when

A finite-duration aperiodic waveform can be expressed as one period of a periodic waveform that has a Fourier series.

But this does not help us in solving for the circuit output in a convenient manner.

However, this point of view will lead us to the Fourier representation of aperiodic waveforms as explained in Sect. 14.2.

$v_1(t)$ is applied and $r_2(t)$ is the output when $v_2(t)$ is applied, $r_1(t) r_2(t)$ is *not* the output when $v_1(t) v_2(t)$ is applied to the circuit. The superposition principle covers only scaling by a constant and linear combinations of inputs. Multiplication of two functions of time is not a linear operation – i.e., it does not obey the superposition principle.

14.2 FOURIER TRANSFORM OF AN APERIODIC WAVEFORM

We could not work out the output of the circuit with an aperiodic input $v(t)$ from the output corresponding to its periodic version $v_-(t)$, since there are infinitely many periods in $v_-(t)$ and $v(t)$ is only one period within it. We can expect the responses to be the same only if $v(t)$ and $v_-(t)$ are identical waveforms. But that is possible if, and only if, $v_-(t)$ contains just one period because $v(t)$ is one period extracted from $v_-(t)$. A periodic waveform can have only one period in it only if its period spans the entire time-domain from $-\infty$ to $+\infty$. Therefore, we want $v(t)$ to be $\lim_{T \rightarrow \infty} v_-(t)$. This limit has to be understood as the limit when the period extends to ∞ on both sides of the time-axis. We expect to obtain the output of the circuit when $v(t)$ is applied by obtaining the output when the Fourier series of $v_-(t)$ is applied to the circuit with the condition that the period of the waveform $v_-(t)$ spans the entire time-axis.

14.2.1 Fourier Transform of a Finite-Duration Aperiodic Waveform

What happens to the Fourier series of a periodic waveform when its period extends to ∞ on both sides of the time-axis? We consider a specific example for this purpose. It is a rectangular pulse of width τ s and height $1/\tau$ enclosing an area of unity.

The choice of zero time-instant with respect to the waveform of a periodic signal affects only the phase spectrum of the signal. It does not affect the amplitude spectrum. Hence, for convenience and with no loss in generality, we assume that the rectangular pulse is located between $-\tau/2$ and $+\tau/2$ in the time-axis. The rectangular pulse as well as its periodic extension is shown in Fig. 14.2-1.

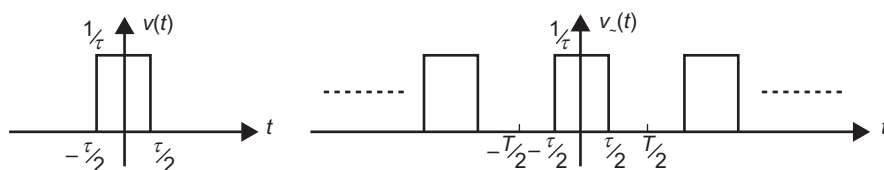


Fig. 14.2-1 A Rectangular Pulse and its Periodic Extension

The exponential Fourier series of $v_-(t)$ is found as

Let $\Delta\omega = \frac{2\pi}{T}$ rad/s, i.e., the fundamental angular frequency of $v_-(t)$

$$\text{Then, } \tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v_-(t) e^{-jn\Delta\omega t} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} \frac{1}{\tau} e^{-jn\Delta\omega t} dt$$

$$= \frac{-1}{jn\Delta\omega T\tau} \left[e^{-jn\Delta\omega\tau/2} - e^{jn\Delta\omega\tau/2} \right]$$

$$e^{-jn\Delta\omega\tau/2} - e^{jn\Delta\omega\tau/2} = -2j \sin \frac{n\Delta\omega\tau}{2} \text{ by Euler's formula} \quad (14.2-1)$$

$$\therefore \tilde{v}_n = \frac{2 \sin n\Delta\omega\tau/2}{n\Delta\omega T\tau} = \frac{\sin n\Delta\omega\tau/2}{n\Delta\omega\tau/2} = \left(\frac{1}{T}\right) \left(\frac{\sin x}{x}\right), \text{ where } x = n\Delta\omega\tau/2$$

For a special case of $\tau = 1$ s,

$$\tilde{v}_n = \left(\frac{1}{T}\right) \left(\frac{\sin x}{x}\right), \text{ where } x = n\Delta\omega/2 = n\pi/T$$

We observe the following in connection with this Fourier series.

- (i) The exponential Fourier series coefficients are real-valued. There are only two possible values of phase – 0 corresponding to a positive real number and π rad corresponding to a negative real number. Therefore, we can combine the amplitude and the phase spectra into a single spectrum which will show both positive and negative components.
- (ii) All frequency components are scaled by $1/T$. Therefore, the amplitude of spectral components will decrease with period.
- (iii) The variation of amplitude of spectral components with the harmonic order, i.e., with the value of frequency, is given by a $\sin x/x$ function. This function appears very frequently in the study of signals and systems and is given a special name – the *sinc*(x/π) function.

The two-sided spectrum of this Fourier series is shown in Fig. 14.2-2 for three different values of $T = 2.5$ s, 5 s and 10 s. The *sinc* function shape is also shown as the envelope of spectral lines. Angular frequency values are used in the horizontal axis. *sinc*(x/π) goes to zero for the first time on two sides of origin when $x = \pm\pi$. This happens when $n\Delta\omega = 2\pi$. There may not be a spectral component at that point since n may not be an integer when $n\Delta\omega = 2\pi$. However, if we treat $\omega = n\Delta\omega$ as a continuous variable and use it in the horizontal axis, the envelope of the spectral lines will cross the axis when $\omega = 2\pi$ rad/s.

Consider a fixed angular frequency interval $[-2\pi, 2\pi]$. The number of spectral components which appear within this frequency range increases with T . That is, more and more frequency components appear within a given band of frequencies as we increase the period of extension. However, the amplitude of spectral components decreases when we do so.

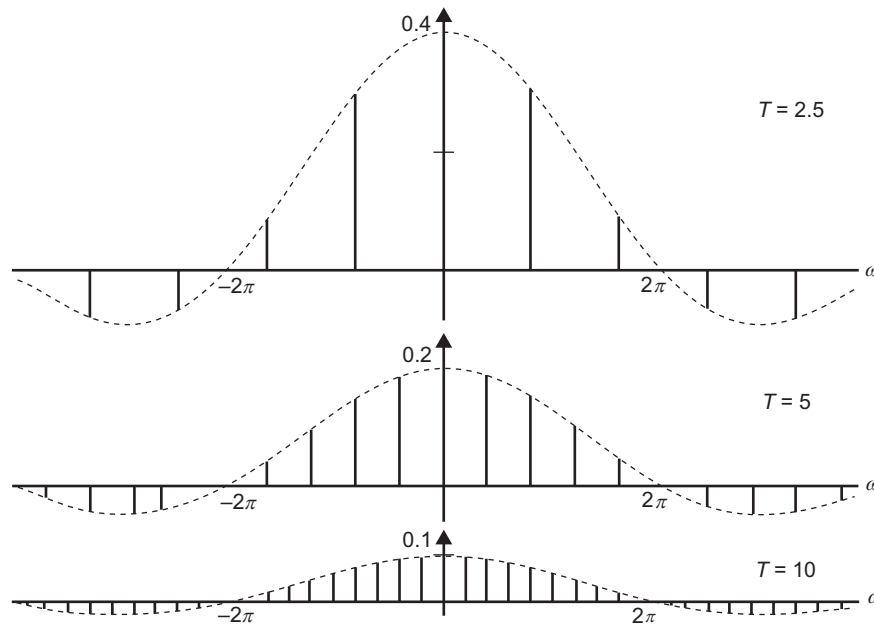


Fig. 14.2-2 Spectral Plots of Periodic Extension of a Rectangular Pulse of Unit Area and Unit Width for Various Periods of Extension

The sinc function is defined as $\text{sinc}(x) = \sin(\pi x)/\pi x$.
 $\sin(x)/x = \text{sinc}(x/\pi)$

We raise a question ...

Figure 14.2-2 shows the spectral components of a rectangular pulse of unit height and unit width subjected to infinite periodic extension.

The spectral components are shown for three values of periods of extension.

We observe that when the period of extension is increased, the number of spectral components appearing in any given band of frequencies increase. This crowding of spectral components is accompanied by a reduction in their amplitude.

When we observe one of the two related quantities increasing while the other is decreasing, we are tempted to ask whether their product remains constant.

We ask that question in this context and the effort to answer it leads us to the notions of 'spectral amplitude density' and Fourier transform.

The separation between the spectral components – $\Delta\omega = 2\pi/T$, the fundamental angular frequency – comes down as we increase T . This is true for any frequency range that we may consider. The envelope of spectral components always has a $\text{sinc}(x/\pi)$ shape, but gets reduced in height as we increase T .

As $T \rightarrow \infty$, a number of spectral components in any given band of frequencies approach infinity with the amplitude of each component approaching zero, i.e., infinitesimal. The spectral components come so close that there is a spectral component at every real value of ω . The spectral envelope curve, while retaining its shape, gets reduced to an infinitesimal height.

However, a spectrum with infinitesimal amplitudes does not look appealing. Hence, we try the following artifice.

For a special case of $\tau = 1$ s,

$$\begin{aligned}\tilde{v}_n &= \left(\frac{1}{T}\right) \left(\frac{\sin x}{x}\right), \text{ where } x = n \Delta\omega / 2 = n\pi/T \\ \text{i.e., } \tilde{v}_n T &= \left(\frac{\sin x}{x}\right), \text{ where } x = n\Delta\omega / 2 = n\pi/T \\ \text{i.e., } \tilde{v}_n \frac{2\pi}{\Delta\omega} &= \left(\frac{\sin x}{x}\right), \text{ where } x = n\Delta\omega / 2 = n\pi/T\end{aligned}\tag{14.2-2}$$

Average spectral
amplitude density
introduced.

Now, we plot the $\tilde{v}_n \frac{2\pi}{\Delta\omega}$ quantity against ω . We divide the amplitude of spectral component by the frequency separation between adjacent spectral components and then scale it by 2π . In doing so, it looks as if we are calculating the *average spectral amplitude density* available in a band of $\Delta\omega$ located at ω . We plot this quantity – i.e., 2π times the *average spectral amplitude density* available in a band of $\Delta\omega$ located at ω – against ω in Fig. 14.2-3.

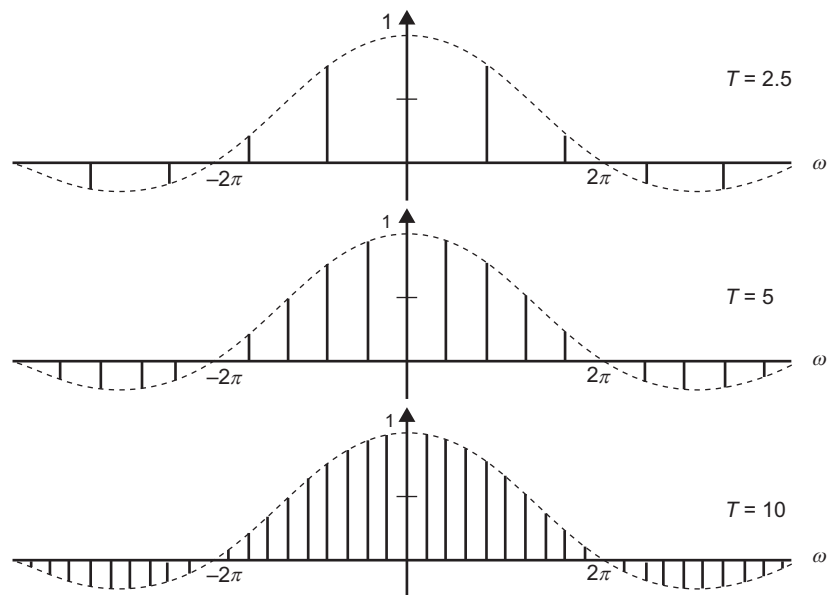


Fig. 14.2-3 Plot of Average Spectral Amplitude Density of a Rectangular Pulse Extension with Different Extension Periods

We get the same envelope in all the three cases. Now, the effect of increasing T is to bring more and more spectral components into the same frequency band. *The fact that the amplitude of spectral components go down as we increase T is now hidden in the quantity that we plot – it is the spectral amplitude multiplied by T that we plot.*

Now, as $T \rightarrow \infty$, the quantity $\Delta\omega \rightarrow 0$, the quantity $n\Delta\omega \rightarrow \omega$ and there is a spectral component of infinitesimal amplitude at every imaginable value of ω . Further, though the spectral component amplitude at any ω is infinitesimal, the quantity $\tilde{v}_n \frac{2\pi}{\Delta\omega}$, which is a ratio of two infinitesimal quantities, approaches a finite limit. This limiting value at a particular value of ω is termed as *the value of Fourier transform of $v(t)$ at ω* .

What we have illustrated in the case of a rectangular pulse applies to any finite-valued finite-duration aperiodic waveform. We formalise the above process for a finite-valued finite-duration aperiodic waveform and define Fourier transform as follows.

Let $\Delta\omega = \frac{2\pi}{T}$ rad/s, i.e., the fundamental angular frequency of $v_n(t)$

Then, $\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v_n(t) e^{-jn\Delta\omega t} dt = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\Delta\omega t} dt$ since $v(t)$ is the same as $v_n(t)$ in this period.

$$\therefore \tilde{v}_n T = \tilde{v}_n \frac{2\pi}{\Delta\omega} = \int_{-T/2}^{T/2} v(t) e^{-jn\Delta\omega t} dt$$

As $T \rightarrow \infty$, $\Delta\omega \rightarrow 0$ and the discrete variable $n\Delta\omega \rightarrow \omega$, the continuous variable and $\tilde{v}_n \frac{2\pi}{\Delta\omega}$ goes to a limiting value.

$$\therefore \lim_{\Delta\omega \rightarrow 0} \tilde{v}_n \frac{2\pi}{\Delta\omega} = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt.$$

The limit on the left side is by definition the Fourier transform $V(j\omega)$ of $v(t)$.

$$\therefore V(j\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt$$

The time-function can be synthesised from the Fourier transform as shown below.

$$V(j\omega) = \lim_{\Delta\omega \rightarrow 0} \tilde{v}_n \frac{2\pi}{\Delta\omega} = \text{The limiting value of spectral amplitude density} \times 2\pi$$

$\therefore$ The sum of spectral amplitude of all spectral components located at ω and in an infinitesimal band of $d\omega$ around $\omega = \frac{V(j\omega)}{2\pi} d\omega$

$$\text{The time-function contributed by these spectral components} = \frac{V(j\omega)}{2\pi} e^{j\omega t} d\omega$$

$v(t) =$ The sum of all such time-function contributions collected from the entire frequency-domain ($-\infty$ to $+\infty$)

$$\therefore v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega) e^{j\omega t} d\omega$$

The following two expressions summarise the forward and inverse Fourier transform process.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt \quad \text{--- The analysis equation}$$

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega) e^{j\omega t} d\omega \quad \text{--- The synthesis equation} \quad (14.2-3)$$

14.2.2 Fourier Transform of Infinite-Duration Aperiodic Waveforms

Infinite periodic extension of a given aperiodic waveform $v(t)$ and the subsequent limiting process of sending the period T of periodic extension to ∞ lie behind the definition of Fourier transform. If $v(t)$ is of infinite duration, the waveform in one period of its periodic extension

Meaning of Fourier transform value at a particular ω value.

Fourier transform pair – Eqn. 14.2-3

The first equation describes the decomposition of an aperiodic waveform into infinite number of sinusoids, each with infinitesimal amplitude, and is called the *analysis equation* of Fourier transform pair.

The second equation describes the construction of a time-function from its spectral decomposition and is called the *synthesis equation* of the Fourier transform pair.

will be different from $v(t)$ for any finite T . However, when $T \rightarrow \infty$, the waveform within one period of $v_T(t)$ will be identical to $v(t)$ since there is only one period now. Therefore, the definition of Fourier transform as given in Eqn. 14.2-3 is valid for such aperiodic waveforms too – provided the Fourier transform converges.

Fourier transform is sure to exist and converge for all finite-valued finite-duration aperiodic waveforms (except for some pathological waveforms which never make an appearance in Circuit Analysis). However, there are semi-infinite aperiodic waveforms that are routinely used in circuit analysis for which Fourier transform, as defined in Eqn. 14.2-3, will not converge, and hence, is not admissible.

If Fourier transform exists and converges for a semi-infinite aperiodic waveform, then, the meaning and interpretation of Fourier transform is the same as in the case of finite-duration aperiodic waveforms.

14.2.3 Interpretation of Fourier Transforms

Differences between spectral descriptions for a periodic waveform and an aperiodic waveform.

Fourier series of a periodic waveform and Fourier transform of an aperiodic waveform are expansions of time-functions along the $j\omega$ -axis in the s -plane.

- (i) A periodic waveform can be resolved into the sum of infinite sinusoidal waveforms. An aperiodic waveform can also be resolved into the sum of infinite sinusoidal waveforms. However, there is a difference between these two ‘infinities’. The ‘infinity’ of the number of spectral components in the decomposition of a periodic waveform is the ‘infinity’ of integers. The ‘infinity’ of the number of spectral components in the decomposition of an aperiodic waveform is the ‘infinity’ of real numbers. The spectrum of the waveform that displays the amplitude and the phase of individual sinusoidal components against the angular frequency is a discrete one in the case of periodic waveform and is a continuous one in the case of aperiodic waveform.
- (ii) Therefore, only sinusoids that have their frequencies as integer multiples of some basic frequency value are present in a periodic waveform, but sinusoids of *all* frequencies are present in an aperiodic waveform.
- (iii) Though sinusoids of all frequencies are present in an aperiodic waveform, the amplitude of sinusoidal component *at any particular frequency* is infinitesimally small. An infinite number of such sinusoids with infinitesimally small amplitudes reinforce each other or cancel each other at various instants to bring forth the waveshape of the aperiodic waveform. *Infinitesimally small quantities are not necessarily negligible when there are infinite numbers of them.*
- (iv) We introduced the concept of *signal space* in an earlier chapter. It is a complex plane in which each point will represent a complex exponential signal with a definite complex frequency. Pure sinusoids are represented by points on the vertical axis (the imaginary axis) – the $j\omega$ -axis. Hence, Fourier series of a periodic waveform is *an expansion of a time-function along the $j\omega$ -axis in the signal space*. Similarly, Fourier transform of an aperiodic waveform is also an expansion of a time-function along the $j\omega$ -axis in the signal space.
- (v) Fourier transform has its roots in complex exponential Fourier series, and hence, it is a two-sided complex function of a real variable ω . The fact that it is a complex function is denoted explicitly by including $j\omega$ as the independent variable in the symbol $V(j\omega)$ – where j indicates the complex nature of the function rather than the complex nature of the independent variable.
- (vi) Fourier transform *does not* give the amplitude and the phase of spectral components. It can not do so since the amplitude of a spectral component at any particular frequency is an infinitesimal. Rather, it gives the *density* of spectral amplitudes. Consider a particular frequency ω rad/s and a small band of frequencies around it denoted by $\Delta\omega$. Collect all the infinitesimal amplitudes of

all the sinusoids in that band and add them up – remember that it is a complex addition. Since there are infinite components in $\Delta\omega$, the sum of infinitesimal amplitudes need not be an infinitesimal. Now, divide the sum by $\Delta\omega$. We get a quantity that has the dimensions of volt per rad/sec (assuming an aperiodic voltage waveform). Repeat the process with a smaller $\Delta\omega$ around the same ω . We will observe that the ratio – which is the *density of spectral amplitudes* at and around ω – approaches a limit as we reduce $\Delta\omega$ to zero. Fourier transform value at ω is this *limiting density of spectral amplitudes* multiplied by 2π . Hence, Fourier transform is a *spectral amplitude density function*. We may express $\Delta\omega$ as $2\pi\Delta f$, where f is the cyclic frequency. Then, Fourier transform value at any ω gives the density of spectral amplitude at that frequency in volts/Hz unit.

- (vii) Therefore, though Fourier transform does not tell us anything about the amplitude of a sinusoid at a particular frequency contained in our aperiodic waveform (except that it is infinitesimally small), it tells us that the sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$ is $\approx V(j\omega_0) \times \Delta\omega/2\pi$, provided $\Delta\omega$ is small.
- (viii) What is the time-function contributed by the sum of all those sinusoidal components located in the frequency band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$? Each frequency component contributes a function of the form $e^{j\omega t}$ with an amplitude that is infinitesimal. However, the value of ω is changing from $\omega_0 - \Delta\omega/2$ to $\omega_0 + \Delta\omega/2$ in the band. Hence, we add sinusoids with slightly different frequencies (we get real sinusoids when we take the corresponding frequency points from the left and right sides of the spectrum). We saw in the previous paragraph that the sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$ is $\approx V(j\omega_0) \Delta\omega/2\pi$. But complex amplitude includes phase information. How do we add phases of *different frequency* sinusoids? Strictly speaking, it is not possible.
- (ix) However, if $\Delta\omega$ is infinitesimally small, i.e., if it is $d\omega$, we can assume that all the sinusoids are virtually at the same frequency and all the spectral components contribute $e^{j\omega_0 t}$. Then, the time-function contributed by the sum of all those sinusoidal components located in the frequency band $[\omega_0 - d\omega/2, \omega_0 + d\omega/2]$ will be $\approx [V(j\omega_0) \times d\omega/2\pi]e^{j\omega_0 t}$ and will appear nearly as a constant amplitude – single frequency sine wave, but the amplitude will be negligibly small.
- (x) But if the width of the band is not so small, we have to divide $\Delta\omega$ into many small intervals, compute the time-function contributed by those small intervals and *add the time-functions point-by-point*. Equivalently, we may evaluate the following integral.

Time-function contributed by all the spectral components in the angular frequency band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$

$$= \frac{1}{2\pi} \left[\int_{-(\omega_0 + \Delta\omega/2)}^{-(\omega_0 - \Delta\omega/2)} V(j\omega) e^{j\omega t} dt + \int_{(\omega_0 - \Delta\omega/2)}^{(\omega_0 + \Delta\omega/2)} V(j\omega) e^{j\omega t} dt \right]$$

- (xi) When two sinusoids of slightly different frequencies combine, they interfere with each other constructively for certain time intervals and destructively in certain other intervals. This results in the well-known *beat phenomenon*. This kind of constructive and destructive interferences takes place to produce wave packets in the time-function contribution under discussion. This interference mechanism is clearly visible in Fig. 14.2-4(b). This figure shows the contribution of spectral components in the 2.75 rad/s to 3.25 rad/s band to the aperiodic waveform that is 1 V rectangular pulse located between -0.5 s and $+0.5$ s in the time-domain. The curve (a) in the same figure shows the contribution of

Fourier transform value at any ω gives the density of spectral amplitude at that frequency in volts/Hz unit.

'The sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$ is $\approx V(j\omega_0) \times \Delta\omega/2\pi$ provided $\Delta\omega$ is small.'

However, this statement is to be interpreted carefully. The amplitudes involved in the sum are amplitudes of *different frequency* sinusoids.

Actually, the contribution from the band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$ to the aperiodic waveform is a wave packet resulting from the interference of many sinusoidal waveforms with different frequencies, rather than a steady sinusoid at ω_0 . Refer to Fig. 14.2-4 and the related discussion.

Fourier transform was derived from Fourier series, and hence, each sinusoidal component in Fourier transform represents a periodic wave, starting from $-\infty$ and lasting up to $+\infty$.

The infinite number of such sinusoids of different frequencies and infinitesimal amplitudes start from $-\infty$ and proceed to evolve in time, interfering constructively at some instants and destructively at certain other instants, in a manner suited to generate the desired aperiodic waveform.

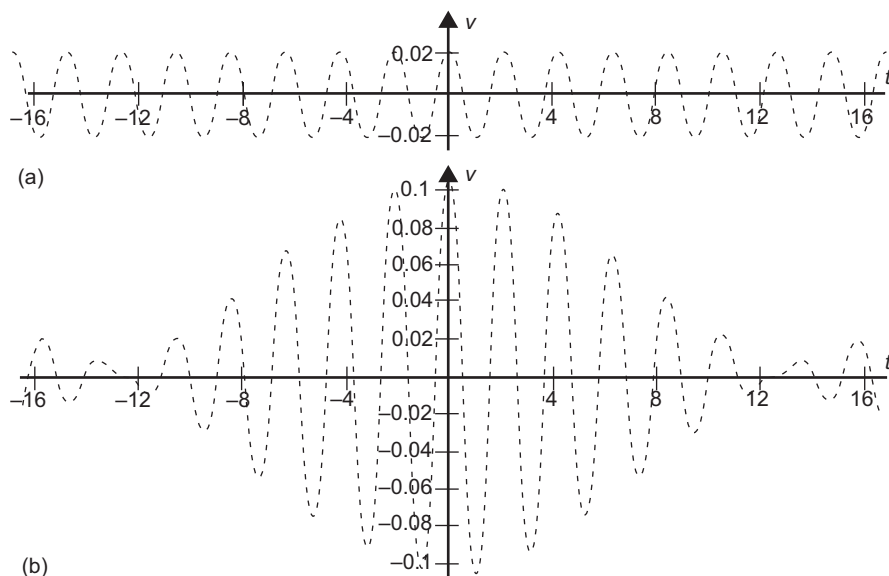


Fig. 14.2-4 Contribution of Frequency Band to a Rectangular Pulse of Unit Amplitude and Unit Width (a) [2.95, 3.05] rad/s Band (b) [2.75, 3.25] rad/s Band

spectral components in the 2.95 rad/s to 3.05 rad/s band to the same rectangular pulse. Since the spread of frequency is smaller in this case, it will take more than 16 s for the interference pattern to manifest. This waveform also will show waxing and waning when plotted for a larger duration. Also, note that the peak amplitudes in the two cases are in the same ratio as that of the frequency bands.

14.3 CONVERGENCE OF FOURIER TRANSFORMS

The Fourier transform of an aperiodic waveform $v(t)$ is given by

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt \quad (14.3-1)$$

It may be possible to evaluate this integral for a given $v(t)$, but that does not make the integral the Fourier transform of $v(t)$. It becomes the Fourier transform of $v(t)$ only if $v(t)$ can be constructed *uniquely* from $V(j\omega)$ by means of the synthesis equation

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega)e^{j\omega t} d\omega \quad (14.3-2)$$

Therefore, the integral in Eqn. 14.3-1 is the Fourier transform of $v(t)$ only if the integral in Eqn. 14.3-2 converges to the value of $v(t)$ for all t . If it does so, we assert that the Fourier transform exists for the signal $v(t)$.

There is a set of conditions to be satisfied by a time-function for its Fourier transform to exist and converge. They are called the *Dirichlet's Conditions*. It is a set of *sufficient but not necessary* conditions for the Fourier transform to exist. This implies that even functions that violate one or more Dirichlet's conditions may have legitimate Fourier transforms. However, if a time-function satisfies all Dirichlet's conditions, the Fourier transform for it is guaranteed to exist and converge.

'Convergence of Fourier transform' refers to the convergence of the *synthesis integral* and not to the convergence of the *analysis integral*. That is, we say a Fourier transform $V(j\omega)$ of a time-function $v(t)$ converges if the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega)e^{j\omega t} d\omega$$

converges to the value of $v(t)$ at the value of t used in the integral.

Dirichlet's conditions require that:

1. $v(t)$ be absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |v(t)| dt < \infty \quad (14.3-3)$$

2. $v(t)$ should have only a finite number of maxima and minima within any finite time interval
3. $v(t)$ should have only a finite number of discontinuities within any finite time interval. Also, these discontinuities must be finite.

If a function $v(t)$ satisfies these three conditions, its Fourier transform defined in Eqn. 14.3-1 exists and the synthesis integral in Eqn. 14.3-2 is sure to converge to the value of $v(t)$ for all t *except at a discontinuity*. At a discontinuity, the synthesis integral will converge to the average of the function before and after the discontinuity.

The second and third Dirichlet's conditions are usually met by all waveforms that have some practical application in circuits (except the impulse function). Only some pathological waveforms manage to violate these conditions and we never encounter them in circuits. However, many commonly used waveforms violate the first Dirichlet's condition – that is, they are not absolutely integrable. However, this does not imply that they have no Fourier transform. Dirichlet's conditions are *sufficient conditions*; but not *necessary conditions*. Therefore, a function which violates one or more of these conditions may still have a Fourier transform. A unit impulse function violates the third Dirichlet's condition, but still has a Fourier transform. A unit step function violates the first Dirichlet's condition, but still has a Fourier transform.

Dirichlet's conditions assure us that the synthesis integral in Eqn. 14.3-2 will converge to the average value of the function before and after the discontinuity at a point of discontinuity. Moreover, convergence to the actual function value is guaranteed at all other instants. But then, it will imply that the synthesis integral in Eqn. 14.3-2 is synthesising a discontinuous time-function if $v(t)$ is discontinuous at some time instant. We note that the synthesis integral is only a short form notation for the sum of infinitely many sinusoids of infinitesimally small amplitudes covering a frequency range from 0 Hz to ∞ Hz. All those sinusoids are *continuous functions of time*. How can we get a discontinuous function by adding continuous functions, even if we add infinite such functions?

The answer requires a more detailed look at the process of convergence of the synthesis integral. We take the example of a rectangular pulse for this purpose.

Dirichlet's conditions are *sufficient conditions*; but not *necessary conditions*, for the existence and convergence of Fourier transforms.

EXAMPLE: 14.3-1

The waveform of a rectangular pulse in time-domain is given in Fig. 14.3-1. (i) Obtain its Fourier transform and plot its continuous spectrum for a special case of unit amplitude and unit pulse-width. (ii) Invert its Fourier transform considering only those frequency components that fall below 50 rad/s and plot the result. (iii) Invert its Fourier transform for the time interval $[-0.52 \text{ s}, -0.48 \text{ s}]$ for (a) considering only those components which fall below 500 rad/s and (b) considering only those components which fall below 2500 rad/s and plot the result.

SOLUTION

- (i) The Fourier transform is obtained by applying the Eqn. 14.3-1 with $v(t) = 1/\tau$ in the interval $(-\tau/2, \tau/2)$.

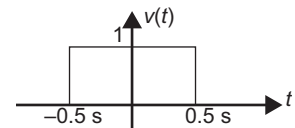


Fig. 14.3-1
Rectangular Pulses for
Example 14.3-1

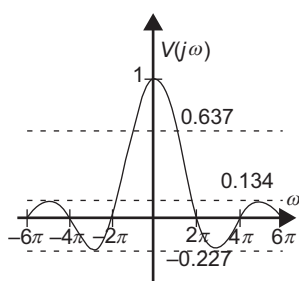


Fig. 14.3-2 Continuous Spectrum of a Rectangular Pulse of Unit Height and Unit Width

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} \frac{1}{\tau} e^{-j\omega t} dt$$

$$\therefore V(j\omega) = \frac{1}{-j\omega\tau} \left[e^{-j\omega\tau/2} - e^{j\omega\tau/2} \right] \quad (14.3-4)$$

$$e^{-j\omega\tau/2} - e^{j\omega\tau/2} = -2j \sin \omega\tau/2 \text{ by Euler's formula}$$

$$\therefore V(j\omega) = \frac{\sin \omega\tau/2}{\omega\tau/2} = \frac{\sin \omega/2}{\omega/2}, \text{ when } \tau = 1 \text{ s.}$$

The plot of the continuous spectrum is shown in Fig. 14.3-2. Since it is a real-valued spectrum, both amplitude (i.e., magnitude) and the phase spectra are combined into one spectrum.

The spectral amplitude density function is found to vary as $\sin(\omega/2)/(\omega/2)$. Sinusoidal components at all frequencies except for $\omega = 2k\pi$ rad/s, where k is a non-zero integer, are present in this rectangular pulse. However, the spectral amplitude density decreases in inverse proportion to the angular frequency.

(ii) We evaluate the partial sum of the infinite number of sinusoidal components which have their frequencies in the band 0 to 50 rad/s to examine how closely will the result of partial synthesis $\hat{v}(t)$ match the original function $v(t)$. We remember that two frequency components at $+\omega$ and $-\omega$ from an exponential Fourier series or Fourier transform will contribute $e^{-j\omega t}$ and $e^{j\omega t}$ terms resulting in a sinusoidal contribution. Therefore, when we accept a spectral component at ω , we have to take its companion component located at $-\omega$ along with it. With this in mind, we write the partial synthesis as shown below.

$$\hat{v}(t) = \frac{1}{2\pi} \int_{-50}^{50} V(j\omega) e^{j\omega t} d\omega$$

Since $V(j\omega)$ is an even real function in this case and $e^{j\omega t} + e^{-j\omega t} = 2 \cos \omega t$ by Euler's formula,

$$\hat{v}(t) = \frac{1}{\pi} \int_0^{50} \frac{\sin(0.5\omega)}{(0.5\omega)} \cos \omega t d\omega.$$

This integral is evaluated numerically (a few lines of code in your favourite programming language will be enough) and the result is plotted in Fig. 14.3-3.

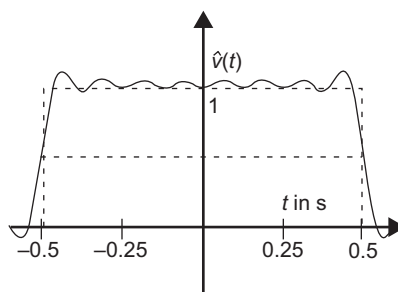


Fig. 14.3-3 Time-domain Waveform Synthesised from 0 to 50 rad/s Spectral Components of the Pulse in Example 14.3-1

This waveform shows that the value at the discontinuity is close to the average value of 0.5; but the frequency components we included in the synthesis are evidently not enough to ensure convergence at all other points. In fact, a simulation study will show that the value of $\hat{v}(t)$ at any t will approach the value of $v(t)$ in an oscillatory manner as we increase the frequency of components included in the synthesis. As we make $\omega \rightarrow \infty$, the amplitude of these oscillations in values go down and the value of $\hat{v}(t)$ converges to the value of $v(t)$, except at neighbourhoods around points of discontinuity.

- (iii) The oscillation in the synthesised waveform shown in Fig. 14.3-3 does not disappear as we increase the number of spectral components included in the synthesis. Rather, these oscillations get crowded towards the points of discontinuity. The result is that there is always a small neighbourhood around the time instants of discontinuity where the difference between the synthesised waveform and the original waveform will oscillate significantly. Increasing the range of frequency included in the synthesis will only reduce the width of this neighbourhood, but will not succeed in reducing it to zero. Thus, there are some time points around the points of discontinuity where the synthesised wave will not converge to the original wave – we can only reduce the range in time-domain where this happens by increasing the range of frequency included in the synthesis. Hence, the synthesised waveform remains continuous even when $\omega \rightarrow \infty$ in the synthesis and can not match the discontinuous waveform we started with. There is always a small neighbourhood of departure around points of discontinuity. This is illustrated by the two curves in Fig. 14.3-4 that show the results of synthesis using components in the 500 rad/s and 2500 rad/s ranges.

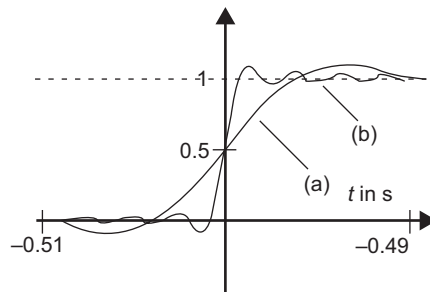


Fig. 14.3-4 Time-domain Waveform Synthesised from (a) 0 to 500 rad/s and (b) 0 to 2500 rad/s Spectral Components of the Rectangular Pulse in Example 14.3-1

Gibb's Oscillations

These oscillations of the difference between $\hat{v}(t)$ and $v(t)$ are called *Gibb's Oscillations*, named after Josiah Gibbs who explained this in 1899.

As $\omega \rightarrow \infty$ in the synthesis, these oscillations will get so crowded that they will contain negligible energy and can be safely ignored.

In any case, we need not worry much about them in circuit analysis. We will not be able to make physical waveforms containing discontinuities, to begin with. We will see the reason for this later in the chapter. One has to generate such a discontinuous waveform before one applies it to some circuit – this is not possible.

14.3.1 Uniqueness of Fourier Transform Pair

Dirichlet's conditions are only *sufficient* conditions for the existence of Fourier transforms. How do we find the Fourier transform for a function that violates one or more of these conditions? For all that we know, it may not exist. One procedure would be to blindly use the analysis integral of Fourier transform pair and study the resulting Fourier transform for its convergence properties. Another method would be to express the troublesome function as a limiting case of some other function which satisfies all of Dirichlet's conditions and try to arrive at the Fourier transform of the troublesome function as a limiting case of Fourier transform of this well-behaved function.

How do we know that the Fourier transform we have arrived at for a function that does not satisfy the Dirichlet's conditions by some means is indeed *the* Fourier transform of that function or, in other words, can there be different Fourier transforms for the same time-function and different time-functions for the same Fourier transform?

The theorem of Uniqueness of Fourier transform Pair assures us that if we have determined a Fourier transform function for a given time-function, then, that is the only Fourier transform that will exist for that function. Similarly, it assures us that if we have determined a time-function by inverting a Fourier transform, then, that is the only time-function for that Fourier transform. This theorem makes our efforts to identify the Fourier transform of functions that do not satisfy Dirichlet's conditions worthwhile. Thus, the time-function $v(t)$ and its Fourier transform (if it exists) $V(j\omega)$ form a *unique pair* and this unique relationship between them is represented symbolically by $v(t) \Leftrightarrow V(j\omega)$.

14.4 SOME BASIC PROPERTIES OF FOURIER TRANSFORMS

Signals are subjected to various operations in the time-domain in electrical circuits and systems. Some of the basic signal operations are scaling by a constant, differentiation, integration, multiplication by another signal (this is called *time-domain modulation*), delaying or advancing in time, etc. When a signal undergoes a time-domain operation, its Fourier transform undergoes a corresponding operation in the frequency-domain. We want to establish this correspondence through the study of properties of Fourier transforms. We cover some basic properties of Fourier transforms in this section and continue to develop more advanced properties that are relevant to circuit analysis in the later sections. Simultaneously, we work out the Fourier transforms of many time-functions of importance to circuit analysis.

14.4.1 Linearity of Fourier Transform

Forward and Inverse Fourier transformation involve mathematical integration in time-domain and frequency-domain, respectively. Integration is a linear mathematical operation and obeys the superposition principle. Therefore, Fourier transforms obey the superposition principle. That is,

$$\text{If } v_1(t) \Leftrightarrow V_1(j\omega) \text{ and } v_2(t) \Leftrightarrow V_2(j\omega), \text{ then, } a_1 v_1(t) + a_2 v_2(t) \Leftrightarrow a_1 V_1(j\omega) + a_2 V_2(j\omega)$$

EXAMPLE: 14.4-1

Obtain the Fourier transform of the waveform in Fig. 14.4-1 and plot its continuous spectrum.

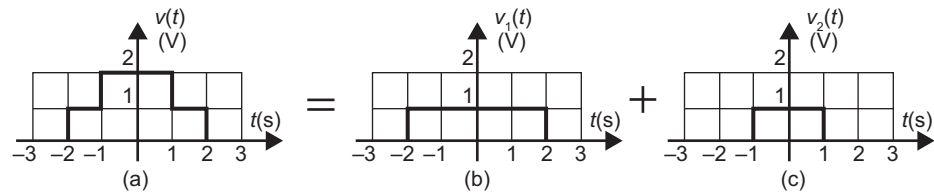


Fig. 14.4-1 Waveform for Example 14.4-1 and its Decomposition

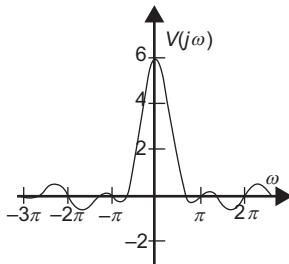


Fig. 14.4-2 Continuous Spectrum of Waveform in Example 14.4-1

SOLUTION

We have already worked out the Fourier transform of a symmetrically positioned rectangular pulse of unit area as $(\sin \omega \tau / 2) / (\omega \tau / 2)$, where τ is the width of the pulse and ω is the angular frequency in rad/s. If the height of the rectangular pulse is V , its area will be $V\tau$ and by linearity property of Fourier transform, its Fourier transform will be $(V\tau)(\sin \omega \tau / 2) / (\omega \tau / 2)$. The waveform $v(t)$ given in Fig. 14.4-1 (a) can be decomposed into the sum of $v_1(t)$ and $v_2(t)$ as shown in (b) and (c) of the same figure. Therefore, the Fourier transform of $v(t)$ is the sum of Fourier transforms of $v_1(t)$ and $v_2(t)$.

$$\therefore V(j\omega) = 4 \frac{\sin 2\omega}{2\omega} + 2 \frac{\sin \omega}{\omega}. \text{ This is plotted in Fig. 14.4-2.}$$

14.4.2 Duality in Fourier Transform

The analysis and synthesis equations of Fourier transform are repeated below. A close look at them reveals that except for a sign change in the exponent and a factor of 2π , the two integrals are similar. This similarity leads to an important property of Fourier transform – the property of *Duality*.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt \quad \text{--- The analysis equation}$$

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega)e^{j\omega t} d\omega \quad \text{--- The synthesis equation}$$

We employ the following mathematical manipulation of variables to arrive at the duality property.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt \quad \text{--- The analysis equation}$$

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega)e^{j\omega t} d\omega \quad \text{--- The synthesis equation}$$

Let two new variables be defined as $t' = -\omega$ and $\omega' = t$.

Rewriting the analysis and synthesis equations in terms of t' and ω' ,

$$V(-jt') = \int_{-\infty}^{\infty} v(\omega')e^{j\omega't'} d\omega' \quad \text{and}$$

$$v(\omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(-jt')e^{-j\omega't'} (-dt') = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(-jt')e^{-j\omega't'} dt'$$

Now, we recast the equation as

$$[2\pi v(\omega')] = \int_{-\infty}^{\infty} V(-jt')e^{-j\omega't'} dt'$$

$$V(-jt') = \int_{-\infty}^{\infty} [2\pi v(\omega')]e^{j\omega't'} d\omega'$$

Now, we get rid of the primed variables and bring back the original variables by substituting $\omega' = \omega$ and $t' = t$.

$$[2\pi v(\omega)] = \int_{-\infty}^{\infty} V(-jt)e^{-j\omega t} dt \quad \text{This is an analysis equation}$$

$$V(-jt) = \int_{-\infty}^{\infty} [2\pi v(\omega)]e^{j\omega t} d\omega \quad \text{This is a synthesis equation}$$

This shows that for every $v(t) \Leftrightarrow V(j\omega)$ pair there exists another Fourier transform pair $V(-jt) \Leftrightarrow 2\pi v(\omega)$.

The $-j$ in $V(-jt)$ indicates that it is a complex function of a real variable t . The interpretation of this result is as follows.

Given a $v(t)$ with a waveshape in t -axis and its $V(j\omega)$ with a waveshape, each for its real part and imaginary part (or magnitude and phase parts) in the ω -axis, visualise a new ω -axis and transfer 2π times the shape that $v(t)$ has in t -axis to this new ω -axis. Imagine this transferred shape as a new Fourier transform of some time-function. If $v(t)$ was a real function of t , this Fourier transform would have only the real part and its imaginary part would be zero. Now, raise the question – which time-function has this Fourier transform?

The *Duality Property* answers this question – visualise a new t -axis. Take the shape of $V(j\omega)$ in the original ω -axis (take the shape of its real and imaginary parts together), reflect the shape on the vertical axis (i.e., get the *mirror image* of the shape) and transfer the reflected shape to the new t -axis. We get a *complex function of a real variable (time)* now. This is the time-function that we wanted.

This relationship is illustrated in the case of a rectangular pulse in Fig. 14.4-3. A rectangular pulse in time-domain has a frequency-domain description which has a *sinc function* shape. By duality property, a rectangular pulse shaped Fourier transform has a time-domain waveform that has a *sinc function* shape.

Duality property of Fourier transforms.

Duality in Fourier transform

For every $v(t) \Leftrightarrow V(j\omega)$ pair, there exists another Fourier transform pair $V(-jt) \Leftrightarrow 2\pi v(\omega)$.

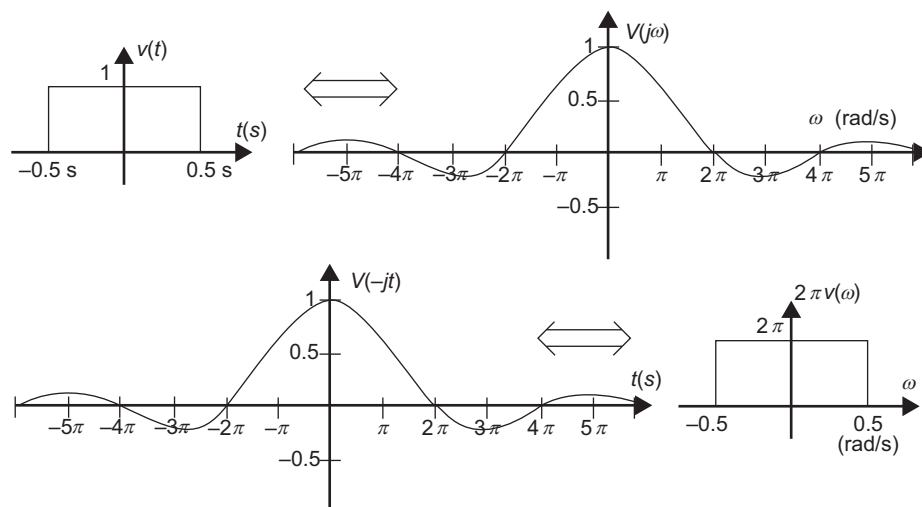


Fig. 14.4-3 Duality Property of Fourier Transforms Illustrated

EXAMPLE: 14.4-2

Find a time-function $v'(t)$ such that its Fourier transform will have the same shape and magnitude as that of the time-function waveform shown in Fig. 14.4-4(a).

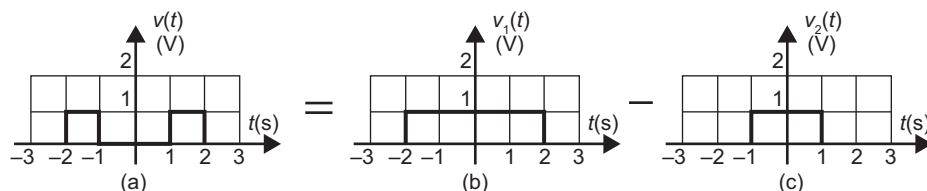
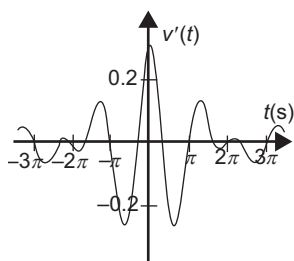
Fig. 14.4-4 Waveform of $v(t)$ for Example 14.4-2 and its Decomposition

Fig. 14.4-5 Plot of the Required Time-Function in Example 14.4-2

SOLUTION

The waveform of $v(t)$ can be decomposed into $v_1(t)$ and $v_2(t)$ as shown in Fig. 14.4-4. $v(t)$ is $v_1(t) - v_2(t)$. The components $v_1(t)$ and $v_2(t)$ are the same as the components which appeared in Example 14.4-1. Therefore, Fourier transform of $v(t)$ will be $V(j\omega) = 4 \frac{\sin 2\omega}{2\omega} - 2 \frac{\sin \omega}{\omega}$ by using the property of linearity of Fourier transforms.

Now, by duality property, 2π times the waveshape of Fig. 14.4-4(a), if treated as a Fourier transform, will have an inverse of $4 \frac{\sin 2(-t)}{2(-t)} - 2 \frac{\sin(-t)}{(-t)} = 4 \frac{\sin 2t}{2t} - 2 \frac{\sin t}{t}$. Therefore, the time-function which will have the waveshape Fig. 14.4-4(a) as its Fourier transform will be given by $v'(t) = \frac{1}{\pi} \left[2 \frac{\sin 2t}{2t} - \frac{\sin t}{t} \right]$. The plot of this function is shown in Fig. 14.4-5.

We take note of a particular trend in the Fourier transform pair – we observe that when the function is sharply limited in one domain, it gets extended from $-\infty$ to ∞ in the

other domain. A rectangular pulse in time-domain gets stretched out over the entire frequency-domain in the form of a sinc function. A function that has a rectangular shape in frequency-domain gets spread out over the entire time-domain in the form of a sinc function. This tendency of functions to get stretched out in one domain when they are confined in the other domain is a general tendency and is not specific to rectangular waveforms. It has many practical implications as far as electrical circuits are concerned. We will look at it in detail in a later section in this chapter.

14.4.3 Time Reversal Property

Assume that a signal $v(t)$ has a Fourier transform $V(j\omega)$. Let $v_1(t)$ be the time-reversed version of $v(t)$, i.e., $v_1(t) = v(-t)$. Then, the Fourier transform of $v_1(t) = V_1(j\omega) = V(-j\omega)$. This may be easily proved, starting from the analysis equation. This implies that a reversal in time-domain is accompanied by a reversal in frequency-domain.

14.4.4 Time Shifting Property

Assume that a signal $v(t)$ has a Fourier transform $V(j\omega)$. Let $v_1(t)$ be the time-shifted version of $v(t)$, i.e., $v_1(t) = v(t - t_d)$, where t_d is a constant time delay. Any value which is taken up by $v(t)$ at t is assumed by $v_1(t)$ at $t + t_d$, and thus, the waveshape of $v_1(t)$ is the same as the waveshape of $v(t)$ shifted forward, i.e., delayed in time-axis. So, when t_d is positive, the waveform gets delayed and when t_d is negative, the waveform gets advanced in time. We obtain an expression for the delayed version as below:

$$V_1(j\omega) = \int_{-\infty}^{\infty} v(t - t_d) e^{-j\omega t} dt. \text{ Substitute } t' = t - t_d. \text{ Then,}$$

$$V_1(j\omega) = e^{-j\omega t_d} \int_{-\infty}^{\infty} v(t') e^{-j\omega t'} dt' = e^{-j\omega t_d} V(j\omega)$$

Time-shifting property of Fourier transforms.

The effect of delaying in time-domain is a multiplication by a factor of unit magnitude and a phase angle that varies linearly with frequency. Therefore, the amplitude of sinusoidal components contained in the signal does not change, but their phases get delayed further by an extent that is directly proportional to their frequency value. The constant of proportionality is t_d . Therefore, $v_1(t)$ will have the same magnitude spectrum as that of $v(t)$, but its phase spectrum will have an additional linear phase delay contribution of $-\omega t_d$ rad.

EXAMPLE: 14.4-3

Find the Fourier transform of the waveform shown in Fig. 14.4-6(a) and plot its continuous magnitude and phase spectra.

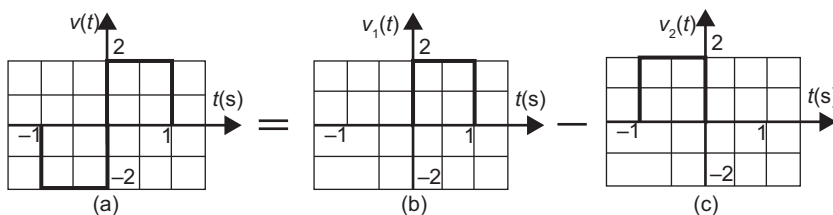


Fig. 14.4-6 Waveform for Example 14.4-3 and its Decomposition

SOLUTION

The decomposition of the given waveform is shown in Fig. 14.4-6(b) and (c). It can be seen that $v(t) = v_1(t) - v_2(t)$, but $v_2(t)$ is $v_1(t)$ reflected about the vertical axis, i.e., $v_2(t) = v_1(-t)$. Therefore, $v(t) = v_1(t) - v_1(-t)$. We find $V_1(j\omega)$ first.

We notice that $v_1(t + 0.5)$ will be a symmetrically placed rectangular pulse of height 2 and width 1 s. It will be placed between -0.5 s and 0.5 s. We have already worked out the Fourier transform for such a pulse and it will be $2 \sin(0.5\omega)/(0.5\omega)$. But this must be $e^{j0.5\omega} \times V_1(j\omega)$ by time-shifting property of Fourier transforms.

$$\therefore V_1(j\omega) = \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] e^{-j0.5\omega}$$

By time-reversal property of Fourier transforms,

$$V_2(j\omega) = \left[2 \frac{\sin(-0.5\omega)}{(-0.5\omega)} \right] e^{-j0.5(-\omega)} = \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] e^{j0.5\omega}$$

By linearity property of Fourier transforms,

$$\begin{aligned} V(j\omega) &= V_1(j\omega) - V_2(j\omega) = \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] [e^{-j0.5\omega} - e^{j0.5\omega}] \\ &= \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] [-2j \sin(0.5\omega)] = -j4 \left[\frac{\sin(0.5\omega)}{(0.5\omega)} \right] \sin(0.5\omega) \end{aligned}$$

The Fourier transform is seen to be purely imaginary and a casual look at it may give us an impression that the phase spectrum is always at $-\pi/2$ rad. The *magnitude spectrum* of a Fourier transform is always positive valued since the magnitude of a complex number is positive. Therefore, if the sign of $\sin^2(0.5\omega)/(0.5\omega)$ changes, then, there has to be an additional contribution of π rad for those values of frequency at which the sign is negative.

The magnitude and phase spectra are plotted in Fig. 14.4-7.

The continuous spectrum of a waveform can also be shown in terms of the real and imaginary parts of its Fourier transform. The real part of Fourier transform is zero in this example and the imaginary part is $-4 \sin^2(0.5\omega)/(0.5\omega)$. The plot of imaginary part of Fourier transform is shown in Fig. 14.4-8.

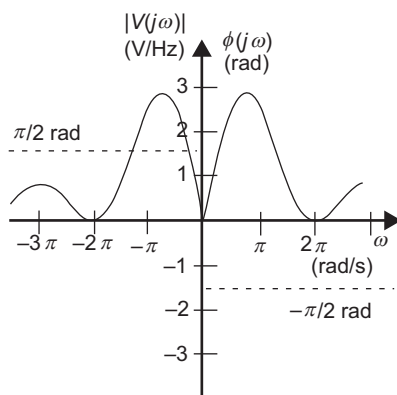


Fig. 14.4-7 Magnitude and Phase Spectra of $v(t)$ in Example 14.4-3

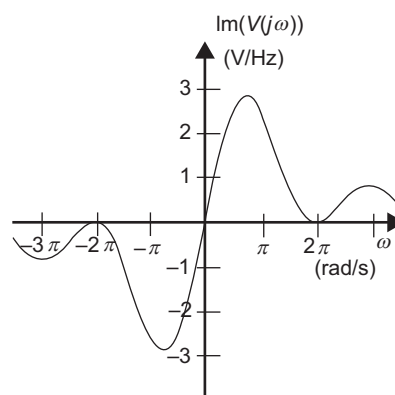


Fig. 14.4-8 Spectrum of Imaginary Part of Fourier Transform in Example 14.4-3

EXAMPLE: 14.4-4

With reference to Fig. 14.4-6, if instead of subtracting $v_2(t)$ from $v_1(t)$, it is added to $v_1(t)$, we get a rectangular pulse of height 2 and width 2 s symmetrically placed between -1 s and 1 s. Find the Fourier transform of $v(t) = v_1(t) + v_2(t)$ and plot its spectrum.

SOLUTION

The Fourier transforms of $v_1(t)$ and $v_2(t)$ have been derived in Example 14.4-3. We get the Fourier transform of $v(t)$ by adding these two transforms.

$$\begin{aligned} V(j\omega) &= V_1(j\omega) + V_2(j\omega) = \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] [e^{-j0.5\omega} + e^{j0.5\omega}] \\ &= \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] [2 \cos(0.5\omega)] = 4 \left[\frac{\sin(0.5\omega)}{(0.5\omega)} \right] \cos(0.5\omega), \end{aligned}$$

but we know that the Fourier transform of a rectangular pulse with a height of V and width τ is $(V\tau) \sin(\omega\tau/2)/(\omega\tau/2)$. It may easily be shown that this is the same as the result arrived above.

$v(t)$ and the plot of the real part of its Fourier transform are shown in Fig. 14.4-9. The imaginary part of Fourier transform is zero.

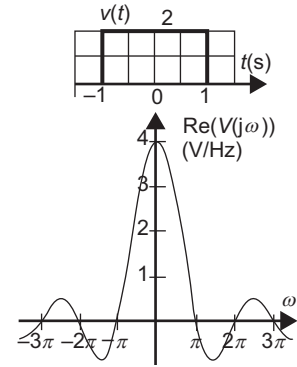


Fig. 14.4-9 Spectrum of Real Part of Fourier Transform in Example 14.4-4

EXAMPLE: 14.4-5

With reference to Fig. 14.4-6, let a new waveform be defined as $v(t) = 1.5v_1(t) + 0.5v_2(t)$, where $v_1(t)$ and $v_2(t)$ are as in Fig. 14.4-6. Find its Fourier transform and plot the real part and the imaginary part of its Fourier transform against ω .

SOLUTION

$$\begin{aligned} V(j\omega) &= 1.5V_1(j\omega) + 0.5V_2(j\omega) = \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] [1.5e^{-j0.5\omega} + 0.5e^{j0.5\omega}] \\ &= \left[2 \frac{\sin(0.5\omega)}{(0.5\omega)} \right] [2 \cos(0.5\omega) - j \sin(0.5\omega)] \end{aligned}$$

The waveform and spectra are shown in Fig. 14.4-10.

The series of examples from Example 14.4-1 to Example 14.4-5 have brought out certain interesting properties of Fourier transform of a real function of time. The first three examples dealt with time-functions which had the property that $v(-t) = v(t)$. Their Fourier transforms had the property that they had only real parts. Moreover, their real parts satisfied the condition $V(-j\omega) = V(j\omega)$.

The fourth example dealt with $v(t)$ which had the property that $v(-t) = -v(t)$ and its Fourier transform had only the imaginary part. Moreover, the imaginary part of Fourier transform satisfied the condition that $V(-j\omega) = -V(j\omega)$.

The fifth example dealt with a waveform that had no symmetry and its Fourier transform was found to have both real and imaginary parts. However, its real part satisfied the condition $\text{Re}(V(-j\omega)) = \text{Re}(V(j\omega))$ and its imaginary part satisfied the condition $\text{Im}(V(-j\omega)) = -\text{Im}(V(j\omega))$.

We look into these symmetry properties of Fourier transform in the next section.

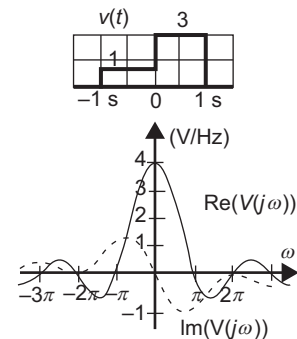


Fig. 14.4-10 Waveform and Spectra for Example 14.4-5

14.5 SYMMETRY PROPERTIES OF FOURIER TRANSFORMS

There is nothing wrong mathematically in $v(t)$ possessing an imaginary part. However, it will not be a *physical* waveform then. We deal with physical waveforms in Circuit Theory. Hence, we deal with *real functions of time* exclusively. The Fourier transform of a real $v(t)$ has many interesting symmetry properties.

14.5.1 Conjugate Symmetry Property

Let $v(t)$ be a real function of t and $V(j\omega)$ its Fourier transform. Then, $V(-j\omega) = V^*(j\omega)$. This can be shown as follows:

$$\begin{aligned}
 V(j\omega) &= \int_{-\infty}^{\infty} v(t)e^{j\omega t} dt \\
 &= \int_{-\infty}^{\infty} v(t)[\cos \omega t + j \sin \omega t] dt \\
 &= \int_{-\infty}^{\infty} v(t) \cos \omega t dt + j \int_{-\infty}^{\infty} v(t) \sin \omega t dt \\
 V(-j\omega) &= \int_{-\infty}^{\infty} v(t) \cos(-\omega t) dt + j \int_{-\infty}^{\infty} v(t) \sin(-\omega t) dt \\
 &= \int_{-\infty}^{\infty} v(t) \cos \omega t dt - j \int_{-\infty}^{\infty} v(t) \sin \omega t dt \\
 \therefore V(-j\omega) &= V^*(j\omega)
 \end{aligned}$$

This implies that $\text{Re}(V(-j\omega)) = \text{Re}(V(j\omega))$ and $\text{Im}(V(-j\omega)) = -\text{Im}(V(j\omega))$. It also implies that $|V(-j\omega)| = |V(j\omega)|$ and $\phi(-j\omega) = -\phi(j\omega)$, where $\phi(j\omega)$ is the angle of $V(j\omega)$.

Hence, for a real function of time (i) the real part of its Fourier transform is an even function of frequency and the imaginary part of its Fourier transform is an odd function of frequency and (ii) the magnitude of its Fourier transform is an even function of frequency while the phase of its Fourier transform is an odd function of frequency.

All the five examples in the previous section illustrate these properties.

Symmetry
properties of Fourier
transform of a
real $v(t)$.

14.5.2 Fourier Transform of an Even Time-Function

If $v(t)$ is an even function of t , then $v(-t) = v(t)$. But, by the time-reversal property of Fourier transforms, we have that Fourier transform of $v(-t)$ is $V(-j\omega)$ which is $V^*(j\omega)$ by conjugate symmetry property. Therefore, for an even $v(t)$, $V(j\omega)$ must be equal to its own conjugate. That will be possible only if $V(j\omega)$ has a zero imaginary part. Therefore, for a real and even $v(t)$, the Fourier transform $V(j\omega)$ is real and even on ω . The first three examples in Sect. 14.4 illustrate this aspect.

Symmetry
properties of Fourier
transform of a real
even $v(t)$.

14.5.3 Fourier Transform of an Odd Time-Function

If $v(t)$ is an odd function of t , then $v(-t) = -v(t)$. But, by the time-reversal property of Fourier transforms, we have that Fourier transform of $v(-t)$ is $V(-j\omega)$ which is $V^*(j\omega)$ by conjugate symmetry property. Therefore, for an odd $v(t)$, $V(j\omega)$ must be equal to the negative of its own conjugate. That will be possible only if $V(j\omega)$ has a zero real part. Therefore, for a real and odd $v(t)$, the Fourier transform $V(j\omega)$ is imaginary and odd on ω . The fourth example in Sect. 14.4 illustrates this aspect.

Symmetry
properties of Fourier
transform of a real
odd $v(t)$.

14.5.4 Fourier Transforms of Even Part and Odd Part of a Real Time-Function

If $v(t)$ is neither even nor odd, its Fourier transform will have both real and imaginary parts. Consider two auxiliary functions $v(t) + v(-t)$ and $v(t) - v(-t)$. The first is obviously even and the second is odd. Moreover, $v(t)$ can be expressed in terms of these two as $0.5[v(t) + v(-t)] + 0.5[v(t) - v(-t)]$. Therefore, any real function $v(t)$ can be decomposed into $v_e(t)$ and $v_o(t)$, where $v_e(t)$ is its even part and $v_o(t)$ is its odd part. These are given by

$$\begin{aligned}
 v &= v_e(t) + v_o(t) \\
 v_e(t) &= \frac{v(t) + v(-t)}{2} \quad \text{and} \quad v_o(t) = \frac{v(t) - v(-t)}{2}
 \end{aligned}$$

We apply linearity property of Fourier transforms to get

$$V(j\omega) = \text{Re}(V(j\omega)) + j \text{Im}(V(j\omega)) = V_e(j\omega) + V_o(j\omega)$$

But, $v_e(t)$ is an *even* function of time, and hence, its Fourier transform must have only a real part. Similarly, $v_o(t)$ is an *odd* function of time, and hence, its Fourier transform must have only an imaginary part. Therefore, we conclude that

If $v(t) \Leftrightarrow V(j\omega)$, then,
 $Ev[v(t)] \Leftrightarrow \text{Re}[V(j\omega)]$ and $Od[v(t)] \Leftrightarrow \text{Im}[V(j\omega)]$,
 where $Ev[v(t)]$ is the even part of $v(t)$ and $Od[v(t)]$ is its odd part.

Fourier transforms of even and odd parts of a real $v(t)$.

14.5.5 $v(0)$ and $V(j0)$

The analysis integral evaluated at $t = 0$ reveals an interesting relation between the Fourier transform value at $\omega = 0$ and the total area content of the time-function. They are equal.

$$V(j0) = \int_{-\infty}^{\infty} v(t)e^{j0t} dt = \int_{-\infty}^{\infty} v(t) dt = \text{Total area content of } v(t).$$

Hence, Fourier transform of a real time-function can not have an imaginary part at $\omega = 0$.

Similarly, the synthesis integral evaluated at $t = 0$ reveals that the value of time-function at $t = 0$ is proportional to the total area content of Fourier transform in the frequency-domain. Duality property is at play here.

$$v(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega)e^{j\omega 0} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega) d\omega = \frac{\text{Total area content of } V(j\omega)}{2\pi}.$$

$V(j\omega)$ for a real $v(t)$ is conjugate symmetric. Hence, the total area under $V(j\omega)$ will be the same as the total area under $\text{Re}(V(j\omega))$ and $v(0) = (\text{Total area under real part of Fourier transform})/2\pi$.

$V(j0) = \text{Area under } v(t)$
 $v(0) = (\text{Area under real part of Fourier transform})/2\pi$

14.6 TIME-SCALING PROPERTY AND FOURIER TRANSFORM OF IMPULSE FUNCTION

We are accustomed to the fact that when recorded music is played back at a higher speed, the treble content in the music increases and the bass content decreases. The operation that takes place in this case is compression of the signal in time-domain.

Consider a time-domain signal $v(t)$ and let $V(j\omega)$ be its Fourier transform. We define a new signal $v'(t)$ such that $v'(t) = v(at)$, where a is a real number. This means that at any t , the value of $v'(t)$ is the value that $v(t)$ assumes at time-instant at . If a is a positive real number greater than 1, the signal $v'(t)$ will be a compressed version of $v(t)$, and, if a is between 0 and 1, it will be an expanded version of $v(t)$. If a is a negative number, the signal $v'(t)$ will undergo a time reversal in addition to compression or expansion. These aspects are illustrated in Fig. 14.6-1. The original signal $v(t)$ in Fig. 14.6-1(a) is time-scaled by 2 in Fig. 14.6-1(b) and by 1/1.5 in Fig. 14.6-1(c). In Fig. 14.6-1(d), it is time-scaled by -2 , and therefore, undergoes compression as well as mirror reflection.

We notice that when a waveform is scaled in time-domain, its normalised energy changes by a factor $1/|a|$. Hence, a compressed waveform will have lesser energy and an expanded one will have higher energy.

What happens in frequency-domain when a signal $v(t)$ is time-scaled?

$$v(at) \Leftrightarrow \int_{-\infty}^{\infty} v(at)e^{j\omega t} dt$$

Substitute $t' = at$. Then, if a is positive

$$v(at) \Leftrightarrow \frac{1}{a} \int_{-\infty}^{\infty} v(t')e^{j\frac{\omega}{a}t'} dt'$$

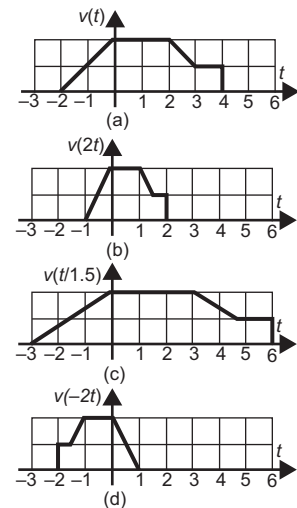


Fig. 14.6-1 Waveforms Illustrating Time-Scaling (a) Original Waveform (b) Time-Scaling by 2 (c) Time-Scaling by 1/1.5 (d) Time-Scaling by -2

If a is negative, $t' \rightarrow \infty$ as $t \rightarrow -\infty$ and $t' \rightarrow -\infty$ as $t \rightarrow \infty$

$$\therefore v(at) \Leftrightarrow -\frac{1}{a} \int_{-\infty}^{\infty} v(t') e^{j\frac{\omega}{a}t'} dt'$$

$$\therefore v(at) \Leftrightarrow -\frac{1}{|a|} V\left(j\frac{\omega}{a}\right)$$

Thus, compression in time-domain ($a > 1$) is accompanied by expansion in frequency-domain, i.e., the spectral amplitude density distribution shifts towards higher frequencies. Similarly, expanding a waveform in time-domain results in the concentration of spectral density towards lower frequencies. We consider an example to illustrate this.

Speed of waveforms and their spectral content

It is instructive to compare the spectrum of a rectangular pulse with a triangular pulse. Refer to the spectrum in Fig. 14.3-2 and compare it with the spectrum in Fig. 14.6-2.

We observe that both spectra are more concentrated in the low frequency range in the principal lobe of the spectral plot. However, the side lobe level is much less in the case of a triangular pulse than the rectangular pulse. Side lobe levels indicate the extent to which high frequency components are needed to synthesise the waveform.

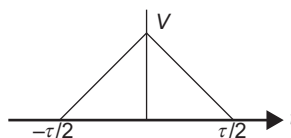
We note that a rectangular pulse needs considerable help from high frequency sinusoids for its synthesis than a triangular pulse needs. This is indeed expected, since a rectangular pulse contains a jump discontinuity – the highest speed at which a waveform can change.

High speed changes in a waveform requires high frequency sinusoids for its construction.

EXAMPLE: 14.6-1

A triangular pulse of amplitude V and duration τ s is placed in the interval $[-\tau/2, \tau/2]$ in the time-domain. (i) Derive an expression for its Fourier transform. (ii) Plot its spectrum for a special case of $V = 1$ and $\tau = 2$ s. (iii) Repeat (ii) if the waveform is time-scaled by a factor of 4.

SOLUTION



This function is an even function of time. Its Fourier transform will have only real part. If $v(t)$ is even, $v(-t) = v(t)$

$$V(j\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt = \int_0^{\infty} [v(t) e^{-j\omega t} + v(-t) e^{j\omega t}] dt$$

$$\therefore V(j\omega) = \int_0^{\infty} v(t) [e^{-j\omega t} + e^{j\omega t}] dt = 2 \int_0^{\infty} v(t) \cos \omega t dt$$

The function $v(t)$ can be expressed as $V(1 - 2t/\tau)$ for $0 \leq t \leq \tau/2$ and zero for $t > \tau/2$.

$$\begin{aligned} \therefore V(j\omega) &= 2 \int_0^{\tau/2} V \left(1 - \frac{2t}{\tau}\right) \cos \omega t dt = 2V \int_0^{\tau/2} \cos \omega t dt - \frac{4V}{\tau} \int_0^{\tau/2} t \cos \omega t dt \\ &= 2V \frac{\sin \omega \tau/2}{\omega} - \frac{4V}{\tau} \left[t \frac{\sin \omega t}{\omega} \Big|_0^{\tau/2} - \int_0^{\tau/2} \frac{\sin \omega t}{\omega} dt \right] \\ &= 2V \frac{\sin \omega \tau/2}{\omega} - \frac{4V}{\tau} \left[t \frac{\sin \omega t}{\omega} \Big|_0^{\tau/2} - \frac{\cos \omega t}{\omega^2} \Big|_0^{\tau/2} \right] = \frac{4V}{\omega^2 \tau} [1 - \cos \omega \tau/2] \\ &= \frac{V\tau}{2} \left(\frac{\sin \omega \tau/4}{\omega \tau/4} \right)^2 = (\text{Pulse Area}) \times \left(\frac{\sin \omega \tau/4}{\omega \tau/4} \right)^2 \end{aligned}$$

The required plots are shown in Fig. 14.6-2.

The spectral plots clearly illustrate that compression in time-domain has resulted in the lowering of spectral content at $\omega = 0$ (we remember that this value must be equal to the area content of $v(t)$) and shifting of spectral content to higher frequency ranges. The spectrum attains its first zero at $4\pi/\tau$ rad/s. When τ is decreased as a result of time-scaling, this value increases as seen in Fig. 14.6-2.

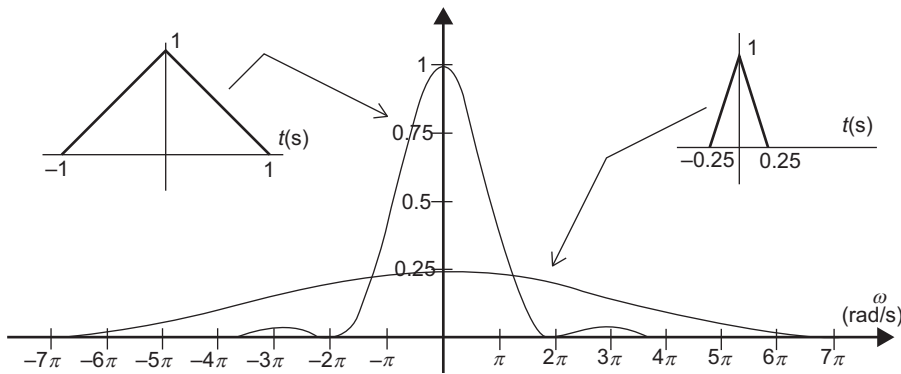


Fig. 14.6-2 Spectral Plots for Triangular Pulses in Example 14.6-1

14.6.1 Compressing a Triangular Pulse in Time-Domain with its Area Content Constant

The scaling factor of $1/|a|$ in $v(at) \Leftrightarrow 1/|a| V(j\omega/a)$ appears due to the fact that we did not keep the area content of the waveform fixed while compressing or expanding it. If we scale the amplitude by the same scaling factor that we use for time-scaling, the waveform we get will be $av(at)$ and its Fourier transform will be $V(j\omega/a)$.

Let $v(t)$ be a triangular pulse of width 2τ and height $1/\tau$, located in the interval $[-\tau, \tau]$ in the time-domain. Its area content is 1. Its Fourier transform $V(j\omega) = 1 \times [(\sin 0.5\omega\tau)/(0.5\omega\tau)]^2$. What happens to the Fourier transform as we let $\tau \rightarrow 0$ and pulse height $\rightarrow \infty$ while keeping its area at unity? Since $(\sin x)/x$ approaches 1 when $x \rightarrow 0$, $V(j\omega) \rightarrow 1$ as $\tau \rightarrow 0$.

However, a time-domain waveform that has infinitesimal width and infinite height with an area content of unity has been defined as unit impulse function. Therefore, the Fourier transform of $\delta(t)$ is 1 for all ω – i.e., $\delta(t) \Leftrightarrow 1$.

The time-scaling property of Fourier transform in effect tells us that we can not *confine* a waveform to narrower regions in time-domain without allowing it to spread out in frequency-domain. *Time-domain waveforms are claustrophobic – they react by leaking out to higher frequency ranges in frequency-domain when they are confined to smaller intervals in time-domain.*

Similarly, waveforms are claustrophobic in frequency-domain too – they react by spreading out in time-domain when they are confined in frequency-domain to narrower ranges of frequency. This must be clear if we consider time-scaling with $|a| < 1$. It follows from the duality property of Fourier transforms.

The narrowest confinement that a waveform can be subjected to in time-domain is to confine it to an infinitesimal interval – this happens in the case of $\delta(t)$. *To synthesise it we need sinusoids of all frequencies from 0 to ∞ with equal strength for all sinusoids.* The narrowest confinement in time-domain results in the widest expansion in frequency-domain.

Duality property of Fourier transform states that for every $v(t) \Leftrightarrow V(j\omega)$ pair, there exists another Fourier transform pair $V(-jt) \Leftrightarrow 2\pi v(j\omega)$. We apply this property to the Fourier transform pair that we had just arrived at, i.e., to $\delta(t) \Leftrightarrow 1$. We get $1 \Leftrightarrow 2\pi \delta(j\omega)$. This implies that the Fourier transform of a *constant time-function* $v(t) = 1$ for all t is an impulse of magnitude 2π , located at $\omega = 0$ in the frequency-domain. Thus, the widest expansion in time-domain is accompanied by the narrowest confinement in frequency-domain.

These two Fourier transform pairs are shown in Fig. 14.6-3.

Unit impulse function and unit constant functions violate Dirichlet's conditions, yet they have Fourier transforms. We worked out the Fourier transform of unit impulse function

Fourier transform of unit impulse function is unity.

Waveforms and claustrophobia?

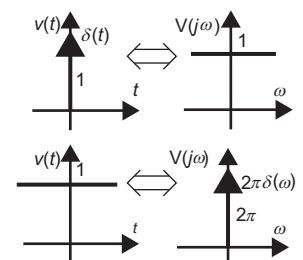


Fig. 14.6-3 Fourier Transforms of Unit Impulse and Unit Constant Functions

as a limiting case of a triangular pulse. However, a straightforward application of analysis integral will also give us the same result. Uniqueness of Fourier transform pair assures us that the synthesis integral should give us $\delta(t)$. Therefore, applying the synthesis equation on Fourier transform of unit impulse function, we get,

$$\int_{-\infty}^{\infty} e^{j\omega t} d\omega = 2\pi\delta(t).$$

By applying analysis equation on Fourier transform of unit constant function, we get,

$$\int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(j\omega).$$

Both the integrals are improper integrals and can not be evaluated otherwise.

14.7 FOURIER TRANSFORMS OF PERIODIC WAVEFORMS

We arrived at the Fourier transform of $\delta(t)$ against the background of time-domain compression of waveforms in the last section. Further, by applying duality principle on this Fourier transform, we derived the Fourier transform for $v(t) = 1$. Then, we tried to obtain the Fourier transform by applying analysis integral on $v(t) = 1$ and ended up with an improper integral. We remembered the theorem of uniqueness of Fourier transforms and assigned $2\pi\delta(j\omega)$ to this integral, since we had seen by other means that this is the Fourier transform of $v(t) = 1$. This integral is

$$\int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(j\omega).$$

We observe that the variable of integration is t , and hence, any substitution for ω in the integrand will simply appear as an argument of impulse on the right side. We choose to substitute $\omega = \omega - \omega_0$, where ω_0 is a specific value of ω . Then,

$$\int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt = 2\pi\delta(j[\omega - \omega_0]).$$

But the integral on the left side is the Fourier analysis integral for the time-function $e^{j\omega_0 t}$ since $\int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt = \int_{-\infty}^{\infty} [e^{j\omega_0 t}] e^{-j\omega t} dt = FT \text{ of } e^{j\omega_0 t}$. Therefore,

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - \omega_0), \text{ and by a similar argument, } e^{-j\omega_0 t} \Leftrightarrow 2\pi\delta(j[\omega + \omega_0])$$

We are only one step away from finding the Fourier transforms of $\sin\omega t$ and $\cos\omega t$.

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}; \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\therefore \cos\omega_0 t \Leftrightarrow \pi\delta(j[\omega - \omega_0]) + \pi\delta(j[\omega + \omega_0])$$

$$\sin\omega_0 t \Leftrightarrow -j\pi\delta(j[\omega - \omega_0]) - j\pi\delta(j[\omega + \omega_0])$$

These are periodic functions with unit amplitude, starting at $-\infty$ and proceeding to ∞ in the time-domain. They possess infinite energy. They are ‘finite power’ waveforms since they have a finite average power of 0.5 W over a cycle. They are not absolutely integrable, and hence, do not satisfy Dirichlet’s conditions. Yet, they have Fourier transforms; only that their Fourier transforms are pairs of impulses located at $\pm\omega_0$, with each impulse magnitude at π V (assuming they are voltage functions).

We remember that Fourier transform gives the density of spectral amplitude at a frequency. A cosine wave at ω_0 rad/s has only one frequency component and all its amplitude, i.e., 1 unit, is concentrated at ω_0 . Hence, the amplitude density at $\pm\omega_0$ (in a two-sided exponential representation) must be ∞ and it must be zero at other frequency points. Density, when integrated, must give the amplitude. Therefore, though the density function is ∞ at $\pm\omega_0$ and zero elsewhere, its integral must yield 1 unit with 2π factor in the inverse Fourier transform accounted. The impulse representation of Fourier transform of periodic functions is justified from this viewpoint too.

Fourier transforms
of $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$.

Fourier transforms
of $\sin\omega_0 t$ and $\cos\omega_0 t$.

A non-sinusoidal periodic waveform can be represented as a sum of harmonically related sine and cosine functions.

Hence, a periodic waveform has a Fourier transform as well as a Fourier series.

Its Fourier transform will be a train of impulse functions located at harmonically related frequency points with the magnitude of impulses equal to the magnitude of Fourier series coefficients.

The Fourier transforms of cosine and sine functions are shown in Fig. 14.7-1.

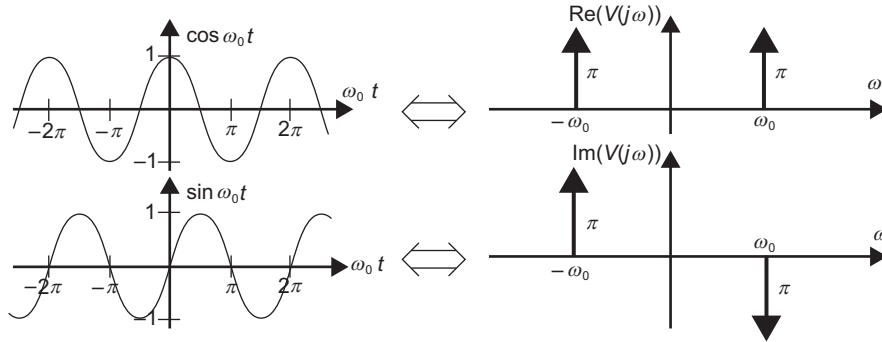


Fig. 14.7-1 Fourier transforms of Unit Cosine and Unit Sine Waveforms

14.8 FOURIER TRANSFORMS OF SOME SEMI-INFINITE DURATION WAVEFORMS

Inputs, even if they are periodic waveforms, are switched on to the circuit at some finite time-instant. This fact is usually brought out by specifying an input function as $f(t)u(t)$, where $f(t)$ is a periodic or aperiodic waveform. A DC voltage of V applied to the circuit from $t = 0$ is specified as $Vu(t)$ and a unit amplitude cosine wave applied to the circuit from $t = 0$ is specified as $\cos(\omega t)u(t)$.

We developed the Fourier transforms for periodic waveforms, including the one with infinite period, i.e., a constant function, in the last section. We study the effect of switching at $t = 0$ on their Fourier transforms in this section.

14.8.1 Fourier Transform of $e^{-\alpha t}u(t)$

Let $v_1(t) = e^{-\alpha t}u(t)$. We find its Fourier transform by the straightforward application of the analysis integral.

$$\begin{aligned}
 V_1(j\omega) &= \int_{-\infty}^{\infty} v_1(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-\alpha t} e^{-j\omega t} dt = \frac{1}{j\omega + \alpha} \\
 \text{Real part} &= \frac{\alpha}{\alpha^2 + \omega^2} \text{ and imaginary part} = \frac{-\omega}{\alpha^2 + \omega^2} \\
 \text{Magnitude} &= \frac{1}{\sqrt{\alpha^2 + \omega^2}} \text{ and phase} = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)
 \end{aligned} \tag{14.8-1}$$

Fourier transform of $e^{-\alpha t}u(t)$.

We note that (i) the real part of Fourier transform is *even*, (ii) the imaginary part of Fourier transform is *odd*, (iii) the magnitude of Fourier transform is *even* and (iv) the phase of Fourier transform is *odd*.

Now, let us consider a function $v_2(t)$ defined as $v_2(t) = e^{\alpha t}u(-t)$. It is easily seen that $v_2(t)$ is $v_1(-t)$, i.e., a mirror reflected version of $v_1(t)$. Using the time-reversal property and the conjugate symmetry property of Fourier transforms, we write its Fourier transform as

$$\begin{aligned}
 V_2(j\omega) &= \frac{1}{-j\omega + \alpha} \\
 \text{Real part} &= \frac{\alpha}{\alpha^2 + \omega^2} \text{ and imaginary part} = \frac{\omega}{\alpha^2 + \omega^2}
 \end{aligned} \tag{14.8-2}$$

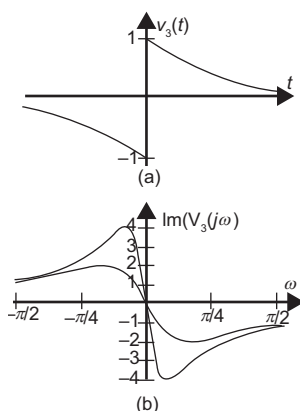


Fig. 14.8-1
(a) Waveform of $[e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$
and (b) its Fourier
Transform for $\alpha = 0.5$
and 0.25

Signum function
defined.

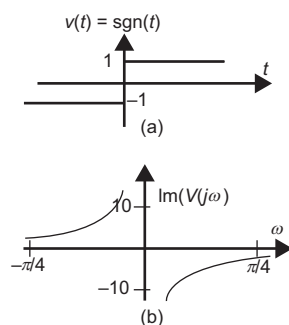


Fig. 14.8-2 Signum
Function and its
Fourier transform

$$\text{Magnitude} = \frac{1}{\sqrt{\alpha^2 + \omega^2}} \text{ and phase} = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

We consider a third function $v_3(t)$ defined as $v_3(t) = v_1(t) - v_2(t) = e^{-\alpha t} u(t) - e^{\alpha t} u(-t)$. $V_3(j\omega)$ is obtained by subtracting $V_2(j\omega)$ from $V_1(j\omega)$. Hence, $V_3(j\omega)$ will have only the imaginary part. We note that $v_3(t)$ is indeed *odd*.

$$\begin{aligned} V_3(j\omega) &= V_1(j\omega) - V_2(j\omega) \\ &= -j \frac{2\omega}{\alpha^2 + \omega^2} \end{aligned} \quad (14.8-3)$$

The waveform $v_3(t)$ and the imaginary part of its Fourier transform for two values of $\alpha = 0.5$ and 0.25 are shown in Fig. 14.8-1. We note that in both cases the imaginary parts of the Fourier transform goes through the origin – in fact this will be true for any non-zero value of α , as Eqn. 14.8-3 would indicate.

14.8.2 Fourier Transform of Signum Function

The function $[e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$ takes on an interesting shape as $\alpha \rightarrow 0$. It becomes $+1$ for $t > 0$ and -1 for $t < 0$. This is defined as the *Signum Function of t* and is indicated by $\text{sgn}(t)$.

$$\text{sgn}(t) = \begin{cases} -1 & \text{for } -\infty < t < 0 \\ \text{undefined} & \text{for } t = 0 \\ 1 & \text{for } 0 < t < \infty \end{cases} \quad (14.8-4)$$

$\text{sgn}(t)$ does not satisfy Dirichlet's conditions, but $v(t) = [e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$ does satisfy it for positive values of α . Therefore, we express $\text{sgn}(t)$ as the limit of $v(t)$ defined as above, as $\alpha \rightarrow 0$. It is only a *limit*, and hence, α never becomes zero; it only approaches zero. Therefore, its Fourier transform given by Eqn. 14.8-3 always goes through the origin in frequency-domain. We now express the Fourier transform of $\text{sgn}(t)$ as the limit of Fourier transform of $v(t)$ as $\alpha \rightarrow 0$.

$$\therefore \text{sgn}(t) \Leftrightarrow \lim_{\alpha \rightarrow 0} \left[-j \frac{2\omega}{\alpha^2 + \omega^2} \right] = \frac{2}{j\omega} \quad (14.8-5)$$

This equation suggests that Fourier transform of $\text{sgn}(t)$ goes to ∞ at $\omega = 0$, but it does not, because the Fourier transform of $[e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$ is a continuous function which goes through the origin for all positive values of α and when we take the limit, we only make α arbitrarily close to 0; but never 0. Therefore, Fourier transform of $\text{sgn}(t)$ is the limit in Eqn. 14.8-5; but its second form, i.e., $(2/j\omega)$ can be used only for $\omega \neq 0$. This is also supported by the fact that the value $V(j0)$ of a Fourier transform is equal to the *total area content* of $v(t)$. The total area content of $\text{sgn}(t)$ is obviously zero. Therefore, Fourier transform of $\text{sgn}(t)$ should have zero value at $\omega = 0$. Signum function and its Fourier transform are shown in Fig. 14.8-2.

14.8.3 Fourier Transform of Unit Step Function

The unit step function $v(t) = u(t)$ is a semi-infinite duration, infinite energy waveform that violates Dirichlet's conditions. It has a Fourier transform if it is interpreted as a limiting case of another time-function.

Consider $v'(t)$ defined as $v'(t) = 0.5 + 0.5[e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$. When we send $\alpha \rightarrow 0$, the second term in this function approaches $0.5 \text{sgn}(t)$ and $v'(t)$ approaches $u(t)$. Therefore, the Fourier transform of $u(t)$ is the limit of Fourier transform of $v'(t)$ as $\alpha \rightarrow 0$. Therefore, we get,

$$u(t) \Leftrightarrow \pi\delta(j\omega) + 0.5 \lim_{\alpha \rightarrow 0} \left[\frac{-j2\omega}{\alpha^2 + \omega^2} \right] = \pi\delta(j\omega) + \frac{1}{j\omega} \quad (14.8-6)$$

We have to remember that α never becomes zero, and hence, the $1/j\omega$ component should never be evaluated at $\omega = 0$. Instead, the value of second term at $\omega = 0$ should be correctly evaluated as zero.

The first component in the Eqn. 14.8-6 indicates that there is a constant component of 0.5 in $u(t)$ – it is indeed so since the average value of $u(t)$ is 0.5. The $u(t)$ function and its Fourier transform magnitude are shown in Fig. 14.8-3.

14.8.4 Fourier transform of Functions of the Form $e^{(-\alpha + j\omega_0)t} \times v(t)$

Let $v'(t) = e^{(-\alpha + j\omega_0)t} \times v(t)$. α is a positive real number. The Fourier transform of $v'(t)$ is

$$\begin{aligned} V'(j\omega) &= \int_{-\infty}^{\infty} v(t) e^{(-\alpha + j\omega_0)t} e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} v(t) e^{-[\alpha + j(\omega - \omega_0)]t} dt \\ &= V[\alpha + j(\omega - \omega_0)] \end{aligned} \quad (14.8-7)$$

This implies that the required Fourier transform can be obtained by replacing every $(j\omega)$ in the Fourier transform of $v(t)$ by $\alpha + j(\omega - \omega_0)$. We identify two special cases here. In the first case, we make $\omega_0 = 0$. Then,

$$e^{-\alpha t} v(t) \Leftrightarrow V(\alpha + j\omega) \quad (14.8-8)$$

We had derived the Fourier transform of $e^{-\alpha t} u(t)$ in an earlier sub-section and the result was $1/(\alpha + j\omega)$. But it should be $\pi\delta(\alpha + j\omega) + 1/(\alpha + j\omega)$ according to Eqn. 14.8-8, because Fourier transform of $u(t)$ is $\pi\delta(\omega) + 1/j\omega$. How do we explain the difference? The function $\pi\delta(\alpha + j\omega)$ is 0 for $(\alpha + j\omega) < 0$ and $(\alpha + j\omega) > 0$. It has an area content of π located at $(\alpha + j\omega) = 0$. However, no value of ω in the range $-\infty$ to $+\infty$ can make $(\alpha + j\omega)$ equal to zero because α is a *real number*. Therefore, this impulse component will never come into play, and hence, need not be carried in the Fourier transform function. Therefore, Fourier transform of $e^{-\alpha t} u(t)$ is $1/(\alpha + j\omega)$ by Eqn. 14.8-8. $u(t)$ is an infinite-energy waveform; $e^{-\alpha t} u(t)$ with positive real α is a finite-energy waveform. This explains the presence of impulse content in the Fourier transform of $u(t)$ and its absence in the Fourier transform of $e^{-\alpha t} u(t)$.

The second case is the one with $\alpha = 0$. Then, $e^{j\omega_0 t} v(t) \Leftrightarrow V[j(\omega - \omega_0)]$. This means that the Fourier transform gets *shifted in the frequency domain* when the time-domain function is multiplied by a complex exponential function. This is termed as the *Frequency Shifting Property* of Fourier transforms.

Now, we multiply $u(t)$ by $0.5(e^{j\omega_0 t} + e^{-j\omega_0 t})$ to get $\cos\omega_0 t u(t)$. The Fourier transform of this function is

$$\cos\omega_0 t \times u(t) \Leftrightarrow \frac{\pi}{2} [\delta(j[\omega - \omega_0]) + \delta(j[\omega + \omega_0])] + \frac{j\omega}{(j\omega)^2 + \omega_0^2} \quad (14.8-9)$$

We have used the Fourier transform of $u(t)$ in deriving the above Fourier transform, and hence, the second term on the right side should not be evaluated at ω_0 . This Fourier transform indicates that there is a pair of impulses located at $\pm\omega_0$. This implies that there is a periodic component $0.5 \cos\omega_0 t$ in this signal. It is indeed so because we can write $\cos\omega_0 t \times u(t)$ as $[0.5 + 0.5\text{sgn}(t)]\cos\omega_0 t$.

Fourier transform of $\sin\omega_0 t u(t)$ can be similarly derived as:

$$\sin\omega_0 t \times u(t) \Leftrightarrow j \frac{\pi}{2} [-\delta(j[\omega - \omega_0]) + \delta(j[\omega + \omega_0])] + \frac{\omega_0}{(j\omega)^2 + \omega_0^2} \quad (14.8-10)$$

Fourier transform of $e^{-\alpha t} \cos\omega_0 t u(t)$ is obtained by using Eqn. 14.8-7 along with Eqn. 14.8-9. The resulting Fourier transform is

$$e^{-\alpha t} \cos\omega_0 t \times u(t) \Leftrightarrow \frac{(\alpha + j\omega)}{(\alpha + j\omega)^2 + \omega_0^2} \quad (14.8-11)$$

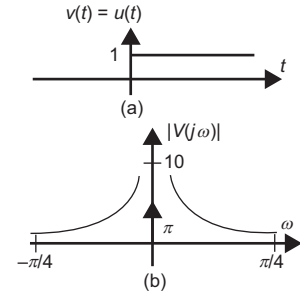


Fig. 14.8-3 Unit Step Function and Magnitude of its Fourier Transform

Fourier transform of $e^{(-\alpha + j\omega)t} v(t)$.

Fourier transform of $e^{-\alpha t} v(t)$.

Frequency shifting property of Fourier transform.

Fourier transform of $[\cos\omega_0 t] u(t)$.

Fourier transform of $[\sin\omega_0 t] u(t)$.

Fourier transform of the function $[e^{-\alpha t} \cos\omega_0 t] u(t)$.

Fourier transform
of the function
 $[e^{-\alpha t} \sin \omega_0 t]u(t)$.

Similarly, the Fourier transform of exponentially damped sine wave can be obtained as

$$e^{-\alpha t} \sin \omega_0 t \times u(t) \Leftrightarrow \frac{\omega_0}{(\alpha + j\omega)^2 + \omega_0^2} \quad (14.8-12)$$

In both cases we have dropped the impulse components in the spectrum since those impulses can not come into action for any legitimate value of ω . We notice that damped sinusoids are aperiodic waveforms with finite energy, whereas, undamped sinusoids are aperiodic waveforms with infinite energy.

14.9 ZERO-STATE RESPONSE BY FREQUENCY-DOMAIN ANALYSIS

We have seen that all practical waveforms, periodic or aperiodic, possess frequency-domain descriptions in the form of Fourier transforms. All waveforms can be synthesised from sinusoids of different frequencies by adding them up. The sinusoids that constitute a periodic waveform as well as an aperiodic waveform are *periodic* and start from $-\infty$ and go to $+\infty$ in the time-domain. *All the details in an aperiodic waveform are constructed by such everlasting sinusoidal signals interfering with each other constructively and destructively at various time-intervals. Thus, an infinite number of periodic sinusoids synthesise the aperiodic nature of an aperiodic waveform.*

The aperiodic nature of a waveform gives rise to the transient response in a linear circuit when it is applied to the circuit. We are considering a linear circuit and such a circuit obeys the superposition principle as far as *zero-state response* is concerned. Hence, the zero-state response to the aperiodic input can be obtained by summing up the zero-state responses that the circuit will display for each component in a set of waveform components that constitute the aperiodic waveform. The set of waveform components we have in mind are those infinite number of sinusoids with frequencies from 0 to ∞ with infinitesimal amplitudes (for finite energy signals) which go into the synthesis of the aperiodic waveform. But, each such component is *periodic and everlasting*, and hence, will produce only a steady-state response. Therefore, we conclude that the *zero-state response* (including the transient response components) in a linear circuit when driven by an aperiodic waveform can be obtained as a superposition of many (possibly infinite) *sinusoidal steady-state response components*.

Let $v_i(t)$ be the input signal to a *linear* circuit and let $v_o(t)$ be its output. We may assume them to be voltage variables for the purpose of being specific. Let $V_i(j\omega)$ and $V_o(j\omega)$ be the corresponding Fourier transforms. Consider an infinitesimal band of frequencies $\Delta\omega$ around a frequency ω . The sum of amplitudes of all complex exponential components contributing to $v_i(t)$ from this band is $V_i(j\omega) \times \Delta\omega/2\pi$. For a sufficiently small $\Delta\omega$, all those components may be taken to have the same frequency ω . Hence, the complex exponential input due to components from this band is $(V_i(j\omega) \times \Delta\omega/2\pi)e^{j\omega t}$. The output due to this component is given by multiplying this input component by the value of *frequency response function* of the circuit. We had represented this function by $H(j\omega)$ earlier. Its magnitude gives the ratio of output amplitude to input amplitude. Its phase angle gives the phase by which the output leads the input.

$$\text{Contribution to } v_o(t) \text{ from } \Delta\omega\text{-band in } v_i(t) = \frac{1}{2\pi} V_i(j\omega) H(j\omega) e^{j\omega t} \Delta\omega$$

$$\therefore v_o(t) = \lim_{\Delta\omega \rightarrow 0} \sum_{\substack{\text{over all such } \Delta\omega \\ \text{bands covering the} \\ \text{entire } \omega\text{-domain}}} \frac{1}{2\pi} V_i(j\omega) H(j\omega) e^{j\omega t} \Delta\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} V_i(j\omega) H(j\omega) e^{j\omega t} d\omega$$

$$\text{But } v_o(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V_o(j\omega) e^{j\omega t} d\omega$$

$$\therefore V_o(j\omega) = V_i(j\omega) H(j\omega) \quad (14.9-1)$$

Where are we?

Two questions were raised in the beginning of this chapter.

- (i) Can an aperiodic waveform be resolved into a sum of sinusoids? If yes, which are the frequencies that are present in the signal decomposition?
- (ii) Can the steady-state frequency response function help us to get the output when such an aperiodic waveform is applied to the circuit? If yes, how do we explain a *steady-state* function yielding a *transient* response term?

The first question was answered in the previous sections. The second question is addressed in Sect. 14.9 and the subsequent sections.

If the circuit is stable, then
Output transform =
Input transform $\times$
Frequency response
function.

From time-domain analysis principles, we know that the zero-state response of $v_o(t)$ for any input can be obtained by convolving the input function with the impulse response $h(t)$. Let us use this principle to work out the relation between the impulse response and the frequency response function. We apply a unit amplitude complex exponential input $e^{j\omega t}$. Then,

$$v_o(t) = e^{j\omega t} * h(t) = h(t) * e^{j\omega t} = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = \left[\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right] e^{j\omega t}$$

But, now we recognise $\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$, as the Fourier transform of $h(t)$. Further, we know that $v_o(t) = H(j\omega) \times e^{j\omega t}$ from our studies on sinusoidal steady-state response.

$$\therefore v_o(t) = e^{j\omega t} * h(t) = \left[\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right] e^{j\omega t} = H(j\omega) \times e^{j\omega t}$$

Therefore, $H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$ i.e., the Fourier transform of $h(t)$ is the same as

the sinusoidal frequency response function $H(j\omega)$.

We had chosen the symbol $H(j\omega)$ for frequency response function earlier in anticipation of this important principle.

Further, we apply time-domain convolution to $v_i(t)$ with $h(t)$ to obtain $v_o(t)$ as below:

$$v_o(t) = \int_{-\infty}^{\infty} h(\tau) v_i(t-\tau) d\tau \quad (14.9-2)$$

Therefore, the inverse Fourier transform of the product of input Fourier transform and the Fourier transform of the impulse response is the same as the time-domain convolution between the input function and the impulse response function. Given two time-domain functions $v_1(t)$ and $v_2(t)$, we have the liberty to assume that one is the input to a circuit and the other is the impulse response of that circuit. Then, we arrive at the following result.

$$\begin{aligned} \text{If } v_1(t) \Leftrightarrow V_1(j\omega), v_2(t) \Leftrightarrow V_2(j\omega) \text{ and } V_3(j\omega) = V_1(j\omega)V_2(j\omega), \text{ then} \\ v_3(t) = \int_{-\infty}^{\infty} v_1(\tau) v_2(t-\tau) d\tau = \int_{-\infty}^{\infty} v_2(\tau) v_1(t-\tau) d\tau \end{aligned} \quad (14.9-3)$$

This is a statement of 'Time-domain Convolution Property of Fourier transforms'.

The ratio of Fourier transform of output to Fourier transform of input is called *System Function*. Equation 14.9-1 shows that system function is the same as frequency response function. Further, it is clear from the discussion above that system function of a linear circuit is the same as the Fourier transform of its impulse response and all are represented by $H(j\omega)$.

An important note

We used frequency response function in Eqn. 14.9-1. An unstable circuit has no steady-state, and hence, it has no frequency response function. Moreover, Fourier transform does not exist for the impulse response of an unstable circuit.

Fourier series and transform technique is applicable only for stable circuits and the results derived in this section are valid only for stable circuits.

If the circuit is stable, then

Frequency response function = Fourier transform of impulse response.

Time-domain Convolution Theorem.

System Function defined.

14.9.1 Why Should the System Function and Fourier Transform of Impulse Response be the Same?

The Fourier transform of $\delta(t)$ is unity. Hence, $\delta(t)$ contains sinusoids of all frequencies with equal strength. Consider a particular frequency value ω_0 and a band of frequencies $\Delta\omega$ around it. Then, the complex exponential contributed by this band is $(\Delta\omega/2\pi)e^{j\omega_0 t}$. When this component goes through the circuit, it comes out as $H(j\omega_0) (\Delta\omega/2\pi) e^{j\omega_0 t}$, where $H(j\omega_0)$ is the frequency response function value at ω_0 . The impulse response will contain this component. The value of Fourier transform of impulse response at ω_0 is 2π times the limiting value of the amplitude density at that frequency. Obviously, it will be $H(j\omega_0)$

(because $\lim_{\Delta\omega \rightarrow 0} \frac{\Delta\omega}{2\pi} H(j\omega_0) = H(j\omega_0)$). The only difference between $H(j\omega_0)$ and Fourier transform of the impulse response at ω_0 will be the unit – a Fourier transform has volt/Hz

A question of
dimension and
units . . .

as its unit, whereas the frequency response function is dimensionless; if $v_i(t)$ and $v_o(t)$ are voltage variables.

We use the system function and Fourier transform of the impulse response interchangeably without paying any attention to the difference in their units. Yet, we will arrive at the correct results. Consider Eqn. 14.9-1. There is no dimensional inconsistency in this equation because $H(j\omega)$ is the frequency response function in it. But if $H(j\omega)$ is interpreted as the Fourier transform of the impulse response, there is an apparent dimensional inconsistency. If there is a dimensional inconsistency in it, there is a similar inconsistency in the convolution integral Eqn. 14.9-2. If we multiply two ‘volts’ and integrate, we should have got ‘volts²’. We immediately remember that in convolution integral, the value of input function is used to *scale* the impulse response and its unit is discarded in this scaling process. Similarly, we discard the unit of Fourier transform of input when we treat $H(j\omega)$ in Eqn. 14.9-1 as the Fourier transform of impulse response.

Time-Domain
Integration Property
of Fourier transforms.

EXAMPLE: 14.9-1

Let $v(t) \Leftrightarrow V(j\omega)$. Find the Fourier transform of $\int_{-\infty}^t v(t)dt$.

SOLUTION

Think of a circuit that can do integration. A single Opamp with one capacitor and one resistor can do it, as we have seen in an earlier chapter. What is the system function of this circuit?

We find the impulse response of the integrator first. That is easy since we know that integral of $\delta(t)$ is $u(t)$. Therefore, the impulse response of an integrator is unit step function. The Fourier transform of $u(t)$ is $\pi\delta(\omega) + 1/j\omega$. Therefore, the system function of the integrator circuit is $\pi\delta(\omega) + 1/j\omega$. Since output transform is given by the product of input transform and system function, Fourier transform of

$$\int_{-\infty}^t v(t)dt = V(j\omega) \times \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] = \frac{V(j\omega)}{j\omega} + \pi V(0)\delta(\omega) \quad (14.9.4)$$

This is the ‘Time-Domain Integration Property of Fourier transforms’.

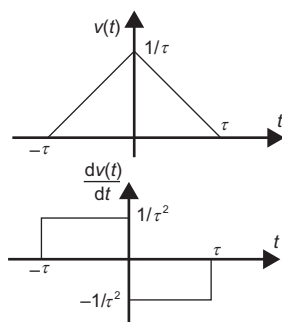


Fig. 14.9-1 A
Triangular Pulse and
its Derivative

EXAMPLE: 14.9-2

Let $v(t) \Leftrightarrow V(j\omega)$. Find the Fourier transform of $\frac{dv(t)}{dt}$.

SOLUTION

Think of a circuit that can do differentiation. What is the system function of this circuit?

We have to find the impulse response of the differentiator circuit first. We do not try to differentiate $\delta(t)$. Fourier transform of $\delta(t)$ was obtained by a limiting process in which the width of a triangular pulse was reduced to infinitesimal with its area maintained at unity in Example 14.6-1. Now, we differentiate this triangular pulse and see what happens when we compress the triangular pulse. The pulse and its first derivative are shown in Fig. 14.9-1.

The Fourier transform of the first derivative is obtained as below:

$$\text{Fourier transform of a symmetrically located rectangular pulse} = \text{Area} \times \frac{\sin \omega \tau / 2}{\omega \tau / 2}$$

When the pulse is delayed by $\tau/2$ its Fourier transform gets multiplied by $e^{-j\omega\tau/2}$

When the pulse is advanced by $\tau/2$ its Fourier transform gets multiplied by $e^{j\omega\tau/2}$

$$\begin{aligned}
 \therefore \frac{dv(t)}{dt} &\Leftrightarrow -\frac{1}{\tau} \times \frac{\sin \omega \tau / 2}{\omega \tau / 2} e^{-j\omega \tau / 2} + \frac{1}{\tau} \times \frac{\sin \omega \tau / 2}{\omega \tau / 2} e^{j\omega \tau / 2} \\
 &= \frac{2j}{\tau} \times \frac{\sin \omega \tau / 2}{\omega \tau / 2} \sin \omega \tau / 2 \text{ (By Euler's formula)} \\
 &= j\omega \times \frac{\sin \omega \tau / 2}{\omega \tau / 2} \times \frac{\sin \omega \tau / 2}{\omega \tau / 2}
 \end{aligned}$$

As $\tau \rightarrow 0$, this Fourier transform $\rightarrow j\omega$ and $v(t) \rightarrow \delta(t)$ simultaneously. Therefore, the system function of a differentiating circuit is $j\omega$.

$$\therefore \text{If } v(t) \Leftrightarrow V(j\omega), \text{ then, } \frac{dv(t)}{dt} \Leftrightarrow j\omega V(j\omega). \quad (14.9-5)$$

This is the 'Time-Domain Differentiation Property of Fourier transforms'. Differentiation in time-domain results in the multiplication by frequency in the frequency-domain.

*Time-Domain
Differentiation
Property of Fourier
transforms.*

EXAMPLE: 14.9-3

Apply the duality property to the result arrived at in Example 14.9-2 and obtain the inverse of $\frac{dV(j\omega)}{d\omega}$, if $v(t) \Leftrightarrow V(j\omega)$.

SOLUTION

$$v(t) \Leftrightarrow V(j\omega)$$

$$\therefore \text{By duality property } v(j\omega) \Leftrightarrow \frac{V(-jt)}{2\pi}$$

$$\text{Let } v'(t) = \frac{V(-jt)}{2\pi}$$

$$\text{Now, } \frac{dv(t)}{dt} \Leftrightarrow j\omega V(j\omega)$$

$$\therefore \text{By duality property } \frac{dv(j\omega)}{d\omega} \Leftrightarrow -jt \frac{V(-jt)}{2\pi} = -jt v'(t)$$

$$\therefore tv'(t) \Leftrightarrow \frac{1}{-j} \times \frac{d(\text{Fourier transform of } v'(t))}{d\omega}, \text{ changing variable } v' \text{ to } v, \text{ we get,}$$

$$tv(t) \Leftrightarrow j \frac{dV(j\omega)}{d\omega} \quad (14.9-6)$$

Equation 14.9-6 is a statement of 'Frequency-Domain Differentiation Property of Fourier transforms'. Differentiation in frequency-domain is accompanied by multiplication by time in time-domain.

*Frequency-
Domain
Differentiation
Property of Fourier
transforms.*

EXAMPLE: 14.9-4

The impulse response of a circuit containing R , L and C is found to be $h(t) = 2e^{-0.5t} + 0.5e^{-t} \sin(100t - \frac{\pi}{3})$; $t \geq 0^+$. (i) What is the order of the circuit? (ii) What is the minimum number of energy storage elements in this circuit? (iii) Can all the energy storage elements in this circuit be of the same type? (iv) Find the steady-state output when $2 \cos(50t + 50^\circ) u(t)$ V is applied to the circuit.

SOLUTION

- (i) There is an exponential with negative real index in the impulse response. It contributes one to the order of the circuit. There is an exponentially damped sinusoidal component in the impulse response. Each such damped sinusoid contributes two to the order of a circuit. Therefore, this circuit is of the third order.
- (ii) The number of independent energy storage elements in a circuit has to be equal to its order. Therefore, the minimum number of energy storage elements in this circuit is three.
- (iii) Sinusoidal oscillatory components in transient response are the result of competing energy storage elements in the circuit. Therefore, all the energy storage elements in this circuit can not be of the same type.
- (iv) We have to find and evaluate the frequency response function at 50 rad/s to calculate the required output. Frequency response function is the same as system function, i.e., the Fourier transform of the impulse response. Impulse response contains two terms. We apply linearity property of Fourier transforms and find the Fourier transform of the impulse response term by term.

$$\text{Fourier transform of } 2e^{-0.5t} u(t) \text{ is } \frac{2}{0.5 + j\omega}.$$

Fourier transform of $0.5e^{-t} \sin(100t - \frac{\pi}{3}) u(t)$ is found as below:

$$0.5e^{-t} \sin(100t - \frac{\pi}{3}) u(t) = 0.5e^{-t} [0.5 \sin 100t - 0.866 \cos 100t] u(t)$$

$$\begin{aligned} \therefore \text{Required Fourier transform} &= \frac{25}{(j\omega + 1)^2 + 100^2} - \frac{0.433(j\omega + 1)}{(j\omega + 1)^2 + 100^2} \\ &= \frac{24.567 - j0.433\omega}{(10001 - \omega^2) + 2j\omega} \end{aligned}$$

Therefore,

$$H(j\omega) = \frac{2}{0.5 + j\omega} + \frac{24.567 - j0.433\omega}{(10001 - \omega^2) + 2j\omega}$$

Evaluating $H(j\omega)$ at $\omega = 50$ rad/s,

$$\begin{aligned} H(j50) &= \frac{2}{0.5 + j50} + \frac{24.567 - j21.65}{7501 + 100j} = 0.0004 - j0.04 + 0.003236 - j0.00293 \\ &= 0.00364 - j0.04293 = 0.0431 \angle -85.15^\circ \end{aligned}$$

Therefore the output when $2 \cos(50t + 50^\circ) u(t)$ V is applied is
 $= 2 \times 0.0431 \cos(50t + 50^\circ - 85.15^\circ) = 0.0862 \cos(50t - 35.15^\circ)$ V.

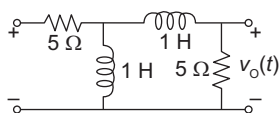


Fig. 14.9-2 Circuit for Example 14.9-5

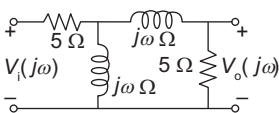


Fig. 14.9-3 Phasor Equivalent Circuit for Example 14.9-5

EXAMPLE: 14.9-5

- (i) Find the step response of $v_o(t)$ in the RL circuit given in Fig. 14.9-2. (ii) Evaluate the area content of $v_o(t)$ and verify that it is equal to $V_o(j0)$.

SOLUTION

- (i) We need to find the system function of the circuit. It can be determined by obtaining the frequency response function of the circuit. Frequency response can be obtained by using the phasor equivalent circuit. This equivalent circuit is shown in Fig. 14.9-3.

Input impedance $Z = 5 + (5 + j\omega) // j\omega$

$$\text{Input current} = \frac{V_i(j\omega)}{Z}$$

$$\text{Current into } 5 \Omega \text{ at the output} = \frac{V_i(j\omega)}{Z} \times \frac{j\omega}{5 + j2\omega}$$

$$\therefore H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{Z} \times \frac{j\omega}{5 + j2\omega} \times 5$$

Substituting for Z and simplifying,

$$H(j\omega) = \frac{5j\omega}{(j\omega)^2 + 15j\omega + 25} = \frac{5j\omega}{(j\omega + 13.09)(j\omega + 1.91)}$$

Fourier transform of output is given by the product of Fourier transform of input and system function. Input is $u(t)$ and its transform is $\pi\delta(j\omega) + 1/j\omega$.

$$\therefore V_o(j\omega) = \frac{5}{(j\omega + 13.09)(j\omega + 1.91)} + \frac{5\pi j\omega\delta(j\omega)}{(j\omega + 13.09)(j\omega + 1.91)}$$

The second term contains an impulse at $\omega = 0$ scaled in magnitude by ω . The scaling factor that comes into effect is zero since $\omega = 0$ is the point at which its area is concentrated. Therefore, this term is zero in effect.

$$\therefore V_o(j\omega) = \frac{5}{(j\omega + 13.09)(j\omega + 1.91)}$$

The right side can be expressed as a sum of fractions as below:

$$V_o(j\omega) = \frac{A}{(j\omega + 13.09)} + \frac{B}{(j\omega + 1.91)}, \text{ where } A \text{ and } B \text{ are to be determined.}$$

$$V_o(j\omega) = \frac{(A + B)j\omega + (1.91A + 13.09B)}{(j\omega + 13.09)(j\omega + 1.91)}$$

Comparing the coefficients of numerator polynomial with those of the actual numerator polynomial, we get, $A + B = 0$ and $1.91A + 13.09B = 5$.

Solving for A and B , we get, $A = -0.447$ and $B = 0.447$.

$$\therefore V_o(j\omega) = \frac{-0.447}{(j\omega + 13.09)} + \frac{0.447}{(j\omega + 1.91)}$$

Now, these two terms are identifiable as a standard Fourier transform of a basic signal. We need not carry out the Fourier synthesis integral in order to invert these two Fourier transforms. Equation 14.8-1 tells us that $1/(j\omega + \alpha)$ is the Fourier transform of $e^{-\alpha t}u(t)$. Therefore, the inverse Fourier transform of the above $V_o(j\omega)$ (and hence, the step response) is

$$v_o(t) = 0.447(e^{-1.91t} - e^{-13.09t})u(t) \text{ V.}$$

(ii) The area content of $v_o(t)$ is obtained by

$$\begin{aligned} \text{Area content} &= \int_{-\infty}^{\infty} v_o(t) dt = 0.447 \int_0^{\infty} (e^{-1.91t} - e^{-13.09t}) dt \\ &= 0.447 \left[\frac{1}{1.91} - \frac{1}{13.09} \right] = 0.2 \text{ volt-sec} \end{aligned}$$

$$\text{The value of } V_o(j\omega) \text{ at } j\omega = 0 \text{ is } \frac{5}{13.09 \times 1.91} = 0.2 \text{ volts/Hz} = 0.2 \text{ volt-sec.}$$

Therefore, the equality between the area content of $v_o(t)$ and zero-frequency value of its Fourier transform is verified.

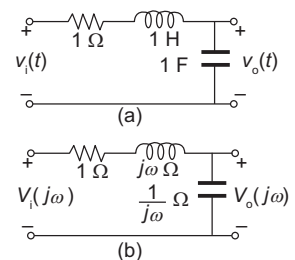


Fig. 14.9-4 (a) Circuit for Example 14.9-6 and (b) Its Phasor Equivalent

EXAMPLE: 14.9-6

Find the step response of the circuit in Fig. 14.9-4(a).

SOLUTION

The phasor equivalent circuit for the circuit in Fig. 14.9-4(a) is shown in Fig. 14.9-4(b). The system function is calculated from this equivalent circuit as

$$\begin{aligned} H(j\omega) &= \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1/j\omega}{1 + j\omega + 1/j\omega} = \frac{1}{(j\omega)^2 + j\omega + 1} \\ &= \frac{1}{[j\omega + (0.5 - j0.866)][j\omega + (0.5 + j0.866)]} \end{aligned}$$

$$V_i(j\omega) = \frac{1}{j\omega} + \pi\delta(j\omega) \quad (\because v_i(t) = u(t))$$

$$\begin{aligned} \therefore V_o(j\omega) &= \frac{1}{[j\omega][j\omega + (0.5 - j0.866)]} + \frac{1}{[j\omega + (0.5 + j0.866)]} + \pi\delta(j\omega) \frac{1}{(j\omega)^2 + j\omega + 1} \\ &= \frac{1}{[j\omega][j\omega + (0.5 - j0.866)]} + \frac{1}{[j\omega + (0.5 + j0.866)]} + \pi\delta(j\omega) \end{aligned}$$

Method of Partial Fractions

We have illustrated one method of inverting Fourier transforms in Example 14.9-5 and Example 14.9-6.

This is the *method of partial fractions*.

continued

The steps in this method of Fourier transform inversion are listed below.

- Obtain $V_o(j\omega)$ and express it as a ratio of rational polynomials in $j\omega$.
- Factorise the denominator polynomial and identify its roots.
- Express $V_o(j\omega)$ as a sum of fractions with one denominator factor assigned to each fraction in the denominator and an unknown constant assigned to the numerator.
- Find the unknown constants in the numerator of fractions. This can be done by two methods. In the first method, we construct the numerator polynomial and compare its coefficients with the actual coefficients. This will give us a system of linear algebraic equations in the unknown constants.
- In the second method, we multiply $V(j\omega)$ by a factor and evaluate the product at the root corresponding to that factor – this is called 'residue evaluation'. The unknown constants we assign to the numerators of fractions are called 'residues at roots of denominator polynomial'.
- Once the numerator constants of partial fractions are obtained, each fraction may be identified as the Fourier transform of some complex exponential function $u(t)$, using the Fourier transform which appears in Eqn. 14.8-1.

continued

The second term on the right was simplified as $\pi\delta(j\omega)$ because only the value of the multiplier at $\omega = 0$ can affect this impulse and that value is 1. $v_o(t)$ is obtained by further recasting of $V_o(j\omega)$ to express it as a sum of easily recognised Fourier transforms as below

$$\begin{aligned}\therefore V_o(j\omega) &= \frac{1}{[j\omega]} + \frac{1}{[j\omega + (0.5 - j0.866)]} + \frac{1}{[j\omega + (0.5 + j0.866)]} + \pi\delta(j\omega) \\ &= \frac{A}{[j\omega]} + \frac{B}{[j\omega + (0.5 - j0.866)]} + \frac{C}{[j\omega + (0.5 + j0.866)]} + \pi\delta(j\omega) \\ A &= V_o(j\omega) \times j\omega \Big|_{j\omega=0} = 1 \\ B &= V_o(j\omega) \times [j\omega + (0.5 - j0.866)] \Big|_{j\omega=-0.5+j0.866} = 0.58 \angle 2.62 \text{ rad} = 0.58e^{j2.62} \\ C &= V_o(j\omega) \times [j\omega + (0.5 + j0.866)] \Big|_{j\omega=-0.5-j0.866} = 0.58 \angle -2.62 \text{ rad} = 0.58e^{-j2.62} \\ \therefore V_o(j\omega) &= \frac{0.58e^{j2.62}}{[j\omega + (0.5 - j0.866)]} + \frac{0.58e^{-j2.62}}{[j\omega + (0.5 + j0.866)]} + \frac{1}{[j\omega]} + \pi\delta(j\omega) \\ \therefore v_o(t) &= [0.58e^{-0.5t}e^{j(0.866t+2.62)} + 0.58e^{-0.5t}e^{-j(0.866t+2.62)} + 1]u(t) \\ &= [1.16e^{-0.5t}\cos(0.866t + 2.62 \text{ rad}) + 1]u(t) \text{ V.}\end{aligned}$$

The two examples considered above dealt with the zero-state responses. The circuits did not have initial energy. Now, we solve an example that involves initial energy and demonstrate how Fourier transform technique can give us the zero-input response along with zero-state response.

EXAMPLE: 14.9-7

Find the total response of the circuit shown in Fig. 14.9-5 when $10u(t)$ V is applied to it. The capacitor C_1 has 10 V across it at $t = 0^-$ with the polarity as shown in the figure. C_2 is initially relaxed.

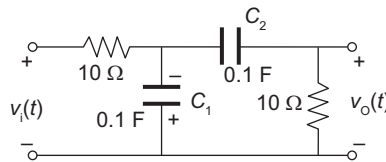


Fig. 14.9-5 Circuit for Example 14.9-7

SOLUTION

The first step is to replace the initial condition in the capacitor by an impulse current source. A 0.1 F capacitor requires 1 C in it for 10 V to pass across it. Hence, we need to connect $\delta(t)$ current source across it with suitable direction. This is shown in Fig. 14.9-6.

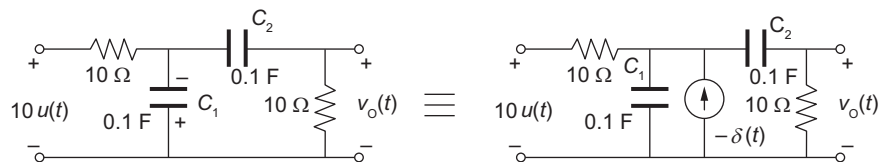


Fig. 14.9-6 Replacing Initial Voltage of Capacitor by Impulse Current Source in Example 14.9-7

Now, the circuit has two inputs at two different locations. Both inputs can be represented as the sum of sinusoids by Fourier transforms. The zero-state response due to multiple sources will obey the superposition principle. We have replaced the initial condition with a source, and hence, the circuit is initially relaxed and has only the zero-state response. Therefore, the total response can be obtained as the sum of zero-state responses to the two sources. Since both sources have Fourier transforms, the zero-state response due to each source is given by the inverse Fourier transform of the product of the source Fourier transform and system function. But the system functions can be different, since the nature and the point of application of the two sources are different. Therefore, in general, we have to obtain the system function for each input source.

However, in this case, there is a simplification possible. We apply the source transformation theorem on $v_i(t)$ and the first $10\ \Omega$ resistor and convert it to a current source of value $0.1v_i(t)$ in parallel with the $10\ \Omega$ connected across C_1 . Then, the two current sources can be replaced with one current source and the circuit becomes a single input circuit. This is shown in Fig. 14.9-7(a). The system function between $v_o(t)$ and this current source value can be obtained from the phasor equivalent circuit shown in Fig. 14.9-7(b).

- Both methods for determination of numerator constants need to be modified when there are roots with multiplicity of more than one in the denominator polynomial. We do not take up this aspect in this chapter. We will deal with partial fractions in greater detail in the next chapter on Laplace transforms.

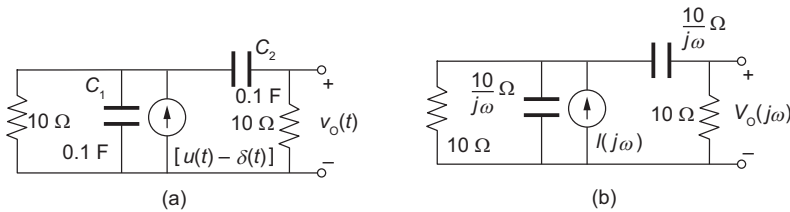


Fig. 14.9-7 (a) Circuit Reduction (b) Phasor Equivalent Circuit for Example 14.9-7

$V_o(j\omega)$ can be obtained by using the current division principle as below:

$$V_o(j\omega) = I(j\omega) \times \frac{\frac{1}{10 + 10/j\omega}}{\frac{1}{10 + 10/j\omega} + \frac{1}{10} + \frac{1}{10}} \times 10 = \frac{10I(j\omega)}{(j\omega)^2 + 3(j\omega) + 1} = \frac{10I(j\omega)}{(j\omega + 2.62)(j\omega + 0.38)}$$

$$I(j\omega) = \left[\pi\delta(j\omega) + \frac{1}{j\omega} \right] - 1$$

$$\begin{aligned} \therefore V_o(j\omega) &= \frac{10 - 10(j\omega)}{(j\omega + 2.62)(j\omega + 0.38)} + \frac{10\pi\delta(j\omega) \times j\omega}{(j\omega + 2.62)(j\omega + 0.38)} \\ &= \frac{10 - 10(j\omega)}{(j\omega + 2.62)(j\omega + 0.38)} \end{aligned}$$

$v_o(t)$ is obtained by partial fraction expansion of this Fourier transform.

$$\begin{aligned} V_o(j\omega) &= \frac{10 - 10(j\omega)}{(j\omega + 2.62)(j\omega + 0.38)} \\ &= \frac{A}{(j\omega + 2.62)} + \frac{B}{(j\omega + 0.38)} \end{aligned}$$

$$\therefore (A + B)j\omega + (0.38A + 2.62B) = 10 - 10j\omega$$

$$\therefore A + B = -10 \text{ and } 0.38A + 2.62B = 10$$

$$\Rightarrow A = -16.16 \text{ and } B = 6.16$$

$$\therefore v_o(t) = [-16.16e^{-2.62t} + 6.16e^{-0.38t}]u(t) \text{ V.}$$

Fourier transform is not really a tool for solving circuits; though it can be used for that purpose. Some Fourier transforms can be very difficult to invert; some applied forcing functions may not possess Fourier transforms at all. All these issues that arise in the

use of Fourier transforms in circuit solution are efficiently solved by Laplace transform. We study Laplace transforms in the next chapter.

However, solving a circuit using Laplace transforms tends to be a mechanical affair. Laplace transforms do not offer the kind of insight that the Fourier transforms offer into the behaviour of signals and circuits. That is the reason why we took up Fourier transforms first.

Fourier transform theory tells us what types of signals and circuits can be made in reality. The theory sets limits to what can be done and what can not be done in linear circuits. For example, Fourier transform theory will tell us that no ideal filter can ever be made.

EXAMPLE: 14.9-8

An ideal low-pass filter is defined as a system which passes on to the output those signal components with a frequency less than a particular value, with no change in their amplitude and phase, and does not pass on those signal components with a frequency greater than that value. This implies that its system function will be $H(j\omega) = 1$ for $|\omega| < \omega_c$ and $= 0$ for $|\omega| > \omega_c$, where ω_c is the so-called cut-off frequency. Find the impulse response of such a filter.

SOLUTION

The frequency response function $H(j\omega)$ of the ideal low-pass filter is a real-valued rectangular pulse in frequency-domain. It has a gain (i.e., magnitude) of unity in the pass band, i.e., in $|\omega| < \omega_c$ band and a gain of zero outside this band. By now, we know that the rectangular pulse in one domain results in $\text{sinc}(x)$ shape in the other domain. We derive the impulse response by inverting the system function.

$$h(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega t} d\omega = \frac{1}{j2\pi t} (e^{j\omega_c t} - e^{-j\omega_c t}) = \frac{j2 \sin \omega_c t}{j2\pi t} = \frac{\omega_c}{\pi} \left(\frac{\sin \omega_c t}{\omega_c t} \right).$$

The system function and the corresponding impulse response with $\omega_c = \pi$ rad/s are shown in Fig. 14.9-8.

The impulse response extends from $-\infty$ to ∞ in time-domain. This implies that the filter circuit started responding from the infinite past, anticipating that an impulse is going to hit it at $t = 0$. This kind of behaviour is called *non-causal* behaviour – it violates *the law of causality* which states that the output of a physical system for an input can appear only after that input was applied to it. Obviously, no physical system can predict the future, and hence, no *physical circuit* with this kind of an impulse response can be made. A necessary condition for realising a physical system with a given $h(t)$ has to be zero for $t < 0$.

Thus, an ideal low-pass filter can not be made since it is a non-causal system. In fact, it is possible to show that no system function with jump discontinuities can be realised since the impulse response of such a system will be non-causal. Therefore, an ideal filter – low-pass, band-pass, high-pass, etc. – can not be made.

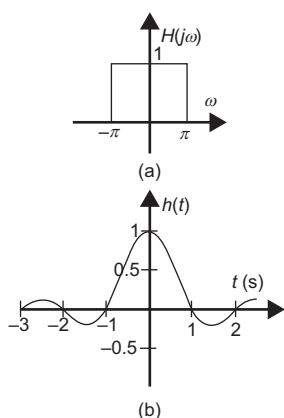


Fig. 14.9-8 (a) Frequency Response and (b) Impulse Response of an Ideal Low-Pass Filter with π rad Cut-Off

EXAMPLE: 14.9-9

Show that the system function $H(j\omega)$ of a physically realisable system can not have a jump discontinuity in real or imaginary parts.

SOLUTION

We do not have the liberty to assume a shape for $H(j\omega)$ since no such shape is mentioned. We assume an arbitrary shape for the real and imaginary parts of $H(j\omega)$ and a jump discontinuity at a frequency ω_j in one of them.

Now, if we differentiate $j \times H(j\omega)$ with respect to ω , the derivative will contain an impulse-pair in the frequency-domain at $\pm\omega_j$. With reference to Eqn. 14.9-6, the time-function corresponding to frequency-domain differentiation is t times the original time-function. An impulse-pair in frequency-domain implies that there is a periodic sinusoidal component with some finite amplitude in the time-domain waveform. Therefore, $t \times h(t)$ will contain a sinusoid at ω_j rad/s from $-\infty$ to ∞ in the time-domain. Therefore, $h(t)$ will contain (a complex sinusoid at ω_j rad/s)/ t . This component can not be identically zero for all $t < 0$. Therefore, $h(t)$ is two-sided, and hence, *non-causal*. Hence, such a system can not be physically realisable.

EXAMPLE: 14.9-10

Show that the system function $H(j\omega)$ of a physically realisable system should have continuously differentiable real and imaginary parts.

SOLUTION

Assume that there is a jump discontinuity in $(n-1)^{\text{th}}$ derivative of either the real or imaginary part of $H(j\omega)$.

Then, $(j)^n \frac{d^n H(j\omega)}{d\omega^n}$ will contain an impulse-pair in the frequency-domain. The time-

function corresponding to this is $t^n h(t)$ and it will contain a finite amplitude sinusoid at the frequency where the impulse-pair is located. Therefore, $h(t)$ will be non-causal.

Therefore, system function of a physically realisable system will be continuous and continuously differentiable in the frequency-domain.

These two examples have effectively shown that we can not design a physical circuit that can limit the frequency content of a signal to a particular band in the frequency-domain. *Signal Band-Limiting* is a routine signal-processing application in the areas of Electronic Communications, Data Communications and Digital Processing of Signals. Analog Filters are employed for this purpose. By now, it should be clear that these filters can only approximate the behaviour of *ideal filters*.

Our circuits and signals did not start at the time instant of Big-Bang. That is why we can not have a precise band-limiting in frequency-domain. However, there is no frequency value in the frequency-domain corresponding to the time instant of Big-Bang in time-domain. Therefore, Fourier transform theory is not averse to a time-function being precisely time-limited in time-domain; only that capacitors and inductors in a circuit will not allow that.

Consider a rectangular pulse in time-domain. Since rectangular pulses do not materialise from thin air, it has to be generated by some circuit. The rectangular pulse appears at the output of that circuit. But between any two terminals supporting a potential difference, there will be the inevitable capacitance present – we may call it parasitic capacitance and ignore it; but it is present. If this capacitance is to tolerate the rectangular pulse voltage, it will demand impulse currents at the rising and falling edges of the pulse. Even if there is a source that can deliver such huge currents, the parasitic inductance present in the current flow path will not allow the impulse currents. Therefore, the rectangular pulse can not appear at the output. Suppose we settle for a trapezoidal pulse instead of a rectangular pulse, then, the parasitic capacitance at the output terminals will demand rectangular pulse currents at the sides of trapezoidal voltage. This results in an attempt to change the current in the parasitic inductance inside the circuit instantaneously. That will not be tolerated even if there is a source that can support infinite voltage, since the capacitance at the terminal of that source will refuse to accept infinite voltage.

This line of reasoning should convince us that all voltage and current variables in physical circuits have to be continuous and continuously differentiable functions of time.

Signal band-limiting

Precise band-limiting of a signal is not possible since the circuit that can do it will have a non-causal impulse response and can not be realised physically.

It is to be particularly noted that we term some circuit elements as parasitic elements only for our convenience. Otherwise, the circuits become too intractable for routine analysis. But, that does not mean that these elements we ignored in our analysis are not playing out their roles in the circuit. *An element is a parasitic element only for the analyst; not for the circuit.* The parasitic elements take their revenge on us for having termed them ‘parasitic’ when we want to make circuit variables change rapidly.

14.10 THE SYSTEM FUNCTION AND SIGNAL DISTORTION

We formally define the system function of a linear circuit below.

The system function for a particular input-output pair is the ratio of Fourier transform of the zero-state response of the output variable to Fourier transform of the input variable. It is the same as frequency response function defined for that input-output pair. It is numerically the same as Fourier transform of the impulse response for that input-output pair.

The system function of a circuit can be obtained by applying circuit analysis principles on the phasor equivalent circuit. The system function of a memoryless circuit will be a real number, independent of frequency. The system function of a lumped linear time-invariant dynamic circuit will be a ratio of rational polynomials in $(j\omega)$ as we have seen in many examples. Therefore, we express the system function $H(j\omega)$ as

Expression for
system function.

$$H(j\omega) = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}$$

where the a 's and b 's are real numbers. The coefficient of the highest power in the denominator is made unity by suitable scaling of polynomials, if required.

What is the relation between the order of the numerator polynomial and the order of the denominator polynomial?

If $m < n$, $H(j\omega)$ may be approximated as $\frac{b_m}{(j\omega)^{n-m}}$ as $\omega \rightarrow \infty$. Therefore, the gain (i.e.,

Low-pass system
functions and
'delaying and
smearing' of signals.

magnitude of system function) of such a circuit approaches zero asymptotically as $\omega \rightarrow \infty$. It will reduce the high frequency components in the output. Such a circuit essentially has a low-pass behaviour and removes high frequency components from the input signal by the time it reaches the output. High frequency components are essential if the signal waveform is to undergo rapid changes. Therefore, it follows that, if a signal goes through a series of circuits with system functions of this type, it will progressively lose its speed and its sharp edges – it gets delayed and smeared. It will become a waveform with progressively lesser rate of change at all t .

If $m = n$, $H(j\omega)$ may be expressed the following way.

$$H(j\omega) = b_m + \frac{c_{m-1}(j\omega)^{m-1} + \cdots + c_1(j\omega) + c_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}$$

where $c_i = b_i / b_m$ for $i = 0$ to $m - 1$

Memoryless
connections
between input and
output and the
corresponding system
function.

Therefore, the impulse response of this circuit will contain an impulse of magnitude b_m . This implies that there is a straight memoryless connection from the input to the output in addition to a parallel path that is dynamic. Such memoryless connections, though theoretically possible, are not practically realisable because even the so-called memoryless circuit elements have parasitic capacitance and inductance associated with them.

A system function with $m > n$ may be approximated as $b_m(j\omega)^{m-n}$ as $\omega \rightarrow \infty$. Therefore, the circuit amplifies high frequency components. This implies that such a circuit sharpens the edges of an applied signal and makes it more rapidly varying. A differentiator circuit is an example. But such a circuit can not be physically made. The inevitable parasitic elements make this type of system function impossible for a physical circuit. Even a well-made

differentiator circuit will have a frequency response that will bend back and go to zero as $\omega \rightarrow \infty$. We should remember again that, *we may choose to ignore the parasitic elements and we may use idealised models for amplifiers, etc., but what we ignore and what we do not ignore do not bind the circuit.*

Thus, all physical circuits will have a system function which tapers down to zero as $\omega \rightarrow \infty$. This means that all physical circuits are low-pass systems in the final analysis.

Parasitic elements force system functions to be of low-pass nature.

14.10.1 The Signal Transmission Context

Consider a situation where we have a circuit producing a voltage signal $v(t)$ at its output. We want to observe the waveform of $v(t)$, and hence, we connect the signal to the oscilloscope using a probe. In this process we have created a *communication channel*. The pair of wires used to connect the signal to the scope is the communication channel and its role is to transmit the signal faithfully to the oscilloscope input. The channel should not do anything to the signal other than taking it from one location to another.

Obviously, the same *signal transmission context* appears whenever we interconnect two electronic or electrical sub-systems where the output of one sub-system has to become the input of another system. In the context of measurement and interconnection, the *transmission channel* that comes up is a sort of inevitable evil. But in a *communications* context, the *transmission channel* is an integral part of the Communication System – whether analog communication or digital communication.

Thus, we encounter the *signal transmission context* in a wide variety of situations in Electrical and Electronic Engineering. The problem in this context is transmitting signal through a wire-channel or a wireless-channel from one location to another without affecting the waveform to be transmitted in any manner.

The *transmission channel* is yet another electrical system that can be modelled by a circuit. But the circuit required to model a channel would be a *distributed parameter circuit* that is described by partial differential equations. We do not take up the study of such circuits in this text. However, lumped parameter circuits can be used for the approximate analysis of such channel circuits. We employ that approach.

We may model the wire-pair used to transmit a signal by the *RLC* circuit in Fig. 14.10-1.

R in the circuit includes the Thevenin's equivalent resistance of the signal source and the series resistance of the wire-pair. L is the self-inductance of the circuit loop formed by connecting the signal source to another circuit or system at the other end of the channel. C is the capacitance of the wire-pair and the input capacitance of the circuit or system connected at the output. R' is the input resistance of the circuit or system connected at the end of the channel. In the context of connecting an electronic level signal to an oscilloscope using a scope probe, R may be in 10's to 100's of Ω , L may be in nH's, C will be about 20–30 pF and R' will be in $M\Omega$ range.

Obviously, this circuit is not memoryless. Therefore, its system function $H(j\omega) = V_o(j\omega)/V_i(j\omega)$ will be a continuous function of ω and will have low-pass nature at high frequency end.

All signals in circuits start at some definite time-point. We have seen that the spectral expansion of such time-signals can not be band-limited. Hence, all signals that we deal with in our circuits will invariably have spectral content spread over the entire frequency-domain. It is possible that the high frequency content is negligible; but there will be some content in any frequency range that we may consider.

Therefore, a signal will invariably lose some of its high frequency spectral content by going through a transmission channel in a signal transmission context.

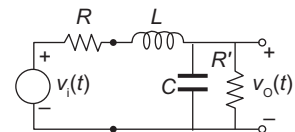


Fig. 14.10-1
Approximate Circuit
Model for a Signal
Transmission Channel

Linear Distortion

A dynamic linear circuit offers different gain and time-delay to different frequency components in the input signal.

The waveshape of the output signal will be different from the waveshape of the input signal due to this *differential treatment* for various frequency components. When the waveshape at output is different from the waveshape at the input, we term the output as a *distorted* output.

This kind of distortion can take place in a linear circuit only if the input signal contains more than one sinusoidal component.

It is called *linear distortion* because it is the distortion caused by a *linear circuit* due to the *differential treatment* it metes out to various sinusoidal components in the signal.

Condition for distortion-free transmission of a signal through a linear transmission channel

A communication channel will transmit a signal without any distortion if the system function $H(j\omega)$ of the channel is $Ce^{j\omega t_d}$, where C is a real constant and t_d is a real number (can be zero too) representing the fixed time-delay in the channel.

Such a channel is called 'distortion-free channel'.

No physical transmission channel can be really distortion-free. This is so because if $H(j\omega) = Ce^{j\omega t_d}$, then the impulse response of the channel will extend from $t = -\infty$ to $t = +\infty$. It will be a non-causal, physically non-realisable channel.

14.10.2 Linear Distortion in Signal Transmission Context

It is extremely important to preserve the *shape* of the signal in a signal transmission context. The shape of the signal usually codes the information that we want to transmit in a communication system. Loss of waveshape results in errors in communication.

Infinite number of sinusoids with finite or infinitesimal amplitudes and frequencies ranging from 0 to ∞ interfere constructively and destructively at various time instants to make the particular waveshape we want to transmit. If these sinusoidal amplitudes undergo the same gain when they go through the channel circuit and if, in addition, they undergo zero time-delay when they go through the channel circuit, the output waveshape will be a replica of the input waveshape with possible scaling of its amplitude. The waveshape is not lost in that case. It is all right if all the sinusoidal components undergo the *same time-delay*. The received waveshape will be a replica of the input waveshape with a fixed time-delay, i.e., the entire waveshape gets shifted in time-axis by a certain delay. The relation between phase-delay and time-delay in a sinusoid is $\phi = \omega t_d$, where t_d is the time-delay suffered by it and ω is its angular frequency. Therefore, the condition that all sinusoidal components in a signal can undergo the *same time-delay* without incurring a loss of its waveshape can be equivalently translated as a *linear phase-delay*, i.e., a phase-delay which is proportional to the angular frequency does not affect the waveshape.

However, if amplitudes of various sinusoidal components get multiplied by various values of gain, the waveshape at the output will be different from the waveshape at the input. Similarly, if various sinusoidal components are subjected to various values of time-delay, the channel circuit *disperses* their relative starting positions and we get a waveshape at the output that is different from the waveshape at the input.

When the waveshape at the output is different from the waveshape at the input, we term the output as a *distorted* output. Obviously, there can be this kind of a distortion only if the input signal contains more than one sinusoidal component. It is called *linear distortion* because it is the distortion caused by a *linear circuit* by the *differential treatment* it metes out to various sinusoidal components in the signal.

14.10.3 Pulse Distortion in First Order Channels

All physical circuits are low-pass systems in the high frequency range. All physical signals have a frequency spectrum covering the entire frequency range. Therefore, ideal distortion-free transmission of signals is impossible. Some degree of distortion in transmission is unavoidable. If that is so, what is the extent of distortion acceptable?

The answer to the question will lead us to another question – how much of the spectrum of the signal can we afford to lose without significant degradation of the waveshape?

Rectangular pulses have a special role in digital communications. They are used to represent 1's and 0's in base-band digital communication systems. Therefore, we study the waveshape distortion when a rectangular pulse is transmitted through a two-wire channel. We simplify the circuit model for such a channel shown in Fig. 14.10-1 further by neglecting the series inductance and shunt resistance. This simplified *RC* circuit model is satisfactory if the wires involved are tightly wound and the channel is only of a short length. We have studied the pulse response of this circuit in earlier chapters. The circuit and its pulse response for various τ/T ratios, where τ is the time constant (*RC*) and T is the pulse width are shown in Fig. 14.10-2.

Specifying the time required for the pulse to rise as a percentage of its width is one way to specify a limit on the acceptable distortion in this context. Rise time of this circuit is 2.2τ . Therefore, if we specify 10% as the limit of distortion we will tolerate, then, we can transmit only rectangular pulses that are at least 22τ s wide.

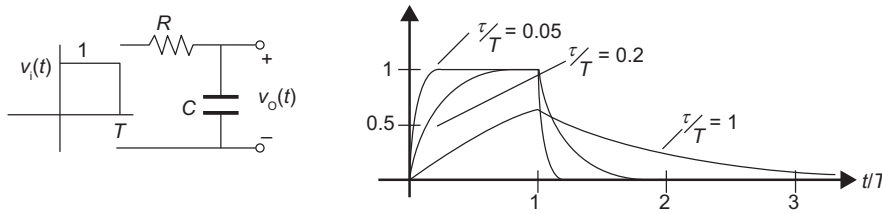


Fig. 14.10-2 Pulse Response of a First Order Channel

The system function of this circuit can be derived from its phasor equivalent circuit and it will be

$$H(j\omega) = \frac{1}{1 + j\omega\tau} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \angle -\tan^{-1} \omega\tau$$

Its gain decreases monotonically to zero and its phase delay increases monotonically to 90° with ω . The half-power cut-off frequency value is $1/\tau$ rad/s, and hence, the bandwidth of this channel is $1/\tau$ rad/s. Note that with the distortion criterion specified as above, we will require an RC channel with a bandwidth of $7\pi/T$ rad/s.

In fact, distortion of waveshape in a communication channel is not a serious problem by itself. We can always design circuits with suitable frequency response function at the receiving end such that the distortion that the signal was subjected to is undone to a large extent, if not entirely. But random noise voltages picked up all along the channel corrupts the signal and detecting the signal in the presence of corrupting noise is the basic issue in communication. Distortion of waveshape makes it more difficult by reducing the normalised energy content of the signal by the time it reaches the receiving end.

The relative proportion of normalised energy of signal received and the normalised power of noise voltages picked up in the channel decides the complexity involved in detecting the signal in a digital communication context. The waveshape is not as important as the energy content of the received signal in digital communication.

Therefore, we have to look for a distortion specification in terms of normalised signal energy rather than in terms of rise time. Again the question comes up – *what are the spectral components we can afford to lose without significant reduction in normalised energy content in the received signal?* Another form of the same question is – how much normalised energy must be there in the signal at the sending end such that we get a pre-specified energy content at the receiving end?

Fourier transform attains its full glory by answering these kinds of questions. We see how it does so in the next section.

14.11 PARSEVAL'S RELATION FOR A FINITE-ENERGY WAVEFORM

Consider a signal $v(t)$ which is of finite energy. All signals are—because we can not have infinite voltages or currents and because signals are switched on at some instant and switched off at another instant. We try to develop a relation between its normalised energy content E_n and its Fourier transform $V(j\omega)$.

$$E_n = \int_{-\infty}^{\infty} [v(t)]^2 dt$$

$v(t)$ in a practical context is a real signal. Therefore, there is no error in expressing $[v(t)]^2$ as $v(t) \times v^*(t)$. $\therefore E_n = \int_{-\infty}^{\infty} v(t) \times v^*(t) dt$

Substituting $v^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V^*(j\omega) e^{-j\omega t} d\omega$, we get,

Parseval's Relation

What are the spectral components that we can afford to lose without significant reduction in energy content in a signal transmission context?

This can be answered if we know the contribution of each frequency component to the normalised energy of the signal.

Parseval's energy relation give us this information.

Parseval's relation
for a finite energy
signal.

Energy Spectral
Density of a
waveform defined.

$$E_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(t) \left[\int_{-\infty}^{\infty} V^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

Reversing the order of integration,

$$E_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} V^*(j\omega) \left[\int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} V^*(j\omega) V(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |V(j\omega)|^2 d\omega$$

$$\text{Therefore, } E_n = \int_{-\infty}^{\infty} [v(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |V(j\omega)|^2 d\omega$$

This relationship is called *Parseval's relation for finite energy signals*.

This relation tells us clearly how much of the total energy content in the signal is contributed by various frequency ranges. It says that given a band of frequencies $\Delta\omega$ around a particular ω , the sinusoidal components falling within that frequency band contribute $|V(j\omega)|^2 \Delta\omega/2\pi$ J to the total energy of $v(t)$.

Therefore, the quantity $\frac{|V(j\omega)|^2}{2\pi}$ is called the *Energy Spectral Density* of $v(t)$ and

has Joule-sec/rad as its unit. It is a real even function of ω . Integrating energy spectral density in the frequency-domain gives the total energy of the signal.

Now, we have acceptable answers to the questions we raised in the context of signal distortion. For example, consider the energy spectral density of the rectangular pulse in

Fig. 14.10-2. Its Fourier transform is $V(j\omega) = T \frac{\sin \omega T/2}{\omega T/2} e^{-j\omega T/2}$. The energy spectral density

plot is shown in Fig. 14.11-1.

We notice that almost the entire signal energy is concentrated in the main lobe. The width of the main lobe is $2\pi/T$ rad/s on one side. If the energy spectral density is integrated from $-2\pi/T$ to $2\pi/T$, we will see that 90.3% of the total energy of pulse is available in that band. Another 4.7% is contributed by the second lobe. Thus, the frequency components with their frequency in the range 0 to $1/T$ Hz contribute 90.3% energy.

Therefore, if we want to preserve 90% of the signal energy intact, the bandwidth of the channel should be more than $2\pi/T$ rad/s. For a moment, assume that the channel behaves like an ideal low-pass filter. In that case, its bandwidth must be a minimum of $2\pi/T$ rad/s. If the channel is a practical system function like the one we saw in the RC circuit model, we will need more than this bandwidth in the channel. The required bandwidth can be found by solving the following integral equation using iterative numerical integration. We have made use of the fact that $V_o(j\omega) = H(j\omega)V_i(j\omega)$ and that the bandwidth ω_c of a first order circuit is $1/\tau$, where τ is its time constant in arriving at this integral equation.

$$0.9T = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{1 + (\omega/\omega_c)^2} \right] \times \left[T \frac{\sin \omega T/2}{\omega T/2} \right]^2 d\omega$$

Solving this equation using iterative numerical integration, we see that the bandwidth of an RC channel has to be about $2.85\pi/T$ rad/s if 90% of the pulse energy is to be made available at the output. Compare this bandwidth with the $7\pi/T$ rad/s for the waveform distortion criterion based on the rise time of the output pulse. With $2.85\pi/T$ rad/s bandwidth in the channel, the output waveshape will be considerably different from the rectangular pulse shape. However, it will have 90% energy intact in it.

Numerical solution may be needed in most of the problems of this kind. But the point is that Fourier transform through Parseval's relation provides answers to many questions which has a direct relevance to design of communication systems.

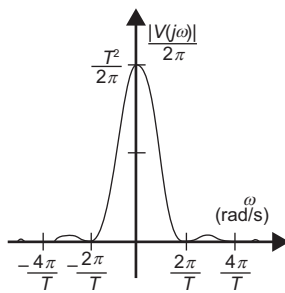


Fig. 14.11-1 The Energy Spectral Density Plot for the Rectangular Pulse in Fig. 14.10-2

EXAMPLE: 14.11-1

$v_i(t) = 2e^{-10000t} u(t)$ V is applied to an RC circuit with $R = 2 \text{ k}\Omega$ and $C = 0.1 \mu$ with zero initial voltage across the capacitor. (i) Find the energy dissipated in the $2 \text{ k}\Omega$ by working entirely in the frequency domain.

SOLUTION

The system function of the circuit is derived as

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega}{j\omega + 1/RC} = \frac{j\omega}{j\omega + \beta}, \text{ where } \beta = \frac{1}{RC}$$

Fourier transform of voltage across the resistor is obtained by taking the product of the input Fourier transform and the system function.

$$v_i(t) = 2e^{-\alpha t} u(t) \text{ with } \alpha = 10000.$$

$$\therefore V_i(j\omega) = 2 \times \frac{1}{j\omega + \alpha} \text{ and } V_o(j\omega) = 2 \times \frac{1}{j\omega + \alpha} \times \frac{j\omega}{j\omega + \beta}.$$

Energy spectral density of output voltage is obtained as

$$\begin{aligned} \text{Energy spectral density} &= \frac{|V_o(j\omega)|^2}{2\pi} = \frac{2}{\pi} \times \left| \frac{1}{j\omega + \alpha} \right|^2 \times \left| \frac{j\omega}{j\omega + \beta} \right|^2 \\ &= \frac{2}{\pi} \frac{1}{\omega^2 + \alpha^2} \frac{\omega^2}{\omega^2 + \beta^2} \end{aligned}$$

Total normalised energy of $v_o(t)$ is given by the area under energy spectral density function in the entire frequency-domain.

$$\therefore E_n \text{ of } v_o(t) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \alpha^2} \frac{\omega^2}{\omega^2 + \beta^2} d\omega$$

$$\text{Let } \frac{1}{\omega^2 + \alpha^2} \frac{\omega^2}{\omega^2 + \beta^2} = \frac{A}{\omega^2 + \alpha^2} + \frac{B}{\omega^2 + \beta^2}$$

$$\text{Then, } (A + B)\omega^2 + A\beta^2 + B\alpha^2 = \omega^2 \Rightarrow A + B = 1, A\beta^2 + B\alpha^2 = 0$$

$$\Rightarrow A = \frac{\alpha^2}{\alpha^2 - \beta^2} \text{ and } B = \frac{-\beta^2}{\alpha^2 - \beta^2}$$

$$E_n \text{ of } v_o(t) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{A}{\omega^2 + \alpha^2} d\omega + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{B}{\omega^2 + \beta^2} d\omega$$

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + \alpha^2} d\omega = \frac{1}{\alpha^2} \int_{-\infty}^{\infty} \frac{1}{1 + (\omega/\alpha)^2} d\omega. \text{ Let } x = \frac{\omega}{\alpha}, \text{ then } d\omega = \alpha dx$$

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + \alpha^2} d\omega = \frac{1}{\alpha} \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx = \frac{1}{\alpha} \tan^{-1} x \Big|_{-\infty}^{\infty} = \frac{\pi}{\alpha}$$

$$\therefore E_n \text{ of } v_o(t) = \frac{2}{\pi} \left[\frac{\pi A}{\alpha} + \frac{\pi B}{\beta} \right] = \frac{2A}{\alpha} + \frac{2B}{\beta}$$

$$\alpha = 10^4 \text{ and } \beta = 5000 \Rightarrow A = 1.33 \text{ and } B = -0.33 \Rightarrow E_n = 1.33 \times 10^{-4} \text{ J}$$

But this is the energy that will be dissipated in a fictitious 1Ω resistance. Therefore, the energy dissipated in $2 \text{ k}\Omega$ resistance will be $1.33 \times 10^{-4} / (2 \times 10^3) = 6.67 \times 10^{-7} \text{ J}$.

Signal bandwidth

We had related the notion of *bandwidth* to circuits until now. It is possible to extend the concept of bandwidth to signals too. We observed in the case of a rectangular pulse that 90% of its energy is contained in the 0 to $2\pi/T$ rad/s band in frequency-domain. Now, we may define *signal bandwidth* as the width of band of frequencies that contribute a specified percentage of total energy of that signal. If we specify 90%, then signal bandwidth of a pulse which is T s wide will be $2\pi/T$ rad/s and since this band starts from 0 rad/s, we will call the signal a *low-pass signal*.

A practical distortion-free transmission criterion is that the channel bandwidth must be more than signal bandwidth with about 50% to 100% margin.

EXAMPLE: 14.11-2

A signal $v_i(t) = 2e^{-0.5t} \sin 100t u(t)$ V is applied to a band-pass filter which may be modelled as an ideal band-pass filter. The filter has a centre frequency of 120 rad/s and a pass band of 40 rad/s. Find the energy content of the filter output as a percentage of the input energy content.

SOLUTION

We have to find the energy spectral density of the input signal first. We see that the input function can be expressed as $2e^{-0.5t} u(t)$ multiplied by $-j0.5(e^{j100t} - e^{-j100t})$. We know from the frequency shifting property of Fourier transforms that $e^{j\omega_0 t} v(t) \Leftrightarrow V[j(\omega - \omega_0)]$.

Therefore, the effect of multiplying $2e^{-0.5t} u(t)$ by $\sin 100t$ will be to shift half its magnitude spectrum, each to $\omega = \pm 100$ rad/s and add. The value of Fourier transform of $2e^{-0.5t} u(t)$ will be negligibly small at 100 rad/s, and hence, the overlap between the two shifted copies will be negligible. Therefore, the new signal will have an energy density spectrum which is the same as the two copies of energy spectral density of $2e^{-0.5t} u(t)$ located at $\omega = \pm 100$ rad/s with a scaling factor of 0.25. Total energy content in a signal is the area under its energy spectral density curve. Hence, $2e^{-0.5t} \sin 100t u(t)$ will have half the energy content of $2e^{-0.5t} u(t)$.

$$\text{Fourier transform of } 2e^{-0.5t} = \frac{2}{j\omega + 0.5}$$

$$\therefore \text{Its energy spectral density} = \frac{1}{\pi} \frac{2}{\omega^2 + 0.25}$$

$$\therefore \text{Its energy content} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + 0.25} d\omega = \frac{4}{\pi} \left[\tan^{-1} \omega / 0.5 \right]_{-\infty}^{\infty} = 4 \text{ J.}$$

Therefore, the total energy content of $2e^{-0.5t} \sin 100t u(t)$ will be 2 J.

The band-pass filter passes frequency components between 80 rad/s and 120 rad/s. This means that the components in the range -20 rad/s to $+20$ rad/s from the original spectrum before frequency shifting are being passed on to output. Therefore, the energy in the filter output = $0.5 \times$ area under energy spectral density of $2e^{-0.5t} u(t)$ between -20 and 20 . This will be $0.5 \times (4/\pi) \times 2 \tan^{-1} 40 = 1.968 \text{ J}$.

Therefore, the energy of filter output is 98.4% of its input signal energy.

14.12 SUMMARY

- Almost all signals employed in circuits – periodic or aperiodic – can be expressed as the sum of infinite number of sinusoids. Finite energy aperiodic signals have continuous Fourier transforms, whereas finite power waveforms will contain impulses in Fourier transform along with the continuous component.
- Given an aperiodic waveform $v(t)$, its Fourier transform is given by the *Analysis Integral* $V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt$. Given a Fourier transform $V(j\omega)$, its inverse transform is obtained by the *Synthesis Integral* $v(t) = \int_{-\infty}^{\infty} V(j\omega)e^{j\omega t} d\omega$. Various properties of the Fourier transform pairs are given in Table 14.12-1.
- Fourier transform is a *spectral complex amplitude density function*. Fourier transform value at any ω gives the density of the complex spectral amplitude at that frequency in volts/Hz unit.
- The sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$ is $\approx V(j\omega_0) \times \Delta\omega/2\pi$, provided $\Delta\omega$ is small.
- Sinusoids of all frequencies are present in an aperiodic waveform, but the amplitude of sinusoidal component at any particular frequency is infinitesimally small. An infinite number of such sinusoids with infinitesimally small amplitudes reinforce each other or cancel each other at various time instants to bring forth the waveshape of the aperiodic waveform.
- The *Synthesis Integral* is not normally used for inverting a Fourier transform. Inversion is done by expressing the Fourier transform in partial fractions and identifying each fraction as the Fourier transform of some basic function. Frequently appearing Fourier transform pairs are listed in Table 14.12-2.
- The Fourier transform of the zero-state response in a *stable* circuit is given by the product of Fourier transform of the input and the circuit frequency response function.
- System function of a linear time-invariant circuit is defined as the ratio of Fourier transform of the output to the Fourier transform of the input. System function and frequency response function of a *stable* circuit are the same.
- The Fourier transform of the impulse response of a *stable* circuit is the same as its system function.
- Since the zero-state response of a *stable* circuit for any input can be obtained by using the system function, it is as complete a characterisation of the *stable* circuit in frequency-domain as the impulse response is in time-domain.
- A signal can not be time-limited and band-limited simultaneously. Narrower confinement of a signal in one domain results in spreading out in the other domain.
- Practical circuits will have system functions with magnitude tapering down to zero with frequency. Practical signals will

be of finite energy and finite duration, and hence, their spectra will extend over the entire frequency-domain. Therefore, any practical signal going through any practical circuit will suffer some extent of waveform distortion.

- A communication channel will transmit a signal without any distortion if the system function $H(j\omega)$ of the channel is $Ce^{j\omega t_d}$, where C is a real constant and t_d is the fixed time-delay in the channel. A practical channel can only approach this performance.
- Parseval's relation describes how the total normalised energy of a signal is apportioned among the sinusoidal components that go into making that signal. It gives an expression for energy density spectrum as $|V(j\omega)|^2/2\pi$ Joules-sec/rad. Parseval's relation helps us to ascertain the required bandwidth in a transmission channel for preserving a stated percentage of signal energy.

Table 14.12-1 Properties of Fourier Transform

Aperiodic signal/property	Fourier transform
$v(t)$	$V(j\omega)$ – Definition
$av_1(t) + bv_2(t)$	$AV_1(j\omega) + bV_2(j\omega)$ – Linearity
$v(t - t_d)$	$e^{-j\omega t_d}V(j\omega)$ – Time-shifting
e^{zt} , z is complex with negative real part	$V(j\omega - z)$ – Multiplication by a complex exponential in time-domain
$e^{j\omega_0 t}v(t)$	$V(j[\omega - \omega_0])$ – Multiplication by a complex exponential
$v(-t)$	$V(-j\omega)$ – Time reversal
$v(at)$	$(1/ a) V(j\omega/a)$ – Time-scaling
$v_1(t) * v_2(t)$	$V_1(j\omega) * V_2(j\omega)/2\pi$ – Frequency-domain convolution
$\frac{dv(t)}{dt}$	$j\omega V(j\omega)$ – Time-domain differentiation
$\int_{-\infty}^t v(t)dt$	$\frac{1}{j\omega} V(j\omega) + \pi V(j0)\delta(j\omega)$ – Time-domain integration
$t v(t)$	$j \frac{dV(j\omega)}{d\omega}$ – Frequency-domain differentiation
$v(t)$ real	$V(-j\omega) = V^*(j\omega)$ – Conjugate symmetry $\text{Re}[V(j\omega)] = \text{Re}[V(-j\omega)]$ and $\text{Im}[V(j\omega)] = -\text{Im}[V(-j\omega)]$ $ V(j\omega) = V(-j\omega) $ and $\angle V(j\omega) = \angle V(-j\omega)$
$v_e(t) = \text{Ev}[v(t)]$	$\text{Re}[V(j\omega)]$
$v_o(t) = \text{Od}[v(t)]$	$\text{Im}[V(j\omega)]$
Duality	$2\pi v(j\omega) \Leftrightarrow V(-jt)$
Parseval's energy relation for a finite energy signal.	$E_n = \int_{-\infty}^{\infty} [v(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\omega) ^2 d\omega$

Table 14.12-2 Basic Fourier Transform Pairs

Signal	Fourier transform	Signal	Fourier transform
1 Constant	$2\pi\delta(j\omega)$	$e^{[-\alpha + j\omega_0]t}u(t)$ Damped complex exponential	$\frac{1}{j\omega + (\alpha - j\omega_0)}$
$e^{j\omega_0 t}$ Complex exponential	$2\pi\delta[j(\omega - \omega_0)]$	$e^{-\alpha t}\cos\omega_0 t u(t)$ Damped cosine	$\frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_0^2}$
$\cos\omega_0 t$ Unit cosine	$\pi[\delta[j(\omega - \omega_0)] + \delta[j(\omega + \omega_0)]]$	$e^{-\alpha t}\sin\omega_0 t u(t)$ Damped sine	$\frac{\omega_0}{(\alpha + j\omega)^2 + \omega_0^2}$
$\cos\omega_0 t u(t)$ switched cosine	$\frac{\pi}{2}[\delta[j(\omega - \omega_0)] + \delta[j(\omega + \omega_0)]] + \frac{j\omega}{(j\omega)^2 + \omega_0^2}$	$e^{-\alpha t}u(t)$ Damped real exponential	$\frac{1}{\alpha + j\omega}$
$\sin\omega_0 t$ Unit sine	$\frac{\pi}{j}[\delta[j(\omega - \omega_0)] - \delta[j(\omega + \omega_0)]]$	$\delta(t)$ Impulse	1
$\sin\omega_0 t u(t)$ Switched sine	$\frac{\pi}{2j}[\delta[j(\omega - \omega_0)] - \delta[j(\omega + \omega_0)]] + \frac{\omega_0}{(j\omega)^2 + \omega_0^2}$	$u(t)$ Step	$\frac{1}{j\omega} + \pi\delta(j\omega)$

14.13 QUESTIONS

- The Fourier transform of a signal $v(t)$ is known to be $V(j\omega) = |\omega|e^{-|\omega|}$ volts/Hz. What will be the approximate amplitude and frequency of the output signal if this signal is applied to an ideal band-pass filter that passes signals of frequencies in the range 0.49 to 0.51 rad/s?
- The signal $v(t) = 2\delta(t) + 3e^{-t} + 2e^{-0.3t}\cos 10t + v_s(t)$ is known to have a Fourier transform of $V(j\omega) = 3 + 2\delta(j\omega)$. What must be $v_s(t)$?
- The Fourier transform values for a signal $v(t)$ at $j2$ rad/s and $-j2$ rad/s are $-0.5 + j0.7$ and $-0.5 - j0.8$, respectively. Explain why $v(t)$ can not be a real function of t .
- The Fourier transform of a rectangular pulse symmetrically located in time-axis is found to cross the frequency-axis for the first time at 100π rad/s. The DC content in its spectrum is 1 volts/Hz. Find its normalised energy.
- A time-function $v(t)$ is defined as $v(t) = \sin(1/t)$ in the interval $[0, 1]$ and zero elsewhere. Will this function satisfy Dirichlet's conditions?
- Does unit ramp function satisfy Dirichlet's conditions?
- The Fourier transform of $v(t)$ has a value of 0.1 volts/Hz at $\omega = 0$. $v(t)$ is applied to an initially relaxed inductor of 0.1 H. What is the inductor current magnitude as $t \rightarrow \infty$?
- The value of a real $v(t)$ at $t = 0$ is known to be zero. Explain why its Fourier transform should have a real part that must be negative in some frequency range.
- The Fourier transform value at $\omega = 0$ for a signal $v(t)$ is $1 - j0.7$. Explain why $v(t)$ can not be a real signal.
- The magnitude spectrum of a signal $v(t)$ is shown in Fig. 14.13-1. Explain why this signal can not be a real function of t .

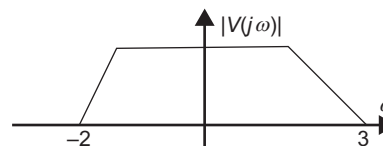


Fig. 14.13-1

- A signal $v(t)$ is shown in Fig. 14.13-2. Find the Fourier transform of its integral.

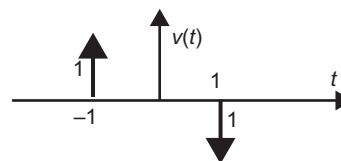


Fig. 14.13-2

12. What is the time-function $v(t)$ if its Fourier transform is $V(j\omega) = 0.5[\cos 2\omega - j\sin 2\omega]$?
13. Find the Fourier transform of $v(t) = 2e^{-3|t|}$.
14. Use the Fourier transform of a rectangular pulse to show that (i) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ and (ii) $\int_0^\infty \text{sinc}(x) dx = \frac{1}{2}$.
15. Find the Fourier transform of $te^{-\alpha t}u(t)$ for a positive real α , starting from the Fourier transform of $e^{-\alpha t}u(t)$.
16. Verify the time-domain differentiation property of Fourier transforms for $v(t) = e^{-\alpha t}u(t)$.
17. A signal $v(t)$ has a normalised energy of 0.01 J. What is the normalised energy of $v(3t)$?
18. If $V_o(j\omega) = 10/(j\omega + 1)$ in the initially relaxed circuit in Fig. 14.13-3, what is the $v_i(t)$ applied to it?

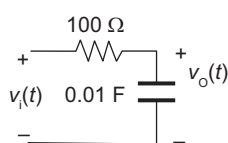


Fig. 14.13-3

19. If $V_o(j\omega) = 10/[(1 - \omega^2) + 2j\omega]$ in the circuit in Fig. 14.13-3, what is the $v_i(t)$ applied to it and what is $v_o(t)$?
20. An ideal high-pass filter has $H(j\omega) = \begin{cases} 1 & \text{for } |\omega| > \omega_c \\ 0 & \text{for } |\omega| < \omega_c \end{cases}$. Find and plot its impulse response.
21. An ideal band-pass filter has $H(j\omega) = \begin{cases} 1 & \text{for } \omega_{c1} < |\omega| < \omega_{c2} \\ 0 & \text{for all other } \omega \end{cases}$. Find and plot its impulse response.

22. Find the percentage of normalised energy contained in the frequency band 0 to α rad/s for a pulse $v(t) = e^{-\alpha t}$ (α is positive real).
23. Show that $\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re}[H(j\omega)] d\omega$ gives the initial slope of the step response of a circuit which has $H(j\omega)$ as its system function.
24. Show that $H(j0)$ gives the steady-state value of the step response of a circuit which has $H(j\omega)$ as its system function.
25. Use the Fourier transform of a rectangular pulse to show that $\int_0^\infty \text{sinc}^2(x) dx = 2$.
26. The energy spectral density functions for two signals $v_1(t)$ and $v_2(t)$ are identical for all ω . Does it imply that $v_1(t) = v_2(t)$ for all t ? Discuss.
27. Show that $v(t)$ and $v(t - t_d)$ will have the same normalised energy by employing frequency-domain reasoning.
28. The voltage across the 1 H in the circuit in Fig. 14.13-4 is passed through a unity gain buffer amplifier to the 5 Ω resistor. Find the total energy dissipated in the 5 Ω resistor, if $v(t) = \delta(t)$ and the circuit was initially relaxed.

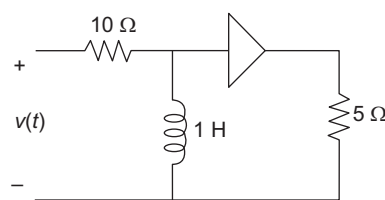


Fig. 14.13-4

14.14 PROBLEMS

1. Express the function $v(t)$ in Fig. 14.14-1 in terms of step functions and obtain its Fourier transform from the defining equations. Verify the equality between the area under $v(t)$ and the value of Fourier transform at $\omega = 0$.

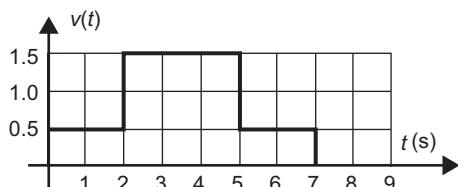


Fig. 14.14-1

2. (i) Find the Fourier transform of the single saw-tooth pulse shown in Fig. 14.14-2 and sketch the magnitude and phase spectrum. (ii) Obtain the area under the real part of its Fourier transform without carrying out the integration.

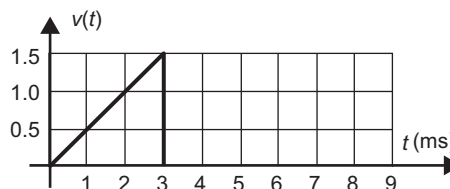


Fig. 14.14-2

3. Obtain the Fourier transform of $v(t)$ in Fig. 14.14-3, starting from the Fourier transform you arrived at in Problem-2 and employing the basic properties of Fourier transforms.

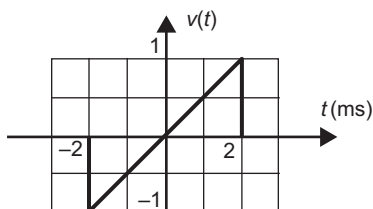


Fig. 14.14-3

4. The signal $v(t)$ has a Fourier transform given by $V(j\omega) = \frac{3}{2 + 3j\omega}$. Find the Fourier transforms of the following signals using the basic properties of Fourier transforms.

- (i) $v_1(t) = \frac{dv(t)}{dt}$, (ii) $v_2(t) = v(-2 - t)$, (iii) $v_3(t) = v(-t/4)$,
 (iv) $v_4(t) = e^{-j3t} v(t - 5)$ (v) $v_5(t) = 3v(t) \cos 6\pi t$ and
 (vi) $v_6(t) = v(4t - 3)$.
5. The Fourier transform $V(j\omega)$ for a signal $v(t)$ is shown in Fig. 14.14-4. Find $v(t)$.

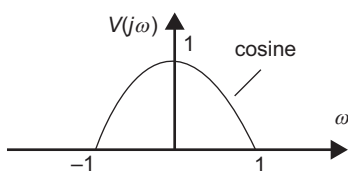


Fig. 14.14-4

6. Determine the time-function corresponding to each of the following Fourier transforms.

(i) $V(j\omega) = 2e^{-2|\omega|} (\cos \omega - j \sin \omega)$

(ii) $V(j\omega) = \begin{cases} j \sin 2\omega & \text{for } |\omega| < \pi \\ 0 & \text{for } |\omega| > \pi \end{cases}$

(iii) $V(j\omega) = \frac{-j\omega}{3 - \omega^2 + 4j\omega}$

(iv) $V(j\omega) = \frac{2j\omega + 1}{9 - \omega^2 + 6j\omega}$

(v) $V(j\omega) = \frac{\omega^2 - 4j\omega - 6}{[2 - \omega^2 + 3j\omega][j\omega + 4]}$

(vi) $V(j\omega) = \text{Im} \left(e^{-j2\omega} \frac{1}{j\omega + 3} \right)$

(vii) $V(j\omega) = \text{Re} \left(e^{-j3\omega} \frac{1}{1 - \omega^2 + j\omega} \right)$

7. Complex Convolution Theorem of Fourier transforms says that if $v_1(t) \Leftrightarrow V_1(j\omega)$ and $v_2(t) \Leftrightarrow V_2(j\omega)$, then, $v_1(t) \times v_2(t) \Leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} V_1(j\theta) V_2(j[\omega - \theta]) d\theta$. Show that the frequency-shifting property of Fourier transforms (i.e., $e^{j\omega_0 t} v(t) \Leftrightarrow$

$V(j[\omega - \omega_0])$) is only a special case of Complex Convolution.

8. (i) Let $H(j\omega)$ be the system function of a low-pass system. Question-23 showed that the area under the real part of $H(j\omega)$ gives the initial slope of the step response. Let us define the rise time t_r for the low-pass system as the time taken by the system to reach the steady-state in the step response if the initial rate of change is maintained throughout. Using the results in Question-23 and Question-24, derive an expression for t_r .
 (ii) There are many definitions for bandwidth of a system. One possible definition is that it is the width of a rectangular Fourier transform with the same value as that of $H(j\omega)$ at $\omega = 0$ and the same area as that of $H(j\omega)$. Derive an expression for bandwidth B defined in this manner using the results of Questions 23 and 24.
 (iii) Show that the product of rise time and bandwidth defined in this manner is a constant, independent of the particular $H(j\omega)$.
 (iv) It was pointed out at many places in the text that confining or limiting a signal in one domain leads to its expansion in the other domain. Interpret the result in (iii) in light of this fact. Also, compare the result with the *uncertainty principle* in Physics.
9. If $v_1(t) = 5e^{-4t} u(t)$ and $v_2(t) = 3e^{-1.5t} u(t)$, find $v_1(t) * v_2(t)$ by working in the frequency-domain.
10. $v(t)$ is a symmetrically located rectangular pulse in time-domain with a height of V and width of τ s. (i) Obtain the waveform of $v_1(t) = v(t) * v(t)$ by carrying out convolution in time-domain. (ii) Find the Fourier transform of $v_1(t)$.
11. Find and sketch the Fourier transform of a time-function $v(t) = \delta(t) + \delta(t - 1) + \delta(t + 1) + \delta(t - 2) + \delta(t + 2)$.
12. One period of a periodic $v(t)$ with a period of 6 s is shown in Fig. 14.14-5. Find and sketch its Fourier transform.

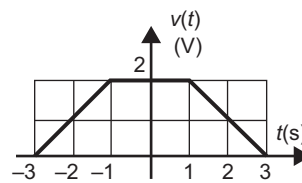


Fig. 14.14-5

13. (i) Find and plot the energy spectral density of $v(t) = (2t)e^{-0.05t}$.
 (ii) Verify Parseval's relation for this signal. (iii) What must be the bandwidth of an ideal low-pass filter if the output of the filter has 90% of input normalised energy when it is driven by this signal?
14. $v(t) = 2 \cos 2\pi \times 10^6 t$ for $0 \leq t \leq 10^{-3}$ s and zero for all other time instants. (i) Find the Fourier transform and energy spectral density for this signal and plot them. (ii) This signal is applied to an ideal band-pass filter with a flat gain of unity in the pass-band. What must be the centre frequency and the bandwidth of this filter if the output is to contain >90% of the input energy?
15. The impulse response of a circuit is seen to be $2e^{-10t}$ V. The output of this circuit is buffered using a buffer amplifier of unity gain and the output of the buffer is connected to an ideal band-reject filter which rejects all frequency components in

the 0 to 10 Hz range and passes all other components with unity gain. What is the normalised energy of the output of the filter?

16. The impulse response of a circuit is seen to be $[2e^{-0.2t} + 0.7e^{-0.05t} \sin(75t + 0.3\pi)]u(t)$ V. The applied impulse is a voltage signal. (i) Find the frequency response function and the system function of the circuit. (ii) Evaluate the final value of its step response without evaluating the step response.
17. The voltage $v_i(t)$ applied to the initially relaxed circuit shown in Fig. 14.14-6 is $[\delta(t) - \delta(t - 0.2)]$ V. Find the voltage across the capacitor as a function of time using the Fourier transform technique.

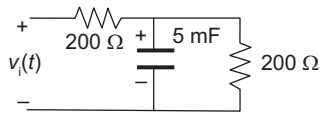


Fig. 14.14-6

18. The applied voltage in the circuit in Fig. 14.14-7 is $10u(t)$ V. The initial current in the inductor was zero and the initial voltage across the capacitor was -10 V with the polarity shown in figure. Find the voltage across the capacitor as a function of time using the Fourier transform technique.

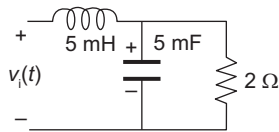


Fig. 14.14-7

19. If $v(t) = 10 \sin 100\pi t u(t)$ in the circuit in Fig. 14.14-8, find the voltage across the $2 \text{ k}\Omega$ resistance as a function of time. The initial current in the inductor was 1 A from left to right.

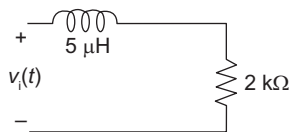


Fig. 14.14-8

20. The circuit of a differentiator with parasitic elements included is shown in Fig. 14.14-9. The operational amplifier may be considered ideal. Find the system function of the circuit and its impulse response.

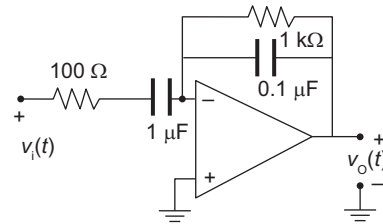


Fig. 14.14-9

21. (i) Find the system function in the circuit in Fig. 14.14-10. (ii) Plot the magnitude of system function. (iii) Find the impulse response and the normalised energy in the circuit output.

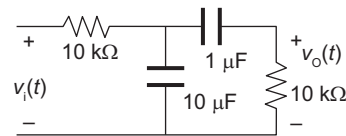


Fig. 14.14-10

22. The 20 pF capacitors in Fig. 14.14-11 represent the parasitic capacitance across the resistors. Find the output of the amplifier when the input is a rectangular pulse of 0.1 V height and $20 \mu\text{s}$ width.

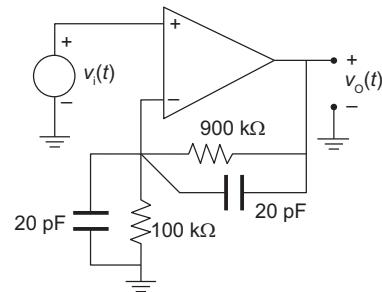


Fig. 14.14-11

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Analysis of Dynamic Circuits by Laplace Transforms

CHAPTER OBJECTIVES

- Expansion of right-sided time-domain waveforms in terms of $e^{\sigma t} \cos(\omega t + \phi)$
- Definition of Laplace transform and inverse integral
- Laplace transform as a generalisation of Fourier transform
- Various properties of Laplace transform of a real function of time
- Convergence of inverse integral and region of convergence in s -plane
- Basic Laplace transform pairs
- Solution of differential equations using Laplace transforms
- Method of partial fractions for inversion of Laplace transforms
- s -domain equivalent circuits and applying them for circuit analysis
- System function $H(s)$ and Laplace transform of impulse response
- Network functions and pole-zero plots
- Graphical interpretation of frequency response function
- Transient response of a coupled coil system and constant flux linkage theorem

This Chapter shows that a time-domain signal can be expanded in terms of general complex exponential functions – even in terms of functions that grow with time. It goes on to show how such a signal expansion can be employed to describe even an unstable linear time-invariant circuit in frequency-domain. Thereby it completes the frequency-domain description of LTI circuits.

INTRODUCTION

A linear time-invariant circuit is completely characterised in time-domain by its impulse response $h(t)$. The sinusoidal steady-state frequency response function $H(j\omega)$ too can be obtained from impulse response with the help of convolution integral. $H(j\omega)$ was seen to contain $h(t)$ in a hidden form in the form of an equation $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$ in Chap. 12. That observation led us to the present part of the book that had the stated aim of showing that $H(j\omega)$ is an equally complete characterisation of a linear time-invariant circuit in frequency-domain.

The sinusoidal steady-state frequency response function $H(j\omega)$ which can be obtained from impulse response with the help of convolution integral is a complete characterisation of a linear time-invariant stable circuit in frequency-domain as far as input functions which possess a Fourier transform are concerned.

However, there are inputs for which Fourier transforms may not exist or are difficult to find. Further, we need a frequency-domain description for unstable circuits too. Thus, Fourier description of linear time-invariant circuits is not a complete description.

Laplace transform completes the frequency-domain description of linear time-invariant circuits.

Sinusoidal waveforms are the basis functions in Fourier description of circuits. Generalised complex exponential waveforms are the basis functions in Laplace transform description of circuits.

Chapters 13 and 14 have more or less settled the issue in the case of stable circuits. Chapter 13 showed that any arbitrary periodic input waveform with certain minimal constraints on it could be expanded in terms of harmonically related sinusoidal waveforms. The chapter further showed how the forced response part (and hence steady-state response part) of the output of a stable circuit can be constructed using this expansion and the $H(j\omega)$ of the circuit. Chapter 14 showed that any arbitrary transient waveform, which satisfies the Dirichlet's criteria, could be expanded in terms of sinusoidal waveforms of all frequencies from 0 to ∞ (Fourier transforms). Further, it showed that zero-state response of a stable circuit to any such input can be constructed from its sinusoidal expansion and the $H(j\omega)$ function. Special emphasis was also placed on the point that even a waveform that does not satisfy Dirichlet's criteria can have a Fourier transform since Dirichlet's criteria are only sufficient conditions and not necessary conditions. The standard unit step function $u(t)$ is an example.

Thus, the Fourier transform technique has shown that the sinusoidal steady-state frequency response function $H(j\omega)$ is a complete characterisation of a linear time-invariant *stable* circuit for all input functions that have Fourier transforms.

However, Fourier transformation technique will not help us in dealing with inputs that are not absolutely integrable since the Fourier transform of such waveforms generally do not converge, and hence, may not exist. Moreover, even if Fourier transform exists for such waveforms, considerable ingenuity will be needed to find out the transform. Observe that we had to deal with $u(t)$ in a roundabout manner in order to find its Fourier transform in Chap. 14. Thus, the class of functions that are not absolutely integrable, i.e., $\int_{-\infty}^{\infty} |v(t)| dt$ does not converge, do pose a problem to frequency-domain analysis of circuits using Fourier transform. What is evidently in need is a more powerful and more general version of signal expansion and a more general version of the system function $H(j\omega)$ so that we can claim the frequency-domain description of linear time-invariant circuits to be as complete as the time-domain description. This chapter deals with such a signal expansion technique – it is called the *Laplace transform*.

Sinusoidal waveforms were the *basis functions* for expanding a signal in Fourier series and Fourier transforms. The reasons for choosing sinusoidal waveforms for this purpose were elucidated in the introductory portion of Chap. 13. We noted that, the choice of *basis function* for expanding an input signal depends on the requirement that the forced response of a linear time-invariant circuit to the chosen function must be easy to determine. We would appreciate it if we could obtain the forced response by a simple multiplication of the forcing function by a number (possibly complex valued). But, then, complex exponential functions are *eigen* functions of linear time-invariant circuits, and, the forced response part of output is decided precisely by a multiplication with a complex number in the case of an *eigen function* input. Therefore, the complex exponential function e^{st} is the *basis function* that we would like to use in expanding an arbitrary input waveform. Based on our decision to try simpler choices first, we used $s = j\omega$ and thereby limited our choice to pure sinusoidal functions drawn from the general class of complex exponential functions in Chaps. 13 and 14. The result was the Fourier transform description of a time-domain waveform. But we meet with certain useful waveforms that refuse to yield to Fourier transformation. Hence, we let the *basis function* be any general complex exponential function e^{st} and try to obtain frequency-domain description (i.e., signal expansion) for much broader class of waveforms now.

The time-domain description of linear time-invariant circuit in the form of convolution integral does not shy away from unstable circuits. Convolution integral and other time-domain techniques apply to unstable and marginally stable circuits too. However, the impulse response of an unstable circuit contains natural response terms that grows with time and hence will not be absolutely integrable. Therefore, impulse response of an unstable circuit may not have a Fourier transform. Therefore, Fourier transform technique cannot handle unstable circuits whereas Laplace transform technique can.

15.1 CIRCUIT RESPONSE TO COMPLEX EXPONENTIAL INPUT

Let the n^{th} order differential equation describing a n^{th} order linear time-invariant circuit be

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \quad (15.1-1)$$

$y(t)$ is some circuit variable identified as the output variable and $x(t)$ is some independent voltage/current source function. Let $x(t) = 1 e^{st}$ be a complex exponential function of unit amplitude and complex frequency $s = \sigma + j\omega$. Let $y(t) = A e^{st}$ be the trial solution where A is a complex number to be determined. Substituting the trial solution in Eqn. 15.1-1, we get,

$$\begin{aligned} \left[s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \right] A e^{st} &= \left[b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \right] e^{st} \\ \therefore A &= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}. \end{aligned}$$

Thus, when the input to a linear time-invariant circuit is a complex exponential function e^{st} , the output is the same complex exponential function multiplied by a complex number. Therefore, the complex frequency of output *remains same as that of input*. The output will have a different phase compared to that of input since A is a complex number in general and has an angle. The value of this complex scaling factor depends on the coefficients of circuit differential equation (i.e., on the circuit parameters) and the complex frequency s of the input. In consonance with the symbol $H(j\omega)$ used for a similar complex number that relates the output to an input of $e^{j\omega t}$, we use the symbol $H(s)$ to represent this number A from this point onwards. Therefore,

when $x(t) = e^{st}$ in a linear time-invariant circuit, $y(t) = H(s) e^{st}$, where

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}.$$

$H(s)$ – the generalised frequency response function of a linear time-invariant circuit.

But, which component of response is this? Since $x(t) = 1 e^{st}$, the complex exponential function was taken to be applied to the circuit from $t = -\infty$ onwards. Therefore, there is only one component in response and that is the forced response. Therefore, the response given above is the forced response as well as the total response. But if $x(t) = 1 e^{st} u(t)$, then, the above expression yields the forced response component only. The natural response terms in zero-state response and the natural response terms in zero-input response have to be found from initial conditions. However, those terms are also expected to be complex exponential functions since natural response terms of a linear time-invariant circuit are complex exponential functions.

The complex function $H(s)$ of a complex variable s can also be written in polar form as $|H(s)| \angle \theta$ and in exponential form as $|H(s)| e^{j\theta}$, where θ is its angle. Therefore, the output $y(t)$ can be expressed as $y(t) = |H(s)| e^{st-j\theta} = |H(s)| e^{\sigma t} e^{j(\omega t + \theta)}$. $H(s)$ may be viewed as a generalised frequency response function. Its magnitude gives the ratio between the amplitude of output complex exponential function and input complex exponential function. Its angle gives the phase angle by which the output complex exponential function leads the input complex exponential function.

EXAMPLE: 15.1-1

A gated input function $v(t) = 2e^{0.2t} \cos 2t u(t)$ V is applied across a series RC circuit with $RC = 2$ s. The voltage across the capacitor is taken as the output. Determine the zero-state response of the output voltage.

We had to add a natural response term $Ae^{-0.5t}$ and adjust the value of A to meet the initial condition in the Example 15.1-1 since the applied voltage was $2e^{0.2t} \cos 2t u(t)$ and not $2e^{0.2t} \cos 2t$. We could have avoided this step and obtained the zero-state response in one step if we could express the function $2e^{0.2t} \cos 2t u(t)$ as a sum of complex exponential functions of type $e^{j\omega t}$ – that would be a Fourier transform.

Fourier transform expresses a transient function as the sum of infinitely many sinusoidal waveforms that start at $-\infty$ and go up to ∞ in time-axis. But this waveform has no Fourier transform. It is a growing function of time.

But, can it be expressed as a sum of complex exponential functions of type $e^{st} = e^{(\sigma + j\omega)t}$ with some non-zero value for σ ? If it can be expressed that way, we can obtain the zero-state response to $2e^{0.2t} \cos 2t u(t)$ as the sum of forced response components to many complex exponential functions with the help of $H(s)$.

SOLUTION

The differential equation governing the voltage $v_o(t)$ across the capacitor in a series RC circuit is $\frac{dv_o}{dt} + \frac{1}{RC} v_o = \frac{1}{RC} v$, where $v(t)$ is the input voltage. Therefore, $H(s) = \frac{1/RC}{s + 1/RC} = \frac{1}{1 + sRC} = \frac{1}{1 + 2s}$.

$$v(t) = 2e^{0.2t} \cos 2t = 2e^{0.2t} \left[\frac{e^{j2t} + e^{-j2t}}{2} \right] = e^{(0.2+j2)t} + e^{(0.2-j2)t}.$$

We apply superposition principle to determine the forced response to these two input components.

$$\text{Forced response to } e^{(0.2+j2)t} = \left. \frac{1}{1+2s} \right|_{s=0.2+j2} \times e^{(0.2+j2)t} = \frac{1}{1.4+j4} \times e^{(0.2+j2)t}$$

$$\text{Forced response to } e^{(0.2-j2)t} = \left. \frac{1}{1+2s} \right|_{s=0.2-j2} \times e^{(0.2-j2)t} = \frac{1}{1.4-j4} \times e^{(0.2-j2)t}$$

$$\begin{aligned} \therefore \text{Forced response to } 2e^{0.2t} \cos 2t &= \frac{1}{1.4+j4} \times e^{(0.2+j2)t} + \frac{1}{1.4-j4} \times e^{(0.2-j2)t} \\ &= 0.236 \left[e^{(0.2+j2)t-j1.234} + e^{(0.2-j2)t+j1.234} \right] \\ &= 0.236 \times e^{0.2t} \times \left[e^{j(2t-1.234)} + e^{-j(2t-1.234)} \right] \\ &= 0.236 \times e^{0.2t} \times 2 \cos(2t - 1.234) \\ &= 0.472 e^{0.2t} \cos(2t - 70.71^\circ) \text{ V.} \end{aligned}$$

We need to find the zero-state response. The initial capacitor voltage is zero since we are trying to solve for zero-state response. Therefore $v_o(t) = Ae^{-0.5t} + 0.472 e^{0.2t} \cos(2t - 70.71^\circ)$ V with $v_o(0^+) = 0$. Therefore, $A = -0.472 \cos(-70.71^\circ) = -0.156$.

Therefore, the zero-state response to $2e^{0.2t} \cos 2t u(t) = -0.156 e^{-0.5t} + 0.472 e^{0.2t} \cos(2t - 70.71^\circ)$ V. (See the side-note)

15.2 EXPANSION OF A SIGNAL IN TERMS OF COMPLEX EXPONENTIAL FUNCTIONS

The signal $2e^{0.2t} \cos 2t u(t)$ does not have a Fourier transform because it is not absolutely integrable. We now formulate the signal decomposition problem in a general form. We define the kind of signals that we are going to use first.

Let $v(t) = f(t) u(t)$ be a right-sided function of time. We consider only right-sided functions of time in the analysis of linear time-invariant circuits since that is the kind of functions we apply to them and that is the kind of functions we get from them. The *underlying function* $f(t)$ is defined for all t and may be non-zero for $t < 0$. For instance, if $f(t) = 1$ for all t then $v(t)$ defined as $f(t) u(t)$ is a unit step function and stands for switching a 1 V DC source on to the circuit at $t = 0$.

If $v(t)$ is absolutely integrable – i.e., if $\int_0^\infty f(t) dt$ is finite – and if $v(t)$ satisfies other Dirichlet's conditions, its Fourier transform will exist. If $v(t)$ is *not* absolutely integrable, its Fourier transform *may not* exist. In any case, we assume that the absolute value of the right-sided signal is *bounded by real exponential function* – i.e., $|v(t)| < Me^{\alpha t}$ for all $t \geq 0$ with some value of M and α . For instance, the signal $v(t) = 1u(t)$ is bounded by $Me^{\alpha t}$ with any $M > 1$ and any $\alpha > 0$. The signal $v(t) = e^{\beta t} u(t)$ is bounded by $Me^{\alpha t}$ with any $M > 1$ and any $\alpha > \beta$. Similarly, the signal $v(t) = e^{\beta t} \cos(\omega t + \theta) u(t)$ is bounded by $Me^{\alpha t}$ with any $M > 1$ and any $\alpha > \beta$. There are signals that cannot be bounded in this sense. We do not deal with them in circuit analysis. However, we assume that the signals we deal with in this chapter are bounded.

Now, we define a new signal $v_1(t)$ by multiplying $v(t)$ with a scaling function $e^{-\sigma t}$ with the value of σ equal to α or greater than α , where α appears in the index of the *bounding exponential* $Me^{\alpha t}$.

$\therefore v_1(t) = v(t)e^{-\sigma t} = f(t)e^{-\sigma t}u(t)$, where $|v(t)| < Me^{\alpha t}$ and $\sigma \geq \alpha$. This results in $v_1(t)$ becoming an absolutely integrable function. In fact $v_1(t)$ goes to zero as $t \rightarrow \infty$ since the factor $e^{-\sigma t}$ is chosen to overpower the maximum growth rate the function $v(t)$ may possibly exhibit and to make it a decaying function thereby. Therefore, Fourier transform exists for $v_1(t)$.

$\therefore V_1(j\omega) = \int_{0^-}^{\infty} v(t)e^{-\sigma t}e^{-j\omega t} dt$, where the lower limit of integration is set at 0^- to handle impulse functions.

$$\begin{aligned} V_1(j\omega) &= \int_{0^-}^{\infty} v(t)e^{-\sigma t}e^{-j\omega t} dt \\ &= \int_{0^-}^{\infty} v(t)e^{-(\sigma+j\omega)t} dt \end{aligned}$$

i.e., $V_1(j\omega) = \int_{0^-}^{\infty} v(t)e^{-st} dt$, where $s = \sigma + j\omega$ = the general complex frequency.

Now, this $V_1(j\omega)$ which is the *Fourier transform* of an exponentially scaled version of $v(t)$ with a particular value of σ that appears in the index of the exponential scaling function is *defined* as the *Laplace transform* $V(s)$ of $v(t)$.

Fourier transform of $v_1(t)$ exists and the Fourier inverse integral converges to $v_1(t)$ for all t only if $\sigma > \alpha$. Therefore, $V_1(j\omega)$ exists only for $\sigma > \alpha$. And since $V(s)$ is just the other name for $V_1(j\omega)$, we conclude that, if $v(t) = f(t)u(t)$ and $|v(t)| < Me^{\alpha t}$ for some M and α , then $V(s) = \int_{0^-}^{\infty} v(t)e^{-st} dt$ is its Laplace transform, where $s = \sigma + j\omega$ is the general complex frequency with $\sigma \geq \alpha$. The Laplace transform exists and the inverse integral converges to $v(t)$ only for those values of s that have $\text{Re}(s) > \alpha$. The region formed by all those values of s in the s -plane for which the Laplace transform of a time-function is defined and is convergent is called the Region of Convergence (ROC) of the Laplace transform. *Obviously the ROC of Laplace transform of a right-sided function is the region to the right of $\text{Re}(s) = \alpha$ line.* This is a vertical straight-line parallel to $j\omega$ axis and crossing σ -axis at α .

$V(s)$ is actually $V_1(j\omega)$ with a particular value of σ chosen. Therefore, the time-function $v(t)$ can be extracted from $V(s)$ by inverse Fourier transform.

$$v_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V_1(j\omega)e^{j\omega t} d\omega \text{ and } v_1(t) = v(t)e^{-\sigma t} \text{ with a particular value of } \sigma$$

$$\begin{aligned} \therefore v(t) &= e^{\sigma t} v_1(t) = e^{\sigma t} \times \frac{1}{2\pi} \int_{-\infty}^{\infty} V_1(j\omega)e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} V(s)e^{\sigma t}e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} V(s)e^{(\sigma+j\omega)t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(s)e^{(s+j\omega)t} d\omega \end{aligned}$$

But $s = \sigma + j\omega \Rightarrow ds = d\sigma + jd\omega = jd\omega$ when the integral is evaluated with a particular value of σ .

$$\therefore v(t) = \frac{1}{j2\pi} \int_{\text{on } \text{Re}(s)=\sigma \text{ line}} V(s)e^{st} ds$$

The Laplace transform defined this way returns the right-side of the underlying function $f(t)$ on inversion. The left-side returned will be zero. In this sense this Laplace transform may be termed as a *unilateral Laplace transform*. We deal only with unilateral Laplace transform in this chapter.

Note that the evaluation of inversion integral has to be performed on a line parallel to $j\omega$ -axis in s -plane with the line crossing the σ -axis within the region of convergence of the Laplace transform.

Laplace transform makes its appearance.

The Laplace Transform Pair

The analysis equation describes the decomposition of $v(t)$ in terms of complex exponential functions.

The synthesis equation describes the construction of $v(t)$ from its complex exponential function components.

That a time-function $v(t)$ has a Laplace transform $V(s)$ does not necessarily imply that it has a Fourier transform $V(j\omega)$ as well.

That a time-function $v(t)$ has a Laplace transform $V(s)$ which does not include $j\omega$ -axis in its ROC does not necessarily imply that $v(t)$ does not have a Fourier transform.

Let $v(t)$ be a right-sided function that is bounded by $Me^{\alpha t}$ with some finite value of M and α . Then the Laplace transform pair is defined as

$$V(s) = \int_0^{\infty} v(t)e^{-st} dt \quad \text{— The analysis equation} \quad (15.2-1)$$

$$v(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} V(s)e^{st} ds \quad \text{— The synthesis equation} \quad (15.2-2)$$

where $s = \sigma + j\omega$ is the complex frequency variable standing for the complex exponential function e^{st} with σ value $> \alpha$. The ROC of $V(s)$ is the entire plane to the right of $\text{Re}(s) = \alpha$ line.

If $j\omega$ -axis is part of the ROC of a Laplace transform, then, substituting $s = j\omega$ in the Laplace transform will give the Fourier transform of the corresponding time-function. This is so because if $j\omega$ -axis is part of the ROC of $V(s)$, the function $v(t)$ will be absolutely integrable, its Fourier transform will exist and its Fourier transform will then be given by its Laplace transform evaluated on $j\omega$ -axis. If $j\omega$ -axis is not a part of the ROC of $V(s)$, the function $v(t)$ will not be absolutely integrable, its Fourier transform may or may not exist and if its Fourier transform exists it will not be the same as its Laplace transform evaluated on $j\omega$ -axis.

15.2.1 Interpretation of Laplace Transform

Laplace transform of a time-domain signal $v(t)$ can be interpreted in a manner similar to the interpretation of Fourier transform we carried out in Sect. 14.2 of Chap. 14.

The Laplace transform $V(s)$ is a complex amplitude density function. Equation 15.2-2 makes it clear that Laplace transform expresses the given time-function as a sum of infinitely many complex exponential functions of infinitesimal complex amplitudes. Thus, Laplace transform is an expansion of $v(t)$ in terms of complex exponential functions. Fourier transform is an expansion of time-function in terms of a special class of complex exponential functions — the ones that are represented as points on $j\omega$ -axis. Therefore, a Fourier transform can be evaluated only on $j\omega$ -axis. But the entire ROC is available for evaluating Laplace transform. Example 15.2-1 illustrates these concepts further.

EXAMPLE: 15.2-1

Find the Laplace transform of $v(t) = u(t)$.

SOLUTION

$$V(s) = \int_0^{\infty} e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{1}{s} \quad \text{for } \text{Re}(s) > 0.$$

Therefore, $V(s) = 1/s$ with ROC of $\text{Re}(s) > 0$.

Thus, the inversion integral can be evaluated on any vertical straight-line on the right-half in s -plane. But does that mean a steady function like $u(t)$ is being synthesised from oscillations that grow with time? It means precisely that. The synthesis equation (Eqn. 15.2-2) reveals that infinite growing complex exponential functions of infinitesimal amplitudes, which start at $-\infty$ and go up to $+\infty$ in time, participate in making the transient time-function $u(t)$. The contribution from a band of complex frequencies around a complex frequency value s is approximately $V(s) \times \Delta s \times e^{st}$, where Δs is the width of complex frequency band. A similar contribution comes from the band located around s^* . These two contributions together will form a growing sinusoidal function as shown below.

$$\begin{aligned} &= \frac{1}{\sigma + j\omega} \times \Delta\omega \times e^{(\sigma + j\omega)t} + \frac{1}{\sigma - j\omega} \times \Delta\omega \times e^{(\sigma - j\omega)t} \\ &= \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} \Delta\omega \end{aligned}$$

Thus, similarly located bands in the two half-sections of the vertical line on which the inversion integral is being evaluated result in a real valued contribution as shown above. Now, the inversion integral for $1/s$ can be written as

$$\begin{aligned} v(t) &= \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s} e^{st} ds \\ &= \frac{1}{j2\pi} \int_0^\infty \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} (j d\omega) \\ &= \frac{1}{2\pi} \int_0^\infty \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} d\omega \end{aligned} \quad (15.2-3)$$

Thus, infinitely many exponentially growing sinusoids of frequencies ranging from zero to infinity, each with infinitesimal amplitude, interfere with each other constructively and destructively from $t = -\infty$ to $t = +\infty$ to synthesise the unit step waveform. Moreover, the exponentially growing sinusoids that participate in this waveform construction process are not unique. The value of σ can be any number >0 . Therefore, each vertical line located in the right-half of s -plane yields a distinct set of infinitely many exponentially growing sinusoids which can construct the unit step waveform.

That infinitely many exponentially growing sinusoids interfere with each other to produce a clean zero for all $t < 0$ and a clean 1 for all $t > 0$ is indeed counter-intuitive and quite surprising when heard first. Maybe we need a little convincing on that. The inversion integral in Eqn. 15.2-3 was evaluated using a short computer program for various values of σ and over finite length sections on the vertical line. In effect, the program calculated the partial integral of the form

$$v(t) \approx \frac{1}{2\pi} \int_0^{\omega_0} \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} d\omega \text{ for various values of } \sigma \text{ and } \omega_0.$$

Figure 15.2-1 shows the resulting waveforms for $\sigma = 0.1$ Np/s and $\omega_0 = 10, 20$ and 50 rad/s.

Even a small range of 10 rad/s shows the tendency of the integral to approach step waveform. With $\omega_0 = 50$ rad/s the integral has more or less yielded step waveform at least in the range -1 s to 4 s. We also observe the familiar Gibbs oscillations at discontinuities. This is expected since Laplace transform, after all, is a kind of generalised Fourier transform. Moreover, observe that the inversion integral returns 0.5 at $t = 0$. That is the value assured by Dirichlet. This was also expected since Laplace transform is a Fourier transform in disguise.

Figure 15.2-2 shows the results of partial evaluation of inversion integral for $\sigma = 1$ Np/s and $\omega_0 = 10, 20$ and 50 rad/s.

This set of simulation result shows that we have to include more and more components in the partial integral to converge to unit step wave-shape in a given time-interval as we let the components grow at a faster rate, i.e., for higher values of σ . And, keeping σ at a fixed value, we would need to include more and more frequency components when we increase the time-range over which we want convergence. However, we have infinite components at our disposal and it will be possible to include enough of them to recover the $u(t)$ shape up to any finite t however large it may be.

Therefore, Laplace transform expands a transient right-sided time-function in terms of infinitely many complex exponential functions of infinitesimal amplitudes. The ROC of such a Laplace transform will include right-half of s -plane and hence the time-domain waveform gets constructed by growing complex exponential functions though it appears counter-intuitive.

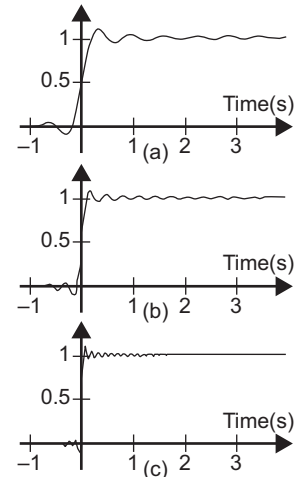


Fig. 15.2-1 Partial Inversion Integral for Unit Step Function for $\sigma = 0.1$ and (a) $\omega_0 = 10$ (b) $\omega_0 = 20$ (c) $\omega_0 = 50$

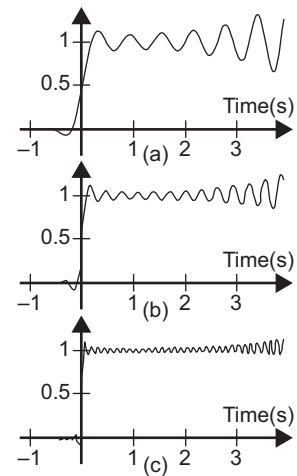


Fig. 15.2-2 Partial Inversion Integral for Unit Step Function for $\sigma = 1$ and (a) $\omega_0 = 10$ (b) $\omega_0 = 20$ (c) $\omega_0 = 50$

15.3 LAPLACE TRANSFORMS OF SOME COMMON RIGHT-SIDED FUNCTIONS

Integral of sum of two functions is the sum of integral of each function. Thus, Laplace transformation is a linear operation. If $v_1(t)$ and $v_2(t)$ are two right-sided functions and a_1 and a_2 are two real numbers, then, $a_1 v_1(t) + a_2 v_2(t) \Leftrightarrow a_1 V_1(s) + a_2 V_2(s)$ is a Laplace transform pair. This is called *Property of Linearity of Laplace transforms*. Now, we work out the Laplace transforms for many commonly used right-sided functions using the defining integral and property of linearity.

Linearity property of Laplace transforms.

Let $v(t) = e^{s_0 t} u(t)$ be a right-sided complex exponential function with a complex frequency of s_0 . Then,

$$V(s) = \int_{0^-}^{\infty} e^{s_0 t} e^{-st} dt = \int_{0^-}^{\infty} e^{-(s-s_0)t} dt = \frac{e^{-(s-s_0)t} \Big|_{0^-}^{\infty}}{(s-s_0)} = \frac{1}{(s-s_0)} \text{ with ROC } \operatorname{Re}(s) > \sigma_0.$$

Therefore, $e^{s_0 t} u(t) \Leftrightarrow 1/(s-s_0)$ is a Laplace transform pair with ROC $\operatorname{Re}(s) > \operatorname{Re}(s_0)$.

The special case of $v(t) = u(t)$ is covered by this transform pair with $s_0 = 0$.

Therefore, $u(t) \Leftrightarrow 1/s$ is a Laplace transform pair with ROC $\operatorname{Re}(s) > 0$.

The special case of $v(t) = \cos \omega_0 t u(t)$ is covered by expressing $v(t)$ as $(e^{j\omega_0 t} + e^{-j\omega_0 t})/2$ by employing Euler's formula and then applying property of linearity of Laplace transforms.

$$\therefore V(s) = \frac{0.5}{s-j\omega_0} + \frac{0.5}{s+j\omega_0} = \frac{s}{s^2 + \omega_0^2}$$

Therefore, $\cos \omega_0 t u(t) \Leftrightarrow \frac{s}{s^2 + \omega_0^2}$ is a Laplace transform pair with ROC $\operatorname{Re}(s) > 0$.

Similarly, $\sin \omega_0 t u(t) \Leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}$ is a Laplace transform pair with ROC $\operatorname{Re}(s) > 0$.

Consider $v(t) = e^{-\alpha t} \cos \beta t u(t)$. This can be expressed as $[e^{(-\alpha+j\beta)t} + e^{(-\alpha-j\beta)t}]/2$ by Euler's formula. Then,

$$\therefore V(s) = \frac{0.5}{s+\alpha-j\beta} + \frac{0.5}{s+\alpha+j\beta} = \frac{(s+\alpha)}{(s+\alpha)^2 + \beta^2}$$

Therefore, $e^{-\alpha t} \cos \beta t u(t) \Leftrightarrow \frac{(s+\alpha)}{(s+\alpha)^2 + \beta^2}$ is a Laplace transform pair with ROC of

$\operatorname{Re}(s) > -\alpha$. Similarly, $e^{-\alpha t} \sin \beta t u(t) \Leftrightarrow \frac{\beta}{(s+\alpha)^2 + \beta^2}$ is a Laplace transform pair with ROC of $\operatorname{Re}(s) > -\alpha$.

Now consider $v(t) = \frac{e^{(s_0+\Delta s)t} - e^{s_0 t}}{\Delta s} u(t)$.

The Laplace transform of this function can be found from the defining integral as

$$V(s) = \frac{1}{\Delta s} \left[\frac{1}{s-s_0-\Delta s} + \frac{1}{s-s_0} \right] = \frac{1}{(s-s_0-\Delta s)(s-s_0)}$$

Now, we send $v(t)$ to a limit as $\Delta s \rightarrow 0$.

$$\lim_{\Delta s \rightarrow 0} v(t) = \lim_{\Delta s \rightarrow 0} \frac{e^{(s_0+\Delta s)t} - e^{s_0 t}}{\Delta s} u(t) = \left(\lim_{\Delta s \rightarrow 0} \frac{e^{(s_0+\Delta s)t} - e^{s_0 t}}{\Delta s} \right) u(t) = t e^{s_0 t} u(t)$$

Therefore, Laplace transform of $t e^{s_0 t} u(t) = \lim_{\Delta s \rightarrow 0} \frac{1}{(s-s_0-\Delta s)(s-s_0)} = \frac{1}{(s-s_0)^2}$

Therefore, $t e^{s_0 t} u(t) \Leftrightarrow 1/(s-s_0)^2$ is a Laplace transform pair with ROC $\operatorname{Re}(s) > \operatorname{Re}(s_0)$.

The special case of $v(t) = t u(t)$ is covered by this transform pair with $s_0 = 0$.

Therefore, $t u(t) \Leftrightarrow 1/s^2$ is a Laplace transform pair with ROC $\operatorname{Re}(s) > 0$.

And finally, we consider $v(t) = \delta(t)$. $V(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^0 dt = 1$. Thus, $\delta(t) \Leftrightarrow 1$ is a Laplace transform pair with ROC of entire s -plane. It requires all complex exponential functions with equal intensity to synthesise an impulse function in time-domain.

These commonly used Laplace transform pairs are listed in the Table 15.3-1. Few are derived in this section, while others will be taken up later.

Table 15.3-1 Basic Laplace Transform Pairs

Time-function	Laplace transform	Region of convergence
$\delta(t)$	1	Entire s-plane
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$e^{s_0 t} u(t)$	$\frac{1}{1-s_0}$	$\text{Re}(s) > \text{Re}(s_0)$
$e^{j\omega_0 t} u(t)$	$\frac{1}{s-j\omega_0}$	$\text{Re}(s) > 0$
$e^{-\alpha t} u(t)$	$\frac{1}{s+\alpha}$	$\text{Re}(s) > -\alpha$
$\cos\omega_0 t u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
$\sin\omega_0 t u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
$e^{-\alpha t} \cos\beta t u(t)$	$\frac{(s+\alpha)}{(s+\alpha)^2 + \beta^2}$	$\text{Re}(s) > -\alpha$
$e^{-\alpha t} \sin\beta t u(t)$	$\frac{\beta}{(s+\alpha)^2 + \beta^2}$	$\text{Re}(s) > -\alpha$
$t u(t)$	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$t^n u(t), n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
$t e^{s_0 t} u(t)$	$\frac{1}{(s-s_0)^2}$	$\text{Re}(s) > \text{Re}(s_0)$
$t^n e^{s_0 t} u(t), n = 1, 2, \dots$	$\frac{n!}{(s-s_0)^{n+1}}$	$\text{Re}(s) > \text{Re}(s_0)$

15.4 THE s-DOMAIN SYSTEM FUNCTION H(S)

We saw in Sect. 15.1 that when an input e^{st} is applied to a linear time-invariant circuit described by an n^{th} order differential equation

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$$

the forced response is given by $H(s)e^{st}$ where

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}.$$

In Sect. 15.2 we observed that a right-sided function $x(t)$ can be expressed as a sum of infinitely many complex exponential functions of frequency between $\sigma - j\infty$ and $\sigma + j\infty$ with the line $\text{Re}(s) = \sigma$ falling within the ROC of Laplace transform of $x(t)$. We combine these two facts along with superposition principle to arrive at the zero-state response of a linear time-invariant circuit to a right-sided input function.

Consider a particular value of complex frequency s and a small band of complex frequency Δs centred on it. This band contributes complex exponential functions of

$H(s)$ in this context is the ratio of complex amplitude of forced response component in output to the complex amplitude of input complex exponential function with a complex frequency of s .

There is only forced response in this context and forced response itself is the total response.

frequencies between $(s - 0.5\Delta s)$ and $(s + 0.5\Delta s)$. For sufficiently small Δs we may take all these complex exponential functions to be evolving at approximately at the centre frequency of the band, i.e., at s itself. In that case, all the infinitesimal contributions coming from this band may be consolidated into a signal $\approx X(s) \Delta s e^{st}$.

This single complex frequency component with complex amplitude of $X(s) \Delta s$ will produce a total response component of $H(s) X(s) \Delta s e^{st}$ in the output. We get the zero-state response of the circuit by adding all such contributions over the line $\text{Re}(s) = \sigma$ falling within the ROC of $X(s)$ and sending the sum to a limit by making $\Delta s \rightarrow 0$. The result will be the following integral.

$$\therefore y(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} H(s)X(s)e^{st} ds. \quad (15.4-1)$$

Comparing the Eqn. 15.4-1 with the synthesis equation of Laplace transform given by Eqn. 15.2-2, it is evident that Eqn. 15.4-1 is the Laplace transform synthesis equation of the Laplace transform $H(s)X(s)$. But then, a synthesis equation which returns $y(t)$ must be synthesising it from the Laplace transform $Y(s)$ of the time-function $y(t)$. Therefore, $Y(s) = H(s)X(s)$. This is an important result that requires restatement. (See side-box)

The ratio of Laplace transform of zero-state response to Laplace transform of input source function is defined as the s-domain System Function.

Therefore, the s -domain System Function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}. \quad (15.4-2)$$

Note carefully that System Function is independent of initial conditions in the circuit since it is the zero-state response to a right-sided input that is employed in its definition. This function is also called a *Transfer Function* when both x and y are similar quantities, i.e., when x and y are voltages or x and y are currents, and is denoted by $T(s)$. It is called an *Input Impedance Function* and is denoted by $Z_i(s)$ if y is the voltage across a terminal pair and x is the current entering the positive terminal. It is called an *Input Admittance Function* and is denoted by $Y_i(s)$ if y is the current into a terminal pair and x is voltage across the terminal pair. These two, i.e., $Z_i(s)$ and $Y_i(s)$ together is at times referred to as *immittance functions*.

If the quantities x and y are voltage/current or current/voltage pair and they refer to different terminal pairs in the circuit, we call the s -domain System Function a *Transfer Impedance Function* or *Transfer Admittance Function* as the case may be. They are represented by $Z_m(s)$ and $Y_m(s)$, respectively.

We have an expression for $H(s)$ as a ratio of rational polynomials in s in Eqn. 15.4-2. Rational polynomials are polynomials containing only integer powers of the independent variable. But then, there is another interesting interpretation possible for $H(s)$.

Let us try to find the impulse response of the circuit by this transform technique. We remember that 'impulse response' means 'zero-state response to unit impulse input' by definition. Hence, we can use the System Function to arrive at the Laplace transform of impulse response as $H(s)X(s)$. But $x(t) = \delta(t)$ and therefore $X(s) = 1$. Hence, for a linear time-invariant circuit, the following statement holds.

Laplace transform of Impulse Response = s-domain System Function, and,

Impulse Response = Inverse Laplace Transform of s-domain System Function

This result was anticipated in naming the System Function as $H(s)$.

Once the System Function and Laplace transform of input source function are known, one can obtain the Laplace transform of zero-state response by inverting the product of input transform and System Function. We will take up the task of inverting Laplace transforms in later sections.

The Laplace transform of zero-state response = Laplace transform of input source function $\times$ Generalised frequency response function.

s -domain System Function defined.

System Function
System Function $H(s)$ is the ratio of Laplace transform of zero-state response to Laplace transform of input source function. Inverse Laplace transform of System Function gives the impulse response of the circuit.

15.5 POLES AND ZEROS OF SYSTEM FUNCTION AND EXCITATION FUNCTION

$H(s)$ is the system function, $X(s)$ is the excitation function and $Y(s)$ is the output function referred to in this section.

We observed that

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}.$$

is a ratio of rational polynomials in complex frequency variable s . Further, we observe from Table 15.3-1 that the excitation functions corresponding to the commonly employed input source functions are also in the form of ratio of rational polynomials in s . Thus, the output function also turns out to be a ratio of rational polynomials in s . Therefore, we can write,

$$H(s) = \frac{Q(s)}{P(s)}, X(s) = \frac{Q_e(s)}{P_e(s)} \text{ and } Y(s) = \frac{Q(s)}{P(s)} \times \frac{Q_e(s)}{P_e(s)}, \text{ where } Q(s) \text{ is an } m^{\text{th}} \text{ order polynomial}$$

polynomial on s and $P(s)$ is an n^{th} order polynomial on s . They are the numerator polynomial and denominator polynomial of System Function, respectively. Similarly, $Q_e(s)$ and $P_e(s)$ are the numerator and denominator polynomials on s for the excitation function.

Let the n roots of $P(s)$ be represented as $p_1, p_2, \dots, p_n$ and the m roots of $Q(s)$ be represented as $z_1, z_2, \dots, z_m$. These roots can be complex in general. $p_1, p_2, \dots, p_n$ are the n values of complex frequency s at which the System Function goes to infinity. They are defined as *poles* of System Function. $z_1, z_2, \dots, z_m$ are the m values of complex frequency s at which the System Function goes to zero value. They are defined as the *zeros* of System Function.

Similarly, the values of s at which $X(s)$ goes to infinity are called the *excitation poles* and the values of s at which $X(s)$ goes to zero are called the *excitation zeros*. They are the same as roots of $P_e(s)$ and $Q_e(s)$, respectively.

Apparently, the System Function poles and excitation function poles together will form the poles of output function. Similarly, the System Function zeros and excitation function zeros together will form the output function zeros. These statements assume that no pole-zero cancellation takes place.

A diagram that shows the complex signal plane, i.e., the s -plane, with all poles of a Laplace transform marked by '×' symbol and all zeros marked by 'o' symbol is called the *pole-zero plot* of that Laplace transform.

Some poles and zeros may have multiplicity greater than 1. In that case, the multiplicity is marked near the corresponding pole or zero in the format ' $r = k$ ', where r indicates the multiplicity and k is the actual value of multiplicity. The default value of $r = 1$ is not marked.

Poles, Zeros,
Pole-Zero Plots.

EXAMPLE: 15.5-1

Obtain the pole-zero plot of the transfer function $V_o(s)/V_s(s)$, excitation function and the output function in the circuit shown in Fig. 15.5-1 with $v_s(t) = 10 e^{-1.5t} \cos 2t u(t)$ V.

SOLUTION

The differential equation describing the second mesh current in this circuit was derived earlier in Example 12-1.3 in Chap. 12. It is reproduced below.

$$\frac{d^3 i_2}{dt^3} + \frac{d^2 i_2}{dt^2} + 2 \frac{di_2}{dt} + i_2 = v_s(t).$$

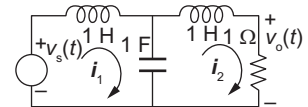


Fig. 15.5-1 Circuit for Example 15.5-1

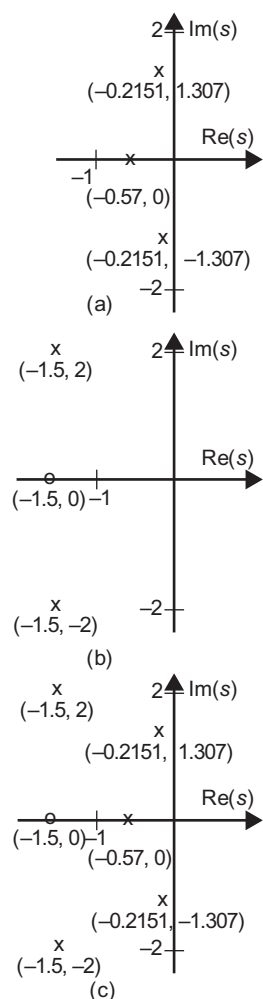


Fig. 15.5-2 Pole-Zero Plots in Example 15.5-1 (a) for System Function (b) for Excitation Function (c) for Output Function

$v_o(t)$ is numerically equal to $i_2(t)$ and hence the differential equation governing $v_o(t)$ is $\frac{d^3 v_o}{dt^3} + \frac{d^2 v_o}{dt^2} + 2 \frac{dv_o}{dt} + v_o = v_s(t)$.

The characteristic equation is $s^3 + s^2 + 2s + 1 = 0$ and its roots are $s_1 = -0.2151 + j1.307$, $s_2 = -0.2151 - j1.307$ and $s_3 = -0.5698$. The roots of a polynomial of degree higher than 2 will normally require the help of root-finding software or numerical methods. There is a pair of *complex conjugate roots*.

$$\text{The System Function } H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{s^3 + s^2 + 2s + 1}$$

$$\text{Laplace transform of } e^{-1.5t} \cos 2t \, u(t) \text{ is } \frac{s + 1.5}{(s + 1.5)^2 + 4}.$$

$$\text{Therefore, } V_s(s) = \frac{10(s + 1.5)}{(s + 1.5)^2 + 4} \text{ and } V_o(s) = \frac{1}{s^3 + s^2 + 2s + 1} \times \frac{10(s + 1.5)}{(s + 1.5)^2 + 4}.$$

We observe that the denominator polynomial is the same as in the left-side of the characteristic equation of the governing differential equation. This will always be so. Hence, *poles of System Function (they are also called 'system poles') will be same as the natural frequencies of the circuit for any linear time-invariant circuit.*

Therefore, the system poles are $p_1 = -0.2151 + j1.307$, $p_2 = -0.2151 - j1.307$ and $p_3 = -0.5698$.

The numerator polynomial of System Function in this case is trivial and there are no 'system zeros'.

The excitation poles are at $p_{e1} = -1.5 + j2$ and $p_{e1} = -1.5 - j2$ and excitation zero is at $z_{e1} = -1.5$.

The pole-zero plots are shown in Fig. 15.5-2.

15.6 METHOD OF PARTIAL FRACTIONS FOR INVERTING LAPLACE TRANSFORMS

Any Laplace transform can be inverted by evaluating the synthesis integral in Eqn. 15.2-2 on a suitably selected vertical line extending from $-\infty$ to ∞ in the s -plane within the ROC of the transform being inverted. But simpler methods based on Residue Theorem in Complex Analysis exist for special Laplace transforms. We do not take up the detailed analysis based on Residue Theorem here. However, the reader has to bear in mind the fact that the 'method of partial fractions' for inverting certain special types of Laplace transforms is based on Residue Theorem in Complex Analysis.

Linear time-invariant circuits are described by linear constant-coefficient ordinary differential equations. All the coefficients are real. Such a circuit will have only real-valued natural frequencies or complex-conjugate natural frequencies. Thus, the impulse response of such a circuit will contain only complex exponential functions. Each complex exponential function will have a Laplace transform of the form $\frac{k}{s - s_o}$, where s_o is the complex frequency of the particular term. Laplace transformation is a linear operation. Hence, Laplace transform of sum of impulse response terms will be a sum of Laplace transform of impulse response terms. Therefore, Laplace transform of impulse response of a linear time-invariant circuit will be sum of finite number of terms of the $\frac{k}{s - s_o}$ type. Such a sum will finally become a ratio of rational polynomials in s . The order of denominator polynomial will be the same as the number of first order terms of $\frac{k}{s - s_o}$ -type that entered the sum.

Many of the normally employed excitation functions in linear time-invariant circuits are also of complex exponential nature. Input functions that can be expressed as linear combinations of complex exponential functions will have Laplace transforms that are ratios of rational polynomials in s as explained above.

Product of Laplace transforms that are ratios of rational polynomials in s will result in a new Laplace transform that is a ratio of rational polynomials in s .

Let $Y(s) = Q(s)/P(s)$ be such a Laplace transform. Let the degree of denominator polynomial be n and that of numerator be m . The degree of numerator polynomial will usually be less than n . If the Laplace transform of output of a linear time-invariant circuit shows $n \leq m$, it usually implies that the circuit model employed to model physical processes has been idealised too much. We assume that $m < n$ in this section. If m is equal to n or more than n , then $Y(s)$ can be written as $Y(s) = k_1 s^{m-n} + \dots + k_{m-n} + \frac{Q'(s)}{P(s)}$, and we employ method of partial fractions on $\frac{Q'(s)}{P(s)}$ only.

Let $p_1, p_2, \dots, p_n$ be the n roots of denominator polynomial. They may be real or complex. If there is a complex root, the conjugate of that root will also be a root of the polynomial. We identify two cases. In the first case all the n roots (i.e., poles of $Y(s)$) are distinct.

Case-1: All the n roots of $P(s)$ are distinct.

Then, we can express $Y(s)$ as a sum of first order factors as below.

$$Y(s) = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_2)} + \dots + \frac{A_n}{(s-p_n)} \quad (15.6-1)$$

Each term in this expansion is called a *partial fraction*. The value of A_i appearing in the numerator of i^{th} partial fraction is called the 'residue at the pole p_i '. The problem of partial fractions involves the determination of these residues. Multiply both sides of the Eqn. 15.6-1 by $(s-p_i)$, where p_i is the pole at which the residue A_i is to be evaluated. Remember that $Y(s)$ will contain $(s-p_i)$ as a factor in the denominator. Hence, the multiplication by $(s-p_i)$ results in cancellation of this factor in $Y(s)$.

$$(s-p_i)Y(s) = \frac{A_1(s-p_i)}{(s-p_1)} + \frac{A_2(s-p_i)}{(s-p_2)} + \dots + \frac{A_i(s-p_i)}{(s-p_i)} + \dots + \frac{A_n(s-p_i)}{(s-p_n)} \quad (15.6-2)$$

Now, we evaluate both sides of Eqn. 15.6-2 at $s = p_i$ to get $A_i = (s-p_i)Y(s)|_{s=p_i}$. This calculation is repeated for $i = 1$ to n to complete all the partial fractions.

Each partial fraction of the type $\frac{A_i}{s-p_i}$ can be recognised as the Laplace transform of $A_i e^{p_i t} u(t)$ by consulting relevant entry in Table 15.3-1. (See the side-note).

Therefore, $y(t) = (A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t})u(t)$, where $A_i = (s-p_i)Y(s)|_{s=p_i}$.

Case-2: One root of multiplicity r and $n-r$ distinct roots for $P(s)$.

In this case the partial fraction expansion is as shown below.

$$Y(s) = \frac{A_1}{(s-p)^r} + \frac{A_2}{(s-p)^{r-1}} + \dots + \frac{A_r}{(s-p)} + \frac{A_{r+1}}{(s-p_{r+1})} + \dots + \frac{A_n}{(s-p_n)} \quad (15.6-3)$$

The first root p is assumed to repeat r times. It may be real or complex. The remaining $(n-r)$ roots are designated as $p_{r+1}, p_{r+2}, \dots, p_n$. Equation 15.6-3 is the partial fraction expansion in this case that can be shown by an application of Residue theorem. We take this as a matter of fact and proceed.

The procedure for evaluating the $(n-r)$ residues at the $(n-r)$ non-repeating poles of $Y(s)$ is the same as in Case-1. Therefore,

The Laplace transform of output of a linear time-invariant circuit excited by an input source function that can be expressed as a linear combination of complex exponential functions will be a ratio of rational polynomials in s .

A Laplace transform that is in the form of a ratio of rational polynomials in s can be inverted by method of partial fractions.

Uniqueness of Laplace Transforms Pair

But, though we know that $e^{p_i t} u(t)$ has a Laplace transform of $1/(s-p_i)$, how do we know that that is the only time-function that will have $1/(s-p_i)$ as its Laplace transform?

It is vital to be sure about that if we want to assert that the time-function is $e^{p_i t} u(t)$ whenever we see a Laplace transform of $1/(s-p_i)$.

The 'Theorem of Uniqueness of Laplace transforms' states that a Laplace transform pair is unique.

That is, if we have, by some method or other, found out that $F(s)$ is the Laplace transform of $f(t)$, then this theorem assures us that only $f(t)$ will have this $F(s)$ as its Laplace transform and no other function will have $F(s)$ as its Laplace transform.

Therefore, whenever we see a $1/(s-p_i)$, we can write $e^{p_i t} u(t)$ as its inverse.

$$A_i = (s - p_i) Y(s) \Big|_{s=p_i} \quad \text{for } i = r+1 \text{ to } n.$$

We multiply both sides of Eqn. 15.6-3 by $(s - p)^r$ for the purpose of evaluating the r residues at the repeating pole. The result is,

$$(s - p)^r Y(s) = A_1 + A_2(s - p) + \cdots + A_r(s - p)^{r-1} + \frac{A_{r+1}(s - p)^r}{(s - p_{r+1})} + \cdots + \frac{A_n(s - p)^r}{(s - p_n)} \quad (15.6-4)$$

$(s - p)^r$ is a factor of denominator of $Y(s)$. Therefore, multiplication of $Y(s)$ by $(s - p)^r$ will cancel this factor in the denominator. Evaluating both sides with $s = p$, we get, $A_1 = (s - p)^r Y(s) \Big|_{s=p}$

Now, we differentiate Eqn. 15.6-4 on both sides with respect to s and substitute $s = p$ to get,

$$A_2 = \frac{d[(s - p)^r Y(s)]}{ds} \Big|_{s=p}.$$

Successive differentiation with respect to s and substitution of $s = p$ leads to

$$\begin{aligned} A_3 &= \frac{d^2[(s - p)^r Y(s)]}{ds^2} \Big|_{s=p} \\ A_4 &= \frac{1}{2!} \frac{d^3[(s - p)^r Y(s)]}{ds^3} \Big|_{s=p} \\ &\vdots \\ A_r &= \frac{1}{r!} \frac{d^{r-1}[(s - p)^r Y(s)]}{ds^{r-1}} \Big|_{s=p}. \end{aligned}$$

The reader may verify that the partial fraction terms corresponding to the non-repeating roots will contribute only zero values in all stages of this successive differentiation.

Once all residues have been calculated, the Eqn. 15.6-4 may be inverted to get the following time-function.

$$y(t) = \left(A_1 \frac{t^{r-1}}{(r-1)!} e^{pt} + A_2 \frac{t^{r-2}}{(r-2)!} e^{pt} + \cdots + A_r e^{pt} + A_{r+1} e^{p_{r+1}t} + \cdots + A_n e^{p_n t} \right) u(t).$$

We have used the Laplace transform pair $t^k e^{pt} u(t) \Leftrightarrow \frac{k!}{(s - p)^{k+1}}$ in arriving at this result. This Laplace transform pair will be proved in a later section.

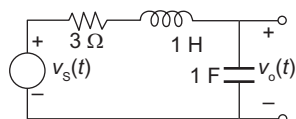


Fig. 15.6-1 Circuit for Example 15.6-1

EXAMPLE: 15.6-1

Determine (i) the impulse response, (ii) the step response and (iii) the zero-state response when $v_s(t) = 2e^{-2t} u(t)$ for $v_o(t)$ in the circuit in Fig. 15.6-1.

SOLUTION

The mesh equation of the circuit is $3i + \frac{di}{dt} = v_s(t) - v_o(t)$, where i is the current flowing in the mesh. But $i = 1 \times \frac{dv_o(t)}{dt}$ flows in the capacitor and $v_o(t)$ is the voltage across capacitor. Therefore, $3 \frac{dv_o(t)}{dt} + \frac{d^2 v_o(t)}{dt^2} = v_s(t) - v_o(t)$. Therefore, the differential equation governing the output voltage is $\frac{d^2 v_o(t)}{dt^2} + 3 \frac{dv_o(t)}{dt} + v_o(t) = v_s(t)$.

$$\text{The System Function } H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{s^2 + 3s + 1}.$$

The roots of denominator polynomial are -2.618 and -0.382 . The factors of the denominator polynomial are $(s + 2.618)$ and $(s + 0.382)$.

(i) The impulse response of a linear time-invariant circuit is same as the inverse transform of its System Function.

$$\begin{aligned} \therefore h(t) &= \text{Inverse of } H(s) \\ &= \text{Inverse of } \frac{1}{(s + 2.618)(s + 0.382)} \\ \frac{1}{(s + 2.618)(s + 0.382)} &= \frac{A_1}{(s + 2.618)} + \frac{A_2}{(s + 0.382)} \\ A_1 &= (s + 2.618) \times \frac{1}{(s + 2.618)(s + 0.382)} \Big|_{s=-2.618} = \frac{1}{(s + 0.382)} \Big|_{s=-2.618} = -0.4472 \\ A_2 &= (s + 0.382) \times \frac{1}{(s + 2.618)(s + 0.382)} \Big|_{s=-0.382} = \frac{1}{(s + 2.618)} \Big|_{s=-0.382} = 0.4472 \\ \therefore \frac{1}{(s + 2.618)(s + 0.382)} &= \frac{-0.4472}{(s + 2.618)} + \frac{0.4472}{(s + 0.382)} \\ \therefore h(t) &= 0.4472(e^{-0.382t} - e^{-2.618t})u(t) \text{ V.} \end{aligned}$$

$$(ii) v_s(t) = u(t) \Rightarrow V_s(s) = 1/s.$$

Therefore, the Laplace transform of step response $= \frac{1}{s(s + 2.618)(s + 0.382)}$.
Expressing this in partial fractions,

$$\begin{aligned} \frac{1}{s(s + 2.618)(s + 0.382)} &= \frac{A_1}{s} + \frac{A_2}{(s + 2.618)} + \frac{A_3}{(s + 0.382)} \\ A_1 &= s \times \frac{1}{s(s + 2.618)(s + 0.382)} \Big|_{s=0} = \frac{1}{2.618 \times 0.382} = 1 \\ A_2 &= (s + 2.618) \times \frac{1}{s(s + 2.618)(s + 0.382)} \Big|_{s=-2.618} = \frac{1}{-2.618 \times -2.236} = 0.1708 \\ A_3 &= (s + 0.382) \times \frac{1}{s(s + 2.618)(s + 0.382)} \Big|_{s=-0.382} = \frac{1}{-0.382 \times -2.236} = 0.1708 \\ \therefore \frac{1}{s(s + 2.618)(s + 0.382)} &= \frac{1}{s} + \frac{0.1708}{(s + 2.618)} + \frac{-1.1708}{(s + 0.382)} \\ \therefore \text{Step response of } v_o(t) &= (1 + 0.1708e^{-2.618t} - 1.1708e^{-0.382t})u(t) \text{ V.} \end{aligned}$$

(iii) $v_s(t) = 2e^{-2t}u(t) \Rightarrow V_s(s) = 2/(s + 2)$. The Laplace transform of zero-state response is given by the product of System Function and Laplace transform of input function.

$$\therefore V_o(s) = \frac{2}{(s + 2)(s + 2.618)(s + 0.382)}$$

Expressing this in partial fractions,

$$\begin{aligned} V_o(s) &= \frac{2}{(s + 2)(s + 2.618)(s + 0.382)} = \frac{A_1}{(s + 2)} + \frac{A_2}{(s + 2.618)} + \frac{A_3}{(s + 0.382)} \\ A_1 &= (s + 2) \times \frac{2}{(s + 2)(s + 2.618)(s + 0.382)} \Big|_{s=-2} = \frac{2}{0.618 \times -1.618} = -2 \\ A_2 &= (s + 2.618) \times \frac{2}{(s + 2)(s + 2.618)(s + 0.382)} \Big|_{s=-2.618} = \frac{2}{-0.618 \times -2.236} = 1.4474 \\ A_3 &= (s + 0.382) \times \frac{2}{(s + 2)(s + 2.618)(s + 0.382)} \Big|_{s=-0.382} = \frac{2}{1.618 \times 2.236} = 0.5528 \\ \therefore \frac{1}{(s + 2)(s + 2.618)(s + 0.382)} &= \frac{-2}{(s + 2)} + \frac{1.4474}{(s + 2.618)} + \frac{0.5528}{(s + 0.382)} \\ \therefore v_o(t) &= (-2e^{-2t} + 1.4474e^{-2.618t} + 0.5528e^{-0.382t})u(t) \text{ V.} \end{aligned}$$

EXAMPLE: 15.6-2

The resistor value in Fig. 15.6-1 under Example 15.6-1 is changed to $2\ \Omega$. (i) Find the step response of $v_o(t)$ (ii) Determine the zero-state response of $v_o(t)$ if $v_s(t) = e^{-t}u(t)$ V.

SOLUTION

The differential equation governing the output voltage with $2\ \Omega$ resistor instead of $3\ \Omega$ is $\frac{d^2 v_o(t)}{dt^2} + 2 \frac{dv_o(t)}{dt} + v_o(t) = v_s(t)$.

$$\text{The System Function } H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}.$$

The factors of the denominator polynomial are $(s+1)$ and $(s+1)$. Therefore, the root at -1 has a multiplicity of 2.

(i) With $v_s(t) = u(t)$ the Laplace transform of output is $V_o(s) = \frac{1}{s(s+1)^2}$. Expressing

this in partial fractions, $V_o(s) = \frac{1}{s(s+1)^2} = \frac{A_1}{s} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)}$. We can find the residues by applying the expressions developed earlier in this section. Alternatively, we may proceed as below:

$$\begin{aligned} \frac{1}{s(s+1)^2} &= \frac{A_1}{s} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)} \\ &= \frac{A_1(s+1)^2 + A_2s + A_3s(s+1)}{s(s+1)^2} \\ &= \frac{s^2(A_1 + A_3) + s(2A_1 + A_2 + A_3) + A_1}{s(s+1)^2}. \end{aligned}$$

Now, comparing the coefficients of various powers of s in the numerator, we get, $A_1 = 1$; $2A_1 + A_2 + A_3 = 0$ and $A_1 + A_3 = 0$

Solving these equations, we get, $A_1 = 1$, $A_2 = -1$ and $A_3 = -1$.

$$\therefore \frac{1}{s(s+1)^2} = \frac{1}{s} + \frac{-1}{(s+1)^2} + \frac{-1}{(s+1)}$$

Therefore, the step response $v_o(t) = (1 - te^{-t} - e^{-t})u(t)$ V.

(ii) With $v_s(t) = e^{-t}u(t)$, $V_s(s) = \frac{1}{(s+1)}$ and $V_o(s) = \frac{1}{(s+1)^3}$.

There is no need for partial fractions in this case and $v_o(t) = \frac{t^2}{2}e^{-t}u(t)$ V.

EXAMPLE: 15.6-3

Find the impulse response of the circuit in Fig. 15.5-1 for Example 15.5-1.

SOLUTION

The System Function was shown to be $\frac{V_o(s)}{V_s(s)} = \frac{1}{s^3 + s^2 + 2s + 1}$ in Example 15.5-1.

The characteristic equation is $s^3 + s^2 + 2s + 1 = 0$ and its roots are $s_1 = -0.215 + j1.307$, $s_2 = -0.215 - j1.307$ and $s_3 = -0.57$. There is a pair of complex conjugate roots. Impulse response is obtained by inverting the System Function.

$$\begin{aligned} H(s) &= \frac{1}{s^3 + s^2 + 2s + 1} = \frac{1}{(s+0.215-j1.307)(s+0.215+j1.307)(s+0.57)} \\ \therefore H(s) &= \frac{A_1}{(s+0.215-j1.307)} + \frac{A_2}{(s+0.215+j1.307)} + \frac{A_3}{(s+0.57)} \\ A_1 &= \left. \frac{1}{(s+0.215+j1.307)(s+0.57)} \right|_{s=-0.215+j1.307} \end{aligned}$$

$$= \frac{1}{(j2.614)(0.355 + j1.307)} = 0.283 \angle -164.8^\circ$$

A_2 will be the conjugate of A_1 and therefore $A_2 = 0.283 \angle 164.8^\circ$

$$\begin{aligned} A_3 &= \frac{1}{(s + 0.215 - j1.307)(s + 0.215 + j1.307)} \Big|_{s=-0.57} \\ &= \frac{1}{(0.355 - j1.307)(0.355 + j1.307)} = 0.545. \\ \therefore h(t) &= 0.283e^{-j164.8^\circ} e^{-(0.215 + j1.307)t} + 0.283e^{j164.8^\circ} e^{-(0.215 - j1.307)t} + 0.545e^{-0.57t} \\ &= 0.283e^{-0.215t} \left[e^{j(1.307t - 164.8^\circ)} + e^{-j(1.307t - 164.8^\circ)} \right] + 0.545e^{-0.57t} \\ &= 0.566e^{-0.215t} \cos(1.307t - 164.8^\circ) + 0.545e^{-0.57t}. \end{aligned}$$

15.7 SOME THEOREMS ON LAPLACE TRANSFORMS

The property of linearity of Laplace transforms was already noted and made use of in earlier sections. We look at other interesting properties of Laplace transform in this section.

15.7.1 Time-shifting Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$ then $v_d(t) = v(t - t_d) = f(t - t_d) u(t - t_d)$ has a Laplace transform $V_d(s) = V(s)e^{-st_d}$.

The shifting operation implied in this theorem is illustrated in Fig. 15.7-1. Note that there is a difference between $f(t - t_d) u(t)$ and $f(t - t_d) u(t - t_d)$. Time-shifting theorem for unilateral Laplace transform works properly for $f(t - t_d) u(t - t_d)$ but not for $f(t - t_d) u(t)$.

This theorem follows from the defining equation for Laplace transforms.

$$V_d(s) = \int_{0^-}^{\infty} v_d(t) e^{-st} dt = \int_{0^-}^{\infty} v(t - t_d) e^{-st} dt$$

Use variable substitution $\tau = t - t_d$

$$V_d(s) = \int_{-t_d}^{\infty} v(\tau) e^{-s(\tau + t_d)} d\tau = e^{-st_d} \int_{-t_d}^{\infty} v(\tau) e^{-s\tau} d\tau.$$

But $v(\tau)$ is zero for all $\tau \leq 0^-$. Therefore, $V_d(s) = e^{-st_d} \int_{0^-}^{\infty} v(\tau) e^{-s\tau} d\tau = e^{-st_d} V(s)$.

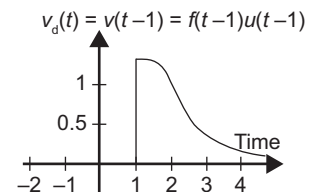
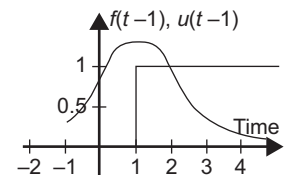
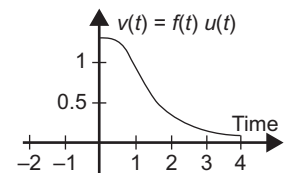
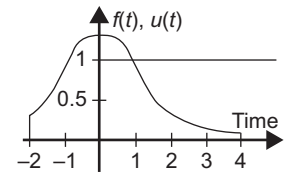


Fig. 15.7-1 Illustrating the Time-Shift Operation Envisaged in Shifting Theorem on Unilateral Laplace Transforms

EXAMPLE: 15.7-1

Find the zero-state response of a series RC circuit with a time constant of 2 s excited by a rectangular pulse voltage of 10 V height and 2 s duration starting from $t = 0$. The voltage across the capacitor is the output variable.

SOLUTION

The differential equation governing the voltage across capacitor in a series RC circuit excited by a voltage source is $\frac{dv}{dt} + \frac{1}{RC}v = \frac{1}{RC}v_s$, where v is the voltage across capacitor and v_s is the source voltage. In this case, the equation is $\frac{dv}{dt} + 0.5v = 0.5v_s$.

Therefore, the System Function is $H(s) = 0.5/(s + 0.5)$.

The rectangular pulse voltage can be expressed as sum of $10u(t)$ and $-10u(t-2)$.

i.e., $v_s = 10[u(t) - u(t-2)]$. Therefore, its Laplace transform is $\frac{10}{s} - \frac{10e^{-2s}}{s} = \frac{10[1 - e^{-2s}]}{s}$.

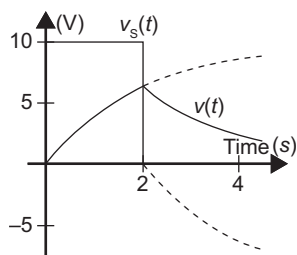


Fig. 15.7-2 Output Response and its Components in the Circuit for Example 15.7-1

$$\text{Therefore, } V(s) = \frac{0.5}{s+0.5} \times \frac{10[1-e^{-2s}]}{s} = \frac{5}{s(s+0.5)} [1-e^{-2s}]$$

We express $\frac{5}{s(s+0.5)}$ in partial fractions as $\frac{A_1}{(s+0.5)} + \frac{A_2}{s}$ and determine A_1 and

$$A_2 \text{ as } A_1 = (s+0.5) \times \frac{5}{s(s+0.5)} \Big|_{s=-0.5} = -10 \text{ and } A_2 = (s) \times \frac{5}{s(s+0.5)} \Big|_{s=0} = 10.$$

$$\therefore V(s) = \frac{5}{s(s+0.5)} [1-e^{-2s}] = \frac{10[1-e^{-2s}]}{s} - \frac{10[1-e^{-2s}]}{(s+0.5)}.$$

Inverse transform of $-10e^{-2s}/s$ is $-10u(t-2)$ by Time-shifting theorem. Similarly, inverse transform of $-10e^{-2s}/(s+0.5)$ is $-10e^{-0.5(t-2)}u(t-2)$. Therefore, the output voltage is given by

$$\begin{aligned} v(t) &= 10[u(t) - u(t-2)] - 10[e^{-0.5t}u(t) - e^{-0.5(t-2)}u(t-2)] \\ &= 10(1 - e^{-0.5t})u(t) - 10(1 - e^{-0.5(t-2)})u(t-2). \end{aligned}$$

Figure 15.7-2 shows the two components of response in dotted curves.

15.7.2 Frequency-shifting Theorem

If $v(t) = f(t)u(t)$ has a Laplace transform $V(s)$, then, $v_d(t) = v(t)e^{s_0 t}$ has a Laplace transform $V_d(s) = V(s - s_0)$.

This theorem follows from the defining equation for Laplace transforms.

$$V_d(s) = \int_{0^-}^{\infty} v_d(t)e^{-st} dt = \int_{0^-}^{\infty} v(t)e^{s_0 t} e^{-st} dt = \int_{0^-}^{\infty} v(t)e^{-(s-s_0)t} dt = V(s-s_0).$$

15.7.3 Time-Differentiation Theorem

If $v(t) = f(t)u(t)$ has a Laplace transform $V(s)$, then, $v_d(t) = \frac{dv(t)}{dt}$ has a Laplace transform

$$V_d(s) = sV(s) - v(0^-). \text{ Note that } \frac{dv(t)}{dt} \neq \frac{df(t)}{dt} \times u(t).$$

The proof follows.

$$V_d(s) = \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt$$

We carry out the integration by parts.

$$\begin{aligned} \frac{d[v(t)e^{-st}]}{dt} &= -sv(t)e^{-st} + \frac{dv(t)}{dt} e^{-st} \\ \therefore \frac{dv(t)}{dt} e^{-st} &= \frac{d[v(t)e^{-st}]}{dt} + sv(t)e^{-st} \\ \therefore V_d(s) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt = s \int_{0^-}^{\infty} v(t)e^{-st} dt + \int_{0^-}^{\infty} \frac{d[v(t)e^{-st}]}{dt} dt = sV(s) + v(t)e^{-st} \Big|_{0^-}^{\infty} \end{aligned}$$

The function $v(t)e^{-st}$ will be a decaying function for any value of s in the ROC of $V(s)$. Otherwise, the Laplace transform will not converge for that value of s . Therefore, it will go to zero as $t \rightarrow \infty$.

$$V_d(s) = sV(s) - v(0^-)$$

Now, by using mathematical induction, we may show that,

Laplace transform of $\frac{d^2 v(t)}{dt^2} = s^2 V(s) - sv(0^-) - \frac{dv(t)}{dt} \Big|_{(0^-)}$ and that, in general,

Laplace transform of $\frac{d^n v(t)}{dt^n} = s^n V(s) - s^{n-1}v(0^-) - s^{n-2} \frac{dv(t)}{dt} \Big|_{(0^-)} - \dots - \frac{d^{n-1}v(t)}{dt^{n-1}} \Big|_{(0^-)}$

Time-differentiation Theorem on Laplace transforms.

15.7.4 Time-integration Theorem

If $v(t) = f(t)$ $u(t)$ has a Laplace transform $V(s)$, then, $v_i(t) = \int_0^t v(t) dt$ has a Laplace transform $V_i(s) = \frac{V(s)}{s}$.

The proof follows.

$$V_i(s) = \int_0^\infty \left(\int_0^t v(t) dt \right) e^{-st} dt$$

We carry out the integration by parts.

$$\frac{d \left(\int_0^t v(t) dt \right) e^{-st}}{dt} = v(t) e^{-st} - s \left(\int_0^t v(t) dt \right) e^{-st}$$

$$\begin{aligned} \therefore V_i(s) &= \int_0^\infty \left(\int_0^t v(t) dt \right) e^{-st} dt = \frac{1}{s} \int_0^\infty v(t) e^{-st} dt + \frac{1}{s} \int_0^\infty \frac{d \left(\int_0^t v(t) dt \right) e^{-st}}{dt} dt \\ &= \frac{1}{s} \int_0^\infty v(t) e^{-st} dt + \frac{1}{s} \left(\int_0^t v(t) dt \right) e^{-st} \Big|_0^\infty \end{aligned}$$

The function $v(t)$ is stated to possess a Laplace transform. This implies that there is an exponential function $Me^{\alpha t}$ with some positive value of M and some real value for α such that $|v(t)| < Me^{\alpha t}$. Otherwise, the $v(t)$ would not have a Laplace transform. Therefore, the function $\int_0^t v(t) dt$ will satisfy the inequality $\left| \int_0^t v(t) dt \right| < |M / \alpha (e^{\alpha t} - 1)|$ and therefore is bounded. Therefore, the Laplace transform of $\int_0^t v(t) dt$ will exist. That is, it is possible to select a value for s such that the function $\int_0^t v(t) dt \times e^{-st}$ is a decaying function. For such a s , i.e., for a value of s in the ROC of Laplace transform of $\int_0^t v(t) dt$, the value of $\left(\int_0^t v(t) dt \right) e^{-st}$ will go to zero as $t \rightarrow \infty$. And, the value of $\left(\int_0^t v(t) dt \right) e^{-st}$ at $t = 0^-$ is zero in any case.

The second term in the last equation is shown to be zero.

Time-integration theorem on Laplace transforms.

$$\therefore V_i(s) = \frac{1}{s} \int_0^\infty v(t) e^{-st} dt = \frac{V(s)}{s}.$$

EXAMPLE: 15.7-2

Find the Laplace transform of $t^n u(t)$.

The function $tu(t)$ is the integral of $u(t)$. Therefore, $tu(t) \Leftrightarrow 1/s^2$. Now, the function $t^2 u(t)$ is two times the integral of $tu(t)$. Therefore, $t^2 u(t) \Leftrightarrow 2/s^3$. Proceeding similarly to the power n , we get, $t^n u(t) \Leftrightarrow n!/s^{n+1}$.

15.7.5 s-Domain-Differentiation Theorem

If $v(t) = f(t)$ $u(t)$ has a Laplace transform $V(s)$, then, $-tv(t)$ has a Laplace transform $\frac{dV(s)}{ds}$.

We show this by determining the Laplace transform of $-tv(t)$ from the defining integral.

$$\int_{0^-}^{\infty} -tv(t)e^{-st} dt = \int_{0^-}^{\infty} -v(t) \left[te^{-st} \right] dt.$$

We have used the limit $\lim_{\Delta\alpha \rightarrow 0} \frac{e^{(\alpha+\Delta\alpha)t} - e^{\alpha t}}{\Delta\alpha} = te^{\alpha t}$ many times before. We use this limit again with $\alpha = -s$ within the integral.

$$\therefore \int_{0^-}^{\infty} -tv(t)e^{-st} dt = \int_{0^-}^{\infty} -v(t) \left[\lim_{\Delta s \rightarrow 0} \frac{e^{-(s+\Delta s)t} - e^{-st}}{-\Delta s} \right] dt.$$

Since the limiting operation is on s and integration is on t we may interchange the order of these two operations.

$$\begin{aligned} \therefore \int_{0^-}^{\infty} -tv(t)e^{-st} dt &= \lim_{\Delta s \rightarrow 0} \int_{0^-}^{\infty} -v(t) \left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{-\Delta s} \right] dt \\ &= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \int_{0^-}^{\infty} v(t) \left[e^{-(s+\Delta s)t} - e^{-st} \right] dt \\ &= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \left(\int_{0^-}^{\infty} v(t)e^{-(s+\Delta s)t} dt - \int_{0^-}^{\infty} v(t)e^{-st} dt \right) \\ &= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} (V(s+\Delta s) - V(s)) \\ &= \lim_{\Delta s \rightarrow 0} \frac{V(s+\Delta s) - V(s)}{\Delta s} \\ &= \frac{dV(s)}{ds}. \end{aligned}$$

15.7.6 s-Domain-Integration Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$, then, has a Laplace transform $\int_s^{\infty} V(s) ds$

$$\int_s^{\infty} V(s) ds = \int_s^{\infty} \left[\int_{0^-}^{\infty} v(t)e^{-st} dt \right] ds = \int_{0^-}^{\infty} v(t) \left[\int_s^{\infty} e^{-st} ds \right] dt = \int_{0^-}^{\infty} v(t) \left[\frac{e^{-st}}{t} - \frac{e^{-\infty t}}{t} \right] dt.$$

If the integration $\int_s^{\infty} V(s) ds$ is carried out in the right-half of s -plane (ROC of right-sided functions will have at least a part of right-half s -plane in it), then $e^{-\infty t}$ in the last step in the equation above will vanish. Then, $\int_s^{\infty} V(s) ds = \int_{0^-}^{\infty} \frac{v(t)}{t} e^{-st} dt = \text{Laplace Transform of } \frac{v(t)}{t}$.

15.7.7 Convolution Theorem

If $x(t)$ and $y(t)$ are two right-sided time-functions with Laplace transforms $X(s)$ and $Y(s)$, respectively and $Z(s) = X(s)Y(s)$, then, $z(t) = x(t)*y(t) = \int_0^t x(\tau)y(t-\tau) d\tau$.

We use the inverse integral to show this.

$$\begin{aligned} z(t) &= \frac{1}{j2\pi} \int_{-\infty}^{\infty} X(s)Y(s)e^{st} ds \\ &= \frac{1}{j2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)e^{-s\tau} d\tau \right] Y(s)e^{st} ds \text{ [Substituting for } X(s)\text{]} \end{aligned}$$

On Convolution Theorem

Convolving the impulse response of a linear time-invariant circuit with its input source function gives the zero-state response.

Hence, the convolution theorem stated here corroborates the fact that Laplace transform of zero-state response is given by the product of Laplace transform of input source function and Laplace transform of impulse response.

This is yet another way to understand why the System Function and Laplace transform of impulse response turn out to be the same.

$$\begin{aligned}
&= \frac{1}{j2\pi} \int_{0^-}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} Y(s) e^{st} e^{-s\tau} ds \right] d\tau \quad [\text{Reversing the order of integration}] \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\frac{1}{j2\pi} \int_{-\infty}^{\infty} Y(s) e^{s(t-\tau)} ds \right] d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau.
\end{aligned}$$

15.7.8 Initial Value Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$ and $\lim_{s \rightarrow \infty} sV(s)$ exists, then, $\lim_{s \rightarrow \infty} sV(s) = v(0^+)$.

We know that the Laplace transform of $\frac{dv(t)}{dt}$ is $sV(s) - v(0^-)$. Therefore,

$$sV(s) - v(0^-) = \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt.$$

We assume that $s \rightarrow \infty$ with its real part always positive. This is consistent with the fact that ROC of Laplace transform of right-sided functions will contain the right-half of s -plane or at least portions of right-half s -plane. Evaluation of the term e^{-st} with $t \rightarrow 0$ and $s \rightarrow \infty$ simultaneously is to be avoided. Hence, we write the integral as below:

$$\begin{aligned}
sV(s) - v(0^-) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
&= \int_{0^-}^{0^+} \frac{dv(t)}{dt} e^{-st} dt + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
&= \int_{0^-}^{0^+} \frac{dv(t)}{dt} e^0 dt + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
&= v(0^+) - v(0^-) + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt.
\end{aligned}$$

Therefore, $sV(s) = v(0^+) + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt$. Now we apply the limit $s \rightarrow \infty$ with its real part always positive. Then the integral vanishes.

Therefore, $\lim_{s \rightarrow \infty} sV(s) = v(0^+)$.

Initial Value theorem on Laplace transforms.

15.7.9 Final Value Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$ and $\lim_{s \rightarrow 0} sV(s)$ exists and all the poles of $sV(s)$ have negative real part, then, $\lim_{s \rightarrow 0} sV(s) = v(\infty)$.

We know that the Laplace transform of $\frac{dv(t)}{dt}$ is $sV(s) - v(0^-)$. Therefore,

$$\begin{aligned}
sV(s) - v(0^-) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt. \\
\lim_{s \rightarrow 0} sV(s) - v(0^-) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^0 dt = v(t) \Big|_{0^-}^{\infty} = v(\infty) - v(0^-) \\
\therefore \lim_{s \rightarrow 0} sV(s) &= v(\infty).
\end{aligned}$$

One has to be very careful in applying this theorem. This theorem works only if all the poles of $sV(s)$ are in the open left-half plane in s -domain. That is, all the poles of $sV(s)$ must have negative real part. Only then will the function $v(t)$ reach a unique and steady final value with time. Otherwise, the value returned by the application of this theorem will

We have to be very careful in applying *Final Value theorem* on Laplace transforms.

not be the final value of $v(t)$. For that matter $v(t)$ may not have a final value at all. Let $v(t)$ be $\sin \omega t$. Then $V(s) = \omega/(s^2 + \omega^2)$ and $sV(s) = s\omega/(s^2 + \omega^2)$. Application of final value theorem says that $v(\infty) = 0$. But there is no unique final value for a sinusoidal waveform. This conflict occurs because of wrong application of the theorem. The $sV(s)$ function in this case has poles on $j\omega$ -axis and hence the final value theorem is not applicable.

15.8 SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

One of the important applications of Laplace transform is in solving linear constant-coefficient ordinary differential equations with initial conditions. The procedure is illustrated below through an example.

EXAMPLE: 15.8-1

Find $y(t)$ for $t > 0^+$ for $x(t) = 2tu(t)$ in the given differential equation with $y(0^-) = 1$, $y'(0^-) = -1$ and $y''(0^-) = 0$.

$$\frac{d^3y}{dt^3} + 2.5\frac{d^2y}{dt^2} + 2.5\frac{dy}{dt} + 1.5y = \frac{dx}{dt} + x.$$

SOLUTION

A differential equation is an equation in which both sides of it can be multiplied by e^{-st} . Since the differential equation is satisfied at all instants of time, both sides of it can be integrated with respect to time from 0^- to ∞ . In short, the Laplace transform operation can be carried out on both sides. Laplace transformation is a linear operation and hence Laplace transform of a sum of terms is equal to sum of Laplace transforms of individual terms. Therefore,

$$\text{LT of } \frac{d^3y}{dt^3} + 2.5 \times \text{LT of } \frac{d^2y}{dt^2} + 2.5 \times \text{LT of } \frac{dy}{dt} + 1.5 \times \text{LT of } y = 1.5 \times \text{LT of } x$$

Now we apply the 'Differentiation in Time Theorem on Laplace transforms' to get, $[s^3 + 2.5s^2 + 2.5s + 1.5]Y(s) = X(s) + y''(0^-) + y'(0^-)[s + 2.5] + y(0^-)[s^2 + 2.5s + 2.5]$

$$\therefore Y(s) = \underbrace{\left\{ \frac{1}{(s^3 + 2.5s^2 + 2.5s + 1.5)} X(s) \right\}}_{\text{Zero-state response terms}} + \underbrace{\left\{ \frac{y''(0^-) + (s + 2.5)y'(0^-) + (s^2 + 2.5s + 2.5)y(0^-)}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \right\}}_{\text{Zero-input response terms}}$$

Since $x(t) = \delta(t)$, $X(s) = 1$ in this example. Substituting the values for initial conditions and Laplace transform of $x(t)$, we get,

$$Y(s) = \underbrace{\left\{ \frac{1}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \right\}}_{\text{Zero-state response terms}} + \underbrace{\left\{ \frac{(s^2 + 1.5s)}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \right\}}_{\text{Zero-input response terms}}$$

We need to factorise the denominator polynomial in order to arrive at the partial fraction expansion. It is a third order polynomial with real coefficients. Complex roots, if any, will have to occur in conjugate pairs for such a polynomial. Therefore, a polynomial of odd degree with real coefficients will necessarily possess a real-valued root. We try to locate that real root by method of bisection.

Try two different values of s such that the polynomial evaluates to a positive and a negative number.

$s^3 + 2.5s^2 + 2.5s + 1.5$ evaluates to 1.5 for $s = 0$ and -1.5 for $s = -2$. Therefore, there must be root between 0 and -2 . We try the mid-value of -1 and see that the polynomial evaluates to 0.5. Therefore, the root must be between -1 and -2 . Hence, we try the

The output transform terms that depend only on the input function result in zero-state response. The transform terms that depend only on the initial conditions on output and its derivatives result in zero-input response.

Bisection method to locate a real root of a polynomial illustrated.

mid-value -1.5 . The polynomial evaluates to 0. Hence, the root is $s = -1.5$. In practice many iteration may be needed to arrive at a root by this technique.

Now, we know that $(s + 1.5)$ is a factor of $s^3 + 2.5s^2 + 2.5s + 1.5$. Therefore, we get the remaining second-order factor by long division as $s^2 + s + 1$. The roots of this factor are $-0.5 \pm j0.866$.

We now expand each response term in partial fractions. Normally we expand a transform in terms of the first-order partial fractions. However, a second-order factor with complex conjugate roots may be expanded more conveniently in a form illustrated below.

$$\begin{aligned}\frac{1}{(s^3 + 2.5s^2 + 2.5s + 1.5)} &= \frac{A}{(s + 1.5)} + \frac{Bs + C}{(s^2 + s + 1)} = \frac{A(s^2 + s + 1) + (Bs + C)(s + 1.5)}{(s + 1.5)(s^2 + s + 1)} \\ &= \frac{(A + B)s^2 + (A + 1.5B + C)s + (A + 1.5C)}{(s + 1.5)(s^2 + s + 1)}.\end{aligned}$$

Comparing the coefficients of various powers of s in the numerator, we get, $A + B = 0$, $A + 1.5B + C = 0$ and $A + 1.5C = 1$. Solving for the unknowns, we get, $A = 0.5715$, $B = -0.5715$ and $C = 0.2857$.

Therefore, the zero-state response = Inverse of $\left(\frac{0.5715}{s + 1.5} + \frac{-0.5715s + 0.2857}{s^2 + s + 1} \right)$

Inverse of $\frac{0.5715}{s + 1.5}$ is $0.5715e^{-1.5t}u(t)$

$$\begin{aligned}\frac{-0.5715s + 0.2857}{s^2 + s + 1} &= \frac{-0.5715s + 0.2857}{(s + 0.5)^2 + (\sqrt{0.75})^2} \\ &= -0.5715 \frac{s + 0.5 - 0.5}{(s + 0.5)^2 + (0.866)^2} + \frac{0.2857}{(s + 0.5)^2 + (0.866)^2} \\ &= -0.5715 \frac{s + 0.5}{(s + 0.5)^2 + (0.866)^2} + \frac{0.5715}{(s + 0.5)^2 + (0.866)^2}\end{aligned}$$

$\therefore$ Inverse of $\frac{-0.5715s + 0.2857}{s^2 + s + 1}$

$$\begin{aligned}&= -0.5715e^{-0.5t} \cos 0.866t u(t) + \frac{0.5715}{0.866} e^{-0.5t} \sin 0.866t u(t) \\ &= -0.5715e^{-0.5t} \cos 0.87t u(t) + 0.66e^{-0.5t} \sin 0.87t u(t)\end{aligned}$$

$\therefore$ Zero-state response = $[0.572e^{-1.5t} - 0.572e^{-0.5t} \cos 0.87t + 0.66e^{-0.5t} \sin 0.87t]u(t)$.

The zero-input response is given by inverse of $\frac{(s^2 + 1.5s)}{(s^3 + 2.5s^2 + 2.5s + 1.5)}$.

$$\begin{aligned}\frac{(s^2 + 1.5s)}{(s^3 + 2.5s^2 + 2.5s + 1.5)} &= \frac{s(s + 1.5)}{(s + 1.5)(s^2 + s + 1)} = \frac{s}{(s^2 + s + 1)} \\ &= \frac{s + 0.5}{(s + 0.5)^2 + 0.866^2} - \frac{0.5}{0.866} \frac{0.866}{(s + 0.5)^2 + 0.866^2}\end{aligned}$$

$\therefore$ Zero-input response = $[e^{-0.5t} \cos 0.866t - 0.5774e^{-0.5t} \cos 0.866t]u(t)$

Total response $y(t)$ is the sum of zero-input response and zero-state response. Therefore,

$$\begin{aligned}\text{Total response } y(t) &= [0.572e^{-1.5t} + 0.429e^{-0.5t} \cos 0.87t + 0.087e^{-0.5t} \sin 0.87t]u(t) \\ &= [0.572e^{-1.5t} + 0.437e^{-0.5t} \cos(0.87t - 11.3^\circ)]u(t)\end{aligned}$$

If we can solve a differential equation with non-zero initial conditions completely in one stroke using Laplace transforms, then, we can indeed solve linear time-invariant circuits with non-zero initial conditions for their total response using Laplace transforms. We have to derive the n^{th} order linear constant-coefficient differential equation describing the circuit in terms of a single chosen variable first. In the second step we have to determine the initial values for that chosen circuit variable and its $(n - 1)$ derivatives from the known initial values of inductor currents and capacitor voltages in the circuit. Then we are ready to employ Laplace transform technique to solve for zero-input response and zero-state response in one step as illustrated in this example.

However, the derivation of differential equation and determination of initial values of chosen variable and its derivatives are the toughest tasks in a circuit analysis

problem. Can Laplace transform technique help us to simplify these two stages of circuit analysis or not?

15.9 THE s -DOMAIN EQUIVALENT CIRCUIT

Yes, it can. Laplace transform technique tells us that we do not even have to derive the circuit differential equation and initial values required to solve it. Let us see how.

The circuit equations arising out of applying KVL in loops and KCL at nodes are equations that remain true at all values of t . Therefore, such equations can be differentiated and integrated with respect to time without changing their truth content. And, of course, being equations they can be multiplied by the same constant or function on both sides.

We choose to multiply all KCL and KVL equations in a linear time-invariant circuit by a function e^{-st} , where s is a complex frequency value drawn from s -plane, with a real part of suitable value such that each term in the equation is converted into an absolutely integrable function of time (so that Laplace transform for that term will converge). And then we choose to integrate the equations from 0^- to ∞ in time-domain. We apply the principle that integral of sum of terms is sum of integrals of individual terms.

The result will be a conclusion – (i) *the algebraic sum of Laplace transforms of element voltages in any loop in a circuit is zero* and (ii) *the algebraic sum of Laplace transforms of element currents at any node in a circuit is zero*.

Or, equivalently, *the Laplace transforms of voltage variables and current variables in a linear time-invariant circuit obey KVL and KCL, respectively*.

Now, suppose we know the relation between the Laplace transform of element voltage and Laplace transform of element current for all circuit elements. Then, the node equations and mesh equations can be written in terms of Laplace transforms of variables straightaway – i.e., we can write the circuit equations in s -domain straightaway instead of writing them in time-domain and transforming them into s -domain at the end of solution process. Hence, we derive the element relationships in s -domain first.

KCL and KVL equations can be written directly in terms of transformed variables.

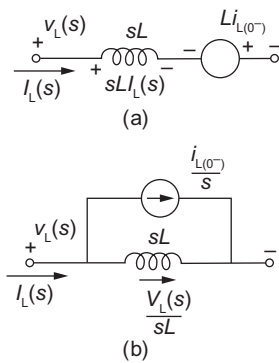


Fig. 15.9-1 s -Domain Equivalent Circuits for an Inductor

15.9.1 s -Domain Equivalents of Circuit Elements

A resistor R is described by the time-domain element equation $v_R(t) = R i_R(t)$. Multiplying both sides by e^{-st} and integrating from 0^- to ∞ , we get, $V_R(s) = R I_R(s)$. Note that we use upper case letters for Laplace transforms. This relation makes it clear that a resistor is represented by a multiplying factor of R that connects the Laplace transform of current through it to the Laplace transform of voltage across it. The ratio of Laplace transform of voltage across an element to the Laplace transform of current through it is defined as the ' s -domain impedance'. Thus, s -domain impedance of a resistor is R itself.

An inductor L is described by the time-domain element equation $v_L(t) = L \frac{di_L(t)}{dt}$, where $v_L(t)$ and $i_L(t)$ are its voltage and current variables as per passive sign convention. Applying Laplace transformation on both sides of this element equation, we get, $V_L(s) = sLI_L(s) - Li_{L(0^-)}$, where $i_{L(0^-)}$ is the initial current in the inductor at $t = 0^-$. The Laplace transform $Li_{L(0^-)}$ represents an impulse voltage $Li_{L(0^-)} \delta(t)$ in time-domain and hence it is consistent with the fact that a non-zero initial condition in an inductor can be replaced with an impulse voltage source in series with the inductor. This s -domain equation for an inductor suggests the following s -domain equivalent circuit shown in Fig. 15.9-1(a) for an inductor

with sL as its s -domain impedance function. The second equivalent circuit shown in Fig. 15.9-1(b) follows from $I_L(s) = \frac{V_L(s)}{sL} + \frac{i_{L(0^-)}}{s}$ and indicates the fact that the s -domain admittance of an inductor is $1/sL$ and that non-zero initial condition in an inductor can be replaced with a step current source in parallel with it.

Note that we use the same graphic symbol for inductor in s -domain as the one we used in time-domain. This will be the case with all other circuit elements too.

A capacitor C is described by the time-domain element equation $i_C(t) = C \frac{dv_C(t)}{dt}$,

where $v_C(t)$ and $i_C(t)$ are its voltage and current variables as per passive sign convention. Applying Laplace transformation on both sides of this element equation, we get, $I_C(s) = sC V_C(s) - C v_{C(0^-)}$, where $v_{C(0^-)}$ is the initial voltage across the capacitor at $t = 0^-$. The Laplace transform $C v_{C(0^-)}$ represents an impulse current $C v_{C(0^-)} \delta(t)$ in time-domain and hence it is consistent with the fact that a non-zero initial condition in a capacitor can be replaced with an impulse current source in parallel with the capacitor. This s -domain equation for a capacitor suggests the following s -domain equivalent circuit shown in Fig. 15.9-2(a) for a capacitor with sC as its s -domain admittance function. The second equivalent circuit shown in Fig. 15.9-2(b) follows from $V_C(s) = \frac{I_C(s)}{sC} + \frac{v_{C(0^-)}}{s}$ and indicates

the fact that the s -domain impedance of a capacitor is $1/sC$ and that non-zero initial condition in a capacitor can be replaced with a step voltage source in series with it.

s -domain impedance is assigned the unit of Ω (ohms) and s -domain admittance is assigned the unit of S (Siemens).

Now, we can construct the *s-domain equivalent circuit* for a circuit in time-domain by replacing all sources by their Laplace transforms and replacing all other circuit elements by their s -domain equivalents. The resulting equivalent circuit will have Laplace transforms of voltages and Laplace transforms of currents as the circuit variables instead of the time-domain variables. Each energy storage element will result in an extra independent source representing its initial condition in s -domain equivalent circuit.

Applying KVL and KCL in this circuit will result in *algebraic equations involving Laplace transforms of voltages and currents, respectively*. Thus, the problem of solving a coupled set of integro-differential equations involving functions of time in the time-domain circuit is translated to solving a coupled set of algebraic equations involving Laplace transforms of variables in the s -domain equivalent circuit. The time-functions may be determined by inverting the Laplace transforms once they have been solved for.

However, *Laplace transform of instantaneous power is not equal to product of Laplace transforms of voltage and current*. In fact, the s -domain convolution of $V(s)$ and $I(s)$ gives the Laplace transform of $p(t) = v(t) i(t)$. Therefore, dealing with power and energy variables in the s -domain is better avoided. They are dealt with in the time-domain itself.

We observe that the s -domain equivalent circuit makes use of the stated initial values of inductor currents and capacitor voltages right at the start. The s -domain equivalent circuit takes care of these initial values in the form of additional source transforms. Therefore, the circuit solution arrived at by analysis of s -domain equivalent circuit will contain both zero-input response components and zero-state response components in one step. Thus, Laplace transform technique yields the total response in a single-step solution process. Of course, Fourier transforms being a special case of Laplace transforms; Fourier transform technique also can do this.

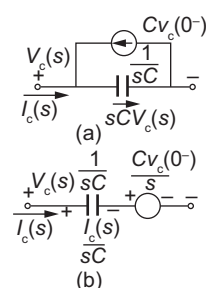


Fig. 15.9-2 s -Domain Equivalent Circuits for a Capacitor

Transforming a time-domain circuit into an s -domain circuit makes it similar to a memoryless circuit with DC excitation since both are described by algebraic equations.

Thus, all concepts and techniques developed in the context of analysis of memoryless circuits in second part of this book (and used later in the analysis of phasor equivalent circuits under sinusoidal steady-state in third part of this book) will be directly applicable in the analysis of s -domain equivalent circuits too.

In particular, (i) the concepts of series and parallel equivalent impedances apply without modification (ii) the concepts of input resistance (i.e., driving-point resistance) and input conductance (i.e., driving-point conductance) apply without modification except that it is 'input impedance function $Z_i(s)$ ' and 'input admittance function $Y_i(s)$ ' in the case s -domain circuits.

Moreover, the techniques of nodal analysis and mesh analysis can be applied in s -domain circuits. All the circuit theorems, except maximum power transfer theorem, can be applied in the context of s -domain equivalent circuits.

15.10 TOTAL RESPONSE OF CIRCUITS USING s -DOMAIN EQUIVALENT CIRCUIT

The application of s -domain equivalent circuit in obtaining the total response of a circuit is illustrated through a set of examples in this section.

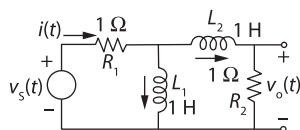


Fig. 15.10-1 Circuit for Example 15.10-1

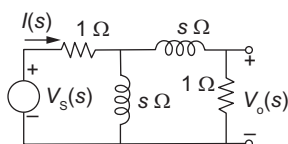


Fig. 15.10-2 The Transformed Equivalent Circuit for Circuit in Example: 15.10-1 with Zero Initial Conditions

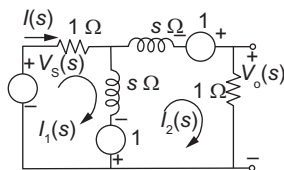


Fig. 15.10-3 Transformed Equivalent Circuit of Circuit in Fig. 15.10-1 with Initial Condition Sources Included

EXAMPLE: 15.10-1

The inductor L_1 has an initial current of 1 A and the inductor L_2 has an initial current of 1 A in the directions marked in the circuit in Fig. 15.10-1. (i) Find the voltage transfer function $V_o(s)/V_s(s)$ and the input impedance function $V_s(s)/I(s)$ (ii) Determine the total response of $v_o(t)$ if $v_s(t) = 2u(t)$ V.

SOLUTION

(i) Transfer functions and immittance functions are defined in the s -domain equivalent circuit. They are defined under zero-state response conditions and as ratios of Laplace transforms of relevant quantities under zero-state response conditions. Therefore, they are defined under zero initial conditions. The s -domain equivalent circuit with zero initial conditions is shown in Fig. 15.10-2.

Series-parallel equivalents and voltage/current division principle may be employed to arrive at the required ratios. Input impedance function $Z_i(s) = V_s(s)/I(s)$.

$$Z_i(s) = (s+1) / \frac{1}{s+1} = \frac{(s+1)s}{2s+1} + 1 = \frac{s^2 + 3s + 1}{2s+1} \Omega.$$

The transformed current in the second 1Ω resistor may be found in terms of $I(s)$. Then $V_o(s)$ may be obtained from that current by multiplying by 1Ω . Let this current transform be called $I_o(s)$. Then,

$$I_o(s) = I(s) \frac{s}{s+s+1} = I(s) \frac{s}{2s+1}$$

$$\text{But } I(s) = \frac{V_s(s)}{Z_i(s)} = V_s(s) \frac{2s+1}{s^2 + 3s + 1}$$

Therefore,

$$\frac{V_o(s)}{V_s(s)} = \frac{I_o(s) \times 1 \Omega}{V_s(s)} = \frac{I(s)}{V_s(s)} \times \frac{s}{2s+1} = \frac{V_s(s)}{V_s(s)} \frac{2s+1}{s^2 + 3s + 1} \times \frac{s}{2s+1} = \frac{s}{s^2 + 3s + 1}.$$

The poles of transfer function are at $s = -2.618$ and $s = -0.382$.

The zero is at $s = 0$.

(ii) The total response of $v_o(t)$ with $v_s(t) = 2u(t)$ V may be solved by mesh analysis or by applying superposition principle. Both methods are illustrated below. The s -domain equivalent circuit with initial conditions accounted and mesh current transforms identified is shown in Fig. 15.10-3.

The mesh equations are:

$$-V_s(s) + I_1(s)[s+1] + I_2(s)[-s] - 1 = 0$$

$$1 + I_1(s)[-s] + I_2(s)[2s+1] - 1 = 0$$

These are expressed in matrix form as below.

$$\begin{bmatrix} s+1 & -s \\ -s & 2s+1 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s)+1 \\ 0 \end{bmatrix}$$

Solving for $I_2(s)$, we get,

$$I_2(s) = \frac{s(V_s(s)+1)}{(s+1)(2s+1)-s^2} = \underbrace{\frac{sV_s(s)}{s^2+3s+1}}_{\text{Zero-state response}} + \underbrace{\frac{s}{s^2+3s+1}}_{\text{Zero-input response}}$$

$$\text{Since } V_o = 1 \times I_2(s), V_o(s) = \underbrace{\frac{sV_s(s)}{s^2+3s+1}}_{\text{Zero-state response}} + \underbrace{\frac{s}{s^2+3s+1}}_{\text{Zero-input response}}$$

The roots of denominator polynomial (i.e., poles of transfer function) are at $s = -2.618$ and $s = -0.382$. The input transform $V_s(s) = 2/s$.

$$\text{Therefore, } \frac{sV_s(s)}{s^2+3s+1} = \frac{2}{s^2+3s+1} = \frac{A}{s+2.618} + \frac{B}{s+0.382}$$

$$A = (s+2.618) \times \frac{2}{s^2+3s+1} \Big|_{s=-2.618} = -0.8945$$

$$B = (s+0.382) \times \frac{2}{s^2+3s+1} \Big|_{s=-0.382} = 0.8945$$

$$\begin{aligned} \text{Therefore, zero-state response} &= \text{Inverse of } \frac{-0.8945}{s+2.618} + \frac{0.8945}{s+0.382} \\ &= 0.8945(e^{-0.382t} - e^{-2.618t})u(t) \text{ V.} \end{aligned}$$

The second component of output is expanded in partial fractions as below.

$$\frac{s}{s^2 + 3s + 1} = \frac{A}{s + 2.618} + \frac{B}{s + 0.382}$$

$$A = (s + 2.618) \times \frac{s}{s^2 + 3s + 1} \Big|_{s=-2.618} = 1.1708$$

$$B = (s + 0.382) \times \frac{s}{s^2 + 3s + 1} \Big|_{s=-0.382} = -0.1708$$

$$\begin{aligned} \text{Therefore, zero-input response} &= \text{Inverse of } \frac{1.1708}{s + 2.618} - \frac{0.1708}{s + 0.382} \\ &= (1.1708e^{-2.618t} - 0.1708e^{-0.382t})u(t) \text{ V.} \end{aligned}$$

Total response is the sum of zero-state response and zero-input response and is given by

$$v_o(t) = (0.2763e^{-0.382t} + 0.7237e^{-2.618t})u(t) \text{ V.}$$

It is not always necessary to split the two components of the response in this manner. It was done here only to demonstrate how the Laplace transform technique brings out both together in one step. Inverting the total response transform $\frac{2+s}{s^2+3s+1}$ would

have resulted in total response straightaway. Indeed $\frac{2+s}{s^2+3s+1} = \frac{0.2763}{s+2.618} + \frac{0.7237}{s+0.382}$.

The same solution can be arrived at by using Superposition theorem. This theorem can be applied only for the zero-state response components due to various inputs. But then, all initial conditions get translated into sources in the transformed equivalent circuit and hence the circuit analysis problem in the s-domain is always a zero-state response problem. Therefore, superposition principle can be freely applied in transformed equivalent circuits. The solution term due to initial condition sources will be understood as the zero-input response once we get back to the time-domain.

There are three sources in this transformed equivalent circuit.

The component circuits required to find the individual response components are shown in Fig. 15.10-4.

The transfer function of the first circuit is already known as $\frac{s}{s^2+3s+1}$ and $V_s(s) = \frac{2}{s}$. Therefore, the output transform in the first circuit is $\frac{2}{s^2+3s+1}$.

$(s+1)/1 \Omega$ shares -1 with $s \Omega$ in series in the second circuit. Therefore, the voltage transform across $(s+1) \Omega$ is $\frac{(s+1)/1}{s+(s+1)/1} \times -1 = \frac{-(s+1)}{s^2+3s+1}$. This voltage transform is divided between $s \Omega$ and 1Ω to yield $\frac{-(s+1)}{s^2+3s+1} \times \frac{1}{s+1} = \frac{-1}{s^2+3s+1}$ at the output.

$(s+1)/s \Omega$ shares 1 in series with 1Ω to produce $\frac{1}{1+(s+1)/s} = \frac{s+1}{s^2+3s+1}$ across the output.

The total output voltage transform is given by the sum of three output voltage transforms. Therefore, $V_o(s) = \frac{2}{s^2+3s+1} + \frac{-1}{s^2+3s+1} + \frac{s+1}{s^2+3s+1} = \frac{2+s}{s^2+3s+1}$. This is the same output transform that we obtained by mesh analysis.

The time-domain function will be

$$v_o(t) = (0.2763e^{-0.382t} + 0.7237e^{-2.618t})u(t) \text{ V.}$$

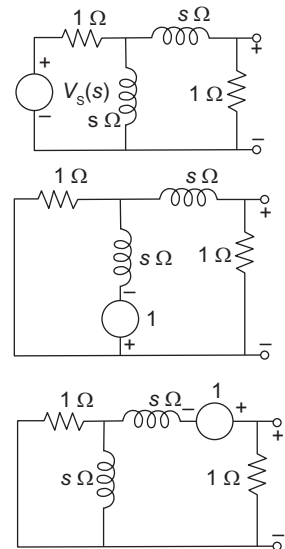


Fig. 15.10-4
Component Circuits
for Applying
Superposition
Theorem for Example
15.10-1

EXAMPLE: 15.10-2

The circuit in Fig. 15.10-5 was in DC steady-state at $t = 0$. The switch in the circuit closes at $t = 0$ introducing a new 1Ω into the circuit. Determine the voltage across the inductor as a function of time.

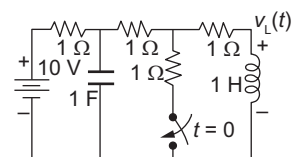


Fig. 15.10-5 Circuit for
Example 15.10-2

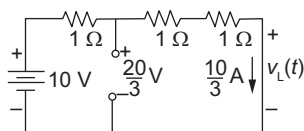


Fig. 15.10-6 Circuit Under DC Steady-State for $t < 0$

SOLUTION

The circuit was in DC steady-state prior to switching at $t = 0$. A capacitor can be modelled by an open-circuit and an inductor by a short circuit for DC steady-state analysis. Hence, the circuit for DC steady-state prior to $t = 0$ is as shown in Fig. 15.10-6.

Therefore, the voltage across the capacitor at $t = 0^-$ was $20/3$ V and the current through inductor at $t = 0^-$ was $10/3$ A. The circuit solution after $t = 0$ can be obtained by solving a new circuit with $10 u(t)$ as input, $v_C(0^-) = 20/3$ V and $i_L(0^-) = 10/3$ A.

The new circuit can be analysed by mesh analysis or nodal analysis techniques. We opt for nodal analysis since the desired output is a node voltage variable straightaway. It will be convenient to use a current source in parallel with capacitor and a current source in parallel with inductor to account for initial conditions since we have opted for nodal analysis. The transformed equivalent circuit required is shown in Fig. 15.10-7.

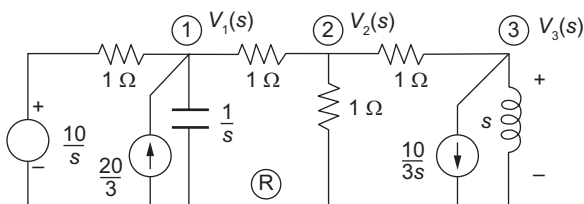


Fig. 15.10-7 Transformed Equivalent Circuit for Example 15.10-2

The first source in series with 1Ω may be replaced by a current source in parallel with 1Ω . The node equations in matrix form will be,

$$\begin{bmatrix} 2+s & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1+\frac{1}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} \frac{10}{s} + \frac{20}{3} \\ 0 \\ -\frac{10}{3s} \end{bmatrix}$$

The determinant of Nodal Admittance Matrix is

$$\Delta(s) = (2+s) \left[3 \left(1 + \frac{1}{s} \right) - 1 \right] - (-1) \left[-1 \left(1 + \frac{1}{s} \right) \right] = (2+s) \left(\frac{2s+3}{s} \right) - \left(\frac{s+1}{s} \right) = \frac{2s^2 + 6s + 5}{s}$$

Solving for $V_3(s)$,

$$\begin{aligned} V_3(s) &= \frac{-10(s+2)}{3(2s^2 + 6s + 5)} = \frac{-1.667(s+2)}{s^2 + 3s + 2.5} \\ &= \frac{-1.667(s+1.5+0.5)}{(s+1.5)^2 + (0.5)^2} = \frac{-1.667(s+1.5)}{(s+1.5)^2 + (0.5)^2} + \frac{-1.667(0.5)}{(s+1.5)^2 + (0.5)^2}. \end{aligned}$$

Therefore,

$$v_3(t) = -1.667e^{-1.5t} (\cos 0.5t + \sin 0.5t) u(t) = -2.36e^{-1.5t} (\cos 0.5t - 45^\circ) u(t) \text{ V.}$$

The voltage across the inductor is same as $v_3(t)$. Note that the final value of inductor voltage is zero. This is expected since under DC steady-state condition the inductor behaves like a short circuit.

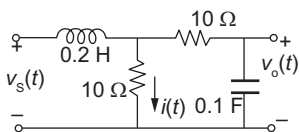


Fig. 15.10-8 Circuit for Example 15.10-3

EXAMPLE: 15.10-3

Verify the initial value theorem and final value theorem on Laplace transforms for $i(t)$ and $v_o(t)$ in the initially relaxed circuit shown in Fig. 15.10-8 when driven by $v_s(t) = u(t)$.

SOLUTION

The s-domain equivalent circuit required for analysis is shown in Fig. 15.10-9.

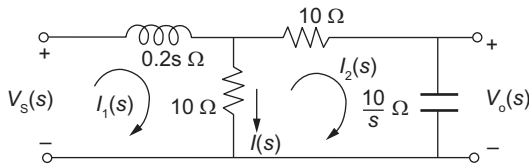


Fig. 15.10-9 The s-Domain Equivalent Circuit of the Circuit in Fig. 15.10-8

Two mesh current transforms are identified in the s-domain equivalent circuit. The mesh equations in matrix form is written by inspection by using the rule that diagonal entry is the sum of all impedances in the corresponding mesh and off-diagonal entries are negative of sum of impedances shared by the two meshes in question.

$$\begin{bmatrix} 0.2s + 10 & -10 \\ -10 & 20 + \frac{10}{s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s) \\ 0 \end{bmatrix}$$

Solving for $I_1(s)$ and $I_2(s)$ by Kramer's rule, we get,

$$I_1(s) = \frac{\left(20 + \frac{10}{s}\right)V_s(s)}{(0.2s + 10)\left(20 + \frac{10}{s}\right) - 100} = \frac{(5s + 2.5)V_s(s)}{s^2 + 25.5s + 25}$$

$$I_2(s) = \frac{10V_s(s)}{(0.2s + 10)\left(20 + \frac{10}{s}\right) - 100} = \frac{2.5sV_s(s)}{s^2 + 25.5s + 25}$$

Now,

$$I(s) = I_1(s) - I_2(s) = \frac{(5s + 2.5)V_s(s)}{s^2 + 25.5s + 25} - \frac{2.5sV_s(s)}{s^2 + 25.5s + 25} = \frac{2.5(s + 1)V_s(s)}{s^2 + 25.5s + 25}$$

$$V_o(s) = \frac{10}{s} \times I_2(s) = \frac{25V_s(s)}{s^2 + 25.5s + 25}$$

The input is a $u(t)$ function and its transform is $1/s$. Therefore,

$$I(s) = \frac{2.5(s + 1)}{s(s^2 + 25.5s + 25)}$$

$$V_o(s) = \frac{25}{s(s^2 + 25.5s + 25)}$$

$$\text{Initial value of } i(t) \text{ at } t = 0^+ = \lim_{s \rightarrow \infty} sI(s) = s \times \frac{2.5(s + 1)}{s(s^2 + 25.5s + 25)} = 0.$$

The poles of $sI(s)$ are in the left-half of s-plane and hence the final value theorem on Laplace transforms is applicable to $I(s)$.

$$\therefore \text{Final value of } i(t) = \lim_{s \rightarrow 0} sI(s) = s \times \frac{2.5(s + 1)}{s(s^2 + 25.5s + 25)} = \frac{2.5}{25} = 0.1 \text{ A.}$$

$$\text{Initial value of } v_o(t) \text{ at } t = 0^+ = \lim_{s \rightarrow \infty} sV_o(s) = s \times \frac{25}{s(s^2 + 25.5s + 25)} = 0.$$

The poles of $sV_o(s)$ are in the left-half of s-plane and hence the final value theorem on Laplace transforms is applicable to $V_o(s)$.

$$\therefore \text{Final value of } v_o(t) = \lim_{s \rightarrow 0} sV_o(s) = s \times \frac{25}{s(s^2 + 25.5s + 25)} = \frac{25}{25} = 1 \text{ V.}$$

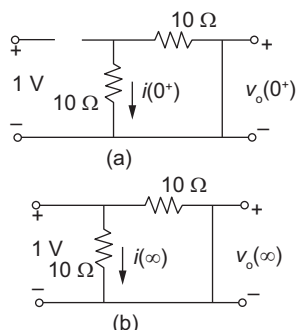


Fig. 15.10-10
Equivalent Circuits
at (a) $t = 0^+$ and
(b) $t \rightarrow \infty$

The initial current at $t = 0^-$ through the inductor was zero and initial voltage across the capacitor at that instant was zero. There was no impulse content in voltage at input. Therefore, the inductor current at $t = 0^+$ remains at zero. There was no impulse current in the circuit. Therefore, the voltage across capacitor remains at zero at $t = 0^+$.

The time-domain equivalent circuit at $t = 0^+$ is shown in Fig. 15.10-10(a). The initial value of current $i(t)$ at 0^+ is clearly zero and initial value of $v_o(t)$ is also zero.

The inductor is replaced by a short circuit and the capacitor by an open-circuit under DC steady-state conditions. The resulting circuit is shown in Fig. 15.10-10(b). Hence, the final value of $i(t)$ $1/10 = 0.1$ A and the final value of $v_o(t)$ is equal to input voltage – i.e., 1 V.

Hence, the initial value theorem and final value theorem on Laplace transforms are verified for $v_o(t)$ and $i(t)$.

EXAMPLE: 15.10-4

(a) Find the transfer function $\frac{V_o(s)}{V_s(s)}$ in the Opamp circuit shown in Fig. 15.10-11 assuming ideal Opamp. (b) Show its pole-zero plot for $k = 2.9$ and $k = 3.1$ and find its zero-state response to $v_s(t) = 0.01\delta(t)$ in both cases with $R = 10 \text{ k}\Omega$ and $C = 1 \text{ }\mu\text{F}$. (c) What is the maximum value of k that can be used in the circuit without making it an unstable one?

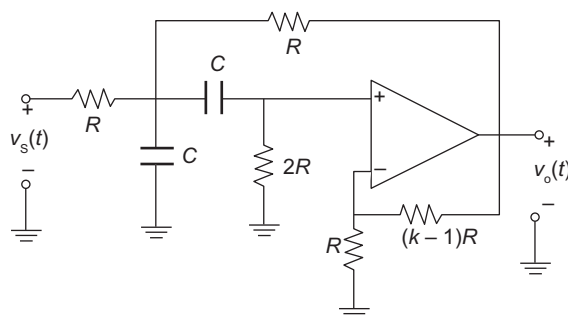


Fig. 15.10-11 The Opamp Circuit for Example 15.10-4

SOLUTION

(a) The Opamp circuit from its non-inverting input to its output is a simple non-inverting amplifier of gain k and can be replaced with a dependent source as shown in the s -domain equivalent circuit in Fig. 15.10-12.

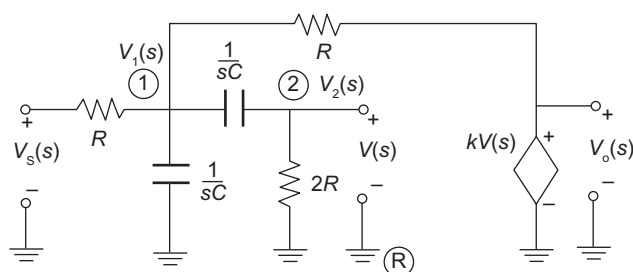


Fig. 15.10-12 Transformed Equivalent Circuit of the Circuit in Fig. 15.10-11

The node voltage transforms $V_1(s)$ and $V_2(s)$ are identified in the transformed equivalent circuit. Writing the node equations at these two nodes, we get,

$$\text{Second Node: } sC[V_2(s) - V_1(s)] + \frac{V_2(s)}{2R} = 0$$

$$\text{Therefore, } V_1(s) = V_2(s) + \frac{V_2(s)}{2sCR} = V_2(s) \left[1 + \frac{1}{2sCR} \right]$$

$$\text{First Node: } \frac{V_1(s) - V_s(s)}{R} + sCV_1(s) + sC[V_1(s) - V_2(s)] + \frac{V_1(s) - kV_2(s)}{R} = 0$$

Substituting for $V_1(s)$ in terms of $V_2(s)$ and simplifying, we get,

$$\frac{V_2(s)}{V_s(s)} = \frac{s \frac{1}{RC}}{s^2 + s(3-k) \frac{1}{RC} + \left(\frac{1}{RC} \right)^2} \quad \text{and} \quad \frac{V_o(s)}{V_s(s)} = \frac{kV_2(s)}{V_s(s)} = \frac{s \frac{k}{RC}}{s^2 + s(3-k) \frac{1}{RC} + \left(\frac{1}{RC} \right)^2}$$

This is the transfer function of the circuit.

(b) The denominator polynomial with $R = 10 \text{ k}\Omega$, $C = 1 \mu\text{F}$ and $k = 2.9$ is $s^2 + 10s + 10,000$. Therefore, the poles are at $s = -5 \pm j 99.88$. The zero of the transfer function is at $s = 0$, i.e., at the origin in the s -plane.

With $k = 3.1$, the denominator polynomial is $s^2 - 10s + 10,000$ and the poles are at $s = 5 \pm j 99.88$. The zero of transfer function is again at $s = 0$. The pole-zero plots are shown in Fig. 15.10-13.

The zero-state response to $v_s(t) = 0.01\delta(t)$ is nothing but the impulse response of the circuit scaled by 0.01.

With $k = 2.9$

The transfer function $H(s) = \frac{290s}{s^2 + 10s + 10,000}$. Inverse transform of $H(s)$ gives the impulse response of the circuit. We complete the squares in the denominator and identify the inverse transforms as shown below.

$$\begin{aligned} \frac{290s}{s^2 + 10s + 10,000} &= 290 \frac{(s+5) - 5}{(s+5)^2 + 99.88^2} \\ &= 290 \left[\frac{(s+5)}{(s+5)^2 + 99.88^2} - \frac{5}{99.88} \frac{99.88}{(s+5)^2 + 99.88^2} \right] \end{aligned}$$

$$\begin{aligned} \therefore h(t) &= 290e^{-5t} (\cos 99.88t - 0.05 \sin 99.88t) u(t) \\ &= 290.36e^{-5t} \cos(99.88t + 2.86^\circ) u(t) \end{aligned}$$

This is a stable impulse response since $\lim_{t \rightarrow \infty} h(t) = 0$

The zero-state response to $0.01\delta(t) = 2.9e^{-5t} \cos(99.88t + 2.86^\circ) u(t)$.

With $k = 3.1$

The transfer function $H(s) = \frac{310s}{s^2 - 10s + 10,000}$. Inverse transform of $H(s)$ gives the impulse response of the circuit. We complete the squares in the denominator and identify the inverse transforms as shown below.

$$\begin{aligned} \frac{310s}{s^2 - 10s + 10,000} &= 310 \frac{(s-5) + 5}{(s-5)^2 + 99.88^2} \\ &= 310 \left[\frac{(s-5)}{(s-5)^2 + 99.88^2} + \frac{5}{99.88} \frac{99.88}{(s-5)^2 + 99.88^2} \right] \end{aligned}$$

$$\begin{aligned} \therefore h(t) &= 310e^{5t} (\cos 99.88t + 0.05 \sin 99.88t) u(t) \\ &= 310.39e^{5t} \cos(99.88t - 2.86^\circ) u(t) \end{aligned}$$

This is an unstable impulse response since it is unbounded. The circuit is an unstable one as evidenced by its poles located in right-half of s -plane.

The zero-state response to $0.01\delta(t) = 3.1e^{5t} \cos(99.88t - 2.86^\circ) u(t)$.

(c) The transfer function
$$\frac{V_o(s)}{V_s(s)} = \frac{s \frac{K}{RC}}{s^2 + s \frac{(3-K)}{RC} + \left(\frac{1}{RC} \right)^2}$$
 has poles on $j\omega$ -axis when

$k = 3$. The poles will lie on the right-half of s -plane for all values of $k > 3$. Therefore, $k < 3$ is the constraint on k value for stability in the circuit.

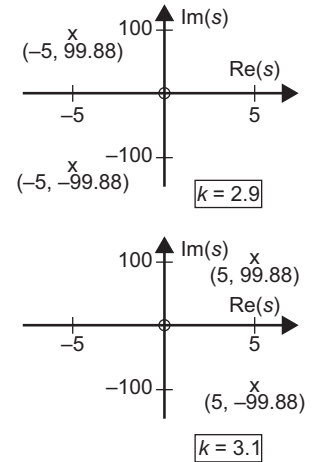


Fig. 15.10-13 Pole-Zero Plots for the Opamp Circuit in Fig. 15.10-11 with $k = 2.9$ and $k = 3.1$

Comments on Example 15.10-4

Note that the circuit in Example 15.10-4 is a pure RC circuit with one dependent source in it.

A passive RC circuit, i.e., a circuit containing only resistors and capacitors and no dependent sources, will have all its poles in the negative real axis. The dependent source is responsible for making the poles complex conjugate in this circuit. Complex conjugate poles are often necessary in filter circuits to tailor the filter frequency response function suitably to meet filtering specifications.

This circuit is used as a band-pass filter in practice. The value of k will be decided by the bandwidth required in the band-pass filter and will be < 3 in any case.

The circuit is marginally stable at $k = 3$. It can function as a sinusoidal oscillator with $k = 3$. But additional circuitry will be needed to stabilise its amplitude of oscillation.

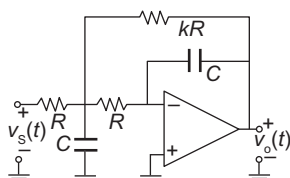


Fig. 15.10-14 The Opamp RC Filter Circuit for Example 15.10-5

EXAMPLE: 15.10-5

(a) Obtain the transfer function of the filter circuit shown in Fig. 15.10-14 and identify the type of filter. (b) Determine the zero-state response for $v_s(t) = 0.1\delta(t)$. $R = 10 \text{ k}\Omega$, $C = 1 \text{ }\mu\text{F}$ and $k = 1$.

SOLUTION

The transformed equivalent circuit is shown in Fig. 15.10-15. Initial condition sources are not required since the circuit is needed for determining transfer function and for evaluating zero-state response. The Opamp is assumed to be ideal. There is only one node that has a free node voltage variable. This node and the assigned node voltage transform are also indicated in the equivalent circuit.

Virtual short across the Opamp input terminals and zero input current drawn by the Opamp inverting pin makes the current in the feedback capacitor equal to $V(s)/R$. This result in the output voltage transforms that is equal to $-V(s)/sCR$.

Now, we write the node equation at the node-1 marked in Fig. 15.10-15.

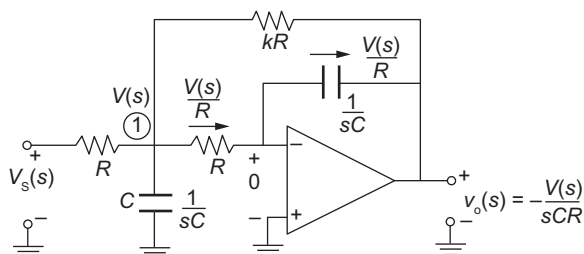


Fig. 15.10-15 The Transformed Equivalent Circuit of Circuit in Fig. 15.10-14

$$\frac{V(s)}{R} = sCV(s) + \frac{V(s)}{R} + \frac{V(s) - \left(\frac{V(s)}{sRC}\right)}{kR} = \frac{V(s)}{R}$$

$$\text{Algebraic simplification leads to } \frac{V(s)}{V_s(s)} = \frac{skRC}{k(sRC)^2 + (2k+1)(sRC) + 1}$$

The output transform $V_o(s) = -V(s)/sCR$. Therefore,

$$\frac{V_o(s)}{V_s(s)} = \frac{-k}{k(sRC)^2 + (2k+1)(sRC) + 1} = \frac{-k \times \frac{1}{k} \left(\frac{1}{RC}\right)^2}{s^2 + s\left(2 + \frac{1}{k}\right)\left(\frac{1}{RC}\right) + \frac{1}{k} \left(\frac{1}{RC}\right)^2}$$

The denominator has its roots in the left-half of s -plane for all positive values of k since a second order polynomial with positive coefficients will have both roots in left-half of s -plane. Therefore, the impulse response is stable and will be absolutely integrable. Therefore, the impulse response will have a Fourier transform. If Fourier transform of a time-function exists, then, the Laplace transform of the same function evaluated on $j\omega$ -axis is its Fourier transform. Fourier transform of impulse response is the frequency response function. Therefore, the sinusoidal steady-state frequency response function of a stable circuit is given by its Laplace transform evaluated with $s = j\omega$.

Therefore, $\frac{V_o(j\omega)}{V_s(j\omega)} = (-k) \frac{\frac{1}{k} \left(\frac{1}{RC}\right)^2}{-\omega^2 + \left(2 + \frac{1}{k}\right) \left(\frac{1}{RC}\right) j\omega + \frac{1}{k} \left(\frac{1}{RC}\right)^2}$. This frequency response

function has a magnitude of unity at $\omega = 0$ and 0 at $\omega = \infty$. Therefore, it is a low-pass filter. Substituting the numerical values, we get,

$$\frac{V_o(s)}{V_s(s)} = \frac{-10^4}{s^2 + 300s + 10^4}$$

The impulse response of the circuit is obtained by inverting the transfer function. The roots of the denominator are at $s = -261.8$ and $s = -38.2$.

$$\frac{V_o(s)}{V_s(s)} = \frac{-10^4}{s^2 + 300s + 10^4} = \frac{A}{s + 261.8} + \frac{B}{s + 38.2}$$

A and B can be evaluated as -44.72 and 44.72 , respectively.

Therefore, $h(t) = 44.72(e^{-38.2t} - e^{-261.8t})u(t)$.

Therefore, zero-state response for $v_s(t) = 0.1\delta(t)$ is $4.472(e^{-38.2t} - e^{-261.8t})u(t)$ V.

EXAMPLE: 15.10-6

A single Opamp, a resistor and a capacitor can form a good integrator circuit. This circuit is shown in Fig. 15.10-16. (i) Derive the transfer function of the circuit and show that it is an integrator. (ii) What is the impulse response of the circuit? (iii) Use the complete offset model of the Opamp and show that the output will enter saturation even without any input. (iv) Suggest a method to overcome the offset problem.

SOLUTION

(i) The s-domain equivalent circuit of the integrator circuit with Opamp assumed to be ideal is shown in Fig. 15.10-17.

The voltage transform at the inverting input terminal is zero due to virtual-short across the ideal Opamp input terminals. This makes the input current transform equal to $V(s)/R$. The input current into Opamp input terminals is zero. Therefore, this current flows into the capacitor as shown in Fig. 15.10-17. The voltage transform developed across the capacitor is $V(s)/sRC$ with positive at virtual ground. Applying KVL in the loop starting from ground end ending at ground and travelling through the capacitor, we get, $V_o(s) = -V(s)/sRC$. Therefore, the transfer function of the circuit is $-1/sRC$. It is an integrator since multiplication by $1/s$ in s-domain corresponds to integration in time-domain by Time-domain Integration Theorem on Laplace transforms. It is an inverting integrator.

(ii) Impulse response of the Integrator circuit = Inverse transform of $-1/sRC = -u(t)/RC$.

(iii) The s-domain equivalent circuit with the Opamp replaced by its offset model is shown in Fig. 15.10-18(a). Note that all DC quantities are treated as step functions in transformed equivalent circuit.

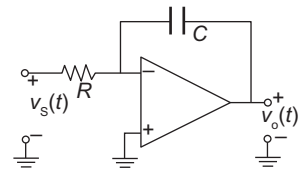


Fig. 15.10-16 The Opamp Integrator Circuit for Example 15.10-6

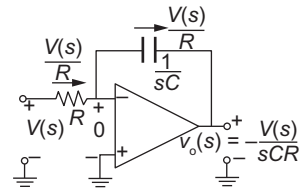


Fig. 15.10-17 Transformed Equivalent Circuit for Opamp Integrator Circuit

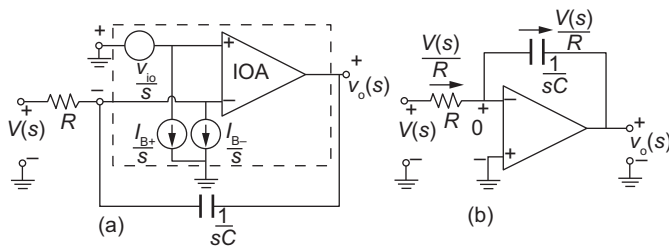


Fig. 15.10-18 (a) Transformed Equipment Circuit of Integrator for Offset Analysis (b) Component Circuit

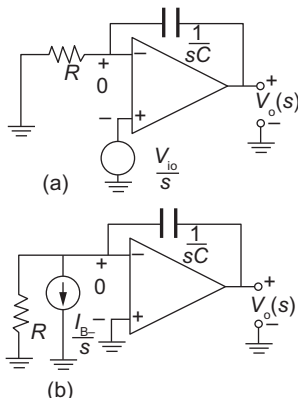


Fig. 15.10-19
Component Circuits
for Offset Analysis in
Integrator Circuit

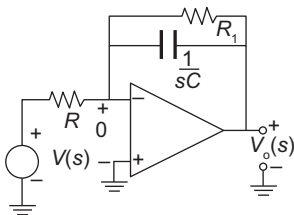


Fig. 15.10-20 A
Practical Opamp
Integrator Circuit

The current source I_{b+} does not contribute to output since it gets shorted by the voltage source V_{io} . The remaining three sources contribute to output. The circuit needed to derive the output contribution due to $V(s)$ acting alone is shown in Fig. 15.10-18(b). But we have already solved this circuit and its output transform is $-V(s)/sRC$.

The circuit to be used for determining output component due to V_{io}/s is shown in Fig. 15.10-19(a). The circuit needed for determining output component due to I_{b-}/s is shown in Fig. 15.10-19(b). The circuit in Fig. 15.10-19(b) shows that this current source comes in parallel with a resistor R . It may be converted into a voltage source in series with R before analysis.

Consider circuit in Fig. 15.10-19(a). The voltage transform at inverting pin is V_{io}/s due to virtual-short across the input pins of an ideal Opamp. Therefore, current through R is V_{io}/sR and this current transform flows through the capacitor since Opamp does not take any current at its input. Therefore, the output is higher by V_{io}/s^2RC compared to inverting pin. Therefore, output transform is $(1 + 1/sRC) (V_{io}/s)$.

Consider circuit in Fig. 15.10-19(b). Once the current source is transformed into a voltage source it becomes an integrator circuit with input transform of $-RI_{b-}/s$.

Therefore, output is I_{b-}/s^2C .

$$\text{Therefore, the total output transform } V_o(s) = \frac{V(s)}{sRC} + \left(1 + \frac{1}{sRC}\right) \frac{V_{io}}{s} - \frac{I_{b-}}{s^2C}$$

$$\text{The output } v_o(t) = -\frac{1}{RC} \int_0^t v(t) dt + V_{io} + \frac{V_{io}}{RC} t - \frac{I_{b-}}{C} t$$

V_{io} could be of either polarity. Even if it is positive the cancellation between the linearly rising terms contributed by V_{io} and I_{b-} need not be perfect. Therefore, this integrator circuit will soon reach one of the saturation levels at the output. Therefore, it cannot perform properly as an integrator.

All Opamps have some non-zero value of V_{io} . Hence, this problem can not be completely solved by changing the Opamp. The problem arises out of infinite gain available in the circuit for DC inputs.

(iv) The solution is to limit the DC steady-state gain by paralleling the capacitor with a resistor as shown in Fig. 15.10-20.

$$\text{But, now the transfer function is } -\frac{R_1 // 1/sC}{R} = -\frac{R_1}{R} \frac{1}{1 + sR_1C} \text{ and it is not a pure integra-}$$

tor. It behaves like a good integrator if the signal expansion for the input signal $v(t)$ contains negligible content at low values of s – i.e., signal contains significant strength only for those values of s such that $sR_1C \gg 1$. This means that the low frequency content in the sinusoidal expansion of the input signal (assuming it has a Fourier transform) must be negligible. Whether this circuit is thought of as an amplifier with a gain of $-R_1/R$ that goes down with frequency or an integrator for signals that have negligible low frequency content is a matter of viewpoint. The value used for R_1 in practice is about 10 to 30 times that of R .

EXAMPLE: 15.10-7

A single Opamp, a resistor and a capacitor can form a good differentiator circuit. This circuit is shown in Fig. 15.10-21.

(i) Derive the transfer function of the circuit and show that it is a differentiator.

(ii) An ideal Opamp is too good to be true. A practical Opamp suffers from many non-idealities. The particular non-ideality that compromises the circuit performance will vary depending on circuit function. For example, we found that integrator circuit is severely compromised by offsets in a practical Opamp. We will see in this example that it is the limited gain and bandwidth of Opamp that compromises the performance of differentiator circuit.

An amplifier contains many capacitors – intentional as well as parasitic – in it and hence represents a high-order dynamic circuit. However, some Opamps like IC 741 can be modelled approximately as a single-time constant amplifier. That is, its gain function

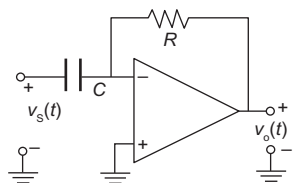


Fig. 15.10-21 An
Opamp Differentiator
Circuit

is of the form $\frac{A}{1+s\tau}$ with $A \approx 250,000$ and $\tau \approx 4$ ms. Obtain the transfer function of differentiator circuit with $R = 10 \text{ k}\Omega$ and $C = 1 \text{ }\mu\text{F}$ using IC 741 and find its step response.

(iii) Suggest a method to modify the oscillatory step response to critically damped step response.

SOLUTION

(i) This is essentially an inverting amplifier structure. The transfer function = $\frac{\text{Impedance in feedback path}}{\text{Impedance in input line}} = -sRC$. It is a differentiator since multiplication by s

in s -domain is equivalent to differentiation in time-domain according to Time-differentiation theorem on Laplace transforms. It is an inverting differentiator.

(ii) The Opamp is to be modelled as a dependent source that senses the voltage transform between non-inverting pin and inverting pin and produces a voltage transform $\frac{AV_d(s)}{1+s\tau}$ at its output with respect to ground where $V_o(s)$ is the transform of voltage of non-inverting pin with respect to inverting pin. The s -domain equivalent circuit incorporating this model for Opamp is shown in Fig. 15.10-22.

Let the node voltage transform at the inverting pin be $V_1(s)$. Writing the node equation at inverting pin, we get,

$$sC[V_1(s) - V(s)] + \frac{1}{R} \left[V_1(s) - \left(-\frac{AV_1(s)}{1+s\tau} \right) \right] = 0$$

$$\text{Simplifying this equation results in } \frac{V_1(s)}{V(s)} = \frac{sC}{sC + \frac{1}{R} \left(1 + \frac{A}{1+s\tau} \right)}$$

$$V_o(s) \text{ is } -\frac{A}{1+s\tau} V_1(s).$$

$$\text{Therefore, } \frac{V_o(s)}{V(s)} = \frac{-AsC}{sC + \frac{1}{R} \left(1 + \frac{A}{1+s\tau} \right)} = (-sRC) \frac{\frac{A}{RC\tau}}{s^2 + s\frac{1}{\tau} + \frac{A+1}{RC\tau}}$$

The DC gain A of any practical Opamp is in thousands and hence $A + 1 \approx A$. Therefore,

$$\frac{V_o(s)}{V(s)} = (-sRC) \frac{\frac{A}{RC\tau}}{s^2 + s\frac{1}{\tau} + \frac{A}{RC\tau}}$$

Note that the order of the circuit is two. The Opamp contributes one extra order to the circuit. Substituting $R = 10 \text{ k}\Omega$, $C = 1 \text{ }\mu\text{F}$, $A = 250,000$ and $t = 4 \text{ ms}$, we get,

$$\begin{aligned} \frac{V_o(s)}{V(s)} &= (-0.01s) \frac{79060^2}{s^2 + 250s + 79060^2} \\ &= (-0.01s) \frac{79060^2}{(s + 125)^2 + 79057^2} \\ &= (-62.5 \times 10^6) \frac{s}{(s + 125)^2 + 79057^2} \end{aligned}$$

Step response is inverse of transfer function multiplied by $1/s$.

$$\begin{aligned} (-62.5 \times 10^6) \frac{1}{(s + 125)^2 + 79057^2} &= \frac{(-62.5 \times 10^6)}{79057} \frac{79057}{(s + 125)^2 + 79057^2} \\ \therefore v_o(t) &\approx -790.57 e^{-125t} \cos 79057t u(t). \end{aligned}$$

Certainly, if a unit step is really applied to this circuit, the output of Opamp will saturate. But what is to be noted is that the response is *highly under-damped*. The oscillation is at 12.6 kHz and oscillation period is about 0.08 ms. But the time constant of damping exponential is 40 ms. It takes about five time constants for an exponential transient to settle down. This implies that the 12.6 kHz transient oscillations

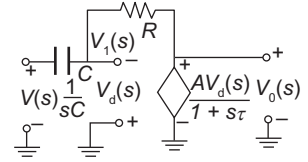


Fig. 15.10-22
Transformed
Equivalent Circuit for
Differentiator Circuit
in Fig. 15.10-21

will last for about 200 ms before they die down. This is really a bad transient performance.

(iii) The solution is to add a little damping by means of a resistor in series with the input capacitor. This will, however, make the differentiator imperfect.

15.11 NETWORK FUNCTIONS AND POLE-ZERO PLOTS

Network function is a ratio of Laplace transforms. It is the ratio of Laplace transform of zero-state response to the Laplace transform of the right-sided input function causing this response. We had called it *s*-domain System Function till now. We use the name '*Network Function*' synonymously from this section onwards.

But ratio of Laplace transform of zero-state response to Laplace transform of *which* input function? There should only be one. That is, a network function can be defined and evaluated only in an *s*-domain equivalent circuit containing one input source transform. Thus, the circuit should have only one independent source active when a network function is evaluated. Hence, there cannot be initial condition sources too. That is why the definition specifies that it is the ratio of Laplace transform of *zero-state response* to Laplace transform of input function.

The response may be measured across or through any circuit element or combinations of such elements in general. Hence, a variety of network functions can be defined in a circuit. In particular, when the response variable and excitation variable pertain to same terminal pair, the network function can only be of driving-point impedance or driving-point admittance type. The two together are referred as immittance functions.

15.11.1 Driving-point Functions and Transfer Functions

Input impedance function or driving-point impedance function is $Z_i(s) = \frac{V(s)}{I(s)}$, where $V(s)$ and $I(s)$ are transforms of voltage and current in a terminal pair as per passive sign convention. Input admittance function or driving-point admittance function is $Y_i(s) = \frac{I(s)}{V(s)}$, where $V(s)$ and $I(s)$ are transforms of voltage and current in a terminal pair as per passive sign convention. They are reciprocals of each other at the same terminal pair. These functions are a special class of network functions.

Transfer impedance function is $Z_m(s) = \frac{V_{ij}(s)}{I_{pq}(s)}$, where $V_{ij}(s)$ is the transform of voltage developed at i^{th} terminal with respect to j^{th} terminal due to a current source $I_{pq}(s)$ delivering current to p^{th} terminal from q^{th} terminal. The terminal pairs p - q and i - j are not the same. Transfer admittance function is $Y_m(s) = \frac{I_{pq}(s)}{V_{ij}(s)}$, where $I_{pq}(s)$ is the transform of current developed from q^{th} terminal to p^{th} terminal due to a voltage source $V_{ij}(s)$ between i^{th} terminal and j^{th} terminal. The terminal pairs p - q and i - j are not the same though they may share a common terminal. These functions are a special class of network functions.

Voltage transfer function is $A_v(s) = \frac{V_{pq}(s)}{V_{ij}(s)}$, where $V_{pq}(s)$ is the Laplace transform of zero-state voltage response developed across terminal pair p - q due to a voltage source transform $V_{ij}(s)$ applied across terminal pair i - j . The terminal pairs p - q and i - j are not the same though they may share a common terminal.

Current transfer function is $A_i(s) = \frac{I_{pq}(s)}{I_{ij}(s)}$, where $I_{pq}(s)$ is the Laplace transform of

zero-state current response developed through terminal pair p–q due to a current source transform $I_{ij}(s)$ applied through terminal pair i–j. The terminal pairs p–q and i–j are not the same though they may share a common terminal.

Thus, there are two types of driving-point network functions and four types of transfer network functions.

We will use the symbol $H(s)$ when we refer to network functions in general and use $Z_i(s)$, $Y_i(s)$, $Z_m(s)$, $Y_m(s)$, $A_v(s)$ and $A_i(s)$ when we refer to specific network functions.

15.11.2 The Three Interpretations for a Network Function $H(s)$

The first interpretation is the definition of a network function itself. That is, a network function is the ratio of Laplace transform of *zero-state response* to the Laplace transform of input source function causing the response.

Two circuit variables are clearly identified in the definition of network function – they are the variable used to measure the zero-state response and the variable that was decided by the input source function. These two variables can be identified in the s -domain equivalent circuit and circuit analysis in s -domain using nodal analysis or mesh analysis can be performed to arrive at the desired network function. The result will be $H(s)$ in the form of a ratio of rational polynomials in s . Thus, from this point of view, we expect to get a $H(s)$ in the following format. We have chosen to make the coefficient of highest power in s in the denominator 1.

$$H(s) = \frac{b'_m s^{m'} + b'_{m-1} s^{m'-1} + \dots + b'_1 s + b'_0}{s^{n'} + a'_{n-1} s^{n'-1} + \dots + a'_1 s + a'_0} \quad (15.11-1)$$

The second interpretation for $H(s)$ comes from the meaning of Laplace transform. Laplace transform of a right-sided function is an expansion of that function in terms of functions of e^{st} type with a value of s ranging from $\sigma - j\infty$ to $\sigma + j\infty$. The value of σ is such that the expansion converges to the time-function at all t . The components in expansion, i.e., signals of type e^{st} , are from $-\infty$ to ∞ in time-domain. Thus, Laplace transform converts a right-sided input into the sum of everlasting complex exponential inputs. Therefore, the problem of zero-state response with a right-sided input is translated into that of forced response with everlasting complex exponential inputs. And, the ratio of Laplace transform of zero-state response to Laplace transform of input source function must then be same as the ratio between forced response to input when input is e^{st} (not $e^{st} \times u(t)$).

Forced response to an everlasting complex exponential input of $1e^{st}$ was seen to be a scaled version e^{st} itself; the scaling factor being a complex number that depends on the complex frequency value s . (Sect. 15.1)

The time-domain circuit can be analysed using nodal analysis or mesh analysis to arrive at the n^{th} order differential equation relating the response variable (y) to excitation variable (x). The result will be

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$$

Then the scaling factor connecting an input of $1e^{st}$ to the output is

$$\frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

Three Interpretations for $H(s)$

$H(s)$, the network function, is a Laplace transform if we invert it to find the impulse response.

$H(s)$, the network function, is a complex gain if we evaluate it at a particular value of s . In that case, it gives the complex amplitude of the forced response with an input of e^{st} with the value of s same as the value at which $H(s)$ was evaluated.

$H(s)$, the network function, functions as a ratio of Laplace transforms when we multiply it by the Laplace transform of input source function and invert the product to determine the zero-state response in time-domain.

Therefore,

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad (15.11-2)$$

In this expression for $H(s)$, the coefficients come from the coefficients of differential equation governing the circuit. In the expression for $H(s)$ in Eqn. 15.11-1, the coefficients were the result of circuit analysis in s -domain. We conclude that $n' = n$, $m' = m$, all a' values are equal to corresponding a values and all b' values are equal to corresponding b values.

The third interpretation comes from definition of network function itself. If the input source function is assumed to be $\delta(t)$, then $H(s)$ becomes a Laplace transform – it becomes the Laplace transform of impulse response. Thus, $H(s)$ is a *ratio of Laplace transforms* and a *Laplace transform* at the same time. It is a Laplace transform when we try to invert it in order to find the impulse response.

15.11.3 Poles and Zeros of $H(s)$ and Natural Frequencies of the Circuit

A network function goes to infinite magnitude at certain values of s . These values are obviously the values of s at which the denominator polynomial evaluates to zero, i.e., at the roots of denominator polynomial. These values of s are called *poles* of the network function. Thus, the *poles* are roots of denominator polynomial of a network function. Similarly, a network function attains zero magnitude at certain values of s . They are roots of numerator polynomial. They are called *zeros* of the network function.

A diagram showing the *pole* points by '×' marking and *zero* points by 'o' marking in complex signal plane (i.e., s -plane) is called the *pole-zero plot* of the network function.

We note from the discussion in Sect. 15.11.2 that the denominator polynomial of a network function apparently has the same order and same coefficients as that of the characteristic polynomial of differential equation describing the linear time-invariant circuit. The roots of characteristic polynomial have been defined as the natural frequencies of the circuit. Does this mean that (i) the degree of denominator polynomial in a network function is the same as the degree of characteristic polynomial (ii) the poles and natural frequencies are the same?

The order of a differential equation is the degree of highest derivative of dependent variable. The order of a circuit and order of the describing differential equation are the same. It will also be equal to the total number of independent inductors and capacitors – number of all-capacitor-voltage source loops – number of all-inductor-current source nodes.

The order of a network function is the degree of denominator polynomial, i.e., the highest power of s appearing in the denominator polynomial.

Thus, we raise the question – is the order of a network function in a linear time-invariant circuit same as the order of the circuit?

The characteristic polynomial of a differential equation is quite independent of right-hand side of the differential equation. But, a network function is very much dependent on the right-hand side of the differential equation. Therefore, there exists a possibility of cancellation of some of the denominator factors by numerator factors in the case of a network function. Therefore, the order of a network function can be lower than the order of the circuit. It can not, however, be higher. This will also imply that the order of two network functions defined within the same network need not be the same.

For instance, let the differential equation describing a linear time-invariant circuit be

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \frac{dx}{dt} + x.$$

The characteristic equation is $s^2 + 3s + 2 = 0$ and the order of circuit is 2. The natural frequencies are $s = -1$ and $s = -2$. The zero-input response *can* contain e^{-t} and e^{-2t} terms. But it may contain only one of them for certain combination of initial conditions.

Consider $y(0) = 1$ and $y'(0) = -1$. Then $y(t) = e^{-t}$ and it will not contain e^{-2t} . Therefore, not all natural response terms need be present in all circuit variables under all initial conditions.

Now consider the network function. It is

$$\frac{Y(s)}{X(s)} = \frac{(s+1)}{(s^2+3s+2)} = \frac{(s+1)}{(s+2)(s+1)} = \frac{1}{(s+2)}.$$

The order of network function is 1. It has one pole at $s = -2$. Therefore, the zero-state response to *any input will not contain* e^{-t} term. This is the effect that a pole-zero cancellation in a network function has on circuit response. But, note that the same circuit may have other network functions that may not involve such pole-zero cancellation. It is only this particular circuit variable denoted by y that refuses to have anything to do with the natural response term e^{-t} .

To conclude,

- The order of a network function and order of the circuit can be different due to possible pole-zero cancellations in a particular network function.
- Poles of any network function defined in a linear time-invariant circuit will be natural frequencies of the circuit.
- However, all natural frequencies need not be present as poles in all network functions defined in that circuit.
- However, all natural frequencies will appear as poles in some network function or other.
- Thus, poles of a network function is a sub-set of natural frequencies of the circuit and natural frequencies will be a union-set of poles of all possible network functions in the circuit.
- A complex frequency that is not a natural frequency of the circuit can not appear as a pole in any network function in that circuit.
- Both the denominator polynomial and numerator polynomial of a network function in a linear time-invariant circuit have real coefficients. Therefore, poles and zeros of a network function will either be real-valued or will occur in complex conjugate pairs.

15.11.4 Specifying a Network Function

A network function $H(s)$ can be specified in three ways. In the first method it is specified as a ratio of rational polynomials in s .

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (15.11-3)$$

In the second method, it is specified as ratio of product of first order factors in numerator and denominator with a gain factor multiplying the entire ratio.

$$H(s) = K \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} \quad (15.11-4)$$

There are m factors in the numerator and n factors in the denominator. $z_1, z_2, \dots, z_m$ are the *zeros* of the network function and $p_1, p_2, \dots, p_n$ are the *poles* of the network function. Note that though the degree of denominator polynomial is shown as n , which is the order of the circuit, pole-zero cancellation may take place leaving the denominator polynomial of network function with a degree less than n .

In the third method of specifying a network function, the *pole-zero plots* along with the *gain factor* K is given. The gain factor K may be directly given or indirectly in the form of value of $H(s)$ evaluated at a particular value of s .

Three different formats in which $H(s)$ may be specified.

EXAMPLE: 15.11-1

The circuit shown in Fig. 15.11-1 is the small signal equivalent circuit of a transistor amplifier for analysis of its behaviour for sinusoidal input at high frequency. Obtain the transfer function between the output voltage and input source voltage.

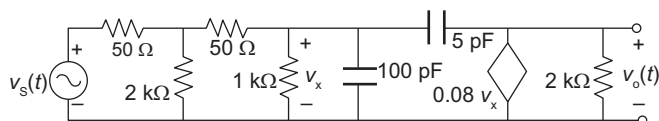


Fig. 15.11-1 Small-signal Equivalent Circuit of a Transistor Amplifier for Example 15.11-1

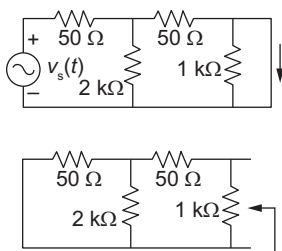


Fig. 15.11-2 Sub-circuits for Determining Norton's Equivalent

SOLUTION

We find the Norton's equivalent of the circuit to the left of 100 pF capacitor first.

The sub-circuits needed for finding this equivalent are shown in Fig. 15.11-2.

The short-circuit current in the first circuit is

$\frac{v_s(t)}{50 + 2000 // 50} \times \frac{2000}{2050} = 9.876 \times 10^{-3} v_s(t)$. The Norton's equivalent resistance is $[(50 \Omega // 2 \text{ k}\Omega) + 50 \Omega] // 1 \text{ k}\Omega = 89.9 \Omega$. Thus, the required Norton's equivalent is $9.876 \times 10^{-3} v_s(t)$ A in parallel with 89.9Ω . The original circuit with this Norton's equivalent in place is shown in circuit in Fig. 15.11-3(a) with $R_1 = 89.9 \Omega$, $R_2 = 2 \text{ k}\Omega$, $C_1 = 100 \text{ pF}$, $C_2 = 5 \text{ pF}$ and $g_m = 0.08$. The corresponding s-domain equivalent circuit is shown in circuit in Fig. 15.11-3(b).

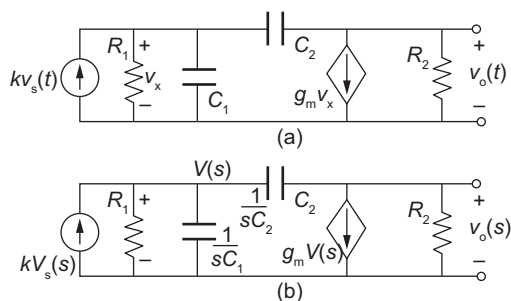


Fig. 15.11-3 (a) Reduced Version of Circuit in Fig. 15.11-1 and (b) its s-Domain Equivalent

The node equations written for the two node voltage transforms $V(s)$ and $V_o(s)$ are as follows.

$$V(s) \left[s(C_1 + C_2) + \frac{1}{R_1} \right] + V_o(s) [-sC_2] = kV_s(s)$$

$$V(s) [-sC_2 + g_m] + V_o(s) \left[sC_2 + \frac{1}{R_2} \right] = 0$$

$$\begin{bmatrix} s(C_1 + C_2) + \frac{1}{R_1} & -sC_2 \\ -sC_2 + g_m & sC_2 + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V(s) \\ V_o(s) \end{bmatrix} = \begin{bmatrix} kV_s(s) \\ 0 \end{bmatrix}$$

Solving for $V_o(s)$ and simplifying the expression, we get,

$$H(s) = \frac{V_o(s)}{V_s(s)} = \frac{k}{C_1} \frac{\left(s - \frac{g_m}{C_2}\right)}{s^2 + s \left(\frac{1+g_m R_1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} \right) + \left(\frac{1}{R_1 R_2 C_1 C_2} \right)}$$

Substituting the numerical values for various parameters, we get,

$$H(s) = \frac{98.76 \times 10^6 (s - 1.6 \times 10^{10})}{s^2 + 1.016 \times 10^9 s + (105.47 \times 10^6)^2}$$

The poles are at $s = -10^9$ Np/s and $s = -11.07 \times 10^6$ Np/s. The zero is at $s = 1.6 \times 10^{10}$ Np/s.

Note that compared to the pole at -11.07×10^6 the other pole and the zero are located two orders away from it. The natural response term contributed by the pole at $s = -11.07 \times 10^6$ Np/s will have a time constant of 90.3 ns whereas the natural response term contributed by the pole at $s = -10^9$ Np/s will have a time constant of 1 ns. Thus, the natural response term contributed by the pole at $s = -10^9$ Np/s will disappear in about 5% of the time constant of the other term. Therefore, the time constant of 90.3 ns is the *dominant time constant* in this amplifier and the corresponding pole at -11.07×10^6 is the *dominant pole*. The amplifier transfer function can be approximated by neglecting the zero and the non-dominant pole to

$$H(s) = \frac{98.76 \times 10^6 (s - 1.6 \times 10^{10})}{(s + 10^9)(s + 11.07 \times 10^6)} \approx \frac{-1.58 \times 10^9}{(s + 11.07 \times 10^6)} = -142.7 \left(\frac{11.07 \times 10^6}{s + 11.07 \times 10^6} \right)$$

EXAMPLE: 15.11-2

Find (i) Input impedance function and (ii) $V_o(s)/V_s(s)$ in the circuit shown in Fig. 15.11-4.

SOLUTION

$$Z_i(s) = (1+s) / \left(1 + \frac{1}{s} \right) = \frac{(1+s) \left(1 + \frac{1}{s} \right)}{2 + s + \frac{1}{s}} = \frac{(s+1)(s+1)}{s^2 + 2s + 1} = 1 \Omega.$$

This is a case of cancellation of all poles by zeros leaving a real value for a network function. The input impedance of the circuit is purely resistive at all values of s . This implies that the current drawn by the circuit behaves as in a memoryless circuit. But, this does not mean that the order of the circuit is zero.

$$\frac{V_o(s)}{V(s)} = \frac{s}{s+1} - \frac{\frac{1}{s}}{1 + \frac{1}{s}} = \frac{s-1}{s+1}.$$

The voltage transfer function has a pole at $s = -1$ and zero at $s = 1$.

This innocuous circuit challenges our notions on order of a circuit. It contains two energy storage elements. Hence, it must be a second-order circuit. But no network function defined in this circuit will be a second-order function if the excitation is a voltage source. Even the zero-input response obtained by shorting the voltage source with initial conditions on inductor and capacitor will contain only e^{-t} . The reader is encouraged to analyse the general situation that develops when many sub-circuits with same set of poles in their input admittance functions are connected in parallel and driven by a common voltage source. Similarly, he is encouraged to ponder over the order of a circuit resulting from series connection of many sub-circuits with the same set

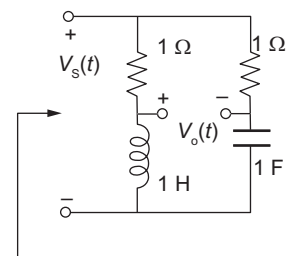


Fig. 15.11-4 Circuit for Example: 15.11-2

of poles in their input impedance functions driven by a common current source. The reader may also note that the current in the circuit in Fig. 15.11-4 will have e^{-t} and $t e^{-t}$ terms in zero-input response due to initial energy storage in inductor and capacitor with input open-circuited (i.e., zero-input response for current source excitation), thereby confirming it is a second-order circuit.

15.12 IMPULSE RESPONSE OF NETWORK FUNCTIONS FROM POLE-ZERO PLOTS

Let $H(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$ be a network function defined in a linear time-invariant circuit. Then the impulse response of this network function is given by its inverse transform. The transform $H(s) \times 1$ (1 is the Laplace transform of $\delta(t)$) can be expressed in partial fractions as below.

$$H(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_2)} + \cdots + \frac{A_n}{(s-p_n)} \quad (15.12-1)$$

We have assumed that all poles are non-repeating ones. If there are repeating poles we may assume that the poles are slightly apart by Δp and evaluate the limit of $h(t)$ as $\Delta p \rightarrow 0$ after we complete the inversion. We will need a familiar limit, $\lim_{\Delta p \rightarrow 0} \frac{e^{(p+\Delta p)t} - e^{pt}}{\Delta p} = te^{pt}$ for this. This strategy will help us to view all poles as non-repeating ones at the partial fractions stage.

$$\therefore H(s) = \sum_{i=1}^n \frac{A_i}{(s-p_i)}; A_i = K \left. \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_{i-1})(s-p_{i+1})\cdots(s-p_n)} \right|_{s=p_i} \quad (15.12-2)$$

Thus, each pole contributes a complex exponential function to impulse response. The complex frequency of the complex exponential function contributed by a pole to impulse response is the same as the value of the pole frequency itself.

A point s in the complex signal space (i.e., the s -plane) stands for the complex exponential signal e^{st} for all t . But, when a point s is marked out as a pole of a network function by a '×' mark, that signal point contributes $e^{st} u(t)$ to the impulse response and not e^{st} . Thus, a point in signal space stands for a two-sided complex exponential signal in general and for a right-sided complex exponential signal when that point is specified as a pole of a network function.

The evaluation of residue A_i at the pole p_i involves the evaluation of the product of terms like $(p_i - z_1)\cdots(p_i - z_l)$ and $(p_i - p_1)\cdots(p_i - p_{i-1})(p_i - p_{i+1})\cdots(p_i - p_n)$. Each of these factors will be a complex number. For instance, consider $(p_i - z_1)$. This is a complex number that can be represented by a directed line drawn from the point $s = z_1$ in s -plane to the point $s = p_i$ in the s -plane with the arrow of the line at $s = p_i$. The length of this line gives the magnitude of the complex number $(p_i - z_1)$ and the angle of the complex number $(p_i - z_1)$ is given by the angle the line makes with positive real axis in the counter-clockwise direction. The magnitude of product of complex numbers is the product of magnitudes of the individual numbers. The angle of product of complex numbers is the sum of angles of individual complex numbers. Therefore, evaluation of residue A_i at the pole p_i reduces to determining certain lengths and angles in the pole-zero plot of the network function.

The reasoning employed in the paragraph above also reveals the roles of poles and zeros of a network function in deciding the impulse response terms. The poles decide the number of terms in impulse response and their complex frequencies. The zeros along with the poles and gain factor K decide the amplitude of each impulse response term.

Geometrical interpretation for residue evaluation.

A network function is a stable one if its impulse response decays to zero with time. This is equivalent to stating that its impulse response must be absolutely integrable, i.e., $\int_0^\infty |h(t)| dt$ must be finite. Therefore, a network function is stable if all the impulse response terms are damped ones. That is, all the poles must have negative real values or complex values with negative real parts. *Therefore, a network function is stable if and only if all its poles are in the left-half of s-plane excluding the j ω -axis.*

Note that a stable network function in a linear time-invariant circuit does not necessarily imply that the circuit itself is stable. A linear time-invariant circuit is stable only if *all* the network functions that can be defined in it are stable ones. That is, a stable circuit will have only stable network functions in it. But, an unstable circuit can have both stable and unstable network functions in it.

The graphical interpretation adduced to impulse response coefficients in this section is illustrated in the examples that follow.

EXAMPLE: 15.12-1

Obtain the pole-zero plots for (i) $H(s) = \frac{\alpha}{s+\alpha}$ (ii) $H(s) = \frac{s}{s+\alpha}$ (iii) $H(s) = \frac{s-\alpha}{s+\alpha}$ for positive and negative values of α and sketch the impulse response for $\alpha = \pm 1$.

SOLUTION

These are standard first-order network functions. They are stable ones for positive values of α and unstable ones for negative values of α . They are important, yet simple, functions.

(i) $H(s) = \frac{\alpha}{s+\alpha}$. Therefore, $h(t) = \alpha e^{-\alpha t} u(t)$. The pole-zero plot and impulse response are shown in Fig. 15.12-1 for $\alpha = \pm 1$.

$$(ii) H(s) = \frac{s}{s+\alpha} = 1 - \frac{\alpha}{s+\alpha}$$

$$\therefore h(t) = \delta(t) - \alpha e^{-\alpha t} u(t)$$

The pole-zero plots and impulse responses for $\alpha = \pm 1$ are shown in Fig. 15.12-2.

$$(iii) H(s) = \frac{s-\alpha}{s+\alpha} = 1 - \frac{2\alpha}{s+\alpha}$$

$$\therefore h(t) = \delta(t) - 2\alpha e^{-\alpha t} u(t)$$

The pole-zero plots and impulse responses for $\alpha = \pm 1$ are shown in Fig. 15.12-3.

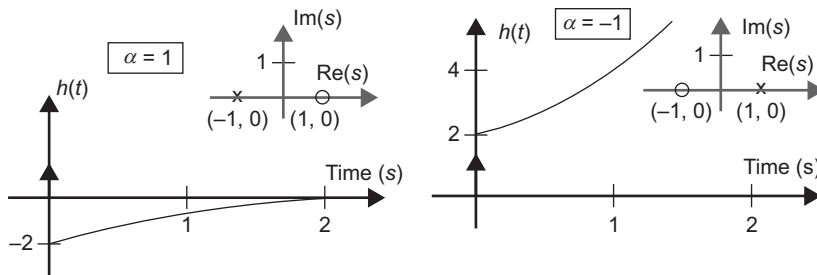


Fig. 15.12-3 Pole-Zero Plot and Impulse Response of $(s - \alpha)/(s + \alpha)$ for $\alpha = \pm 1$

EXAMPLE: 15.12-2

A second order low-pass network function in standard form is given as $H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, where ξ is the damping factor and ω_n is the undamped natural

Stability and pole location.

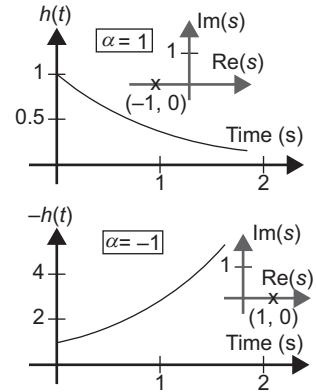


Fig. 15.12-1 Pole-Zero Plot and Impulse Response of $\alpha/(s + \alpha)$ for $\alpha = \pm 1$

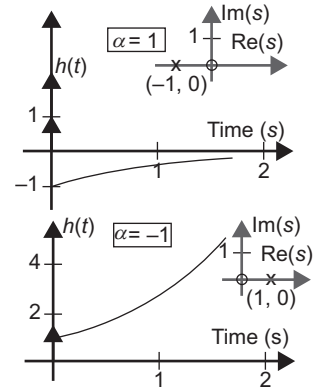


Fig. 15.12-2 Pole-Zero Plot and Impulse Response of $s/(s + \alpha)$ for $\alpha = \pm 1$

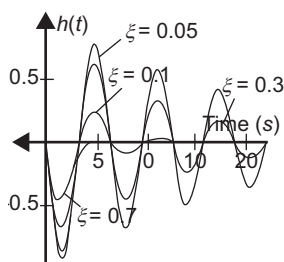


Fig. 15.12-5 Impulse Response of Standard Second-Order Network Function for Various Damping Factors

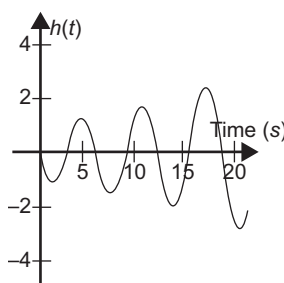


Fig. 15.12-6 Impulse Response of Standard Second-Order Network Function for $\xi = -0.05$

frequency as defined in Sect. 11.6 in Chap. 11. Obtain expressions for impulse response of the network function for positive and negative values of ξ in the range $-1 < \xi < 1$.

SOLUTION

The poles are at $s = -\xi\omega_n \pm j\sqrt{1-\xi^2}\omega_n$. They are complex conjugate poles for $-1 < \xi < 1$. They are located in the right-half s -plane for $-1 < \xi < 0$ and in the left-half s -plane for $0 < \xi < 1$. They are located on $j\omega$ -axis at $\pm j\omega_n$ when $\xi = 0$. The poles have a magnitude of ω_n for all values of ξ in the range $(-1, 1)$. The pole line of the pole $s = -\xi\omega_n + j\sqrt{1-\xi^2}\omega_n$ makes an angle of $\cos^{-1}(\xi)$ with negative real axis in the case of positive ξ and with positive real axis in the case of negative ξ . Thus, the damping factor magnitude is given by cosine of pole angle (Fig. 15.12-4).

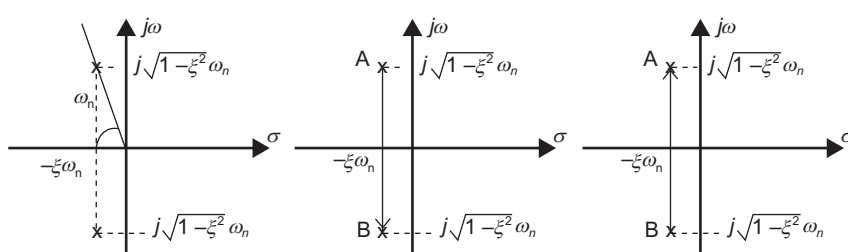


Fig. 15.12-4 Pole-Zero Plots for a Standard Second Order Low-Pass Network Function with Positive Damping

The residue at the pole marked as B is given by ω_n^2 divided by the complex number represented by the line connecting A and B in Fig. 15.12-4 with an arrow towards B. This line is seen to be $2\sqrt{1-\xi^2}\omega_n$ in length and it makes -90° with a positive real axis. Similarly, the residue at the pole marked as B is given by ω_n^2 divided by the complex number represented by the line connecting A and B in Fig. 15.12-4 with an arrow towards B. This line is seen to be $2\sqrt{1-\xi^2}\omega_n$ in length and it makes 90° with positive real axis.

Therefore,

$$\begin{aligned} h(t) &= \frac{\omega_n^2}{2(\sqrt{1-\xi^2})\omega_n} \left[e^{j90^\circ} e^{(-\xi + j\sqrt{1-\xi^2})\omega_n t} + e^{-j90^\circ} e^{(-\xi - j\sqrt{1-\xi^2})\omega_n t} \right] u(t) \\ &= \frac{\omega_n \Theta^{-\xi t}}{2(\sqrt{1-\xi^2})} \left[e^{j(\sqrt{1-\xi^2}\omega_n t + 90^\circ)} + e^{-j(\sqrt{1-\xi^2}\omega_n t + 90^\circ)} \right] u(t) \\ &= \frac{\omega_n \Theta^{-\xi t}}{\sqrt{1-\xi^2}} \cos(\sqrt{1-\xi^2}\omega_n t + 90^\circ) u(t) = \frac{-\omega_n \Theta^{-\xi t}}{\sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\omega_n t u(t). \end{aligned}$$

This response is shown in Fig. 15.12-5 for $\omega_n = 1$ and $\xi = 0.7, 0.3, 0.1$ and 0.05 .

We observe that as the poles get closer and closer to $j\omega$ -axis, the impulse response oscillations become more and more under-damped and last for many cycles.

The impulse response for $\xi = -0.05$ as in Fig. 15.12-6 shows the unbounded nature of impulse response of a network function with poles on right-half s -plane.

15.13 SINUSOIDAL STEADY-STATE FREQUENCY RESPONSE FROM POLE-ZERO PLOTS

Let $H(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$ be a network function defined in a linear time-invariant circuit and let all the poles of this network function be in the left-half of s -plane excluding $j\omega$ -axis. The zeros can be located anywhere in s -plane. Then the network function

is stable and has an absolutely integrable impulse response. Therefore, the Fourier transform of its impulse response exists. Fourier transform of impulse response, if it exists, is the sinusoidal steady-state frequency response function $H(j\omega)$ for the network function $H(s)$. Note that sinusoidal steady-state frequency response function is definable and relevant only in the case of a *stable* network function. Therefore, $H(s)$, *evaluated* with $s = j\omega$, gives a frequency response function provided the network function is stable. Thus,

$$H(j\omega) = K \frac{(j\omega - z_1)(j\omega - z_2) \cdots (j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_n)} \quad (15.13-1)$$

$H(j\omega)$ is a complex function of a real variable ω . The plots of $|H(j\omega)|$ versus ω and $H(j\omega)$ versus ω yield the *frequency response plots* of the network function. The first is called the *magnitude plot* and the second is called the *phase plot*.

Frequency response function in factorised form.

15.13.1 Three Interpretations for $H(j\omega)$

The network function $H(s)$ can be interpreted in three ways as discussed in Sect. 15.11. Three interpretations of $H(j\omega)$ follow from this. They have already been dealt with in Chap. 14 on Fourier transforms.

- (i) $H(j\omega)$ is the ratio of complex amplitudes of output complex exponential and input complex exponential when input complex exponential is of the form $Ae^{j\omega t}$. Equivalently, $H(j\omega)$ is the complex amplitude of output when input is $e^{j\omega t}$ (not $e^{j\omega t}u(t)$). The signal $e^{-j\omega t}$ is a signal that is different from $e^{j\omega t}$. Hence, $H(j\omega)$ is a *two-sided* function from this point of view.
If input is $e^{j\omega t}u(t)$, then, $H(j\omega) e^{j\omega t}$ gives the forced response component (same as the steady-state response component).
- (ii) $H(j\omega)$ is the ratio of Laplace transform of zero-state response to Laplace transform of input when input is of the form $Ae^{j\omega t}u(t)$. The signal $Ae^{-j\omega t}u(t)$ is not the same as $Ae^{j\omega t}u(t)$. Hence, $H(j\omega)$ is a *two-sided* function from this point of view also.
- (iii) $H(j\omega)$ is the Fourier transform of $h(t)$. A Fourier transform is a *two-sided complex function* of ω . It expands the time-domain signal into sum of complex exponential functions with ω value ranging from $-\infty$ to ∞ . Hence, $H(j\omega)$ is a *two-sided* function from this point of view.

The three interpretations for $H(j\omega)$.

There are two ways to solve the problem of finding the *steady-state output* when input variable is $\cos\omega_0 t u(t)$. The first method is to express $\cos\omega_0 t u(t)$ as $\text{Re}(e^{j\omega_0 t})$ and express the output as $\text{Re}[H(j\omega_0)e^{j\omega_0 t}]$. This method was called *Phasor Method* in Chap. 8. This method results in the steady-state response component and is based on the first interpretation of $H(j\omega)$.

$$\text{Re}[H(j\omega_0)e^{j\omega_0 t}] = \text{Re}[|H(j\omega_0)|e^{\angle H(j\omega_0)}e^{j\omega_0 t}] = |H(j\omega_0)|\cos[\omega_0 t + \angle H(j\omega_0)].$$

Now, there is no harm if $H(j\omega)$ is thought of as a *single-sided function* of ω provided we interpret the magnitude of $H(j\omega)$ as the amplitude of output sinusoidal waveform with input amplitude of 1 and the phase of $H(j\omega)$ as the phase angle by which the output sinusoidal waveform leads the input sinusoidal waveform.

The second way is to express $\cos\omega_0 t u(t)$ as $0.5 e^{j\omega_0 t}u(t) + 0.5 e^{-j\omega_0 t}u(t)$ by applying Euler's formula and expressing the *steady-state output* as $0.5[H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}]$. We have seen in Chap. 14 that $H(-j\omega) = [H(j\omega)]^*$. Therefore, the steady-state output will be $\text{Re}[H(-j\omega_0)\cos\omega_0 t - \text{Im}[H(j\omega_0)\sin\omega_0 t] = |H(j\omega_0)|\cos[\omega_0 t + \angle H(j\omega)]$; same as in the first method. This method is also based on the first interpretation of $H(j\omega)$ but uses a two-sided version of $H(j\omega)$.

Two methods for arriving at steady-state response when input is a sinusoid – One-sided $H(j\omega)$ versus two-sided $H(j\omega)$.

Frequency response function is not new to us. We had dealt with the frequency response of first-order and second-order circuits in detail in Sect. 10.9 in Chap. 10, in Sects. 11.5, 11.11 and 11.12 in Chap. 11 and in Sect. 14.10 in Chap. 14 on Fourier transforms. But the observation that $H(j\omega)$ can be evaluated by evaluating $H(s)$ on $j\omega$ -axis leads to a graphical interpretation for sinusoidal steady-state frequency response function based on the pole-zero plot of $H(s)$. This interpretation affords an insight into the variation of magnitude and phase of $H(j\omega)$ without evaluating it at all values of ω . It helps us to visualise the salient features of the frequency response function without extensive calculations.

15.13.2 Frequency Response from Pole-Zero Plot

Each factor of the form $(j\omega - z_i)$ in the numerator of Eqn. 15.13-1 is a complex number that can be thought of as a line directed from $s = z_i$ to $s = j\omega$ in s -plane. The magnitude of $(j\omega - z_i)$ is equal to the length of the line and the angle of $(j\omega - z_i)$ is the angle that the line makes with positive real axis in a counter-clockwise direction. A similar interpretation is valid for factors of the type $(j\omega - p_i)$ in the denominator of Eqn. 15.13-1 too. Let d_{z_i} be the length of the line joining the zero at $s = z_i$ to the excitation signal point $s = j\omega$ on the imaginary axis in s -plane. Let the angle that the line makes with positive real axis in counter-clockwise be θ_{z_i} . Similar quantities for a pole at $s = p_i$ are d_{p_i} and θ_{p_i} . Then, the frequency response function $H(j\omega)$ can be expressed in terms of these lengths and angles as

$$H(j\omega) = K \frac{d_{z_1} d_{z_2} \dots d_{z_m}}{d_{p_1} d_{p_2} \dots d_{p_n}} \angle (\theta_{z_1} + \theta_{z_2} + \dots + \theta_{z_m}) - (\theta_{p_1} + \theta_{p_2} + \dots + \theta_{p_n})$$

$$= |H(j\omega)| \angle \theta(j\omega),$$

$$\text{where } |H(j\omega)| = K \frac{d_{z_1} d_{z_2} \dots d_{z_m}}{d_{p_1} d_{p_2} \dots d_{p_n}} \text{ and } \theta(j\omega) = \sum_{i=1}^m \theta_{z_i} - \sum_{i=1}^n \theta_{p_i}$$

$$|H(j\omega)| = \text{Gain factor} \times \frac{\text{Product of zero-distances}}{\text{Product of pole-distances}}$$

$$\theta(j\omega) = \angle(\text{Sum of zero angles}) - (\text{Sum of pole angles})$$

Hence, we can make a rough sketch of the frequency response function by visualising how the various zero-distances and pole-distances vary when ω is taken from $-\infty$ to $+\infty$ in the s -plane.

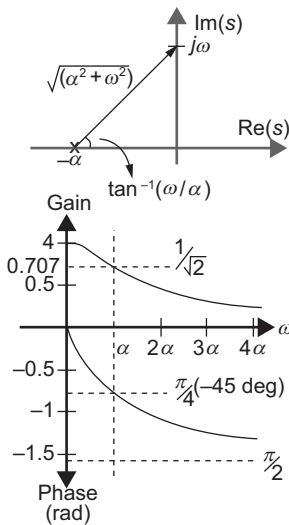


Fig. 15.13-1 Pole-Zero Plot and Frequency Response of $H(s) = \alpha/(s + \alpha)$

EXAMPLE: 15.13-1

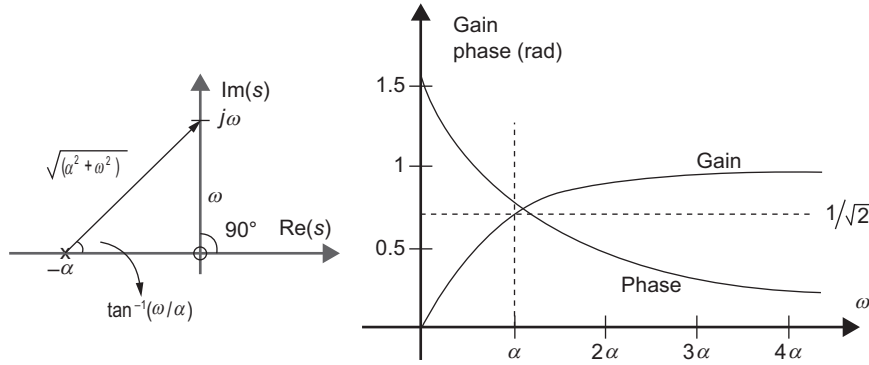
Obtain the frequency response plots for (i) $H(s) = \alpha/(s + \alpha)$ (ii) $H(s) = s/(s + \alpha)$ using geometrical interpretation of frequency response and obtain expressions for bandwidth in both cases.

SOLUTION

(i) Figure 15.13-1 shows the pole-zero plot and frequency response function. The pole distance and the pole angle are marked in Fig. 15.13-1. Obviously, the gain magnitude goes to $1/\sqrt{2}$ times initial gain when $\omega = \alpha$ and the phase at that point is -45° . Therefore, the bandwidth is α rad/s and the function is a low-pass function.

(ii) Figure 15.13-2 shows the pole-zero plot and frequency response function. The pole distance and the pole angle are marked in Fig. 15.13-2. The zero distance is same as the excitation frequency value ω and the zero angle is 90° . Obviously, the gain magnitude goes to $1/\sqrt{2}$ times the final gain when $\omega = \alpha$ and the phase at that point is 45° . Therefore, the bandwidth is α rad/s and the function is a high-pass function.

$$|H(j\omega)| = \frac{\omega}{\sqrt{\alpha^2 + \omega^2}} \text{ and } \theta(j\omega) = \left(\frac{\pi}{2} - \tan^{-1} \frac{\omega}{\alpha} \right) \text{ rad.}$$

Fig. 15.13-2 Gain and Phase Plots of $H(s) = s/(s + \alpha)$ **EXAMPLE: 15.13-2**

The biquadratic network function $H(s) = \frac{as^2 + bs + c}{s^2 + 2\xi\omega_n s + \omega_n^2}$ is a second order low-pass function if $a = 0$, $b = 0$ and $c = \omega_n^2$. It is a second order high-pass function if $a = 1$ and $b = c = 0$. It is a band-pass function if $a = c = 0$ and $b = 2\xi\omega_n$ and it is a band-reject function if $a = 1$, $b = 0$ and $c = \omega_n^2$. The frequency response functions for low-pass, high-pass and band-pass second order functions were studied in detail in Sect. 11.11 in Chap. 11 in the context of frequency response of series RLC circuit.

Consider the band-pass function and band-reject function and sketch their frequency response plots for $\xi \ll 1$.

SOLUTION

The poles of the function are at $s = -\xi\omega_n \pm j\sqrt{1-\xi^2}\omega_n$. Let us consider the band-pass function first.

$$H(s) = \frac{2\xi\omega_n s}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

The zero of this function is at $s = 0$.

The distance of the pole at $s = -\xi\omega_n + j\sqrt{1-\xi^2}\omega_n$ to the excitation frequency point

$j\omega$ is denoted by d_1 and the distance of the pole at $s = -\xi\omega_n - j\sqrt{1-\xi^2}\omega_n$ to $j\omega$ is d_2 . The distance of zero at $s = 0$ to $j\omega$ is itself. The pole angles θ_1 and θ_2 are also shown in the pole-zero diagram in Fig. 15.13-3.

The magnitude function is $2\xi\omega_n\omega/(d_1 d_2)$ and the phase function is $(\pi/2) - (\theta_1 + \theta_2)$. The distances d_1 and d_2 are equal to ω_n at $\omega = 0$ and the sum of the angles θ_1 and θ_2 at that frequency is 360° . Therefore, the gain at zero frequency is 0 (due to the zero-distance of zero) and angle is 90° . As $\omega \rightarrow \infty$, all the three distances go to ∞ , and hence magnitude goes to zero. θ_1 and θ_2 go to 90° as $\omega \rightarrow \infty$ and hence the phase angle of the frequency response function goes to -90° as $\omega \rightarrow \infty$.

As ω increases from 0, the distance d_1 decreases and the distances d_2 and d_3 increase. Consider a pair of ω values equal to $[\sqrt{(1-\xi^2)} - \xi]\omega_n$ and $[\sqrt{(1-\xi^2)} + \xi]\omega_n$ i.e., two ω values separated by $\pm$ real part of the pole from the imaginary part of the pole. The distance d_1 undergoes a variation from $\sqrt{2}\xi\omega_n$ to $\xi\omega_n$ and again to $\sqrt{2}\xi\omega_n$ as ω varies from $[\sqrt{(1-\xi^2)} - \xi]\omega_n$ to $[\sqrt{(1-\xi^2)} + \xi]\omega_n$ passing through the point $[\sqrt{(1-\xi^2)}]\omega_n$. The

distances d_2 and d_3 also vary. However, if $\xi \ll 1$, the variation in these two quantities will be negligible over this frequency range and approximation $d_2 \approx 2\sqrt{(1-\xi^2)}\omega_n$ and $d_3 \approx \omega_n$ will be satisfactory.

Therefore, the magnitude of frequency response function will vary from $2\xi\omega_n\omega_n \div 2\sqrt{2}\xi\omega_n\sqrt{(1-\xi^2)}\omega_n \approx 1/\sqrt{2}$ to $2\xi\omega_n\omega_n \div \xi\omega_n \times 2\sqrt{(1-\xi^2)}\omega_n \approx 1$ and again to $1/\sqrt{2}$ as ω

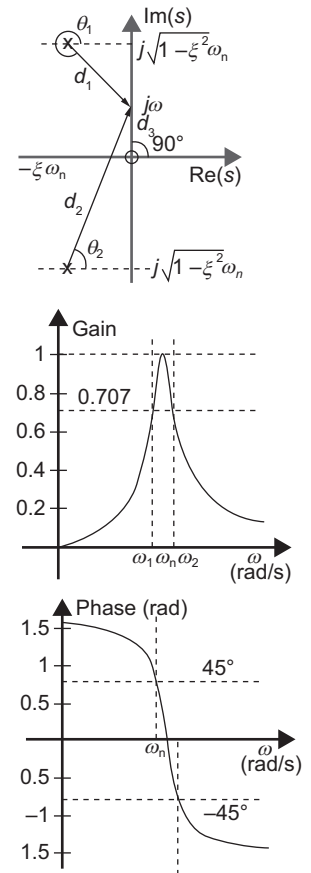


Fig. 15.13-3 Pole-Zero Plot and Frequency Response Plot for a Second Order Band-Pass Function

varies from $\left[\sqrt{1-\xi^2}-\xi\right]\omega_n$ to $\left[\sqrt{1-\xi^2}+\xi\right]\omega_n$ passing through the point $\left[\sqrt{1-\xi^2}\right]\omega_n$. The imaginary part of poles can be taken as approximately ω_n itself for $\xi \ll 1$. Therefore, the maximum gain is 1 at $\omega = \omega_n$ and the gain goes through $1/\sqrt{2}$ at $\omega_{1,2} = (1 \pm \xi)\omega_n$. The phase angle at $\omega = \omega_n$ is zero (Fig. 15.13-3).

The centre frequency of the narrow band-pass function is seen to be $\approx \omega_n$ and the bandwidth is $\approx 2\xi\omega_n$. Thus, the ratio of centre frequency to bandwidth of a narrow band-pass second-order network function is $1/2\xi$ or Q of the denominator polynomial.

Let us consider the band-reject function now.

$$H(s) = \frac{s^2 + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

The poles are at $s = -\xi\omega_n \pm j\sqrt{1-\xi^2}\omega_n$ and zeros are at $s = \pm j\omega_n$. The distance of the pole at $s = -\xi\omega_n + j\sqrt{1-\xi^2}\omega_n$ to the excitation frequency point $j\omega$ is denoted by d_1 and the distance of the pole at $s = -\xi\omega_n - j\sqrt{1-\xi^2}\omega_n$ to $j\omega$ is d_2 . The distance of zero at $s = j\omega_n$ to $j\omega$ is d_3 and distance of zero at $s = -j\omega_n$ to $j\omega$ is d_4 . The pole angles θ_1 and θ_2 are also shown in the pole-zero diagram in Fig. 15.13-4. The zero-angles are -90° and 90° for all $\omega < \omega_n$ and 90° and 90° for all $\omega > \omega_n$. The gain is given by $d_3 d_4 / d_1 d_2$ and starts at 1 at $\omega = 0$ since $d_1 = d_2$ and $d_3 = d_4$. The gain goes to 1 as $\omega \rightarrow \infty$ since $d_1 \approx d_2$ and $d_3 \approx d_4$ under that condition. The gain is zero at $\omega = \omega_n$ since d_3 is zero under that condition. Therefore, it is a band-reject function.

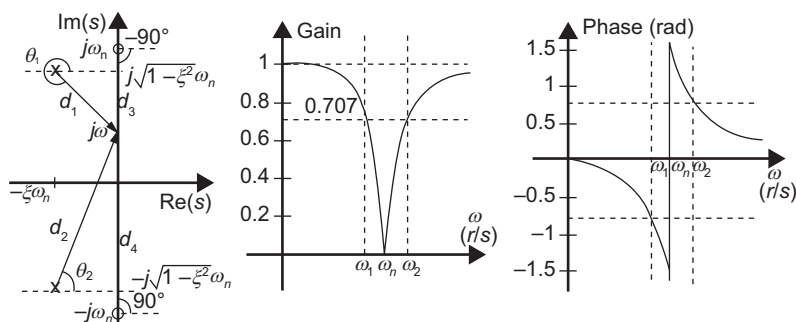


Fig. 15.13-4 Pole-Zero Plot and Frequency Response Plot for a Second Order Band-Reject Function

The pole-zero plots and frequency response plots are shown in Fig. 15.13-4. For $\xi \ll 1$, it may be shown that the gain crosses $1/\sqrt{2}$ level at $\omega_{1,2} = (1 \pm \xi)\omega_n$ and that the phase angles at those frequencies are -45° and 45° . The centre frequency of the narrow band-reject function is seen to be $\approx \omega_n$ and the bandwidth is $\approx 2\xi\omega_n$. Thus, the ratio of centre frequency to bandwidth of a narrow band-reject second-order network function is $1/2\xi$ or Q of the denominator polynomial.

The frequency response of higher order network functions can similarly be sketched. A higher order $H(j\omega)$ can be expressed as product of first-order factors and biquadratic factors. The magnitude response for first-order factors and biquadratic factors may be sketched separately first and then multiplied together to get the magnitude response curve of $H(j\omega)$. Phase curves will have to be added.

Poles on negative real axis contribute a monotonically decreasing magnitude response. Poles close to $j\omega$ -axis render resonant peaks in magnitude response and zeros on $j\omega$ -axis produce zero gain response at the excitation frequencies equal to the zero locations. Thus, the graphical interpretation of frequency response function is a valuable aid to a circuit designer who wants to locate poles and zeros of a network function to tailor the frequency response function to meet design specifications.

15.14 ANALYSIS OF COUPLED COILS USING LAPLACE TRANSFORMS

We had taken up the sinusoidal steady-state analysis of circuits containing coupled coils in Sect. 8.11 of Chap. 8. We had developed two equivalent circuits in time-domain for a set of two coupled coils. One of them represented the magnetic coupling by using dependent sources. The second one was a conductively coupled equivalent circuit. These equivalents are shown in Fig. 15.14-1 for both the relative polarities that can exist in a two-coil system.

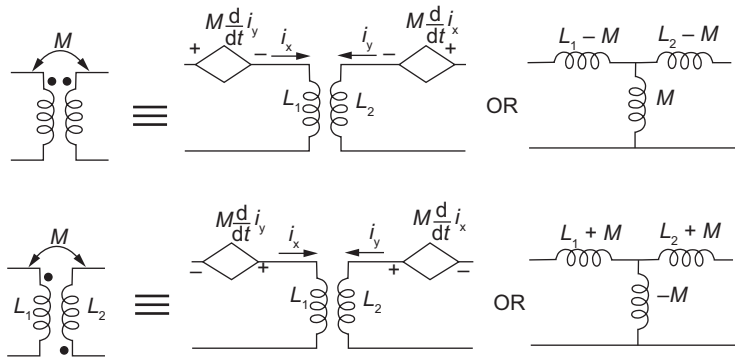


Fig. 15.14-1 Equivalent Circuits for a Coupled Two-Coil System

A coupled set of two coils is also called a *two-winding transformer* from the application perspective.

In the first part of this section we generalise the conclusions we arrived at in Sect. 8.11 in Chap. 8 for arbitrary inputs. Specifically, we show that (i) the input impedance function of a *passively-terminated* two-winding transformer is independent of the relative polarity of windings (ii) the secondary voltage of a transformer with unity coupling coefficient ($k = 1$) and zero winding resistances is turns ratio times primary voltage, quite independent of the wave-shape of input (iii) a transformer with $k = 1$ reflects an impedance $Z(s)$ connected in the secondary side to the primary side as $Z(s)/n^2$, where n is the ratio of secondary turns to primary turns (iv) a practical two-winding transformer with finite coil inductance values, imperfect coupling ($k \neq 1$) and non-zero winding resistances will be a band-pass system with a lower cut-off frequency decided by winding resistances and an upper cut-off frequency decided by the coupling coefficient.

In the second part of this section we take up the issue of instantaneous changes in currents in a coupled-coil system and show that such changes are possible in inductors involved in a coupled-coil system under certain specific circuit conditions. We show that a coil with low resistance kept shorted will *expel magnetic flux* in the common magnetic path. This is the principle that works behind *electromagnetic shielding* of electrical and electronic equipment.

In the third part of this section, we look briefly at a practical problem involving coupled-coils – that of breaking the primary side circuit while it is carrying current. This turns out to be a stressful operation for the load connected at the secondary side and specific design measures are incorporated in the load circuit to protect it against failure under this condition.

15.14.1 Input Impedance Function and Transfer Function of a Two-Winding Transformer

Consider a two-winding transformer with *zero winding resistances*. Let the self-inductance of primary and secondary windings be L_p and L_s . Let n be the ratio of secondary turns to primary turns. Let the secondary be terminated in a load circuit that has an input impedance of $Z(s)$ (Fig. 15.14-2).

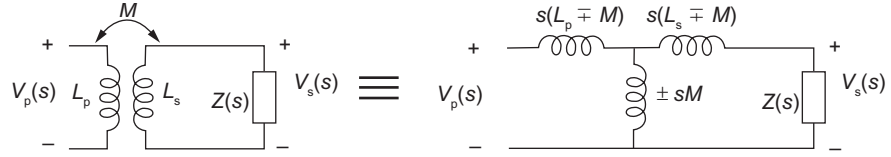


Fig. 15.14-2 A Passively Terminated Two-Winding Transformer (Winding Resistances Negligible)

We do not specify the relative polarity. Instead, we cover both options by using proper signs for the inductance values in the equivalent circuit shown.

Now the primary-side input impedance function $Z_p(s)$ can be derived as below.

Input impedance of a passively terminated two-winding transformer with negligible winding resistance.

$$\begin{aligned} Z_p(s) &= s(L_p \mp M) + \frac{[s(L_s \mp M) + Z(s)] \times [\pm sM]}{s(L_s \mp M) + Z(s) \pm sM} \\ &= s(L_p \mp M) + \frac{[s(L_s \mp M) + Z(s)] \times [\pm sM]}{sL_s + Z(s)} \\ &= \frac{s(L_p \mp M) \times sL_s + s(L_p \mp M) \times Z(s) \pm s(L_s \mp M) \times sM \pm sMZ(s)}{sL_s + Z(s)} \\ &= \frac{s^2(L_p L_s - M^2) + sL_p Z(s)}{sL_s + Z(s)} \end{aligned}$$

We observe that this impedance function is independent of the relative polarity of windings.

Now, assume that the coupling coefficient is unity. Thus, for a perfectly coupled two-winding transformer with zero winding resistances, the input impedance function is given by,

Input impedance of a passively terminated two-winding transformer with negligible winding resistance and perfect magnetic coupling.

$$\begin{aligned} Z_p(s) &= \frac{sL_p Z(s)}{sL_s + Z(s)} = \frac{sL_p Z(s)}{s(n^2 L_p) + Z(s)} \left(\because \frac{L_s}{L_p} = n^2 \right) \\ &= \frac{sL_p [Z(s) / n^2]}{sL_p + [Z(s) / n^2]} = L_1 // Z(s) \end{aligned}$$

This impedance is shown in Fig. 15.14-3.

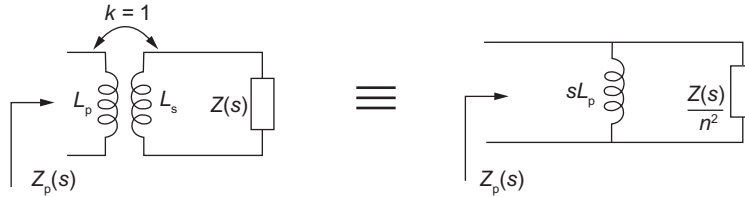


Fig. 15.14-3 Reflection of Secondary Impedance Function to Primary with $k = 1$

Now, we can incorporate the primary winding resistance r_p as an extra impedance in series at the input and the secondary winding resistance r_s as a part of the load.

We arrive at the primary side impedance function as $Z_p(s) = r_p + sL_p // \left(\frac{r_s + Z(s)}{n^2} \right)$ as shown in Fig. 15.14-4.

Now, we look at the voltage transfer function of a two-winding transformer. Let $I_p(s)$ and $I_s(s)$ be the Laplace transforms of the primary and secondary mesh currents in the s -domain equivalent circuit shown in Fig. 15.14-2. Then the mesh equations in matrix form is

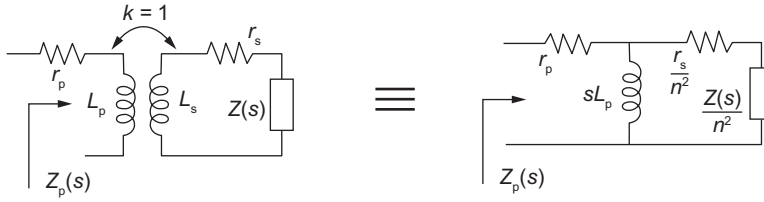


Fig. 15.14-4 Primary-side Input Impedance in a Perfectly Coupled Transformer

$$\begin{bmatrix} sL_p & \mp sM \\ \mp sM & sL_s + Z(s) \end{bmatrix} \begin{bmatrix} I_p(s) \\ I_s(s) \end{bmatrix} = \begin{bmatrix} V_p(s) \\ 0 \end{bmatrix}$$

$$\text{Solving for } I_s(s), I_s(s) = \frac{\pm sMV_p(s)}{s^2(L_pL_s - M^2) + sL_pZ(s)}$$

$$\text{Since } V_s(s) = Z(s) \times I_s(s),$$

$$\frac{V_s(s)}{V_p(s)} = \frac{\pm sMZ(s)}{s^2(L_pL_s - M^2) + sL_pZ(s)} = \frac{\pm sMZ(s)}{s(L_pL_s - M^2) + L_pZ(s)} \text{ for } r_p = r_s = 0. \quad (15.14-1)$$

$L_pL_s = M^2$ for a perfectly coupled transformer (i.e., $k = 1$). Therefore, for a perfectly coupled transformer with $r_p = r_s = 0$, the voltage transfer function is given by:

$$\frac{V_s(s)}{V_p(s)} = \frac{\pm sMZ(s)}{sL_pZ(s)} = \frac{M}{L_p} = \pm \frac{\sqrt{L_pL_s}}{L_p} = \pm \sqrt{\frac{L_s}{L_p}} = \pm n \text{ for } k=1, r_p = r_s = 0 \quad (15.14-2)$$

Equation 15.14-2 reveals that a perfectly coupled transformer with zero winding resistances generates an exact scaled replica of the applied primary voltage waveform across its secondary winding quite independent of the shape of that waveform or the type of load connected to the secondary. The scaling factor involved is the turns-ratio n defined as the ratio of secondary winding turns to primary winding turns. The $\pm$ sign in Eqn. 15.14-2 indicates that there is a polarity inversion in secondary voltage with respect to primary voltage for one relative polarity in windings.

Various factors prevent a practical transformer from meeting this ideal. Winding resistances and imperfect coupling are two such factors. If the mesh equations are written again with r_p and r_s included, the transfer function can be shown as

$$H(s) = \frac{V_s(s)}{V_p(s)} = \frac{\pm sMZ(s)}{s^2(L_pL_s - M^2) + s(L_pZ(s) + r_sL_p + r_pL_s) + r_p(r_s + Z(s))} \quad (15.14-3)$$

We carry out further analysis of the transfer function in Eqn. 15.14-3 for a special load – resistive load. Let $Z(s) = R$. Then,

$$H(s) = \frac{V_s(s)}{V_p(s)} = \frac{\pm sMR}{s^2(L_pL_s - M^2) + s(L_pR + r_sL_p + r_pL_s) + r_p(r_s + R)} \quad (15.14-4)$$

This is a second order band-pass function (note the s term in the numerator polynomial).

We express this transfer function as the product of the ideally expected value ($= \pm n$) and a factor involving s as follows.

$$H(s) = \frac{V_s(s)}{V_p(s)} = \frac{\pm M}{L_p} \times \frac{s}{s^2 \left(\frac{(1-k^2)L_s}{R} \right) + s \left(1 + \frac{r_s}{R} + \frac{r_p}{R} \frac{L_s}{L_p} \right) + \frac{r_p}{L_p} \left(\frac{r_s}{R} + 1 \right)} \quad (15.14-5)$$

Ideal Transformer

A perfectly coupled transformer is called an ideal transformer if L_p , L_s and M have infinitely high values.

An ideal transformer can model practical iron-cored transformers approximately. Such an ideal transformer will have input impedance that is a turns-ratio ratio transformed version of the secondary load impedance.

We have discussed the application of such transformers in impedance matching applications in Chap. 8.

Voltage Transfer Function of a passively terminated transformer with negligible winding resistance and perfect coupling.

Voltage Transfer Function of a passively terminated transformer with negligible winding resistance.

Voltage Transfer Function of a passively terminated transformer with winding resistance included.

Voltage Transfer Function of a resistively terminated and imperfectly coupled transformer with winding resistance included. It is a band-pass function.

$$M = k\sqrt{L_p L_s}. \text{ Therefore, } \frac{M}{L_p} = k\sqrt{\frac{L_s}{L_p}} = nk. \text{ Therefore,}$$

$$H(s) = \pm n \times k \times \frac{s}{s^2 \left(\frac{(1-k^2)L_s}{R} \right) + s \left(1 + \frac{r_s}{R} + \frac{r_p}{R} \frac{L_s}{L_p} \right) + \frac{r_p}{L_p} \left(\frac{r_s}{R} + 1 \right)} \quad (15.14-5)$$

We want to determine the low frequency cut-off and high frequency cut-off for the sinusoidal steady-state frequency response function of this transfer function. In general, these two frequencies will be some complex functions of k, n, r_s, r_p, R and L_p . But, we can use some approximations in the case of practically important iron-cored transformer.

In an iron-cored transformer the coupling coefficient is nearly unity. The load resistance is usually much larger than the transformer winding resistances. Hence, the effect of leakage flux and the effect of winding resistance are only second-order effects. *The aggregate result of the two second-order effects can be taken as the product of effects when each is acting alone.* Therefore, when we consider the effect of winding resistance we assume that $k = 1$ and when we consider the effect of imperfect coupling we assume that $r_p = r_s = 0$.

$$H(s) (\text{with } k = 1) = \pm n \frac{s}{s \left(1 + \frac{r_s}{R} + \frac{r_p}{R} \frac{L_s}{L_p} \right) + \frac{r_p}{L_p} \left(\frac{r_s}{R} + 1 \right)}.$$

Further algebraic manipulation of the above expression leads to,

$$H(s) = \pm n \times \frac{1}{1 + \frac{r_s}{R} + n^2 \frac{r_p}{R}} \times \frac{s}{s + \frac{r_p \left(1 + \frac{r_s}{R} \right)}{L_p \left(1 + \frac{r_s}{R} + n^2 \frac{r_p}{R} \right)}} = n \times \beta \times \frac{s}{s + \alpha} \quad (15.14-7)$$

β is a voltage reduction factor due to part of the applied voltage getting dropped across the winding resistances. $-\alpha$ is the pole of the first order transfer function. The transfer function is seen to be a first order high-pass function. The DC gain (evaluated by substituting $s = j0$) – i.e., the ratio between the steady-state response under DC conditions to the DC input value – is zero. See Example 15.13-1 and Fig. 15.13-2 for the frequency response function plots of the function $s/(s + \alpha)$. The upper cut-off frequency is

$$\alpha \text{ rad/s, where } \alpha = \frac{r_p \left(1 + \frac{r_s}{R} \right)}{L_p \left(1 + \frac{r_s}{R} + n^2 \frac{r_p}{R} \right)} \approx \frac{r_p}{L_p} \text{ for practical values of } R.$$

We observe that both the voltage reduction factor β and the cut-off frequency α can be obtained from the equivalent circuit shown in Fig. 15.14-4 with $Z(s)$ replaced with R .

The high frequency behaviour of a tightly coupled transformer is studied after neglecting the winding resistances. Then, Eqn. 15.14-6 simplifies to

$$H(s) = \pm n \times k \times \frac{1}{s \left(\frac{(1-k^2)L_s}{R} \right) + 1} = \pm n \times k \times \frac{\frac{R}{(1-k^2)L_s}}{s + \frac{R}{(1-k^2)L_s}} = \pm n \times k \times \frac{\lambda}{s + \lambda} \quad (15.14-8)$$

Waveform Distortion in Transformers

Coil resistances prevent a two-winding transformer from passing DC voltage and low frequency AC voltage from primary to secondary.

Hence, a signal containing low frequency content will suffer waveform distortion when it goes through a transformer.

Imperfect coupling ($k < 1$) prevents a transformer from passing high frequency AC voltage from primary to secondary. The two-winding transformer exhibits low-pass behaviour in the high-frequency end due to this.

Hence, a signal containing high frequency content will suffer waveform distortion when it goes through a transformer.

This is a low-pass transfer function with a cut-off frequency of

$$\lambda = \frac{R}{(1-k^2)L_s} = \frac{R/n^2}{(1-k^2)L_p} \text{ rad/s.}$$

The product of right-sides of Eqn. 15.14-7 and Eqn. 15.14-8 provides an approximation for Eqn. 15.14-6 in the case of a tightly coupled two-winding transformer with winding resistances very small compared to load resistance. Thus,

$$H(s) = \underbrace{\pm n}_{\text{Ideal gain expected}} \times \underbrace{k}_{\text{voltage-drop factor due to imperfect magnetic coupling}} \times \underbrace{\beta}_{\text{voltage-drop factor due to winding resistances}} \times \underbrace{\frac{s}{s+\alpha}}_{\text{high-pass factor}} \times \underbrace{\frac{\lambda}{s+\lambda}}_{\text{low-pass factor}}. \quad (15.14-9)$$

Approximate voltage transfer function of a resistively terminated transformer.

This transfer function has a band-pass frequency response function with lower cut-off frequency of α rad/s and upper cut-off frequency of λ rad/s. Primary winding resistance and primary winding inductance decide the lower cut-off frequency. The upper cut-off frequency is decided by the coupling coefficient, the primary winding inductance and the load resistance.

Lower the winding resistances and higher the coupling coefficient, higher will be the bandwidth of the transformer. Bandwidth of a transformer is load dependent.

15.14.2 Flux Expulsion by a Shorted Coil

One does not make a good transformer and keep its secondary shorted. However, there are situations and applications in which the electrical system can be modelled as a transformer with a low resistance secondary winding kept shorted. Therefore, we look at the input impedance of a transformer with secondary shorted as in Fig. 15.14-5.

$Z_p(s)$ may be determined by mesh analysis. The result will be,

$$Z_p(s) = \frac{s^2 L_p L_s (1-k^2) + s(L_p r_s + L_s r_p) + r_p r_s}{r_s + s L_s}. \quad (15.14-10)$$

The impedance at DC frequency is $r_p \Omega$. The impedance at high frequency is $\approx s(1-k^2)L_p$ indicating that the transformer behaves as an inductance of $(1-k^2)L_p$ at high frequencies when its secondary is kept shorted.

Assume that the shorted coil has negligibly small resistance. Taking $r_s = 0$, we get,

$$Z_p(s) = r_p + s(1-k^2)L_p.$$

Now, if the transformer is perfectly coupled, (i.e., $k = 1$), then the input impedance becomes equal to primary winding resistance at all frequencies. Hence, the short-circuit impedance of a good transformer with $k = 1$, $r_p = r_s = 0$ is zero at all frequencies.

The expressions for $I_p(s)$ and $I_s(s)$ with $r_s = 0$ may be derived as

$$I_p(s) = \frac{V_p(s)}{s(1-k^2)L_p + r_p} \quad \text{and} \quad I_s(s) = \frac{\frac{M}{L_s} V_p(s)}{s(1-k^2)L_p + r_p} = \frac{k}{n} I_p(s).$$

Currents entering the dots generate positive flux linkages in coils. Let ψ_p be the flux linkage in primary coil and ψ_s be the flux linkage in the secondary coil. Then,

$$\begin{aligned} \psi_p(s) &= L_p I_p(s) - M I_s(s) \\ &= L_p I_p(s) - \frac{k}{n} M I_s(s) \\ &= L_p I_p(s) - \frac{k^2}{n} \sqrt{L_p L_s} I_s(s) \\ &= L_p I_p(s) [1 - k^2] \end{aligned}$$

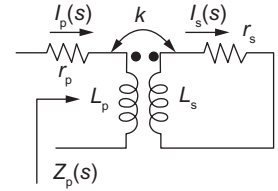


Fig. 15.14-5 A Transformer with Secondary Shorted

$$\psi_s(s) = -L_s I_s(s) + M I_p(s) = \frac{-M V_p(s)}{s(1-k^2)L_p + r_p} + \frac{M V_p(s)}{s(1-k^2)L_p + r_p} = 0.$$

Therefore, the secondary coil that started with zero flux linkage continues to hold zero flux linkage at all times after that. Thus,

A shorted coil with zero resistance expels flux completely at all frequencies in the magnetic path where it is located.

It does this by allowing suitable current flow in it in order to cancel the mutual flux generated by current in the driven coil. Both primary and secondary currents rise

exponentially with a time constant of $\frac{(1-k^2)L_p}{r_p}$ s in step response.

But what if the windings are perfectly coupled in addition? Then,

$$I_p(s) = \frac{V_p(s)}{r_p} \text{ and } I_s(s) = \frac{1}{n} \frac{V_p(s)}{r_p}.$$

If $v_p(t)$ is a unit step function, the current rises instantaneously from 0 to $1/r_p$ in the primary winding and from 0 to $1/nr_p$ in the secondary winding. And the flux linkages in both windings continue to be at zero.

There was no impulse voltage in the circuit. How was it possible for the inductor currents to change instantaneously without impulse voltages in the circuit?

Currents in individual coils can change instantaneously in a coupled-coil system. Faraday's law indicates that the induced e.m.f. across a coil is proportional to the rate of change of flux linkage in it. Therefore, flux linkage of a coil can not change instantaneously unless impulse voltage is applied to the coil. This is equivalent to the statement that current in an inductor can not change instantaneously unless impulse voltage is applied in the case of inductors that are not magnetically coupled. However, there is a distinction between instantaneous change of flux linkage and current if the flux linkage in a coil can be produced by more than one current as in the case of coupled inductors. Therefore, the more general principle of 'no instantaneous change in flux linkage in a coil unless impulse voltage is applied' should be applied in the case of coupled-coils.

The flux linkage in a shorted coil with zero resistance remains constant in time. This statement is sometimes referred to as '*Constant Flux Linkage Theorem*'.

Note that in the step response of a perfectly coupled transformer with shorted secondary ($r_s = 0$), the flux linkage of secondary winding remained at zero though the current increased instantaneously. The magnetic coupling was perfect, and the primary winding can not have flux linkage in it in that case unless secondary winding has it. Therefore, the primary winding flux linkage also remained at zero despite step change in primary current.

We may derive a relationship between the instantaneous changes in primary and secondary windings as below.

$$\begin{aligned} \psi_p(t) &= L_p i_p(t) - M i_s(t) \\ \psi_s(t) &= -L_s i_s(t) + M i_p(t). \text{ Therefore,} \\ \Delta \psi_p(t) &= L_p \Delta i_p(t) - M \Delta i_s(t) \\ \Delta \psi_s(t) &= L_s \Delta i_s(t) - M \Delta i_p(t) \end{aligned}$$

But, there can not be instantaneous changes in flux linkages in the absence of impulse voltages. Hence, $\Delta \psi_p(t)$ and $\Delta \psi_s(t)$ are zero. Therefore,

$$\begin{bmatrix} L_p & -M \\ M & -L_s \end{bmatrix} \begin{bmatrix} \Delta i_p \\ \Delta i_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of the matrix on the left-side is non-zero for all $k < 1$. Therefore, Δi_p and Δi_s can only be zero with imperfect coupling. That is, there can be no instantaneous

How can
currents in inductive
coils change
instantaneously?

Constant Flux
linkage Theorem.

change in coil currents in a two-coil system if the coils are imperfectly coupled and there is no impulse voltage in the circuit.

However, if $k = 1$, the determinant of the square matrix in the left-side of the equation is zero and there can be a non-zero solution for Δi_p and Δi_s . In fact, any pair of values that satisfy the constraint that $\Delta i_s = \Delta i_p/n$ will be permitted. The exact value by which the primary current jumps will be decided by the jump in primary voltage and the primary resistance. Since the flux linkage in primary winding does not change, all the primary voltage will have to be absorbed by the primary resistance at all t .

Its resistance compromises the effectiveness of flux expulsion from shorted coil. The expression for flux linkage in shorted coil when $k \neq 1$ and $r_s \neq 0$ is

$$\frac{\psi_s(s)}{V_p(s)} = \frac{Mr_s}{s^2(1-k^2)L_pL_s + s(L_p r_s + L_s r_p) + r_p r_s}.$$

The DC value of this ratio is M/r_p . Therefore, DC flux will not be expelled under steady-state. Similarly, low frequency AC flux will also manage to get into the shorted coil under steady-state conditions. However, the ratio goes to low values at high frequency. Therefore, non-zero resistance in shorted coil results in DC and low frequency fluxes penetrating into the coil. High frequency flux is expelled more or less effectively.

The principle of flux expulsion detailed in this sub-section is employed in *shielding* sensitive electronic equipment from electromagnetic interference.

Flux Expulsion by Shorted Coils

A shorted coil with zero resistance expels flux completely at all frequencies in the magnetic path where it is located.

Currents in individual coils can change instantaneously in a coupled-coil system with perfect coupling.

Non-zero resistance in shorted coil makes flux-expulsion partial at DC and low frequencies.

15.14.2 Breaking the Primary Current in a Transformer

Let I_p and I_s be the currents in the primary and secondary windings of the transformer at the instant at which the switch S is opened in the circuit shown in Fig. 15.14-6.

Let us denote the instant of switching as $t = 0$. Then, the flux linkages in the windings just before switching are:

$$\begin{aligned}\psi_p(0^-) &= L_p I_p - M I_s \\ \psi_s(0^-) &= -L_s I_s + M I_p\end{aligned}$$

and the flux linkages at $t = 0^+$ are

$$\begin{aligned}\psi_p(0^+) &= L_p 0 - M i_s(0^+) = -M i_s(0^+) \\ \psi_s(0^+) &= -L_s i_s(0^+) + M 0 = -L_s i_s(0^+).\end{aligned}$$

Since there is no impulse in the closed secondary loop, the flux linkage in the secondary winding will have to be continuous. Therefore,

$$\psi_s(0^+) = \psi_s(0^-) \Rightarrow -L_s i_s + M I_p = -L_s i_s(0^+)$$

Therefore, $i_s(0^+) = I_s - \frac{M}{L_s} I_p = I_s - \frac{I_p}{n}$. This current is usually in a direction opposite to that of $i_s(0^-)$. The corresponding flux linkage in primary winding at $t = 0^+$ is

$$\psi_p(0^+) = -M i_s(0^+) = -M I_s + \frac{M^2}{L_s} I_p. \text{ The initial flux linkage in this coil was } \psi_p(0^-) = -L_p I_p - M I_s.$$

Therefore, the instantaneous decrease in flux linkage that took place in this coil is

$$(L_p I_p - M I_s) - \left(M I_s + \frac{M^2}{L_s} I_p \right) = I_p \left(L_p - \frac{M^2}{L_s} \right) = (1 - k^2) L_p I_p.$$

Hence, an impulse voltage of area content equal to $(1 - k^2) L_p I_p$ volt-sec will appear across the primary winding and will have to be absorbed by the switch. In fact, there will be arcing across the switch due to this very large voltage trying to establish across it.

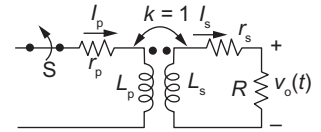


Fig. 15.14-6 Breaking the Primary Current in a Transformer

But arcing involves energy. Where does the energy come from? It is possible to show that the energy that gets dissipated across the switch is $0.5(1 - k^2)L_p I_p^2$ J. This comes from the magnetic energy storage in the coupled coil system.

The sudden change of secondary current from I_s to $I_s - \frac{I_p}{n}$ results in a sudden change across the load voltage from RI_s to $RI_s - \frac{RI_p}{n}$. There are practical applications in which the second term dominates and makes the load voltage a negative voltage with enough magnitude to destroy the load if the load happens to be sensitive electronic equipment. This large negative voltage decays exponentially with a time constant of L_s/R s. By the time the transient is over, the remaining initial magnetic energy stored in the coil system would have got dissipated in R .

15.15 SUMMARY

- Let $v(t)$ be a right-sided function that is bounded by $Me^{\alpha t}$ with some finite value of M and α . Then the Laplace transform pair is defined as

$$V(s) = \int_0^\infty v(t)e^{-st} dt \quad \text{The analysis equation}$$

$$v(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} V(s)e^{st} ds \quad \text{The synthesis equation,}$$

where $s = \sigma + j\omega$ is the complex frequency variable standing for the complex exponential function e^{st} with σ value $\geq \alpha$. The ROC of $V(s)$ is the entire plane to the right of $\text{Re}(s) = \alpha$ line.

- Laplace transform expands a transient right-sided time-function in terms of infinitely many complex exponential functions of infinitesimal amplitudes. The ROC of such a Laplace transform will include the right-half of s -plane and hence the time-domain waveform gets constructed by the growing complex exponential functions.
- For a linear time-invariant circuit described by

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y =$$

$$b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$$

the ratio of rational polynomials in s defined as

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

has three interpretations.

1. It may be viewed as a generalised frequency response function. Its magnitude gives the ratio between the amplitudes of the output complex exponential function and input complex exponential function when the input is of the form Ae^{st} . Its angle gives the phase angle by which the output complex exponential function leads the input complex exponential function.

2. It is also the ratio of Laplace transform of zero-state response to Laplace transform of input source function called 'The s -domain System Function'.

3. Further, it is the Laplace transform of Impulse Response.

- Laplace transforms can be inverted by the method of partial fractions.
- Laplace transformation of both sides of a linear constant-coefficient ordinary differential equation converts it into an algebraic equation on transforms. Thus, Laplace transform affords a convenient way to solve such differential equations with initial conditions.
- All linear circuit elements have s -domain equivalents. The s -domain equivalent of the complete circuit can be constructed by replacing each element with its s -domain equivalent. KVL and KCL are directly applicable to the transformed quantities.
- The s -domain equivalent circuits may be analysed by nodal analysis or mesh analysis procedures. All circuit theorems developed in the context of memoryless circuits are applicable to the s -domain equivalent circuits.
- The s -domain System Function is also called the *Network Function*. Immittance functions, transfer functions and transfer immittance functions are three classes of network functions usually employed in circuit analysis. Those complex frequency values at which a network function goes to ∞ are called its *poles* and those complex frequency values at which the network function goes to zero are called its *zeros*. The following points have to be kept in mind in the context of network functions.
 1. The order of a network function and the order of the circuit can be different due to possible pole-zero cancellations in a particular network function.

2. Poles of any network function defined in a linear time-invariant circuit will be natural frequencies of the circuit.
 3. All natural frequencies need not be present as poles in all network functions defined in that circuit.
 4. However, all natural frequencies will appear as poles in some network function or other.
 5. Thus, poles of a network function is a sub-set of natural frequencies of the circuit and the natural frequencies will be a union-set of poles of all possible network functions in the circuit.
 6. A complex frequency that is not a natural frequency of the circuit cannot appear as a pole in any network function in that circuit.
 7. Both the denominator polynomial and the numerator polynomial of a network function in a linear time-invariant circuit have real coefficients. Therefore, poles and zeros of a network function will either be real-valued or will occur in complex conjugate pairs.
- Pole-zero plot along with a gain factor K will specify a network function uniquely. The impulse response of the network function may be obtained from its pole-zero plot. The poles decide the number of terms in the impulse response and their complex frequencies. The zeros along with the poles and a gain factor K decide the amplitude of each impulse response term.
 - A network function is a stable one if its impulse response decays to zero with time. This is equivalent to stating that its impulse response must be absolutely integrable, i.e., $\int_0^\infty |h(t)| dt$ must be finite. Therefore, a network function is stable if all the impulse response terms are damped ones. That is, all the poles must have negative real values or complex values with negative real parts. Therefore, a network function is stable if and only if all its poles are in the left-half of s -plane excluding the $j\omega$ -axis.
 - Sinusoidal steady-state frequency response function $H(j\omega)$ can be obtained by evaluating $H(s)$ on $j\omega$ -axis. This evaluation can also be carried out in a graphical manner.
 - The DC steady-state gain of a network function is $H(0)$.
 - Important properties of Laplace transforms are summarised in Table 15.15.1

Table 15.15-1 Some Important Properties of Laplace Transforms

Signal/property	Laplace transform property
$v(t) = f(t) u(t)$	$V(s)$ – Definition
$av_1(t) + bv_2(t)$	$aV_1(s) + bV_2(s)$ – Linearity
$v(t - t_d)$	$e^{-st_d} V(s)$ – Time-shifting
$e^{s_0 t} v(t)$, s_0 is a complex number	$V(s - s_0)$ – Multiplication by a complex exponential in time-domain – Frequency Shifting
$\frac{dv(t)}{dt}$	$sV(s) - v(0^-)$ Time-domain differentiation
$\int_0^t v(t) dt$	$\frac{1}{s} V(s)$ – Time-domain integration
$-t v(t)$	$\frac{dV(s)}{ds}$ – Frequency-domain differentiation
$\frac{v(t)}{t}$	$\int_s^\infty V(s) ds$
$v_1(t) * v_2(t)$	$V_1(s) V_2(s)$ – Time-domain convolution
$v(0^+)$	$\lim_{s \rightarrow \infty} sV(s) = v(0^+)$, if the limit exists
$v(\infty)$	$\lim_{s \rightarrow 0} sV(s) = v(\infty)$, if the limit exists and all poles of $sV(s)$ are in left-half of s -plane

15.16 PROBLEMS

- Evaluate the voltage across the capacitor in a series RC circuit with $R = 1 \text{ k}\Omega$ and $C = 1000 \text{ }\mu\text{F}$ at $t = 0^+$ and $t = 20 \text{ s}$ if the applied voltage to the circuit is (a) $2 e^{0.01t} \cos t \text{ V}$ (b) $2 e^{0.01t} \cos t u(t) \text{ V}$. Use differential equation approach.
- A current source with $i_s(t) = f(t) u(t) \text{ A}$ is applied to a parallel RL circuit with $R = 1 \text{ }\Omega$ and $L = 1 \text{ H}$. The voltage across the combination is found to be $= 2e^{-t} \sin t u(t) \text{ V}$. Find $f(t)$ and the initial current in the inductor. Do not use the Laplace transform technique.
- The output variable y in a linear time-invariant circuit is related to the input variable x by the following differential equation $\frac{d^3 y}{dt^3} + 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \frac{dx}{dt} + x$. Its zero-input response is found to contain $e^{-0.5t}$ term among other terms. If $x(t) = 3e^{0.01t} \sin t$, find the instantaneous value of y at $t = 10 \text{ s}$. Do not use Laplace transform technique.
- A voltage source of $v_s(t) = 2e^{-0.2t} \cos(t - 45^\circ) u(t) \text{ V}$ is applied to a series RLC circuit with $R = 1 \text{ }\Omega$, $L = 1 \text{ H}$ and $C = 1 \text{ F}$. The circuit is initially relaxed. Determine the total response of current in the circuit by solving the circuit differential equation without using Laplace transforms.
- A bounding exponential $Me^{\alpha t}$ is to be determined for each of the functions listed below. Find the minimum value of α and the corresponding value of M for each. (i) $tu(t)$ (ii) $u(t) - u(t - 2)$ (iii) $e^{3t} u(t)$ (iv) $e^{-3t} u(t)$ (v) $e^{3t} u(t - 3)$ (vi) $e^{-3t} u(t - 2)$ (vii) $e^{-2t} \cos 2t u(t)$ (viii) $e^{2t} \cos 2t u(t)$
- The signals listed in Problem 5 are applied to a parallel RL circuit with $R = 2 \text{ }\Omega$ and $L = 1 \text{ H}$ as current sources. Find the instantaneous voltage across the combination at $t = 2 \text{ s}$ in each case by using Laplace transforms.
- The Laplace transform of impulse response of a linear time-invariant circuit is given by $H(s) = \frac{2}{s^2 + 3s + 2}$. (i) Find the differential equation describing the circuit, assuming that the output variable is y and input variable is x . (ii) Find the total response of the circuit if $x(t) = 3u(t)$ with $y(0^-) = 1$ unit and $y'(0^-) = 1$ unit/s by Laplace transform technique.
- The Laplace transform of current drawn by an initially relaxed dynamic circuit from unit impulse voltage source is a $I(s) = \frac{2}{s^2 + 3s + 2}$. (i) Find the differential equation relating the current drawn by the circuit to the voltage applied. (ii) Find the total response of current if the circuit is initially relaxed and $v_s(t) = 2e^{-0.5t} \cos 2t u(t) \text{ V}$ is applied to it by using Laplace transforms.
- Find the total response of current in the circuit in Problem 8 if the circuit is initially relaxed and $v_s(t) = 2e^{-0.5t} \cos 2t u(t - 0.5) \text{ V}$ is applied to it.
- Let $f(t)$ be a periodic waveform with a period of $T \text{ s}$. Let $v(t) = f(t) u(t)$ and $v_p(t) = f(t) [u(t) - u(t - T)]$. That is, $v(t)$ is the right-side of a periodic waveform and $v_p(t)$ is one period of $f(t)$. Develop an expression for Laplace transform of $v(t)$ in terms of Laplace transform of $v_p(t)$ using time-shifting theorem. What is the ROC of Laplace transform of $v(t)$?
- Find the Laplace transforms of (i) a symmetric square wave of unit amplitude and unit period (ii) a rectangular pulse waveform of unit amplitude with the first pulse located between 0 s and 0.5 s and pulses repeating every 2 s using the result derived in Problem 10.
- Solve the system of differential equation $\frac{di_1}{dt} + 2i_1 - i_2 = v_s$ and $\frac{di_2}{dt} + 3i_2 - i_1 = 0$ for $v_s(t) = 2\sin 3t u(t) \text{ V}$ with $i_1(0^-) = 1$ and $i_2(0^-) = 1$.
- If $\frac{d^3 y}{dt^3} + 2 \frac{d^2 y}{dt^2} = 0$ and $y(0^-) = 0$, $y'(0^-) = 1$ and $y''(0^-) = -1$, find $y(t)$ by using Laplace transform technique.
- Let $x(t) = 3tu(t)$ and $y(t) = 2u(t)$. Find $x(t)*y(t)$ by inverting $X(s)Y(s)$ and verify by time-domain convolution.
- The impulse response of a linear time-invariant circuit is $2e^{-0.05t} u(t)$. Find the zero-state response when input is $3e^{-0.1t}$ by convolution theorem on Laplace transforms.
- Let $x(t) = t[u(t) - u(t - 2)]$ and $y(t) = [u(t) - u(t - 2)]$. Find $x(t)*y(t)$ by inverting $X(s)Y(s)$ and verify by time-domain convolution.
- Find the Laplace transform of (i) $\cos \pi t [u(t) - u(t - 2)]$ (ii) $2\sinh 0.2t [u(t) - u(t - 1)]$ by using shifting theorem.
- Find the Laplace transform of (i) $2t^3 e^{-t} u(t)$ (ii) $\frac{e^{-3t} - e^{-t}}{t} u(t)$
- Let $V(s) = \frac{4}{s^2 + 4}$. (i) Find and its inverse transform. (ii) Find its derivative and its inverse transform.
- Using Laplace transforms, find the value of R in the circuit in Fig. 15.16-1 such that the damping factor of the circuit for voltage input is 0.2 . Find the step response for $v_o(t)$ with this R and verify the initial value and final value theorems on step response.

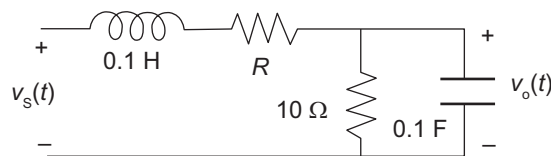


Fig. 15.16-1

- (i) Obtain the input impedance function and input admittance function for the circuits shown in Fig. 15.16-2 and prepare pole-zero plots for these immittance functions. (ii) Determine the zero-state input current as a function of time when input is $u(t) \text{ V}$ and verify initial value and final value theorems in each case. (iii) Determine the zero-state input voltage as a function of time when input is $u(t) \text{ A}$. All circuit elements have unit values.

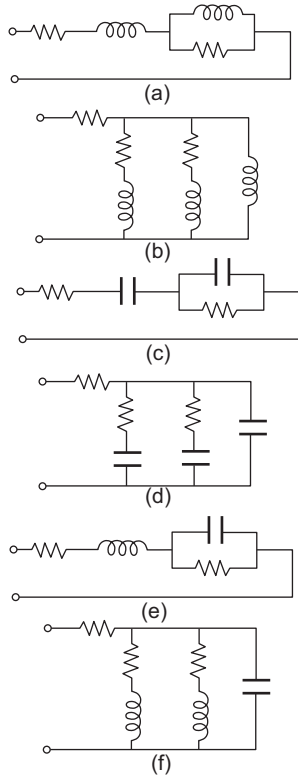


Fig. 15.16-2

22. (a) Find the voltage transfer function $V_o(s)/V_s(s)$ and driving-point impedance function in the circuit in Fig. 15.16-3. (b) Prepare the pole-zero plot for both network functions. (c) Determine the step response for $v_o(t)$ and $i_s(t)$. (d) Verify initial value theorem and final value theorem on Laplace transforms in the case of $v_o(t)$ and $i_s(t)$.

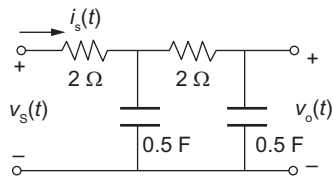


Fig. 15.16-3

23. The initial current in the inductor is 0.5 A and the initial voltage across the capacitor is 1 V in the circuit in Fig. 15.16-4. A single

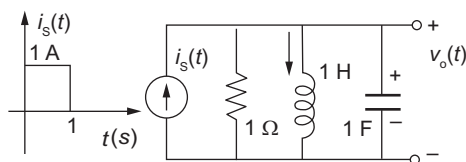


Fig. 15.16-4

rectangular pulse of current is applied to the circuit as shown in the figure. Solve for $v_o(t)$ by s -domain equivalent circuit method.

24. (i) Show that the voltage transfer function in the circuit in Fig. 15.16-5 is a real number if $R_1 C_1 = R_2 C_2$. (ii) Obtain the input impedance function with $R_1 C_1 = R_2 C_2$.

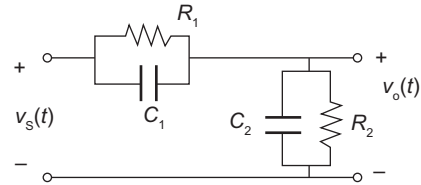


Fig. 15.16-5

25. The impulse response of i_x in the circuit in Fig. 15.16-6 contains a real exponential term that has a time constant of 1.755 s. (i) Show the pole-zero plots for $I_x(s)$ and $V_o(s)$ when $v_s(t) = u(t)$ V and the circuit is initially relaxed. (ii) Find $i_x(t)$ and $v_o(t)$ for $t \geq 0^+$ if $v_s(t) = u(t)$ and both capacitors have 1 V across them with bottom plate positive at $t = 0^-$ and inductor has zero current at $t = 0^-$.

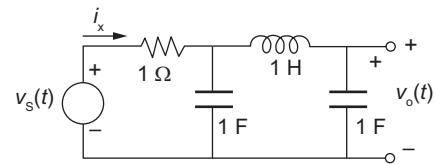


Fig. 15.16-6

26. The impulse response of input current in the circuit in Fig. 15.16-7 contains a $(1/6)\delta(t)$ component. (i) Find the value of R . (ii) Find the driving-point impedance function and show its pole-zero plot. (iii) Find the time-function describing the current delivered by source for $t \geq 0^+$ if $v_s(t) = 2\cos(2t + 30^\circ)u(t)$, $i_1(0^-) = 1$ A and $i_2(0^-) = -1$ A.

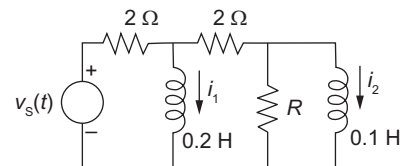


Fig. 15.16-7

27. Find the zero-state response for $v_x(t)$ by nodal analysis in s -domain in the circuit in Fig. 15.16-8 if $v_{s1}(t) = 2u(t)$ V and $v_{s2}(t) = 2e^{-t}u(t)$ V.

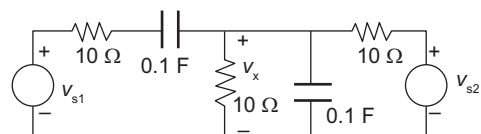


Fig. 15.16-8

flux linkages in coils and the total energy stored in the transformer under step response steady-state condition?

37. The transformer used in the circuit in Fig. 15.16-13 is the same as the one in Problem 35. Switch was in closed position for a long time prior to $t = 0$ and it is opened at $t = 0$. (i) Find and plot $i_2(t)$, $v_o(t)$ and $v_s(t)$ for $t \geq 0^+$ (ii) Calculate the energy dissipated in the switch and the $2\ \Omega$ resistance after at $t = 0$.

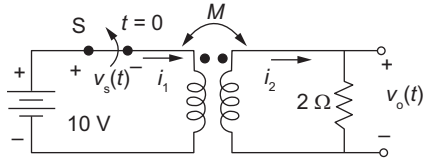


Fig. 15.16-13

38. Two coupled coils can be connected in series in two ways. The connection in Fig. 15.16-14(a) is *differential series* and connection in Fig. 15.16-14(b) is *cumulative series*. Find the equivalent inductance between A and B in both cases.

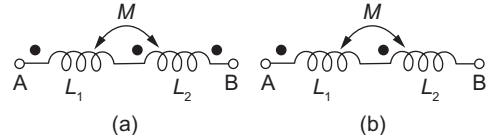


Fig. 15.16-14

39. Find the equivalent inductance between A and B in the two possible parallel connections of two coupled inductors in Fig. 15.16-15.

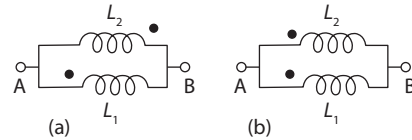


Fig. 15.16-15

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Part Six

Introduction to Network Analysis

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Two-Port Networks and Passive Filters

CHAPTER OBJECTIVES

- To employ the concepts of linear circuit analysis to arrive at efficient representations for a linear time-invariant two-port network.
- To illustrate the procedure to evaluate various parameter sets of a two-port network through a set of solved examples.
- To derive and explain various properties exhibited by parameter sets of symmetric reciprocal two-port networks.
- To apply image parameter formulation to symmetric reactive T and Π networks, and thereby, arrive at passive filter designs.
- To explain and illustrate the need for m -derived filter sections and termination sections for a passive filter.
- To illustrate passive filter design through a set of solved examples.

This chapter introduces linear time-invariant two-port networks and derives external descriptions for them. Two such descriptions – the transmission parameter description and the image parameter description – are applied in Passive Filter Design using Symmetric Reactive Two-Port T and Π Networks.

INTRODUCTION

We, at this point in our study of circuit analysis, know how to analyse a linear time-invariant circuit with initial conditions to obtain its transient and steady-state performance by employing time-domain techniques or frequency-domain techniques. Further, we know how to do it efficiently by judicious application of some powerful circuit theorems.

Of course, Circuit Theory does not end there. In fact, it is only the beginning. However, a close study of the previous 15 chapters of the text would have imparted to the reader the capability to analyse any linear time-invariant circuit of reasonable complexity.

There is nothing special about a two-port network from this point of view. It is just another linear time-invariant circuit that can be analysed with the techniques evolved in the first 15 chapters of this text.

However, evolving the special analysis procedures for two-port networks (we will see soon what they are) makes the circuit analysis procedure more efficient and more elegant.

There are practical application circuits in which a major portion of the circuit does not undergo any change in structure or in parameters and the remaining portion undergoes changes in structure and/or parameters. The circuit is to be analysed for various parameter sets. A case in point would be an electronic amplifier. Maybe we have designed an amplifier and we would like to analyse its performance (voltage gain, input impedance, bandwidth, efficiency, etc.) for different loading conditions at the output and for different driving conditions at the input. The structure, interconnections and parameters inside the amplifier do not change, but what is connected at the input and at the output of the amplifier change from one circuit analysis problem to another. In this case, should we really set up all the circuit equations for each configuration and solve the entire set of equations repeatedly for each input drive – output load configuration? Can we not abstract the circuit behaviour of the unchanging portion of the circuit (i.e., the amplifier itself) in terms of a condensed set of equations and use this condensed set of equations to solve the entire circuit repeatedly with different input and output conditions? If we could, that would make the analysis procedure efficient.

In fact, we can do precisely that, provided the unchanging portion of the circuit satisfies certain conditions. The amplifier in this discussion has a pair of terminals at which an input source can be applied. It has another terminal pair across which a load or a two-terminal load network can be connected. These are the two terminal-pairs at which the remaining portion of the integral circuit is allowed to interact with the amplifier. *We generalise these ideas to define a linear time-invariant two-port network as a linear network that contains no independent sources and has two pairs of terminals at which it can interact with the external world with no interaction permitted with the external world except through these two terminal pairs.*

The port at which the input source is customarily applied is called the input port and the port at which the output is usually measured is termed as the output port. Usually, the left side port is taken to be the input port and the right side port is taken to be the output port while drawing the circuit diagrams of two-port networks, unless specified otherwise.

The input port and the output port can have one terminal common between them. In that case it will be a three-terminal two-port network. Otherwise, it will be a four-terminal two-port network.

Note carefully that the only interaction that the definition of a two-port network permits is the interaction through the ports. This implies that the outside world can not interact with the network except through these two ports. For instance, the external circuitry should not have magnetic or electric coupling with any element inside the two-port network. Similarly, the external circuitry should not have any coupling with the two-port network through dependent sources. In fact, no kind of coupling – optical, electro-acoustic, piezo-electric etc., – is permitted between the external circuit and the two-port network. If such coupling exists, it is not possible to partition the circuit into a two-port network interacting with an external network.

The external variables i.e., the port-voltages and the port-currents of the two-port network satisfying the above conditions can be described by a condensed set of equations. In fact, there are only four variables – two voltages and two currents and there will only be two equations tying up these four variables. These two equations will describe the behaviour of the two-port network as far as its interaction with the external circuit is concerned. A detailed modelling of the internal details of the two-port network is needed just once – and that is for finding the two describing equations for the two-port network. Once these two equations are obtained, the two-port network may subsequently be replaced by this equation set for repeated analysis with varying external circuit parameters.

This chapter deals with two-port network equations and their applications to passive two-port electric filters.

Two-port Networks

A 'linear time-invariant two-port network' contains linear, lumped circuit elements and does not contain independent sources.

It has two terminal pairs identified for interaction with other circuits external to it. These two terminal pairs may be accessed in order to apply excitation source to the network or to measure the response of the network.

Each such terminal-pair at which an excitation may be applied or a response may be measured is called a 'port'.

A two-port network can interact with the external world only through the ports.

16.1 DESCRIBING EQUATIONS AND PARAMETER SETS FOR TWO-PORT NETWORKS

A linear time-invariant two-port network with all the *port variables* identified is shown in Fig. 16.1-1.

The port-currents are shown to enter the network at both ports. This is a matter of tradition and we have no reason to depart from this tradition. The two port-voltage variables and the two port-current variables are instantaneous variables and are functions of time, in general. However, they could very well be phasors or Laplace Transforms if the two-port network is drawn as a phasor equivalent circuit or an s -domain equivalent circuit, respectively.

The current entering a port through a terminal will leave through the other terminal (Why?)

Now, we consider a general network that can be viewed as a cascade of three sub-networks. The first network, N_A , drives a linear time-invariant two-port network N_B and the third network N_C loads the two-port network at its second port. Both N_A and N_C may contain independent sources and, for simplicity, we take them to be linear. We assume that (i) this cascaded network has a unique solution and (ii) N_A , N_B and N_C interact only through the terminals of interconnection. This cascade is shown in Fig. 16.1-2.

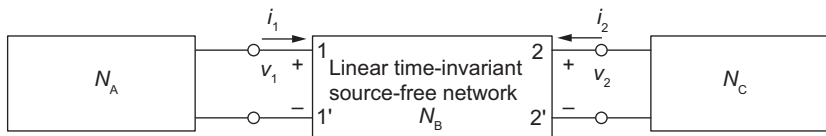


Fig. 16.1-2 A Linear Two-Port Embedded in a Cascaded Network

16.1.1 Short-Circuit Admittance Parameters for a Two-Port Network

This network cascade in Fig. 16.1-2 satisfies the conditions for applying the Substitution Theorem. We apply this theorem to replace the network N_A with an independent voltage source to arrive at the circuit shown in Fig. 16.1-3. We assume that the circuit in Fig. 16.1-3 has a unique solution. Linear circuits generally have a unique solution except in rare instances of degeneracy. We forget about such rare degenerate circuits. (After all, this is an introductory text on Circuits).

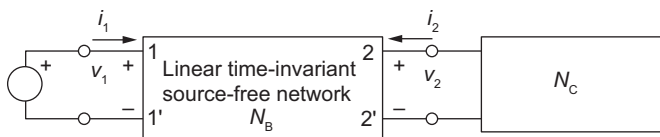


Fig. 16.1-3 Circuit in Fig. 16.1-2 After Applying Substitution Theorem

But, if this circuit has a unique solution, we can apply the Substitution Theorem once more to replace the network N_C with an independent voltage source v_2 without affecting the circuit solution anywhere in network N_B , as shown in Fig. 16.1-4.

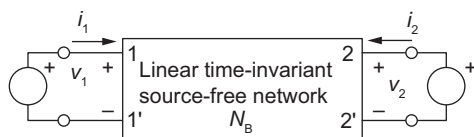


Fig. 16.1-4 Circuit in Fig. 16.1-2 Reduced to a Linear Two-Port Driven By Two Voltage Sources

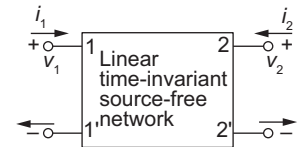


Fig. 16.1-1 A Linear Time-Invariant Two-Port Network

Therefore, we can solve for i_1 and i_2 in the network cascade in Fig. 16.1-2 by developing equations for the voltage-driven two-port network in Fig. 16.1-4.

Any response variable in a linear time-invariant circuit can be expressed as a linear combination of excitation functions. This is a consequence of linearity of the circuit and we have been using this very important principle right from Chap. 4 of this text. The excitation functions are v_1 and v_2 in this case. The response variables are i_1 and i_2 . Therefore, we can express i_1 and i_2 as

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 \end{aligned} \quad (16.1-1)$$

where, y_{11} , y_{12} , y_{21} , and y_{22} , are real-valued proportionality constants in the case of memoryless circuits, functions of $j\omega$ in the case of phasor equivalent circuits and functions of s in the case of s -domain equivalent circuits. They connect voltage variables to current variables, and hence, have a dimension of admittance. This was anticipated in using ‘ y ’ to designate them. They are called *y-parameters* of the two-port network and the network description in Eqn. 16.1-1 is called *y-parameter description* for the two-port.

Obviously, the *y-parameters* of a two-port can be evaluated by analysing the two-port in isolation. Once the parameters are obtained, the same set of parameters can be employed to describe the two-port in terms of Eqn. 16.1-1 to analyse the network cascade in Fig. 16.1-2 for a variety of networks – N_A and N_C .

How do we obtain these parameters? We interpret Eqn. 16.1-1 in the following manner to obtain them.

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 \\ \therefore y_{11} &= \left. \frac{i_1}{v_1} \right|_{v_2=0}, y_{12} = \left. \frac{i_1}{v_2} \right|_{v_1=0}, y_{21} = \left. \frac{i_2}{v_1} \right|_{v_2=0}, y_{22} = \left. \frac{i_2}{v_2} \right|_{v_1=0} \end{aligned} \quad (16.1-2)$$

Hence, we note that determination of the *y-parameters* involve shorting one or the other port. This is why they are also called *Short-Circuit Admittance Parameters* of the two-port.

The defining equations in Eqn. 16.1-2 assumed a memoryless linear time-invariant two-port network. The equations get modified as

$$y_{11}(j\omega) = \left. \frac{I_1}{V_1} \right|_{V_2=0}, y_{12}(j\omega) = \left. \frac{I_1}{V_2} \right|_{V_1=0}, y_{21}(j\omega) = \left. \frac{I_2}{V_1} \right|_{V_2=0}, y_{22}(j\omega) = \left. \frac{I_2}{V_2} \right|_{V_1=0}$$

for sinusoidal steady-state analysis of dynamic circuits with bold-faced italic quantities representing phasors.

Further, they get modified as

$$y_{11}(s) = \left. \frac{I_1(s)}{V_1(s)} \right|_{V_2(s)=0}, y_{12}(s) = \left. \frac{I_1(s)}{V_2(s)} \right|_{V_1(s)=0}, y_{21}(s) = \left. \frac{I_2(s)}{V_1(s)} \right|_{V_2(s)=0}, y_{22}(s) = \left. \frac{I_2(s)}{V_2(s)} \right|_{V_1(s)=0}$$

for analysis of dynamic circuits in s -domain.

s -domain description is the most general description for a linear time-invariant circuit – memoryless or dynamic. Hence, we use s -domain description for two-port network parameters from this point onwards. Further, we express the two-port describing equation in matrix form as $\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$, where $\begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix}$ is the *Short-Circuit Admittance Function Matrix* for the two-port network.

Determination of the four *y-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-5.

The first circuit has to be solved for i_1 and i_2 by employing standard circuit analysis procedures. Once these two variables have been obtained, we can work out y_{11} and y_{21} from them. Similarly, the solution for i_1 and i_2 by using the standard circuit analysis procedure on the second circuit yields y_{22} and y_{12} . Obviously, the circuit within the box must be known completely for carrying out the required circuit analysis.

Defining
equation for
y-parameters.

y-Parameters

y_{11} is the input admittance at port-1 with port-2 kept shorted.

y_{12} is the reverse transfer admittance from port-2 to port-1 with port-1 kept shorted.

y_{22} is the input admittance at port-2 with port-1 kept shorted.

y_{21} is the forward transfer admittance from port-1 to port-2 with port-2 kept shorted.

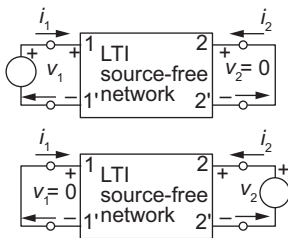


Fig. 16.1-5 Circuits to be Solved for Obtaining *y-Parameters*

16.1.2 Open-Circuit Impedance Parameters for a Two-Port Network

The cascaded network in Fig. 16.1-2 was reduced to Fig. 16.1-4 for the purpose of relating the port-voltage variables and port-current variables of the two-port network in the previous sub-section. We were a little partial in deciding to replace the left-side and the right-side networks with independent voltage sources. We could have used independent current sources also for this purpose. We apply the Substitution Theorem on the circuit in Fig. 16.1-2 and replace the networks N_A and N_C with suitable independent current sources now to arrive at the circuit shown in Fig. 16.1-6.

We can now express the port-voltage variables v_1 and v_2 as linear combinations of excitation functions i_1 and i_2 since the network is a linear one. This leads to the next description of a two-port network. This description will involve impedance functions in the s -domain since a current excitation function is transformed into a voltage response function only through an impedance function. Therefore, the required describing equations in s -domain will be,

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} \quad (16.1-3)$$

where $\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix}$ is the *Open-Circuit Impedance Parameter Matrix* for the two-port network. That they are *open-circuit parameters* is evident from the defining equations for the parameters as shown below:

$$z_{11}(s) = \left. \frac{V_1(s)}{I_1(s)} \right|_{I_2(s)=0}, \quad z_{12}(s) = \left. \frac{V_1(s)}{I_2(s)} \right|_{I_1(s)=0}, \quad z_{21}(s) = \left. \frac{V_2(s)}{I_1(s)} \right|_{I_2(s)=0}, \quad z_{22}(s) = \left. \frac{V_2(s)}{I_2(s)} \right|_{I_1(s)=0}$$

These are called *z-parameters* in shortened form. Determination of the four *z-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-7.

The first circuit has to be solved for v_1 and v_2 by employing standard circuit analysis procedures. Once these two variables have been obtained, we can work out z_{11} and z_{21} from them. Similarly, the solution for v_1 and v_2 by using standard circuit analysis procedure on the second circuit yields z_{22} and z_{12} .

EXAMPLE: 16.1-1

Find the *y-parameters* and *z-parameters* for the resistive linear time-invariant two-port network in Fig. 16.1-8 with $R_1 = 2 \Omega$, $R_2 = 4 \Omega$, $R_3 = 4 \Omega$ and $R_4 = 1 \Omega$.

SOLUTION

We determine *y-parameters* first. The circuit for determining y_{11} and y_{21} is shown in the circuit in Fig. 16.1-9(a).

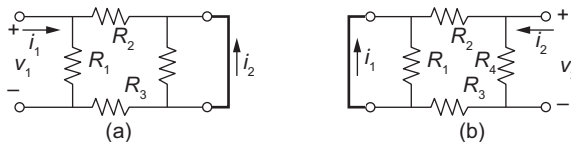


Fig. 16.1-9 Circuits for Determining *y-Parameters* in Example 16.1-1

$$i_1 = \frac{v_1}{R_1} + \frac{v_1}{R_2 + R_3} = \frac{v_1}{2\Omega} + \frac{v_1}{8\Omega} = v_1 \times 0.625S; \quad i_2 = -\frac{v_1}{R_2 + R_3} = -\frac{v_1}{8\Omega} = -v_1 \times 0.125S$$

$$\therefore y_{11} = \left. \frac{i_1}{v_1} \right|_{v_2=0} = 0.625S \text{ and } y_{21} = \left. \frac{i_2}{v_1} \right|_{v_2=0} = -0.125S$$



Fig. 16.1-6 Circuit in Fig. 16.1-3 Reduced To a Linear Two-Port Driven by Two Current Sources

Defining equation for *z-parameters*.

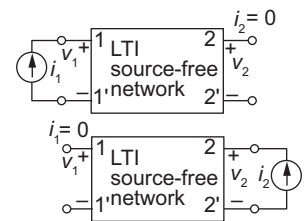


Fig. 16.1-7 Circuits to be Solved For Determining *z-Parameters*

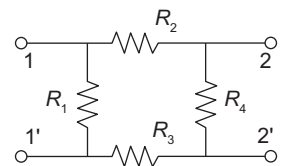


Fig. 16.1-8 Circuit for Example 16.1-1

z-Parameters

z_{11} is the input impedance at port-1 with port-2 kept open.

z_{12} is the reverse transfer impedance from port-2 to port-1 with port-1 kept open.

z_{22} is the input impedance at port-2 with port-1 kept open.

z_{21} is the forward transfer impedance from port-1 to port-2 with port-2 kept open.

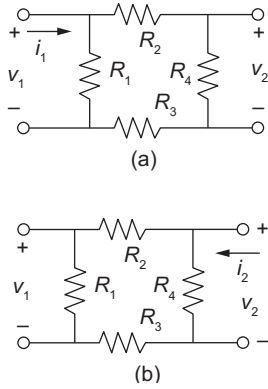


Fig. 16.1-10 Circuits for Determining z-Parameters in Example 16.1-1

The circuit for determining y_{12} and y_{22} is shown in the circuit in Fig. 16.1-9(b).

$$i_2 = \frac{v_2}{R_4} + \frac{v_2}{R_2 + R_3} = \frac{v_2}{1\Omega} + \frac{v_2}{8\Omega} = v_2 \times 1.125\text{S}; i_1 = \frac{v_2}{R_2 + R_3} = -\frac{v_2}{8\Omega} = -v_2 \times 0.125\text{S}$$

$$\therefore y_{12} = \left. \frac{i_1}{v_2} \right|_{v_1=0} = -0.125\text{S} \text{ and } y_{22} = \left. \frac{i_2}{v_2} \right|_{v_1=0} = 1.125\text{S}$$

We determine z-parameters next. The circuit for determining z_{11} and z_{21} is shown in the circuit in Fig. 16.1-10(a).

$$v_1 = [R_1 // (R_2 + R_3 + R_4)] i_1 = i_1 \times 2\Omega // 9\Omega = i_1 \times 1.6364\Omega$$

$$v_2 = R_4 \times \frac{R_1}{R_1 + R_2 + R_3 + R_4} \times i_1 = i_1 \times 1\Omega \times \frac{2\Omega}{11\Omega} = i_1 \times 0.1818\Omega$$

$$\therefore z_{11} = \left. \frac{v_1}{i_1} \right|_{i_2=0} = 1.6364\Omega \text{ and } z_{21} = \left. \frac{v_2}{i_1} \right|_{i_2=0} = 0.1818\Omega$$

The circuit for determining z_{12} and z_{22} is shown in the circuit in Fig. 16.1-10(b).

$$v_2 = [R_4 // (R_2 + R_3 + R_1)] i_2 = i_2 \times 1\Omega // 10\Omega = i_2 \times 0.9091\Omega$$

$$v_1 = R_1 \times \frac{R_4}{R_1 + R_2 + R_3 + R_4} \times i_2 = i_2 \times 2\Omega \times \frac{1\Omega}{11\Omega} = i_2 \times 0.1818\Omega$$

$$\therefore z_{12} = \left. \frac{v_1}{i_2} \right|_{i_1=0} = 0.1818\Omega \text{ and } z_{22} = \left. \frac{v_2}{i_2} \right|_{i_1=0} = 0.9091\Omega$$

Therefore, the short-circuit admittance parameter matrix is $\begin{bmatrix} 0.625 & -0.125 \\ -0.125 & 1.125 \end{bmatrix} \text{S}$

and the open-circuit impedance parameter matrix is $\begin{bmatrix} 1.6364 & 0.1818 \\ 0.1818 & 0.9091 \end{bmatrix} \Omega$. The

corresponding describing equations for the linear time-invariant two-port network is

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0.625 & -0.125 \\ -0.125 & 1.125 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1.6364 & 0.1818 \\ 0.1818 & 0.9091 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}.$$

We note that (i) y-matrix and z-matrix are inverses of each other (ii) $y_{12} = y_{21}$ and $z_{12} = z_{21}$ in this example.

EXAMPLE: 16.1-2

The two-port network in Example 16.1-1 is terminated in a resistor of value $R\Omega$ at its second port. Obtain an expression for its input resistance using z-parameters.

SOLUTION

The describing equation in terms of z-parameters is

$$v_1 = z_{11}i_1 + z_{12}i_2$$

$$v_2 = z_{21}i_1 + z_{22}i_2$$

Connecting a resistor R across the output port imposes a constraint among v_2 and i_2 . This constraint equation is given by $v_2 = -Ri_2$. Substituting this in the second z-parameter equation, we get,

$$-Ri_2 = z_{21}i_1 + z_{22}i_2 \Rightarrow i_2 = -\frac{z_{21}}{R + z_{22}}i_1$$

Substituting this expression for i_2 in the first z-parameter equation, we get,

$$v_1 = z_{11}i_1 + z_{12}i_2 = z_{11}i_1 - \frac{z_{12}z_{21}}{R + z_{22}}i_1 = i_1 \left[z_{11} - \frac{z_{12}z_{21}}{R + z_{22}} \right]$$

Therefore, input resistance is given by

$$R_{in} = \frac{v_1}{i_1} = \left[z_{11} - \frac{z_{12}z_{21}}{R + z_{22}} \right] = 1.6364 - \frac{0.1818^2}{R + 0.9091} = \frac{1.4546 + 1.6364R}{0.9091 + R}\Omega$$

EXAMPLE: 16.1-3

(i) Find the y -parameters in s -domain for the circuit shown in Fig. 16.1-11. (ii) Obtain the transfer function between port-voltages when the second port is terminated in a resistance of $R \Omega$.

SOLUTION

(i) The circuit needed for determining $y_{11}(s)$ and $y_{21}(s)$ is shown in the circuit in Fig. 16.1-12(a).

$$\begin{aligned} V_1(s) &= \left[0.5s + \frac{0.5s \times 1/s}{0.5s + 1/s} \right] \times I_1(s) = \frac{s(0.25s^2 + 1)}{(0.5s^2 + 1)} \times I_1(s) \\ \therefore I_1(s) &= \frac{(0.5s^2 + 1)}{s(0.25s^2 + 1)} \times V_1(s) \\ \text{and } I_2(s) &= -\frac{1/s}{0.5s + 1/s} \times \frac{(0.5s^2 + 1)}{s(0.25s^2 + 1)} \times V_1(s) = -\frac{1}{s(0.25s^2 + 1)} \times V_1(s) \\ \therefore y_{11}(s) &= \frac{(0.5s^2 + 1)}{s(0.25s^2 + 1)} \text{ S and } y_{21}(s) = -\frac{1}{s(0.25s^2 + 1)} \text{ S} \end{aligned}$$

We observe that there is essentially no difference between the circuit in Fig. 16.1-12(a) and the circuit in Fig. 16.1-12(b) except that the excitation and the response ports are interchanged. Thus, we expect that $y_{22}(s)$ will be the same as $y_{11}(s)$ and $y_{12}(s)$ will be the same as $y_{21}(s)$.

$$\therefore y_{22}(s) = \frac{(0.5s^2 + 1)}{s(0.25s^2 + 1)} \text{ S and } y_{12}(s) = -\frac{1}{s(0.25s^2 + 1)} \text{ S}$$

(ii) Connecting a resistor R across the output port imposes a constraint between $V_2(s)$ and $I_2(s)$. This constraint equation is $V_2(s) = -RI_2(s)$. Substituting this constraint equation in the second y -parameter equation, we get,

$$\begin{aligned} I_2(s) &= y_{21}(s)V_1(s) + y_{22}(s)V_2(s) \\ &= y_{21}(s)V_1(s) - Ry_{22}(s)I_2(s) \\ \therefore I_2(s) &= \frac{y_{21}(s)V_1(s)}{1 + Ry_{22}(s)} \\ \therefore V_2(s) &= -RI_2(s) = \frac{-Ry_{21}(s)V_1(s)}{1 + Ry_{22}(s)} \\ \therefore H(s) &= \frac{V_2(s)}{V_1(s)} = \frac{-Ry_{21}(s)}{1 + Ry_{22}(s)} \end{aligned}$$

Substituting for $y_{22}(s)$ and $y_{21}(s)$, we get the transfer function as

$$H(s) = \frac{4R}{s^3 + s^2(2R) + 4s + 4R} V/V.$$

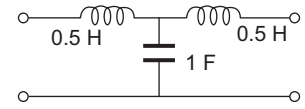


Fig. 16.1-11 Circuit for Example 16.1-3

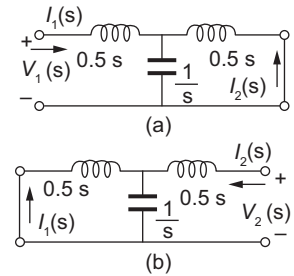


Fig. 16.1-12 Circuits for Determining y -Parameters in Example 16.1-3

EXAMPLE: 16.1-4

Find the z -parameters of the two-port network shown in Fig. 16.1-13.

SOLUTION

The circuit to be analysed to determine z_{11} and z_{21} is shown as Fig. 16.1-14(a).

We solve this circuit to obtain v_1 and v_2 in terms of i_1 .

Applying KCL at the first terminal at left-side port in circuit (a), we get,

$$i_1 = \frac{v_x}{1} - 2v_y.$$

Further $v_y = -0.5v_x$ by Ohm's law for the 0.5Ω resistor across the second port. Combining these two equations, we get,

$$i_1 = v_x - 2(-0.5v_x) = 2v_x$$

$$\text{But } v_x = v_1, \text{ and hence, } z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0} = 0.5 \Omega$$

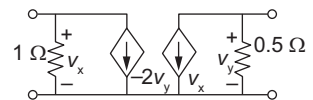


Fig. 16.1-13 Circuit for Example 16.1-4

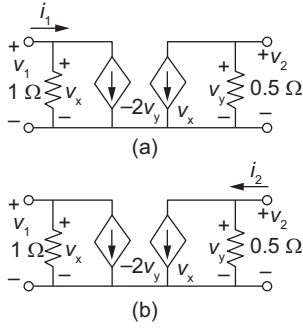


Fig. 16.1-14 Circuits for Determination of z -Parameters in Example 16.1-4

Now we have $v_y = -0.5v_x$ and $i_1 = 2v_x$.

Therefore, $v_2 = v_y = -0.25i_1$, and hence, $z_{21} = \left. \frac{v_2}{i_1} \right|_{i_2=0} = -0.25 \Omega$

The circuit to be analysed to determine z_{11} and z_{21} is shown as in Fig. 16.1-14(b). We solve this circuit to obtain v_1 and v_2 in terms of i_2 .

Applying KCL at the first terminal at the right-side port in the circuit in Fig. 16.1-14(b), we get, $i_2 = v_x + 2v_y$.

Further, $v_x = 2v_y$ by Ohm's law for the 1Ω resistor across the first port. Combining these two equations, we get, $i_2 = 2v_y + 2v_y = 4v_y$

But $v_2 = v_y$ and hence $z_{22} = \left. \frac{v_2}{i_2} \right|_{i_1=0} = 0.25 \Omega$

Now, we have $v_x = 2v_y$ and $i_2 = 4v_y$

Therefore, $v_1 = v_x = 0.5i_2$ and hence $z_{12} = \left. \frac{v_1}{i_2} \right|_{i_1=0} = 0.5 \Omega$.

Therefore, the open-circuit impedance matrix of this two-port network is

$$\begin{bmatrix} 0.5 & 0.5 \\ -0.25 & 0.25 \end{bmatrix} \Omega.$$

16.1.3 Hybrid Parameters and Inverse-Hybrid Parameters for a Two-Port Network

With reference to Fig. 16.1-2, we apply the Substitution Theorem to replace the network N_A with a current source of value i_1 first. The Substitution Theorem is applied again to the resulting network to replace the network N_C with a voltage source of value v_2 to arrive at the reduced network shown in Fig. 16.1-15.

We can now express the input port-voltage variable v_1 and the output port-current variable i_2 as linear combinations of excitation functions i_1 and v_2 , since the network is a linear one. This leads to the next description of a two-port network. This description will involve impedance function, admittance function and transfer functions in s -domain since it is a mixed excitation set and a mixed response set. Therefore, the required describing equations in s -domain will be,

$$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix}$$

where $\begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix}$ is the *Hybrid Parameter Matrix* for the two-port network. The defining equations for the parameters are as shown below:

$$h_{11}(s) = \left. \frac{V_1(s)}{I_1(s)} \right|_{v_2(s)=0}, h_{12}(s) = \left. \frac{V_1(s)}{V_2(s)} \right|_{I_1(s)=0}, h_{21}(s) = \left. \frac{I_2(s)}{I_1(s)} \right|_{V_2(s)=0}, h_{22}(s) = \left. \frac{I_2(s)}{V_2(s)} \right|_{I_1(s)=0}$$

These are called *h-parameters* in shortened form. $h_{11}(s)$ is the input impedance function with output port shorted. $h_{12}(s)$ is the reverse voltage transfer ratio with input open and $h_{21}(s)$ is the forward current transfer ratio with output shorted. $h_{22}(s)$ is the admittance seen from the second port with the first port kept open.

Determination of the four *h-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-16.

With reference to Fig. 16.1-2 again, we apply the Substitution Theorem to replace the network N_A with a voltage source of value v_1 first. Substitution Theorem is applied again to the resulting network to replace the network N_C with a current source of value i_2 to arrive at the reduced network shown in Fig. 16.1-17.

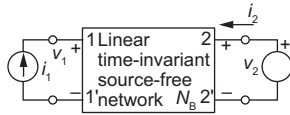


Fig. 16.1-15 Circuit in Fig. 16.1-3 Reduced to a Two-Port Circuit Driven by a Current Source and a Voltage Source

Defining equation for h -parameters.

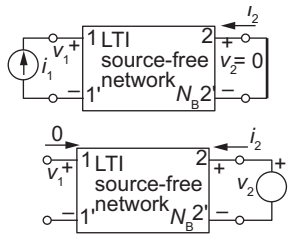


Fig. 16.1-16 Circuits that are to be Solved for Determination of h -Parameters

We can now express the input port-current variable i_1 and the output port-voltage variable v_2 as linear combinations of excitation functions v_1 and i_2 , since the network is a linear one. This leads to the next description of a two-port network. This description will involve the impedance function, the admittance function and the transfer function in the s -domain since it is a mixed excitation set and a mixed response set. Therefore, the required describing equations in s -domain will be,

$$\begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix}$$

where $\begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$ is the *Inverse-Hybrid Parameter Matrix* for the two-port network.

The defining equations for the parameters are as shown below:

$$g_{11}(s) = \left. \frac{I_1(s)}{V_1(s)} \right|_{I_2(s)=0}, \quad g_{12}(s) = \left. \frac{I_1(s)}{I_2(s)} \right|_{V_1(s)=0}, \quad g_{21}(s) = \left. \frac{V_2(s)}{V_1(s)} \right|_{I_2(s)=0}, \quad g_{22}(s) = \left. \frac{V_2(s)}{I_2(s)} \right|_{V_1(s)=0}$$

These are called *g-parameters* in shortened form. Determination of the four *g-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-18.

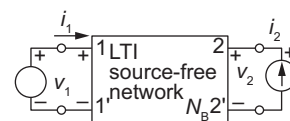


Fig. 16.1-17 Circuit in Fig. 16.1-3 Reduced to a Two-Port Network Driven by a Voltage Source and a Current Source

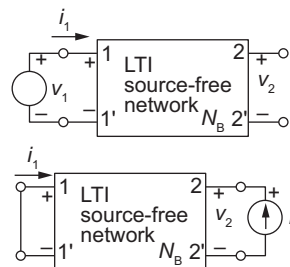


Fig. 16.1-18 Circuits that are to be Solved for Determination of *g-Parameters*

EXAMPLE: 16.1-5

Find the *h-parameters* and *g-parameters* of the two-port network shown in Fig. 16.1-19. $R_s = 1 \text{ k}\Omega$, $R_i = 98.01 \text{ k}\Omega$, $R_o = 1 \text{ k}\Omega$, $R_1 = 1 \text{ k}\Omega$, $R_2 = 99 \text{ k}\Omega$ and $K = 100$.

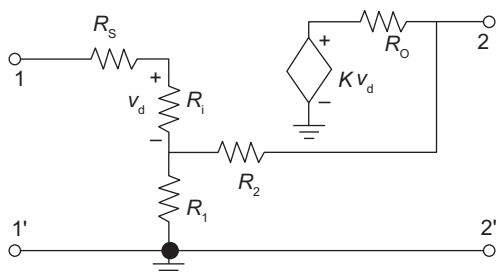


Fig. 16.1-19 Circuit for Example 16.1-5

SOLUTION

We determine *h-parameters* first. The defining equations of *h-parameters* for a memoryless two-port network are

$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0}, \quad h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2=0}, \quad h_{12} = \left. \frac{V_1}{V_2} \right|_{I_1=0} \quad \text{and} \quad h_{22} = \left. \frac{I_2}{V_2} \right|_{I_1=0}$$

The circuit to be analysed for determining h_{11} and h_{21} is shown as the circuit in Fig. 16.1-20(a). The circuit to be analysed for determining h_{12} and h_{22} is shown as the circuit in Fig. 16.1-20(b).

We take up the circuit in Fig. 16.1-20(a) first. The short across the second port connects the resistor R_2 directly across R_1 .

h-Parameters

$h_{11}(s)$ is the input impedance function with output port shorted.

$h_{12}(s)$ is the reverse voltage transfer ratio with input port open.

$h_{21}(s)$ is the forward current transfer ratio with output port shorted.

$h_{22}(s)$ is the admittance seen from output port with input port kept open.

g-Parameters

$g_{11}(s)$ is the input admittance function with output port open.

$g_{12}(s)$ is the reverse current transfer ratio with input port shorted.

$g_{21}(s)$ is the forward voltage transfer ratio with output port open.

$g_{22}(s)$ is the impedance seen from output port with input port kept shorted.

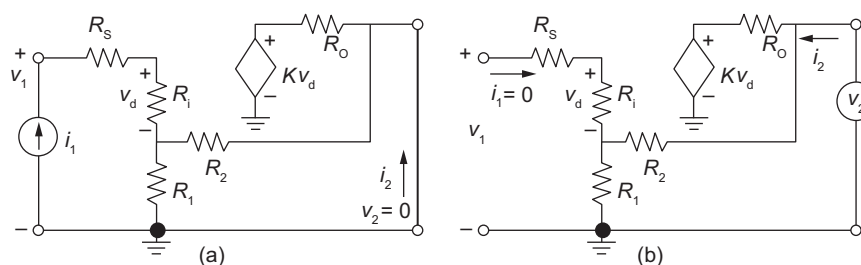


Fig. 16.1-20 Circuits To Be Solved for Determining h -Parameters in Example 16.1-5

$$\therefore v_1 = i_1 \times [R_s + R_1 + R_1 \parallel R_2] = i_1 \times [1 + 98.01 + 1 / .99] \text{ k}\Omega$$

$$\therefore h_{11} = [1 + 98.01 + 1 / .99] \text{ k}\Omega = 100 \text{ k}\Omega$$

Further, the voltage v_d across R_1 is $R_1 i_1$, and hence, the dependent source at the second port produces $K R_1 i_1$ V across its output. This source delivers $K R_1 i_1 / R_0$ A of current into the short across the second port. The second component of current in the short is the one coming from the R_2 line. Its value is $i_1 \times R_1 / (R_1 + R_2)$. Therefore,

$$i_2 = - \left(\frac{i_1 R_1}{R_1 + R_2} + \frac{K R_1 i_1}{R_0} \right) \Rightarrow h_{21} = - \left(\frac{R_1}{R_1 + R_2} + \frac{K R_1}{R_0} \right) = -9801.01$$

Now, consider the circuit in Fig. 16.1-20(b). Since the first port is left open, the voltage v_d is zero. Hence, the dependent voltage source at the second port is a short. Therefore, the resistor R_0 and the series combination of R_1 and R_2 come directly across the voltage source v_2 .

$$\therefore h_{22} = \left. \frac{i_2}{v_2} \right|_{i_1=0} = \frac{1}{R_0} + \frac{1}{R_1 + R_2} = 0.00101 \text{ S}$$

Further, since i_1 is zero, there is no voltage drop across R_s and R_1 . Therefore, v_1 is the same as the voltage across the resistor R_1 .

$$\therefore v_1 = \frac{R_1}{R_1 + R_2} v_2 \Rightarrow h_{12} = \frac{R_1}{R_1 + R_2} = 0.01$$

Therefore, the h -parameter matrix for this linear time-invariant two-port network is given by $\begin{bmatrix} 0.1 \text{ M}\Omega & 0.01 \\ -9801.01 & 1.01 \text{ mS} \end{bmatrix}$.

We determine g -parameters next. The defining equations of g -parameters for a memoryless two-port network are $g_{11} = \left. \frac{i_1}{v_1} \right|_{v_2=0}$, $g_{21} = \left. \frac{v_2}{v_1} \right|_{v_2=0}$, $g_{12} = \left. \frac{i_1}{i_2} \right|_{v_1=0}$ and $g_{22} = \left. \frac{v_2}{i_2} \right|_{v_1=0}$.

The circuit to be analysed for determining g_{11} and g_{21} is shown in Fig. 16.1-21(a) and the circuit to be analysed for determining g_{12} and g_{22} is shown in Fig. 16.1-21(b). Note that

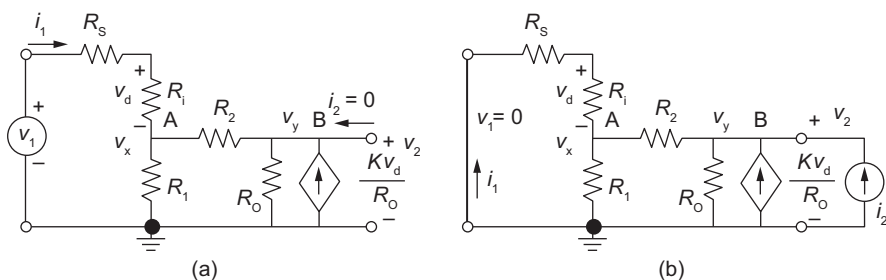


Fig. 16.1-21 Circuits to be Solved for Determination of g -Parameters in Example 16.1-5

the dependent voltage source in series with R_o at the second port has been converted into a dependent current source in parallel with R_o in both circuits.

We take up the circuit in Fig. 16.1-21(a) first. Two nodes A and B and the corresponding node voltage variables v_x and v_y are identified in this circuit. We write the node equations at these two nodes.

$$\text{Node A: } v_x \left[\frac{1}{R_s + R_i} + \frac{1}{R_2} + \frac{1}{R_1} \right] - v_y \frac{1}{R_2} = v_1 \frac{1}{R_s + R_i}$$

$$\text{Node B: } v_y \left[\frac{1}{R_2} + \frac{1}{R_o} \right] - v_x \frac{1}{R_2} = K \frac{R_i}{R_i + R_s} \frac{(v_1 - v_x)}{R_o}$$

Note that we have substituted for v_d in terms of v_1 and v_x using voltage division principle relevant to series connected resistors. Substituting the component values and simplifying the expressions, we get,

$$\begin{bmatrix} 1.0201 & -0.01 \\ 98.99 & 1.01 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0.0101 \\ 99 \end{bmatrix} v_1.$$

Solving for the node voltage variables, we get,

$$v_x = 0.495v_1 \text{ and } v_y = 49.5v_1$$

$$\therefore v_2 = v_y = 49.5v_1 \Rightarrow g_{21} = 49.5$$

$$\text{and } i_1 = \frac{v_1 - v_x}{R_s + R_i} = \frac{v_1 - 0.495v_1}{R_s + R_i} = \frac{0.505v_1}{R_s + R_i} \Rightarrow g_{11} = \frac{0.505}{R_s + R_i} = 5.1 \times 10^{-6} \text{ S.}$$

Now, we consider the circuit in Fig. 16.1-21(b). The node voltage variables assigned are v_x and v_y at node-A and node-B, respectively. We write the node equations as

$$\text{Node A: } v_x \left[\frac{1}{R_s + R_i} + \frac{1}{R_2} + \frac{1}{R_1} \right] - v_y \frac{1}{R_2} = 0$$

$$\text{Node B: } v_y \left[\frac{1}{R_2 + R_o} \right] - v_x \frac{1}{R_2} = i_2 - K \frac{R_i}{R_i + R_s} \frac{v_x}{R_o}.$$

Note that we have substituted for v_d in terms of v_x using voltage division principle relevant to series connected resistors.

Substituting the component values and simplifying the expressions, we get,

$$\begin{bmatrix} 1.0201 & -0.01 \\ 98.99 & 1.01 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1000i_2 \end{bmatrix}$$

Solving for the node voltage variables, we get,

$$v_x = 4.95i_2 \text{ and } v_y = 504.95i_2 \Rightarrow v_2 = 504.95i_2 \Rightarrow g_{22} = 504.95 \Omega.$$

$$\text{Further, } i_1 = \frac{v_x}{R_s + R_i} = \frac{4.95i_2}{R_s + R_i} = -5 \times 10^{-5} i_2 \Rightarrow g_{12} = -5 \times 10^{-5}.$$

Therefore, the g -parameter matrix for this linear time-invariant two-port network is given by $\begin{bmatrix} 5.1 \mu\text{S} & -5 \times 10^{-5} \\ 49.5 & 504.95 \Omega \end{bmatrix}$.

16.2 EQUIVALENT CIRCUITS FOR A TWO-PORT NETWORK

The describing equations for a memoryless linear time-invariant two-port network using y -parameters is given by

$$i_1 = y_{11}v_1 + y_{12}v_2$$

$$i_2 = y_{21}v_1 + y_{22}v_2$$

Interpreting
two-port parameter
defining equations
to arrive at an
equivalent circuit
for the network.

The left side of the equations consists of currents. Hence, the terms on the right-side must also be currents. Consider the first equation. We can see that the term $y_{11}v_1$ represents the current drawn by an admittance of y_{11} from the port-voltage v_1 . Hence, this term can be accounted for by connecting an admittance of y_{11} across the first port. However, the second term represents a current drawn from the first port with the value of this current decided by the *second port-voltage*. Therefore, it can not be taken into account by connecting any admittance across the first port. But then, it can be accounted for by connecting a dependent source – a voltage-controlled current source across the first port with the second port-voltage v_2 as the controlling variable. Thus, an admittance of y_{11} in parallel with a dependent current source at the input port will satisfy the first equation. The second equation also can be interpreted in a similar manner as the parallel combination of an admittance of y_{22} and a dependent current source $y_{21}i_1$ across the second port. This interpretation leads to an equivalent circuit shown in Fig. 16.2-1(a) for the linear time-invariant two-port network. The voltage and current variables are instantaneous variables and the parameters are real numbers for a memoryless two-port. The voltage and current variables are Laplace Transforms and parameters are ratios of rational polynomials in s for a general linear time-invariant dynamic circuit.

Similar interpretations for the remaining three descriptions of a linear time-invariant two-port network in terms of z -, h - and g -parameters lead to three other equivalent circuits shown in Fig. 16.2-1 (b), (c) and (d), respectively.

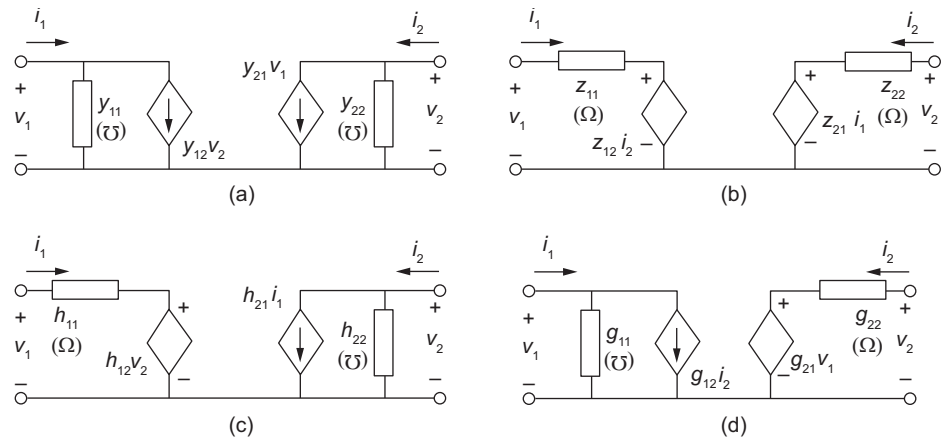


Fig. 16.2-1 Equivalent Circuits for a Linear Time-Invariant Two-Port Network

But this does not imply that any given two-port will have all the four parameter sets and the corresponding equivalent circuits. These equivalent circuits exist only if the corresponding parameters exist. It is possible that a given linear time-invariant two-port network has only some parameter sets. For instance, the circuit in Fig. 16.2-2(a) has no z and h parameters (i.e., all z -parameters are infinite) but has y - and g -parameters. The circuit in Fig. 16.2-2(b) has no y -, h - or g -parameters; but it has z -parameters.

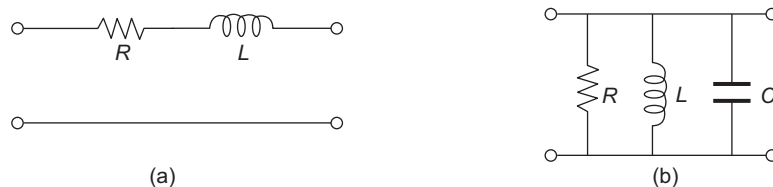


Fig. 16.2-2 (a) A Two-Port with No z -Parameters (b) A Two-Port with No y -Parameters

16.3 TRANSMISSION PARAMETERS (ABCD PARAMETERS) OF A TWO-PORT NETWORK

Two-port network theory finds application in solving transmission line problems in Electrical and Electronics Engineering. Power System Engineers often have to solve for the magnitude of sending-end voltage to be maintained in the line in order to ensure a pre-specified voltage magnitude at the receiving-end when the receiving-end is delivering a pre-specified power at a certain power factor. In addition, they would like to solve for the sending end line current, sending end active and reactive power etc., under this condition. The problem essentially involves determination of v_1 and i_1 of a two-port network given v_2 and i_2 of the network. A similar problem arises in filtering context. The filter engineer wants to study the attenuation and phase shift suffered by a signal when it goes through a linear time-invariant two-port network.

The four parameter-sets we discussed in the previous sub-sections are the *basic parameter sets* of a two-port. None of the *basic parameter sets* are suited for the analysis problem described above. A new parameter set, which is a *derived parameter set*, is used for this kind of *transmission analysis problems*. This new parameter set is called ‘Transmission Parameter Set’ or *ABCD Parameters* for the two-port. One method to arrive at this set is shown below.

We want to express v_1 and i_1 in terms of v_2 and i_2 . We start with *y-parameter* description of the two-port.

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 \end{aligned}$$

Algebraic manipulation of the second equation leads to $v_1 = -\frac{y_{22}}{y_{21}}v_2 + \frac{1}{y_{21}}i_2$.

Substituting this in the first equation, we get, $i_1 = \left(y_{12} - \frac{y_{11}y_{22}}{y_{21}} \right) v_2 + \frac{y_{11}}{y_{21}}i_2$. Thus, the required expressions that express the first port-variables in terms of second port-variables are,

$$\begin{aligned} v_1 &= -\frac{y_{22}}{y_{21}}v_2 + \frac{1}{y_{21}}i_2 \\ i_1 &= \left(y_{12} - \frac{y_{11}y_{22}}{y_{21}} \right) v_2 + \frac{y_{11}}{y_{21}}i_2. \end{aligned}$$

A passively terminated two-port will usually *deliver* current into the passive load connected across the second port. Therefore, it is convenient to consider $(-i_2)$ as the port-current variable of interest in analysing transmission problems. Hence, the above set of equations have been traditionally written as

$$\begin{aligned} v_1 &= Av_2 + B(-i_2) \\ i_1 &= Cv_2 + D(-i_2) \end{aligned} \quad \text{or as} \quad \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} \quad \text{in matrix form. We see that}$$

$A = -\frac{y_{22}}{y_{21}}, B = -\frac{1}{y_{21}}, C = \left(\frac{y_{12}y_{21} - y_{11}y_{22}}{y_{21}} \right)$ and $D = -\frac{y_{11}}{y_{21}}$. We could have started with

z-parameter description and arrived at equivalent expressions for transmission parameters in terms of *z-parameters*. Similarly, *ABCD parameters* can be expressed in terms of *h-* or *g-parameters* too.

Note that the equation $\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$ does not imply that the variables v_2 and

i_2 are the independent variables, i.e., the sources, and the variables v_1 and i_1 are response variables. Obviously, both the port-voltage and the port-current of a port can not be constrained by the independent voltage source and the independent current source, respectively, at the

same time. Thus, *ABCD parameter* description of a two-port is to be understood as a doctored form of one of the four *basic parameter descriptions* for the same two-port. In this sense, the *ABCD parameter set* is a derived set and not a basic set. *ABCD parameter* based description of a two-port network does not yield an equivalent circuit for it due to the same reason.

The transmission parameters can be interpreted as follows.

$$A = \text{Reverse voltage transfer ratio} = \left. \frac{v_1}{v_2} \right|_{i_2=0}$$

$$B = \text{Short-circuit transfer impedance} = \left. \frac{v_1}{(-i_2)} \right|_{v_2=0} (\Omega)$$

$$C = \text{Open-circuit transfer admittance} = \left. \frac{i_1}{v_2} \right|_{i_2=0} (\text{S})$$

$$D = \text{Reverse current transfer ratio} = \left. \frac{i_1}{(-i_2)} \right|_{v_2=0}$$

All these parameters are ratios of rational polynomials in s for a general dynamic two-port network.

EXAMPLE: 16.3-1

A 50 Hz, 250 km transmission line can be modelled approximately by the Π -equivalent circuit shown in Fig. 16.3-1. The receiving-end load is 40 MW at 220 kV with 0.8 lagging power factor. Find (i) *ABCD parameters* of the line and (ii) the sending-end voltage magnitude and angle, sending-end current, active and reactive power at the sending-end, sending-end power factor and line transmission efficiency using two-port transmission equations.

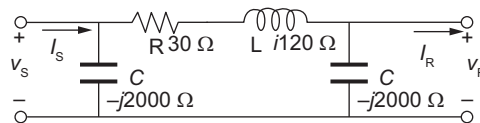


Fig. 16.3-1 Equivalent Circuit of Transmission Line in Example 16.3-1

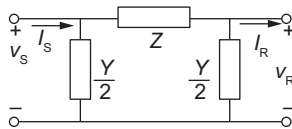


Fig. 16.3-2 A Π Two-Port Network

SOLUTION

(i) We derive the *ABCD parameters* for the two-port network shown in Fig. 16.3-2 first and substitute the relevant numbers later. $\mathbf{Z}$ is the series phasor impedance and $\mathbf{Y}$ is the total admittance of the two shunt branches.

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$

The circuits to be solved for determination of *ABCD parameters* are shown in Fig. 16.3-3.

Solving the first circuit, we get, $V_r = V_s \frac{2/Y}{Z + 2/Y} = V_s \frac{1}{1 + 1/2 ZY}$ and

$$I_s = V_s \left(\frac{Y}{2} + \frac{1}{Z + 2/Y} \right) = V_s Y \left(\frac{4 + ZY}{2(2 + ZY)} \right) = V_s Y \left(\frac{4 + ZY}{4(1 + 1/2 ZY)} \right) = V_s Y \left(1 + 1/4 ZY \right).$$

Comparing with the *ABCD equations*, we get, $A = 1 + 1/2 ZY$ and $C = Y(1 + 1/4 ZY)$

Solving the second circuit in Fig. 16.3-3, we get,

$$I_R = V_s \frac{1}{Z} \Rightarrow V_s = Z I_R,$$

$$I_R = I_s \frac{\frac{2}{Y}}{Z + \frac{2}{Y}} = I_s \frac{1}{1 + \frac{1}{2}ZY} \Rightarrow I_s = I_R \left(1 + \frac{1}{2}ZY\right)$$

Comparing with the ABCD equations, we get, $B = Z$ and $D = 1 + \frac{1}{2}ZY$. Therefore,

the ABCD matrix for the Π -network in Fig. 16.3-2 is $\begin{bmatrix} 1 + \frac{1}{2}ZY & Z \\ Y \left(1 + \frac{1}{4}ZY\right) & 1 + \frac{1}{2}ZY \end{bmatrix}$. Substituting

$$Z = 30 + j120 \Omega \text{ and } Y = j10^{-3} S, \text{ we get, } \begin{bmatrix} 0.94 \angle 0.91^\circ & 123.69 \angle 75.96^\circ \Omega \\ 0.00097 \angle 90.44^\circ S & 0.94 \angle 0.91^\circ \end{bmatrix}.$$

(ii) The receiving-end active power is 40 MW and power factor is 0.8 lag. Therefore, the receiving-end current magnitude is $40 \times 10^6 / (0.8 \times 220 \times 10^3) = 227.27$ A. The phase angle of current with respect to voltage is $-\cos^{-1}(0.8) = -36.87^\circ$. Taking the receiving-end voltage as the reference phasor, we now have $V_R = 220 \times 10^3 \angle 0^\circ$ V and $I_R = 227.27 \angle -36.87^\circ$ A. Substituting these values in the ABCD equations, we have,

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} 0.94 \angle 0.91^\circ & 123.69 \angle 75.96^\circ \Omega \\ 0.00097 \angle 90.44^\circ S & 0.94 \angle 0.91^\circ \end{bmatrix} \begin{bmatrix} 220 \times 10^3 \\ 227.27 \angle -36.87^\circ \end{bmatrix}$$

Therefore, $V_s = 229.6 \angle 5.26^\circ$ kV and $I_s = 192.56 \angle 27.18^\circ$ A. The sending-end power $S_s = V_s I_s^* = (41.012 - j16.504)$ MVA. Hence, active power at the sending-end = 41.012 MW, reactive power = -16.504 MVar (i.e., capacitive), power factor is $\cos(27.18^\circ \sim 5.26^\circ) = 0.928$ lead and transmission efficiency is $= 100 \times 40/41.012 = 97.32\%$.

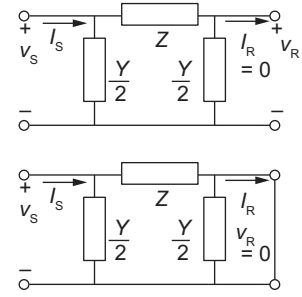


Fig. 16.3-3 Phasor Equivalent Circuits for Determination of ABCD Parameters in Example 16.3-1

16.4 INTER-RELATIONSHIPS BETWEEN VARIOUS PARAMETER SETS

We have discussed five different parameter sets for a two-port network till now. Table 16.4-1 abstracts these five parameter sets.

Table 16.4-1 Various Parameter Sets for a Linear Time-Invariant Two-Port Network

Name of the parameter Set	Defining equation
Short-circuit admittance parameter, $[y]$	$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$
Open-circuit impedance parameter set, $[z]$	$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$
Hybrid parameter set, $[h]$	$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix}$
Inverse-hybrid parameter set, $[g]$	$\begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix}$
Transmission or ABCD parameter set, $[ABCD]$	$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2(s) \\ -I_2(s) \end{bmatrix}$

All these parameter sets describe the same two-port network port-variables. Therefore, the various parameter sets must be inter-related. The conversion equations between various parameter sets will be of help in determining the four parameter sets from the measured values of the fifth parameter set in practical contexts. Some of the inter-relations are easy to see. For instance, it is clear that *z-parameter* and *y-parameter matrices* are inverses of each other (provided the inverses exist). Similarly, the *h-parameter* and the *g-parameter* matrices are inverses of each other. *z*, *y*, *h* and *ABCD* parameters are the most frequently used parameter sets. The conversion between them is shown in Table 16.4-2.

Table 16.4-2 Conversion Between *z*, *y*, *h* and *ABCD* Parameters

	[z]	[y]	[ABCD]	[h]
[z]	$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\frac{1}{\Delta y} \begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix}$	$\begin{bmatrix} \frac{A}{C} & \frac{AD-BC}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$	$\frac{1}{h_{22}} \begin{bmatrix} \Delta h & h_{12} \\ -h_{21} & 1 \end{bmatrix}$
[y]	$\frac{1}{\Delta z} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & -\frac{AD-BC}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$	$\frac{1}{h_{11}} \begin{bmatrix} 1 & -h_{12} \\ -h_{21} & \Delta h \end{bmatrix}$
[ABCD]	$\frac{1}{z_{21}} \begin{bmatrix} z_{11} & \Delta z \\ 1 & z_{22} \end{bmatrix}$	$\frac{1}{y_{21}} \begin{bmatrix} -y_{22} & -1 \\ \Delta y & -y_{11} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\frac{1}{h_{21}} \begin{bmatrix} -\Delta h & -h_{11} \\ -h_{22} & -1 \end{bmatrix}$
[h]	$\frac{1}{z_{22}} \begin{bmatrix} \Delta z & z_{12} \\ -z_{21} & 1 \end{bmatrix}$	$\frac{1}{y_{11}} \begin{bmatrix} 1 & -y_{12} \\ y_{21} & \Delta y \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & \frac{AD-BC}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$

16.5 INTERCONNECTIONS OF TWO-PORT NETWORKS

The very purpose of defining and evaluating the five different sets of describing parameter sets for a linear time-invariant two-port network lies in the relative ease of combining them for various types of interconnections between two two-port networks. We discuss five different ways of interconnecting two linear time-invariant two-port networks. The first interconnection is the *cascade connection* in which the second port of the first two-port network feeds the first port of the second two-port network. This interconnection is shown in Fig. 16.5-1.

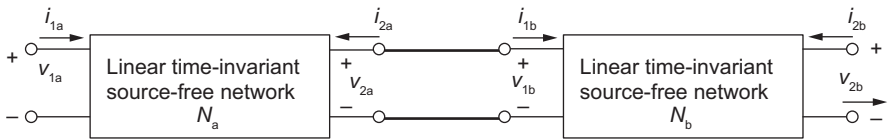


Fig. 16.5-1 Cascade Connection of Two Two-Port Networks

We show that the $ABCD$ matrix of the overall two-port network is the product of $ABCD$ matrices of N_a and N_b .

$$\begin{aligned} \begin{bmatrix} v_{1b} \\ i_{1b} \end{bmatrix} &= \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} v_{2b} \\ -i_{2b} \end{bmatrix} \\ \therefore \begin{bmatrix} v_{1a} \\ i_{1a} \end{bmatrix} &= \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} v_{2a} \\ -i_{2a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} v_{1b} \\ i_{1b} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} v_{2b} \\ -i_{2b} \end{bmatrix} \end{aligned}$$

$$\therefore ABCD \text{ matrix of the overall two-port} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

This result can be extended to n networks connected in cascade to arrive at the general result $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \dots \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$.

No other parameter set (i.e., y , z , h or g) will combine so elegantly for a cascade connection. Obviously, transmission parameter based description is the most convenient description for a two-port network in cascading context. Note that N_a followed by N_b is not the same as N_b followed by N_a – the ordering of cascade matters. Matrix product is not commutative.

The remaining four interconnections are shown in Fig. 16.5-2. The connection in Fig. 16.5-2(a) is termed *Series Connection* and in this connection both the input and the output ports are connected in series. It can be shown that the z -parameters of the overall two-port network is the sum of corresponding z -parameters of N_a and N_b . The connection in Fig. 16.5-2(b) is called *Parallel Connection*. Both the input ports and the output ports are tied in parallel in this connection. The y -parameters get added in this case.

The connection in Fig. 16.5-2(c) is a *Series-Parallel Connection* in which the input ports are connected in series and the output ports are connected in parallel. The h -parameters get added in this case. The connection in Fig. 16.5-2(d) is called a *Parallel-Series Connection*. The input ports are in parallel and the output ports are in series in this connection. The g -parameters get added in this case. These four connections appear in the context of analysis of negative feedback amplifiers in the area of Analogue Electronic Circuits.

The various two-port interconnections and the interconnection equations are summarised in Table 16.5-1.

Table 16.5-1 Two-Port Network Interconnections and Interconnection Equations

Connection name	Interconnection equation
Cascade [Fig. 16.5-1]	$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$
Series [Fig. 16.5-2 (a)]	$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} z_{11a} & z_{12a} \\ z_{21a} & z_{22a} \end{bmatrix} + \begin{bmatrix} z_{11b} & z_{12b} \\ z_{21b} & z_{22b} \end{bmatrix}$
Parallel [Fig. 16.5-2 (b)]	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} y_{11a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix} + \begin{bmatrix} y_{11b} & y_{12b} \\ y_{21b} & y_{22b} \end{bmatrix}$
Series-Parallel [Fig. 16.5-2 (c)]	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} h_{11a} & h_{12a} \\ h_{21a} & h_{22a} \end{bmatrix} + \begin{bmatrix} h_{11b} & h_{12b} \\ h_{21b} & h_{22b} \end{bmatrix}$
Parallel-Series [Fig. 16.5-2 (d)]	$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11a} & g_{12a} \\ g_{21a} & g_{22a} \end{bmatrix} + \begin{bmatrix} g_{11b} & g_{12b} \\ g_{21b} & g_{22b} \end{bmatrix}$

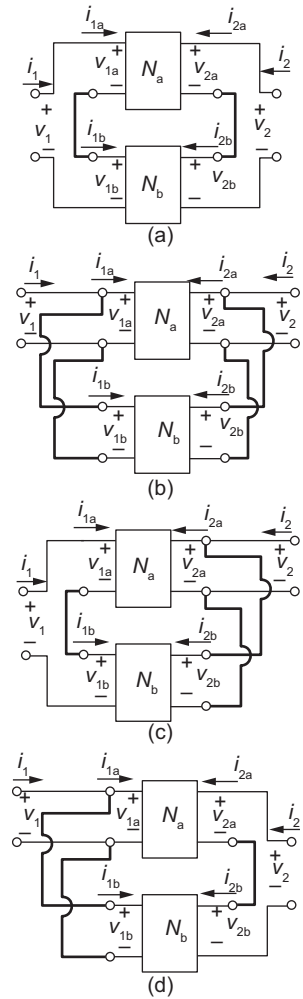


Fig. 16.5-2 Two-Port Network Interconnections (a) Series (b) Parallel (c) Series-Parallel (d) Parallel-Series

16.6 RECIPROCITY AND SYMMETRY IN TWO-PORT NETWORKS

Various forms of
Reciprocity
Theorem.

What kind of
circuits can be
reciprocal?

For a 'reciprocal'
LTI two-port network
(i) $y_{12} = y_{21}$
(ii) $z_{12} = z_{21}$
(iii) $h_{12} = -h_{21}$
(iv) $g_{12} = -g_{21}$
(v) $AD - BC = 1$
For a 'symmetric' LTI
two-port network
(i) $y_{11} = y_{22}$
(ii) $z_{11} = z_{22}$
(iii) $h_{11}h_{22} - h_{12}h_{21} = 1$
(iv) $g_{11}h_{22} - g_{12}g_{21} = 1$
(v) $A = D$
For a 'symmetric
reciprocal' LTI two-port
network
(i) $y_{11} = y_{22}; y_{12} = y_{21}$
(ii) $z_{11} = z_{22}; z_{12} = z_{21}$
(iii) $h_{12} = -h_{21};$
 $h_{11}h_{22} - h_{12}h_{21} = 1$
(iv) $g_{12} = -g_{21};$
 $g_{11}h_{22} - g_{12}g_{21} = 1$
(v) $A = D;$
 $AD - BC = 1$

We had discussed Reciprocity Theorem for memoryless circuits in Sect. 5.7 of Chap. 5. The first two forms of the theorem are reproduced below for ease of reference.

'The ratio of voltage measured across a pair of terminals to the excitation current applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.'

'The ratio of current measured in a short-circuit across a pair of terminals to the excitation voltage applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.'

We noted in Sect. 5.7 of Chap. 5 that the Reciprocity Theorem is essentially dependent on the symmetry of nodal conductance matrix or mesh resistance matrix. The proof for this theorem was given for memoryless circuits in Chap. 5. This theorem can be easily extended to dynamic circuits too, since the nodal admittance and mesh impedance matrices in s -domain share the same symmetry properties that the corresponding matrices for memoryless circuits exhibit.

A dynamic circuit obeys the Reciprocity Theorem if its nodal admittance and mesh impedance matrices in s -domain are symmetric. A circuit containing R, L, C and M (hence, transformers) will have symmetric admittance and impedance matrices. Therefore, such a circuit will be *reciprocal* unconditionally.

Dependent sources *can* destroy the symmetry of these matrices. However, the circuit can have symmetric admittance and impedance matrices for certain special values of dependent source coefficients. Therefore, one can not state that a circuit containing dependent sources is necessarily *non-reciprocal*. Its admittance or impedance matrix will have to be examined for symmetry in order to decide the issue.

Some texts on circuit theory apparently give an impression that *passive circuits* are necessarily reciprocal. This is not strictly true. It is true that the elements R, L, C and M are *passive elements* and circuits containing only these elements are indeed *reciprocal*. However, there are multi-terminal elements that are *passive*, but make the circuit admittance and impedance matrices asymmetric. An element called '*gyrator*' is one such element. Thus, *passivity* does not imply *reciprocity*. But *symmetric nodal admittance matrix or mesh impedance matrix* does imply reciprocity.

If a two-port network is reciprocal (it is necessarily reciprocal if it contains only R, L, C and M), then, a straightforward application of Reciprocity Theorem quoted in the beginning of this section leads to the following conclusions.

For a reciprocal linear time-invariant two-port network $y_{12} = y_{21}$ and $z_{12} = z_{21}$.

It then follows from the conversion equations in Table 16.4-2 that for a reciprocal linear time-invariant two-port network $h_{12} = -h_{21}$, $g_{12} = -g_{21}$ and $AD - BC = 1$.

Let a voltage source v_s be connected to the first port of a *reciprocal* linear time-invariant two-port network with the second port shorted (or open). We note the current drawn from the voltage source. Subsequently, we connect v_s to the second port with the first port kept shorted (or open) and observe the current drawn from the source. If the currents drawn from the source remain the same in the two sets of observations, we call it a *symmetric two-port network*. The input impedance and admittance from one port with the other port kept open or short remain the same when the roles of ports are interchanged in a symmetric two-port network. This implies that $y_{11} = y_{22}$ (and therefore, $z_{11} = z_{22}$, $h_{11}h_{22} - h_{12}h_{21} = 1$, $g_{11}h_{22} - g_{12}g_{21} = 1$ and $A = D$) for a symmetric two-port.

For a *symmetric reciprocal linear time-invariant two-port network* (i) $y_{11} = y_{22}$ and $y_{12} = y_{21}$, (ii) $z_{11} = z_{22}$ and $z_{12} = z_{21}$, (iii) $h_{12} = -h_{21}$ and $h_{11}h_{22} - h_{12}h_{21} = 1$, (iv) $g_{12} = -g_{21}$ and $g_{11}h_{22} - g_{12}g_{21} = 1$ and (v) $A = D$ and $AD - BC = 1$.

16.7 STANDARD SYMMETRIC T AND PI EQUIVALENTS

Figure 16.7-1 shows a general T-network and Π -network using three independent impedances. They are not symmetric networks, but we assume that the impedances are passive impedances. From this section onwards, the qualifier ‘passive’ implies that the network contains only R , L , C and M type of elements. Then, they are reciprocal networks.

Any passive reciprocal two-port network requires only three parameters to describe it. The fourth parameter will be provided by the reciprocity condition. The network in Fig. 16.7-1(a), being a reciprocal one, has three parameters. It has three impedances also. Therefore, it must be possible to determine $Y_A(s)$, $Y_B(s)$ and $Z(s)$ such that the resulting network has a pre-specified set of parameters (z , y , h , g or $ABCD$). Therefore, it follows that any reciprocal network will have a T-equivalent – i.e., given any reciprocal network, it will be possible to find a T-network with suitable values of $Y_A(s)$, $Y_B(s)$ and $Z(s)$ such that the T-network and the original two-port network will have the same two-port parameter sets.

A similar reasoning helps us to conclude that any reciprocal network will have a Π -equivalent too – i.e., given any reciprocal network, it will be possible to find a Π -network with suitable values of $Z_A(s)$, $Z_B(s)$ and $Y(s)$ such that the Π -network and the original two-port network will have the same two-port parameter sets.

If, in addition to being reciprocal, the passive two-port network is symmetric too, then, its characterisation will require only two parameters. Moreover, the T and Π equivalents also will be symmetric, i.e., $Y_A(s) = Y_B(s)$ and $Z_A(s) = Z_B(s)$. This leads to the standard symmetric passive T and Π equivalents for any ‘passive symmetric reciprocal’ linear time-invariant two-port network shown in Fig. 16.7-2.

The reason for assigning $Y(s)/2$ as the shunt branch in Fig. 16.7-2(a) and $Z(s)/2$ as the series branch in Fig. 16.7-2(b) will become clear in subsequent discussions in this chapter. The point to be noted at this juncture is that the two equivalent networks are symmetric.

We have already obtained the $ABCD$ matrix of the symmetric Π -network in Example 16.3-1.

$ABCD$ Matrix of the Π -network in Fig. 16.7-2(a) is

$$\begin{bmatrix} 1 + \frac{1}{2} Z(s)Y(s) & Z(s) \\ Y(s)(1 + \frac{1}{4} Z(s)Y(s)) & 1 + \frac{1}{2} Z(s)Y(s) \end{bmatrix}$$

EXAMPLE: 16.7-1

Find the $ABCD$ Matrix of a symmetric T-network.

SOLUTION

The s -domain circuits that are to be solved to determine the $ABCD$ parameters are shown in Fig. 16.7-3.

Consider the first circuit.

$$V_2(s) = V_1(s) \frac{1/Y(s)}{1/Y(s) + Z(s)/2} = V_1(s) \frac{1}{1 + \frac{1}{2} Z(s)Y(s)}$$

$$\therefore A(s) = \left. \frac{V_1(s)}{V_2(s)} \right|_{I_2(s)=0} = 1 + \frac{1}{2} Z(s)Y(s)$$

Further,

$$I_1(s) = V_2(s)Y(s) \quad (\because \text{Drop across } \frac{1}{2} Z(s) \text{ on the right side is zero})$$

$$\therefore C(s) = \left. \frac{I_1(s)}{V_2(s)} \right|_{I_2(s)=0} = Y(s)$$

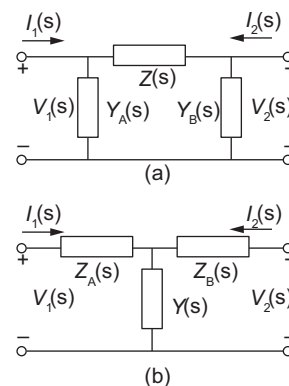


Fig. 16.7-1 General Passive T and Π Networks in s -Domain

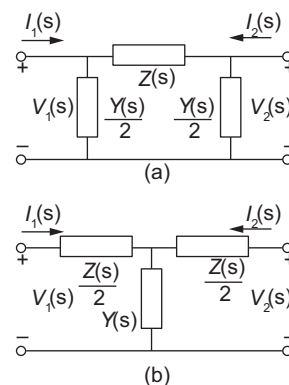


Fig. 16.7-2 Standard Symmetric Passive T and Equivalent Circuits for an Arbitrary Passive Symmetric Two-Port Network

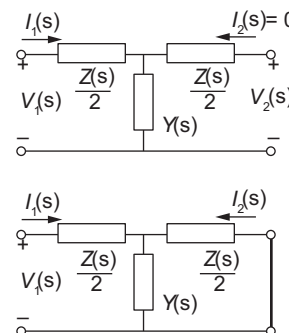


Fig. 16.7-3 Circuits to be Solved for Determining $ABCD$ Parameters of a Symmetric T-Network

Impedance Z and admittance Y can be employed to construct a T- or Π -network as in Fig. 16.7-2. They are then called the 'underlying impedances' of T- or Π -network.

A T and a Π formed from the same underlying impedances will not be equivalent to each other – they will have different parameter sets.

The T-equivalent and the Π -equivalent of a given two-port network will not have the same underlying impedances.

Now, we can obtain $B(s)$ and $D(s)$ from the second circuit in Fig. 16.7-3. However, we know that $A = D$ and $AD - BC = 1$ for a symmetric reciprocal network. We can use these two constraints to work out $B(s)$ and $D(s)$.

$$\therefore D(s) = 1 + \frac{1}{2} Z(s)Y(s) \text{ and}$$

$$B = \frac{AD - 1}{C} = \frac{1 + Z(s)Y(s) + \frac{1}{4} [Z(s)Y(s)]^2 - 1}{Y(s)} = Z(s) \left(1 + \frac{1}{4} Z(s)Y(s) \right)$$

Therefore, the ABCD Matrix of T-network in Fig. 16.7-2(b) is

$$\begin{bmatrix} 1 + \frac{1}{2} Z(s)Y(s) & Z(s) \left(1 + \frac{1}{4} Z(s)Y(s) \right) \\ Y(s) & 1 + \frac{1}{2} Z(s)Y(s) \end{bmatrix}$$

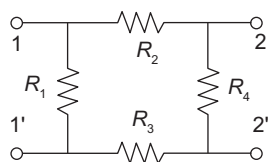


Fig. 16.7-4 Resistive Two-Port Network in Example 16.7-2

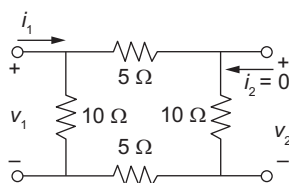


Fig. 16.7-5 Circuit for Determining A and C in Example 16.7-2

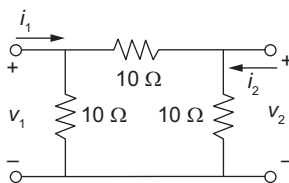


Fig. 16.7-6 The Π -Equivalent for the Resistive Two-Port Network in Example 16.7-2

EXAMPLE: 16.7-2

(i) Find the ABCD Parameters of the resistive two-port shown in Fig. 16.7-4 with $R_1 = R_4 = 10 \Omega$ and $R_2 = R_3 = 5 \Omega$. (ii) Find the T- and Π - equivalents for this two-port network.

SOLUTION

(i) The circuit to be solved for determining A and C is shown in Fig. 16.7-5.

$$v_2 = \frac{10 \Omega}{(5 + 10 + 5) \Omega} \times v_1 = 0.5 \times v_1 \Rightarrow A = \left. \frac{v_1}{v_2} \right|_{i_2=0} = 2$$

$$i_1 = \frac{v_1}{10 \Omega // (5 + 10 + 5) \Omega} = \frac{3v_1}{20} \text{ A}$$

$$\text{But } v_1 = 2v_2$$

$$\therefore i_1 = \frac{3}{20} \times 2v_2 \Rightarrow C = \left. \frac{i_1}{v_2} \right|_{i_2=0} = 0.3 \text{ S}$$

Consider the resistance seen from the first port with the second port open in the circuit in Fig. 16.7-5. It is $10 \Omega // 20 \Omega$. The resistance seen from the second port with the first port kept open is $10 \Omega // 20 \Omega$. These two resistance values are equal, and hence, the network in Fig. 16.7-5 is a symmetric reciprocal one.

Therefore, $D = A = 2$ and $B = (AD - 1)/C = 10 \Omega$.

Therefore, the ABCD matrix of the network is $\begin{bmatrix} 2 & 10 \\ 0.3 & 2 \end{bmatrix}$.

$$(ii) \text{ ABCD matrix of the resistive } \Pi\text{-equivalent is } \begin{bmatrix} 1 + \frac{1}{2} ZY & Z \\ Y \left(1 + \frac{1}{4} ZY \right) & 1 + \frac{1}{2} ZY \end{bmatrix}$$

Comparing this with the ABCD matrix of the resistive network, we get, $1 + 0.5 ZY = 2$, $Z = 10 \Omega \Rightarrow Y = 0.2 \text{ S}$.

$\therefore$ The Π -equivalent is as shown in Fig. 16.7-6. Note that Π -equivalent uses Z in the series arm and $0.5Y$ in the shunt arms.

$$\text{ABCD matrix of the resistive T-equivalent is } \begin{bmatrix} 1 + \frac{1}{2} ZY & Z \left(1 + \frac{1}{4} ZY \right) \\ Y & 1 + \frac{1}{2} ZY \end{bmatrix}$$

Comparing this with the $ABCD$ matrix of the resistive network, we get,

$$1 + 0.5ZY = 2, Y = 0.3 \text{ S} \Rightarrow Z = \frac{20}{3} \Omega.$$

$\therefore$ The T-equivalent is as shown in Fig. 16.7-7.

Note that the T-equivalent uses $0.5Z$ in series arms and Y in the shunt arm.

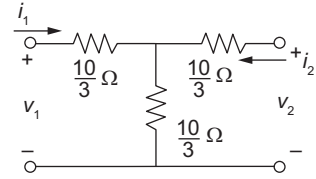


Fig. 16.7-7 The T-Equivalent for the Resistive Two-Port Network in Example 16.7-2

16.8 IMAGE PARAMETER DESCRIPTION OF A RECIPROCAL TWO-PORT NETWORK

A reciprocal two-port requires three independent parameters to describe it. For instance, the set of parameters (A , B , C) will describe it since the fourth parameter D can be obtained from the constraint $AD - BC = 1$ arising out of the reciprocal nature of the network.

$ABCD$ parameters are widely applied in the analysis of transmission lines under sinusoidal steady-state. However, analysts prefer another equivalent formulation when it comes to analysis of transmission lines under transient conditions or with aperiodic inputs. Similarly, filter designers favour this alternative formulation to $ABCD$ parameters. This alternative way to describe a *reciprocal two-port network* is called the ‘*Image Parameter Description*’. We take up this description in this section.

Two impedances called *image impedances* (also called *iterative impedances*) designated by Z_{im1} and Z_{im2} and a constant called *image transfer constant* designated by γ describe a reciprocal linear time-invariant two-port network in this formulation.

Image impedances are a pair of impedances – Z_{im1} and Z_{im2} – such that the input impedance at port-1 with port-2 terminated in Z_{im2} is Z_{im1} and the input impedance at port-2 with port-1 terminated in Z_{im1} is Z_{im2} . There exists such a pair of impedances for every linear time-invariant two-port network. This is shown in Fig. 16.8-1.

Image impedances defined.

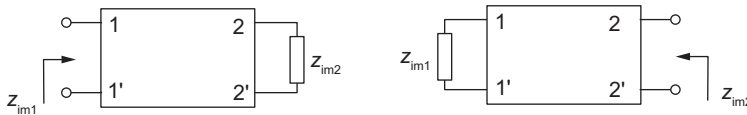


Fig. 16.8-1 Definition of Image Impedances

Image impedances can be obtained from $ABCD$ parameters. Consider the first circuit in Fig. 16.8-1.

$$\begin{aligned} v_1 &= Av_2 - Bi_2 \\ i_1 &= Cv_2 - Di_2 \end{aligned}$$

But $v_2 = -Z_{im2}i_2$. Using this in the second $ABCD$ equation, we get,

$$i_1 = [-CZ_{im2} - D]i_2 \Rightarrow i_2 = \frac{1}{[-CZ_{im2} - D]}i_1.$$

Substituting this in the first $ABCD$ equation, we get,

$$v_1 = Av_2 - Bi_2 = \frac{[AZ_{im2} + B]}{[CZ_{im2} + D]}i_1. \text{ Therefore,}$$

Image Impedances

Image impedances are a pair of impedances – Z_{im1} and Z_{im2} – such that the input impedance at port-1 with port-2 terminated in Z_{im2} is Z_{im1} and the input impedance at port-2 with port-1 terminated in Z_{im1} is Z_{im2} .

They are given in terms of $ABCD$ parameters as

$$\begin{aligned} Z_{im1} &= \sqrt{\frac{AB}{CD}} \\ Z_{im2} &= \sqrt{\frac{DB}{CA}} \end{aligned}$$

Image impedances of an LTI two-port network in terms of its ABCD parameters.

$$Z_{im1} = \frac{v_1}{i_1} = \frac{[AZ_{im2} + B]}{[CZ_{im2} + D]} \quad (16.8-1)$$

A similar derivation using second circuit in Fig. 16.8-1 leads to

$$Z_{im2} = \frac{v_1}{i_1} = \frac{[DZ_{im1} + B]}{[CZ_{im1} + A]} \quad (16.8-2)$$

Substituting Eqn. 16.8-2 in Eqn. 16.8-1 and solving for Z_{im1} , we get,

$$Z_{im1} = \sqrt{\frac{AB}{CD}} \quad (16.8-3)$$

Substituting Eqn. 16.8-3 in Eqn. 16.8-2, we get,

$$Z_{im2} = \sqrt{\frac{DB}{CA}} \quad (16.8-4)$$

Image transfer constant defined.

The third parameter needed to complete the description is obtained by determining the ratios $\frac{v_1}{v_2}$ and $\frac{i_1}{(-i_2)}$ when the second port is terminated in Z_{im2} and v_1 is applied at the first port. The geometric mean of these two ratios is expressed as the exponential of a number γ and that γ is called the *Image Transfer Constant*.

$$e^\gamma = \sqrt{\frac{v_1}{v_2} \times \frac{i_1}{(-i_2)}} \text{ with } v_2 \text{ developed across } Z_{im2}. \quad (16.8-5)$$

Image transfer constant can be expressed in terms of the ABCD parameters.

$$v_1 = Av_2 - Bi_2$$

$$i_1 = Cv_2 - Di_2$$

But, with Z_{im2} termination, $-i_2 = \frac{v_2}{Z_{im2}}$. Substituting this in the first ABCD equation, we get, $\frac{v_1}{v_2} = (A + \frac{B}{Z_{im2}})$. Substituting $v_2 = -Z_{im2}i_2$ in the second ABCD equation, we get,

$$\frac{i_1}{(-i_2)} = (D + CZ_{im2}). \text{ But } Z_{im2} = \sqrt{\frac{DB}{CA}}.$$

$$\therefore \frac{v_1}{v_2} = A + \frac{B}{Z_{im2}} = A + \frac{\sqrt{ABCD}}{D} \text{ and } \frac{i_1}{(-i_2)} = D + CZ_{im2} = D + \frac{\sqrt{ABCD}}{A}$$

$$\begin{aligned} \therefore e^\gamma &= \sqrt{\frac{v_1}{v_2} \times \frac{i_1}{(-i_2)}} = \sqrt{AD + BC + 2\sqrt{ABCD}} = \sqrt{(\sqrt{AD} + \sqrt{BC})^2} \\ &= \sqrt{AD} + \sqrt{BC} = \sqrt{AD} + \sqrt{AD - 1} \end{aligned}$$

The last step in the above derivation made use of the fact that $AD - BC = 1$ for a reciprocal network. The number γ essentially contains information on how effectively the applied voltage and current are transmitted to the output side by the two-port network. It is possible to show that transmission in the reverse direction – i.e., if voltage is applied at the second port and the first port is terminated in Z_{im1} – is also governed by the same γ factor.

Z_{im1} , Z_{im2} and γ will be real numbers for a memoryless two-port network. They will be functions of ω for sinusoidal steady-state analysis of a dynamic linear time-invariant two-port network and they will be functions of complex frequency s for a dynamic circuit in s -domain.

Note that $v_1 i_1$ is the power delivered to the input port and $-v_2 i_2$ is the power delivered to the terminating impedance (i.e., the load impedance). Hence, the number e^γ can be viewed as a measure of power transfer in the two-port network.

$e^\gamma = (P_i/P_o)^{0.5}$, where P_i is the input power into the network and P_o is the output power delivered by the network when the load impedance is equal to the image impedance.

The ratios v_1/v_2 and $i_1/(-i_2)$ are the same as e^γ for a symmetric network.

16.8.1 Image Parameters for a Symmetric Reciprocal Two-Port Network

In this case, we need only two parameters. Eqn. 16.8-3 and Eqn. 16.8-4 reveal that $Z_{im1} = Z_{im2}$ if $A = D$. For a symmetric network $A = D$. Therefore, the two image impedances are equal and are equal to $\sqrt{\frac{B}{C}}$. This impedance is now given a new name – it is called the *Characteristic Impedance* of the symmetric two-port network and is usually designated by Z_o .

$$\therefore \text{Characteristic impedance of a symmetric two-port network} = Z_o = \sqrt{\frac{B}{C}}$$

The two transfer ratios $\frac{v_1}{v_2} = A + \frac{\sqrt{ABCD}}{A}$ and $\frac{i_1}{-i_2} = D + \frac{\sqrt{ABCD}}{A}$, when the second

port is terminated in Z_{im2} and v_1 is applied at the first port, will be equal for a symmetric network. Therefore, the *image transfer constant* may be interpreted as the ratio of applied voltage to output voltage with characteristic impedance termination in the case of symmetric

networks. That is, $e^\gamma = \frac{\text{Input voltage}}{\text{Output voltage}}$ with Z_o termination $= A + \frac{\sqrt{ABCD}}{D} = A + \sqrt{BC}$ for a

symmetric two-port network. With this interpretation of e^γ , the number γ gets a new name applicable only in the case of symmetric networks. It is then called *Propagation Constant*. Real part of γ has the unit of '*nepers*' and imaginary part is in '*radians*'.

16.8.2 Image Parameters in terms of Open-Circuit and Short-Circuit Impedances

Consider the four circuits shown in Fig. 16.8-2.

Z_{1o} is the input impedance measured from port-1 with port-2 kept open. Z_{1s} is the input impedance measured from port-1 with port-2 kept shorted. Z_{2o} is the input impedance measured from port-2 with port-1 kept open. Z_{2s} is the input impedance measured from port-2 with port-1 kept shorted.

$$\begin{aligned} v_1 &= Av_2 - Bi_2 \\ i_1 &= Cv_2 - Di_2 \end{aligned}$$

With the second port open, $\frac{v_1}{i_1} = \frac{A}{C}$ and with the second port shorted $\frac{v_1}{i_1} = \frac{B}{D}$.

Similarly, with the first port open $\frac{v_2}{i_2} = \frac{D}{C}$ and with the first port shorted $\frac{v_2}{i_2} = \frac{B}{A}$. Therefore,

$$Z_{1o} = \frac{A}{C}, Z_{1s} = \frac{B}{D}, Z_{2o} = \frac{D}{C} \text{ and } Z_{2s} = \frac{B}{A} \quad (16.8-6)$$

Comparing Eqn. 16.8-6 with Eqn. 16.8-3 and Eqn. 16.8-4, we get,

$$Z_{im1} = \sqrt{\frac{AB}{CD}} = \sqrt{Z_{1o}Z_{1s}} \text{ and } Z_{im2} = \sqrt{\frac{DB}{CA}} = \sqrt{Z_{2o}Z_{2s}} \quad (16.8-7)$$

$A = D$ for a symmetric network. Therefore, Eqn. 16.8-6 shows that for a symmetric linear time-invariant two-port network $Z_{1o} = Z_{2o} = \frac{A}{C}$ and $Z_{1s} = Z_{2s} = \frac{B}{A}$ and Eqn. 16.8-7

shows that for such a network $Z_{im1} = Z_{im2} = Z_o = \sqrt{\frac{B}{C}} = \sqrt{Z_{1o}Z_{1s}} = \sqrt{Z_{2o}Z_{2s}}$. The equal values

Image Parameters

(i) General LTI Two-Port Network

$$Z_{im1} = \sqrt{\frac{AB}{CD}}; Z_{im2} = \sqrt{\frac{DB}{CA}}$$

$$\text{and } e^\gamma = \sqrt{AD} + \sqrt{BC}$$

(ii) Symmetric Reciprocal Two-Port Network

$$Z_o = Z_{im1} = Z_{im2} = \sqrt{\frac{B}{C}};$$

$$\text{and } e^\gamma = \sqrt{AD} + \sqrt{AD-1} = A + \sqrt{A^2-1}$$

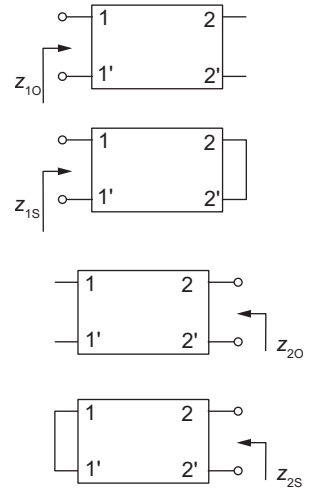


Fig. 16.8-2 Definition of Z_{1o} , Z_{1s} , Z_{2o} and Z_{2s}

of Z_{1o} and Z_{2o} is designated by a symbol Z_{oc} and the equal values of Z_{1s} and Z_{2s} is designated by a symbol Z_{sc} in the case of a symmetric two-port. Thus, for a symmetric linear time-invariant two-port network,

$$Z_o = \sqrt{\frac{B}{C}} = \sqrt{Z_{oc} Z_{sc}} \quad (16.8-8)$$

Z_{oc} and Z_{sc} are independent of the port from where they are measured in the case of a symmetric network.

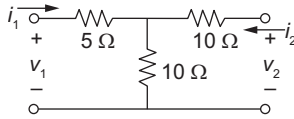


Fig. 16.8-3 Circuit for Example 16.8-1

EXAMPLE: 16.8-1

(i) Find the ABCD parameters for the asymmetric T-network in Fig. 16.8-3 and obtain its image parameters. (ii) Calculate the image impedances from open-circuit and short-circuit impedances and verify the values.

SOLUTION

(i) With the second port open $v_2 = \frac{10}{5+10}v_1 = \frac{2}{3}v_1$ and $i_1 = \frac{v_2}{10} = 0.1v_2$. Therefore, $A = 1.5$ and $C = 0.1 \text{ S}$. With the second port shorted $i_2 = -i_1 \times \frac{10}{10+10} = -0.5i_1$. Therefore, $D = 2$. The network is a reciprocal one. Therefore, $AD - BC = 1$. Hence, $B = 20 \Omega$.

$$\text{Image impedance } Z_{im1} = \sqrt{\frac{AB}{CD}} = \sqrt{\frac{1.5 \times 20 \Omega}{0.1 \text{ S} \times 2}} = 12.25 \Omega$$

$$\text{Image impedance } Z_{im2} = \sqrt{\frac{DB}{CA}} = \sqrt{\frac{2 \times 20 \Omega}{0.1 \text{ S} \times 1.5}} = 16.33 \Omega$$

$$e^{\gamma} = \sqrt{AD} + \sqrt{AD - 1} = \sqrt{1.5 \times 2} + \sqrt{(1.5 \times 2) - 1} = 3.15$$

$$\therefore \text{Image transfer constant} = \ln 3.15 = 1.15 \text{ Np}.$$

In a resistive network with a single independent voltage source, the voltage across any terminal pair can only be $\leq$ the applied voltage. Therefore, the voltage ratio $\frac{v_1}{v_2}$ has to be a real number ≥ 1 . Similarly, the current flow at any point in a single-source

resistive network can only be $\leq$ the source current. Therefore, the current ratio $\frac{i_1}{-i_2}$

has to be a real number ≥ 1 . Therefore, e^{γ} , the image transfer constant, has to be a positive real number for a resistive two-port. The circuit with image impedance termination is shown in Fig. 16.8-4.

The impedance seen at the input is the other image impedance – i.e., 12.25Ω . $\therefore i_1 = 0.08163v_1$. This current divides into 10Ω and 26.33Ω in parallel. $\therefore -i_2 = 0.08163v_1 \times 10/36.33 = 0.02247v_1$ and $v_2 = 0.02247v_1 \times 16.33 = 0.367v_1$. Therefore, $\frac{v_1}{v_2} = 1/0.367 =$

2.7254 and $\frac{i_1}{-i_2} = 36.33/10 = 3.633$. Therefore, $e^{\gamma} = \sqrt{2.7254 \times 3.633} = 3.15$. This is the same

that value we obtained using ABCD parameters.

(ii) The open-circuit and short-circuit impedances are calculated as

$$Z_{1o} = 5 \Omega + 10 \Omega + 15 \Omega$$

$$Z_{1s} = 5 \Omega + 10 \Omega // 10 \Omega = 10 \Omega$$

$$Z_{2o} = 10 \Omega + 10 \Omega = 20 \Omega$$

$$Z_{2s} = 10 \Omega + 5 \Omega // 10 \Omega = 13.33 \Omega$$

$$\text{Image impedance } Z_{im1} = \sqrt{Z_{1o} Z_{1s}} = \sqrt{15 \Omega \times 10 \Omega} = 12.25 \Omega$$

$$\text{Image impedance } Z_{im2} = \sqrt{Z_{2o} Z_{2s}} = \sqrt{20 \Omega \times 13.33 \Omega} = 16.33 \Omega$$

They are the same as the ones calculated from ABCD parameters.

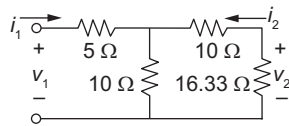


Fig. 16.8-4 Circuit in Example 16.8-1 with Image Impedance Termination

EXAMPLE 16.8-2

- (i) Find the $ABCD$ parameters of the network in Fig. 16.8-5 for $\omega = 1$ rad/s and $\omega = 4$ rad/s. (ii) Obtain the image parameters at these two frequencies. (iii) Find the steady-state output voltage as a function of time if $v_1(t) = \cos t$ V and the output port is terminated in characteristic impedance. (iv) Repeat part (iii) if $v_1(t) = \cos 4t$ V.

SOLUTION

(i) This is a symmetric Π -network with $Z(s) = s$ and $Y(s) = s$. We have already obtained the $ABCD$ matrix of the symmetric Π -network in Example 16.3-1.

$$ABCD \text{ Matrix of the } \Pi\text{-network is } \begin{bmatrix} 1 + \frac{1}{2}Z(s)Y(s) & Z(s) \\ Y(s)(1 + \frac{1}{4}Z(s)Y(s)) & 1 + \frac{1}{2}Z(s)Y(s) \end{bmatrix}$$

Substituting $Z(s) = s$ and $Y(s) = s$, we get, $A = 1 + 0.5s^2$ and $B = s$.

We get the $ABCD$ parameters required for sinusoidal steady-state analysis by substituting $s = j\omega$. Then, $A = D = 1 - 0.5\omega^2$, $B = j\omega$ and $C = j\omega(1 - 0.25\omega^2)$.

ω	A	B (Ω)	C (S)	D
1 rad/s	0.5	$j1$	$j0.75$	0.5
4 rad/s	-7	$j4$	$-j12$	-7

(ii) The network is a symmetric one. Therefore, there are only two image parameters. They are the characteristic impedance Z_o and the propagation constant γ .

$$Z_o = \sqrt{\frac{B}{C}} \text{ and } e^\gamma = A + \sqrt{BC}$$

ω	$Z_o (\Omega)$	e^γ	γ
1 rad/s	± 1.1547	$0.5 \pm j0.866$	$\pm j\pi/3$
4 rad/s	$\pm j0.5774$	$-0.0718, -13.93$	$-2.634 + j\pi, 2.634 + j\pi$

The values for Z_o and γ are calculated and tabulated in the table above. Many comments on the calculation procedure and the nature of values arrived at are in order at this point.

The equation for Z_o admits two possible values for it, since it is a square root. The value of Z_o at $\omega = 1$ rad/s is real. There is no confusion as to which value of Z_o is to be accepted at this frequency. It is 1.1547Ω of resistance. We rule out -1.1547Ω , since no physical resistor can have a negative resistance value.

The equation $e^\gamma = A + \sqrt{BC}$ was obtained by substituting $Z_o = \sqrt{\frac{B}{C}}$ in the equation $e^\gamma = A + \frac{B}{Z_o}$. Therefore, the sign of root to be accepted in $e^\gamma = A + \sqrt{BC}$ is the same as the

sign of root accepted in $Z_o = \sqrt{\frac{B}{C}}$. Therefore, at $\omega = 1$ rad/s, $e^\gamma = 0.5 + j0.866$ if the value of Z_o accepted is 1.1547Ω .

We expect γ to come out as a complex number in this case. Let $\gamma = \alpha + j\beta$. Then, $e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta$. Therefore, magnitude of e^γ must be e^α and angle of e^γ (in radians) must be β . Magnitude of e^γ in the present case is $\sqrt{0.5^2 + 0.866^2} = 1$ and the angle is $\tan^{-1} \frac{0.866}{0.5} = \frac{\pi}{3}$.

Therefore, $\alpha = 0$ and $\beta = \pi/3$ and $\gamma = 0 + j\pi/3$.

Thus, the propagation constant is a pure imaginary number. If propagation constant is purely imaginary, then, the e^γ factor must have a magnitude of unity. That will imply that the amplitude of input and output at 1 rad/s are equal – i.e., there is no loss of amplitude when a 1 rad/s sinusoidal waveform goes through this two-port network, provided the termination impedance is a resistance of 1.1547Ω .

We consider the second frequency of 4 rad/s now. At this frequency, the characteristic impedance is a pure reactance. It can be an inductive reactance of 0.5774Ω corresponding to a positive sign in the square root operation or it can be a capacitive reactance of 0.5774Ω corresponding to a negative sign in the square root operation. Both values are physically realisable. If an inductive reactance of 0.5774Ω is connected across the output, the value of e^γ is -0.0718 . If a capacitive reactance of 0.5774Ω is connected across the output, the value of e^γ is -13.93 .

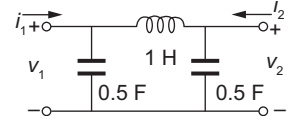


Fig. 16.8-5 Circuit for Example 16.8-2

In the first case, $e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 0.0718 \angle \pi$. Therefore, $\alpha = \ln 0.0718 = -2.634 \text{ Np}$ and $\beta = \pi \text{ rad}$ leading to $\gamma = -2.634 + j\pi$. But this value of γ implies that the input voltage amplitude is much less than the output voltage amplitude, since $\frac{V_1}{V_2} = e^\gamma$. Thus, there is amplification of voltage amplitude. This happens due to partial cancellation of $j2 \text{ S}$ admittance of capacitor at 4 rad/s by $j0.5774 \Omega$ of inductive impedance. The resultant impedance of parallel combination is $-j3.732 \Omega$ at 4 rad/s . This negative impedance cancels most of $j4 \Omega$ impedance in the series path leading to a large flow of current towards output and consequent voltage amplification.

In the second case, $e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 13.93 \angle \pi$. Therefore, $\alpha = \ln 13.93 = 2.634 \text{ Np}$ and $\beta = \pi \text{ rad}$ leading to $\gamma = 2.634 + j\pi$. This value of γ implies that the output voltage amplitude is only $1/13.93 = 0.0718$ times the input amplitude at 4 rad/s . There is considerable attenuation of the signal at 4 rad/s .

Thus, if attenuation is the aim, the termination has to be done by connecting a $-j0.5774 \Omega$ at 4 rad/s – i.e., a capacitor of 0.433 F – across the output.

(iii) $v_1 = \cos t \text{ V}$. Therefore $\omega = 1 \text{ rad/s}$. The ratio $\frac{V_1}{V_2} = e^\gamma = 0.5 + j0.866 = 1 \angle 60^\circ$ at this frequency. This ratio has to be understood as a phasor ratio in the context of sinusoidal steady-state analysis. Therefore $v_2 = \cos(t - 60^\circ) \text{ V}$.

(iv) $v_1 = \cos 4t \text{ V}$. Therefore, $\omega = 4 \text{ rad/s}$. The ratio $\frac{V_1}{V_2} = e^\gamma = -0.0718$ at this frequency if termination is by a 0.1444 H inductor and it is -13.93 if termination is by a 0.433 F capacitor. This ratio has to be understood as a phasor ratio in the context of sinusoidal steady-state analysis. Therefore $v_2 = -13.93 \cos(4t) \text{ V}$ if terminated by 0.1444 H inductor and $v_2 = -0.0718 \cos(4t) \text{ V}$ if terminated by a 0.433 F capacitor.

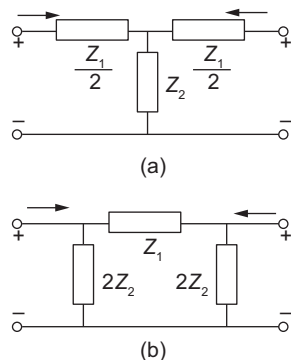


Fig. 16.9-1
(a) Symmetric
T-Network
(b) Symmetric
II-Network

Characteristic
impedance of
a symmetric
T-network

16.9 CHARACTERISTIC IMPEDANCE AND PROPAGATION CONSTANT OF SYMMETRIC T AND PI NETWORKS UNDER SINUSOIDAL STEADY-STATE

Symmetric reactive T- and Π -networks are used in designing passive filters. This section discusses the characteristic impedance and propagation constant of T- and Π -networks to set the background for the discussion on passive filters in the subsequent sections.

The standard T- and Π -networks employing the same underlying phasor impedances Z_1 and Z_2 are shown in Fig. 16.9-1(a) and (b).

Consider the T-network first. Let Z_{OT} be the characteristic impedance of the T-network.

$$Z_{oc} = \frac{Z_1}{2} + Z_2 = \frac{Z_1 + 2Z_2}{2}$$

$$Z_{sc} = \frac{\frac{Z_1}{2} \times Z_2}{\frac{Z_1}{2} + Z_2} + \frac{Z_1}{2} = \frac{Z_1 Z_2}{Z_1 + 2Z_2} + \frac{Z_1}{2} = \frac{Z_1^2 + 4Z_1 Z_2}{2(Z_1 + 2Z_2)}$$

$$\therefore Z_{OT} = \sqrt{Z_{oc} Z_{sc}} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} \quad (16.9-1)$$

$$A = 1 + \frac{Z_1}{2Z_2} \text{ and } B = \frac{Z_1 Z_2 + \frac{Z_1^2}{4}}{Z_2} = \frac{Z_{OT}^2}{Z_2^2} \quad (\text{see Example 16.7-1})$$

Let γ be the propagation constant of T-network. We expect γ to be a complex number when we evaluate it for sinusoidal steady-state analysis. Hence, we assume $\gamma = \alpha + j\beta$.

$$e^\gamma = A + \frac{B}{Z_{OT}} \text{ for a symmetric T-network. Therefore,}$$

$$e^{\gamma} = e^{\alpha + j\beta} = e^{\alpha} \angle \beta = 1 + \frac{Z_1}{2Z_2} + \frac{Z_{OT}}{Z_2} \quad (16.9-2)$$

Now, consider the Π -network in Fig. 16.9-1. Let $Z_{O\pi}$ be the characteristic impedance of the Π -network.

$$\begin{aligned} Z_{OC} &= (Z_1 + 2Z_2) / 2Z_2 = \frac{(Z_1 + 2Z_2)2Z_2}{(Z_1 + 4Z_2)} \\ Z_{SC} &= Z_1 / 2Z_2 = \frac{2Z_1Z_2}{(Z_1 + 2Z_2)} \\ \therefore Z_{O\pi} &= \sqrt{Z_{OC}Z_{SC}} = \sqrt{\frac{4Z_1Z_2^2}{Z_1 + 4Z_2}} = Z_1Z_2 \frac{1}{\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}} = \frac{Z_1Z_2}{Z_{OT}} \\ Z_{O\pi} &= \frac{Z_1Z_2}{Z_{OT}} \end{aligned} \quad (16.9-3)$$

Equation 16.9-3 shows the relation between the characteristic impedances of a T-network and a Π -network that use the same Z_1 and Z_2 . The two characteristic impedances are different.

We require A and B of the Π -network in order to determine its propagation constant, for a symmetric Π -network. Therefore,

$$\begin{aligned} A &= 1 + \frac{Z_1}{2Z_2} \text{ and } B = Z_1 \\ e^{\gamma} &= A + \frac{B}{Z_{O\pi}} \text{ for a symmetric } \Pi\text{-network. Therefore,} \\ e^{\gamma} &= e^{\alpha + j\beta} = e^{\alpha} \angle \beta = 1 + \frac{Z_1}{2Z_2} + \frac{Z_{OT}}{Z_2} \end{aligned} \quad (16.9-4)$$

Equation 16.9-4 and Eqn. 16.9-2 show that the propagation constants of a T-network and a Π -network that use the same Z_1 and Z_2 are the same.

Propagation characteristics of a symmetric T-network.

Characteristic impedance of a Π -network.

Propagation characteristics of the Π -network is the same as that of T-network.

16.9.1 Attenuation Constant α and Phase Constant β

The propagation constant γ of a symmetric network will be a complex function of the angular frequency variable ω for sinusoidal steady-state analysis. The real part of this complex propagation constant is called the *attenuation constant* and the imaginary part is called the *phase constant*.

The ratio of input voltage phasor to output voltage phasor in a symmetric network under the single-frequency sinusoidal steady-state condition is given by e^{γ} with γ calculated at the corresponding frequency. The ratio of input current phasor to output current phasor is given by the same number. The magnitude of e^{γ} gives the ratio of input sinusoid amplitude to output sinusoid amplitude and the angle part of e^{γ} gives the *phase angle by which the input sinusoidal waveform leads the output sinusoidal waveform under the steady-state condition*. The magnitude of e^{γ} , is e^{α} and its angle is β rad. Thus, the amplitude of output is $1/e^{\alpha}$ times the amplitude of input and the output sinusoidal waveform *lags behind* the input sinusoidal waveform by β rads.

We are interested in the *gain* provided to an input signal when we deal with an amplifier network. A filter is expected to pass sinusoidal signal components falling within a specified frequency range with no change in amplitude or phase and to stop signal components falling outside that range completely. Hence, we will be interested in the *loss of amplitude* rather than the *gain in amplitude* when it comes to a filter network. Therefore, an electric filter is described by its (1/gain) characteristic rather than by its gain characteristic. *Attenuation* of a network is defined as (1/gain) of that network.

Attenuation and Phase Constants

Attenuation of a network is defined as (1/gain) of that network, where 'gain' is the magnitude of sinusoidal steady-state frequency response function.

If γ is the propagation constant evaluated at ω rad/s, then, $\gamma = \alpha + j\beta$, where α is the 'Attenuation Constant' and β is the 'Phase Constant' at ω rad/s.

Attenuation = e^{α}
Attenuation in Nepers = α
Attenuation in db = 8.686α db

Therefore, we characterise the symmetric T-network and Π -network by *attenuation* rather than by *gain*. *Attenuation* in absolute value is given by e^α . However, it has become customary in filter engineering to specify attenuation in dB (decibel) units.

$$\text{Attenuation in dB} = 20 \log_{10} e^\alpha \text{ dB} = \frac{20 \ln e^\alpha}{\ln 10} = 8.686\alpha \text{ dB}$$

Thus, the attenuation in dB units in a two-port network is directly proportional to the real part of its propagation constant. This explains why the real part (α) of γ is called the *attenuation constant*. The unit of ‘*neper*’ is sometimes used for α .

16.10 CONSTANT-k LOW-PASS FILTER

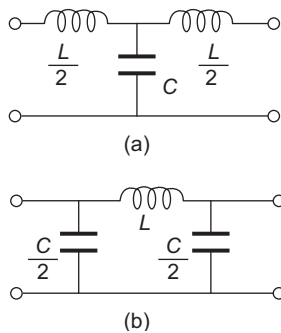


Fig. 16.10-1 (a) Constant-k T-Section Low-pass Filter (b) Constant-k Π -Section Low-pass Filter

We commence our discussion on passive filters designed using symmetric reactive two-port networks from this section onwards. The constant-k low-pass filter is taken up first. It is also called the ‘*Prototype LPF Section*’. The T-network and Π -network versions of prototype low-pass filter section are shown in Fig. 16.10-1.

The characteristic impedance and propagation constant of these two prototype sections can be determined by substituting $Z_1 = j\omega L$ and $Z_2 = 1/j\omega C$ in the corresponding expressions arrived at in Sect. 16.9.

Note that the product of Z_1 and Z_2 in both sections is a constant, independent of angular frequency. This is the reason behind the name ‘constant-k low-pass filter’. Let this constant be designated as R_o^2 . That is,

$$R_o = \sqrt{Z_1 Z_2} = \sqrt{\frac{L}{C}} \quad (16.10-1)$$

R_o has dimensions of resistance. Now, the characteristic impedance of the T-section low-pass filter can be expressed as

$$Z_{or} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = \sqrt{R_o^2 - \frac{\omega^2 L^2}{4}} = R_o \sqrt{1 - \frac{\omega^2 L^2}{4R_o^2}} = R_o \sqrt{1 - \frac{\omega^2 LC}{4}}$$

Let ω_c be defined as equal to $\frac{2}{\sqrt{LC}}$. Then,

$$Z_{or} = R_o \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} = R_o \sqrt{1 - \left(\frac{f}{f_c}\right)^2} \quad (16.10-2)$$

where, $R_o = \sqrt{\frac{L}{C}} \Omega$, $\omega_c = \frac{2}{\sqrt{LC}}$ rad/s and $f_c = \frac{1}{\pi \sqrt{LC}}$ Hz

The characteristic impedance of Π -section is known to be given by $Z_{o\pi} = \frac{Z_1 Z_2}{Z_{or}}$.

Therefore,

$$Z_{o\pi} = \frac{R_o^2}{R_o \sqrt{1 - \left(\frac{f}{f_c}\right)^2}} = \frac{R_o}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}} \text{ where } R_o = \sqrt{\frac{L}{C}} \Omega \text{ and } f_c = \frac{1}{\pi \sqrt{LC}} \text{ Hz} \quad (16.10-3)$$

Z_{or} and $Z_{o\pi}$ are *resistive* for $0 \leq f < f_c$ and are *reactive* for $f_c < f < \infty$. The variation of Z_{or} and $Z_{o\pi}$ for $0 \leq f < f_c$ is shown in Fig. 16.10-2.

The quantity R_o gets another interpretation now – it is the value of characteristic impedance of T-section and Π -section prototype low-pass filters at zero frequency – i.e., for DC input. The characteristic impedance of both sections vary with frequency and

Constant-k Filters

A constant-k filter is a reactive T- or Π -section such that the product of Z_1 and Z_2 (the impedances used to derive the series arm and shunt arm, respectively) is a constant, independent of frequency.

$\therefore Z_1 = jX_1, Z_2 = jX_2$ and

$Z_1 Z_2 = -X_1 X_2 = \alpha$
constant $= R_o^2$

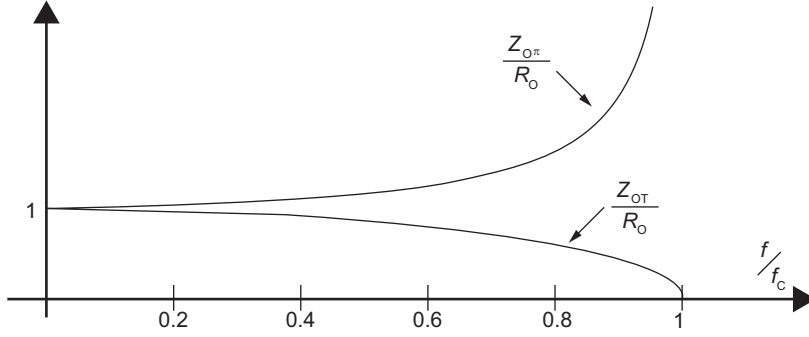


Fig. 16.10-2 Variation of Z_{OT} and $Z_{O\pi}$ of Low-Pass Sections in the Frequency Range $0 \leq f < f_c$

terminating the filter with a fixed resistance will not result in the characteristic impedance termination at all frequencies in the range $0 \leq f < f_c$. Characteristic impedance is reactive outside this frequency range, and hence, resistive termination will not result in the characteristic impedance termination in any case as far as $f_c < f$ range is concerned.

We take up the attenuation features of these two prototype sections now. We had noted in Sect. 16.9 that T-section and Π -section using the same Z_1 and Z_2 will have the same propagation constant. The propagation constant γ is governed by Eqn. 16.9-4.

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 + \frac{Z_1}{2Z_2} + \frac{Z_{OT}}{Z_2}$$

But the reader is reminded of the fact that this equation is valid only if the network is terminated in its characteristic impedance at the frequency at which this expression is evaluated.

Substituting $Z_1 = j\omega L$ and $Z_2 = 1/j\omega C$ in the above equation and simplifying the expressions using the previously defined quantities Z_{OT} and f_c , we get,

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 - 2x^2 + j2x\sqrt{1-x^2} \text{ where } x = \frac{f}{f_c} \quad (16.10-4)$$

We study this function e^γ for the two frequency ranges in detail.

Behaviour of e^γ in the range $0 \leq f < f_c$

The value of x in this range is between 0 and 1, and therefore, the function e^γ evaluates to a complex number with non-zero imaginary part. The sign of $\sqrt{1-x^2}$ that we accept has to be the same as the sign we accepted in evaluating the characteristic impedance in this frequency range. We had accepted a positive sign so that the characteristic impedance in this frequency range will turn out to be a positive-valued resistance. Therefore, we accept the positive root of $(1-x^2)$ in Eqn. 16.10-4. The magnitude of the right-side of Eqn. 16.10-4 gives the attenuation in the filter (i.e., e^α).

$$\therefore \text{Attenuation} = e^\alpha = \sqrt{\left[(1-2x^2)^2 + \left(2x\sqrt{1-x^2} \right)^2 \right]}, \text{ where } x = \frac{f}{f_c} \quad (16.10-4)$$

Simplifying the expression, we get,

$$\text{Attenuation} = e^\alpha = 1$$

Therefore, the *attenuation constant* α is 0 in this frequency range. Therefore, if a sinusoidal signal with frequency $< f_c$ is applied to the filter, the output amplitude will be the same as the input amplitude, *provided the filter is terminated in its characteristic impedance*

at that frequency – i.e., $R_o \sqrt{(1-x^2)} \Omega$ in the case of *T*-section and $\frac{R_o}{\sqrt{(1-x^2)}} \Omega$ in the case

of *Π*-section, where $x = f/f_c$.

Thus, the frequency range $0 \leq f < f_c$ constitutes the *pass-band* of the filter. But zero attenuation (and unity gain) is possible if, and only if, the filter is terminated in its characteristic impedance at all frequencies.

What is the behaviour of the phase angle part of attenuation function (i.e., e^γ) in the frequency range $0 \leq f < f_c$?

$$\beta = \text{Angle of } [1 - 2x^2 + j2x\sqrt{1-x^2}], \text{ where } x = \frac{f}{f_c} \quad (16.10-4)$$

We have seen that the magnitude of $[1 - 2x^2 + j2x\sqrt{1-x^2}]$ is 1 in the range $0 \leq x < 1$.

Therefore, $\sin \beta$ must be $2x\sqrt{1-x^2}$ and is positive in this range. Therefore, β is expected to be between 0 and π rads. Moreover, $\cos \beta$ must be $1 - 2x^2$.

Phase response
of constant- k
low-pass filter in
pass-band
(Z_o termination is
assumed).

$$\begin{aligned} \cos \beta &= 1 - 2x^2 \Rightarrow 1 - 2 \sin^2 \frac{\beta}{2} = 1 - 2x^2 \\ \therefore \sin \frac{\beta}{2} &= x \quad (\because \text{we expect } \beta \text{ to be positive}) \\ \therefore \beta &= 2 \sin^{-1} x = 2 \sin^{-1} \frac{f}{f_c} \text{ rads} \end{aligned} \quad (16.10-5)$$

Thus, a sinusoidal signal with its frequency $< f_c$ undergoes no change in its amplitude, but suffers a phase lag of $2 \sin^{-1} \frac{f}{f_c}$ rads when it goes through the prototype low-pass filter, *provided the filter is terminated in its characteristic impedance at that frequency.*

Behaviour of e^γ of prototype low-pass filter in the range $f_c < f$ (i.e., $x > 1$)

$Z_{oT} = R_o \sqrt{(1-x)^2}$ is a pure reactance in this range. We express it as a reactance in the following manner.

$$Z_{oT} = \pm jR_o \sqrt{(x^2 - 1)} \quad (16.10-6)$$

The positive sign corresponds to inductive termination and the negative sign corresponds to capacitive termination. Now, the attenuation function is

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 - 2x^2 + j2x\sqrt{1-x^2} = 1 - 2x^2 \pm j^2 2x\sqrt{x^2 - 1} = 1 - 2x^2 \mp 2x\sqrt{x^2 - 1}$$

with a negative sign for inductive termination and a positive sign for capacitive termination. The filter is expected to provide a large attenuation to signals in the frequency range $f > f_c$. Therefore, we must choose a negative sign in the expression above. *That is, the T-section low-pass filter must be terminated in an inductive reactance in the $f > f_c$ band for good attenuation.* Then,

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 - 2x^2 - 2x\sqrt{x^2 - 1} \quad (16.10-7)$$

We expect this number to be a negative real number since $x > 1$ in the frequency range we are considering now. Therefore,

$$e^\alpha = 2x^2 - 1 + 2x\sqrt{x^2 - 1} \text{ and } \beta = \pi \text{ rads} \quad (16.10-8)$$

What follows now is an algebraic manipulation to express α in a convenient form.

Let $\sqrt{z} = 2x^2 - 1$ where z is a temporary variable. Then

$$\begin{aligned} z &= (2x^2 - 1)^2 = 4x^4 - 4x^2 + 1 \\ \therefore z - 1 &= 4x^4 - 4x^2 = 4x^2(x^2 - 1) \\ \therefore \sqrt{z-1} &= 2x\sqrt{(x^2 - 1)} \end{aligned}$$

Now, we express e^α in terms of z as

$$e^\alpha = \sqrt{z} + \sqrt{z-1} \quad (16.10-9)$$

Further, we write e^α as below.

$$e^\alpha = \frac{e^\alpha + e^{-\alpha}}{2} + \frac{e^\alpha - e^{-\alpha}}{2} = \cosh \alpha + \sinh \alpha$$

An identity involving hyperbolic functions helps us at this point.

$$\begin{aligned} \cosh^2 \alpha - \sinh^2 \alpha &= 1 \text{ (Identity)} \Rightarrow \sinh \alpha = \sqrt{\cosh^2 \alpha - 1} \\ \therefore e^\alpha &= \cosh \alpha + \sqrt{\cosh^2 \alpha - 1} \end{aligned} \quad (16.10-10)$$

Comparing Eqn. 16.10-9 and Eqn. 16.10-10, we see that,

$$\cosh \alpha = \sqrt{z} = 2x^2 - 1$$

$$\text{But } \cosh \alpha = 2 \cosh^2 \frac{\alpha}{2} - 1 \text{ (Identity)}$$

$$\therefore 2 \cosh^2 \frac{\alpha}{2} - 1 = 2x^2 - 1 \Rightarrow \cosh \frac{\alpha}{2} = x$$

$$\therefore \alpha = 2 \cosh^{-1} x = 2 \cosh^{-1} \frac{f}{f_c} \quad (16.10-11)$$

The attenuation in dB is given by

$$\text{Attenuation} = 17.372 \cosh^{-1} \frac{f}{f_c} \text{ dB} \quad (16.10-12)$$

The attenuation constant α is positive for all $f > f_c$, since inverse hyperbolic cosine is positive in that range. Thus, the filter reduces the amplitude of signals in this frequency range and adds a phase of $-\pi$ rad to them. Therefore, the frequency range $f > f_c$ is the *stop-band* of the filter. The frequency value that separates the *pass-band* and *stop-band* – i.e., f_c – is called the *cut-off frequency* of the filter.

Note that Eqn. 16.10-11 gives the attenuation constant and Eqn. 16.10-12 yields the dB attenuation correctly if, and only if, the filter is terminated in an *inductive reactance* of value $R_o \sqrt{(x^2 - 1)} \Omega$ in the case of T-section filter, where $x = f/f_c$. The reader may verify that

the Π -section filter has to be terminated in a *capacitive reactance* of value $\frac{R_o}{\sqrt{(x^2 - 1)}} \Omega$ for the same purpose.

Figure 16.10-3 shows the dB attenuation and phase constant in the pass-band and stop-band for the prototype low-pass filter terminated in its characteristic impedance at all frequencies.

Attenuation provided by constant-k low-pass filter in the stop-band (Z_o termination is assumed).

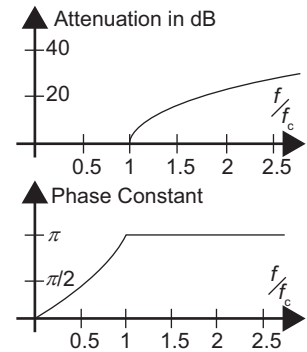


Fig. 16.10-3 dB Attenuation and Phase Constant for a Prototype Low-Pass Filter

16.10.1 Ideal Low-pass Filter Versus Constant-k Low-pass Filter

An ideal low-pass filter is expected to provide 0 dB attenuation and 0 rad phase shift to all sinusoidal components with frequencies below the cut-off frequency f_c . Further, it is

A practical constant-k low-pass filter has non-zero attenuation in pass-band.

expected to provide infinite attenuation (i.e., zero gain) for all sinusoidal components with frequency $>f_c$. A constant-k low-pass filter fails in both due to two reasons.

A constant-k low-pass filter can provide zero attenuation in the pass-band, provided it is terminated in its characteristic impedance, but its characteristic impedance in the pass-band is a *frequency dependent resistance*. The load on the filter is usually a fixed resistor. Hence, the filter is terminated in its characteristic impedance at only one frequency, at the best. This frequency is chosen to be DC usually. Failure to terminate the filter in its characteristic impedance in the pass-band results in non-zero attenuation for all frequency components except for the DC component in the input signal.

The rise of attenuation with frequency in the stop-band is gradual even if the filter is terminated in its characteristic impedance. If the filter is not terminated at Z_o , the rise of attenuation with frequency will be still slower. In actual practice, the filter is terminated at Z_o only for $\omega = 0$. This makes the stop-band attenuation far from satisfactory in practical applications.

Thus, the major issues with the prototype low-pass filter are (i) problem of non-zero pass-band attenuation due to termination and (ii) unsatisfactory stop-band attenuation.

There are satisfactory solutions to these two issues. But before we take them up, let us develop design equations for a prototype low-pass filter from specifications.

16.10.2 Prototype Low-pass Filter Design

The design specifications are the values of cut-off frequency f_c and the values of load resistance R_L . We make the filter experience characteristic impedance termination at $\omega = 0$. Both T-section and Π -section have characteristic impedance of R_o at this frequency.

$$\therefore R_o = \sqrt{\frac{L}{C}} = R_L$$

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

Design equations for a constant-k low-pass prototype filter.

Solving these two equations for L and C , we get,

$$L = \frac{R_L}{\pi f_c} \text{ H and } C = \frac{1}{\pi R_L f_c} \text{ F}$$

The designs are given in Fig. 16.10-4.

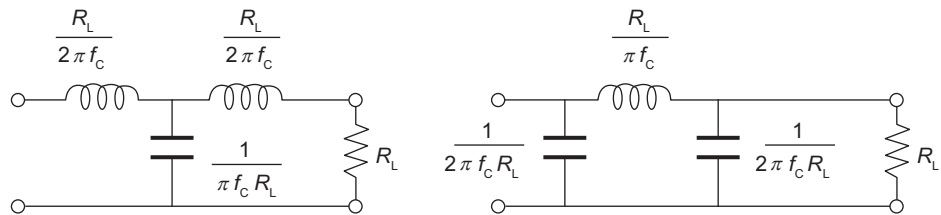


Fig. 16.10-4 Prototype Low-Pass Filter Designs

Now, we address the issue of unsatisfactory rise in attenuation in the stop-band in the prototype filter and arrive at a solution to this problem in the next section

16.11 m -DERIVED LOW-PASS FILTER SECTIONS FOR IMPROVED ATTENUATION

An m -derived LPF (Low-Pass Filter) section is a circuit section that is cascaded with the prototype section in order to improve the performance of the prototype section. More than one such m -derived section may be employed in a practical filter. Each such section results in the overall filter, providing zero gain (infinite attenuation) at a frequency above the cut-off frequency of the prototype section. We use T-sections in our discussion almost exclusively. Results for Π -sections will be arrived at by comparison.

Consider the T-network shown in Fig. 16.11-1.

The shunt branch comprising an inductor and a capacitor will resonate at $f_\infty = \frac{1}{2\pi\sqrt{LC}}$ Hz. Hence, the output signal component at f_∞ will be zero, thanks to the short-circuit caused by the series resonance in the shunt connected branch. *This conclusion is independent of the termination impedance at the output.*

However, the attenuation characteristics of the prototype filter section will get modified when we cascade an m -derived section with it unless (i) the input impedance of the m -derived section is equal to characteristic impedance of the prototype filter and (ii) the m -derived section is terminated in its characteristic impedance. The m -derived T-section is shown in Fig. 16.11-2.

Thus, m -derived T-section is obtained by scaling the series impedance of a prototype T-section by a real positive number ' m ' ($0 < m < 1$) and using an impedance Z_2' in the shunt branch such that the characteristic impedance of m -derived T-section is the same as the characteristic impedance of prototype T-section. Let Z_{om} be the characteristic impedance of the m -derived section. We want it to be equal to Z_{oT} .

$$Z_{om} = Z_{oT} \Rightarrow \sqrt{mZ_1Z_2' + \frac{m^2Z_1^2}{4}} = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}$$

Solving this equation for Z_2' , we get,

$$Z_2' = \frac{Z_2}{m} + \frac{1-m^2}{4m}Z_1 \quad (16.11-1)$$

m -derived Π -section is formed by scaling the shunt branch admittance by m and changing the series branch impedance suitably such that the resulting section has a characteristic impedance that is equal to Z_{om} . The derivation of this impedance is skipped and the result is shown in Fig. 16.11-3.

Z_1 is an inductor and Z_2 is a capacitor in the case of a low-pass filter. Thus, the m -derived LPF sections will be as in Fig. 16.11-4.

The series-tuned shunt branch in the m -derived T-section in Fig. 16.11-4 goes short at a frequency f_∞ and provides infinite attenuation at that frequency, quite independent of what is connected at the output. The value of m decides this frequency.

$$f_\infty = \frac{1}{2\pi\sqrt{(1-m^2)LC}} = \frac{1}{\pi\sqrt{LC}} \times \frac{1}{\sqrt{1-m^2}} = f_c \times \frac{1}{\sqrt{1-m^2}}$$

The parallel-tuned series branch in m -derived Π -section in Fig. 16.11-4 goes open at a frequency f_∞ and provides infinite attenuation at that frequency, quite independent of what is connected at the output. The reader may verify that f_∞ in Π -section is the same as f_∞ in T-section.

$$\therefore \frac{f_\infty}{f_c} = \frac{1}{\sqrt{1-m^2}} \text{ for an } m\text{-derived LPF section} \quad (16.11-2)$$

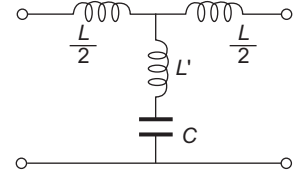


Fig. 16.11-1 A Low-pass Filter with zero gain at one frequency

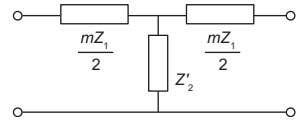


Fig. 16.11-2 An m -Derived T-Section

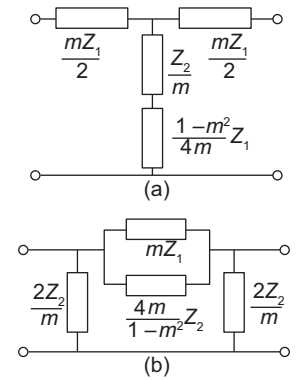


Fig. 16.11-3 (a) m -Derived T-Section and (b) m -Derived Π -Section

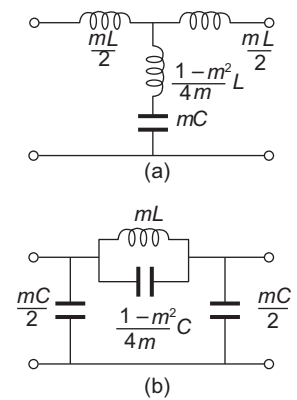


Fig. 16.11-4 (a) m -Derived T-Section Low-Pass Filter (b) m -Derived Π -Section Low-Pass Filter

Design specification will include the values of frequency at which infinite attenuation is desired in the filter. There has to be one m -derived section for each frequency at which infinite attenuation is desired. The value of m to be employed in each m -derived section can be worked out from the corresponding frequency at which infinite attenuation is needed and the filter cut-off frequency by employing Eqn. 16.11-2. The final filter design will have a prototype section cascaded with as many m -derived sections as the frequencies at which infinite attenuation is desired.

The pass-band of a filter section is the range of frequency for which the characteristic impedance is real-valued – i.e., resistive. An m -derived section has the same characteristic impedance as that of a prototype section. Therefore, the m -derived section and the prototype section will have the same pass-band.

The pass-band attenuation of the cascade will remain zero, *provided the filter is terminated in its characteristic impedance at its output*. But we have seen that this can not be done unless the filter load is a frequency-dependent resistance. Thus, the problem arising out of non- Z_o termination will continue to be present even in a filter with m -derived sections cascaded with prototype section.

However, the stop-band attenuation will be much improved in a cascaded filter. Each m -derived section forces the stop-band attenuation to reach infinite value at a certain frequency in the stop-band. The attenuation characteristic in stop-band can be tailored to meet stringent specifications by pulling down the gain to zero at a few judiciously selected stop-band frequency values with the help of m -derived sections. An m -derived section with m close to zero will result in stop-band attenuation rising rapidly after the cut-off frequency f_c .

However, if an m -derived section has the same pass-band performance as that of the prototype section, and, in addition, if it can introduce infinite attenuation at some stop-band frequency, why use a prototype filter section at all? Why do not we make the filter using m -derived sections exclusively?

Consider the T-section prototype low-pass filter. The shunt-connected branch is a pure capacitor. The impedance of the capacitor approaches zero at high frequencies. Therefore, a prototype section provides zero gain (infinite attenuation) at high frequencies. The m -derived T-section contains a series combination of inductor and capacitor in the shunt branch. Though the capacitor goes short at high frequencies, the shunt branch does not; thanks to the inductor. Therefore, the m -derived section provides a non-zero finite gain at high frequencies. Thus, the attenuation characteristic of m -derived section is unsatisfactory in the high frequency range. This is why at least one prototype section is needed in a cascaded filter structure.

A low-pass filter design terminates the filter at its Z_o value at $\omega = 0$. This is done by making its $R_o = \sqrt{L/C}$ equal to the load resistance R_L . Thus, the filter works under mismatched termination condition at all other frequencies. Consequently, the pass-band attenuation will be non-zero except at $\omega = 0$ in a filter cascade.

Why do we need a prototype section at all?

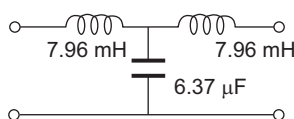


Fig. 16.11-5 The Prototype Section in Example 16.11-1

EXAMPLE: 16.11-1

Design a low-pass filter for a cut-off frequency of 1 kHz and a load resistance of 50 Ω . The filter should provide infinite attenuation at 1.5 kHz and 2 kHz. Use T-sections.

SOLUTION

One prototype section and two m -derived sections are needed for this design.

Step-1: Design of prototype low-pass section

Using the design formula arrived at in Sect. 16.10, we calculate L and C as

$$L = \frac{R_L}{\pi f_c} = \frac{50}{\pi \times 1000} = 15.92 \text{ mH}$$

$$C = \frac{1}{\pi R_L f_c} = \frac{1}{\pi \times 50 \times 1000} = 6.37 \mu\text{F}$$

The prototype section is shown in Fig. 16.11-5.

Step-2: Design of an m -derived T-section for $f_{\infty} = 1.5$ kHz

$$f_{\infty} = \frac{f_c}{\sqrt{1-m^2}} \Rightarrow m = \sqrt{1 - \left(\frac{f_c}{f_{\infty}}\right)^2}$$

$$\therefore m = \sqrt{1 - \frac{1}{1.5^2}} = 0.7454; \frac{mL}{2} = 5.93 \text{ mH}; mC = 4.75 \text{ } \mu\text{F}$$

$$\therefore \frac{1-m^2}{4m} = 0.149; \frac{1-m^2}{4m} L = 2.37 \text{ mH}$$

Therefore, the m -derived section required is as shown in Fig. 16.11-6.

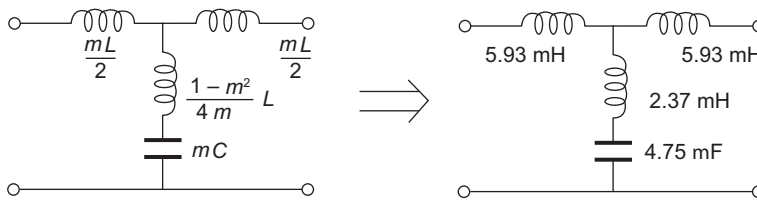


Fig. 16.11-6 m -Derived Section for $f_{\infty} = 1.5$ kHz in Example 16.11-1

Step-3: Design of an m -derived T-section for $f_{\infty} = 2$ kHz

$$\therefore m = \sqrt{1 - \frac{1}{2^2}} = 0.866; \frac{mL}{2} = 6.89 \text{ mH}; mC = 5.52 \text{ } \mu\text{F}$$

$$\therefore \frac{1-m^2}{4m} = 0.072; \frac{1-m^2}{4m} L = 1.15 \text{ mH}$$

Therefore, the m -derived section required is as shown in Fig. 16.11-7.

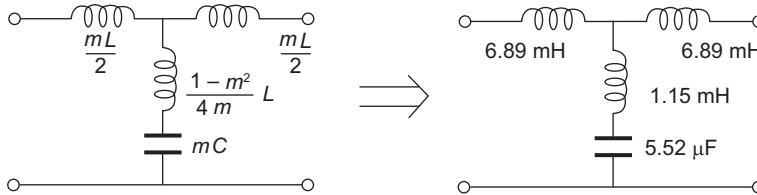


Fig. 16.11-7 m -Derived Section for $f_{\infty} = 2$ kHz in Example 16.11-1

Step-4: Simplify the final design by combining elements

The cascade of one prototype and two m -derived sections with the filter terminated at 50Ω load resistance is shown in Fig. 16.11-8.

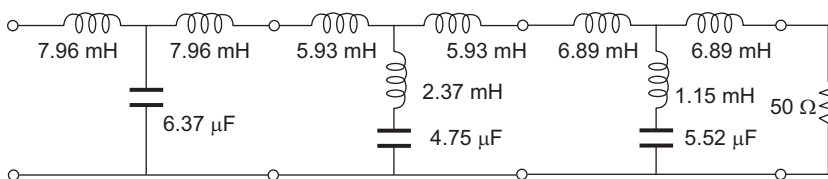


Fig. 16.11-8 Cascade of Prototype and m -Derived Sections in Example 16.11-1

The inductors that appear in series can be replaced with equivalents to arrive at the final design shown in Fig. 16.11-9.

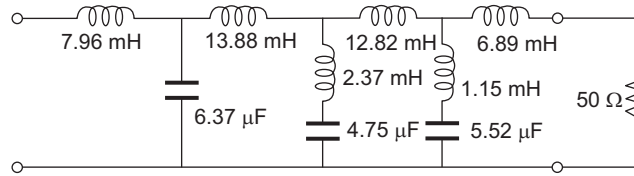


Fig. 16.11-9 Final Design Solution in Example 16.11-1

16.12 m -DERIVED HALF-SECTIONS FOR FILTER TERMINATION

The problem of unsatisfactory attenuation performance in the stop-band in the prototype filter has been addressed satisfactorily in the previous section. The problem of non-zero attenuation in the pass-band due to imperfect termination at the filter output remains. We are constrained to load the filter by a load resistance R_L from a practical point of view. We will, of course, make R_o equal to R_L , but that will result in Z_o -termination only at DC frequency. We address this issue in the current section.

The solution lies in interposing a two-port reactive network between the filter output and the load resistance. This network should have a property such that its input impedance is Z_{oT} (we assume a T-section filter) at least in the frequency range $0 \leq f < f_c$ (i.e., in the pass-band) when it is terminated at a resistance of R_o value. An m -derived half-section has this property to a satisfactory extent for a particular value of m – that value is $m = 0.6$.

An m -derived half-T section is formed as illustrated in Fig. 16.12-1 – Fig. 16.12-1(c) shows the m -derived half-T section that we want.

The m -derived half-T section is used to terminate a T-section filter cascade as shown in Fig. 16.12-2. Note that the half-T section is an *asymmetric network*, and therefore, it will have two image impedances – Z_{im1} and Z_{im2} . We expect to see that Z_{im2} is at least approximately equal to R_o in the frequency range $0 \leq f < f_c$ for some value of m and that Z_{im1} is equal to Z_{oT} .

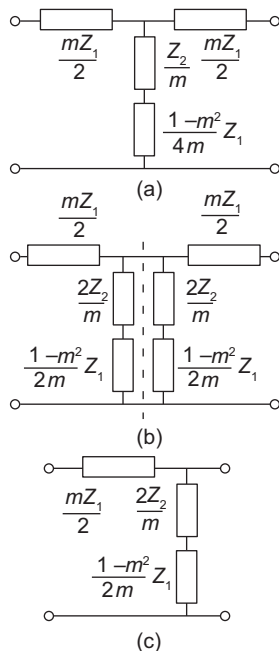


Fig. 16.12-1 (a) An m -Derived Section
(b) Bisection of m -Derived Section
(c) An m -Derived Half-T Section

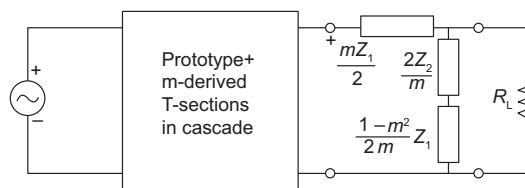


Fig. 16.12-2 Use of m -Derived Half-T Section as Termination Section

$$Z_{im1} = \sqrt{Z_{1o}Z_{1s}} = \frac{mZ_1}{2} \left(\frac{mZ_1}{2} + \frac{1-m^2}{2m}Z_2 + \frac{2Z_2}{m} \right) = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} = Z_{oT}$$

$$Z_{im2} = \sqrt{Z_{2o}Z_{2s}}$$

Substituting for Z_{2o} and Z_{2s} and simplifying,

$$Z_{im2} = \frac{Z_1Z_2}{\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}} \times \left[1 + \frac{1-m^2}{4} \frac{Z_1}{Z_2} \right] = Z_{o\pi} \times \left[1 + \frac{1-m^2}{4} \frac{Z_1}{Z_2} \right]$$

Substituting $Z_1 = j\omega L$, $Z_2 = 1/j\omega C$ and employing definitions of ω_c and f_c , we get,

$$\begin{aligned}\frac{Z_1}{Z_2} &= -\omega^2 LC = -4 \left(\frac{\omega}{\omega_c} \right)^2 = -4 \left(\frac{f}{f_c} \right)^2 = -4x^2 \text{ with } x = \left(\frac{f}{f_c} \right) \\ \therefore Z_{\text{im}2} &= Z_{o\pi} [1 - (1 - m^2)x^2] \\ \text{But } Z_{o\pi} &= \frac{R_o}{\sqrt{1 - x^2}} \Rightarrow \therefore Z_{\text{im}2} = \frac{R_o}{\sqrt{1 - x^2}} [1 - (1 - m^2)x^2]\end{aligned}$$

This impedance is resistive for $0 \leq x < 1$ (i.e., in the pass-band) for any $m < 1$. The variation of this resistance with x for different values of m is shown in Fig. 16.12-3.

The curves in Fig. 16.12-3 show that $Z_{\text{im}2}$ is more or less constant at R_o for about 85% range in the pass-band if $m = 0.6$. Therefore, if the half-T section is designed with $m = 0.6$ and is terminated at R_o at its output side, the input impedance will be Z_{oT} (i.e., $= R_o \sqrt{1 - x^2} \Omega$) for $0 \leq f \leq 0.85f_c$. Thus, the preceding filter sections will experience Z_o -termination for most of the pass-band range. Hence, the attenuation contributed by the prototype section and the m -derived sections will be zero for 85% of the pass-band. Therefore, $m = 0.6$ is the value used for the termination section.

All the filter sections – the prototype section and the m -derived sections – contribute zero attenuation in $0 \leq f \leq 0.85f_c$ range. However, the terminating section itself will contribute some non-zero attenuation in this band. This is so since *it is not a symmetric section* and its α will not be zero except at $\omega = 0$. However, even with this attenuation, the pass-band performance will be much better with the termination section than without it. All sections will cause pass-band attenuation if the termination section is not used.

The m -derived half-T section needed for terminating a T-section filter cascade is calculated with $m = 0.6$ and is shown in Fig. 16.12-4. L and C are the same values that are calculated in the design of a prototype filter. It may also be noted that like any other m -derived section, the terminating half-section will also contribute a frequency of infinite attenuation. This frequency at which infinite attenuation is available will be $1.25f_c$.

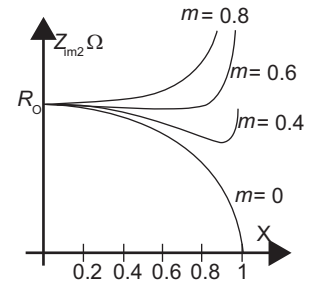


Fig. 16.12-3 Variation of Image Impedance $Z_{\text{im}2}$ of Half-T Section with Frequency and m

m -derived half-sections are designed with $m = 0.6$. Fig. 16.12-3 explains why.

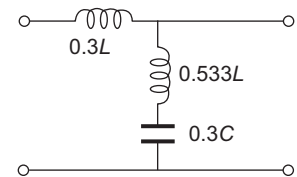


Fig. 16.12-4 m -Derived Half-T Output Terminating Section for a Low-Pass Filter

16.12.1 m -Derived Half-Sections for Input Termination

We have been ignoring the inevitably present source resistance in the analysis of filters till now. We take the source resistance of input source into account now. Refer to Fig. 16.12-5, which shows a T-section low-pass filter with output terminating section driven by a voltage source through a source resistance of R_s .

The input impedance of the filter is resistive and is equal to $R_o \sqrt{1 - x^2}$, where $x = f/f_c$ in the range $0 \leq x \leq 0.85$ due to the output termination. Therefore, the source voltage gets shared between R_s and $R_o \sqrt{1 - x^2}$ for almost the entire pass-band range of frequencies.

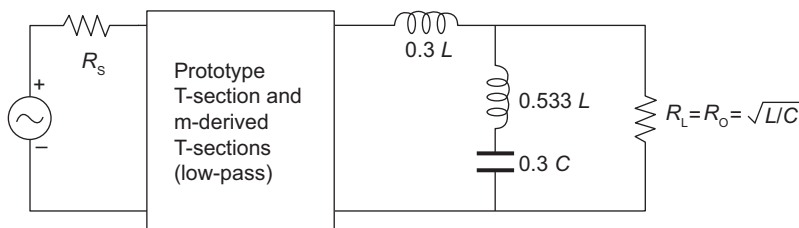


Fig. 16.12-5 A Low-pass Filter with Output Termination being Driven by a Source through a Resistance

This introduces an additional factor $\frac{R_o \sqrt{1-x^2}}{R_s + R_o \sqrt{1-x^2}}$ with $x = \frac{f}{f_c}$ in the gain

magnitude of the filter. The filter structure introduces only a small attenuation in the pass-band. However, this factor brings in a frequency-dependent attenuation in the overall filter. Frequency-dependent attenuation within the pass-band of a filter results in distortion of waveform – if a signal, which has all its frequency components within the pass-band, is applied to the filter, the output waveform is different from the input waveform. Therefore, we have to make the voltage-sharing ratio at the input independent of frequency. This can be done only by making the impedance faced by the source independent of frequency at least within the pass-band. This is possible by using a half-T section as an interface between the source and the filter as shown in Fig. 16.12-6.

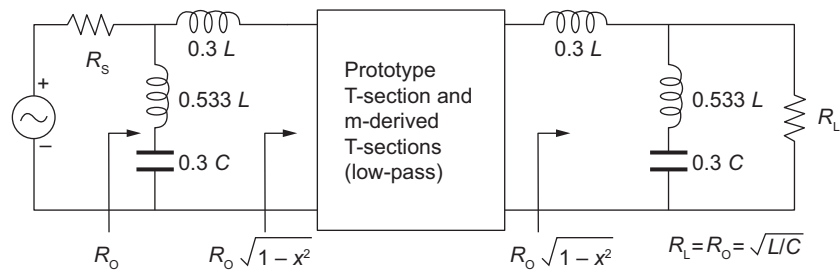


Fig. 16.12-6 Low-Pass Filter Cascade with Input and Output Terminating Half-Sections

The impedance levels at various ports for $0 \leq x \leq 0.85$ (i.e., $0 \leq f \leq 0.85f_c$) are marked in the figure. If $R_s = R_L$, then, the filter will be matched for maximum power transfer over 85% of pass-band.

16.12.3 Half- Π Termination Sections for Π -Section Filters

The m -derived Π -section low-pass filter is shown in Fig. 16.12-7(a) and the half- Π terminating section derived from it is shown in Fig. 16.12-7(b). The component values with $m = 0.6$ are also marked. The typical application context is shown in Fig. 16.12-7(c). If $R_s = R_L$, then, the filter is matched for maximum power transfer in the pass-band.

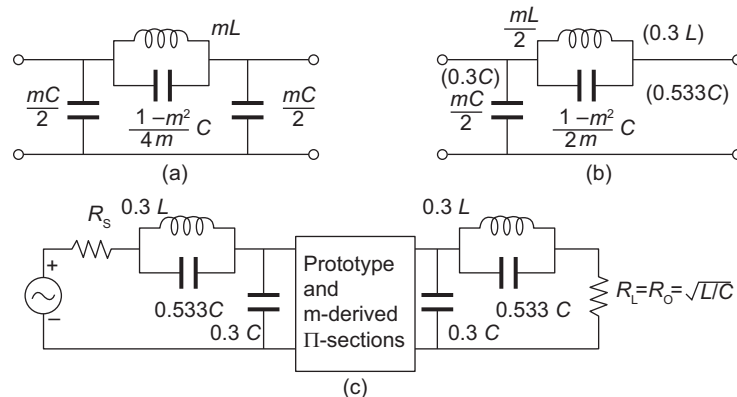


Fig. 16.12-7 (a) m -Derived Π -Section (b) Corresponding Half-Section for Filter Cascade in (c)

EXAMPLE: 16.12-1

Design a low-pass filter using Π -sections to cut-off at 1 kHz. The filter load impedance is a resistance of $300\ \Omega$ and the driving source has $120\ \Omega$ source resistance. The filter must provide zero gain at 1.25 kHz. Use terminating half-sections.

SOLUTION

The terminating sections are m -derived half-sections with $m = 0.6$. The value of m required in an m -derived section to provide infinite attenuation at $1.25f_c$ is also 0.6. Thus, no separate m -derived section is needed to provide zero gain at 1.25 kHz in this problem. The terminating sections will provide zero gain at that frequency. Thus, the filter will have a prototype section, an input terminating section and an output terminating section.

Design of Prototype Π -section

Using the design formula arrived at in Sect. 16.10, we calculate L and C as

$$L = \frac{R_L}{\pi f_c} = \frac{300}{\pi \times 1000} = 95.5\ \text{mH} \text{ and } C = \frac{1}{\pi R_L f_c} = \frac{1}{\pi \times 300 \times 1000} = 1.061\ \mu\text{F}$$

The prototype Π -section is shown in Fig. 16.12-8.

The output terminating half- Π section is shown in Fig. 16.12-9.

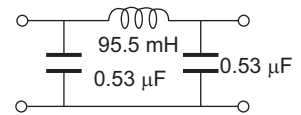


Fig. 16.12-8 Prototype Π -Section for Low-Pass Filter Design in Example 16.12-1

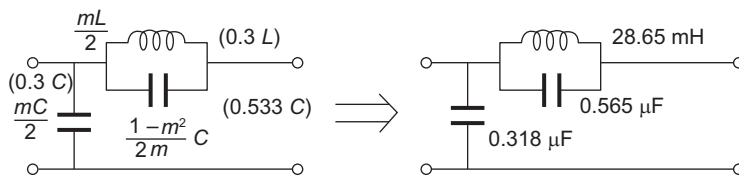


Fig. 16.12-9 Output Terminating Section in Example 16.12-1

The input termination section is shown in Fig. 16.12-10.

The cascade of the three sections is shown in Fig. 16.12-11.

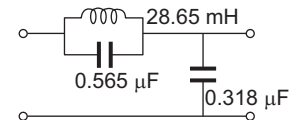


Fig. 16.12-10 Input Termination Section in Example 16.12-1

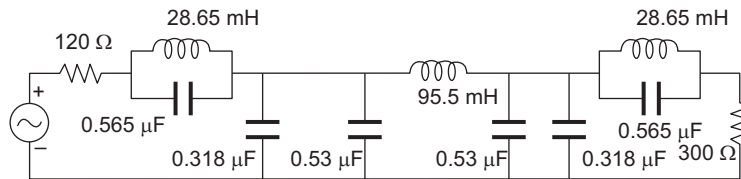


Fig. 16.12-11 Cascade of Terminating Sections and Prototype Filter in Example 16.12-1

The capacitors appearing in parallel can be combined. The final design is shown in Fig. 16.12-12.

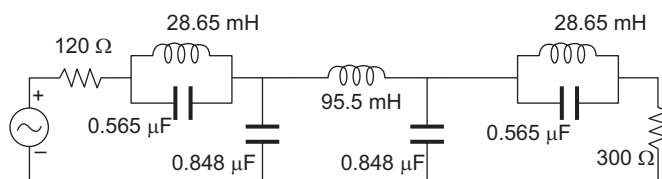


Fig. 16.12-12 Final Design for Low-Pass Filter in Example 16.12-1

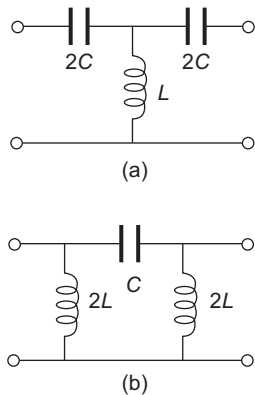


Fig. 16.13-1 (a) Constant-k T-Section High-Pass Filter (b) Constant-k Pi-Section High-Pass Filter

Characteristic impedance of a T-section constant-k prototype high-pass filter.

Characteristic impedance of a Pi-section constant-k prototype high-pass filter.

16.13 CONSTANT- k AND m -DERIVED HIGH-PASS FILTERS

Constant- k T-section and Π -section prototype filter circuits are shown in Fig. 16.13-1.

Consider the T-section filter. The first capacitor presents high impedance at low frequency, while the inductor presents low impedance. This results in considerable attenuation of the signal at the inductor node. The filter is usually terminated at a load resistance. The high impedance of the second capacitor (high compared to load resistance) further chokes even the little voltage developed at that point at low frequencies. Thus, the signal appearing across the output is a doubly attenuated version and is quite small at low frequencies.

The capacitors go short and the inductor goes open at high frequencies, thereby, passing on the input signal without significant attenuation. Therefore, the circuit is indeed of high-pass nature.

We study the characteristic impedance and propagation constant of the T-section high-pass prototype to arrive at expressions for cut-off frequency and stop-band attenuation. The algebraic manipulations involved in derivation of relevant expressions are similar to the ones we carried out in the case of low-pass filter. Hence, many steps in the derivation are skipped in the subsequent development. The reader is encouraged to verify the derivations.

$$Z_1 = \frac{1}{j\omega C} \text{ and } Z_2 = j\omega L$$

$$\therefore Z_1 Z_2 = \frac{L}{C} = \text{a constant}$$

$$Z_{oT} = \sqrt{Z_{oC} Z_{sC}} = \sqrt{\frac{L}{C}} \times \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = R_o \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = R_o \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad (16.13-1)$$

where $R_o = \sqrt{\frac{L}{C}} \Omega$; $\omega_c = \frac{1}{2\sqrt{LC}} \text{ rad/s}$ and $f_c = \frac{1}{4\pi\sqrt{LC}} \text{ Hz}$

Equation 16.13-1 shows that the characteristic impedance of T-section is resistive and increasing for $f > f_c$ and reactive for $0 \leq f < f_c$. It is zero at $f = f_c$. The characteristic impedance starts at 0Ω (resistive) at f_c and increases with f to approach R_o asymptotically with frequency. We know that $Z_{o\pi} = \frac{Z_1 Z_2}{Z_{oT}}$ for a symmetric T- Π pair using the same Z_1 and Z_2 and $Z_1 Z_2 = R_o^2$. Therefore,

$$Z_{o\pi} = \frac{R_o}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad (16.13-2)$$

$Z_{o\pi}$ is resistive in the range $f > f_c$ and reactive for $0 \leq f < f_c$. The characteristic impedance starts at $\infty \Omega$ (resistive) at f_c and decreases with frequency to approach R_o asymptotically.

We take up the propagation constant of prototype high-pass filter now. Remember that a symmetric T- Π pair using the same Z_1 and Z_2 will have the same propagation constant.

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 + \frac{Z_1}{2Z_2} + \frac{Z_{oT}}{Z_2}$$

$$\text{Substituting } Z_1 = \frac{1}{j\omega C} \text{ and } Z_2 = j\omega L \text{ and using } f_c = \frac{1}{4\pi\sqrt{LC}} \text{ and defining } x = \frac{f_c}{f},$$

we get the following expression for e^γ after some algebraic manipulation.

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 - 2x^2 - j2x\sqrt{1 - x^2} \quad (16.13-2)$$

Equation 16.13-3 shows that e^γ is real for $x > 1$ (i.e., $0 \leq f < f_c$) and it is complex for $x < 1$ (i.e., for $f > f_c$).

The expression for e^γ can be written as $e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 - 2x^2 - 2x\sqrt{x^2 - 1}$ in the range $0 \leq f < f_c$. It is real and negative and its magnitude is infinite at $f = 0$ and decreases to 1 at $f = f_c$. The angle β is π or $-\pi$ rad in the entire range. Therefore, this must be the stop-band since the signal is attenuated (e^γ with a magnitude > 1 indicates that gain is < 1) in this band.

$e^\alpha = 2x^2 - 1 + 2x\sqrt{x^2 - 1}$ and $\beta = \pi$ or $-\pi$ in the stop-band. Expression for α may be obtained by a procedure similar to the one we employed in the case of a low-pass filter. The result is

$$\alpha = 2 \cosh^{-1} \left(\frac{f_c}{f} \right) \text{ for } 0 \leq f < f_c$$

e^γ is complex for $x > 1$ (i.e., for $f > f_c$). Then, its magnitude (i.e., e^α) is given by $e^\alpha = \sqrt{(1 - 2x^2)^2 + 4x^2(1 - x^2)} = 1 \Rightarrow \alpha = 0$. Therefore, there is no attenuation in this band and the attenuation constant α is zero in this band. Hence, $f > f_c$ is the pass-band of the filter and $f_c = \frac{1}{4\pi\sqrt{LC}}$ is the cut-off frequency of the filter. The phase constant β in the

pass-band is given by $\beta = -2 \sin^{-1} \left(\frac{f_c}{f} \right)$ rad. Thus, for a prototype high-pass filter,

$$\alpha = \begin{cases} 2 \cosh^{-1} \left(\frac{f_c}{f} \right) & \text{for } 0 \leq f < f_c \\ 0 & \text{for } f > f_c \end{cases} \text{ and } \beta = \begin{cases} -\pi \text{ rad} & 0 \leq f < f_c \\ -2 \sin^{-1} \left(\frac{f_c}{f} \right) & \text{rad for } f > f_c \end{cases}$$

Attenuation characteristics of constant-k high-pass filter in the stop-band.

Attenuation and phase characteristics of a constant-k prototype high-pass filter.

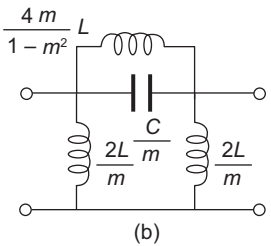
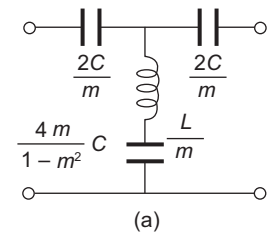
16.13.1 Design Equations for Prototype High-Pass Filter

Design specifications will be the load resistance R_L and the cut-off frequency f_c . The filter can be Z_0 -terminated only at a particular frequency. This frequency is the frequency at infinity in the case of a high-pass filter. The Z_0 of T-section and Π -section high-pass prototype approaches R_0 at infinite frequency. R_0 is made equal to R_L by design.

$$\therefore \sqrt{\frac{L}{C}} = R_L \text{ and } \frac{1}{4\pi\sqrt{LC}} = f_c. \text{ Solving for } L \text{ and } C,$$

$$L = \frac{R_L}{4\pi f_c} \text{ H and } C = \frac{1}{4\pi f_c R_L} \text{ F} \quad (16.13-4)$$

Design equations for a constant-k prototype high-pass filter.



16.13.2 m -Derived Sections for Infinite Attenuation

The T-section and Π -section m -derived high-pass filters used for providing infinite attenuation at a particular frequency in the stop-band are shown in Fig. 16.13-2.

The frequency at which infinite attenuation will be provided is the resonance frequency of the shunt branch in the case of T-section and the resonance frequency of the series branch in the case of Π -section. This frequency (f_∞) can be obtained as

$$f_\infty = \sqrt{1 - m^2} f_c \quad (16.13-4)$$

16.13.3 Termination Sections for High-Pass Filter

The principles behind the application of m -derived half-sections for input and output termination of high-pass filters are the same as in the case of low-pass filters. The value of

Fig. 16.13-2 (a) m -Derived T-Section High-Pass Filter (b) m -Derived Π -Section High-Pass Filter

m to be used is also the same – i.e., $m = 0.6$ for termination sections. The termination sections will contribute infinite attenuation at $0.8f_c$.

The termination section to be used for T-section high-pass filter is shown as the circuit in Fig. 16.13-3(a) and the termination section to be used for Π -section high-pass filter is shown as the circuit in Fig. 16.13-3(b). The component values with $m = 0.6$ are also marked.

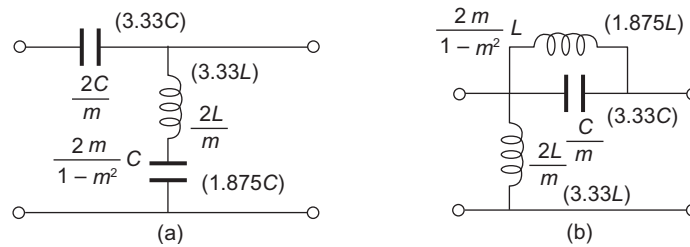


Fig. 16.13-3 (a) Half-Section for Terminating a T-Section High-Pass Filter
(b) Half-Section for Terminating a Π -Section High-Pass Filter

EXAMPLE: 16.13-1

Design a high-pass filter to cut-off at 1 kHz using T-sections. The load on the filter is a $300\ \Omega$ resistor and the filter is driven by a voltage source with a source resistance of $300\ \Omega$. The filter should provide zero gain to a 900 Hz component. Use terminating sections.

SOLUTION

Step-1: Prototype design

Using Eqn. 16.13-4 along with $R_L = 300\ \Omega$ and $f_c = 1000\ \text{Hz}$, we get L and C as

$$L = \frac{R_L}{4\pi f_c} \text{ H} = \frac{300}{4\pi \times 1000} = 23.9\ \text{mH} \text{ and } C = \frac{1}{4\pi f_c R_L} \text{ F} = \frac{1}{4\pi \times 300 \times 1000} = 265\ \text{nF}$$

Step-2: Calculate m for $f_\infty = 900\ \text{Hz}$ and components in the m -derived section

$$f_\infty = \sqrt{1-m^2} f_c \Rightarrow m = \sqrt{1 - \left(\frac{f_\infty}{f_c}\right)^2} = \sqrt{1 - 0.9^2} = 0.436$$

$$\therefore \frac{2C}{m} = 1.216\ \mu\text{F}; \frac{L}{m} = 54.82\ \text{mH}; \frac{4m}{1-m^2} C = 0.57\ \mu\text{F}$$

Step-3: Calculate components in m -derived half-section for termination

$$3.33C = 0.883\ \mu\text{F}; 3.33L = 79.6\ \text{mH}; 1.875C = 0.5\ \mu\text{F}$$

Step-4: Draw the circuit diagram of the cascade

The filter cascade is shown in Fig. 16.13-4.

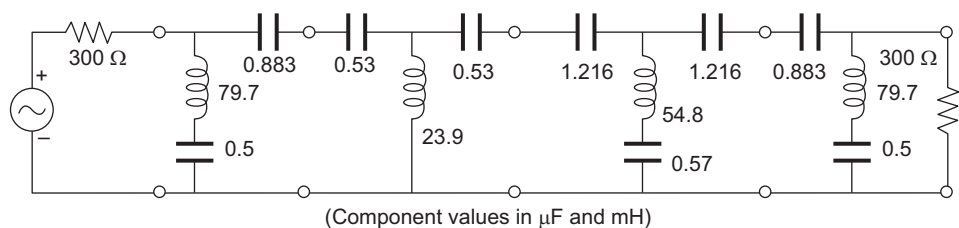


Fig. 16.13-4 Cascade of Prototype, m -Derived and Terminating Sections in Example 16.13-1

Step-5: Use series-parallel combinations to reduce the number of elements. The circuit after this reduction has been carried out is shown in Fig. 16.13-5.

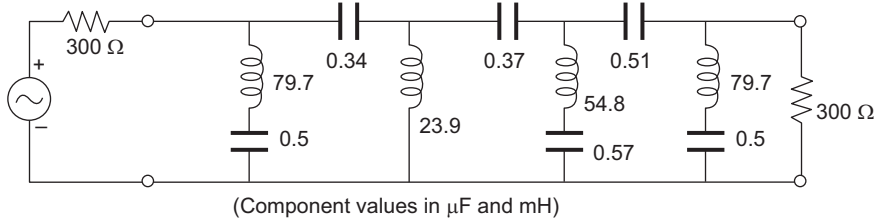


Fig. 16.13-5 Final Design for High-Pass Filter in Example 16.13-1

16.14 CONSTANT-k BAND-PASS FILTER

Constant-k T-section and Π -section band-pass filters are shown in Fig. 16.14-1.

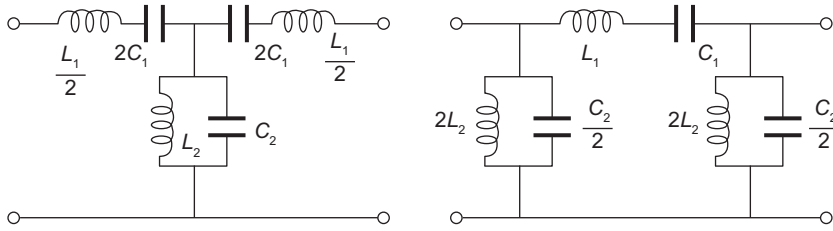


Fig. 16.14-1 (a) Constant-k T-Section Band-Pass Filter (b) Constant-k Π -Section Band-Pass Section

‘Constant-k’ implies that the product of underlying impedances Z_1 and Z_2 must be a constant independent of frequency.

$$\begin{aligned}
 Z_1 &= j\omega L_1 + \frac{1}{j\omega C_1} = -j \frac{(1 - \omega^2 L_1 C_1)}{\omega C_1} \\
 Z_2 &= j\omega L_2 \parallel \frac{1}{j\omega C_2} = j \frac{\omega L_2}{(1 - \omega^2 L_2 C_2)} \\
 \therefore Z_1 Z_2 &= \frac{(1 - \omega^2 L_1 C_1)}{\omega C_1} \times \frac{\omega L_2}{(1 - \omega^2 L_2 C_2)} = \frac{L_2}{C_1} \times \frac{(1 - \omega^2 L_1 C_1)}{(1 - \omega^2 L_2 C_2)}
 \end{aligned}$$

Therefore, $L_1 C_1$ has to be equal to $L_2 C_2$ if the filter is to be of a constant-k type. This constraint is imposed on band-pass filter design.

$$\therefore L_1 C_1 = L_2 C_2 \text{ by design.}$$

$$\text{Then, } Z_1 Z_2 = \frac{L_2}{C_1}$$

This product is defined as R_o^2 of the filter.

$$\therefore R_o = \sqrt{Z_1 Z_2} = \sqrt{\frac{L_2}{C_1}} = \sqrt{\frac{L_1}{C_2}} \quad (\because L_1 C_1 = L_2 C_2)$$

Note that the constraint $L_1 C_1 = L_2 C_2$ results in both series branch and shunt branch resonating at the same frequency. We define this frequency as ω_o rad/s.

$$\therefore \omega_o = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}} \text{ and } f_o = \frac{1}{2\pi \sqrt{L_1 C_1}} = \frac{1}{2\pi \sqrt{L_2 C_2}}$$

Consider the T-section in Fig. 16.14-1. The series branch is open at DC and the shunt branch is short at DC. Therefore, the DC steady-state gain of the filter is zero. The series branch becomes a short-circuit and the shunt branch becomes an open-circuit at f_o . Therefore, the gain at that frequency is unity. The series branch is open at high frequencies and the shunt branch is short at high frequencies. Therefore, the gain of the filter goes to zero as $f \rightarrow \infty$. Thus, this circuit is indeed of a band-pass nature.

It is clear that the centre frequency of the filter is ω_o . We proceed to determine the upper and lower cut-off frequencies of the filter by analysing its propagation constant. A constant-k filter terminated at characteristic impedance has a propagation constant that is decided by the following equation.

$$e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta = 1 + \frac{Z_1}{2Z_2} + \frac{Z_{oT}}{Z_2} \quad (16.14-1)$$

Both Z_1 and Z_2 are pure reactances at all ω , and hence, we can express them as $Z_1 = jX_1$ and $Z_2 = jX_2$. X_1 and X_2 can be positive or negative depending on the ω value. For instance, X_1 – the series branch reactance – is negative for $\omega < \omega_o$ and positive for $\omega > \omega_o$. X_2 – the shunt branch reactance – is positive for $\omega < \omega_o$ and negative for $\omega > \omega_o$.

$$\text{Now, } \frac{Z_1}{2Z_2} = \frac{jX_1}{j2X_2} = \frac{X_1}{2X_2} = \text{a real number}$$

Therefore, the only way in which e^γ can be a complex number is by the term $\frac{Z_{oT}}{Z_2}$ becoming imaginary.

$$\begin{aligned} Z_{oT} &= \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = \sqrt{R_o^2 + \frac{(jX_1)^2}{4}} = \sqrt{R_o^2 - \frac{X_1^2}{4}} \\ \therefore \frac{Z_{oT}}{Z_2} &= \frac{\sqrt{R_o^2 - \frac{X_1^2}{4}}}{jX_2} = -j \frac{\sqrt{R_o^2 - \frac{X_1^2}{4}}}{X_2} \end{aligned}$$

This term can become a pure imaginary number only if $R_o^2 - \frac{X_1^2}{4}$ is positive. X_1 is the reactance of series combination of L_1 and C_1 . It starts at $-\infty$ at $\omega = 0$, crosses zero at $\omega = \omega_o$ and increase further towards ∞ as $\omega \rightarrow \infty$. Then, there must be a range of frequency values $f_1 < f_o < f_2$ in which $\frac{X_1^2}{4} < R_o^2$. In that frequency range, e^γ will be a complex number

and we can e^γ express as $e^\gamma = e^\alpha \angle \beta = 1 + \frac{X_1}{2X_2} - j \frac{\sqrt{R_o^2 - \frac{X_1^2}{4}}}{X_2}$. The attenuation suffered by a signal in this frequency range is given by the magnitude of e^γ (i.e., e^α).

$$e^{2\alpha} = \left(1 + \frac{X_1}{2X_2}\right)^2 + \left(\frac{\sqrt{R_o^2 - \frac{X_1^2}{4}}}{X_2}\right)^2 = 1 + \frac{X_1^2}{4X_2^2} + \frac{X_1}{X_2} + \frac{R_o}{X_2^2} - \frac{X_1^2}{4X_2^2}$$

$$\text{But } R_o^2 = Z_1 Z_2 = (jX_1)(jX_2) = -X_1 X_2$$

$$\therefore e^{2\alpha} = 1 + \frac{X_1^2}{4X_2^2} + \frac{X_1}{X_2} - \frac{X_1X_2}{X_2^2} - \frac{X_1^2}{4X_2^2} = 1$$

This implies that, in the frequency range $f_1 < f_o < f_2$ in which $\frac{X_1^2}{4} < R_o^2$, there is no attenuation in the filter (provided it is always Z_o -terminated). The attenuation constant α is zero in that frequency range. Hence, that range must be the pass-band of the filter. We obtain the edges of pass-band – i.e., f_1 and f_2 by solving $\frac{X_1^2}{4} < R_o^2$ with the equality sign.

$$\frac{X_1^2}{4} = R_o^2 \Rightarrow X_1 = \pm 2R_o$$

Substituting for X_1 , $\frac{1 - \omega^2 L_1 C_1}{\omega C_1} = \pm 2R_o$. Solving this quadratic equation on ω , we get

ω_1 and ω_2 as

$$\omega_1 = \sqrt{\omega_o^2 + \frac{R_o^2}{L_1^2}} - \frac{R_o}{L_1} \text{ and } \omega_2 = \sqrt{\omega_o^2 + \frac{R_o^2}{L_1^2}} + \frac{R_o}{L_1} \quad (16.14-2)$$

These two values give the lower cut-off angular frequency and the higher cut-off angular frequency. The bandwidth is the difference between the two.

We develop an interesting relationship between the three frequencies ω_1 , ω_o and ω_2 by multiplying ω_1 and ω_2 .

$$\omega_1 \omega_2 = \left(\sqrt{\omega_o^2 + \frac{R_o^2}{L_1^2}} - \frac{R_o}{L_1} \right) \times \left(\sqrt{\omega_o^2 + \frac{R_o^2}{L_1^2}} + \frac{R_o}{L_1} \right) = \omega_o^2 + \frac{R_o^2}{L_1^2} - \frac{R_o^2}{L_1^2} = \omega_o^2$$

Thus, the centre frequency of this band-pass filter is the geometric mean of its cut-off frequencies.

16.14.1 Design Equations of Prototype Band-Pass Filter

Design specifications will be the value of load resistance R_L and the lower and upper cut-off frequencies f_1 and f_2 .

$$\omega_2 - \omega_1 = \frac{2R_o}{L_1} \text{ (from Eqn. 16.14-2)}$$

R_o will be the value of characteristic impedance at ω_o and the usual design choice is to make the filter Z_o -terminated at that frequency. Therefore, R_o is chosen to be equal to R_L . Therefore,

$$L_1 = \omega_2 - \omega_1 = \frac{2R_o}{\omega_2 - \omega_1} = \frac{R_L}{\pi(f_2 - f_1)} \text{ H}$$

$$\text{Then, } C_2 = \frac{L_1}{R_o^2} = \frac{L_1}{R_L^2} = \frac{1}{\pi R_L(f_2 - f_1)} \text{ F}$$

$$f_o = \sqrt{f_1 f_2} = \frac{1}{2\pi \sqrt{L_1 C_1}} = \frac{1}{2\pi \sqrt{L_2 C_2}}$$

$$\therefore C_1 = \frac{1}{4\pi^2 L_1 f_1 f_2} = \frac{(f_2 - f_1)}{4\pi R_L f_1 f_2} \text{ F and } L_2 = \frac{1}{4\pi^2 C_2 f_1 f_2} = \frac{R_L (f_2 - f_1)}{4\pi f_1 f_2} \text{ H}$$

Pass-band edge frequency values for any constant-k filter can be obtained by solving the equation $X_1^2 = 4R_o^2$, where X_1 is the series arm reactance and $R_o^2 = X_1 X_2$, where X_2 is the shunt arm reactance.

Cut-off frequencies of a constant-k prototype band-pass filter.

Design equations for constant-k prototype band-pass filter.

Therefore, the design equations are

$$L_1 = \frac{R_L}{\pi(f_2 - f_1)} \text{ H}; C_1 = \frac{(f_2 - f_1)}{4\pi R_L f_1 f_2} \text{ F}; L_2 = \frac{R_L(f_2 - f_1)}{4\pi f_1 f_2} \text{ H and } C_2 = \frac{1}{\pi R_L(f_2 - f_1)} \text{ F}$$

16.15 CONSTANT-k BAND-STOP FILTER

Constant-k T-section and Π -section prototype band-stop filters are shown in Fig. 16.15-1.

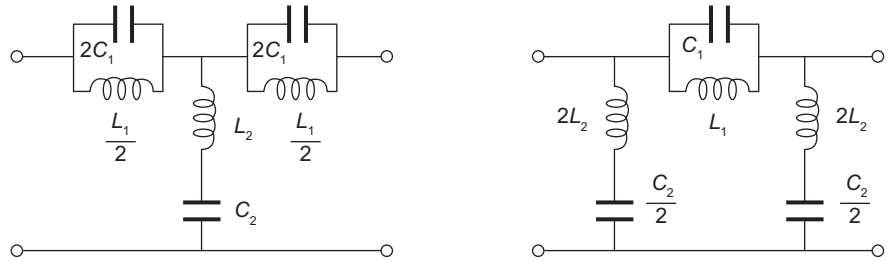


Fig. 16.15-1 (a) Prototype Band-Stop T-Section (b) Prototype Band-Stop Π -Section

$$Z_2 = j\omega L_2 + \frac{1}{j\omega C_2} = -j \frac{(1 - \omega^2 L_2 C_2)}{\omega C_2}$$

$$Z_1 = j\omega L_1 // \frac{1}{j\omega C_1} = j \frac{\omega L_1}{(1 - \omega^2 L_1 C_1)}$$

$$\therefore Z_1 Z_2 = \frac{(1 - \omega^2 L_2 C_2)}{\omega C_2} \times \frac{\omega L_1}{(1 - \omega^2 L_1 C_1)} = \frac{L_1}{C_2} \times \frac{(1 - \omega^2 L_2 C_2)}{(1 - \omega^2 L_1 C_1)}$$

Therefore, $L_1 C_1$ has to be equal to $L_2 C_2$ if the filter is to be of constant-k type. This constraint is imposed on band-pass filter design.

$$\therefore L_1 C_1 = L_2 C_2 \text{ by design.}$$

$$\text{Then, } Z_1 Z_2 = \frac{L_1}{C_2}$$

This product is defined as R_o^2 of the filter.

$$\therefore R_o = \sqrt{Z_1 Z_2} = \sqrt{\frac{L_1}{C_2}} = \sqrt{\frac{L_2}{C_1}} (\because L_1 C_1 = L_2 C_2)$$

Note that the constraint $L_1 C_1 = L_2 C_2$ results in both series branch and shunt branch resonating at the same frequency. We define this frequency as ω_o rad/s.

$$\therefore \omega_o = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}} \text{ and } f_o = \frac{1}{2\pi \sqrt{L_1 C_1}} = \frac{1}{2\pi \sqrt{L_2 C_2}}$$

Consider the T-section in Fig. 16.15-1. The series branch is short at DC and the shunt branch is open at DC. Therefore, the DC steady-state gain of the filter is unity. The series branch becomes an open-circuit and the shunt branch becomes a short-circuit at f_o . Therefore, the gain at that frequency is zero. The series branch is short at high frequencies and the shunt branch is open at high frequencies. Therefore, the gain of the filter goes to unity as $f \rightarrow \infty$. Thus, this circuit is indeed of a band-stop nature.

A frequency is in the pass-band of this filter if X_1 at that frequency satisfies $\frac{X_1^2}{4} < R_o^2$

condition as in the case of band-pass filter. X_1 starts at zero for $\omega = 0$ (due to inductor), goes to ∞ as $\omega \rightarrow \omega_o$ from left, starts at $-\infty$ on the right of ω_o and ends up at zero as $\omega \rightarrow \infty$. Therefore, $0 < f < f_1$ and $f_2 < f$ must be the pass-band of the filter, where f_1 and f_2 are solutions of the equation $\frac{X_1^2}{4} = R_o^2$. Solving this equation, we get,

$$f_1 = \frac{1}{2\pi} \sqrt{\omega_o^2 + \frac{1}{16R_o^2 C_1^2}} - \frac{1}{4R_o C_1} \text{ and } f_2 = \frac{1}{2\pi} \sqrt{\omega_o^2 + \frac{1}{16R_o^2 C_1^2}} + \frac{1}{4R_o C_1}.$$

$[0, f_1)$ and (f_2, ∞) are the pass-bands and (f_1, f_2) is the stop-band. It can be shown that $f_o = \sqrt{f_1 f_2}$ in the case of a band-stop filter also.

The design specifications will be the values of load resistance R_L , f_1 and f_2 . The design equations can be derived using the same procedure we employed in the case of a band-pass filter.

The design equations are

$$L_1 = \frac{R_L (f_2 - f_1)}{\pi f_1 f_2} \text{ H; } C_1 = \frac{1}{4\pi R_L (f_2 - f_1)} \text{ F; } L_2 = \frac{R_L}{4\pi (f_2 - f_1)} \text{ H and } C_2 = \frac{(f_2 - f_1)}{\pi R_L f_1 f_2} \text{ F.}$$

The filter will be Z_o -terminated only at $\omega = 0$ and $\omega = \infty$ with this design. In practice, m -derived sections and m -derived half-sections will be needed to tailor the pass-band and stop-band attenuation characteristics in the case of band-pass filters and band-stop filters too. We skip these topics, as they will represent too great a detail for an introductory textbook.

However, we can not close this chapter without a brief mention on an important circuit element that we have neglected in our analysis of filters. It is the winding and core loss resistance of inductors used in the filter designs. Obviously, they will affect the pass-band attenuation. The attenuation constant will no longer be correctly described by the compact expression that we derived for it in this chapter. A practical design can not ignore the winding resistance of inductors and this ugly reality will often dictate the computer simulation for design validation.

Cut-off frequencies of a constant-k prototype band-stop filter.

Design equations for a constant-k prototype band-stop filter.

16.16 RESISTIVE ATTENUATORS

Resistive Attenuators are resistive two-port networks designed for providing attenuation to the source voltage waveform while providing impedance matching at the input port and the output port. It is a memoryless linear time-invariant two-port network, and therefore, does not produce any distortion in the waveshape. The design objective of a resistive attenuator is to provide a constant attenuation to all frequency components present in the input waveform.

We distinguish two cases. In the first case, the source resistance and load resistance are equal. In the second case, these two resistances are unequal.

The attenuator may be designed by employing a symmetrical resistive two-port network when the source resistance and the load resistances are equal. When they are unequal, we require asymmetrical two-port networks for realising the attenuator.

16.16.1 Attenuation provided by a Symmetric Resistive Attenuator

The propagation constant of a resistive two-port will have zero imaginary part since such a network is memoryless and can not delay the frequency components in time-domain. Thus,

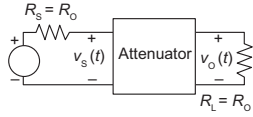


Fig. 16.16-1
Symmetrical
Attenuator Design
Context

Note that N is the attenuation provided by the network interposed between the source and the load. The ratio of the open-circuit voltage of source to the load voltage will be $2N$.

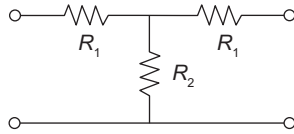


Fig. 16.16-2 A
Symmetrical T-Section
Attenuator

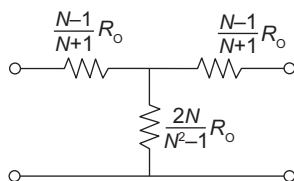


Fig. 16.16-3 Design
Values for a
Symmetrical T-Section
Attenuator

γ for a symmetric resistive attenuator is a real number α . Attenuation in absolute value is given by e^α . However, it is usually specified in dB (decibel) units in the context of the design of resistive attenuators.

$$\text{Attenuation in dB} = 20 \log_{10} e^\alpha \text{ dB} = \frac{20 \ln e^\alpha}{\ln 10} = 8.686\alpha \text{ dB}$$

Thus, the attenuation in absolute value is e^α ; it is 8.686α in dB units and it is α in Np. The symmetrical attenuator design context is shown in Fig. 16.16-1.

The design specifications are (i) $\frac{v_s(t)}{v_o(t)} = N (N > 1)$ and (ii), where N is the required

value of attenuation in absolute units and R_{im1} and R_{im2} are the image impedances at port-1 and port-2, respectively. They will be equal, since it is a symmetrical network. The design requires that they be equal to the common value of source resistance and load resistance.

This symmetrical attenuator may be designed using a T-section, a Π -section, a Lattice section or a Bridged-T section.

16.16.2 The Symmetrical T-Section Attenuator

The symmetrical T-section attenuator is shown in Fig. 16.16-2. The expressions for R_1 and R_2 are derived in terms of the design specifications N and R_o as follows.

$$N = e^\alpha = A + \sqrt{(A^2 - 1)} \text{ and } R_{im1} = R_{im2} = \sqrt{Z_{sc} Z_{oc}} \text{ for a symmetrical two-port}$$

$$Z_{sc} = R_1 + \frac{R_1 R_2}{R_1 + R_2} \text{ and } Z_{oc} = R_1 + R_2$$

$$\therefore R_o = \sqrt{(R_1^2 + 2R_1 R_2)}$$

$$A = \frac{R_1 + R_2}{R_2}$$

$$\therefore N = A + \sqrt{(A^2 - 1)}$$

$$= \frac{R_1 + R_2}{R_2} + \sqrt{\left(\frac{R_1 + R_2}{R_2}\right)^2 - 1}$$

$$= \frac{R_1 + R_2}{R_2} + \frac{\sqrt{(R_1^2 + 2R_1 R_2)}}{R_2}$$

$$= 1 + \frac{R_1 + R_o}{R_2}$$

$$\therefore R_2 = \frac{R_1 + R_o}{N - 1}$$

Substituting this expression for R_2 in the expression for R_o^2 , we get,

$$R_o^2 = R_1^2 + 2R_1 R_2 = R_1^2 + 2R_1 \frac{R_1 + R_o}{N - 1}.$$

This yields a quadratic equation on R_1 . The solution will be $R_1 = \frac{N-1}{N+1} R_o$.

$$\text{Therefore, } R_2 = \frac{R_1 + R_o}{N - 1} = \frac{2N}{N^2 - 1} R_o$$

The T-section attenuator with the design values marked is shown in Fig. 16.16-3. The spread of resistors R_1/R_2 is $(N - 1)^2/2N$ and is $\sim 0.5N$ for large attenuation.

16.16.3 The Symmetrical Π -Section Attenuator

The symmetrical Π -section attenuator is shown in Fig. 16.16-4(a). The expressions for R_1 and R_2 can be derived in terms of the design specifications N and R_o by employing a procedure similar to the one employed for a T-section attenuator. The resulting expressions are:

$$R_1 = \frac{N^2 - 1}{2N} R_o$$

$$R_2 = \frac{N + 1}{N - 1} R_o$$

The Π -section attenuator with the design values marked is shown in Fig. 16.16-4(b). The spread of resistors R_1/R_2 is $(N - 1)^2/2N$ and is $\sim 0.5N$ for large attenuations.

16.16.4 The Symmetrical Lattice-Section Attenuator

A symmetrical lattice attenuator is shown in Fig. 16.16-5(a). Note that the lattice is drawn in the form of a bridge.

$$Z_{OC} = \frac{R_A + R_B}{2} \text{ and } Z_{SC} = 2 \frac{R_A R_B}{R_A + R_B}$$

A symmetric, reciprocal network needs only two parameters to describe it completely. Hence, Z_{oc} and Z_{sc} will form a complete set of parameters for such a network. That is, all other two-port network parameters can be worked out from these two impedances for a symmetric, reciprocal network.

$$\text{Now, let } x = 2Z_{oc} = R_A + R_B \text{ and } y = \frac{Z_{sc}}{2} = \frac{R_A R_B}{R_A + R_B}$$

The equations shown above may be employed to form a quadratic equation on R_A after eliminating R_B , with x and y treated as known constants. Solving the resulting equations, we get,

$$R_A = \frac{x \pm \sqrt{(x^2 - 4xy)}}{2} \text{ and } R_B = \frac{xy}{R_A}$$

Now, we set Z_{oc} and Z_{sc} of this network to be the same as Z_{oc} and Z_{sc} of the T-section attenuator design. Then, all the two-port parameters of this lattice network will be the same as those of the T-section network since Z_{oc} and Z_{sc} will decide all two-port parameters for a symmetric, reciprocal network. Therefore, a symmetrical lattice network and a symmetrical T-network with the same Z_{oc} and Z_{sc} will have the same R_o and α .

$$\therefore x = 2Z_{oc} = 2 \left[\frac{N-1}{N+1} + \frac{2N}{N^2-1} \right] R_o = 2 \frac{N^2+1}{N^2-1} R_o \text{ and}$$

$$y = \frac{Z_{sc}}{2} = \frac{1}{2} \frac{R_o^2}{Z_{oc}} = \frac{1}{2} \frac{N^2-1}{N^2+1} R_o$$

$$\text{Now, } R_A \text{ and } R_B \text{ may be found as } R_A = \frac{N-1}{N+1} R_o \text{ and } R_B = \frac{N+1}{N-1} R_o$$

The symmetric lattice attenuator design is shown in Fig. 16.16-5(b). The resistor spread in this case is $\frac{R_B}{R_A} = \left(\frac{N+1}{N-1} \right)^2$ and is ~ 1 for large N .

The choice of attenuator design for a given application depends on the resistor-spread among other factors. Note that T-section and Π -section attenuators are suitable for low and moderate values of attenuation ($1 < N < 10$), whereas Lattice attenuator is suitable for large attenuations from the resistor-spread point of view.

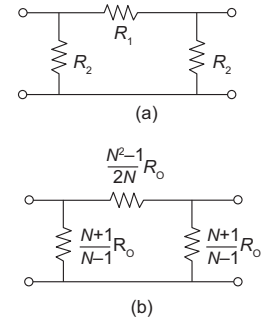


Fig. 16.16-4 (a) A Π -Section Symmetrical Attenuator (b) Design Values for Π -Section Attenuator

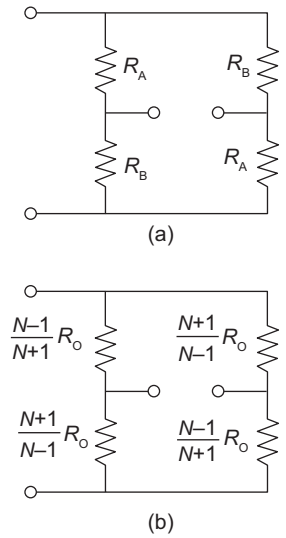


Fig. 16.16-5 (a) A Symmetrical Lattice Attenuator (b) Design Values for Lattice Attenuator

Large spread of resistors in a circuit tends to accentuate the problems arising out of parasitic capacitance across the resistors too. Large spread in resistor values is better avoided from this point of view.

16.16.5 The Symmetrical Bridged-T-Section Attenuator

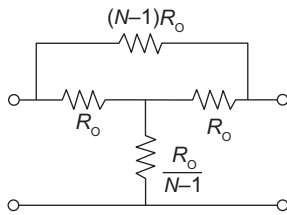


Fig. 16.16-6 The Symmetrical Bridged-T Attenuator

The symmetrical Bridged-T attenuator is shown with the design values marked in the Fig. 16.16-6.

The condition of symmetry requires that the series arms in the T-section be equal. Then, there are three resistors to be decided from two specifications (N and R_o). Thus, the design is under-determined.

One possible design strategy is to set the series arms at R_o . Then, the shunt-connected resistor has to be $\frac{R_o}{N-1}$ and the bridging resistor has to be $(N-1)R_o$.

Independent control of attenuation and characteristic resistance are possible in this attenuator. The attenuation value N may be varied by varying the shunt resistance and the bridging resistance such that their product remains constant at R_o^2 . Such a variation will not affect the characteristic resistance of the network.

16.16.6 Asymmetrical T-Section and Π -Section Attenuators

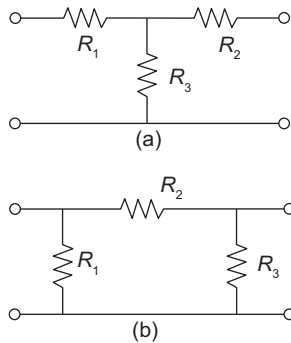


Fig. 16.16-7 (a) Asymmetrical T-Section Attenuator
(b) Asymmetrical Π -Section Attenuator

Consider a voltage source with a source resistance of R_s connected to a load resistance of R_L with $R_s \neq R_L$. An asymmetrical attenuator interposed between the source and the load in this case has the objective of providing attenuation to the signal while matching the load resistance R_L to the source resistance R_s at the input port and matching the source resistance R_s to the load resistance R_L at the output port. Thus, R_{im1} of the attenuator must be equal to R_s and R_{im2} of the attenuator must be equal to R_L .

Thus, there are three specifications for designing an asymmetrical attenuator. They are $R_{im1} = R_s$, $R_{im2} = R_L$ and attenuation N . Note that the attenuation N specified is the attenuation provided by the attenuating network from its input to its output. The ratio of the open-circuit voltage of source to the load voltage will be $2N$.

Both T-section and Π -section asymmetrical attenuators contain three non-equal resistors. With three design specifications, the three resistor values can be solved for. The asymmetrical sections are shown in Fig. 16.16-7(a) and Fig. 16.16-7(b).

The design equations for an asymmetrical T-section attenuator are:

$$\begin{aligned} R_1 &= R_s \frac{N^2 + 1}{N^2 - 1} - 2\sqrt{R_s R_L} \frac{2N}{N^2 - 1} \\ R_2 &= 2\sqrt{R_s R_L} \frac{2N}{N^2 - 1} \\ R_3 &= R_L \frac{N^2 + 1}{N^2 - 1} - 2\sqrt{R_s R_L} \frac{2N}{N^2 - 1} \end{aligned}$$

The design equations for an asymmetrical Π -section attenuator are:

$$\begin{aligned} R_1 &= R_s \frac{N^2 - 1}{N^2 - 2N\sqrt{R_s/R_L} + 1} \\ R_2 &= \frac{\sqrt{R_s R_L}}{2} \frac{N^2 - 1}{N} \\ R_3 &= R_L \frac{N^2 - 1}{N^2 - 2N\sqrt{R_L/R_s} + 1} \end{aligned}$$

16.17 SUMMARY

- A linear time-invariant two-port network is a linear network that contains no independent sources and has two pairs of terminals at which it can interact with the external world with no interaction permitted with the external world except through these two terminal pairs.
- A two-port network is described by equations relating the two port-voltages and two port-currents. Five such descriptions were dealt with in this chapter. Each description leads to a set of four parameters for the network. y , z , h , g and $ABCD$ were the parameter sets described in this chapter. All the parameter sets may not exist for a given two-port network.
- For a 'symmetric reciprocal linear time-invariant two-port network' (i) $y_{11} = y_{22}$ and $y_{12} = y_{21}$ (ii) $z_{11} = z_{22}$ and $z_{12} = z_{21}$ (iii) $h_{12} = -h_{21}$ and $h_{11}h_{22} - h_{12}h_{21} = 1$ (iv) $g_{12} = -g_{21}$ and $g_{11}g_{22} - g_{12}g_{21} = 1$ and (v) $A = D$ and $AD - BC = 1$
- $ABCD$ matrix of a two-port network that is a cascade of many two-port networks is the matrix product of $ABCD$ matrices of individual two-port networks.
- z -parameters get added in series connection of two two-port networks. y -parameters get added in a parallel connection of two two-port networks. h -parameters get added in a series-parallel-connection of two two-port networks. g -parameters get added in a parallel-series connection of two two-port networks.
- Any physical, passive, reciprocal, linear two-port network has a passive, reciprocal, T-network equivalent and a passive, reciprocal, Π -network equivalent.
- Two impedances called *image impedances* (also called *iterative impedances*) designated by Z_{im1} and Z_{im2} and a constant called *image transfer constant* designated by γ describe a reciprocal, linear, time-invariant two-port network in an image impedance formulation.
- Image impedances are a pair of impedances Z_{im1} and Z_{im2} such that the input impedance at port-1 with port-2 terminated at Z_{im2} is Z_{im1} and the input impedance at port-2 with port-1 terminated at Z_{im1} is Z_{im2} . There exists such a pair of impedances for every linear time-invariant two-port network. They are given by

$$Z_{im1} = \sqrt{\frac{AB}{CD}} = \sqrt{Z_{1o}Z_{1s}} \text{ and } Z_{im2} = \sqrt{\frac{DB}{CA}} = \sqrt{Z_{2o}Z_{2s}}$$
 where Z_{1o} and Z_{1s} are impedances measured at port-1 with port-2 open and short, respectively, and Z_{2o} and Z_{2s} are impedances measured at port-2 with port-1 open and short, respectively.
- *Image transfer constant* γ is such that

$$e^{\gamma} = \sqrt{\frac{v_1}{v_2} \times \frac{i_1}{(-i_2)}} = \sqrt{AD} + \sqrt{AD-1} \text{ with } v_2 \text{ developed across } Z_{im2}.$$
 γ is a complex number $\alpha + j\beta$ for sinusoidal steady-state analysis. Both α and β will be functions of frequency. α is called the *attenuation constant* and β is called the *phase constant*.
- Image impedances are equal for a symmetric two-port. Image impedance is also called *characteristic impedance* Z_o for such a network. γ for a symmetric network is also called its *propagation constant*. $Z_o = \sqrt{\frac{B}{C}} = \sqrt{Z_{oc}Z_{sc}}$ and $e^{\gamma} = A + \sqrt{BC}$ for a symmetric network.
- For a T- Π pair made using the same Z_1 and Z_2 ,

$$Z_{oT} = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} \text{ and } Z_{o\Pi} = \frac{Z_1Z_2}{Z_{oT}}.$$
 Both T-network and Π -network will have the same propagation characteristics given by $e^{\gamma} = e^{\alpha + j\beta} = e^{\alpha} \angle \beta = 1 + \frac{Z_1}{2Z_2} + \frac{Z_{oT}}{2}$.
- Symmetric reactive T- and Π -networks terminated at their Z_o can be used for designing passive filters. Constant-k passive filters use $Z_1 = jX_1$ and $Z_2 = jX_2$ such that $Z_1Z_2 = R_o^2 = \text{a constant}$, where Z_1 and Z_2 are the underlying impedances in the T- and Π -networks. The pass-band of the filter is that range of frequencies for which its Z_o is resistive. Equivalently, pass-band is that range of frequencies for which $\frac{X_1^2}{4} < R_o^2$. Constant-k filters will have zero attenuation in the pass-band, provided they are Z_o -terminated in the entire pass-band.
- m -derived T-sections and Π -sections are used along with prototype filter sections in order to provide infinite attenuation at specific stop-band frequencies without affecting the pass-band behaviour of the prototype filter significantly.
- m -derived half-T sections and half- Π sections are employed at the output of the filter to provide Z_o -termination to the filter over most of the pass-band range. Similar half-sections are employed at the input of the filter to provide an input impedance which is more or less constant and resistive over most of the pass-band range to the source.
- Winding resistance and core losses in the inductors employed in the passive filter affect its attenuation characteristics detrimentally; especially in the pass-band.

16.18 QUESTIONS

1. A two-port network has $[y] = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ S. Find one network that satisfies this specification and explain why that network will have no $[z]$ matrix.
2. A two-port network has $[y] = \begin{bmatrix} 2 & -0.5 \\ -1 & 2 \end{bmatrix}$ S. What is the nature of the circuit – memoryless or dynamic? Is it a passive network? If a 0.5Ω resistor is connected directly across the input port, find the new $[y]$ matrix.
3. A resistor R is connected in series with the input port of a two-port network. Derive expressions for y -parameters of the new network in terms of y -parameters of the original network and R .
4. A two-port network has $[z] = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Omega$. Find one network that satisfies this specification and explain why that network will have no $[y]$ matrix.
5. A two-port network has $[z] = \begin{bmatrix} j2 & -j0.5 \\ -j & j2 \end{bmatrix} \Omega$ at 50 Hz. Is it a passive network? If a 0.5Ω resistor is connected in series to the input port, find the new $[z]$ matrix.
6. A resistor R is connected in parallel with the output port of a two-port network. Derive expressions for $[z]$ of the new network in terms of $[z]$ of the original network and R .
7. A resistor R is connected in series with the input port of a two-port network. Derive $[h]$ of the new network in terms of $[h]$ of original network and R .
8. Which are the g -parameters that change when a resistor R is connected directly across the input port and what are the new values?
9. Which are the h -parameters that change when a resistor R is connected directly across the output port and what are the new values?
10. Express $ABCD$ parameters in terms of z -parameters.
11. Express $ABCD$ parameters in terms of h -parameters.
12. Express $ABCD$ parameters in terms of g -parameters.
13. Show that if two two-port networks are connected in series at input and output, the resulting two-port network will have a $[z]$ matrix that is the sum of $[z]$ matrices of the individual networks.
14. Show that if two two-port networks are connected in parallel at input and output, the resulting two-port network will have a $[y]$ matrix that is the sum of $[y]$ matrices of the individual networks.
15. Show that if two two-port networks are connected in series at input and in parallel at output, the resulting two-port network will have a $[h]$ matrix that is the sum of $[h]$ matrices of the individual networks.
16. Show that if two two-port networks are connected in parallel at input and in series at output, the resulting two-port network will have a $[g]$ matrix that is the sum of $[g]$ matrices of the individual networks.
17. Find the $ABCD$ parameters for the two two-port networks shown in Fig. 16.18-1.

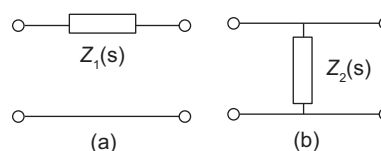


Fig. 16.18-1

18. Find the $ABCD$ parameters of (i) a symmetric-T network and (ii) a symmetric- Π network using the results arrived at in Question-17.
19. The $ABCD$ matrix of a network N_B is $\begin{bmatrix} 2 & 10 \\ 0.3 & 2 \end{bmatrix}$. When this network is driven by the output port of another network N_A , the cascade is found to have an $ABCD$ matrix of $\begin{bmatrix} 2 & 10 \\ 0.3 & 2 \end{bmatrix}$. What is the $ABCD$ matrix of cascade when the network N_B drives the network N_A ?
20. (i) Derive expressions for $ABCD$ parameters of a new network formed by connecting a resistor R directly across the input port in terms of $ABCD$ parameters of the original network and R . (ii) Repeat, if the resistor R is connected across the output port.
21. Two-port parameters are to be employed for analysis of an electronic amplifier that has an external capacitor connected between the input and output terminals. Which parameter set will you use and why?
22. Show that the image impedances of a two-port network are given by $Z_{im1} = \sqrt{\frac{AB}{CD}}$ and $Z_{im2} = \sqrt{\frac{DB}{CA}}$.
23. Show that the Z_o of a symmetric reactive two-port network has to be either purely resistive or purely reactive at any frequency.
24. Does the order of cascading of the prototype section and the m -derived sections in a multi-stage passive filter affect its attenuation characteristics? Explain.
25. Which are the two-port parameter sets that exist for an ideal transformer with a primary-to-secondary turns ratio of n ? Find all of them.
26. Explain why the resistor spread in a lattice attenuator is close to unity for large attenuations by physical reasoning.
27. A voltage source with a source resistance of R_s is to be impedance-matched to a load resistance R_L by connecting an L -Section as shown in Fig. 16.18-2. (i) Derive expressions for R_1 and R_2 in terms of R_s and R_L . (ii) Derive an expression for the attenuation offered by the L -Section with these values of R_1 and R_2 .

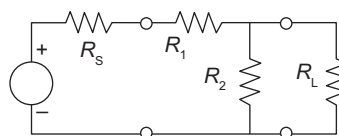


Fig. 16.18-2

16.19 PROBLEMS

1. Find the y -parameters of the network in Fig. 16.19-1 by considering it as a parallel connection of a T-network and a Π -network.

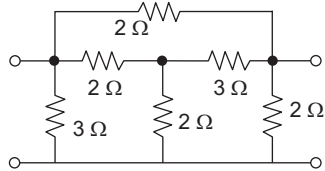


Fig. 16.19-1

2. The $[y]$ matrix of a resistive network is $\begin{bmatrix} 0.625 & -0.125 \\ -0.125 & 1.125 \end{bmatrix}$ S.

Find the new $[y]$ matrix if a $2\ \Omega$ resistor is connected (i) across the input port (ii) across the output port and (iii) between the upper input terminal and the upper output terminal.

3. Find $[y(s)]$ matrix for the network shown in Fig. 16.19-2 by considering it as a parallel connection of two two-port networks.

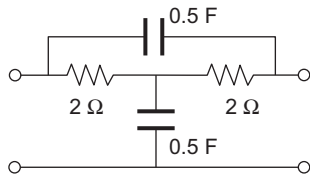


Fig. 16.19-2

4. (i) Consider the circuit in Fig. 16.19-3 as a series connection of two two-port networks, and thereby, determine its $[z]$ matrix. (ii) If the input is driven by a voltage source v_s through a source resistance of $1\ \text{k}\Omega$ and output is loaded with a $1\ \text{k}\Omega$ resistance, determine the voltage gain of the circuit.

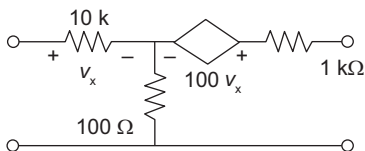


Fig. 16.19-3

5. (i) Obtain $[y(s)]$ of the network in Fig. 16.19-4 by treating it as a parallel connection of two two-port networks. (ii) If the input

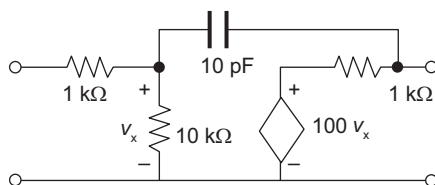


Fig. 16.19-4

is driven by a voltage source v_s through a source resistance of $1\ \text{k}\Omega$ and the output is loaded with a $10\ \text{k}\Omega$ resistance, find the voltage transfer function of the network and find its approximate bandwidth.

6. (i) Obtain $[g]$ of the network in Fig. 16.19-5 by treating it as the parallel-series connection of two two-port networks. (ii) Obtain the transfer ratio of output voltage to source current when the input is driven by a current source and the output is loaded with a resistor of $10\ \text{k}\Omega$.

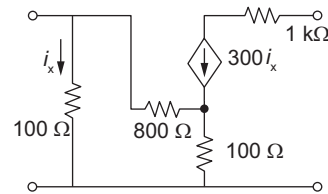


Fig. 16.19-5

7. The $[z]$ matrix of a resistive network is $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ Ω . Find the new $[z]$ matrix if a $2\ \Omega$ resistor is connected (i) in series with the input port, (ii) in series with the output port (iii) between the upper input terminal and the upper output terminal (iv) across the input port and (v) across the output port.
8. The network N in Fig. 16.19-6 is a resistive symmetric two-port with $y_{11} = 0.2\ \text{S}$ and $y_{12} = -0.05\ \text{S}$. Find the port voltages.

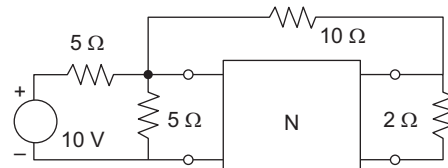


Fig. 16.19-6

9. The network in Fig. 16.19-7 is called a symmetric lattice network. Find Z_{oc} , Z_{sc} , Z_0 , $[ABCD]$ and γ in terms of Z_A and Z_B .

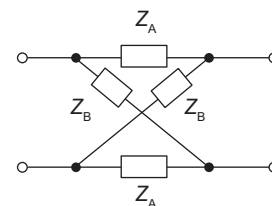


Fig. 16.19-7

10. Find the T-equivalent and Π -equivalent of the symmetric lattice network in Problem 9.

11. A two-port network is terminated at impedance $Z_L(s)$ at the output port. Derive expressions for voltage gain and input impedance functions in terms of (i) y -parameters and Z_L , (ii) z -parameters and Z_L , (iii) h -parameters and Z_L , (iv) g -parameters and Z_L and (v) $ABCD$ -parameters and Z_L .
12. Find the transmission parameters of the network in Fig. 16.19-8.

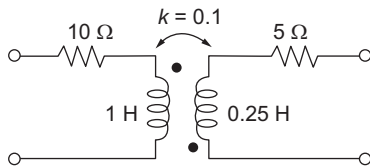


Fig. 16.19-8

13. The $[z]$ matrix of a network is $\begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix} \Omega$. Find the $ABCD$ matrix of a new network formed by connecting a 1 H inductor in series with the input port.
14. The h -parameter equivalent circuit for small-signal operation of a transistor is given in Fig. 16.19-9. $h_{11} = 2 \text{ k}\Omega$, $h_{12} = 0.0005$, $h_{21} = 200$ and $h_{22} = 0.05 \text{ mS}$. An amplifier is formed by loading the output port with $2 \text{ k}\Omega$ and driving the input by voltage source v_s with a source resistance of $0.6 \text{ k}\Omega$. Find (i) the voltage gain v_2/v_s (ii) the input resistance experienced by the voltage source and (iii) the Thevenin's equivalent at the output.

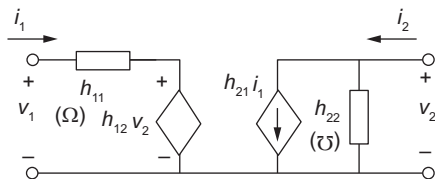


Fig. 16.19-9

15. Find the A parameter of the twin-T network shown in Fig. 16.19-10 in s -domain, and thereby obtain the transfer function $V_2(s)/V_1(s)$ when the output is open. Show that this circuit can function as a band-stop filter (also called *notch filter*).

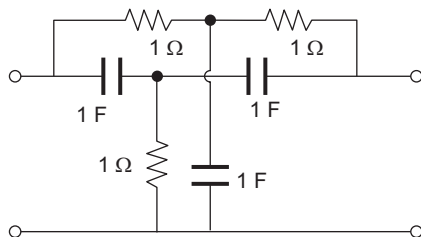


Fig. 16.19-10

16. The two-port network N in Fig. 16.19-11 has $h_{11} = 10 \Omega$, $h_{12} = 0$, $h_{21} = 10$ and $h_{22} = 0.01 \text{ S}$. Find the transfer function $V_2(s)/V_1(s)$.

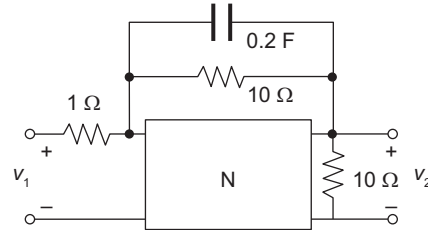


Fig. 16.19-11

17. (i) Find the $ABCD$ parameters of the network in Fig. 16.19-12. (ii) A load impedance $Z_L(s)$ is connected across the output port 2-2'. Find the ratio of Laplace Transform of current through $Z_L(s)$ to the voltage applied at the input port 1-1' using $ABCD$ parameters. (Hint: It is a lattice network redrawn in the form of a bridge circuit.)

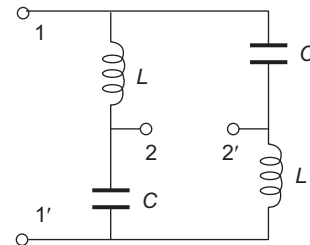


Fig. 16.19-12

18. Find the characteristic impedance and the propagation constant of the network in Fig. 16.19-13 at 1 rad/s.

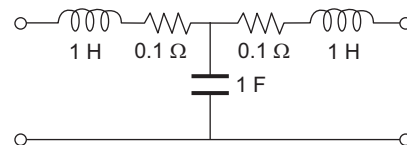


Fig. 16.19-13

19. (i) Design a prototype T-section low-pass filter to cut-off at 100 Hz with a load resistance of 75Ω . (ii) Calculate the attenuation in Np and in dB at 200 Hz and 1 kHz. (iii) Find the phase shift suffered by the output signal for 10 Hz and 50 Hz.
20. (i) Design a prototype Π -section high-pass filter to cut-off at 100 Hz with a load resistance of 75Ω . (ii) Calculate the attenuation in Np and in dB at 20 Hz and 30 Hz. (iii) Find the phase shift suffered by the output signal for 150 Hz and 500 Hz.
21. Design a T-section low-pass filter to meet the following specifications. Cut-off frequency = 5 kHz, source resistance = 75Ω , load resistance = 75Ω and frequencies at

which infinite attenuation is desired = 5.5 kHz and 8 kHz. Use termination sections.

22. Repeat the design in Problem 19 using Π -sections.
23. Design a T-section high-pass filter for a cut-off frequency of 5 kHz. Source and load resistances are $100\ \Omega$ each. Infinite attenuation is desired at 4.5 kHz. Use termination sections.
24. Repeat the design in Problem 21 using Π -sections.
25. Design a constant-k band-pass filter for passing signals in the frequency range between 1 kHz and 1.2 kHz into a load resistance of $75\ \Omega$.
26. Design a band-stop filter to reject signals in the frequency range between 10 kHz and 11 kHz. The load resistance is $300\ \Omega$.
27. The open-circuit voltage observed across a signal source varies between $\pm 100\ \text{mV}$. The voltage across a $600\ \Omega$ resistance connected across this source is found to vary between $\pm 50\ \text{mV}$. Design a T-Section attenuator such that the voltage across a $600\ \Omega$ load connected across the output of the attenuator varies between $\pm 5\ \text{mV}$.
28. Repeat Problem 27 using a Π -section attenuator.
29. Repeat Problem 27 using a lattice-section attenuator.
30. Repeat Problem 27 if the load resistance connected across the attenuator output is $1200\ \Omega$.

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Introduction to Network Topology

CHAPTER OBJECTIVES

- To introduce and illustrate topological analysis of Electrical Networks.

This chapter provides an introduction to the study of topological properties of electrical networks. The reader is taken through an introduction to linear graphs, incidence matrix, circuit matrix and cut-set matrix and KCL/KVL equations in terms of topological matrices followed by Nodal Analysis, Loop Analysis and Node-Pair Analysis of networks.

INTRODUCTION

An electrical network is an interconnection of two-terminal elements and/or multi-terminal elements. Two sets of laws govern the electrical behaviour of such a network. The first set comprises Kirchhoff's Current Law and Kirchhoff's Voltage Law. The second set comprises element relations for the two-terminal elements and multi-terminal elements. Element relationship encodes the electrical behaviour of the physical device that is being modelled by the two-terminal element model or the multi-terminal model in the form of a constraint between terminal voltage and terminal current variables. 'Electric circuit' and 'electrical network' mean more or less the same. However, the word *network* usually implies a *complex interconnection of large number of electrical elements*.

Kirchhoff's Current Law imposes constraints among the various currents leaving any node in the network. Kirchhoff's Voltage Law imposes constraints among the various voltage variables that appear within any loop in the network. These constraint equations do not depend on the particular nature of elements involved in the network. They depend only on the way the elements are interconnected – i.e., KCL and KVL equations depend only on the *structure* of the network and do not depend on the nature of elements employed to form the *network structure*. Properties that depend entirely on the *structure* of a network are called the *topological properties* of that network.

Topological
properties of a
network.

The study of topological properties of a network employing a branch of Mathematics called *Linear Graph Theory* is termed *Network Topology*. This chapter provides a brief introduction of network topology.

17.1 LINEAR ORIENTED GRAPHS

Consider the three networks shown in Fig. 17.1-1(a), (b) and (c). They all have the same network structure. Six two-terminal elements are interconnected between four nodes to form six loops in all the three networks. However, the nature of elements is different in the case of networks in Fig. 17.1-1(a) and (b). The nature of elements in the network in Fig. 17.1-1(c) is not specified. The generic symbol for a two-terminal element is used for all the six elements in that network.

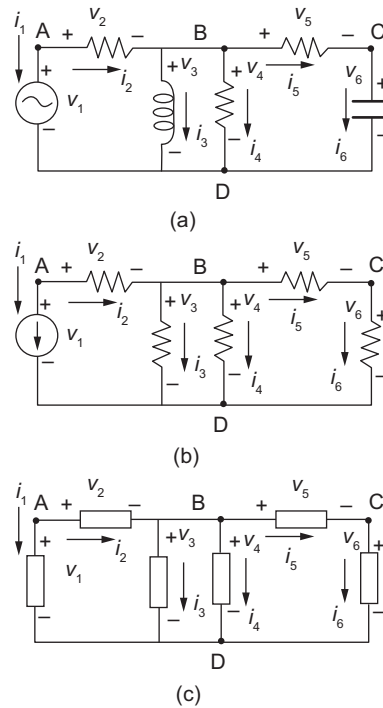


Fig. 17.1-1 Three Electrical Networks with the Same Network Structure

However, the KCL equations written for the four nodes and the KVL equations written for the six loops will be identical for the three networks. They are:

$$\begin{aligned}
 i_1 + i_2 &= 0 \\
 i_3 + i_4 + i_5 - i_2 &= 0 \\
 i_6 - i_5 &= 0 \\
 -i_1 - i_3 - i_4 - i_6 &= 0 \\
 -v_1 + v_2 + v_3 &= 0 \\
 -v_3 + v_4 &= 0 \\
 -v_4 + v_5 + v_6 &= 0 \\
 -v_1 + v_2 + v_5 + v_6 &= 0 \\
 -v_1 + v_2 + v_4 &= 0 \\
 -v_3 + v_5 + v_6 &= 0
 \end{aligned}$$

Thus, the particular nature of elements in a network is of no consequence in preparing the topological equations of an electrical network. Then, there is no reason to carry the symbols of elements all through the topological analysis. We may suppress the nature of elements and represent all the elements in a network by the generic symbol of an element. Still better, we may choose the generic symbol to be a simple *line segment*. The interconnection of electrical elements results in junctions or nodes in the network. We represent this fact by assigning a *small bubble* symbol at the junction points (nodes) in the network. Thus, we represent an electrical network by a model obtained by replacing the element symbols with *line segments* and nodes with *bubbles* for topological analysis. The resulting model is the same as what mathematicians call a *linear graph*. The line-segments connected between bubbles are called *branches* of the *graph* and the bubbles themselves are called *nodes* of the *graph*.

A 'linear graph' becomes an *oriented linear graph* if we associate a unique direction with each *branch* of a linear graph. The direction associated with a *branch* in a linear graph is taken to represent the *reference direction for the current variable in that electrical element* when the linear graph represents an electrical network. *The reference polarity for the voltage variable of the corresponding element is understood to be as per the passive sign convention.*

The oriented linear graph representing all the networks in Fig. 17.1-1 is shown in Fig. 17.1-2. Branches in a linear graph are numbered in a sequential order as shown. The nodes are also numbered in a similar manner.

A branch has two ends. A branch is said to be *incident* at a *node* if that branch either starts at that node or ends at that node. For instance, Branch-2 in the linear graph in Fig. 17.1-2 is incident at node-1. It is incident at node-2 also. The branches incident at node-2 are Branch-2, Branch-3, Branch-4 and Branch-5.

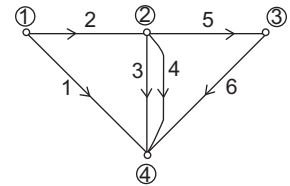


Fig. 17.1-2 Oriented Linear Graph Representation for All the Networks in Fig. 17.1-1

17.1.1 Connected Graph, Subgraphs and Some Special Subgraphs

A *linear graph* is a collection of points called *nodes* and line segments called *branches* with *branches* interconnecting *nodes*. A *subgraph* of a *linear graph* is a subset of its *nodes* and *branches*. A *subgraph* is a *proper subgraph* if the number of nodes and branches that it consists of is less than that of its parent graph.

A *path* is a special subgraph of a graph. A *path* contains two nodes at which there is only one branch incident. These are called the *terminal nodes* of the *path*. There are exactly two branches incident at all the other nodes present in the *path*. No proper subgraph of a *path* will have these two properties. For instance, the branches 2, 3 and 6 along with the nodes 1, 2, 4 and 3 constitute a subgraph that is a *path* of the graph in Fig. 17.1-2. However, the branches 1, 2 and 3 along with the nodes 1, 2 and 4 constitute a proper subgraph of the same graph that does not qualify to be termed as a *path*. This subgraph does not contain terminal nodes. Similarly, branches 1, 3 and 6 along with nodes 1, 2, 3 and 4 constitute a proper subgraph that is not a *path*. This subgraph contains three nodes with one branch incident at them and one node with three branches incident at it.

A graph is a *connected graph* if there is at least one *path* between any two nodes. That is, for any two nodes in the graph, a set of nodes and branches can be found such that a *path* is formed with the specified nodes as its terminal nodes.

A *loop* is another special subgraph of a graph. It is a connected subgraph of the graph with exactly two branches incident at each node in the subgraph. It can be called a *closed path*. If the terminal nodes of a *path* are made to coincide, the result will be a *loop*. A *loop* can be specified by listing the branches appearing in it. For instance, the set of branches 1, 2 and 3 (shown as in Fig. 17.1-3(a)) will constitute a *loop* in the graph in Fig. 17.1-2. However, the set of branches 1, 2, 3 and 5 (shown as in Fig. 17.1-3(b)) will not qualify as a *loop* since this set of branches constitutes a subgraph that contains a node (node-3) which has only one branch incident at it. This is despite the fact that this subgraph itself contains a subgraph (the branches 1, 2 and 3) that qualifies as a *loop* embedded within it. The set of

Definition of a subgraph.

A special subgraph called *path* and its properties.

Definition of connected graph.

Definition of a loop in a linear graph.

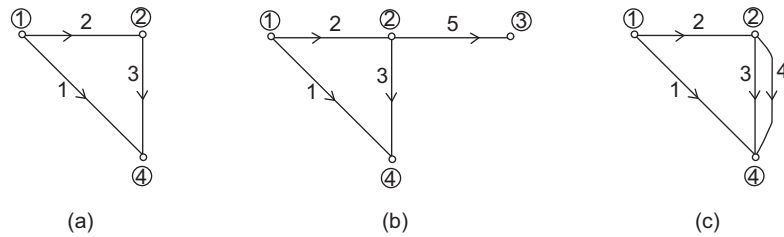


Fig. 17.1-3 Some Subgraphs of the Graph in Fig. 17.1-2 (a) Is a Loop, (b) and (c) Are Not

Definition of a tree.

Twigs, Links,
Co-tree.

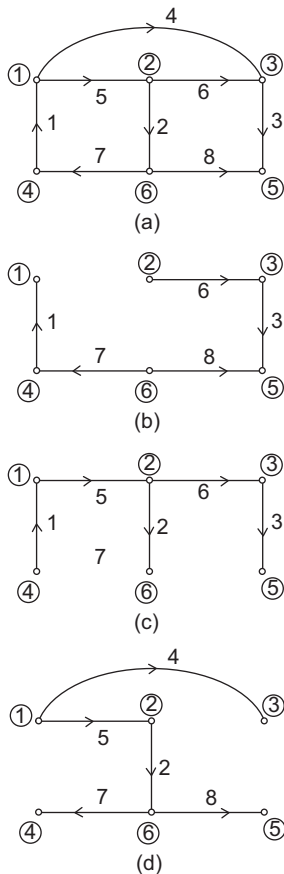


Fig. 17.1-4 A Linear Graph and Three Trees for the Graph

branches 1, 2, 3 and 4 (shown as in Fig. 17.1-3(c)) does constitute a proper subgraph of the graph in Fig. 17.1-2. However, it is not a *loop* since it contains two nodes (node-2 and node-4) with three branches incident at them.

A connected graph can contain many loops, but it is not necessary that it should contain any. That is, it is very well possible for a graph to remain connected even without a single loop in it. That raises an interesting question – given a connected graph that contains one or many loops in it, is it possible to find a *connected subgraph* of this graph that contains all the nodes but no loops?

Such a *connected subgraph* of a graph is a special subgraph. It is called a *tree* of the graph. Thus, a *tree* of a graph is a *connected subgraph of that graph with all the nodes of the graph present in it but no loops present in it*. The branches that appear in the *tree* are called *twigs*. Those branches that do not appear in a *tree* are called *links*. Obviously, a branch that happens to be a *link* for some particular choice of a *tree* can very well turn out to be a *twig* for some other choice of a *tree*. All the *links* together will form another subgraph of the parent graph. This subgraph is the complement of *tree* and is called *co-tree*. A *tree* is a *connected subgraph*; a *co-tree* need not be a *connected subgraph*.

Figure 17.1-4(a) shows a linear graph with 6 nodes and 8 branches. Three *trees* for this graph are also shown in the same figure under (b), (c) and (d). Many more *trees* can be selected for this graph. A *tree* can be specified by listing its *twigs*. For instance, {1, 3, 6, 7, 8} specifies the *tree* in Fig. 17.1-4(b), whereas {1, 2, 3, 5, 6} specifies the one in Fig. 17.1-4(c) and {2, 4, 5, 7, 8} specifies the one in Fig. 17.1-4(d). Listing of nodes in a *tree* is superfluous. A subgraph qualifies to be called a *tree* if, and only if, it contains all the nodes of the parent graph.

We note that all the three *trees* shown in Fig. 17.1-4 contain exactly 5 *twigs* and that the parent graph contains 6 nodes. It can be verified that any *tree* of this graph will contain exactly 5 *twigs*. This, in fact, is a general result. *The number of branches appearing in a tree of an n-node linear graph will be (n – 1)*. In other words, it requires exactly $(n - 1)$ branches to keep an n -node graph connected without even a single loop in it. Conversely, a graph with n nodes and $< (n - 1)$ branches can not be a *connected graph*. However, any arbitrary set of $(n - 1)$ branches drawn from a graph with n nodes and $> (n - 1)$ branches need not constitute a *tree* for that graph. Moreover, a graph with n nodes and $> (n - 1)$ branches need not necessarily be a *connected graph*.

The fact that the number of *twigs* in any *tree* of an n -node, connected graph is $(n - 1)$ can be proved by mathematical induction. However, a different line of proof is offered below.

Consider node-1 in the graph shown in Fig. 17.1-4(a). This node is connected to node-2 by a single branch. Similarly, node-1 is connected to node-4 and node-3 by single-branch connections. However, the *path* between node-1 and node-5 requires traversal of at least two branches. Thus, node-5 and node-6 are away from node-1 by at least two branches, whereas node-3 and node-4 are away from node-1 by one branch only. We introduce the

concept of *neighbouring nodes* at this point. *Neighbouring nodes* of a node in a graph are those nodes that are one branch away from the node under consideration. Thus, nodes 2, 3 and 4 are the neighbouring nodes of node-1 in the graph in Fig. 17.1-4(a).

A node in a graph should have at least one neighbouring node if the graph is a *connected graph*. Therefore, a *tree* for a *connected graph* can be constructed by the following algorithm.

Select an arbitrary node, say node- j , as the seed node. Poll the remaining nodes one by one, starting with node-1 and reaching up to node- n but excluding node- j . When a particular node is polled, check to see whether it is the neighbouring node of any node in the already connected group of nodes. If so, connect it up by using a suitable branch. If not, skip that node and go to the next. Once all the nodes are polled in this manner, repeat the polling process on the nodes that were skipped in the first pass. Repeat the polling passes until all the nodes are connected up. This process will terminate successfully since a node in a connected graph will have at least one neighbouring node. The number of branches used in this process will be exactly $(n - 1)$. If we choose to add one more branch to the subgraph formed at the end of this process, we can not do so without creating a loop since all the nodes are connected up already. Thus, the number of branches needed to connect up all the nodes of a graph without forming a loop is exactly $(n - 1)$, where n is the number of nodes in the graph.

This algorithm is illustrated for the graph shown in Fig. 17.1-4(a). Let us select node-5 as the seed node. Now, consider node-1. Node-1 is not a neighbour of node-5. So, we skip it in the first pass. Similarly, node-2 also skipped in the first pass for the same reason. Node-3 is seen to be a neighbour of node-1. Hence, it is connected up to node-1 by accepting Branch-3 as a *twig*. Node-4 is skipped in the first pass since it is not a neighbour of node-5 or node-3. Node-6 is a neighbour of node-5 and is connected up by accepting Branch-8 as a *twig*. This completes the first pass. At the end of the first pass, we get three nodes – 3, 5 and 6 – connected up by two branches – 3 and 8 – and three more nodes to be connected up.

The first node to be considered in the second pass is node-1. Node-1 is the neighbour of node-3. Hence, node-1 can be connected up by accepting Branch-4 as a *twig*. The next node to be considered is node-2. It is the neighbour of node-3 and node-6. Hence, node-2 can be connected up by accepting either Branch-6 or Branch-2 as a *twig*. We accept Branch-2 as a *twig*. (The other choice will lead us to a different *tree*). The next node to be considered in the second pass is node-4. It is the neighbour of node-6. We connect it up by accepting Branch-7 as a *twig*. That completes the second pass. At the end of second pass, we get all the six nodes connected up by five branches – 3, 8, 4, 2 and 7. These branches form a *tree*. The addition of one more branch to this subgraph will result in the formation of a loop. The removal of one or more branches from this subgraph will make it a non-connected sub-graph.

The number of *twigs* in a tree of an n -node linear graph is $(n - 1)$.

17.2 THE INCIDENCE MATRIX OF A LINEAR ORIENTED GRAPH

A linear oriented graph can be described in graphical form as in Fig. 17.2-1. It can also be described in tabular form with the branch number – starting node for that branch and ending node for that branch as entries in the rows of the table. The same information can be given in the form of a matrix called *All Incidence Matrix* $\mathbf{A}_a$ defined as below.

The *All Incidence Matrix* $\mathbf{A}_a$ of an oriented linear graph has as many rows, as there are nodes in the graph. It has as many columns, as there are branches in the graph. Let a_{ij} represent the entry in $\mathbf{A}_a$ at i^{th} row- j^{th} column position. Then,

- $a_{ij} = 1$, if the j^{th} branch is incident at the i^{th} node and is *oriented away* from that node.
- $a_{ij} = -1$, if the j^{th} branch is incident at the i^{th} node and is *oriented towards* that node.
- $a_{ij} = 0$ if the j^{th} branch is not incident at the i^{th} node.

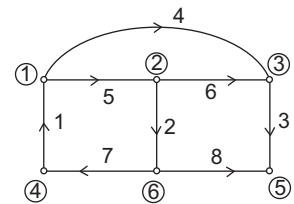


Fig. 17.2-1 An Oriented Linear Graph for Illustrating Incidence Matrix

The all incidence matrix $\mathbf{A}_a$ for the linear graph shown in Fig. 17.2-1 is determined below:

$$\begin{array}{c} \text{Nodes} \\ \downarrow \end{array} \quad \begin{array}{c} \text{Branches} \rightarrow \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \end{array}$$

$$\mathbf{A}_a = \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{matrix} \begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Incidence Matrix
 $\mathbf{A}$ of a Linear Graph.

A branch can be incident only at two nodes and it has to be incident at exactly two nodes. Since a branch has a direction associated with it, it will have to be incident at one node with orientation away from that node and at another node with orientation towards it. Therefore, a column in $\mathbf{A}_a$ can contain only two non-zero entries and those entries have to be +1 and -1. Hence, the sum of all rows of $\mathbf{A}_a$ will be a row containing zeros. This implies that we do not need all the n rows of $\mathbf{A}_a$ matrix. Any one row can be suppressed. The information contained in the suppressed row can be obtained by using the principle that the sum of entries in any column of $\mathbf{A}_a$ matrix must be zero. The sub-matrix of $\mathbf{A}_a$ matrix obtained by discarding the row corresponding to any one node is called the *Reduced Incidence Matrix* (or simply, the *Incidence Matrix*) and is represented by $\mathbf{A}$. The node that was discarded in preparing the *Incidence Matrix* $\mathbf{A}$ is called the *reference node*. For instance, if node-6 is chosen as the reference node for the graph in Fig. 17.2-1, its incidence matrix will be

$$\begin{array}{c} \text{Nodes} \\ \downarrow \end{array} \quad \begin{array}{c} \text{Branches} \rightarrow \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \end{array}$$

$$\mathbf{A} = \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{matrix} \begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The size and rank
of incidence matrix
of a connected
linear graph.

The order of incidence matrix $\mathbf{A}$ is $(n - 1) \times b$, where n is the number of nodes in the graph and b is the number of branches in it. If the graph is a connected graph, b will be $\geq (n - 1)$. Therefore, the order of the largest *square sub-matrix* that can be drawn from the incidence matrix of a connected linear graph will be $(n - 1)$. The *rank* of a matrix is the order of the largest square sub-matrix drawn from the matrix such that the determinant of at least one such sub-matrix is non-zero. Thus, the rank of $\mathbf{A}$ of a connected linear graph can be at the most $(n - 1)$. $\mathbf{A}$ is a sub-matrix of $\mathbf{A}_a$. Therefore, the rank of the complete incidence matrix $\mathbf{A}_a$ of a connected linear graph too can only be $\leq (n - 1)$.

Consider a *tree* specified by the set of branches $\{1, 3, 6, 7, 8\}$ in the graph shown in Fig. 17.2-1. The graph and the *tree* are shown in Fig. 17.2-2(a) and (b).

The columns of the incidence matrix are now arranged in such a manner that the *tree twigs* appear first, followed by *tree links*. The $\mathbf{A}$ matrix is:

$$\begin{array}{c} \text{Nodes} \\ \downarrow \end{array} \quad \begin{array}{c} \text{Twigs} \rightarrow \\ 1 \quad 3 \quad 6 \quad 7 \quad 8 \end{array} \quad \begin{array}{c} \text{Links} \rightarrow \\ 2 \quad 4 \quad 5 \end{array}$$

$$\mathbf{A} = [\mathbf{A}_t \mid \mathbf{A}_l] = \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{matrix} \left[\begin{array}{ccccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{array} \right]$$

A matrix can be partitioned into two sub-matrices $\mathbf{A}_t$ and $\mathbf{A}_p$, where $\mathbf{A}_t$ is of the order $(n-1) \times (n-1)$ and $\mathbf{A}_p$ is of the order $(n-1) \times (b-n+1)$. We observe that the determinant of $\mathbf{A}_t$ in this example is $+1$. Therefore, the rank of $\mathbf{A}$ matrix in this example is $5 - 1$ one less than the number of nodes.

The result, in fact, is a general one. The determinant of sub-matrix $\mathbf{A}_t$ of the Incidence matrix $\mathbf{A}$ for a chosen *tree* of a connected graph is ± 1 . Hence, the rank of incidence matrix of a connected graph with n nodes is $(n-1)$. This can be proved as below:

One node (the *reference node*) is suppressed in forming the $\mathbf{A}$ matrix. This node has to remain connected to other nodes at least by one branch in any *tree* of the graph, since a *tree* is a connected subgraph of the parent graph. Therefore, at least one *twig* of any *tree* of the graph has to be incident at the *reference node*. Then, the column corresponding to this *twig* in $\mathbf{A}_t$ will contain only one non-zero entry. This entry can be a $+1$ or -1 . Thus, the determinant of $\mathbf{A}_t$ can now be obtained as $\pm 1 \times$ (determinant of a sub-matrix of $\mathbf{A}_t$ obtained by removing the column corresponding to the *twig* that is incident at the *reference node* and the row corresponding to the node at which the second end of this *twig* is incident).

This sub-matrix is the incidence matrix of a *tree* containing $(n-1)$ nodes with the *reference node* being the same as the node that corresponds to the eliminated row. The reasoning explained in the last paragraph is applicable to this sub-matrix too.

Proceeding in this manner, we arrive at a sub-matrix representing the incidence matrix of a *tree* containing just two nodes. Such an incidence matrix will contain just one entry – either $+1$ or -1 .

Thus, the determinant of $\mathbf{A}_t$ for any chosen *tree* will be the product of many factors of ± 1 value and will be non-zero. Thus, $\mathbf{A}_t$ for any chosen *tree* of a connected graph will be non-singular and will have a value of ± 1 . Therefore, the rank of incidence matrix $\mathbf{A}$ (and that of $\mathbf{A}_a$ too) of a connected graph with n nodes is $(n-1)$.

Further, it is possible to prove that the columns of an $(n-1) \times (n-1)$ square and the non-singular sub-matrix drawn from the *Incidence Matrix* $\mathbf{A}$ of a *connected linear graph* will correspond to the *twigs* of some *tree* of that graph.

Therefore, the number of *trees* for a given connected graph will be equal to the number of square non-singular sub-matrices of the Incidence Matrix $\mathbf{A}$ of that graph. It is possible to show with the help of Binet-Cauchy Theorem in Matrix Algebra that this number is equal to $\det(\mathbf{A}\mathbf{A}^T)$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. The example graph in Fig. 17.2-2 has 35 *trees*.

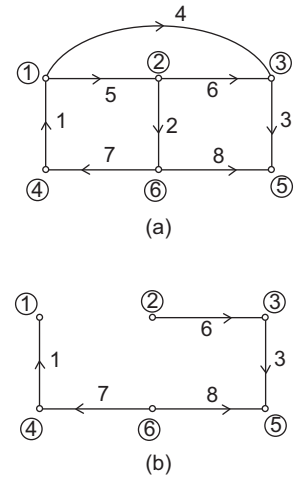


Fig. 17.2-2 (a) A Linear Graph and (b) A Chosen Tree

The rank of Incidence Matrix $\mathbf{A}$ (and that of $\mathbf{A}_a$ too) of a connected graph with n nodes is $(n-1)$.

Number of *trees* of a connected graph = determinant of $(\mathbf{A}\mathbf{A}^T)$.

17.2.1 Path Matrix and its Relation to Incidence Matrix

Path Matrix $\mathbf{P}$ for a connected linear graph is defined with reference to a chosen *tree* of that graph. Given a connected graph, select a *tree* and a *reference node* for the incidence matrix. The $\mathbf{P}$ matrix with respect to the selected *tree* and the *reference node* is defined as $\mathbf{P} = [p_{ij}]$, where

$p_{ij} = +1$, if the *twig-j* is in the unique directed *path* in the tree from node- i to the *reference node* and its orientation agrees with that of the *path*.

$p_{ij} = -1$, if the *twig-j* is in the unique directed *path* in the tree from node- i to the *reference node* and its orientation disagrees with that of the *path*.

$p_{ij} = 0$, if the *twig-j* is not in the unique directed *path* in the tree from node- i to the *reference node*.

It is a square matrix of the order $(n-1) \times (n-1)$ whose rows correspond to *paths* from the corresponding nodes to the *reference node* and whose columns correspond to *twigs*. Rows and columns in the $\mathbf{P}$ matrix are ordered in the same manner as in the $\mathbf{A}$ matrix.

With reference to Fig. 17.2-2, node-6 was selected as the *reference node* for preparing matrix $\mathbf{A}$ earlier. Consider node-1. This node is connected to the *reference node* by branches 1 and 7. The orientation of the *path* is from node-1 to the *reference node*. Branch-1 is oriented in a direction that is opposite to the orientation of *path*, so is Branch-7. Therefore,

Definition of *Path Matrix* $\mathbf{P}$.

the first row of $\mathbf{P}$ matrix will have -1 in the first and fourth columns and 0 in the remaining columns. The complete $\mathbf{P}$ matrix and its transpose are shown below:

$$\mathbf{P} = \begin{array}{c} \text{Paths} \\ \downarrow \\ \begin{array}{c} \text{Twigs} \rightarrow \\ \begin{array}{ccccc} & 1 & 3 & 6 & 7 & 8 \\ (1) & -1 & 0 & 0 & -1 & 0 \\ (2) & 0 & 1 & 1 & 0 & -1 \\ (3) & 0 & 1 & 0 & 0 & -1 \\ (4) & 0 & 0 & 0 & -1 & 0 \\ (5) & 0 & 0 & 0 & 0 & -1 \end{array} \end{array} \end{array} \quad \mathbf{P}^T = \begin{array}{c} \text{Twigs} \\ \downarrow \\ \begin{array}{c} \text{Paths} \rightarrow \\ \begin{array}{ccccc} & (1) & (2) & (3) & (4) & (5) \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 & -1 & 0 \\ 8 & 0 & 1 & -1 & 0 & -1 \end{array} \end{array} \end{array}$$

Consider the i^{th} row of $\mathbf{A}_t$ matrix and the i^{th} column of $\mathbf{P}^T$ matrix. The i^{th} row of $\mathbf{A}_t$ matrix reveals the *twigs* that are incident at the i^{th} node. The i^{th} column of $\mathbf{P}^T$ matrix reveals the *twigs* that constitute the *path* from the i^{th} node to the *reference node*. Obviously, such a *path* can contain only one *twig* that is incident at the i^{th} node. Therefore, i^{th} row of $\mathbf{A}_t$ matrix and i^{th} column of $\mathbf{P}^T$ matrix can contain only one pair of non-zero entries at similar locations. Thus, when i^{th} row of $\mathbf{A}_t$ matrix and the i^{th} column of $\mathbf{P}^T$ matrix are multiplied together, there will be only one non-zero product entry. If the *twig* that is common to the i^{th} row of $\mathbf{A}_t$ matrix and the i^{th} column of $\mathbf{P}^T$ matrix is incident at the i^{th} node with orientation away from the node, it will enter the $\mathbf{A}_t$ matrix and $\mathbf{P}^T$ matrix with a positive sign. Otherwise, it will enter both matrices with a negative sign. Therefore, the product of i^{th} row of $\mathbf{A}_t$ matrix and i^{th} column of $\mathbf{P}^T$ matrix will be 1. Thus, all the diagonal entries of the matrix product $\mathbf{A}_t \mathbf{P}^T$ will be equal to 1.

Now, consider the i^{th} row of $\mathbf{A}_t$ matrix and j^{th} column of $\mathbf{P}^T$ matrix with $i \neq j$. They can have non-zero entries in similar positions only if the i^{th} node is in the *path* of the j^{th} node. In that case, exactly two *twigs* that are in the *path* from node- j to the *reference node* must be incident at node- i . If those two *twigs* have relatively same orientation with respect to node- i , they will have a relatively opposite orientation with respect to the *path* of the j^{th} node. If they have relatively opposite orientation with respect to node- i , they will have relatively same orientation with respect to the *path* of the j^{th} node. Therefore, the product of i^{th} row of $\mathbf{A}_t$ matrix and the j^{th} column of $\mathbf{P}^T$ matrix will be 0 for all $i \neq j$.

Therefore, the matrix product $\mathbf{A}_t \mathbf{P}^T$ is an identity matrix of the order $(n - 1) \times (n - 1)$. This fact can be used to calculate the inverse of $\mathbf{A}_t$ matrix. This inverse will be needed under various circumstances in topological analysis of networks.

$$\therefore \mathbf{A}_t^{-1} = \mathbf{P}^T$$

Let us verify this for the graph and the *tree* in Fig. 17.2-2.

$$\mathbf{A}_t = \begin{array}{c} \text{Nodes} \\ \downarrow \\ \begin{array}{c} \text{Twigs} \rightarrow \\ \begin{array}{ccccc} & 1 & 3 & 6 & 7 & 8 \\ (1) & -1 & 0 & 0 & 0 & 0 \\ (2) & 0 & 0 & 1 & 0 & 0 \\ (3) & 0 & 1 & -1 & 0 & 0 \\ (4) & 1 & 0 & 0 & -1 & 0 \\ (5) & 0 & -1 & 0 & 0 & -1 \end{array} \end{array} \end{array} \quad \mathbf{P}^T = \begin{array}{c} \text{Twigs} \\ \downarrow \\ \begin{array}{c} \text{Paths} \rightarrow \\ \begin{array}{ccccc} & (1) & (2) & (3) & (4) & (5) \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 & -1 & 0 \\ 8 & 0 & 1 & -1 & 0 & -1 \end{array} \end{array} \end{array}$$

$$\therefore \mathbf{A}_t \mathbf{P}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Transpose of the
Path Matrix gives the
inverse of sub-matrix
of $\mathbf{A}$ corresponding
to *tree twigs*.

17.3 KIRCHHOFF'S LAWS IN INCIDENCE MATRIX FORMULATION

17.3.1 KCL Equations from A Matrix

Kirchhoff's Current Law states that the sum of currents leaving a node in a network is zero on an instant-to-instant basis at all nodes in the network. The orientation of a particular branch in the graph of a network was taken to be the same as the reference direction for current in that branch. The incidence matrix entry is +1 if a branch is incident at a node with orientation away from the node. It is -1 if a branch is incident at a particular node and is oriented towards it. Therefore, the KCL equation at a node can be obtained by multiplying the entries in the row of $\mathbf{A}$ corresponding to that node by the elements of a column vector of branch currents. Therefore, the set of KCL equations at all nodes (except at the node used as the *reference node* for preparing the $\mathbf{A}$ matrix) can be expressed in terms of $\mathbf{A}$ matrix as:

$$\mathbf{A}\mathbf{i}(t) = \mathbf{0} \quad (17.3-1)$$

where $\mathbf{A}$ is the $(n - 1) \times b$ incidence matrix of the graph of the network, $\mathbf{i}(t)$ is the $b \times 1$ column vector containing the instantaneous value of branch currents and $\mathbf{0}$ is an $(n - 1) \times 1$ column vector containing zero-valued entries.

KCL equations can also be expressed in terms of transformed currents in s -domain as $\mathbf{A}\mathbf{I}(s) = \mathbf{0}$, where $\mathbf{I}(s)$ is the $b \times 1$ column vector containing the Laplace Transforms of branch currents. They can also be expressed as $\mathbf{A}\mathbf{I}(j\omega) = \mathbf{0}$ for sinusoidal steady-state analysis, where $\mathbf{I}(j\omega)$ is the $b \times 1$ column vector containing the phasor branch currents.

The incidence matrix $\mathbf{A}$ can be partitioned into two sub-matrices $-\mathbf{A}_t$ and $\mathbf{A}_l$, where columns of $\mathbf{A}_t$ correspond to the *twigs* of a *tree* chosen for the graph and columns of $\mathbf{A}_l$ correspond to the *links* of the same chosen *tree*. $\mathbf{A}_t$ will be non-singular and invertible. Let the branch current column vector also be partitioned in the same manner. Then,

$$\begin{aligned} \mathbf{A}\mathbf{i}(t) &= [\mathbf{A}_t \mid \mathbf{A}_l] \begin{bmatrix} \mathbf{i}_t(t) \\ \mathbf{i}_l(t) \end{bmatrix} = \mathbf{0} \\ \therefore \mathbf{i}_l(t) &= -\mathbf{A}_l^{-1} \mathbf{A}_t \mathbf{i}_t(t) \\ \text{and } \mathbf{i}(t) &= \begin{bmatrix} -\mathbf{A}_l^{-1} \mathbf{A}_t \\ \mathbf{U} \end{bmatrix} \mathbf{i}_t(t) \end{aligned} \quad (17.3-2)$$

This shows that the set of *link currents* in a network for any chosen *tree* of its graph can serve as a *basis set* for all branch currents in that network. *Twig currents* can be expressed as linear combinations of *link currents*.

Consider a linear time-invariant circuit containing two-terminal passive elements, independent voltage sources and independent current sources. How many independent current sources can it contain?

An independent current source is a two-terminal element that has a current variable that is a specified function of time. Thus, the current flowing through a current source is already decided, quite independent of any other network variable. Such an independent function of time can not be expressed as a linear combination of other independent functions of time. Therefore, it must be possible to select a *tree* that does not contain any independent current source in it. If there are more than $(b - n + 1)$ independent current sources in the network, then, it will not be possible to select a *tree* that contains no independent current sources as its *twigs*. In that case, one or more independent current sources will appear as *twigs* even after assigning $(b - n + 1)$ current sources as *links*. But then, KCL implies that *twig currents* can be expressed as a linear combination of *link currents*. This implies that the source function of one or more independent current sources can be expressed as a linear combination of source functions of other independent current sources only. That leads to a contradiction, since a time-function that is expressible as a linear combination of a set of independently specified time-functions can not be an *independent time-function*. Hence,

The row size and the rank of $\mathbf{A}$ matrix are $(n - 1)$. Therefore, there are only $(n - 1)$ independent KCL equations in a network containing n nodes.

Twig currents can be expressed as linear combinations of *link currents* in a network for any choice of *tree* from its graph.

The maximum number of independent current sources that can be present in a network containing two-terminal passive elements, independent voltage sources and independent current sources is $(b - n + 1)$, where b is the total number of elements and n is the number of nodes in the network.

The statement above assumes that each element is represented by a branch in the network graph.

the maximum number of *independent current sources* that can be present in a network containing two-terminal passive elements, independent voltage sources and independent current sources is $(b - n + 1)$. A more detailed modelling of elements in the network will be required before the solution of the network is attempted if a network contains more than $(b - n + 1)$ independent current sources.

It is necessary that a *tree* that does not contain any independent current source as its *twig* exists in the network graph for an electrical network to have a unique solution. However, this is only a necessary but not a sufficient condition.

This does not mean that we will find it impossible to select a tree that does not contain any independent current source in it only when there are more than $(b - n + 1)$ independent current sources in a network. Consider a circuit that has less than $(b - n + 1)$ independent current sources; but with a node where only independent current sources are incident. Obviously, we must allow at least one independent current source to become a *twig* since this particular node too must remain connected to the rest of the nodes in the *tree*. Then, the current in that *twig* will be expressed as a linear combination of *link* currents and the only *link* currents that contribute to the linear combination will be the ones corresponding to the remaining current sources incident at that node. This will result in an attempt to equate an independently specified time-function to a linear combination of other independently specified time-functions. Such an attempt will lead to a contradiction and the network will require a more detailed modelling before a solution can be attempted. Thus, a network containing two-terminal passive elements, independent voltage sources and independent current sources will have a unique solution only if it is possible to select a *tree* that contains no independent current sources as *twigs*. But that does not mean that we have to choose only that kind of a *tree* for solving the network; only that it must be possible to choose such a *tree* if we set our mind to do so.

Further, the condition arrived at in the last paragraph is only a necessary condition for the existence of a unique solution for the network. It is not sufficient. A similar constraint on the number of independent voltage sources too will have to be satisfied. We will deal with this constraint in a later section in this chapter.

17.3.2 KVL Equations and the A Matrix

Kirchhoff's Voltage Law states that the algebraic sum of voltages in any loop in an electrical network is zero on an instant-to-instant basis.

The incidence matrix of the graph of an electrical network contains all the information on the structure of the network. Therefore, it contains the information on how various branches form various loops in the network. However, this information is available in **A** matrix in an indirect form. Thus, the KVL loop equations can not be derived from **A** matrix in a direct manner.

Therefore, we resort to defining a new set of voltage variables for the network. These variables are voltages of nodes with one particular node chosen as the *reference node* for measuring the voltages of other nodes. They are called *node voltage variables* and they were already introduced earlier in Chap. 4 in the context of Nodal Analysis of electrical circuits.

We decide to make the node chosen as the *reference node* for defining *node voltage variables* in a network the same as the node that is chosen as the *reference node* for preparing the incidence matrix **A** of the graph of the network. Now, consider the k^{th} branch that is incident at i^{th} and j^{th} nodes. Consider a loop containing the *reference node*, i^{th} node, the Branch- k , j^{th} node and back to *reference node*. Apply KVL in this loop. Assuming that the Branch- j is oriented away from i^{th} node, we get,

$$-v_{ni} - v_k + v_{nj} = 0 \Rightarrow v_k = v_{ni} - v_{nj} = a_{ik}v_{ni} + a_{jk}v_{nj}$$

where v_{ni} and v_{nj} are node voltages at i^{th} and j^{th} nodes respectively, v_k is the branch voltage of k^{th} branch. a_{ik} and a_{jk} are elements of **A** matrix. If Branch- j is oriented away from i^{th} node, $a_{ik} = 1$ and $a_{jk} = -1$.

If the Branch- j is oriented towards i^{th} node, we get,

$$-v_{ni} - v_k + v_{nj} = 0 \Rightarrow v_k = -v_{ni} + v_{nj} = a_{ik}v_{ni} + a_{jk}v_{nj}.$$

Thus, $v_k = a_{ik}v_{ni} + a_{jk}v_{nj}$ gives the correct value for v_k in terms of end node voltages irrespective of the orientation of k^{th} branch. The signs of entries in $\mathbf{A}$ matrix take care of the polarity of the branch voltage. Extending this argument for all branches in the graph, we can write the following matrix equation that expresses the KVL equations in the network.

$$\mathbf{v}(\mathbf{t}) = \mathbf{A}^T \mathbf{v}_n(\mathbf{t}) \quad (17.3-3)$$

where $\mathbf{v}(\mathbf{t})$ is the $b \times 1$ column vector of instantaneous branch voltage variables, $\mathbf{v}_n(\mathbf{t})$ is the $(n - 1) \times 1$ column vector of instantaneous node voltage variables at all nodes with respect to the *reference node* and $\mathbf{A}^T$ is the transpose of incidence matrix. Instantaneous variables can be replaced by Laplace Transforms in the context of s -domain analysis and with phasor variables in the context of sinusoidal steady-state analysis. Equation 17.3-3 is called the *node transformation equation*.

Branch voltages can be expressed as a linear combination of node voltages.

17.4 NODAL ANALYSIS OF NETWORKS

Equation 17.3-1 and Eqn. 17.3-3 help us to evolve a procedure for solving an electrical network by determining its node voltages first. First, we consider a network containing two-terminal passive elements and independent current sources only.

The network has to be prepared for nodal analysis first. The preparation required in this case consists of replacing series/parallel combination of similar elements with a single element, wherever such a replacement is possible. In addition, the circuit may have to be transformed into an s -domain equivalent circuit or a phasor equivalent circuit depending on the analysis context. The initial condition sources in the s -domain equivalent circuit have to be in the current source format.

The second step is to draw the network graph and label the nodes and branches. Numbering of nodes can be done in an arbitrary manner except that the last node to be numbered is the *reference node*. However, the numbering of branches has to follow a scheme. All the passive branches are numbered first and the independent current source branches are numbered last. Let b_p be the number of passive branches and b_g be the number of independent current source branches.

The next step is to prepare $\mathbf{A}$ matrix in the partitioned form. The columns of $\mathbf{A}$ matrix are arranged in sequential order. The first partition of $\mathbf{A}$ will contain all columns corresponding to the passive branches. The second partition will contain all columns corresponding to the independent current source branches. They are denoted by $\mathbf{A}_p$ and $\mathbf{A}_g$, respectively. The branch current transform vector $\mathbf{I}(s)$ and the branch voltage transform vector $\mathbf{V}(s)$ are also partitioned in a similar manner (s -domain analysis is assumed).

$$\text{Now, } \mathbf{A} = [\mathbf{A}_p \mid \mathbf{A}_g] \text{ and } \mathbf{A}^T = \begin{bmatrix} \mathbf{A}_p^T \\ \mathbf{A}_g^T \end{bmatrix}.$$

Applying Eqn. 17.3-1 and Eqn. 17.3-3, we get,

$$[\mathbf{A}_p \mid \mathbf{A}_g] \begin{bmatrix} \mathbf{I}_p(s) \\ \mathbf{I}_g(s) \end{bmatrix} = \mathbf{0} \quad (17.4-1)$$

$$\begin{bmatrix} \mathbf{V}_p(s) \\ \mathbf{V}_g(s) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_p^T \\ \mathbf{A}_g^T \end{bmatrix} [\mathbf{V}_n(s)] \quad (17.4-2)$$

Kirchhoff's Current Law.

Node Transformation equation.

Each passive branch voltage variable can be related to its current variable by the relation $I_k(s) = y_k(s) V_k(s)$, where $y_k(s)$ is the admittance function of the k^{th} passive branch.

Thus, the column vectors $\mathbf{I}_p(s)$ and $\mathbf{V}_p(s)$ can be related through a $b_p \times b_p$ square diagonal matrix $\mathbf{Y}_p(s)$ with $Y_{p,ii} = y_i(s)$, the admittance of i^{th} branch and $Y_{p,ij} = 0$ for $i \neq j$.

$$\mathbf{I}_p(s) = \mathbf{Y}_p(s) \times \mathbf{V}_p(s) \quad (17.4-3)$$

Substituting this relation in Eqn. 17.4-1 and using Eqn. 17.4-2, we get,

$$[\mathbf{A}_p \mathbf{Y}_p(s) \mathbf{A}_p^T] \mathbf{V}_n(s) = -[\mathbf{A}_g \mathbf{I}_g(s)]$$

Solving for the vector of node voltage transforms, we get,

$$\mathbf{V}_n(s) = -[\mathbf{A}_p \mathbf{Y}_p(s) \mathbf{A}_p^T]^{-1} [\mathbf{A}_g \mathbf{I}_g(s)]$$

The matrix $\mathbf{A}_p \mathbf{Y}_p(s) \mathbf{A}_p^T$ will have the dimension of admittance and will be of the order $b_p \times b_p$, where b_p is the number of passive elements in the network. This matrix is called the *Nodal Admittance Matrix* of the network and is denoted by $\mathbf{Y}_n(s)$.

$$\mathbf{Y}_n(s) = \mathbf{A}_p \mathbf{Y}_p(s) \mathbf{A}_p^T$$

Thus, the network solution is given by:

$$\mathbf{Y}_n(s) = \mathbf{A}_p \mathbf{Y}_p(s) \mathbf{A}_p^T \text{ and } \mathbf{V}_n(s) = -[\mathbf{Y}_n(s)]^{-1} [\mathbf{A}_g \mathbf{I}_g(s)]$$

The branch voltages and the passive branch currents can be now obtained with the help of Eqn. 17.4-1 and Eqn. 17.4-2.

Note that nodal analysis does not require the identification of a *tree* in the network graph.

The $\mathbf{Y}_n(s)$ matrix

The nodal admittance matrix arrived at in this formulation is the same as the nodal admittance matrix that appeared in the context of Nodal Analysis in Chap. 4.

It will be a symmetric matrix for a network containing no dependent sources. Diagonal entries will be equal to the sum of all admittances connected at the corresponding node. Off-diagonal entries will be equal to the negative of the sum of all admittances connected between the relevant nodes.

EXAMPLE: 17.4-1

Determine the power delivered by each current source in the circuit in Fig. 17.4-1 by nodal analysis.

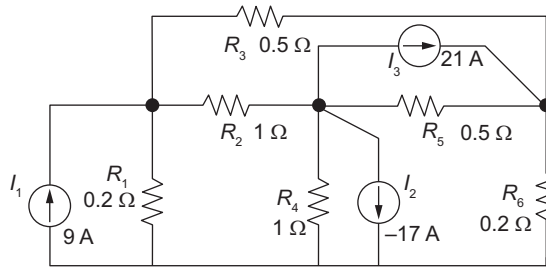


Fig. 17.4-1 Circuit for Example 17.4-1

SOLUTION

Step-1: Assign reference directions for currents and voltages for all branches as per the passive sign convention and prepare the network graph. Number the branches and nodes. Passive branches are numbered first.

Step-2: Form the $\mathbf{A}$ matrix and partition it into $\mathbf{A}_p$ and $\mathbf{A}_g$. The reference node selected is shown in Fig. 17.4-2. The $\mathbf{A}$ matrix is

$$\mathbf{A} = \left[\begin{array}{cccccc|cccc} 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \end{array} \right]$$

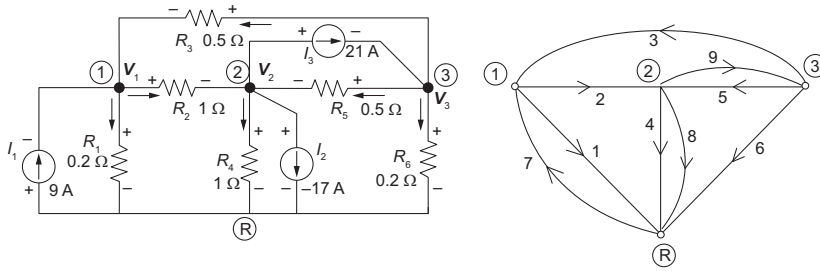


Fig. 17.4-2 Reference Direction Assignment and Network Graph in Example 17.4-1

$$\mathbf{A}_p = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{A}_g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Step-3: Form $\mathbf{Y}_p$ and calculate $\mathbf{Y}_n$.

$$\mathbf{Y}_p = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix} \text{ S}$$

$$\mathbf{Y}_n = \mathbf{A}_p \mathbf{Y}_p \mathbf{A}_p^T$$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{bmatrix} \text{ S}$$

Step-4: Identify $\mathbf{I}_g$ vector and determine $\mathbf{V}_n$ vector.

$$\mathbf{I}_g = \begin{bmatrix} 9 \\ -17 \\ 21 \end{bmatrix} \text{ A and } \mathbf{A}_g \mathbf{I}_g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ -17 \\ 21 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \\ -21 \end{bmatrix} \text{ A}$$

$$\mathbf{V}_n = -\begin{bmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 4 \\ -21 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ V}$$

Step-5: Determine $\mathbf{V}_g$ vector and obtain power delivered by current sources.

$$\mathbf{V}_g = \mathbf{A}_g^T \mathbf{V}_n = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \text{ V}$$

$$\text{Power delivered by 9A Source} = -(-2\text{V}) \times (9\text{A}) = 18 \text{ W}$$

$$\text{Power delivered by } -17\text{A Source} = -(1\text{V}) \times (-17\text{A}) = 17 \text{ W}$$

$$\text{Power delivered by 21A Source} = -(-2\text{V}) \times (21\text{A}) = 42 \text{ W.}$$

17.4.1 The Principle of v-Shift

The v-shift principle

An unaccompanied voltage source connected between node- i and node- j in a network may be shifted into all the other branches connected at either node- i or node- j with a short-circuit applied between node- i and node- j in such a manner that KVL equations in no loop is affected.

Currents through other elements and voltages across them will not be affected by this shifting operation.

We now consider nodal analysis of networks containing two-terminal passive elements and independent current sources and voltage sources.

An independent voltage source is called an *accompanied voltage source* if it has a series impedance along with it. Such an *accompanied voltage source* can be converted into an independent current source in parallel with an impedance prior to nodal analysis. However, if an independent voltage source is unaccompanied, we need to carry out an additional manipulation on the network before we can apply the nodal analysis outlined in this section. We first shift the location of the independent voltage source by employing the v-shift principle and then apply the source transformation theorem to convert such sources into independent current sources before applying the nodal analysis procedure. This is illustrated in the following example.

EXAMPLE: 17.4-2

Find the power delivered by the independent voltage sources in the circuit in Fig. 17.4-3.

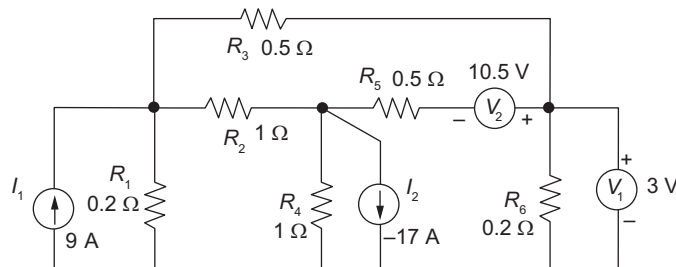


Fig. 17.4-3 Network in Example 17.4-2

SOLUTION

The network, after the unaccompanied independent voltage source V_1 has been shifted into all the branches connected at the positive terminal of this source, is shown in Fig. 17.4-4.

Note the polarity of the sources that appear in the three branches after the v-shift – it is as if the voltage source V_1 was given a push at its bottom and was slid into the branches.

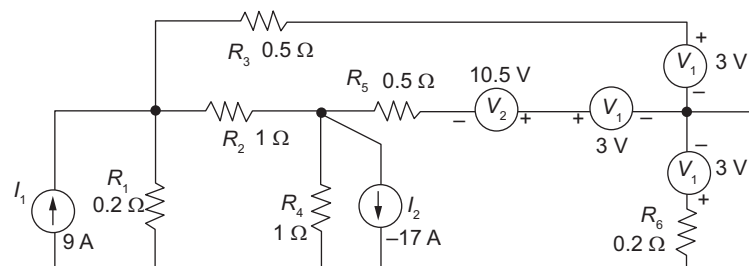


Fig. 17.4-4 Applying v-Shift Principle on the Network in Fig. 17.4-3

The source V_1 has been shifted into R_3 , R_5 and R_6 lines in such a manner that KVL equation in no loop is affected. The source value at the original location is set to zero – that is, the source is replaced with a short-circuit at its original location.

Now, two voltage sources in series with R_5 may be combined into one source. Further, all voltage sources that appear in series with a resistance can be replaced with a current source and a parallel resistance combination by applying the source transformation theorem. The final network that contains only resistors and independent current sources is shown in Fig. 17.4-5(a).

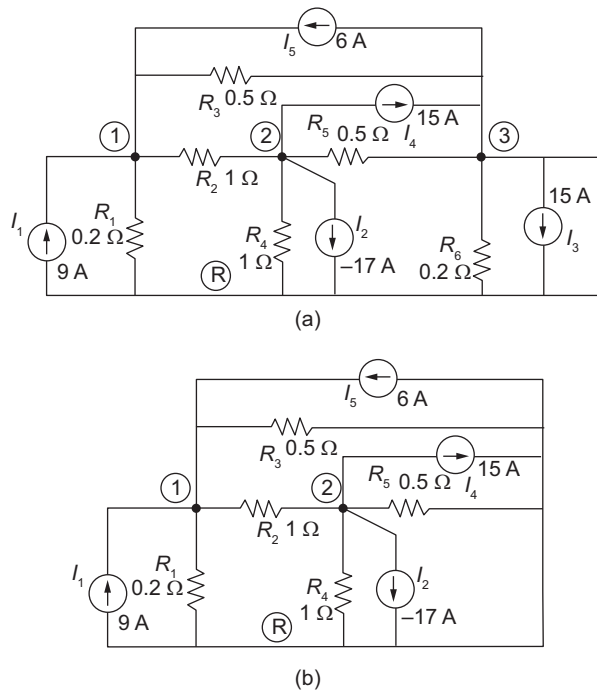


Fig. 17.4-5 Network Manipulation in Example 17.4-2

But node-3 is shorted to the reference node in the network in 17.4-5(a). Therefore, the current source I_3 and the resistance R_3 will not affect the node voltages at node-1 and node-2. Thus, the network to be solved is the one in Fig. 17.4-5(b).

Now, the network is ready for nodal analysis using the procedure outlined before. However, this is a simple network and the node voltages may be obtained by combining the current sources connected in parallel at node-1 and node-2 and applying the superposition theorem. The node voltages are $v_1 = 2$ V and $v_2 = 1$ V.

Once the two node voltages are obtained, we go back to Fig. 17.4-4 and move the three copies of source V_1 into the short-circuit between node-3 and the reference node. This will make the node voltage at node-3 assume a value of 3 V.

The currents through the voltage sources have to be determined by applying Kirchhoff's Laws at suitable nodes.

Current in $R_5 = [1 - (-7.5)] / 0.5 = 17$ A from left to right. Therefore, the current delivered by the 10.5 V source is 17 A and the power delivered is 178.5 W.

Similarly, KCL at node-3 will show that the current delivered by the 3 V source is zero. Therefore, the power delivered by the 3 V source is 0 W.

17.4.2 Nodal Analysis of Networks Containing Ideal Dependent Sources

Ideal dependent sources are four-terminal elements. However, the input side of such an ideal dependent source is either an open-circuit (in the case of voltage-controlled sources)

or a short-circuit (in the case of current-controlled sources). Therefore, they can be represented in the network graph by a single branch corresponding to their output circuit. The controlling variable of the dependent source has to be identified in terms of branch voltage variables and an admittance representation has to be prepared for each dependent source in the network prior to nodal analysis of networks containing dependent sources.

The nodal analysis procedure is illustrated in the following examples.

EXAMPLE: 17.4-3

Find the node voltages in the circuit in Fig. 17.4-6. Prepare the network graph with the reference direction for currents as marked in the figure.

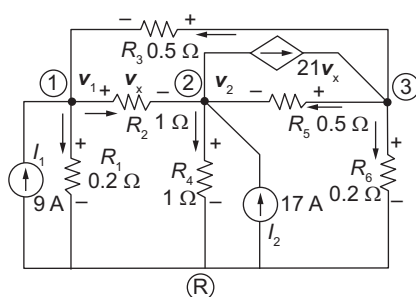


Fig. 17.4-6 Circuit for Example 17.4-3

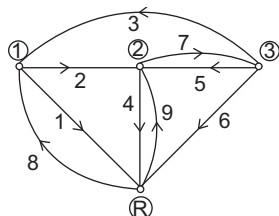


Fig. 17.4-7 Oriented Graph for the Network in Example 17.4-3

SOLUTION

The network contains a VCCS. The controlling variable is identified as $v_x = v_2$, the branch voltage of Branch-2. The dependent source output is then $21v_2$. The oriented graph of the network is shown in Fig. 17.4-7. The passive branches are numbered first, followed by the dependent source and independent current sources.

The $\mathbf{A}$ matrix is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_p = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{A}_g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Note that the branch representing the output port of the dependent source appears as the last column of $\mathbf{A}_p$.

The $\mathbf{Y}_p$ matrix will contain a row and a column (seventh row and seventh column) corresponding to the dependent source. Non-zero entries in the seventh row will be at the first column position. The entry will be 21.

$$\mathbf{i}_p = \mathbf{Y}_p \mathbf{V}_p \Rightarrow \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 21 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

Network manipulation for nodal analysis with VCCS

When the controlling variable of a dependent current source is a voltage variable, the controlling variable has to be expressed as a combination of branch voltage variables before setting up the branch admittance matrix $\mathbf{Y}_p$.

The voltage variable v_r represents the voltage across the dependent source output terminals according to the passive sign convention. Now, the nodal admittance matrix $\mathbf{Y}_n$ may be found from

$$\mathbf{Y}_n = \mathbf{A}_p \mathbf{Y}_p \mathbf{A}_p^T = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 21 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 & -2 \\ -1 & 25 & -2 \\ -2 & -23 & 9 \end{bmatrix} \text{ S}$$

$$\mathbf{I}_g = \begin{bmatrix} 9 \\ 17 \end{bmatrix} \text{ A and } \mathbf{A}_g \mathbf{I}_g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 17 \end{bmatrix} = \begin{bmatrix} -9 \\ -17 \\ 0 \end{bmatrix} \text{ A}$$

$$\mathbf{V}_n = - \begin{bmatrix} 8 & -1 & -2 \\ -1 & 25 & -2 \\ -2 & -23 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ -17 \\ 0 \end{bmatrix} = - \begin{bmatrix} 0.1406 & 0.0432 & 0.0408 \\ 0.0102 & 0.0534 & 0.0141 \\ 0.0573 & 0.1461 & 0.1563 \end{bmatrix} \begin{bmatrix} -9 \\ -17 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ V.}$$

EXAMPLE: 17.4-4

Find the node voltages in the circuit in Fig. 17.4-8. Prepare the network graph with the reference direction for currents as marked in the figure.

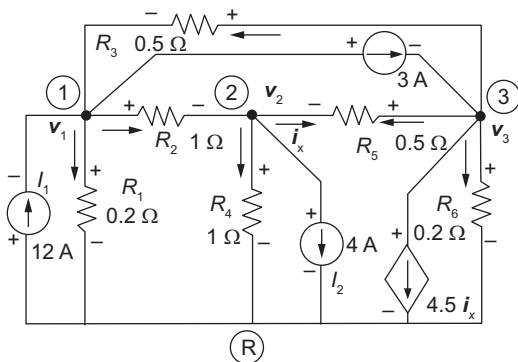


Fig. 17.4-8 Network for Example 17.4-4

SOLUTION

The oriented network for this network is shown in Fig. 17.4-9. The resistors are numbered from 1 to 6. The output port of the dependent source is represented by Branch-7. The independent current sources are represented by Branch-8, Branch-9 and Branch-10.

The controlling variable i_x is the current through R_5 in a direction opposite to the reference direction. This current can be expressed in terms of node voltage variables

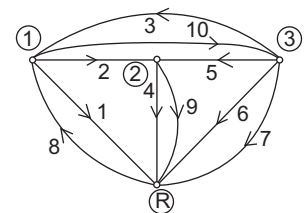


Fig. 17.4-9 Oriented Graph for the Network in Example 17.4-4

**Network manipulation
for nodal analysis with
CCCS**

When the controlling variable of a dependent current source is a current, the controlling variable has to be expressed as a branch voltage multiplied by an admittance before setting up the branch admittance matrix $\mathbf{Y}_p$.

as $i_x = -v_5/0.5 \text{ A} = -2 v_5 \text{ A}$, where v_5 is the branch voltage of Branch-5. Therefore, the output current of the dependent source is $-9v_5 \text{ A}$. Hence, the admittance matrix $\mathbf{Y}_p$ will now be

$$\mathbf{i}_p = \mathbf{Y}_p \mathbf{v}_p \Rightarrow \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & -9 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}.$$

The incidence matrix and its partitions are obtained as:

$$\mathbf{A} = \left[\begin{array}{ccccccc|ccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \end{array} \right]$$

$$\mathbf{Y}_n = \mathbf{A}_p \mathbf{Y}_p \mathbf{A}_p^T$$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & -9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -7 & 0 \end{bmatrix} \text{ S}$$

$$\mathbf{I}_g = \begin{bmatrix} 12 \\ 4 \\ 3 \end{bmatrix} \text{ A and } \mathbf{A}_g \mathbf{I}_g = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \\ -3 \end{bmatrix} \text{ A}$$

$$\mathbf{V}_n = - \begin{bmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -7 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ V.}$$

EXAMPLE: 17.4-5

Find the current delivered by the dependent voltage source in the circuit shown in Fig. 17.4-10. Employ nodal analysis and prepare the network graph with the reference direction for currents as marked in the figure.

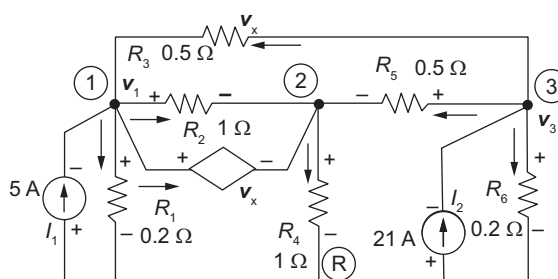


Fig. 17.4-10 Circuit for Example 17.4-5

SOLUTION

The dependent source is an unaccompanied voltage source and has to be shifted into R_2 , R_4 and R_5 lines as shown in Fig. 17.4-11.

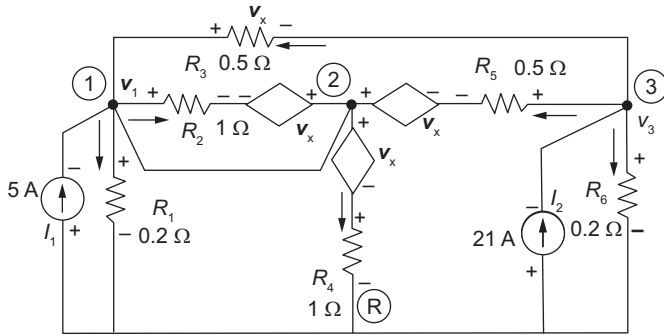


Fig. 17.4-11 Circuit in Fig. 17.4-10 After v-Shifting

Now, the elements connected across the short-circuit between node-1 and node-2 may be ignored and the remaining two branches connected at node-2 can be source-transformed into dependent current sources as shown in Fig. 17.4-12.

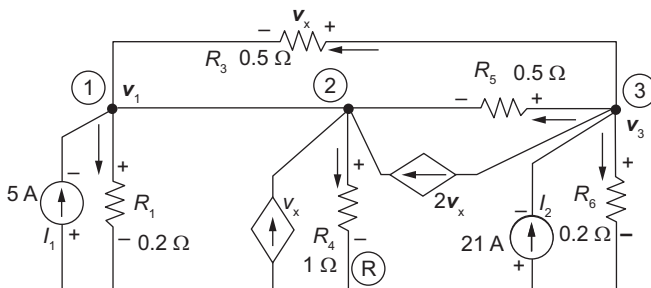


Fig. 17.4-12 Circuit in Fig. 17.4-11 After Source Transformation

The network graph for the network in Fig. 17.4-12 is shown in Fig. 17.4-13.

Branches 1 to 5 represent the resistors, Branch-6 and Branch-7 are the dependent current source output ports and Branch-8 and Branch-9 represent the independent current sources.

The controlling variable of both dependent sources is v_x and $v_x = v_2$, the branch voltage variable of Branch-2.

A matrix and its partition are shown below:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

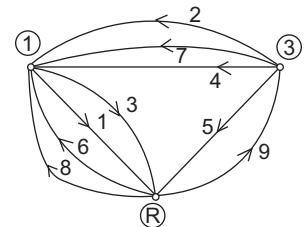


Fig. 17.4-13 Network Graph for the Network in Fig. 17.4-12

**Network manipulation
for nodal analysis with
VCVS and/or CCVS**

If the controlled variable of a dependent source is a voltage, the output port of the dependent source has to be converted into a current source by combining it with some passive impedance and applying the source transformation.

If the dependent voltage source is an unaccompanied source, v-shifting has to be performed first before carrying out the step described above.

$$\begin{aligned}
 \mathbf{i}_p = \mathbf{Y}_p \mathbf{v}_p &\Rightarrow \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} \\
 \mathbf{Y}_n &= \mathbf{A}_p \mathbf{Y}_p \mathbf{A}_p^T \\
 &= \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 13 & -7 \\ -6 & 11 \end{bmatrix} \text{ S} \\
 \mathbf{I}_g &= \begin{bmatrix} 5 \\ 21 \end{bmatrix} \text{ A and } \mathbf{A}_g \mathbf{I}_g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 21 \end{bmatrix} = \begin{bmatrix} -5 \\ -21 \end{bmatrix} \text{ A} \\
 \mathbf{v}_n &= -\begin{bmatrix} 13 & -7 \\ -6 & 11 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ -21 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ V}
 \end{aligned}$$

Now, the value of $\mathbf{v}_x$ is $3 - 2 = 1$ V. Refer to Fig. 17.4-11, using this value for $\mathbf{v}_x$ and shifting the three dependent voltage sources back to the original location, we get the node-2 voltage as $2 \text{ V} - 1 \text{ V} = 1 \text{ V}$.

Apply KCL at node-1.

Current delivered to the dependent voltage source $= 5 - 2/0.2 - (2 - 1)/1 + (3 - 2)/0.5 = -4$ A.

$\therefore$ Current delivered by the dependent voltage source $= 4$ A.

17.5 THE CIRCUIT MATRIX OF A LINEAR ORIENTED GRAPH

The *All Circuit Matrix* $\mathbf{B}_a$ of an oriented graph gives information about the way in which various branches constitute the various loops in the network.

There is one row for each loop in the network graph in the *All Circuit Matrix*. There is one column for each branch in the network graph in the *All Circuit Matrix*. Thus, the row order of $\mathbf{B}_a$ is equal to the number of loops in the network graph and the column order of $\mathbf{B}_a$ is equal to the number branches b in the network graph.

The *All Circuit Matrix* $\mathbf{B}_a$ is prepared by assigning a direction of traversal for each loop. This direction can be *clockwise* or *counter-clockwise* in the loop. The direction chosen for different loops need not be the same. Once a direction of traversal is selected for a loop, the row for that loop is completed according to the following rule.

$b_{ij} = +1$, if j^{th} branch is in the i^{th} loop and its orientation agrees with the chosen direction of traversal in the i^{th} loop.

$b_{ij} = -1$, if j^{th} branch is in the i^{th} loop and its orientation disagrees with the chosen direction of traversal in the i^{th} loop.

$b_{ij} = 0$, if j^{th} branch is not in the i^{th} loop.

The number of loops in a network graph can be very large. Some loops are, in fact, a combination of two or more other loops. The oriented graph shown in Fig. 17.5-1(a) has

seven loops. They are listed below. All the loops are assigned a clockwise direction and the list of branches that constitute the loop is used to designate the loop.

- Loop-1: {1, 5, 2, 7}
 Loop-2: {2, 6, 3, 8}
 Loop-3: {5, 4, 6}
 Loop-4: {1, 4, 3, 8, 7}
 Loop-5: {1, 5, 6, 3, 8, 7}
 Loop-6: {2, 5, 4, 3, 8}
 Loop-7: {1, 4, 6, 2, 7}

The *All Circuit Matrix* for this network graph will have seven rows and eight columns. The matrix is obtained as

$$\mathbf{B}_a = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} \text{Loop-1} \\ \text{Loop-2} \\ \text{Loop-3} \\ \text{Loop-4} \\ \text{Loop-5} \\ \text{Loop-6} \\ \text{Loop-7} \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Loop-4 is the union of the first three loops. Hence, the row corresponding to Loop-4 in the $\mathbf{B}_a$ matrix can be obtained by summing up the rows corresponding to the first three loops. Similarly, the row corresponding to Loop-5 can be obtained by adding the first two rows.

Thus, not all rows of $\mathbf{B}_a$ matrix are independent. Thus, the rank of $\mathbf{B}_a$ matrix can not be equal to its column or row order.

A *tree* of a connected graph is a connected subgraph containing all its nodes; but without any loops. A *tree* will have $(n - 1)$ *twigs* and $(b - n + 1)$ *links*. Replacing a *link* in the *tree* will form one loop. A loop produced by replacing a *link* in a selected *tree* of a graph is called the *fundamental loop (or circuit) associated with that link*. The direction of traversal for such a *fundamental circuit* is chosen in such a way that the orientation of the associated *link* agrees with this direction. The term *f-circuit* is used to designate such a loop.

17.5.1 The Fundamental Circuit Matrix $\mathbf{B}_f$

A circuit matrix constructed by considering only the *f-circuits* of a graph with respect to a selected *tree* is called *Fundamental Circuit Matrix* or *f-circuit matrix* in short. A *tree* has $(b - n + 1)$ *links*, and hence, the *f-circuit matrix* will have $(b - n + 1)$ rows and b columns. It will be a sub-matrix of the *All Circuit Matrix*, provided the same direction is used for *f-circuits* in both matrices. *f-circuit matrix* is denoted by $\mathbf{B}_f$.

Let the order in which the *f-circuits* are arranged in the rows of $\mathbf{B}_f$ be the same as the order in which the *links* are arranged in the columns of the same matrix. Then, the $\mathbf{B}_f$ matrix can be partitioned column-wise in such a way that the first partition is a $(b - n + 1) \times (n - 1)$ sub-matrix corresponding to the *twigs* of the tree and the second partition is a $(b - n + 1) \times (b - n + 1)$ identity matrix $\mathbf{U}$ corresponding to the *links* in the *tree*.

$$\mathbf{B}_f = \left[\begin{array}{c|c} \mathbf{B}_f & \mathbf{U} \\ \hline (b-n+1) \times (n-1) & (b-n+1) \times (b-n+1) \end{array} \right]$$

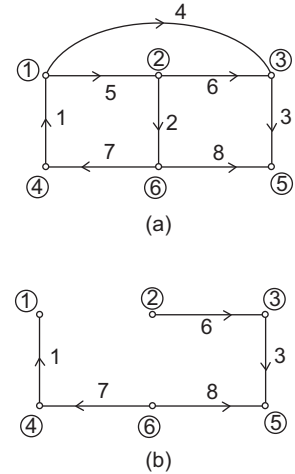


Fig. 17.5-1 (a) A Linear Oriented Graph and (b) A Tree for this Graph

An *f-circuit* is a loop created when a *link* is replaced in a *tree* with the direction of traversal set such that the orientation of the *link* agrees with this direction.

The largest square sub-matrix that can be drawn from $\mathbf{B}_f$ is of the order $(b - n + 1) \times (b - n + 1)$, since the row order of $\mathbf{B}_f$ is $(b - n + 1)$ and the column order is b . There is at least one such square sub-matrix that is non-singular – the identity sub-matrix of the order $(b - n + 1) \times (b - n + 1)$ corresponding to the columns representing the *links*. Hence, the rank of *f-circuit* matrix is $(b - n + 1)$. This implies that rows of $\mathbf{B}_f$ are linearly independent.

$\mathbf{B}_f$ is a sub-matrix of the *All Circuit Matrix* $\mathbf{B}_a$. Therefore, the rank of $\mathbf{B}_a$ is at least $(b - n + 1)$. We will soon show that it is exactly $(b - n + 1)$.

With reference to Fig. 17.5-2, a *tree* for the graph in Fig. 17.5-2(a) is shown as Fig. 17.5-2(b). The *f-circuits* formed by replacing the *links* one by one are marked in Fig. 17.5-2(c), (d) and (e). The directions of traversal for the three *f-circuits* are also marked. Note that the direction of traversal of *f-circuits* coincides with the orientation of the *link* that defines it. The *f-circuit matrix* $\mathbf{B}_f$ for this choice of a *tree* is given below.

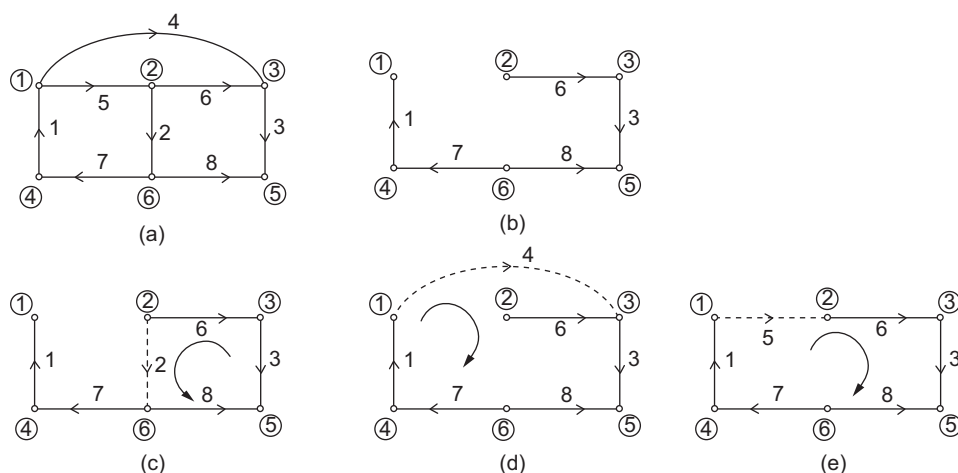


Fig. 17.5-2 (a) A Network Graph (b) A Chosen Tree (c), (d), (e) The *f-circuits*

$$\mathbf{B}_f = \begin{matrix} & \begin{matrix} 1 & 3 & 6 & 7 & 8 & 2 & 4 & 5 \end{matrix} \\ \begin{matrix} \text{f-loop-1} \\ \text{f-loop-2} \\ \text{f-loop-3} \end{matrix} & \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}. \text{ Rank of } \mathbf{B}_f \text{ in this example is 3.}$$

17.5.2 Relation between All Incidence Matrix $\mathbf{A}_a$ and All Circuit Matrix $\mathbf{B}_a$

Let the columns of the two matrices $\mathbf{A}_a$ and $\mathbf{B}_a$ of a given graph be arranged in the same order. A row of $\mathbf{A}_a$ gives us the information on the branches that are incident at the node corresponding to that row. A column of $\mathbf{B}_a^T$ gives us the information on which branches appear in a particular loop represented by that column.

Consider the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$. Assume that the i^{th} node is present in the j^{th} loop. Then, there can be exactly two branches that are incident at the i^{th} node in the j^{th} loop. If these two branches agree (or disagree) with the direction of traversal of the loop, then, one of them will be oriented towards the node and the other will be oriented away from that node. That is, if these two branches enter the circuit matrix with +1 or -1, one of them will enter with +1 and the other with -1 in the incidence matrix. Therefore, the product of the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$ will be equal to zero in that case. If one of the two branches enter with +1 and the other with -1 in the circuit matrix, they will

enter with same sign in the incidence matrix. Therefore, the product of the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$ will be equal to zero in that case.

Now, assume that the i^{th} node is not present in the j^{th} loop. Then, no branch that is incident at the i^{th} node can be present in the j^{th} loop. Then, the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$ will not contain non-zero entries in similarly placed locations. Therefore, the product of the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$ will be zero.

Thus, the product of the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$ will always be equal to zero for any i and j . We get an important result from this reasoning.

$$\mathbf{A}_a \mathbf{B}_a^T = 0 \quad (17.5-1)$$

This is called the *orthogonality relation* between incidence and circuit matrices of a linear graph. The reasoning presented above was based on considering the i^{th} row of $\mathbf{A}_a$ and the j^{th} column of $\mathbf{B}_a^T$ at a time. Therefore, the orthogonality relation is valid for any sub-matrices of $\mathbf{A}_a$ and $\mathbf{B}_a$. In particular, the *reduced incidence matrix* $\mathbf{A}$ and the *f-circuit matrix* for any choice of a *tree* will be orthogonal. That is,

$$\mathbf{A} \mathbf{B}_f^T = 0 \quad (17.5-2)$$

Taking the matrix transpose on both sides yield the remaining two orthogonality relations $-\mathbf{B}_a \mathbf{A}_a^T = 0$ and $\mathbf{B}_f \mathbf{A}^T = 0$.

$\mathbf{B}_f$ matrix can be prepared only after selecting a particular *tree* for the graph. Then, both $\mathbf{B}_f$ and $\mathbf{A}$ can be partitioned column-wise by arranging the *twigs* first, followed by the *links*.

$$\mathbf{B}_f = \begin{bmatrix} \mathbf{B}_{ft} & \mathbf{U} \\ (b-n+1) \times (n-1) & (b-n+1) \times (b-n+1) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_t & \mathbf{A}_l \\ (n-1) \times (n-1) & (n-1) \times (b-n+1) \end{bmatrix}$$

Now, applying the orthogonality relation in Eqn. 17.5-2, we get,

$$\mathbf{A}_t \mathbf{B}_{ft}^T + \mathbf{A}_l = 0.$$

But, $\mathbf{A}_t$ is non-singular and invertible. Therefore,

$$\mathbf{B}_{ft} = -[\mathbf{A}_t^{-1} \mathbf{A}_l]^T \text{ and } \mathbf{B}_f = \begin{bmatrix} -[\mathbf{A}_t^{-1} \mathbf{A}_l]^T & \mathbf{U} \end{bmatrix} \quad (17.5-3)$$

Equation 17.5-3 expresses the *f-circuit matrix* in terms of sub-matrices of *incidence matrix*.

The rank of *All Circuit Matrix* can now be ascertained with the help of Sylvester's law of nullity. This law states that if the product of two matrices is zero, then, the sum of the ranks of the two matrices is not greater than the number of columns of the first matrix in the product. Therefore, from Eqn. 17.5-1, it follows that $(\text{rank of } \mathbf{A}_a + \text{rank of } \mathbf{B}_a) \leq b$. But rank of $\mathbf{A}_a$ is known to be $(n - 1)$. Therefore, rank of $\mathbf{B}_a \leq (b - n + 1)$. However, $\mathbf{B}_f$, a sub-matrix of $\mathbf{B}_a$, is known to have a rank of $(b - n + 1)$. Therefore, rank of $\mathbf{B}_a \geq (b - n + 1)$. The only value that satisfies the ' $\leq (b - n + 1)$ ' condition and the ' $\geq (b - n + 1)$ ' condition simultaneously is $(b - n + 1)$. Therefore, the rank of $\mathbf{B}_a$ is $(b - n + 1)$.

Orthogonality relation between $\mathbf{A}_a$ and $\mathbf{B}_a$.

Orthogonality between $\mathbf{A}$ and $\mathbf{B}_f$.

Rank of All Circuit Matrix $\mathbf{B}_a$ is $(b - n + 1)$.
Rank of f-Circuit Matrix $\mathbf{B}_f$ is $(b - n + 1)$.

17.6 KIRCHHOFF'S LAWS IN FUNDAMENTAL CIRCUIT MATRIX FORMULATION

17.6.1 Kirchhoff's Voltage Law and the $\mathbf{B}_f$ Matrix

Kirchhoff's Voltage Law states that the algebraic sum of voltage variables around any loop in an electrical network is zero on an instant-to-instant basis. A loop is assigned a direction of traversal before the row corresponding to this loop is filled in the $\mathbf{B}_a$ matrix. If a branch

Kirchoff's Voltage
Law in terms of All
Circuit Matrix $\mathbf{B}_a$.

is traversed in the same direction as its orientation, its voltage variable enters the KVL sum with a positive sign. If it is traversed in the direction opposite to its orientation, its voltage variable enters the KVL sum with a negative sign. But these signs are already available in the $\mathbf{B}_a$ matrix entries. Thus, the KVL equation for a loop can be obtained by multiplying the row of $\mathbf{B}_a$ corresponding to that loop by the column vector of branch voltage variables arranged in the same order as in columns of $\mathbf{B}_a$. Thus, KVL equations in all the loops of the circuit will be given by

$$\mathbf{B}_a \mathbf{v}(t) = \mathbf{0} \quad (17.6-1)$$

where $\mathbf{v}(t)$ is the $b \times 1$ column vector containing the instantaneous value of branch voltages and $\mathbf{0}$ is a column vector containing zero-valued entries.

KVL equations can also be expressed in terms of transformed currents in s -domain as $\mathbf{B}_a \mathbf{V}(s) = \mathbf{0}$, where $\mathbf{V}(s)$ is the $b \times 1$ column vector containing the Laplace Transforms of branch voltages. They can also be expressed as $\mathbf{B}_a \mathbf{V}(j\omega) = \mathbf{0}$ for sinusoidal steady-state analysis, where $\mathbf{V}(j\omega)$ is the $b \times 1$ column vector containing the phasor branch currents.

The rank of $\mathbf{B}_a$ is only $(b - n + 1)$. This means that only $(b - n + 1)$ KVL equations in a network can be linearly independent of each other. The f -circuit matrix $\mathbf{B}_f$ for some chosen *tree* for the network graph is one choice of sub-matrix of $\mathbf{B}_a$ with a rank and row order of $(b - n + 1)$. Thus, selecting a *tree* and preparing an f -circuit matrix $\mathbf{B}_f$ solves the problem of selecting a set of $(b - n + 1)$ independent KVL loop equations for the given network. Thus, the independent set of KVL equations available for a network can be expressed as

$$\mathbf{B}_f \mathbf{v}(t) = \mathbf{0} \quad (17.6-2)$$

The f -circuit matrix $\mathbf{B}_f$ can be partitioned into two sub-matrices – $\mathbf{B}_f$ and $\mathbf{U}$ – where columns of $\mathbf{B}_f$ correspond to *twigs* of a *tree* chosen for the graph and columns of $\mathbf{U}$ correspond to the *links* of the same chosen *tree*. Let the branch voltage column vector also be partitioned in the same manner. Then,

$$\begin{aligned} \mathbf{B}_f \mathbf{v}(t) &= [\mathbf{B}_f \mid \mathbf{U}] \begin{bmatrix} \mathbf{v}_t(t) \\ \mathbf{v}_l(t) \end{bmatrix} = \mathbf{0} \\ \therefore \mathbf{v}_l(t) &= -\mathbf{B}_f \mathbf{v}_t(t) \\ \text{and } \mathbf{v}(t) &= \begin{bmatrix} \mathbf{U} \\ -\mathbf{B}_f \end{bmatrix} \mathbf{v}_t(t) \end{aligned}$$

Link voltages can be
expressed as linear
combinations of twig
voltages.

This shows that the set of *twig voltages* in a network for any chosen *tree* of its graph can serve as a *basis set* for all branch voltages in that network. *Link voltages* can be expressed as linear combinations of *twig voltages*.

Consider a linear time-invariant circuit containing two-terminal passive elements, independent voltage sources and independent current sources. How many independent voltage sources can it contain?

An independent voltage source is a two-terminal element that has a voltage variable that is a specified function of time. Thus, the voltage appearing across a voltage source is already decided, quite independent of any other network variable. Such an independent function of time can not be expressed as a linear combination of other independent functions of time. Therefore, it must be possible to select a *tree* that contains all the independent voltage sources present in the network. If there are more than $(n - 1)$ independent voltage sources in the network, then, it will not be possible to select a *tree* that contains all the independent voltage sources as its *twigs*. In that case, one or more independent voltage sources will appear as *links* even after assigning $(n - 1)$ voltage sources as *twigs*. But then, KVL implies that *link voltages* can be expressed as a linear combination of *twig voltages*.

The set of
 $(b - n + 1)$
independent KVL loop
equations are given
by this equation.

The maximum
number of independent
voltage sources that
can be present in a
network containing
two-terminal passive
elements, independent
voltage sources and
independent current
sources is $(n - 1)$, where
 n is the number of
nodes in the network.

This implies that the source function of one or more independent voltage sources can be expressed as a linear combination of source functions of other independent voltage sources alone. That leads to a contradiction, since a time-function that is expressible as a linear combination of a set of independently specified time-functions can not be an *independent time-function*. Hence, the maximum number of *independent voltage sources* that can be present in a network containing two-terminal passive elements, independent voltage sources and independent current sources is $(n - 1)$. A more detailed modelling of elements in the network will be required before the solution of the network is attempted if a network contains more than $(n - 1)$ independent voltage sources.

This does not mean that we will find it impossible to select a tree that contains all independent voltage sources present in the network only when there are more than $(n - 1)$ independent voltage sources in a network. Consider a circuit that has less than $(n - 1)$ independent voltage sources; but with a loop that has only independent voltage sources in it. Obviously, we must allow at least one independent voltage source to become a *link*. Otherwise, the '*tree*' will contain a loop, and hence, will not be a *tree*. Then, the voltage of that *link* will be expressed as a linear combination of *twig* voltages and the only *twig* voltages that contribute to the linear combination will be the ones corresponding to the remaining voltage sources present in the *f-loop* formed by that *link*. This will result in an attempt to equate an independently specified time-function to a linear combination of other independently specified time-functions. Such an attempt will lead to a contradiction and the network will require a more detailed modelling before a solution can be attempted. Thus, a network containing two-terminal passive elements, independent voltage sources and independent current sources will have a unique solution only if it is possible to select a *tree* that contains all the independent voltage sources present in the network as its *twigs*. But that does not mean that we have to choose only that kind of a *tree* for solving the network; only that it must be possible to choose such a *tree* if we wish to do so.

Further, the condition arrived at in the last paragraph is only a necessary condition for the existence of a unique solution for the network. It is not sufficient. A similar constraint on the number of independent current sources too will have to be satisfied. This was already pointed out in Sect. 17.3.

It is necessary that a *tree* that contains all independent voltage sources that are present in the network as its *twigs* exists in the network graph for an electrical network to have a unique solution. However, this is only a necessary but not a sufficient condition.

17.6.2 Kirchhoff's Current Law and the $\mathbf{B}_f$ Matrix

Kirchhoff's Current Law equations for an electrical network are given by

$\mathbf{A}\mathbf{i}(t) = [\mathbf{A}_t \mid \mathbf{A}_l] \begin{bmatrix} \mathbf{i}_t(t) \\ \mathbf{i}_l(t) \end{bmatrix} = \mathbf{0}$. $\mathbf{A}_t$ is invertible. Therefore, all branch currents can be expressed in terms of *link currents* alone as shown below:

$$\mathbf{i}(t) = \begin{bmatrix} -\mathbf{A}_t^{-1}\mathbf{A}_l \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t)$$

However, $\mathbf{B}_f = -[\mathbf{A}_t^{-1}\mathbf{A}_l]^T$ and $\mathbf{B}_f = \begin{bmatrix} -[\mathbf{A}_t^{-1}\mathbf{A}_l]^T \\ \mathbf{U} \end{bmatrix}$ as a result of orthogonality relation that exists between $\mathbf{A}$ and $\mathbf{B}_f$. Therefore,

$$\mathbf{i}(t) = \begin{bmatrix} -\mathbf{A}_t^{-1}\mathbf{A}_l \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t) = \begin{bmatrix} \mathbf{B}_f^T \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t) = \mathbf{B}_f^T \mathbf{i}_l(t) \quad (17.6-3)$$

Equation 17.6-3 represents the KVL equations for an electrical network in terms of its *f-circuit matrix* for a chosen *tree*.

Branch currents in a network can be expressed in terms of *link currents* for a chosen *tree*.
 $\mathbf{i}(t) = \mathbf{B}_f^T \mathbf{i}_l(t)$

17.7 LOOP ANALYSIS OF ELECTRICAL NETWORKS

Equation 17.6-2 and Eqn. 17.6-3 help us to evolve a procedure for solving an electrical network by determining its *link currents* first. First, we consider a network containing two-terminal passive elements and independent voltage sources only.

The network has to be prepared for loop analysis first. The preparation required in this case consists of replacing series/parallel combination of similar elements with a single element, wherever such a replacement is possible. In addition, the circuit may have to be transformed into an s -domain equivalent circuit or a phasor equivalent circuit, depending on the analysis context. The initial condition sources in the s -domain equivalent circuit have to be in the voltage source format.

The second step is to draw the network graph and label the nodes and branches. The numbering of nodes can be done in an arbitrary manner except that the last node to be numbered is the reference node. However, numbering of branches has to follow a scheme. All the independent voltage source branches are numbered first and the passive branches are numbered last. Let b_p be the number of passive branches and b_g be the number of independent voltage source branches.

The next step is to prepare $\mathbf{B}_f$ matrix in a partitioned form. This requires that a *tree* be chosen first. There is no constraint on the nature of branches selected as *twigs* or *links*. Any *tree* is good enough for the purpose of loop analysis. A *tree* is chosen only for the purpose of choosing $(b - n + 1)$ independent KVL equations. However, the network is assumed to possess a unique solution.

The columns of $\mathbf{B}_f$ matrix are arranged in a sequential order. The first partition of $\mathbf{B}_f$ will contain all columns corresponding to the independent voltage source branches. The second partition will contain all columns corresponding to the passive branches. They are denoted by $\mathbf{B}_{fg}$ and $\mathbf{B}_{fp}$, respectively. The branch current transform vector $\mathbf{I}(s)$ and the branch voltage transform vector $\mathbf{V}(s)$ are also partitioned in a similar manner (s -domain analysis is assumed).

Now, $\mathbf{B}_f = [\mathbf{B}_{fg} \mid \mathbf{B}_{fp}]$ and $\mathbf{B}_f^T = \begin{bmatrix} \mathbf{B}_{fg}^T \\ \mathbf{B}_{fp}^T \end{bmatrix}$. Applying Eqn. 17.6-2, we get,

$$[\mathbf{B}_{fg} \mid \mathbf{B}_{fp}] \begin{bmatrix} \mathbf{V}_g(s) \\ \mathbf{V}_p(s) \end{bmatrix} = \mathbf{0}. \quad (17.7-1)$$

Therefore,

$$\mathbf{B}_{fp} \mathbf{V}_p(s) = -\mathbf{B}_{fg} \mathbf{V}_g(s) \quad (17.7-2)$$

Each passive branch voltage variable can be related to its current variable by the relation $V_k(s) = z_k(s) I_k(s)$, where $z_k(s)$ is the impedance function of the k^{th} passive branch. Thus, the column vectors $\mathbf{I}_p(s)$ and $\mathbf{V}_p(s)$ can be related through a $b_p \times b_p$ square diagonal matrix $\mathbf{Z}_p(s)$ with $Z_{pii} = z_i(s)$, the impedance of the i^{th} branch and $Z_{pij} = 0$ for $i \neq j$.

$$\mathbf{V}_p(s) = \mathbf{Z}_p(s) \times \mathbf{I}_p(s) \quad (17.7-3)$$

$\mathbf{Z}_p(s)$ is called the *branch impedance matrix*.

Using this relation in Eqn. 17.7-2, we get,

$$\mathbf{B}_{fp} \mathbf{Z}_p(s) \mathbf{I}_p(s) = -\mathbf{B}_{fg} \mathbf{V}_g(s) \quad (17.7-4)$$

Now, we use Eqn. 17.6-3 to express $\mathbf{I}_p(s)$ in terms of the *link current transform* vector $\mathbf{I}_l(s)$.

$$\begin{aligned} \mathbf{I}(s) &= \begin{bmatrix} \mathbf{I}_g(s) \\ \mathbf{I}_p(s) \end{bmatrix} = \mathbf{B}_f^T \mathbf{I}_l(s) = \begin{bmatrix} \mathbf{B}_{fg}^T \\ \mathbf{B}_{fp}^T \end{bmatrix} \mathbf{I}_l(s) \\ \therefore \mathbf{I}_p(s) &= \mathbf{B}_{fp}^T \mathbf{I}_l(s) \end{aligned} \quad (17.7-5)$$

Substituting Eqn. 17.7-5 in Eqn. 17.7-4, we get the loop analysis equation:

$$[\mathbf{B}_{fp}\mathbf{Z}_p(s)\mathbf{B}_{fp}^T]\mathbf{I}_l(s) = -\mathbf{B}_{fg}\mathbf{V}_g(s) \quad (17.7-6)$$

The matrix product $\mathbf{B}_{fp}\mathbf{Z}_p(s)\mathbf{B}_{fp}^T$ will be an impedance matrix. It is called *the loop impedance matrix* of the network and is denoted by $\mathbf{Z}_L(s)$. If the *branch impedance matrix* is symmetric, then, the *loop impedance matrix* will also be symmetric.

The diagonal entries of $\mathbf{Z}_L(s)$ will be equal to the sum of the impedance functions of all branches in the corresponding loop. The off-diagonal entries will be the sum of the impedance functions of all branches that are common to the two loops involved, with a positive or a negative sign. The sign is positive if the common branch is traversed in the same direction in both the loops. The sign is negative if the common branch is traversed in different directions in the two loops.

$\mathbf{Z}_L(s)$ is a square matrix of order $(b - n + 1)$. It is invertible. Therefore,

$$\mathbf{I}_l(s) = -[\mathbf{Z}_L(s)]^{-1}\mathbf{B}_{fg}\mathbf{V}_g(s) \quad (17.7-7)$$

$\mathbf{I}_p(s)$, the current transform vector for passive branches and $\mathbf{I}_g(s)$, the current transform vector for the independent voltage sources, can be obtained by applying Eqn. 17.7-5, once $\mathbf{I}_l(s)$ is determined. $\mathbf{V}_p(s)$, the voltage transform vector for the passive branches can be obtained by applying Eqn. 17.7-3.

The Loop Analysis Equation.

EXAMPLE: 17.7-1

Find the power delivered by the independent voltage sources in the network in Fig. 17.7-1 by loop analysis. Prepare the network graph using the reference directions marked in the figure.

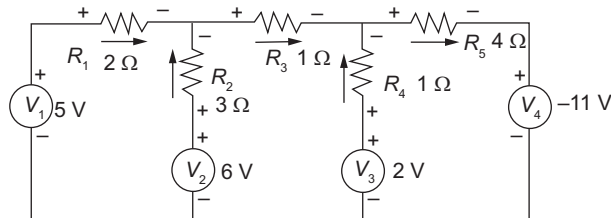


Fig. 17.7-1 Network for Example 17.7-1

SOLUTION

The graph of the network is shown in Fig. 17.7-2(a). A *tree* chosen for preparing the *f-circuit matrix* is shown in Fig. 17.7-2(b). The graph has seven nodes and its *tree* has six *twigs*. The number of *links* is three and there are three *f-circuits*.

The *f-circuit matrix* is constructed as below:

$$\mathbf{B}_f = \begin{array}{c} \text{Voltage sources} \\ \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 6 & 8 & 5 & 7 & 9 \end{array} \\ \begin{array}{l} \text{f-loop-1} \\ \text{f-loop-2} \\ \text{f-loop-3} \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

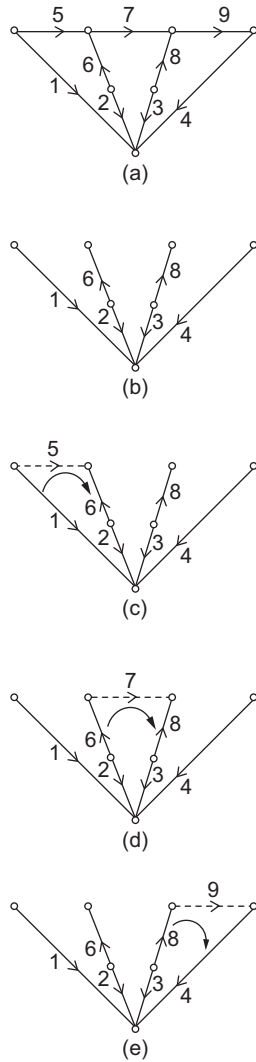


Fig. 17.7-2 Graph, Tree and f -circuits in Example 17.7-1

$$\mathbf{B}_{fg} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{B}_{fp} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Z}_p = \begin{bmatrix} 6 & 8 & 5 & 7 & 9 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \Omega$$

$$\begin{aligned} \therefore \mathbf{Z}_L = \mathbf{B}_{fp} \mathbf{Z}_p \mathbf{B}_{fp}^T &= \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Omega \\ &= \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \Omega \end{aligned}$$

$$\mathbf{V}_g = \begin{bmatrix} 5 \\ 6 \\ 2 \\ -11 \end{bmatrix} \text{ V and } \mathbf{B}_{fg} \mathbf{V}_g = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 2 \\ -11 \end{bmatrix} \text{ V} = \begin{bmatrix} 1 \\ -4 \\ -13 \end{bmatrix} \text{ V}$$

$$\therefore \mathbf{I}_l = -\mathbf{Z}_L^{-1} \mathbf{B}_{fg} \mathbf{V}_g = -\begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ A}$$

$$\text{Now, } \mathbf{I}_g = \mathbf{B}_{fg}^T \mathbf{I}_l = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ A} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \text{ A}$$

Therefore, power delivered by $V_1 = -(5 \text{ V} \times -1 \text{ A}) = 5 \text{ W}$

Power delivered by $V_2 = -(6 \text{ V} \times -1 \text{ A}) = 6 \text{ W}$

Power delivered by $V_3 = -(2 \text{ V} \times -1 \text{ A}) = 2 \text{ W}$

Power delivered by $V_4 = -(-11 \text{ V} \times 3 \text{ A}) = 33 \text{ W}.$

17.7.1 The Principle of i -Shift

We now consider loop analysis of networks containing two-terminal passive elements and independent current sources and voltage sources.

An independent current source is called an *accompanied current source* if it has an impedance connected in parallel across its terminals. Such an *accompanied current source* can be converted into an independent voltage source in series with an impedance prior to loop analysis. However, if an independent current source is unaccompanied, we need to carry out an additional manipulation on the network before we can apply the nodal analysis outlined in this section. We first shift the location of the independent current source by employing the i -shift principle and then apply the source transformation theorem to convert such sources into independent voltage sources before applying the loop analysis procedure. This is illustrated in the following example.

EXAMPLE: 17.7-2

Determine the power delivered by the current source in the circuit shown in Fig. 17.7-3 by loop analysis.

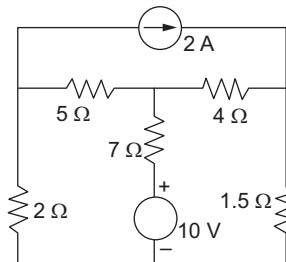


Fig. 17.7-3 Circuit for Example 17.7-2

SOLUTION

The independent current source in this circuit is unaccompanied. It may be shifted by placing copies of this source in parallel with all branches in any one loop that contains this current source in such a manner that KCL at all nodes remains unaffected. The original location of the current source has to be left open after shifting the source from there. In the present circuit, the 2 A source can be shifted across the 5 Ω and the 4 Ω resistors or across the 2 Ω and the 1.5 Ω. These two alternatives are shown in Fig. 17.7-4.

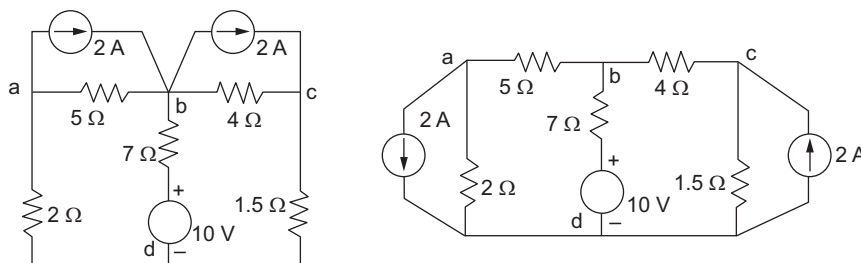


Fig. 17.7-4 Illustrating *i*-Shifting Operation

Note that the polarity of the shifted current sources are adjusted in such a way that the KCL equations at nodes a, b, c and d are not affected by the shifting operation in both alternatives shown in Fig. 17.7-4. Therefore, the circuit solution – that is, the voltage and the current variables of all elements other than the current source that was shifted – will not be affected by the shifting operation.

The next step would be to replace the current sources in parallel with resistors with the voltage sources in series with resistors. We select the second alternative in Fig. 17.7-4 and carry out this step to obtain the circuit shown in Fig. 17.7-5.

Loop analysis of the circuit in Fig. 17.7-5 may be performed now by following the procedure outlined in Example 17.7-1. However, this is a simple circuit and a standard

The *i*-shift principle

An unaccompanied current source connected between node-*i* and node-*j* in a network may be shifted across all the other branches that appear in any loop that contains both node-*i* and node-*j*, with an open-circuit applied in place of the original current source, in such a manner that the KCL equation in no node is affected.

Currents through other elements and voltages across them will not be affected by this shifting operation.

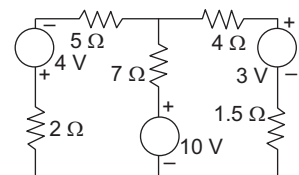


Fig. 17.7-5 Circuit in Fig. 17.7-3 Prepared for Loop Analysis

mesh analysis will yield -1 A in the first mesh and 0 A in the second mesh. The solution is translated to the original circuit as illustrated in the sequence of the circuit solution steps shown in Fig. 17.7-6.

The voltage across the current source is 5 V with a positive sign at the current delivery point. Hence, the power delivered by the current source is 10 W.

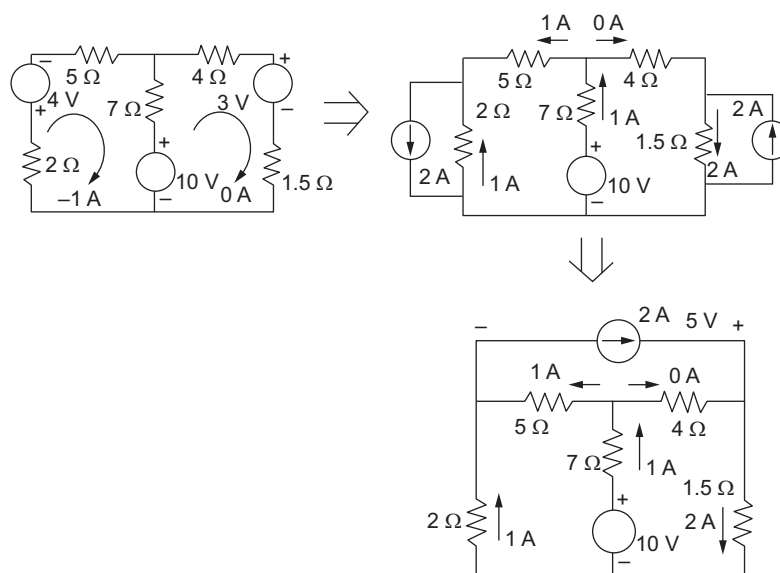


Fig. 17.7-6 Circuit Solution in Example 17.7-2

17.7.2 Loop Analysis of Networks Containing Ideal Dependent Sources

It is necessary to arrive at impedance representations for ideal dependent sources in the network before the loop analysis procedure outlined in this section can be applied to networks containing such sources.

The network has to be prepared for loop analysis by converting all the independent current sources into independent voltage sources by applying i -shifting, if necessary. The resulting circuit will then contain passive branches, independent voltage sources and dependent sources, if any.

The independent voltage source branches are numbered first and passive branches follow. The dependent source output port branches are numbered last.

A CCVS does not pose any special problem for loop analysis. If the controlling variable of a CCVS can be identified as the branch current (or its negative) of a passive branch of impedance z_k , then, it can be represented by a suitable impedance entry in an off-diagonal position in the *Branch Impedance Matrix* $\mathbf{Z}_p$. The entry will be z_k (or $-z_k$), scaled by the scaling factor of dependent source, in the column corresponding to the passive branch and the row corresponding to the dependent source output branch.

In the case of a CCCS, the output current source has to be converted into a dependent voltage source by applying the source transformation with the help of i -shifting, if necessary. The resulting CCVS or CCVS's can be handled using the same procedure outlined in the previous paragraph.

If the controlling variable of a VCVS can be identified as a passive branch voltage variable or as a sum of voltages of passive branches, then, it can be represented by suitable

entries in the $\mathbf{Z}_p$ matrix. For instance, let Branch-7, Branch-8 and Branch-9 in a network be passive branches and let Branch-11 be the output port of a VCVS. Let the VCVS output be $2v_x$, where v_x is identified to be equal to $v_7 + v_8 - v_9$. Then, this source can be represented by making $\mathbf{Z}_{p11,7} = 2z_7$, $\mathbf{Z}_{p11,8} = 2z_8$ and $\mathbf{Z}_{p11,9} = -2z_9$.

In the case of a VCCS, the output current source has to be converted into a dependent voltage source by applying the source transformation with the help of i -shifting, if necessary. The resulting VCVS or VCVS's can be handled using the same procedure outlined in the previous paragraph.

EXAMPLE: 17.7-3

Find the power delivered by each dependent source in the circuit in Fig. 17.7-7.

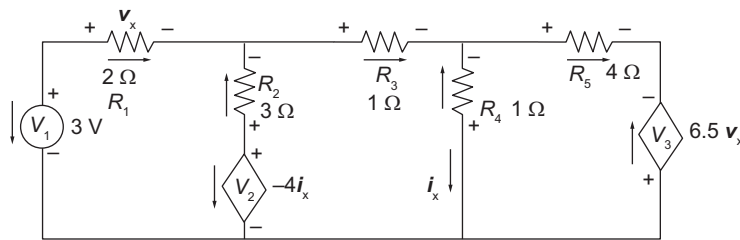


Fig. 17.7-7 Circuit for Example 17.7-3

SOLUTION

The network graph is shown in Fig. 17.7-8.

The controlling variable of a dependent source represented by Branch-7 is identified as $-i_5$. Therefore, the voltage variable of Branch-7 is $4i_5$.

The controlling variable of dependent source represented by Branch-8 is identified as $2i_2$. Therefore, the voltage variable of Branch-8 is $13i_2$.

Now, the *branch impedance matrix* is constructed as

$$\mathbf{Z}_p = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 4 & 0 \\ 7 & 0 & 0 & 0 & 4 & 0 & 0 \\ 8 & 13 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Omega$$

The *tree* selected and the *f-circuits* are shown in Fig. 17.7-9.

$$\mathbf{B}_f = \begin{bmatrix} \text{f-loop-1} & \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \text{f-loop-2} & \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 \end{bmatrix} \\ \text{f-loop-3} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{bmatrix}$$

$$\mathbf{B}_{ig} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{B}_{fp} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

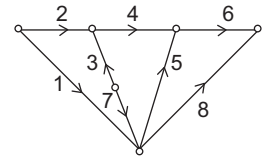


Fig. 17.7-8 Network Graph of Network in Example 17.7-3

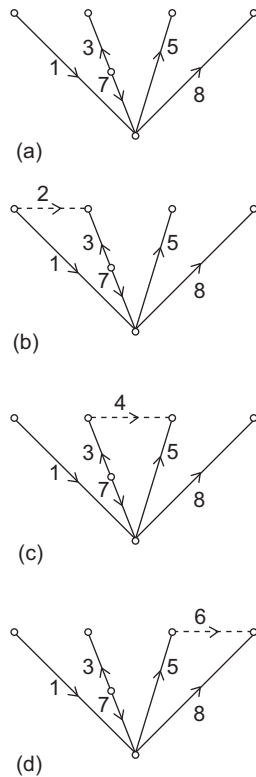


Fig. 17.7-9(a) A Tree for the Graph in Fig. 17.7-8 (b), (c), (d) Corresponding f -Circuits

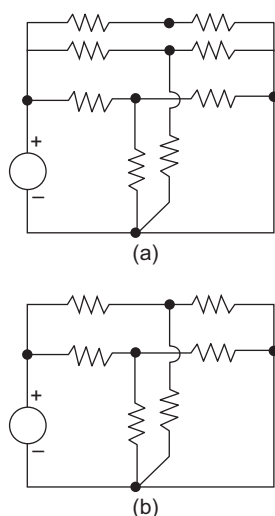


Fig. 17.7-10 (a) A Non-Planar Network (b) A Planar Network that Appears to be Non-Planar

$$\mathbf{Z}_L = \mathbf{B}_{fg} \mathbf{Z}_p \mathbf{B}_{fg}^T = \begin{bmatrix} 5 & -7 & 4 \\ -3 & 9 & -5 \\ -13 & -1 & 5 \end{bmatrix} \Omega$$

$$\mathbf{V}_g = [3]$$

$$\therefore \mathbf{I}_l = -\mathbf{Z}_L^{-1} \mathbf{B}_{fg} \mathbf{V}_g = -\begin{bmatrix} 5 & -7 & 4 \\ -3 & 9 & -5 \\ -13 & -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} [3] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ A}$$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \mathbf{B}_f^T \mathbf{I}_l = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ A} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ -1 \\ -3 \end{bmatrix}$$

$$v_7 = 4i_5 = 4 \text{ V and } v_8 = 13i_2 = 13 \text{ V.}$$

$$\therefore \text{Power delivered by the CCVS} = -(4 \text{ V} \times -1 \text{ A}) = 4 \text{ W}$$

$$\text{Power delivered by the VCVS} = -(13 \text{ V} \times -3 \text{ A}) = 39 \text{ W.}$$

17.7.3 Planar Graphs and Mesh Analysis

Topological graphs of electrical networks can be drawn on a plane. It may or may not be possible to draw a graph on a plane such that no branch crosses each other. A *planar graph* is a graph that *can be* drawn such that no two branches cross each other.

With reference to Fig. 17.7-10, the first network is a *non-planar* network, as there is no way to avoid a pair of branches crossing each other when the network graph is drawn on a plane. The second network shows a pair of branches crossing each other when it is drawn as in Fig. 17.7-10(b). However, it is possible to redraw the network such that the crossing of branches is avoided. Hence, it is a *planar* network and its graph is a *planar graph*.

The branches of a planar graph separate the plane into non-overlapping regions (see Fig. 17.7-11). Five nodes and seven branches separate the two-dimensional plane into four non-overlapping regions in graph 17.7-11(a). Three regions formed by the branches $\{1, 2, 3\}$, $\{3, 4, 5\}$ and $\{5, 6, 7\}$ are 'interior' to the graph. These regions are called the *meshes* of the network. The fourth region is the region exterior to the graph. It is called the *outside mesh*. It is not considered in arriving at the number of meshes in a network. Thus, the number of meshes in graph 17.7-11(a) is three. The graph 17.7-11(b) contains four meshes.

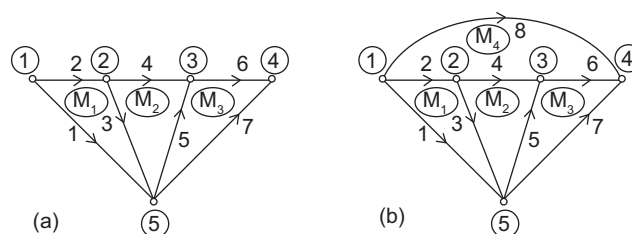


Fig. 17.7-11 (a) A Planar Graph with all Meshes Containing at Least One Branch that does not Appear in any Other Mesh (b) A Planar Graph with One Mesh that has only Shared Branches

Mesh is a sequence of branches that form a *loop* in a planar network with no other branch enclosed within the region bounded by the *loop*.

All the three meshes in Fig. 17.7-11(a) have one branch that is not common to any other mesh. Branch-2 appears only in mesh-1, Branch-4 appears only in mesh-2 and Branch-6 appears only in mesh-3. Such branches are said to be ‘*wholly owned*’ by their respective meshes.

The second mesh in Fig. 17.7-11(b) has no such *wholly owned* branch. All its branches appear in common with some other meshes.

Consider the loop formed by branches {1, 2, 4, 5} in Fig. 17.7-11(a). This loop encloses Branch-3 within it. Thus, it can be considered as the *union loop* of the loop formed by {1, 2, 3} and the loop formed by {3, 4, 5}. Therefore, any loop in a planar graph can be expressed as a *union loop* of its *meshes* if the loop under consideration is not a *mesh*. There are only two kinds of loops in planar graphs – *meshes* and loops that are *union loops* of *meshes*.

Consider the KVL equations written for *meshes* in a planar graph. Assume that all *meshes* in the planar graph have at least one ‘*wholly owned*’ branch. Then, the KVL equation for each *mesh* will have at least one voltage variable that appears only in that equation. Therefore, KVL equation of a *mesh* can not be expressed as a linear combination of KVL equations of other *meshes*. Then, the set of KVL equations written for *meshes* in the graph must be an independent set of equations. A loop that is not a *mesh* will be a ‘*union loop*’ of two or more *meshes* in the case of a planar graph. Therefore, KVL for a loop that is not a *mesh* can be expressed as a linear combination of KVL equations for those *meshes* that constitute the ‘*union loop*’. Hence, the number of independent KVL equations in a planar network can only be equal to the number of *meshes* in its graph.

A planar graph may contain one or more *meshes* that do not ‘*wholly own*’ even a single branch. However, a *mesh*, by definition, does not enclose any branch within it. Therefore, a *mesh* is not a *union loop* of other loops. Hence, KVL equation for a *mesh* that contains only shared branches can not be determined as a linear combination of KVL equations for other loops in the network.

Therefore, KVL equations written for *meshes* in a planar graph will form a set of independent equations.

However, we know that the rank of a circuit matrix of any network that has a unique solution is $(b - n + 1)$, where b is the number of branches and n is the number of nodes in the graph. That is, the number of independent KVL equations available in a uniquely solvable network has to be $(b - n + 1)$. Hence, it follows that the number of *meshes* in a uniquely solvable planar network graph must be exactly $(b - n + 1)$.

We have seen that the *link currents* for a chosen *tree* will form a *basis set* for all branch currents in the network – i.e., all branch currents in a network can be expressed as linear combinations of *link currents* for any chosen *tree*. Loop analysis results in an equation for determination of these *link currents*.

However, the set of *link currents* is not the only *basis set* for branch currents in a network. Many such *basis sets* do exist. A set of *loop currents* defined for a set of $(b - n + 1)$ loops such that no loop is a ‘*union loop*’ of two or more loops from the set will be a *basis set* for branch currents. *Loop current* is a fictitious current that is considered to flow in the periphery of a loop. Consider the graph in Fig. 17.7-12. Three loops are assigned loop currents. Branches {1, 5, 2, 7} form the first loop and the loop current assigned to this loop is i_1 . Branches {2, 6, 3, 8} form the first loop and the loop current assigned to this loop is i_2 . Branches {1, 4, 3, 8, 7} form the first loop and the loop current assigned to this loop is i_3 . All loop currents are assumed to flow in the clockwise direction.

Now, each branch current can be expressed in terms of one or two loop currents. If that branch appears only in one loop, then, its branch current will be $(\pm \text{loop current})$ of that loop. If the branch appears as a common branch of two loops, then, the branch current is $(\pm \text{sum or difference between the two loop currents})$. We express the relation between the branch currents and loop currents in matrix form as below. Also, shown by the side of this

Two kinds of meshes in a planar graph.

Union loops and meshes.

KVL equations for meshes in a planar network will form a set of $(b - n + 1)$ independent equations.

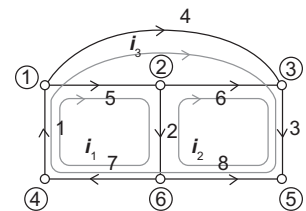


Fig. 17.7-12 Illustrating Loop Currents

relationship is the $\mathbf{B}$ matrix for the three loops, *assuming the loops are traversed in the same direction as the loop currents*. The ordering of branches in the columns of $\mathbf{B}$ matrix is the same as the ordering in the branch current vector. It can be seen that the connection matrix between the branch current vector and the loop current vector is the transpose of $\mathbf{B}$ matrix. This is a general result.

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Thus, loop currents assigned in $(b - n + 1)$ loops chosen in such a way that the $\mathbf{B}$ matrix for these loops will have a rank of $(b - n + 1)$ will form a *basis set* for branch currents in a network. If the direction of traversal in a loop for writing the $\mathbf{B}$ matrix coincides with the direction assigned to the loop current in that loop, then, the relation between the branch current vector and the loop current vector can be expressed as $\mathbf{i}(\mathbf{t}) = \mathbf{B}^T \mathbf{i}_L(\mathbf{t})$, where $\mathbf{i}(\mathbf{t})$ is the instantaneous branch current vector and $\mathbf{i}_L(\mathbf{t})$ is the instantaneous loop current vector. This relationship is called the *loop transformation*.

Mesh analysis of planar circuits was discussed in detail in Sect. 4.7, 4.8 and 4.9 in Chap. 4. The definition, meaning and interpretation of *mesh current variable* were dealt with in that context. *Mesh current* is a special case of *loop current*.

Mesh currents are usually assumed to flow in a clockwise direction in the mesh analysis context. Let $\mathbf{B}_m$ be the circuit matrix written for *meshes* in a planar network with the direction of traversal of the *mesh* the same as the direction assigned to the *mesh current* – i.e., clockwise direction. Then, $\mathbf{i}(\mathbf{t}) = \mathbf{B}_m^T \mathbf{i}_m(\mathbf{t})$, where $\mathbf{i}(\mathbf{t})$ is the instantaneous branch current vector and $\mathbf{i}_m(\mathbf{t})$ is the instantaneous *mesh current* vector. This relationship is called *mesh transformation*.

Let the *mesh circuit matrix* $\mathbf{B}_m$ be partitioned into $\mathbf{B}_m = [\mathbf{B}_{mg} \mid \mathbf{B}_{mp}]$, where $\mathbf{B}_{mg}$ contains columns corresponding to the voltage source branches and $\mathbf{B}_{mp}$ contains columns corresponding to the passive branches. Then, it may be shown by a development similar the one that led to Eqn. 17.7-6 that:

$$[\mathbf{B}_{mp} \mathbf{Z}_p(s) \mathbf{B}_{mp}^T] \mathbf{I}_m(s) = -\mathbf{B}_{mg} \mathbf{V}_g(s) \quad (17.7-8)$$

The *mesh currents* can be obtained from this equation and the branch currents may be determined subsequently by using the mesh transformation equation.

$\mathbf{B}_{mp} \mathbf{Z}_p(s) \mathbf{B}_{mp}^T$ is called the *mesh impedance matrix* and is denoted by $\mathbf{Z}_m(s)$. It is a symmetric matrix for a network that does not contain dependent sources. The diagonal entries are the sum of all impedances in the corresponding *mesh* and the off-diagonal entries are negative of the sum of impedances that are common between the corresponding *meshes*. The *mesh voltage vector* is given by $-\mathbf{B}_{mg} \mathbf{V}_g(s)$ and may be interpreted as the vector of the *sum of voltage rises* encountered in traversing the *meshes* in the clockwise direction.

17.7.4 Duality

Consider the *planar graph* in Fig. 17.7-13(a). It has six nodes and eight branches. Therefore, it will have a 5×5 $\mathbf{A}$ matrix and 3×3 $\mathbf{B}_m$ matrix. Correspondingly, it will have five node equations in node formulation and three mesh equations in mesh formulation.

We construct a second graph from this graph by following the procedure described next. (i) Assign a node each for each *mesh* in the graph. Draw a node outside the graph.

Loop
transformation.

Mesh
transformation.

This is to be the reference node of the second graph. (ii) Interconnect the nodes such that the interconnecting branches intersect the branches in the first graph. Thus, there will be one branch in the second graph for each branch in the first graph. Orient the branch away from the node if the intersected branch agrees in orientation with the mesh current. Otherwise, orient it towards the node. (iii) Label the nodes in the second graph such that the label number assigned to the mesh in the first graph and the label number assigned to the node of the second graph that was kept within that *mesh* are the same. (iv) Label the branches in the new graph such that the label number agrees with that of the intersected branch in the first graph.

This graph construction procedure is illustrated in Fig. 17.7-13(b). Dotted curves represent the branches in the newly constructed graph. The new graph is redrawn in Fig. 17.7-13(c). Obviously, the number of nodes in the new graph constructed in this manner will be one more than the number of meshes in the first graph. Also, note that the first graph in Fig. 17.7-13(a) can be constructed from the graph in Fig. 17.7-13(c) by following the same procedure. Therefore, the number of meshes in the new graph (the one in Fig. 17.7-13(c)) must be one less than the number of nodes in the first graph. Thus, the number of mesh equations in one graph is equal to the number of node equations in the other.

Further, the procedure employed in the graph construction ensures that (i) both graphs have the same number of branches (ii) the number of branches appearing in each *mesh* in one of them will be equal to the number of branches incident at the corresponding node in the other and (iii) a row in the *mesh circuit matrix* $\mathbf{B}_m$ of one graph will be identical to the row for the corresponding node in the incidence matrix $\mathbf{A}$ of the other graph. Thus, $\mathbf{B}_m$ of one graph is the same as $\mathbf{A}$ of the other. Therefore, KCL equations of one graph will be the same as the KVL equations of the other except that all i 's get replaced by v 's in the equations. Two oriented graphs are called *dual graphs* if they are related in this manner and one can be constructed from the other by following the procedure outlined in this section. Only *planar graphs* have *dual graphs*.

If two networks are represented by *dual graphs*, then, the node equations of one will have the same format as the mesh equations of the other. This does not imply that the solutions of these equations will be identical. Let us denote the branch admittance matrix of the first network by $\mathbf{Y}_{p1}(s)$, node voltage vector by $\mathbf{V}_{n1}(s)$ and source current vector by $\mathbf{I}_{g1}(s)$. Similarly, let the branch impedance matrix of the second network be $\mathbf{Z}_{p2}(s)$, mesh current vector $\mathbf{I}_{m2}(s)$ and source voltage vector $\mathbf{V}_{g2}(s)$. Then, the node equations of the first network are given by:

$$[\mathbf{A}_p \mathbf{Y}_{p1}(s) \mathbf{A}_p^T] \mathbf{V}_{n1}(s) = -\mathbf{A}_g \mathbf{I}_{g1}(s)$$

The mesh equations of the second network are given by:

$$[\mathbf{B}_{mp} \mathbf{Z}_{p2}(s) \mathbf{B}_{mp}^T] \mathbf{I}_{m2}(s) = -\mathbf{B}_{mg} \mathbf{V}_{g2}(s)$$

Incidence matrix of one network is equal to the *mesh circuit matrix* of the other since the two networks are represented by *dual graphs*. However, $\mathbf{V}_{n1}(s)$ and $\mathbf{I}_{m2}(s)$ can be equal (except in units) only if (i) $\mathbf{Y}_{p1}(s) = \mathbf{Z}_{p2}(s)$ (except in units) and (ii) $\mathbf{i}_{g1}(s) = \mathbf{V}_{g2}(s)$ (except in units). The first condition implies that the branch corresponding to a passive branch in one network has to be a passive branch in the other network too. Moreover, the impedance function of a branch in one network has to be the same as the admittance function of the corresponding branch in the other network. The second condition implies that a source branch in one network must have a corresponding source branch in the other network. Further, the voltage source in one network has to correspond to a current source in the other network. The source functions of two such corresponding source branches must be the same.

Two networks satisfying these requirements in addition to possessing *dual graphs* are called *dual networks*. Branch currents in one of them will be numerically the same as the branch voltages in the other at all t .

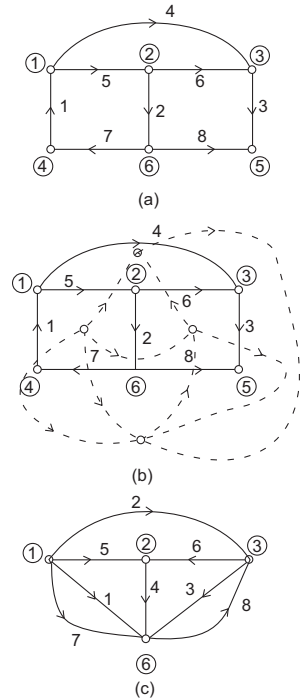


Fig. 17.7-13 Illustrating Dual Graphs

Branch requirements to be satisfied by a pair of dual networks.

Dual network is constructed by constructing the *dual graph* first. The branches of the graph are translated into network elements subsequently by applying the conditions enumerated in the preceding paragraph.

Consider the network shown in Fig. 17.7-14(a).

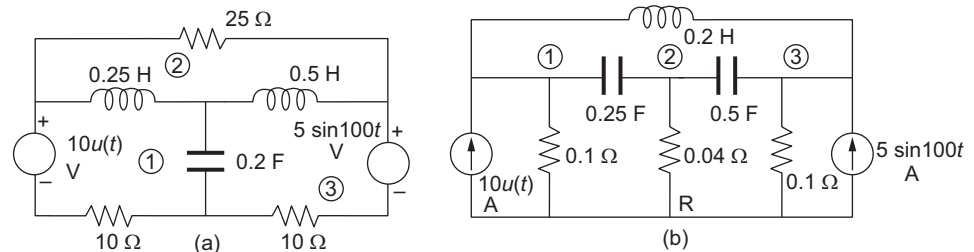


Fig. 17.7-14 Illustrating Dual Networks

The graph of this network is shown in Fig. 17.7-13(a). The dual graph is shown in Fig. 17.7-13(b). Now, the dual network may be constructed by taking up similarly labelled pairs of branches, one by one. For instance, consider the branches labelled 4 in both networks. It is a $25\ \Omega$ resistance in the network in Fig. 17.7-13(a). Therefore, it has to be a $25\ \text{S}$ conductance in the network in Fig. 17.7-13(b). Hence, $0.04\ \Omega$ is connected between the second node and the reference node in the network in Fig. 17.7-13(b).

Consider the branches labelled 5 in the two networks. It is a $0.25\ \text{H}$ inductance in the network in Fig. 17.7-13(a). Therefore, it has to be an element that has $0.25\ \text{s}$ as its admittance function. That will be a capacitance of $0.25\ \text{F}$. Hence, there will be a $0.25\ \text{F}$ capacitance between the first and the second nodes in the network in Fig. 17.7-13(b).

$10u(t)\ \text{V}$ source in the network in Fig. 17.7-13(a) calls for $10u(t)\ \text{A}$ current source between the first node and the reference node in the network Fig. 17.7-13(b). $10\sin 100t\ \text{V}$ source in network Fig. 17.7-13(a) calls for $10\sin 100t\ \text{A}$ current source between the third node and the reference node in the network in Fig. 17.7-13(b).

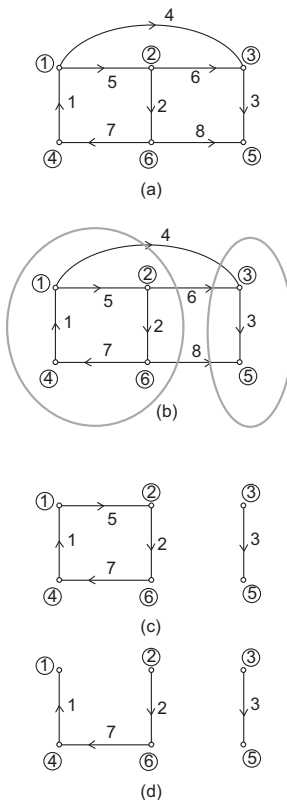


Fig. 17.8-1 (a) A Connected Graph (b) Node Grouping in the Graph (c), (d) Two Subgraphs with no Connection Between the Two Node Groups

17.8 THE CUT-SET MATRIX OF A LINEAR ORIENTED GRAPH

The branches connected between various nodes keep a graph connected. Removal of one or more judiciously selected branches can separate a connected graph into two connected subgraphs with no connection between them. The structural information of a graph can be given in terms of such sets of branches that will separate the graph into two connected subgraphs on their removal from the original graph.

Removal of a branch means that we will remove the branch; but leave the nodes intact. We may group the nodes into two sub-groups of nodes. Then, the graph will split into two connected subgraphs if we remove all the branches that are connected between one group of nodes and the second group of nodes. We need to remove only those branches that are connected between the two groups of nodes to make the original graph into two connected subgraphs. However, it may be possible to remove more branches from both subgraphs and yet keep them connected. Thus, the branches that are connected between one group of nodes and the second group of nodes is the *minimum set of branches* that have to be removed to split the original graph into two connected subgraphs for a given choice of node grouping.

With reference to Fig. 17.8-1, nodes 1, 2, 4 and 6 are chosen as group-1 and nodes 3 and 5 are chosen as group-2. The minimum set of branches to be removed to break all connections between the two node groups will be $\{4, 6, 8\}$. Removal of this set gives the subgraphs in Fig. 17.8-1(c). They are connected subgraphs, but the two subgraphs in

Fig. 17.8-1(d) are also connected subgraphs that split the nodes into two groups without any connection between the two groups. The subgraphs in Fig. 17.8-1(d) are obtained by removing the set of branches $\{4, 6, 8, 5\}$. We note that this set contains the set $\{4, 6, 8\}$ as a proper sub-set.

17.8.1 Cut-sets

A cut-set is a set of minimum number of branches that have to be removed from a connected graph to make the graph unconnected into exactly two connected subgraphs. A cut-set divides the nodes into two groups with no connection between them. One of the groups may contain a single node. That is, a single node is treated as a connected subgraph in the definition of a cut-set. Thus, $\{4, 6, 8\}$ is a cut-set of the graph in Fig. 17.8-1, whereas $\{4, 6, 8, 5\}$ is not a cut-set since this set contains more than the minimum number of branches to be removed. The branch set $\{4, 6, 8, 2, 5\}$ is not a cut-set. It splits the nodes into three groups and the graph into three connected subgraphs.

A cut-set divides the nodes into two groups with each group of nodes remaining connected within the group. However, this does not imply that given an arbitrary grouping of the nodes of a graph into two non-overlapping subgroups we will be able to find a cut-set which will cut the graph into two connected subgraphs with the specified node grouping. Such a cut-set may not exist. For instance, a cut-set which splits the graph in Fig. 17.8-1(a) into two connected subgraphs with nodes $1, 2$ and 5 in one of them and nodes $3, 4$ and 6 in the other does not exist. However, a cut-set which splits the same graph into two connected subgraphs with nodes $1, 2$ and 4 in one of them and nodes $3, 5$ and 6 in the other exists – it is $\{2, 4, 6, 7\}$.

Removal of all the branches incident at a particular node splits the graph into two connected graphs one of which will be an isolated node. There are n such cut-sets.

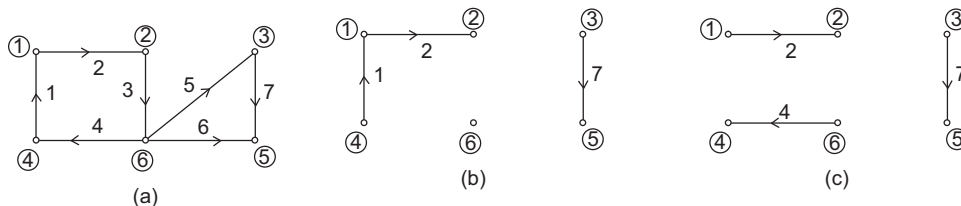


Fig. 17.8-2 A Hinged Graph

However, there is a special case in which the reasoning in the previous paragraph is erroneous. See the graph in Fig. 17.8-2(a). Node-6 is a special node – it is a *hinge node*. The graph splits into three subgraphs when all branches incident at that node are removed as shown in Fig. 17.8-2(b). Similarly, removing all branches that connect the node-group containing node-4 and node-6 from the other group results in the formation of three subgraphs, as shown in Fig. 17.8-2(c). Thus, certain node groupings will not result in cut-sets in the case of *hinged graphs*.

We consider only *non-hinged graphs* from this point onwards. Removal of all branches incident at a node will split a non-hinged graph into two connected subgraphs. Hence, the set of all branches incident at any node is a cut-set of the graph for a non-hinged graph.

17.8.2 The All Cut-set Matrix Q_a

The cut-set information for a graph can be organised in matrix form. A cut-set has to be associated with an orientation for this purpose. A cut-set divides the nodes into two groups

with no branch running from one group to the other. There are two choices for a cut-set orientation – either from node group-1 to node group-2 or the other way.

All *Cut-set Matrix* $\mathbf{Q}_a$ is a matrix with as many rows as the cut-sets and as many columns as the branches. It is constructed by filling in entries as per the following rule.

$q_{ij} = +1$, if the j^{th} branch is present in the i^{th} cut-set and its orientation agrees with the cut-set orientation.

$q_{ij} = -1$, if the j^{th} branch is present in the i^{th} cut-set and its orientation disagrees with the cut-set orientation.

$q_{ij} = 0$, if the j^{th} branch is not present in the i^{th} cut-set.

With this rule for constructing $\mathbf{Q}_a$, n rows of this matrix will be identical to the rows of the *all incidence matrix* in the case of a non-hinged graph. That is, $\mathbf{A}_a$ matrix is a sub-matrix of $\mathbf{Q}_a$ matrix. The rank of $\mathbf{A}_a$ is $(n - 1)$. Therefore, the rank of $\mathbf{Q}_a$ has to be $\geq (n - 1)$.

17.8.3 Orthogonality Relation Between Cut-set Matrix and Circuit Matrix

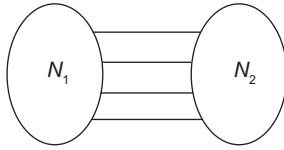


Fig. 17.8-3 A Cut-Set Divides the Nodes Into Two Groups

A cut-set divides the nodes into two groups. Let the first group of nodes and all the branches that are connected between them be enclosed in N_1 and the second group of nodes and all the branches that are connected between them be enclosed in N_2 , as shown in Fig. 17.8-3.

The branches interconnecting N_1 and N_2 are the cut-set branches. If a loop is completely within N_1 or N_2 , then, the row corresponding to this cut-set in $\mathbf{Q}_a$ matrix and the row corresponding to such a loop in $\mathbf{B}_a$ matrix will not have non-zero entries in similarly placed locations. Therefore, the product of the row corresponding to this cut-set in $\mathbf{Q}_a$ and the *transpose* of the row corresponding to such a loop in $\mathbf{B}_a$ matrix will be zero.

If a loop spans over both N_1 and N_2 , then, the loop will have to go from N_1 to N_2 and back (or the other way). Of course, the loop may go back and forth between N_1 and N_2 many times. Therefore, such a loop will contain an even number of branches in common with the cut-set. Consider two such branches corresponding to one excursion of the loop from N_1 to N_2 and back. If the orientation of both these branches agrees (or disagrees) with that of the cut-set, then, the orientation of one of them will agree with that of the loop and that of the other will disagree with the loop orientation. Therefore, the product of the row corresponding to the cut-set in $\mathbf{Q}_a$ and the *transpose* of the row corresponding to such a loop in $\mathbf{B}_a$ matrix will contain terms of the kind $[(\pm 1 \times \mp 1) + (\pm 1 \times \pm 1)]$. Therefore, the product of the row corresponding to the cut-set in $\mathbf{Q}_a$ and the *transpose* of the row corresponding to such a loop in $\mathbf{B}_a$ matrix will be zero for loops that span over N_1 and N_2 .

This argument is valid for any cut-set–loop pair. Therefore,

$$\mathbf{Q}_a \mathbf{B}_a^T = \mathbf{0} \quad (17.8-1)$$

This relation is known as the *orthogonality relation* between the circuit matrix and the cut-set matrix. Obviously, $\mathbf{B}_a \mathbf{Q}_a^T = \mathbf{0}$ will also be true.

Now, Sylvester's Law of Nullity can be used to show that $(\text{rank of } \mathbf{Q}_a + \text{rank of } \mathbf{B}_a) \leq b$. Therefore, $\text{rank of } \mathbf{Q}_a \leq b - (b - n + 1) = (n - 1)$. We have shown earlier that $\text{rank of } \mathbf{Q}_a \geq (n - 1)$. Combining these two conditions, we conclude that the rank of $\mathbf{Q}_a$ matrix is exactly $(n - 1)$.

17.8.4 The Fundamental Cut-set Matrix $\mathbf{Q}_f$

A *tree* of a connected graph contains the minimum number of branches required to keep it connected. Therefore, if one *twig* is removed from a tree, the nodes in the *tree* split into two groups and the *tree* splits into two connected subgraphs. The cut-set formed by the *twig* that was removed along with all those *links* that were connected between the two node groups

Orthogonality relation between cut-set matrix and circuit matrix for a non-hinged graph.

is called a *fundamental cut-set* or an *f-cut-set*. A fundamental cut-set is uniquely associated with a *twig* in a chosen *tree*. An *f-cut-set* will contain only one *twig* – the *twig* that defines it. All the remaining branches in an *f-cut-set* will be *links*. The orientation of an *f-cut-set* is always chosen to agree with the orientation of the defining *twig*.

Figure 17.8-4 shows a linear graph in (a) and a chosen *tree* for this graph in (b). The grouping of nodes that results from the removal of *twigs* one by one is shown in (c), (d) and (e).

The rank of all cut-set matrix $\mathbf{Q}_c$ is $(n - 1)$.
The rank of *f-cut-set* matrix $\mathbf{Q}_f$ is $(n - 1)$.

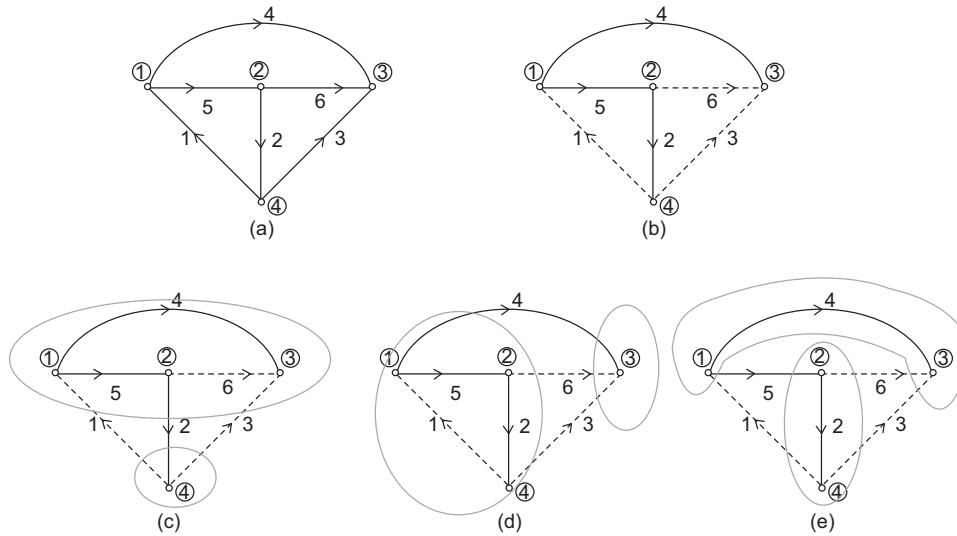


Fig. 17.8-4 (a) A Graph (b) A Chosen Tree (c), (d), (e) *f*-Cut-Sets

The *f-cut-set matrix* $\mathbf{Q}_f$ is a cut-set matrix prepared by including only *f-cut-sets* with respect to a chosen *tree*. The columns of this matrix are usually arranged in such a manner that the *twigs* come first followed by the *links*. Further, the *f-cut-sets* are arranged in rows as per the same order employed in arranging the *twigs* in columns.

This leads to the first $(n - 1)$ columns of the matrix forming an $(n - 1) \times (n - 1)$ identity matrix. Therefore, the rank of $\mathbf{Q}_f$ matrix will be $(n - 1)$. That is expected since each *f-cut-set* will include one branch that is not there in any other *f-cut-set* – that branch will be the defining *twig* for that cut-set. Thus, with the scheme of ordering just described, the $(n - 1) \times b$ *f-cut-set matrix* $\mathbf{Q}_f$ may be partitioned column-wise into an $(n - 1) \times (n - 1)$ unit matrix $\mathbf{U}$ and an $(n - 1) \times (b - n + 1)$ $\mathbf{Q}_\ell$ matrix. Therefore, the rank of $\mathbf{Q}_f$ is $(n - 1)$.

The $\mathbf{Q}_f$ matrix for the example in Fig. 17.8-4 is:

$$\mathbf{Q}_f = \begin{array}{c} \text{f-cut-set-1} \\ \text{f-cut-set-2} \\ \text{f-cut-set-3} \end{array} \begin{array}{cc} \text{Twigs} & \text{Links} \\ \begin{array}{ccc} 2 & 4 & 5 \end{array} & \begin{array}{ccc} 1 & 3 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array} \right] \end{array}$$

$$\mathbf{Q}_f \text{ can be partitioned as } \mathbf{Q}_f = \left[\begin{array}{c|c} \mathbf{U} & \mathbf{Q}_\ell \\ \hline \end{array} \right] \begin{array}{l} (n-1) \times (n-1) \\ (n-1) \times (b-n+1) \end{array}$$

17.8.5 Relation Between $\mathbf{Q}_\ell$, $\mathbf{A}$ and $\mathbf{B}_\ell$

The *all cut-set matrix* $\mathbf{Q}_a$ and the *all circuit matrix* $\mathbf{B}_a$ for a graph have been shown to be orthogonal to each other. The line of reasoning employed to arrive at the result makes it

clear that even sub-matrices of $\mathbf{Q}_a$ and $\mathbf{B}_a$ will be orthogonal to each other. In particular, $\mathbf{Q}_f$ and $\mathbf{B}_f$ for the same choice of a tree will be orthogonal to each other. That is, $\mathbf{Q}_f \mathbf{B}_f^T = \mathbf{0}$. Using the partitioned form of both matrices, we get,

$$\begin{aligned}\mathbf{Q}_f \mathbf{B}_f^T &= \begin{bmatrix} \mathbf{U} & \mathbf{Q}_f \end{bmatrix} \begin{bmatrix} \mathbf{B}_f^T \\ \mathbf{U} \end{bmatrix} = \mathbf{0} \\ \therefore \mathbf{Q}_f &= -\mathbf{B}_f^T\end{aligned}$$

but, we know from Eqn. 17.5-3 that $\mathbf{B}_f^T = -\mathbf{A}_t^{-1} \mathbf{A}_r$. Therefore, $\mathbf{Q}_f$, $\mathbf{A}$ and $\mathbf{B}_f$ matrices are related by the following equations.

Inter-relations
among incidence
matrix, f-circuit matrix
and f-cut-set matrix.

$$\begin{aligned}\mathbf{Q}_f &= -\mathbf{B}_f^T = \mathbf{A}_t^{-1} \mathbf{A}_r \\ \mathbf{Q}_f &= \begin{bmatrix} \mathbf{U} & -\mathbf{B}_f^T \end{bmatrix} = \begin{bmatrix} \mathbf{U} & \mathbf{A}_t^{-1} \mathbf{A}_r \end{bmatrix} \\ \mathbf{B}_f &= \begin{bmatrix} -\mathbf{Q}_f^T & \mathbf{U} \end{bmatrix} = \begin{bmatrix} -(\mathbf{A}_t^{-1} \mathbf{A}_r)^T & \mathbf{U} \end{bmatrix}\end{aligned}\quad (17.8-2)$$

17.9 KIRCHHOFF'S LAWS IN FUNDAMENTAL CUT-SET FORMULATION

Kirchhoff's Voltage Law and Kirchhoff's Current Law equations for a network were expressed in terms of $\mathbf{A}$ matrix in Sect. 17.3 and in terms of $\mathbf{B}_f$ matrix in Sect. 17.6. We make use of these representations to arrive at KCL and KVL equations in terms of $\mathbf{Q}_f$ matrix in this section.

17.9.1 Kirchhoff's Current Law and the $\mathbf{Q}_f$ Matrix

Equation 17.6-3 gives KCL in terms of $\mathbf{A}$ matrix and $\mathbf{B}_f$ matrix. This equation is repeated below:

$$\mathbf{i}(t) = \begin{bmatrix} -\mathbf{A}_t^{-1} \mathbf{A}_r \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t) = \begin{bmatrix} \mathbf{B}_f^T \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t) = \mathbf{B}_f^T \mathbf{i}_l(t)$$

This equation reveals that all branch currents in a network can be expressed in terms of a reduced set of currents – the set of *link currents* for a chosen *tree*. We substitute the relation between $\mathbf{Q}_f$ and $\mathbf{B}_f$ from Eqn. 17.8-2 in the above equation to complete the description of KCL equations in terms of *f-cut-set matrix*. We get,

$$\begin{aligned}\mathbf{i}(t) &= \begin{bmatrix} \mathbf{i}_r(t) \\ \mathbf{i}_l(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_t^{-1} \mathbf{A}_r \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t) \\ &= \begin{bmatrix} \mathbf{B}_f^T \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t) = \mathbf{B}_f^T \mathbf{i}_l(t) \\ &= \begin{bmatrix} -\mathbf{Q}_f \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t)\end{aligned}\quad (17.9-1)$$

Kirchhoff's Current
Law equations for a
network in terms of
various topological
matrices.

It may be verified that the relation $\mathbf{i}(t) = \begin{bmatrix} \mathbf{i}_r(t) \\ \mathbf{i}_l(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{Q}_f \\ \mathbf{U} \end{bmatrix} \mathbf{i}_l(t)$ is equivalent to the

relation $\mathbf{Q}_f \mathbf{i}(t) = \mathbf{0}$. This is yet another form of KCL equations in cut-set formulation. These equations are nothing but linear combinations of KCL equations at $(n - 1)$ nodes.

17.9.2 Kirchhoff's Voltage Law and the $\mathbf{Q}_f$ Matrix

We know that KVL equations of a network can be expressed in terms of the *f-circuit matrix* $\mathbf{B}_f$ for some chosen *tree* as $\mathbf{B}_f \mathbf{v}(t) = [\mathbf{B}_f \mid \mathbf{U}] \begin{bmatrix} \mathbf{v}_t(t) \\ \mathbf{v}_l(t) \end{bmatrix} = \mathbf{0}$. This equation shows that all branch voltages in a network can be expressed in terms of a reduced set of voltages – the set of *twig voltages* for some chosen *tree*. This fact is restated in the following manner.

$$\mathbf{v}_l(t) = -\mathbf{B}_f \mathbf{v}_t(t) \text{ and } \mathbf{v}(t) = \begin{bmatrix} \mathbf{U} \\ -\mathbf{B}_f \end{bmatrix} \mathbf{v}_t(t)$$

We substitute for $\mathbf{B}_f$ in terms of $\mathbf{Q}_f$ to complete the description of KVL equations in terms of $\mathbf{Q}_f$ matrix.

$$\begin{aligned} \mathbf{v}(t) &= \begin{bmatrix} \mathbf{v}_t(t) \\ \mathbf{v}_l(t) \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ -\mathbf{B}_f \end{bmatrix} \mathbf{v}_t(t) \\ &= \begin{bmatrix} \mathbf{U} \\ -\mathbf{B}_f \end{bmatrix} \mathbf{v}_t(t) \\ &= \begin{bmatrix} \mathbf{U} \\ \mathbf{Q}_f^T \end{bmatrix} \mathbf{v}_t(t) = \mathbf{Q}_f^T \mathbf{v}_t(t) = \mathbf{Q}_f^T \mathbf{v}_t(t) \end{aligned} \quad (17.9-2)$$

Kirchhoff's Voltage Law equations for a network in terms of various topological matrices.

17.10 NODE-PAIR ANALYSIS OF NETWORKS

$$\mathbf{Q}_f \mathbf{i}(t) = \mathbf{0} \quad (17.10-1)$$

$$\mathbf{v}(t) = \mathbf{Q}_f^T \mathbf{v}_t(t) = \mathbf{0} \quad (17.10-2)$$

Two equations – Eqn. 17.10-1 and Eqn. 17.10-2 – help us to evolve a procedure for solving an electrical network by determining its *twig voltages* first. First, we consider a network containing two-terminal passive elements and independent current sources only. *Twig voltages* are not necessarily node voltages. They can be; but they need not be. They are, in general, voltages that appear across pairs of nodes. Hence, this analysis is called *node-pair analysis*.

The network has to be prepared for node-pair analysis first. The preparation required in this case consists of replacing series/parallel combination of similar elements with a single element wherever such a replacement is possible. In addition, the circuit may have to be transformed into an *s*-domain equivalent circuit or a phasor equivalent circuit depending on the analysis context. The initial condition sources in the *s*-domain equivalent circuit have to be in the voltage source format.

The second step is to draw the network graph and number the nodes and branches. The numbering of nodes can be done in an arbitrary manner except that the last node to be numbered is the reference node. However, the numbering of branches has to follow a scheme. All the independent current source branches are numbered first and the passive branches are numbered last. Let b_p be the number of passive branches and b_g be the number of independent current source branches.

The next step is to prepare $\mathbf{Q}_f$ matrix in a partitioned form. This requires that a *tree* be chosen first. There is no constraint on the nature of branches selected as *twigs* or *links*. Any *tree* is good enough for the purpose of node-pair analysis. A *tree* is chosen only for the purpose of choosing $(n - 1)$ independent KCL equations. However, the network is assumed to possess a unique solution.

KCL equations in cut-set formulation.
KVL equations in cut-set formulation.

The columns of $\mathbf{Q}_f$ matrix are arranged in sequential order. The first partition of $\mathbf{Q}_f$ will contain all columns corresponding to the independent current source branches. The second partition will contain all columns corresponding to the passive branches. They are denoted by $\mathbf{Q}_{fg}$ and $\mathbf{Q}_{fp}$, respectively. The branch current transform vector $\mathbf{I}(s)$ and the branch voltage transform vector $\mathbf{V}(s)$ are also partitioned in a similar manner (s -domain analysis is assumed).

Now, $\mathbf{Q}_f = [\mathbf{Q}_{fg} \mid \mathbf{Q}_{fp}]$ and $\mathbf{Q}_f^T = \begin{bmatrix} \mathbf{Q}_{fg}^T \\ \mathbf{Q}_{fp}^T \end{bmatrix}$. Applying Eqn. 17.10-1, we get,

$$\text{Therefore, } \begin{bmatrix} \mathbf{Q}_{fg} & \mathbf{Q}_{fp} \end{bmatrix} \begin{bmatrix} \mathbf{I}_g(s) \\ \mathbf{I}_p(s) \end{bmatrix} = 0.$$

$$\mathbf{Q}_{fp} \mathbf{I}_p(s) = -\mathbf{Q}_{fg} \mathbf{I}_g(s)$$

Each passive branch current variable can be related to its voltage variable by the relation $I_k(s) = y_k(s) V_k(s)$, where $y_k(s)$ is the admittance function of k^{th} passive branch. Thus, the column vectors $\mathbf{I}_p(s)$ and $\mathbf{V}_p(s)$ can be related through a $b_p \times b_p$ square diagonal matrix $\mathbf{Y}_p(s)$ with $Y_{pii} = y_i(s)$, the admittance of i^{th} branch, and $Y_{pij} = 0$ for $i \neq j$.

$$\mathbf{I}_p(s) = \mathbf{Y}_p(s) \times \mathbf{V}_p(s) \quad (17.10-3)$$

$\mathbf{Y}_p(s)$ is called the *branch admittance matrix*.

Using this relation in, we get,

$$\mathbf{Q}_{fp} \mathbf{Y}_p(s) \mathbf{V}_p(s) = -\mathbf{Q}_{fg} \mathbf{I}_g(s) \quad (17.10-4)$$

Now, we use Eqn. 17.10-2 to express $\mathbf{V}_p(s)$ in terms of the *twig voltage transform* vector $\mathbf{V}_t(s)$.

$$\mathbf{V}(s) = \begin{bmatrix} \mathbf{V}_g(s) \\ \mathbf{V}_p(s) \end{bmatrix} = \mathbf{Q}_f^T \mathbf{V}_t(s) = \begin{bmatrix} \mathbf{Q}_{fg}^T \\ \mathbf{Q}_{fp}^T \end{bmatrix} \mathbf{V}_t(s)$$

$$\therefore \mathbf{V}_p(s) = \mathbf{Q}_{fp}^T \mathbf{V}_t(s)$$

Substituting in Eqn. 17.10-4, we get the node-pair analysis equation:

$$[\mathbf{Q}_{fp} \mathbf{Y}_p(s) \mathbf{Q}_{fp}^T] \mathbf{V}_t(s) = -\mathbf{Q}_{fg} \mathbf{I}_g(s) \quad (17.10-5)$$

The matrix product $\mathbf{Q}_{fp} \mathbf{Y}_p(s) \mathbf{Q}_{fp}^T$ will be an admittance matrix. It is called the *node-pair admittance matrix* of the network and is denoted by $\mathbf{Y}_t(s)$. If the *branch admittance matrix* is symmetric, then, the *node-pair admittance matrix* will also be symmetric.

The diagonal entries of $\mathbf{Y}_t(s)$ will be equal to the sum of admittance functions of all branches in the corresponding cut-set. The off-diagonal entries will be the sum of the admittance functions of all branches that are common to the two cut-sets involved, with a positive or negative sign. The sign is positive if the branch orientation is the same with respect to both cut-sets. Otherwise, it is negative.

$\mathbf{Y}_t(s)$ is a square matrix of order $(n - 1)$. It is invertible. Therefore,

$$\mathbf{V}_t(s) = -[\mathbf{Y}_t(s)]^{-1} \mathbf{Q}_{fg} \mathbf{I}_g(s) \quad (17.10-6)$$

$\mathbf{V}_p(s)$, the voltage transform vector for passive branches and $\mathbf{V}_g(s)$, the voltage transform vector for the independent current sources can be obtained by applying Eqn. 17.10-2, once $\mathbf{V}_t(s)$ is determined. $\mathbf{I}_p(s)$, the current transform vector for the passive branches can be obtained by applying Eqn. 17.10-3.

If the network contains independent voltage sources, they have to be converted into independent current sources prior to node-pair analysis. It may be necessary to employ v -shifting for this purpose.

The node-pair
analysis equation.

Node-pair
analysis solution.

17.10.1 Node-Pair Analysis of Networks Containing Ideal Dependent Sources

It is necessary to arrive at the admittance representations for ideal dependent sources in the network before the node-pair analysis procedure outlined in this section can be applied to networks containing such sources.

The network has to be prepared for node-pair analysis by converting all independent voltage sources into independent current sources by applying v -shifting, if necessary. The resulting circuit will then contain passive branches, independent current sources and dependent sources, if any.

The independent current source branches are numbered first and passive branches follow. The dependent source output port branches are numbered last.

A VCCS does not pose a special problem for node-pair analysis. If the controlling variable of a VCCS can be identified as the branch voltage or a combination of branch voltages of passive branches, then, it can be represented by suitable admittance entries in off-diagonal positions in the *branch admittance matrix* $\mathbf{Y}_p$. For instance, let Branch-7, Branch-8 and Branch-9 in a network be passive branches and let Branch-11 be the output port of a VCCS. Let the VCCS output be $2v_x$, where v_x is identified to be equal to $v_7 + v_8 - v_9$. Then, this source can be represented by making $\mathbf{Y}_{p11,7} = 2$, $\mathbf{Y}_{p11,8} = 2$ and $\mathbf{Y}_{p11,9} = -2$.

In the case of a VCVS, the output voltage source has to be converted into a dependent voltage source/s by applying the source transformation with the help of v -shifting if necessary. The resulting VCCS or VCCS's can be handled using the same procedure outlined in the last paragraph.

If the controlling variable of a CCCS can be identified as the current (or its negative) in a passive branch of admittance y_k , then, it can be represented by suitable entries in the $\mathbf{Y}_p$ matrix. The entry will be y_k (or $-y_k$) scaled by the scaling factor of dependent source in the column corresponding to the passive branch and row corresponding to the dependent source output branch.

In the case of a CCVS, the output voltage source has to be converted into a dependent current source/s by applying the source transformation with the help of v -shifting if necessary. The resulting CCCS or CCCSs can be handled using the same procedure outlined in the previous paragraph.

EXAMPLE: 17.10-1

Find the total power delivered to the resistors in the circuit shown in Fig. 17.10-1. Use node-pair analysis.

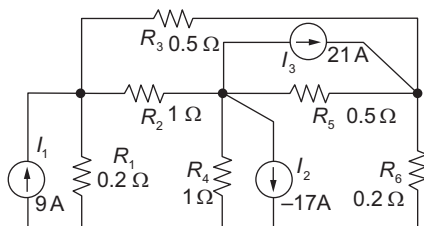


Fig. 17.10-1 Network for Example 17.10-1

SOLUTION

Reference directions assigned for various branches are shown in Fig. 17.10-2(a). Passive sign convention is used throughout. The independent current source branches are

numbered first and the passive branches are numbered after that. The network graph for the circuit is shown in Fig. 17.10-2(b). A tree selected for the purpose of preparing *f-cut-set matrix* is also shown within the graph. Solid lines represent *twigs* and dotted lines represent *links*. The tree is chosen arbitrarily.

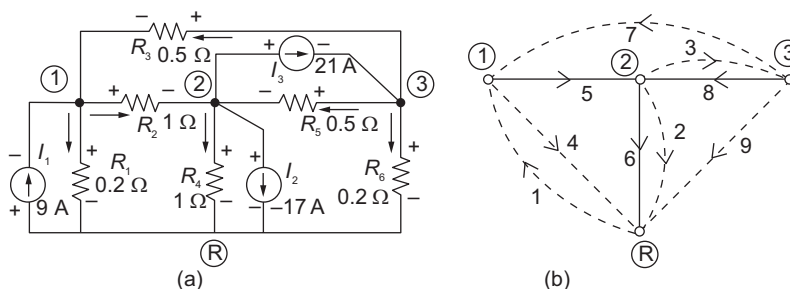


Fig. 17.10-2 (a) Network in Example 17.10-1 with Reference Directions Assigned (b) Network Graph with the Chosen Tree

Note the symmetry in node-pair admittance matrix.

$$\mathbf{Q}_f = \begin{matrix} & \text{sources} & & \text{resistors} \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \begin{matrix} (5) \\ (6) \\ (8) \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$\mathbf{Q}_{fg} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } \mathbf{Q}_{fp} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \mathbf{Y}_p = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \text{ S}$$

$$\mathbf{Y}_t = \mathbf{Q}_{fp} \mathbf{Y}_p \mathbf{Q}_{fp}^T$$

$$= \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & -2 \\ 5 & 11 & 5 \\ -2 & 5 & 9 \end{bmatrix} \text{ S}$$

$$\mathbf{I}_g = \begin{bmatrix} 9 \\ -17 \\ 21 \end{bmatrix} \text{ A and } \mathbf{Q}_{fg} \mathbf{I}_g = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ -17 \\ 21 \end{bmatrix} = \begin{bmatrix} -9 \\ -26 \\ -21 \end{bmatrix} \text{ A}$$

$$\mathbf{V}_t = -\mathbf{Y}_t^{-1} \mathbf{Q}_{fg} \mathbf{I}_g = - \begin{bmatrix} 8 & 5 & -2 \\ 5 & 11 & 5 \\ -2 & 5 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ -26 \\ -21 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ V}$$

$$\mathbf{V}_p = \mathbf{Q}_{fp}^T \mathbf{V}_t = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} \text{ V}$$

$$\mathbf{I}_p = \mathbf{Y}_p \mathbf{V}_p = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 1 \\ 2 \\ 4 \\ 15 \end{bmatrix} \text{ A}$$

Total power dissipated in the resistors = $\mathbf{V}_p^T \times \mathbf{I}_p = \mathbf{I}_p^T \times \mathbf{V}_p = 77 \text{ W}$.

EXAMPLE: 17.10-2

Find the power delivered by the dependent source in the circuit in Fig. 17.10-3.

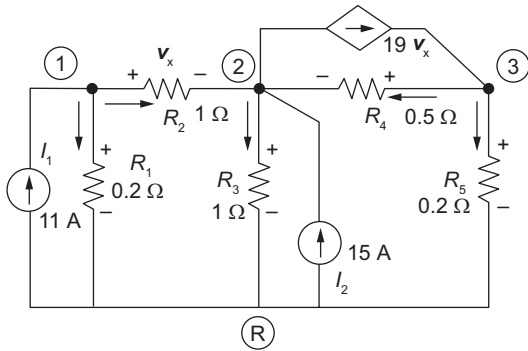


Fig. 17.10-3 Circuit for Example 17.10-2

SOLUTION

The graph of the network along with the chosen tree is shown in Fig. 17.10-4. Branches 3, 5 and 6 are the *twigs* of the *tree* and branches 1, 2, 4, 7, and 8 are the *links*.

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \begin{matrix} (3) \\ (5) \\ (6) \end{matrix} \end{matrix} \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \mathbf{Q}_{fg} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \mathbf{Q}_{fp} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

The control variable of the dependent source is v_x . It is identified as the voltage variable of Branch-4. Therefore, output current of dependent source is $19v_x$ A. i.e., $i_8 = 19v_x$ A. Therefore, there has to be an entry ($= 19$) in the row corresponding to the Branch-8 and column corresponding to the Branch-4 in the branch admittance matrix $\mathbf{Y}_p$.

$$\mathbf{Y}_p = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 5 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 2 & 0 \\ 8 & 0 & 19 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{v}$$

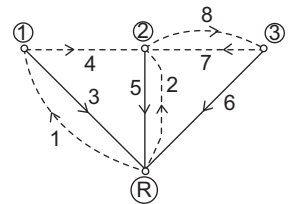


Fig. 17.10-4 Network Graph in Example 17.10-2

Note that the dependent source has produced asymmetry in the node-pair admittance matrix.

$$\begin{aligned}
 \mathbf{Y}_t &= \mathbf{Q}_{tp} \mathbf{Y}_p \mathbf{Q}_{tp}^T \\
 &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 19 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 0 \\ 18 & -15 & -2 \\ -19 & 17 & 7 \end{bmatrix} \text{ S} \\
 \mathbf{I}_g &= \begin{bmatrix} 11 \\ 15 \end{bmatrix} \text{ A and } \mathbf{Q}_{tg} \mathbf{I}_g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 11 \\ 15 \end{bmatrix} = \begin{bmatrix} -11 \\ -15 \\ 0 \end{bmatrix} \text{ A} \\
 \mathbf{V}_t &= -\mathbf{Y}_t^{-1} \mathbf{Q}_{tp} \mathbf{I}_g = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ V, } \mathbf{V}_p = \mathbf{Q}_{tp}^T \mathbf{V}_t = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} \text{ V and } \mathbf{I}_p = \mathbf{Y}_p \mathbf{V}_p = \begin{bmatrix} 10 \\ 1 \\ 1 \\ 15 \\ 4 \\ 19 \end{bmatrix} \text{ A.}
 \end{aligned}$$

Therefore, the voltage across dependent source is $v_g = -2$ V and the current in the dependent source is $i_g = 19$ A, as per passive sign convention. The power delivered by the dependent source is $(-v_g \times i_g) = 38$ W.

17.11 ANALYSIS USING GENERALISED BRANCH MODEL

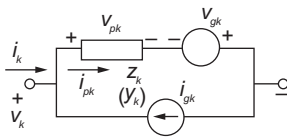


Fig. 17.11-1
Generalised Branch
Model for Network
Analysis

The most general model for a branch in a network is shown in Fig. 17.11-1.

This combination of a passive impedance in series with a voltage source with a current source paralleling the series combination is represented by a branch in the network graph. The branch voltage variable is v_k and the branch current variable is i_k . These are the variables that enter the KVL and KCL equations, respectively. v_{pk} is the voltage variable across the passive impedance component in the k^{th} generalised branch and i_{pk} is the corresponding current variable as per the passive sign convention. v_{gk} is the source voltage in series with the passive impedance in the k^{th} generalised branch and i_{gk} is the current source value in it. The following relations hold good between various variables.

$$\begin{aligned}
 v_k &= v_{pk} - v_{gk} \\
 i_k &= i_{pk} - i_{gk} \\
 v_k &= z_k i_k; \quad i_k = y_k v_k
 \end{aligned} \tag{17.11-1}$$

v -shifting and i -shifting are employed in the given network to convert all branches of the network into generalised branches. It is possible that in some branches v_k is zero or i_k is zero or both are zero. Those are only special cases of the generalised branches and do not pose any problem to modelling the network. However, either impedance z_k or admittance y_k must exist and should be non-zero in the generalised branch model.

Let there be n nodes and b generalised branches in the network graph. Let the following vectors and matrices be defined.

- $\mathbf{v}(t)$ = Column vector of instantaneous branch voltage variables.
- $\mathbf{v}_n(t)$ = Column vector of instantaneous node voltage variables.
- $\mathbf{v}_p(t)$ = Column vector of instantaneous voltage variable across the passive impedance within the generalised branches.
- $\mathbf{v}_g(t)$ = Column vector of instantaneous voltage variable across the voltage source within the generalised branches.
- $\mathbf{v}_t(t)$ = Column vector of instantaneous *twig* voltage variables for a chosen *tree*.

- $\mathbf{v}_l(t)$ = Column vector of instantaneous *link* voltage variables for a chosen *tree*.
 $\mathbf{i}(t)$ = Column vector of instantaneous branch current variables.
 $\mathbf{i}_p(t)$ = Column vector of instantaneous current variable through the passive impedance within the generalised branches.
 $\mathbf{i}_g(t)$ = Column vector of instantaneous current variable of the current source within the generalised branches.
 $\mathbf{i}_t(t)$ = Column vector of instantaneous *twig* current variables for a chosen *tree*.
 $\mathbf{i}_l(t)$ = Column vector of instantaneous *link* current variables for a chosen *tree*.
 $\mathbf{Z}_p$ = $b \times b$ branch impedance matrix.
 $\mathbf{Y}_p$ = $b \times b$ branch admittance matrix.

17.11.1 Node Analysis

KCL equations in terms of incidence matrix are given by $\mathbf{A}\mathbf{i}(t) = \mathbf{0}$ and KVL equations in terms of incidence matrix are given by $\mathbf{v}(t) = \mathbf{A}^T \mathbf{v}_n(t)$.

$$\begin{aligned}
 \mathbf{v}(t) &= \mathbf{v}_p(t) - \mathbf{v}_g(t) \text{ and } \mathbf{i}(t) = \mathbf{i}_p(t) - \mathbf{i}_g(t) \\
 \therefore \mathbf{A}\mathbf{i}(t) = \mathbf{0} &\Rightarrow \mathbf{A}\mathbf{i}_p(t) = \mathbf{A}\mathbf{i}_g(t) \\
 \text{Substituting } \mathbf{i}_p(t) &= \mathbf{Y}_p \mathbf{v}_p(t), \text{ we get,} \\
 \mathbf{A}\mathbf{Y}_p \mathbf{v}_p(t) &= \mathbf{A}\mathbf{i}_g(t) \\
 \mathbf{v}(t) = \mathbf{A}^T \mathbf{v}_n(t) &\Rightarrow \mathbf{v}_p(t) - \mathbf{v}_g(t) = \mathbf{A}^T \mathbf{v}_n(t) \\
 \therefore \mathbf{v}_p(t) &= \mathbf{A}^T \mathbf{v}_n(t) + \mathbf{v}_g(t) \\
 \therefore \mathbf{A}\mathbf{Y}_p \mathbf{v}_p(t) &= \mathbf{A}\mathbf{i}_g(t) \Rightarrow \mathbf{A}\mathbf{Y}_p \mathbf{A}^T \mathbf{v}_n(t) = \mathbf{A}[\mathbf{i}_g(t) - \mathbf{Y}_p \mathbf{v}_g(t)]
 \end{aligned}$$

$\mathbf{A}\mathbf{Y}_p \mathbf{A}^T$ is the node admittance matrix of the network. The *node voltage* variables are given by the following equation.

$$\mathbf{v}_n(t) = [\mathbf{A}\mathbf{Y}_p \mathbf{A}^T]^{-1} \mathbf{A}[\mathbf{i}_g(t) - \mathbf{Y}_p \mathbf{v}_g(t)]$$

Branch voltage vector can be obtained after node voltage vector is obtained by employing the relation $\mathbf{v}(t) = \mathbf{A}^T \mathbf{v}_n(t)$. Then, $\mathbf{v}(t) = \mathbf{v}_p(t) - \mathbf{v}_g(t)$ can be used to determine the voltage across the passive impedances. The current through the passive impedances can then be obtained by using $\mathbf{i}_p(t) = \mathbf{Y}_p \mathbf{v}_p(t)$. Then, the branch currents can be determined by $\mathbf{i}(t) = \mathbf{i}_p(t) - \mathbf{i}_g(t)$.

Node Equation
using generalised
branch model.

17.11.2 Loop Analysis

Select a *tree* for the network graph. KCL equations in terms of *f-circuit matrix* are given by $\mathbf{i}(t) = \mathbf{B}_f^T \mathbf{i}_l(t)$ and KVL equations in terms of *f-circuit matrix* are given by $\mathbf{B}_f \mathbf{v}(t) = \mathbf{0}$. These two equations can be used to arrive at the loop analysis equation given below:

$$\mathbf{B}_f \mathbf{Z}_p \mathbf{B}_f^T \mathbf{i}_l(t) = \mathbf{B}_f [\mathbf{v}_g(t) - \mathbf{Z}_p \mathbf{i}_g(t)]$$

$\mathbf{B}_f \mathbf{Z}_p \mathbf{B}_f^T$ is the loop impedance matrix of the network. The *link current* variables are given by the following equation.

$$\mathbf{i}_l(t) = [\mathbf{B}_f \mathbf{Z}_p \mathbf{B}_f^T]^{-1} \mathbf{B}_f [\mathbf{v}_g(t) - \mathbf{Z}_p \mathbf{i}_g(t)]$$

Loop Equation
using generalised
branch model.

17.11.3 Node-pair Analysis

Select a *tree* for the network graph. Then, KCL equations in terms of *f-cut-set matrix* are given by $\mathbf{Q}_f \mathbf{i}(t) = \mathbf{0}$ and KVL equations in terms of *f-cut-set matrix* are given by $\mathbf{v}(t) = \mathbf{Q}_f^T \mathbf{v}_t(t)$. These two equations can be used to arrive at the node-pair analysis equation given below:

$$\mathbf{Q}_f \mathbf{Y}_p \mathbf{Q}_f^T \mathbf{v}_t(t) = \mathbf{Q}_f [\mathbf{i}_g(t) - \mathbf{Y}_p \mathbf{v}_g(t)]$$

Node-pair
equation using
generalised branch
model.

$\mathbf{Q}_f \mathbf{Y}_p \mathbf{Q}_f^T$ is the node-pair matrix of the network. The *twig voltage* variables are given by the following equation.

$$\mathbf{v}_t(t) = [\mathbf{Q}_f \mathbf{Y}_p \mathbf{Q}_f^T]^{-1} \mathbf{Q}_f [\mathbf{i}_g(t) - \mathbf{Y}_p \mathbf{v}_g(t)]$$

Instantaneous variables were used in all the equations. Phasor variables, phasor impedances and admittances have to be used for sinusoidal steady-state analysis and s -domain transforms and impedance/admittance functions have to be used for analysis in the s -domain.

We have noted that certain preliminary network manipulations are needed to prepare a network for node analysis, loop analysis and node-pair analysis when dependent sources (and ideal transformers) are present in it. A more advanced formulation called ‘mixed variable formulation’ avoids this problem and systematises the network equation formulation still further. However, this formulation is beyond the scope of this introductory text.

The examples employed to illustrate the various techniques of network analysis using topological formulations were memoryless circuits. However, the analysis equations developed are of general nature and they can be employed in sinusoidal steady-state analysis and s -domain analysis too.

The examples used were simple networks. They did not really call for topological methods of solution. However, graph theoretic formulation is needed for analysing complex networks containing a large number of nodes and branches. An electrical power system and the internal circuit of an Operational Amplifier are two examples that need such a systematic analysis procedure. Topological formulation systematises the solution procedure and renders an *algorithmic* nature to the procedure. Thus, topological analysis procedures can be converted into *network analysis and simulation software*. Moreover, topological analysis sets a strong theoretical basis for advanced studies on Electrical and Electronic Networks. For instance, consider the power of an important general network theorem that arises from topological analysis – Tellegen’s theorem.

Tellegen's Theorem

Consider an arbitrary lumped network with n nodes and b branches in its graph.

Let $v_k(t)$ be the branch voltage variable and $i_k(t)$ be the branch current variable of Branch- k according to the passive sign convention with the reference direction for $i_k(t)$ chosen arbitrarily.

If the branch voltage variables satisfy all the constraints imposed by KVL and if the branch current variables satisfy all the constraints imposed by KCL, then,

$$\sum_{k=1}^b v_k(t_2) i_k(t_1) = 0$$

where t_1 and t_2 need not be the same time-instant.

17.12 TELLEGEN'S THEOREM

Consider a *lumped parameter* network with n nodes and b branches in it. The branches in the network may be linear or non-linear, time-invariant or time-variant, active or passive. Let the reference directions for the branch currents be chosen *arbitrarily* for all the branches in the network. However, the reference polarity for the branch voltage will be chosen as per the passive sign convention, once the reference direction for branch current is chosen. The oriented graph for the network may now be prepared. Select one of the nodes in the graph as the reference node.

Let $\mathbf{A}$ be the *reduced incidence matrix* of the network graph. Then, we know that KCL equations at all nodes except the reference node will be given by the matrix equation $\mathbf{A}\mathbf{i}(t) = \mathbf{0}$, where $\mathbf{i}(t)$ is the $b \times 1$ column vector of instantaneous branch currents. Further, we know that KVL equations in the network can be restated as $\mathbf{v}(t) = \mathbf{A}^T \mathbf{v}_n(t)$, where $\mathbf{v}(t)$ is the $b \times 1$ column vector of instantaneous branch voltages and $\mathbf{v}_n(t)$ is the $(n-1) \times 1$ column vector of instantaneous node voltages with reference to the *reference node*.

$$\mathbf{v}(t) = \mathbf{A}^T \mathbf{v}_n(t) \Rightarrow \mathbf{v}(t)^T = \mathbf{v}_n(t)^T \mathbf{A}$$

Post-multiply both sides of this equation by the branch current vector $\mathbf{i}(t)$.

$$\mathbf{v}(t)^T \mathbf{i}(t) = \mathbf{v}_n(t)^T \mathbf{A} \mathbf{i}(t)$$

But $\mathbf{A}\mathbf{i}(t) = \mathbf{0}$ by Kirchhoff’s Current Law. Therefore, we get,

$$\mathbf{v}(t)^T \mathbf{i}(t) = \sum_{k=1}^b v_k(t) i_k(t) = \sum_{k=1}^b p_k(t) = 0 \quad (17.12-1)$$

Equation 17.12-1 is the statement of Tellegen's theorem. This equation is essentially a restatement of the fact that energy is conserved in a lumped parameter electrical network. Each term in the summation in Eqn. 17.12-1 is a power and the summation represents the total power delivered to *all branches* of the network at t . The total power delivered to *all branches* in a network at any instant has to be zero. Thus, Tellegen's theorem for a single network does not seem to reveal anything that we did not know already.

However, Tellegen's theorem is a powerful general network theorem. It depends only on two Kirchhoff's laws and remains valid quite independent of the nature of the lumped elements that make the network.

Kirchhoff's laws are valid on an instant-to-instant basis. Consider two time-instants t_1 and t_2 . We write the KCL equation at t_1 and KVL equation at t_2 .

$$\begin{aligned}\mathbf{A}\mathbf{i}(t_1) &= \mathbf{0} \\ \mathbf{v}(t_2) &= \mathbf{A}^T \mathbf{v}_n(t_2)\end{aligned}$$

$$\text{Now, } \mathbf{v}(t_2) = \mathbf{A}^T \mathbf{v}_n(t_2) \Rightarrow \mathbf{v}(t_2)^T = \mathbf{v}_n(t_2)^T \mathbf{A}$$

Post-multiply both sides of this equation by the branch current vector $\mathbf{i}(t_1)$.

$$\mathbf{v}(t_2)^T \mathbf{i}(t_1) = \mathbf{v}_n(t_2)^T \mathbf{A} \mathbf{i}(t_1)$$

But $\mathbf{A} \mathbf{i}(t_1) = \mathbf{0}$ by Kirchhoff's Current Law. Therefore, we get,

$$\mathbf{v}(t_2)^T \mathbf{i}(t_1) = \sum_{k=1}^b v_k(t_2) i_k(t_1) = 0 \quad (17.12-2)$$

Now, *this is not a restatement of conservation of energy*. The terms in the summation are not instantaneous power terms. Tellegen's theorem now tells us something that is new and significant.

That is not all. Consider two networks that have the same number of nodes and branches in their graphs. Let the same numbering scheme be used in both graphs. Also, let the same reference directions be assigned to the corresponding branches in the two graphs. Further, let the corresponding nodes be accepted as the *reference nodes* in both graphs. Now, the two network graphs will have the same incidence matrix $\mathbf{A}$. Let $i_k(t)$, $v_k(t)$ for $k = 1$ to b represent the branch current and the voltage instantaneous variables in the first network and $\hat{i}_k(t)$, $\hat{v}_k(t)$ for $k = 1$ to b represent the branch current and the voltage instantaneous variables in the first network.

We write the KCL equation at t_1 for the first network and KVL equation at t_2 for the second network.

$$\begin{aligned}\mathbf{A}\mathbf{i}(t_1) &= \mathbf{0} \\ \hat{\mathbf{v}}(t_2) &= \mathbf{A}^T \hat{\mathbf{v}}_n(t_2)\end{aligned}$$

$$\text{Now, } \hat{\mathbf{v}}(t_2) = \mathbf{A}^T \hat{\mathbf{v}}_n(t_2) \Rightarrow \hat{\mathbf{v}}(t_2)^T = \hat{\mathbf{v}}_n(t_2)^T \mathbf{A}$$

Post-multiply both sides of this equation by the branch current vector $\mathbf{i}(t_1)$ of the first network at t_1 .

$$\hat{\mathbf{v}}(t_2)^T \mathbf{i}(t_1) = \hat{\mathbf{v}}_n(t_2)^T \mathbf{A} \mathbf{i}(t_1)$$

But $\mathbf{A} \mathbf{i}(t_1) = \mathbf{0}$ by Kirchhoff's Current Law. Therefore, we get,

$$\hat{\mathbf{v}}(t_2)^T \mathbf{i}(t_1) = \sum_{k=1}^b \hat{v}_k(t_2) i_k(t_1) = 0 \quad (17.12-3)$$

By similar reasoning, it is possible to show that

$$\mathbf{v}(t_2)^T \hat{\mathbf{i}}(t_1) = \sum_{k=1}^b v_k(t_2) \hat{i}_k(t_1) = 0 \quad (17.12-4)$$

These equations definitely do not deal with conservation of energy. The terms in the summation are not power terms – the voltage and current variables in a product term are taken at different instants from two different networks. Tellegen's theorem, as stated in

terms of Eqn. 17.12-3 and Eqn. 17.12-4, is a *topological theorem* and depends only on the fact that any lumped parameter network has to satisfy KVL and KCL constraints at all instants of time.

The fact that instantaneous branch voltage variables in one network at one instant of time enter into a relation with instantaneous branch current variables in another network at a different instant of time does look very surprising; even when considering the topological similarity between the two networks and the common incidence matrix. A theorem that asserts that this surprising conclusion has to be true must indeed be of much theoretical importance in Network Theory. And, it is.

However, this book will not cover such advanced topics. It is time for you to put this textbook down (if you have not done that already long back!) and take up excellent works on Electrical Network Theory written by stalwarts in the area for advanced studies.

17.13 SUMMARY

- The study of topological properties of a network employing a branch of Mathematics called *Linear Graph Theory* is termed *Network Topology*.
- An *oriented linear graph* is a collection of *nodes* and *branches* with *branches* interconnecting the *nodes*. A direction is assigned to each *branch*. The interconnection information for such a graph is contained in its *all incidence matrix* $\mathbf{A}_a$. The rows of $\mathbf{A}_a$ correspond to the nodes and the columns of $\mathbf{A}_a$ correspond to the branches. The entry a_{ij} is +1, if the j^{th} branch is incident at the i^{th} node with its orientation away from the node; -1, if the orientation is towards the node and 0 if the j^{th} branch is not incident at the i^{th} node.
- Order of $\mathbf{A}_a$ is $n \times b$, where n is the number of nodes and b is the number of branches in the graph. Rank of $\mathbf{A}_a$ is $(n - 1)$. The equation $\mathbf{A}_a \mathbf{i}(t) = \mathbf{0}$ represents KCL equations at all nodes, where $\mathbf{i}(t)$ is an $n \times 1$ column vector containing branch currents.
- The *reduced incidence matrix* $\mathbf{A}$ for a graph is a sub-matrix of $\mathbf{A}_a$ obtained by discarding one row corresponding to a chosen node termed as *reference node*. Its rank is $(n - 1)$ and $\mathbf{A} \mathbf{i}(t) = \mathbf{0}$ represents $(n - 1)$ independent KCL equations available for the network.
- A *tree* of a connected graph is a connected subgraph of the graph with all the nodes present in it but no loop present. It will contain $(n - 1)$ branches. The branches in a *tree* are called *twigs* and the branches that do not appear in a *tree* are called its *links*. There are $\det(\mathbf{A} \mathbf{A}^T)$ *trees* for a connected graph.
- A matrix can be partitioned column-wise into $\mathbf{A}_t$ and $\mathbf{A}_l$ with $\mathbf{A}_t$ containing all columns corresponding to the *twigs* of a *tree* and $\mathbf{A}_l$ containing columns corresponding to *links*. Then, $\mathbf{A}_t$ will be invertible.
- All branch currents in a network can be expressed in terms of *link currents* for a chosen *tree* by the equation $\mathbf{i}(t) = -\mathbf{A}_l^{-1} \mathbf{A}_t \mathbf{i}_l(t)$.
- Applying KVL to the network leads to the conclusion that the relation $\mathbf{v}(t) = \mathbf{A}^T \mathbf{v}_n(t)$ can express all branch voltages of a network in terms of node voltages with respect to a chosen *reference node*.
- *Nodal analysis* of a network containing passive branches and current source branches results in the node equation set $\mathbf{V}_n(s) = -[\mathbf{Y}_n(s)]^{-1} [\mathbf{A}_s \mathbf{I}_s(s)]$, where $\mathbf{Y}_n(s) = \mathbf{A}_p \mathbf{Y}_p(s) \mathbf{A}_p^T$ is the *nodal admittance matrix* of the network and $\mathbf{I}_s(s)$ is the vector of Laplace transforms of source currents. $\mathbf{V}_n(s)$ is the vector of Laplace transforms of node voltages. $\mathbf{Y}_p(s)$ is the *branch admittance matrix*. $\mathbf{A}_s$ and $\mathbf{A}_p$ are the column-wise partitions of $\mathbf{A}$ matrix containing columns corresponding to the current source branches and columns corresponding to the passive branches, respectively.
- A *loop* (or *circuit*) is a connected subgraph of the graph with exactly two branches incident at each node in the subgraph.
- *Fundamental circuit matrix* $\mathbf{B}_f$ is a circuit matrix prepared with reference to a chosen *tree*. It is a $(b - n + 1) \times b$ matrix that is partitioned column-wise into $\mathbf{B}_{ft}$ and $\mathbf{B}_{fl}$, with $\mathbf{B}_{ft}$ containing columns corresponding to *twigs* and $\mathbf{B}_{fl}$ containing columns corresponding to *links*. The rows of *f-circuit matrix* correspond to *f-circuits* formed when *links* are replaced on the *tree*, one at a time. The orientation of an *f-circuit* coincides with that of the *link* that produces the circuit. The entry b_{ij} will be +1, if the j^{th} branch is a part of the i^{th} *f-circuit* and its orientation agrees with that of the *f-circuit*; -1 if its orientation disagrees with that of the circuit and 0 if the j^{th} branch is not a part of the i^{th} *f-circuit*. $\mathbf{B}_f$ is of full rank. $\mathbf{B}_{fl}$ can be made equal to

an identity matrix by suitable arrangement of rows and columns.

- The *f-circuit matrix* is orthogonal to the *incidence matrix*. This leads to the relations

$$\mathbf{B}_n = -[\mathbf{A}_t^{-1} \mathbf{A}_l]^T \text{ and } \mathbf{B}_t = \begin{bmatrix} -[\mathbf{A}_t^{-1} \mathbf{A}_l]^T & \mathbf{U} \end{bmatrix}.$$

- $\mathbf{B}_f \mathbf{v}(t) = \begin{bmatrix} \mathbf{B}_f & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{v}_t(t) \\ \mathbf{v}_l(t) \end{bmatrix} = \mathbf{0}$ represents KVL equations for the network. This equation reveals that all branch voltages in a network can be expressed in terms of *twig* voltages for a chosen *tree* through the relation $\mathbf{v}(t) = \begin{bmatrix} \mathbf{U} \\ -\mathbf{B}_f^T \end{bmatrix} \mathbf{v}_t(t)$.

- $\mathbf{i}(t) = \mathbf{B}_f^T \mathbf{i}_l(t)$ represents KCL equations for the network.
- Loop analysis* of a network containing passive branches and voltage source branches results in the loop equation $\mathbf{I}_l(s) = -[\mathbf{Z}_L(s)]^{-1} \mathbf{B}_{fg}^T \mathbf{V}_g(s)$, where $\mathbf{Z}_L(s) = \mathbf{B}_{fp} \mathbf{Z}_p(s) \mathbf{B}_{fp}^T$ is the *loop impedance matrix* of the network and $\mathbf{V}_g(s)$ is the vector of Laplace transforms of source voltages. $\mathbf{I}_l(s)$ is the vector of Laplace transforms of link currents. $\mathbf{Z}_p(s)$ is the *branch impedance matrix*. $\mathbf{B}_{fg}$ and $\mathbf{B}_{fp}$ are the column-wise partitions of $\mathbf{B}_f$ matrix containing columns corresponding to voltage source branches and columns corresponding to passive branches, respectively.
- A planar graph will contain $(b - n + 1)$ *meshes*. KVL in these *meshes* will yield a set of $(b - n + 1)$ independent equations.
- A *tree* for a planar graph can be chosen in such a manner that the *f-circuits* for this choice of *tree* will coincide with the *meshes* in the graph if, and only if, all meshes possess at least one *wholly owned* branch.
- Mesh analysis for a planar network will be the same as loop analysis only if (i) a *tree* with *f-circuits* coincide with *meshes* and (ii) orientation of *links* coincide with the direction of mesh currents.
- A *cut-set* for a network graph is the set of minimum number of branches that have to be removed from the graph to split the graph exactly into two connected subgraphs.
- Fundamental cut-set matrix* $\mathbf{Q}_f$ is a circuit matrix prepared with reference to a chosen *tree*. It is an $(n - 1) \times b$ matrix that is partitioned column-wise into $\mathbf{Q}_n$ and $\mathbf{Q}_l$ with $\mathbf{Q}_n$ containing columns corresponding to *twigs* and $\mathbf{Q}_l$ containing columns corresponding to *links*. The rows of *f-cut-set matrix* correspond to *f-cut-sets* formed when *twigs* are removed from the *tree*, one at a time. The orientation of an *f-cut-set* coincides with that of the *twig* that produces the cut-set. The entry q_{ij} will be +1 if the j^{th} branch is a part of the i^{th} *f-cut-set* and its orientation agrees with that of the *f-cut-set*; -1 if its orientation disagrees with that of the circuit and 0 if the j^{th} branch is not a part of the i^{th} *f-cut-set*. $\mathbf{Q}_f$ is of full rank. $\mathbf{Q}_n$

can be made equal to an identity matrix by suitable arrangement of rows and columns.

- The *f-cut-set matrix* is orthogonal to the *f-circuit matrix*. This leads to the following relations between various network matrices.

$$\mathbf{Q}_n = -\mathbf{B}_n^T = \mathbf{A}_t^{-1} \mathbf{A}_l$$

$$\mathbf{Q}_f = \begin{bmatrix} \mathbf{U} & -\mathbf{B}_f^T \end{bmatrix} = \begin{bmatrix} \mathbf{U} & \mathbf{A}_t^{-1} \mathbf{A}_l \end{bmatrix}$$

$$\mathbf{B}_f = \begin{bmatrix} -\mathbf{Q}_n^T & \mathbf{U} \end{bmatrix} = \begin{bmatrix} -(\mathbf{A}_t^{-1} \mathbf{A}_l)^T & \mathbf{U} \end{bmatrix}$$

- $\mathbf{Q}_f \mathbf{i}(t) = \mathbf{0}$ represents KCL equations for the network in terms of *f-cut-set matrix*.

- $\mathbf{v}(t) = \mathbf{Q}_f^T \mathbf{v}_l(t)$ represents KVL equations for the network in terms of *f-cut-set matrix*.

- Node-pair analysis* of a network containing passive branches and current source branches results in the loop equation $\mathbf{V}_l(s) = -[\mathbf{Y}_l(s)]^{-1} \mathbf{Q}_{fp} \mathbf{I}_s(s)$ where $\mathbf{Y}_l(s) = \mathbf{Q}_{fp} \mathbf{Y}_p(s) \mathbf{Q}_{fp}^T$ is the *node-pair admittance matrix* of the network and $\mathbf{I}_s(s)$ is the vector of Laplace transforms of source currents. $\mathbf{V}_l(s)$ is the vector of Laplace transforms of *twig voltages*. $\mathbf{Y}_p(s)$ is the *branch impedance matrix*. $\mathbf{Q}_{fg}$ and $\mathbf{Q}_{fp}$ are the column-wise partitions of $\mathbf{Q}_f$ matrix containing columns corresponding to the current source branches and columns corresponding to the passive branches, respectively.

- v-shifting* and *i-shifting* are useful in preparing a network for nodal analysis, loop analysis or node-pair analysis.

- Dependent sources, if present in a network, will call for certain network manipulations before nodal analysis, loop analysis or node-pair analysis can be carried out. Dependent sources can make *nodal admittance matrix*, *loop impedance matrix* and *node-pair admittance matrix* asymmetric.

- The *generalised branch model* for a branch in a network systematises the nodal analysis, loop analysis and node-pair analysis equations still further.

- Consider two networks that have the same number of nodes and branches in their graphs. Let the same numbering scheme be used in both graphs. Also, let the same reference directions be assigned to the corresponding branches in the two graphs. Further, let the corresponding nodes be accepted as the *reference nodes* in both graphs. Now, the two network graphs will have same the incidence matrix $\mathbf{A}$. Let $i_k(t)$, $v_k(t)$ for $k = 1$ to b represent the branch current and the voltage instantaneous variables in the first network and $\hat{i}_k(t)$, $\hat{v}_k(t)$ for $k = 1$ to b represent the branch current and the voltage instantaneous variables in the first network.

Then, Tellegen's theorem states that $\sum_{k=1}^b \hat{v}_k(t_2) i_k(t_1) = 0$ and

$\sum_{k=1}^b v_k(t_2) \hat{i}_k(t_1) = 0$ for any t_1 and t_2 . It is a theorem of great theoretical importance.

17.14 PROBLEMS

1. The reduced incidence matrix $\mathbf{A}$ of an oriented graph is given below. Verify whether the following sets of branches can constitute *trees* for this graph without drawing the graph.
(i) {1, 6, 7, 8} (ii) {1, 3, 4, 6} (iii) {2, 3, 4, 5}

$$\mathbf{A} = \begin{array}{c|cccccccc} \text{branches} \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \text{node-1} & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \text{node-2} & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \text{node-3} & 0 & 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ \text{node-4} & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}$$

2. Draw the graph represented by the $\mathbf{A}$ matrix given in Problem-1.
3. Select a *tree* in the graph with $\mathbf{A}$ matrix given in Problem-1 such that all *twigs* of the *tree* are incident at one node. Obtain the path matrix for this *tree* and find $\mathbf{A}_t^{-1}$.
4. A graph with its incidence matrix as given in Problem-1 is the graph of an electrical network. The branches constituting the outer loop are independent current source branches. All the current sources have their branch current variable at 1 A. Find the currents in all the remaining branches.
5. For the graph in Problem-1, find (i) the number of *trees*, (ii) the number of *trees* that do not contain Branch-5 as a *twig*, (iii) the number of *trees* that do not contain Branch-5 and Branch-2 as *twigs*, (iv) the number of *trees* that do not contain branches 2, 3, 4 and 5 as *twigs*, (v) the number of *trees* that contain Branch-5 as a *twig* and (vi) the number of *trees* that contain Branch-5 and Branch-2 as *twigs*.
6. Obtain the incidence matrix of the graph represented by the $\mathbf{A}$ matrix given in Problem-1 by selecting node-2 as the *reference node* instead of node-5 as in Problem-1.
7. A graph with its incidence matrix as given in Problem-1 is the graph of an electrical network. All branches incident at node-3 are independent voltage sources with voltage variable at 1 V. Find the voltage variable values for the remaining branches in the network. [Hint: Take node-3 as the *reference node* and use node transformation equation.]
8. An electrical network represented by the graph described in Problem-1 has independent current sources constituting all the branches in the outer loop and independent voltage sources constituting all the branches that are incident at node-3. The current variables of all independent current source branches are equal to 1 A and the voltage variables of all independent voltage source branches are equal to 1 V. Draw the network diagram and mark the values of voltage and current in all the branches.
9. The incidence matrix of an oriented graph is given as

$$\mathbf{A} = \begin{array}{c|ccccccc} \text{branches} \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \text{node-1} & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \text{node-2} & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ \text{node-3} & 0 & 0 & -1 & 0 & -1 & 1 & -1 \end{array}$$

for a chosen *tree* is known to be $\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Identify the

tree and determine the *twig currents* assuming that all *link currents* are at 1 A.

10. Find the total power dissipated in the circuit shown in Fig. 17.14-1 by nodal analysis.

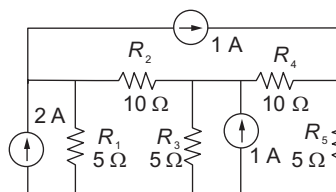


Fig. 17.14-1

11. Find the current delivered by the independent voltage sources in the circuit in the figure by nodal analysis. Use *v*-shifting. [Hint: A voltage source in series with a current source may be shorted out.]

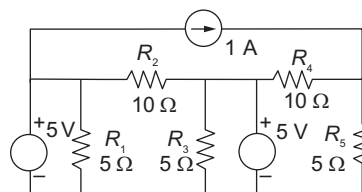


Fig. 17.14-2

12. Find all branch voltages and branch currents in the network shown in Fig. 17.14-3 by nodal analysis.

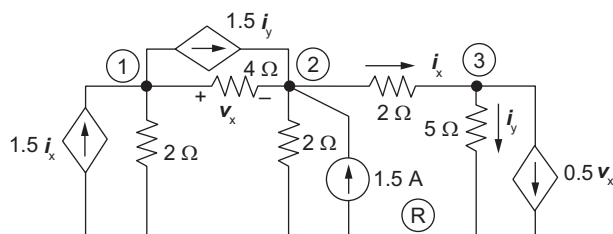


Fig. 17.14-3

13. Find the total power dissipated in the circuit in Fig. 17.14-4 by nodal analysis ($k = 2$).

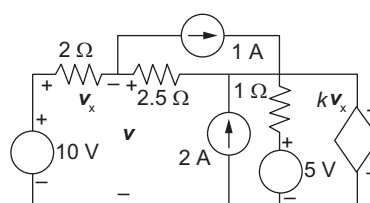


Fig. 17.14-4

14. Determine the $\mathbf{B}_f$ matrix for a graph with

$$\begin{array}{c} \text{branches} \rightarrow 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\ \text{node-1} \left[\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ \text{node-2} \left[\begin{array}{ccccccc} 0 & -1 & 0 & 1 & 1 & 0 & 0 \end{array} \right] \\ \text{node-1} \left[\begin{array}{ccccccc} 0 & 0 & -1 & 0 & -1 & 1 & -1 \end{array} \right] \end{array}$$

without drawing the graph. Use $\{2, 3, 5\}$ as *twigs*.

15. The branches $\{5, 6, 7\}$ in the network with $\mathbf{A}$ matrix, as given in Problem-14, are independent voltage source branches of voltage variable equal to 1 V. Find the branch voltages for all the remaining branches in the network using a suitably constructed *f-circuit matrix*.
16. Three identical independent voltage sources and four identical resistors are used to construct a network with

$$\begin{array}{c} \text{branches} \rightarrow 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\ \text{node-1} \left[\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ \text{node-2} \left[\begin{array}{ccccccc} 0 & -1 & 0 & 1 & -1 & 0 & 0 \end{array} \right] \\ \text{node-1} \left[\begin{array}{ccccccc} 0 & 0 & -1 & 0 & 1 & 1 & -1 \end{array} \right] \end{array} \quad \text{Draw all the}$$

possible networks that possess a unique solution using these seven elements.

17. The *f-circuit matrix* of a network graph for a chosen *tree* is

$$\mathbf{B}_f = \begin{array}{c} \begin{array}{ccccccccc} & 1 & 3 & 6 & 7 & 8 & 2 & 4 & 5 \\ \text{f-loop-1} & \left[\begin{array}{ccccccccc} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right] \\ \text{f-loop-2} & \left[\begin{array}{ccccccccc} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right] \\ \text{f-loop-3} & \left[\begin{array}{ccccccccc} 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \\ \text{f-loop-4} & \left[\begin{array}{ccccccccc} 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \end{array} \end{array} \quad \text{Find its}$$

incidence matrix without drawing the graph.

18. If the branches $\{8, 2, 4, 5\}$ in the graph with an *f-circuit matrix*, as given in Problem-17, are independent current source branches with 1 A branch current value and all the other branches are 10Ω resistor branches, find the total power dissipated in the circuit without drawing the graph and without using any matrix other than $\mathbf{B}_f$ matrix.
19. Find the *mesh circuit matrix* $\mathbf{B}_m$ of the graph that has an *f-circuit matrix* $\mathbf{B}_f$ given by

$$\mathbf{B}_f = \begin{array}{c} \begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{f-loop-1} & \left[\begin{array}{ccccccccc} -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{array} \right] \\ \text{f-loop-2} & \left[\begin{array}{ccccccccc} 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \end{array} \right] \\ \text{f-loop-3} & \left[\begin{array}{ccccccccc} 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \end{array} \end{array} \quad \text{Also,}$$

express all the branch currents in terms of the mesh currents.

20. Three loops formed by branch sets $\{1, 6, 8\}$, $\{1, 6, 9, 7\}$ and $\{1, 2, 3, 9, 6\}$ are chosen to prepare KVL equations for a network that has the graph in Fig. 17.14-5. Will these three KVL equations be independent equations? How many choices for loops exist for obtaining one more KVL equation so that a complete set of independent KVL equations will result? What are the choices?

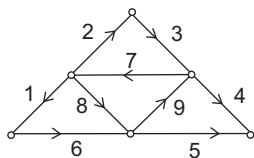


Fig. 17.14-5

21. Is it possible to select a *tree* for the graph in Fig. 17.14-5 such that its *f-circuits* are the same as its *meshes*? If yes, determine *tree*. If no, explain why it is not possible?
22. Assign mesh currents in a clockwise direction in the *meshes* formed by branch sets $\{1, 6, 8\}$ and $\{2, 3, 7\}$ and mesh currents in a counter-clockwise direction in the remaining two *meshes* in the graph in Fig. 17.14-5 and obtain the matrix that connects the branch currents to the mesh currents.
23. Obtain the phasor voltages across the capacitors with polarity marked in Fig. 17.14-6 by loop analysis. Each element should be represented as a branch in the network graph and a *tree* that has all the capacitor branches as its *links* should be chosen. Use *link currents* as the *basis set* in loop analysis.

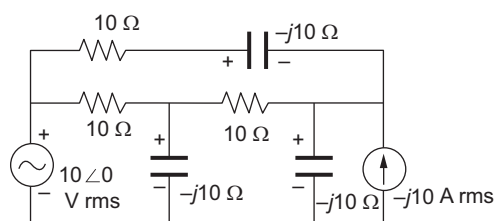


Fig. 17.14-6

24. (i) Assign reference directions and draw the network graph of the network in Fig. 17.14-7. (ii) Obtain the connection matrix between branch currents and the loop currents in the three loops identified in the network diagram. (iii) Determine the loop impedance matrix of the network.

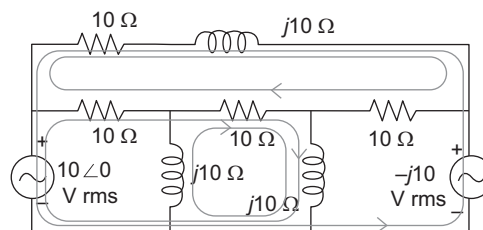


Fig. 17.14-7

25. Find v_1 in the circuit in Fig. 17.14-8 by loop analysis using *loop currents* in the loops indicated in the circuit diagram as the *basis set* for branch currents. Assume that loop currents flow in the clockwise direction in all the loops.

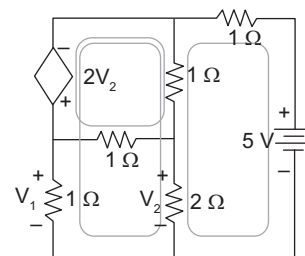


Fig. 17.14-8

26. Determine the dual network for the network in Fig. 17.14-7.
 27. Draw the graph that has

$$\mathbf{A} = \begin{array}{l} \text{branches} \rightarrow 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\ \text{node-1} \left[\begin{array}{cccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ \text{node-2} \left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right] \\ \text{node-3} \left[\begin{array}{cccccccc} 0 & 0 & -1 & 0 & -1 & 1 & 0 & -1 \end{array} \right] \\ \text{node-4} \left[\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{array} \right] \end{array}$$

and construct its dual network.

28. Draw the diagram of a network that is the dual of the network in Fig. 17.14-6.
 29. With reference to the graph in Fig. 17.14-5, determine which of the following sets of branches will form cut-sets and explain why. (i) {1, 6}, (ii) {1, 2, 7}, (iii) {1, 8, 4, 9}, (iv) {1, 2, 3, 6} and (v) {2, 4, 5, 7, 9}.
 30. With reference to the graph in Fig. 17.14-5, write the cut-set equations (i.e., KCL equations) for the following cut-sets – {1, 6}, {1, 2, 7, 8}, {5, 6, 8, 9} and {2, 5, 7, 9}. Will this set of equations form an independent set of equations? If not, why?
 31. The cut-sets {2, 3, 5, 7}, {3, 4, 6} and {1, 2, 6} are selected for preparing the KCL equations of a network with the graph shown in Fig. 17.14-9. (i) Assign cut-set orientations arbitrarily and write the cut-set equations for the three chosen cut-sets. (ii) Are these equations independent? (iii) Do they form a complete set of independent KCL equations for the network? (iv) Enumerate the many choices exist for a cut-set which will yield a fourth KCL equation that is independent of the three KCL equations already obtained?

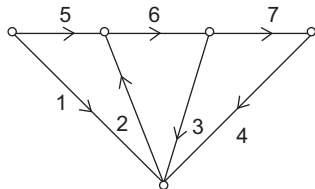


Fig. 17.14-9

32. With reference to the graph in Fig. 17.14-9, select a *tree* such that all *twigs* are incident at a node. Determine the *f-cut-set matrix* for this *tree*, and thereby, determine the *f-circuit matrix* for the same *tree*.
 33. (i) Select branches {4, 5, 6, 7} as the *twigs* of a *tree* for the graph in Fig. 17.14-9 and determine its $\mathbf{Q}_f$ matrix in the partitioned form. (ii) If all the *twigs* are independent voltage source branches with branch voltage variable value of 1 V, find the voltages across the remaining branches.
 34. A connected network has the *f-circuit matrix* given as

$$\mathbf{B}_f = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \text{ for some choice of } \textit{tree}. \text{ Obtain}$$

the *f-cut-set matrix* for the same *tree* without drawing the graph.

35. Repeat Problem-10 by using node-pair analysis. Choose a convenient node as the *reference node*.
 36. Find the phasor voltage across the capacitors with the polarity as shown in the circuit in Fig. 17.14-10 by node-pair analysis. The *reference node* is marked as R in the circuit diagram.

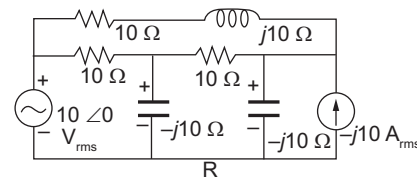


Fig. 17.14-10

37. Repeat Problem-12 by node-pair analysis.
 38. Obtain the node-pair admittance matrix of the circuit in Fig. 17.14-7.
 39. Prepare the network graph for the network in Fig. 17.14-11 using *generalised branch model* for the branches. Solve for v by node analysis. The *reference node* to be used is marked as R in the diagram.

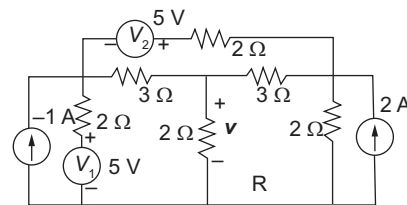


Fig. 17.14-11

40. Prepare the network graph for the network in Fig. 17.14-12 using *generalised branch model* for the branches. Solve for v_1 , v_2 and v_3 by node-pair analysis.

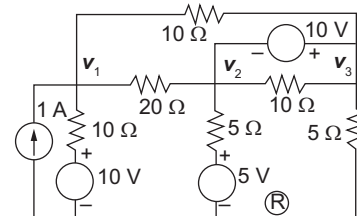


Fig. 17.14-12

41. Repeat Problem-40 by loop analysis using *link currents* as the *basis set* for branch currents. Select a suitable *tree*.

Answers to Selected Problems

Chapter 1

- (a) 432,000 Coulombs (b) 11.66 V (c) 88.18 AH (d) 3.76×10^6 J, 1.045 kWh
- (a) 28 AH (b) 9 A (c) 2.94×10^6 J, 0.817 kWh 8.9 A, 71.2 AH
- (a) 10 AH (b) 60 AH (c) 10 A (d) 2.133×10^6 J, 0.593 kWh
- (a) a 100 Ω resistor (b) 0.36 mC (c) 1 mC (d) 5 mJ (e) 23 ms
- (a) a 0.1 μ F capacitor (b) 1 μ C (c) 5 μ J (d) 3.75 μ J
- (a) a 0.6 H inductor
 - $q(t) = \begin{cases} 0.1(1 - \cos 100t) \text{ coul} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$
 - $p(t) = \begin{cases} 3000 \sin 200t \text{ W} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$
 - $E(t) = \begin{cases} 30 \sin^2 100t \text{ J} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$
- (a) -240 W (b) 120 J, increases
- It is a resistor of 5 Ω and the current through it is $(2 - e^{-100t})$ A for $t \geq 0$.
- (d) 0.85 A
 - $v(t) = \begin{cases} 0 \text{ V} & \text{for } t \leq 0 \\ 20t \text{ V} & \text{for } 0 < t \leq 4 \text{ s} \\ (160 - 20t) \text{ V} & \text{for } 4 \text{ s} < t \leq 8 \text{ s} \\ 0 \text{ V} & \text{for } t > 8 \text{ s} \end{cases}$
 - $v(t) = \begin{cases} 0 \text{ V} & \text{for } t \leq 0 \\ 1 \text{ V} & \text{for } 0 < t \leq 4 \text{ s} \\ -1 \text{ V} & \text{for } 4 \text{ s} < t \leq 8 \text{ s} \\ 0 \text{ V} & \text{for } t > 8 \text{ s} \end{cases}$
 - $v(t) = \begin{cases} 0 \text{ V} & \text{for } t \leq 0 \\ 100t^2 \text{ V} & \text{for } 0 < t \leq 4 \text{ s} \\ (1600t - 100t^2 - 3200) \text{ V} & \text{for } 4 \text{ s} < t \leq 8 \text{ s} \\ 3200 \text{ V} & \text{for } t > 8 \text{ s} \end{cases}$
- $v(t) = \begin{cases} 10 \text{ V} & \text{for } t < 0 \\ (10 - \frac{2}{\pi} \sin 1000\pi t) \text{ V} & \text{for } t \geq 0 \end{cases}$
- 9.97 A, 1.994 Wb-T, 9.94 J
- (b) $v_x = -10$ V (c) Yes, it is a DC source.
- (a) (15 V, -3 A) and (-15 V, 3 A) (b) It is an active element and is a DC source. (c) No
- 4 H, 1.98 H
- 1 H, 0.7 H
- 1 H, 0.5

- (a) $\psi_1 = 0.139 \sin(100\pi t + 30^\circ)$ Wb-T
 $\psi_2 = 0.224 \sin(100\pi t + 42^\circ)$ Wb-T
 - $v_1 = 43.53 \cos(100\pi t + 30^\circ)$ Wb-T
 $v_2 = 70.35 \cos(100\pi t + 42^\circ)$ Wb-T
 - $\psi_1 = 0.139 \sin(100\pi t - 30^\circ)$ Wb-T
 $\psi_2 = 0.224 \sin(100\pi t + 138^\circ)$ Wb-T
 $v_1 = 43.53 \cos(100\pi t - 30^\circ)$ Wb-T
 $v_2 = 70.35 \cos(100\pi t + 138^\circ)$ Wb-T

Chapter 2

- $v_2 = -10$ V, $v_4 = 15$ V, $v_5 = -15$ V, $i_1 = -3$ A, $i_5 = 2$ A
- (i) $v_1 = -15$ V, $v_5 = 15$ V, $v_7 = 10$ V, $i_2 = 3$ A, $i_3 = -5$ A, $i_4 = -8$ A, $i_6 = -5$ A
 (ii) Elements a , e and g (iii) Elements b , c , d and f
 (iv) Elements b , c , d and f . Total power = 140 W (v) Elements a , e and g . Power absorbed = -140 W
- (i) $i_2 = -i_3 - i_4 + i_8$; $i_1 = i_3 + i_4 + i_7 - i_8$; $i_6 = -i_3 - i_7$; $i_5 = -i_4 - i_3 - i_7$
 (ii) $v_1 = v_3 - v_7 - v_8$; $v_2 = -v_3 + v_7$; $v_5 = v_4 + v_8$; $v_6 = v_3 - v_4$
- (i) The elements are designated as $[a // (b+c)] + d + [(e+f)//g]$ where $//$ is parallel connection and $+$ is series connection. Then, $i_c = -2$ A, $i_d = 1$ A, $i_f = -2$ A, $i_g = 3$ A, $v_b = 5$ V, $v_d = -5$ V, $v_e = 5$ V.
 (ii) 3
 (iii) $[5 \Omega // (2.5 \Omega + 10 \text{ V})] + [-1.5 \text{ V}] + [3.3333 \Omega // (2.5 \Omega + 15 \text{ V})]$ where $//$ stands for parallel connection and $+$ stands for series connection.
 (iv) $[5 \Omega // (2.5 \Omega + 2.5 \text{ A})] + [1 \text{ A}] + [3.3333 \Omega // (2.5 \Omega + 2 \text{ A})]$ where $//$ stands for parallel connection and $+$ stands for series connection.
- (i) The elements are designated as $[a // (b+c)] + d + [(e+f)//g]$ where $//$ is parallel connection and $+$ is series connection. Then, $i_c = -2$ A, $i_d = 1$ A, $i_f = -2$ A, $i_g = 3$ A, $v_b = 5$ V, $v_d = -5$ V, $v_e = 5$ V.
 (ii) Power delivered by a (1 ACS) = -5 W, Power delivered by d (1 ACS) = 5 W, Power delivered by g (3 ACS) = -30 W, Power delivered by b (5 V VS) = -10 W, Power delivered by c (10 V VS) = 20 W, Power delivered by e (5 V VS) = -10 W, Power delivered by f (15 V VS) = 30 W.
- (i) The elements are designated as $[a // b] + c + d + [e // f]$ where $//$ is parallel connection and $+$ is series connection. Then, $i_b = 1$ A, $i_d = 1$ A, $i_f = -3$ A, $v_b = -10$ V, $v_c = 5$ V, $v_f = 10$ V.

- (ii) $a \rightarrow 10 \text{ V VS}$, $b \rightarrow 1 \text{ A CS}$, $c \rightarrow 5 \text{ V VS}$, $d \rightarrow 1 \text{ A CS}$, $e \rightarrow 10 \text{ V VS}$, $f \rightarrow 3 \text{ A CS}$; $a \rightarrow 2 \text{ A CS}$, $b \rightarrow 10 \text{ V VS}$, $c \rightarrow 5 \text{ V VS}$, $d \rightarrow 5 \text{ V VS}$, $e \rightarrow 2 \text{ A CS}$, $f \rightarrow 3 \text{ A CS}$; $a \rightarrow 2 \text{ A CS}$, $b \rightarrow 10 \text{ V VS}$, $c \rightarrow 1 \text{ A CS}$, $d \rightarrow 1 \text{ V VS}$, $e \rightarrow 2 \text{ A CS}$, $f \rightarrow 10 \text{ V VS}$ – where CS is an independent current source and VS is an independent voltage source.
9. R must be between 15Ω and 31.67Ω .
10. R must be between 7.43Ω and 9.33Ω .
11. $R_1 = 2 \Omega$, $R_2 = 2.5 \Omega$, $R_3 = 5 \Omega$, $R_4 = 5 \Omega$, $R_5 = 10 \Omega$, Total power dissipated = 125 W .
12. (i) 36 V (ii) 120 V/s .
13. (i) -0.4 A (ii) -52 A/s
14. Current in $R_{AD} = 2 \text{ A}$, Current in $R_{AB} = 0 \text{ A}$, Current in $R_{BD} = 5 \text{ A}$, Current in $R_{AC} = 0 \text{ A}$, Current in $R_{BC} = 0 \text{ A}$, Current in $R_{CD} = 2 \text{ A}$.
15. Power consumed by 10Ω resistor = 0 W , Power consumed by 5Ω resistor = 20 W , Power delivered by independent voltage source = 0 W , Power delivered by dependent source = 20 W .
16. $V_{AB} = 0 \text{ V}$, Total positive power dissipated in the circuit = 15 W , Power delivered by independent source = 10 W , Power delivered by dependent source = 5 W .
17. Power delivered by $2i_x$ source = 2.556 W , Power delivered by 2Ω resistor = -2.556 W , Power delivered by 3Ω resistor = -3.834 W , Power delivered by 6Ω resistor = -7.667 W , Power delivered by $0.5v_x$ source = -3.195 W , Power delivered by 13 V source = 14.696 W .
18. 9 V ; Power absorbed by $0.5i_x$ source = -10.125 W , Power absorbed by 2Ω resistors = 10.125 W each, Power absorbed by 4Ω resistor = 20.25 W , Power absorbed by 3Ω resistor = 27 W , Power absorbed by $0.75v_x$ source = -30.375 W , Power absorbed by 3 A source = -27 W .
19. Energy delivered to 9Ω resistor in $[0, 2\text{s}] = 6.75 \text{ J}$, Energy delivered to 2Ω resistor in series with v_{s1} in $[0, 2\text{s}] = 37.625 \text{ J}$, Energy delivered to 2Ω resistor in series with v_{s2} in $[0, 2\text{s}] = 40.375 \text{ J}$.
20. 3.775 coulombs
21. $k_1 = 0.5 \text{ V/A}$, $k_2 = 21 \text{ A/V}$
22. $\alpha = -4 \text{ V/A}$, $\beta = 6.5 \text{ V/V}$

Chapter 3

1. $11.43:6.43$
2. $i_x = 0.4 \text{ A}$
3. Power in $3 \Omega = 3 \text{ W}$; $V_{DC} = 57 \text{ V}$; Power delivered by $V_{DC} = 171 \text{ W}$.
4. 109Ω in series with 29Ω with output taken across 29Ω
5. (i) $R_1 = 2.5 \Omega$, $R_2 = 5 \Omega$
6. (i) $R = 189.9 \Omega$ (ii) 170.1Ω
7. $R = 60 \Omega$
8. $R = 12 \Omega$
9. $v(t) = \delta(t - 2) \text{ V}$
10. (ii) $v(t) = [1.25 \delta(t) + 6.25 u(t)] \text{ V}$
11. (i) $v(t) = 10 [u(t - 0.02) - u(t - 0.03)] \text{ V}$

$$(ii) i(t) = \begin{cases} 1 \text{ A} & \text{for } 10 \text{ ms} < t \leq 20 \text{ ms} \\ \left(1 + \frac{0.1(t-20)}{3}\right) \text{ A} & \text{for } 20 \text{ ms} < t \leq 50 \text{ ms} \\ 2 \text{ A} & \text{for } t > 50 \text{ ms} \end{cases}$$

$$(iii) p(t) = \begin{cases} 0 \text{ W} & \text{for } 10 \text{ ms} < t \leq 20 \text{ ms} \\ \left(10 + \frac{(t-20)}{3}\right) \text{ W} & \text{for } 20 \text{ ms} < t \leq 50 \text{ ms} \\ 0 \text{ W} & \text{for } t > 50 \text{ ms} \end{cases}$$

$$E(t) = \begin{cases} 0.15 \text{ J} & \text{for } 10 \text{ ms} < t \leq 20 \text{ ms} \\ \left(0.15 + 9.9933 \times 10^{-3}t + 0.167 \times 10^{-6}t^2\right) \text{ J} & \text{for } 20 \text{ ms} < t \leq 50 \text{ ms} \\ 0.6 \text{ J} & \text{for } t > 50 \text{ ms} \end{cases}$$

12. $361:1$

13. (i) $-1/3 \text{ A}$ (iii) 0 W

14. (i)

$$i(t) = \begin{cases} 0.5 - \cos(400t + \frac{\pi}{3}) \text{ A} & ; t \geq 0 \end{cases}$$

$$p(t) = \begin{cases} 5 \sin(400t + \frac{\pi}{3}) - 5 \sin(800t + \frac{2\pi}{3}) \text{ W} & ; t \geq 0 \end{cases}$$

$$E(t) = \begin{cases} 9.375 - 12.5 \cos(400t + \frac{\pi}{3}) + 6.25 \cos(800t + \frac{2\pi}{3}) \text{ J} & ; t \geq 0 \end{cases}$$

(ii) 0.5 A (iii) -0.5 A (iv) 800 rad/s .

$$15. (i) i_L(t) = \begin{cases} (20e^{0.5t} - 18) \text{ A} & ; 0 \leq t \leq 2 \text{ s} \\ 36.37 \text{ A} & ; t > 2 \text{ s} \end{cases}$$

(ii) 36.37 A , 330.7 J

16. Average value of applied voltage in the time interval $[1.5\text{s}, 1.7\text{s}]$ is 6 V .

17. 50 mH

18. (i) 110 V (iii) $(v_L)_{av} = 0 \text{ V}$, $(i_L)_{av} = 5 \text{ A}$, $(i_s)_{av} = 4.545 \text{ A}$, $(i_{DC})_{av} = 0.455 \text{ A}$; 50 W (iv) $(v_L)_{av} = 0 \text{ V}$, $(i_L)_{av} = 10 \text{ A}$, $(i_s)_{av} = 9.09 \text{ A}$, $(i_{DC})_{av} = 0.91 \text{ A}$; 100 W

19. (i) $V_{DC} = 500 \text{ V}$ (iii) $(v_L)_{av} = 0 \text{ V}$, $(i_L)_{av} = 5 \text{ A}$, $(i_{s1})_{av} = 4.902 \text{ A}$, $(i_{s2})_{av} = 0.098 \text{ A}$; Average power delivered by 10 V source = 49 W ; Average power delivered by 500 V source = 49 W

$$20. \text{ Current drawn from source} = \frac{5}{2\pi} \{1 - \cos 2000\pi t\} \text{ A}; t \geq 0\text{s};$$

$$\text{Current through } L_4 = \frac{5}{2\pi} \{1 - \cos 2000\pi t\} \text{ A}; t \geq 0\text{s}; \text{ Voltage}$$

across $L_4 = 2.5 \sin 2000\pi t \text{ V}; t \geq 0\text{s}$.

21. (i) 4 V , 2 V , 6 V ; 0.6 Wb-T , 0.3 Wb-T , 0.9 Wb-T (ii) 0.133 V (iii) 1.8 J , 0.9 J , 2.7 J (iv) 4.05 J , 1.35 W

22. (i) 0.0267 J (ii) 12 A , 6 A , 8 A ; 0.24 Wb-T , 0.3 Wb-T , 0.267 Wb-T (iii) 20 V (iv) 1.44 J , 0.9 J , 1.067 J , 3.407 J (v) 3.2 J , 640 W .

23. $i_s(t) = 0.1 \delta(t - 3) \text{ A}$

$$24. i(t) = [5 \times 10^{-5} \delta(t) + 2 \times 10^{-5} u(t)] \text{ A}$$

$$25. (i) i_s(t) = 10 [u(t - 0.01) - u(t - 0.035)] \text{ A}$$

$$(ii) v(t) = \begin{cases} [-10 + 1000(t - 0.01)] \text{ V} & ; 0.01 \text{ s} \leq t \leq 0.035 \text{ s} \\ 15 \text{ V} & ; t > 0.035 \text{ s} \end{cases}$$

$$(iii) p(t) = \begin{cases} (10000t - 200) \text{ W} & ; 0.01 \text{ s} \leq t \leq 0.035 \text{ s} \\ 0 \text{ W} & ; t > 0.035 \text{ s} \end{cases}$$

$$E(t) = \begin{cases} (2 - 200t + 5000t^2) \text{ J} & ; 0.01 \text{ s} \leq t \leq 0.035 \text{ s} \\ 1.125 \text{ J} & ; t > 0.35 \text{ s} \end{cases}$$

26. $36:100$

27. (i) $-20/3 \text{ V}$, $200/9 \text{ J}$ (iii) 0

28. (i) $v_c(t) = \begin{cases} 125 - 250 \cos\left(400t + \frac{\pi}{3}\right) \text{ V}; & t \geq 0 \\ 250 \sin\left(400t + \frac{\pi}{3}\right) - 250 \sin\left(800t + \frac{2\pi}{3}\right) \text{ W}; & t \geq 0 \\ E(t) = \frac{5}{8} \left[1 - \cos\left(400t + \frac{\pi}{3}\right) + \cos\left(800t + \frac{2\pi}{3}\right) \right] \text{ J}; & t \geq 0 \end{cases}$
- (ii) 125 V (iii) -125 V (iv) 800 rad/s
29. (i) $v_c(t) = \begin{cases} [7.092e^{1.5t} - 5.092] \text{ V}; & 0 \leq t \leq 1 \text{ s} \\ 26.692 \text{ V}; & t > 1 \text{ s} \end{cases}$
- (ii) $v_c(4) = 26.7 \text{ V}$, $E(4) = 0.168 \text{ J}$
30. Average current delivered in [2 s, 2.5 s] is 2 A.
31. (i) 5 A (ii) 12.5 mF
32. (i) $0.6 \cos(300t) \text{ A}; t \geq 0$.
(ii) $0.3 \cos(300t) \text{ A}; t \geq 0$ and $50 \sin(300t) \text{ V}; t \geq 0$
33. (i) $133.33 \sin(300t) \text{ V}; t \geq 0$.
(ii) $33.33 \sin(300t) \text{ V}; t \geq 0$ and $1 \cos(300t) \text{ A}; t \geq 0$.
34. (i) 3.692 V, 68.15 μJ , 149.94 μJ , 224.91 μJ
(ii) $v_c(t) = \begin{cases} (3.69 + 76923t^2) \text{ V}; & 0 \leq t \leq 0.002 \text{ s} \\ (3.38 + 307.7t) \text{ V}; & 0.002 \text{ s} < t \leq 0.006 \text{ s} \\ (3.33 + 326.15t - 1538.46t^2) \text{ V}; & 0.006 \text{ s} < t \leq 0.008 \text{ s} \\ 5.54 \text{ V}; & t > 0.008 \text{ s} \end{cases}$
- (iii) Total energy delivered by source = 2.69 mJ; Energy dissipated in the resistor = 2.133 mJ; 85.2 μJ , 187.4 μJ , 281.14 μJ .
35. (i) 0.665 mJ (ii) 0 V, 6 V, 4 V; 0 C, 0.3 mC, 0.133 mC.
(iii) -66.67 μA (iv) 1.17 mJ; 0 J, 0.9 mJ, 0.27 mJ
(v) -4 mJ; -1.33 mW.

Chapter 4

- 176.3 W
- $R_1 = 5 \Omega$, $R_2 = 10 \Omega$, $R_3 = 1.25 \Omega$, $I = -2 \text{ A}$
- $R_{12} = 50 \Omega$, $R_{23} = 50 \Omega$, $R_{1R} = 25 \Omega$, $R_{2R} = 50 \Omega$,
 $R_{3R} = 16.67 \Omega$, $R_{13} = 25 \Omega$
- (i) $\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0.3636 & 0.1818 & 0.1818 \\ 0.4545 & -0.1616 & -0.0505 \\ 0 & 0.2222 & -0.5556 \\ 0.4545 & 0.6162 & 0.5051 \\ 0.1818 & 0.202 & 0.3131 \\ 0.1818 & -0.202 & -0.1313 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$
(ii) $I_1 = x \text{ A}$, $I_2 = 2.5x \text{ A}$ and $I_3 = x \text{ A}$, where x is an arbitrary real value.
- (i) $\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0.0905 & 0.1699 & 0.0548 \\ 0.0027 & 0.0658 & -0.3014 \\ 0.0877 & 0.1041 & 0.3562 \\ 0.0411 & -0.0137 & 0.4795 \\ -0.1288 & -0.0904 & 0.1644 \\ 0.0384 & -0.0795 & -0.2192 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}$
(ii) $V_1 = x \text{ V}$, $V_2 = -0.6671x \text{ V}$ and $I_3 = 0.4166x \text{ A}$, where x is an arbitrary real value.

- $V_1 = 0 \text{ V}$; $V_2 = 35 \text{ V}$; $V_3 = 10 \text{ V}$
- (i) $k_1 = 0.2 \text{ A/A}$; $k_2 = 0.15 \text{ A/A}$; $k_3 = 0.1 \text{ A/A}$
- (i) $k_1 = -1.6 \text{ A/A}$; $k_2 = -0.2 \text{ A/A}$; $k_3 = 0.5 \text{ A/A}$
(ii) Power delivered by 10 V source = 11 W, Power delivered by 5 V source = -0.7625 W, Power delivered by $k_1 i_x$ source = -1.9 W, Power delivered by $k_2 v_x$ source = 0.513 W, Power delivered by $k_3 i_y$ source = -1.335 W.
- $k_1 = -0.2 \text{ A/V}$; $k_2 = -0.4 \text{ A/V}$; $k_3 = -1.6 \text{ A/V}$; $k_4 = 0.2 \text{ A/V}$; $k_5 = -0.2 \text{ A/V}$; $k_6 = 0.2 \text{ A/V}$.
- $k = 0.5 \text{ V/V}$
- $k = -1 \text{ V/V}$
- Node voltages are 10 V, 20 V and 12 V.
- Shunt arms -4 Ω , 4 Ω , 5 Ω . Series arms -6 Ω , 3 Ω , 4 Ω .
- $\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} -0.0433 & -0.0099 & 0.0542 \\ 0.1006 & -0.057 & -0.0436 \\ -0.0436 & -0.0732 & 0.1168 \\ -0.057 & 0.1302 & -0.0732 \\ -0.0007 & 0.0633 & -0.0626 \\ 0.0563 & -0.0669 & 0.0106 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$
- (i) $v = (0.2222V_1 + 0.4444I + 0V_2) \text{ V}$ (ii) -2.5 A
(iii) All mesh currents are equal to 2.5 A.
- Voltage across current sources = 6 V with +ve at current delivery point.
- Current delivered to 12.5 V source = 4.375 A; Power delivered by 13.5 V source = 16.875 W; Power delivered by 13.6 V source = 20.4 W; Power delivered by 13.7 V source = 22.263 W; Power delivered by 12.5 V source = -54.688 W; Power dissipated in 0.1 Ω = 0.156 W; Power dissipated in 0.15 Ω = 0.338 W; Power dissipated in 0.2 Ω in the 13.7 V source = 0.528 W; Power dissipated in 0.2 Ω in the 12.5 V source = 3.828 W.
- Same as above.
- Same as in Problem-5
- Mesh currents are 0.3 A, -1 A, -1.7 A and -2 A in meshes in the clockwise order. However, the voltage across current sources cannot be determined uniquely. The circuit has infinite solutions.
- This circuit cannot be solved uniquely. It has infinite possible solutions. Two possible solutions for current source voltages are (i) -2 V, 0 V, -9 V and (ii) 0 V, 2 V, -11 V.
- Same as in Problem-11
- $i_1 = (0.2V_1 - 0.2V_2) \text{ A}$; $i_2 = (0.2V_1 - 0.0333V_2 + 0.5I) \text{ A}$; $i_3 = (-0.1667V_2 + 0.5I) \text{ A}$, where i_1 is the current through 2 Ω from left to right, i_2 is the current through the 3 Ω in parallel with the current source and i_3 is the current in 3 Ω in series with the -3 i_x dependent voltage source from left to right.

Chapter 5

- 2 V
- (i) 0.9 A (ii) -4 V
- (i) 45 Ω (ii) 8.333 V/A; 0.4167 V/V (iii) 19 V/s
- $R_1 = R$, $R_2 = 2R$, $R_3 = R$ for any real +ve R .
- 1 V/A

6. (i) 11.727 V (ii) Power dissipated in $0.8 \Omega = 0.095$ W, Power dissipated in $0.5 \Omega = 0.149$ W, Power dissipated in $1 \Omega = 0.075$ W, Power dissipated in $10 \Omega = 13.75$ W.
7. (i) -1.936 A (ii) Power dissipated in 20Ω resistor = 0.084 W, Power dissipated in $30 \Omega = 33.98$ W, Power dissipated in $10 \Omega = 86.18$ W.
8. 337.6 ms
9. $v_0 = \frac{-k}{4+k}(v_1 + v_2 + v_3)$ and
as $k \rightarrow \infty$, $v_0 \rightarrow -(v_1 + v_2 + v_3)$
11. 1.433 V
12. -42.02 mA
14. (i) $v_x = 81.82(v_1 - v_2)$ V
(ii) $\Delta v_x = \pm 148.76(v_1 - v_2) \times |\Delta R|$ V
15. (i) 10Ω (ii) 8.9909Ω
16. 35 V + 11.1Ω in series
17. 22.5Ω
18. $\frac{20}{11}i_s(t)$ voltage source in series with -0.909Ω ; $-2i_s(t)$ current source in parallel with 0.909Ω .
19. $4i_s(t)$ voltage source in series with 10Ω ; $0.4i_s(t)$ current source in parallel with 10Ω
20. $0.06 v_s(t)$ voltage source in series with 25Ω ; $0.0024v_s(t)$ current source in parallel with 10Ω
21. $-\frac{2}{3}v_s(t)$ voltage source in series with $\frac{40}{3} \Omega$; $-\frac{1}{20}v_s(t)$ current source in parallel with $\frac{40}{3} \Omega$.
22. (i) $\frac{200k}{200k+201}v_s(t)$ voltage source in series with $\frac{1}{k+1.005} \text{ k}\Omega$.
(ii) $\frac{200k}{200k+201}v_s(t) \rightarrow v_s(t)$ as $k \rightarrow \infty$; $\frac{1}{k+1.005} \text{ k}\Omega \rightarrow 0 \Omega$
as $k \rightarrow \infty$; $\frac{v_s(t)}{i_s(t)} = (201 + 200k) \text{ k}\Omega$ as $k \rightarrow \infty$.
23. $40v_s(t)$ voltage source in series with $2 \text{ k}\Omega$.
24. $-1.973v_s(t)$ voltage source in series with $2 \text{ k}\Omega$.
25. 2 A
26. 10 V
27. 3 A
28. -2 V; No.
30. 10 V; No
31. (i) $\frac{40}{3} \Omega$; 1.875 W (ii) Efficiency 20%
32. (i) 5 Ω ; 3.94 A (ii) 5 A
33. 2.222 $\text{k}\Omega$; -179.4 V/V
34. (i) 0.169Ω (ii) -33.33 mA
35. There is no positive real value of R for which power transfer goes to a maximum.

Chapter 6

2. $i_L = 100$ mA, $i_s = 91.67$ mA, $i_0 = 16.67$ mA, $v_0 = 10$ V
3. $v_0 = 2$ V, $i_L = 100$ mA, $i_s = 91.67$ mA, $i_0 = 16.67$ mA

4. (ii) $i_L = -11.94$ mA, $i_s = 0.18$ mA, $i_0 = -12.18$ mA, $v_0 = -6.09$ V
7. (i) 0.2 V (ii) 5×10^5 rad/s
8. $v_{01} = -10$ V, $v_{02} = (1.035 \pm 0.011)$ V
9. (ii) $R = 4.7 \text{ k}$, $R_1 = 10 \text{ k}$, $R_2 = 90 \text{ k}$, $R_3 = 470 \text{ k}$, $R_4 = 465 \text{ k}$
will be a satisfactory design. (iii) $\frac{v_{01}}{v_s} = -0.0989$; $\frac{v_{02}}{v_s} = -9.89$
10. 3.02 mV
11. (i) 0.8
12. (ii) 7.6 V
13. -0.9988

Chapter 7

1. (i) 13.555, 250 Hz, 500π rad/s, -72.62° (ii) 2.705, 60 Hz, 120π rad/s, -112.43° (iii) 7.3107, 25 Hz, 50π rad/s, 25.27° .
2. (i) 90° , -15° , $y(t)$ lags $x(t)$ by 105° (ii) 64.3° , 15° , $y(t)$ lags $x(t)$ by 49.3° (iii) 20.1° , undefined, undefined.
3. 0.4 s, 2.5 Hz
4. -17 W, nature of element cannot be decided.
5. (i) No (ii) Must be a composite element. (iii) 10.61 W (iv) No, 10.38 W
6. (ii) 0 W
7. 0 W
8. 3.33 W
9. 13.33 W
10. (i) 0.8 J (ii) 320 ms
11. (i) 0 W (ii) 0.8 J
12. (i) 433 W (ii) 2.106 J (iii) 486.4 ms
13. (i) 6.25 A; 8.9 A (ii) 300 W; 292.1 W (iii) DC, 6.25 A
14. (i) 31.25 W (ii) Cyclic average = 12.5 A, RMS value = 12.6 A; Cyclic average = 2.5 V, RMS value = 3.536 V.
15. 0.25
16. Cyclic average = 2.5 V; RMS Value = 4.1 V
17. 0 W
18. 25 W
19. 1.066; 1.1921
20. (i) 7.07 V (ii) 11.1 V (iii) 5.55 V
21. 2.6; 3.873
22. 90 W
23. 120 W
24. 62.84 V
25. 152.4 W
26. 10 W

Chapter 8

1. $649.9 \angle -0.82^\circ \Omega$; $0.00154 \angle -0.82^\circ$ S and $181.86 \angle -1.2^\circ \Omega$; $0.0056 \angle 1.2^\circ$ S.
2. (i) $(200 + j188.5) \Omega$, $274.83 \angle 43.3^\circ \Omega$ (ii) 377.65Ω in parallel with 1.063 H inductor. (iii) $250.15 \angle 48.5^\circ \Omega$, No
3. (i) $(13.15 - j49.56) \Omega$, $51.3 \angle -75.1^\circ \Omega$ (ii) 13.15Ω resistor in series with 54 μ F capacitor (iii) $60.91 \angle -77.5^\circ \Omega$, No.
4. $Z = (10 + j17.32) \Omega$, Series equivalent $10 \Omega + 45.94 \text{ mH}$, Parallel equivalent $40 \Omega // 61.3 \text{ mH}$, No.
5. (i) $7.78 \angle -45^\circ$ A, $7.78 \angle -165^\circ$ A, $7.78 \angle 75^\circ$ A

- (ii) $(605 + j605.14)$ VA each at 0.707 lag power factor
 (iii) 0 V_{rms} (iv) 1815.4 W, 1815.4 Var (v) 1815.4 W
6. Same as in Problem-5
7. $7.78\angle-45^\circ \text{ A}$, $7.78\angle-165^\circ \text{ A}$, $7.78\angle75^\circ \text{ A}$, 0 A.
8. (a) $4.92\angle26.57^\circ \text{ A}$, $7.78\angle-75^\circ \text{ A}$, $7.78\angle165^\circ \text{ A}$
 $I_n = 3.48\angle-108.4^\circ \text{ A}$
 (i) Source V_1 : $(484.04 - j242.07)$ VA at 0.894 lead, Source V_2 : $(605.14 - j605.14)$ VA at 0.707 lead, Source V_3 : $(605.14 - j605.14)$ VA at 0.707 lead (ii) $[1694 - 382.6 \cos(240\pi t + 71.56^\circ)] \text{ W}$
 [Note : All the three capacitors in the question have 265 μF value]
9. (i) $18.2\angle-31^\circ \text{ A}$, $19.6\angle-163.2^\circ \text{ A}$, $15.3\angle78.1^\circ \text{ A}$
 (ii) $(3586.13 + j2154.76)$ VA, $(3279.65 + j3079.79)$ VA, $(2625.6 + j2353.34)$ VA (iii) $28.66\angle27^\circ \text{ A}$ V_{rms} (iv) 9.49 kW, 7.59 kVar (v) $[9491.2 + 847.5 \cos(200\pi t + 41.8^\circ)] \text{ W}$
10. (i) $8.4\angle-33.3^\circ \text{ A}$, $8.65\angle175.55^\circ \text{ A}$, $6.35\angle57^\circ \text{ A}$ (ii) $(1615 + j1060.4)$ VA at 0.836 lag, $(844.5 + j603.5)$ VA at 0.814 lag, $(420.4 + j557.8)$ VA 0.602 lag (iii) $2.66\angle66.2^\circ \text{ V}_{\text{rms}}$
 (iv) $2.32\angle36.8^\circ \text{ A}_{\text{rms}}$
11. $2.87\angle34.3^\circ \text{ V}_{\text{rms}}$ voltage source in series with $0.93\angle32^\circ \Omega$ impedance, $2.87\angle34.3^\circ \text{ A}_{\text{rms}}$
12. (i) 20 Ω , 151.25 W (ii) No change, the reactance will have to be zero.
13. $8.23\angle-17^\circ \text{ A}_{\text{rms}}$
14. An ideal independent current source of $10\angle-90^\circ \text{ A}_{\text{rms}}$.
15. (i) $44.44\angle25.84^\circ \text{ A}_{\text{rms}}$ in $2.25\angle-25.84^\circ \Omega$ and $111.83\angle-57^\circ \text{ A}_{\text{rms}}$ in $0.894\angle57^\circ \Omega$ (ii) 0.545 lag; 9.38 kVar
 (iii) $39.64\angle5.25^\circ \text{ A}_{\text{rms}}$ at 0.996 lead delivering $(3.95 - j0.363)$ kVA, $(3.18 - j1.54)$ kVA goes into $2.25\angle-25.84^\circ \Omega$ and $(0.77 + j1.18)$ kVA goes into $0.894\angle57^\circ \Omega$.
16. $\frac{89.5}{\sqrt{2}}\angle-123.1^\circ \text{ V}_{\text{rms}}$ voltage source in series with $25.7\angle-88.3^\circ \Omega$ impedance.
17. (ii) Complex power delivered to $Z_1 = (36.6 + j48.8)$ VA;
 Complex power delivered to $Z_2 = (44.8 - j36.6)$ VA;
 Complex power delivered to $Z_3 = (122 + j121.9)$ VA.
18. $(14.7 + j14.7)$ VA
19. (i) $227.2\angle-0.33^\circ \text{ V}_{\text{rms}}$, $10.7\angle-45.33^\circ \text{ A}_{\text{rms}}$
 (ii) $(1.73 + j1.75)$ kVA, 0.7 lag, $(1.72 + j1.72)$ kVA, 99.5%.
20. (i) $108.05\angle0.22^\circ \text{ V}_{\text{rms}}$, $9.7\angle-63.7^\circ \text{ A}_{\text{rms}}$ (ii) 239 μF (iii) $(485 - j118)$ VA at 0.972 lead at source, $(484 - 122.1)$ VA at load, efficiency without capacitor = 97.46%, with capacitor = 99.8%.
21. $16.75\angle56.9^\circ \Omega$
22. (i) $i_1(t) = 13.1 \cos(120\pi t - 27.9^\circ) \text{ A}$; $i_2(t) = 4.02 \cos(120\pi t + 118.5^\circ) \text{ A}$ (ii) $11.88\angle72.9^\circ \Omega$ and $38.67\angle-73.5^\circ \Omega$
23. (i) 2.07 kVA, 1.43 kVar, 0.725 lag (ii) 88 μF (iii) $6.52 \text{ A}_{\text{rms}}$.
24. $(6.79 + j3.77)$ VA
25. 100 rad/s
26. Complex power delivered by first source = $(0.575 + j0.791)$ VA at 0.59 lag; Complex power delivered by second source = $(0.378 + j0.245)$ VA at 0.84 lag; Current delivered by first source = $0.9314\angle-49^\circ \text{ A}_{\text{rms}}$; Current delivered by second source = $0.45\angle-29^\circ \text{ A}_{\text{rms}}$; Voltage at node-C = $0.9762\angle1.8^\circ \text{ V}_{\text{rms}}$.
27. (i) 0 VAr (ii) $(0.612 + j0.543)$ VA and $(0.388 - 0.467)$ VA
 (iii) 1.66°

28. (i) 2 (ii) $110 \cos(120\pi t - 89.8^\circ) \text{ mA}$; $55 \cos(120\pi t + 91.9^\circ) \text{ mA}$; $0.566 \cos(120\pi t + 91.9^\circ) \text{ V}$.
29. 63.125 rad/s, 70.71 rad/s and 83.03 rad/s.
30. Power dissipated in the 100 Ω primary side resistor = 5.15 W, Power dissipated in the 100 Ω secondary side resistor = 19.3 W.
31. 0.5
32. 0.1826; $\sim 2 \text{ W}$.
33. 1:4 ($n = 0.25$); 0.955 μF in the secondary, in series with the load.

Chapter 9

- $I_R = 10\sqrt{5}\angle-56.6^\circ \text{ A}_{\text{rms}}$
1. (i) $I_Y = 10\sqrt{5}\angle-165^\circ \text{ A}_{\text{rms}}$ (ii) 8 kW, 2.15 kVar
 $I_B = 10\sqrt{5}\angle86.6^\circ \text{ A}_{\text{rms}}$
 $Z_{RY} = (20\sqrt{3} + j20) \Omega$
 (iii) $Z_{YB} = 40 \Omega$
 $Z_{BR} = (27.32 + j7.32) \Omega$
2. $V_{BR} = 235.31\angle103^\circ \text{ V}_{\text{rms}}$, $V_{YN} = 100\angle136.8^\circ \text{ V}_{\text{rms}}$,
 $V_{BN} = 237.51\angle74.9^\circ \text{ V}_{\text{rms}}$
3. $I_Y = 7.82\angle-169^\circ \text{ A}_{\text{rms}}$, $I_{BY} = 3.64\angle-24.17^\circ \text{ A}_{\text{rms}}$,
 $I_{RB} = 11.83\angle122.2^\circ \text{ A}_{\text{rms}}$.
4. 4392.5 W, -4314.8 VAr
5. (i) 4063.5 W, 819.3 VAr (capacitive)
 (ii) $Z_{RY} = (22 - j39.63) \Omega$
 $Z_{YB} = (17.45 - j5.53) \Omega$
 $Z_{BR} = (29.35 + j11.62) \Omega$
6. (i) $I_R = 9.68\angle26.57^\circ \text{ A}_{\text{rms}}$, $I_Y = 9.68\angle-93.43^\circ \text{ A}_{\text{rms}}$
 (ii) $(21.34 + j10.67) \Omega$, $I_B = 9.68\angle146.57^\circ \text{ A}_{\text{rms}}$
 $I_B = 9.68\angle146.57^\circ \text{ A}_{\text{rms}}$ per phase.
 (iv) $W_1 = 3865 \text{ W}$, $W_2 = 2134 \text{ W}$
7. $(9.07 + j9.07) \text{ KVA}$
8. Line currents are $34.64\angle6.87^\circ \text{ A}_{\text{rms}}$, $34.64\angle-113.13^\circ \text{ A}_{\text{rms}}$ and $34.64\angle126.87^\circ \text{ A}_{\text{rms}}$. Load branch currents are $20\angle36.87^\circ \text{ A}_{\text{rms}}$, $20\angle-83.14^\circ \text{ A}_{\text{rms}}$ and $20\angle156.87^\circ \text{ A}_{\text{rms}}$. Three-phase complex power delivered to load = $(9.6 - j7.2) \text{ kVA}$.
9. Line currents are $24.36\angle69.26^\circ \text{ A}_{\text{rms}}$, $24.36\angle170.73^\circ \text{ A}_{\text{rms}}$ and $24.36\angle50.73^\circ \text{ A}_{\text{rms}}$. Load branch currents are $14.08\angle-39.3^\circ \text{ A}_{\text{rms}}$, $14.08\angle-159.3^\circ \text{ A}_{\text{rms}}$ and $14.08\angle80.7^\circ \text{ A}_{\text{rms}}$. Three-phase complex power delivered to load = $(4.75 - j3.56) \text{ kVA}$.
10. Line currents are $6.67\angle53.13^\circ \text{ A}_{\text{rms}}$, $11.55\angle-96.87^\circ \text{ A}_{\text{rms}}$ and $11.55\angle143.13^\circ \text{ A}_{\text{rms}}$. Load branch currents are same as line currents. Three-phase complex power delivered to load = $(4.4 - j6.8) \text{ kVA}$.
11. Line currents are $15.36\angle3.81^\circ \text{ A}_{\text{rms}}$, $15.36\angle-116.19^\circ \text{ A}_{\text{rms}}$ and $15.36\angle123.81^\circ \text{ A}_{\text{rms}}$. Load branch currents in Y-connected load are $15.36\angle40.7^\circ \text{ A}_{\text{rms}}$, $15.36\angle-79.3^\circ \text{ A}_{\text{rms}}$ and $15.36\angle160.7^\circ \text{ A}_{\text{rms}}$. Load branch currents in Δ -connected load are $5.61\angle-37.73^\circ \text{ A}_{\text{rms}}$, $5.61\angle-157.73^\circ \text{ A}_{\text{rms}}$ and $5.61\angle82.27^\circ \text{ A}_{\text{rms}}$. Three-phase complex power delivered to Y-connected load = $(7.08 - j7.08)$.
 Three-phase complex power delivered to Δ -connected load = $(2.83 + j5.66) \text{ kVA}$.
 Three-phase complex power delivered by the source = $(10.62 - j0.71)$ at 0.998 lead power factor.

12. (i) $17.52\angle 72.4^\circ \text{ V}_{\text{rms}}$ (ii) Line currents are $9.22\angle -21.22^\circ \text{ A}_{\text{rms}}$, $7.8\angle -127.3^\circ \text{ A}_{\text{rms}}$ and $10.3\angle 112.04^\circ \text{ A}_{\text{rms}}$. Load Line voltages are $136.23\angle 12.2^\circ \text{ V}_{\text{rms}}$, $137.54\angle -97.2^\circ \text{ V}_{\text{rms}}$ and $158.35\angle 137.1^\circ \text{ V}_{\text{rms}}$. Load Phase voltages are $92.15\angle -21.22^\circ \text{ V}_{\text{rms}}$, $78\angle -127.3^\circ \text{ V}_{\text{rms}}$ and $80.3\angle 112.04^\circ \text{ V}_{\text{rms}}$. (iii) Active power delivered to $10\ \Omega = 849.1\text{ W}$, Active power delivered to $15\ \Omega = 912.6\text{ W}$, Active power delivered to $8\ \Omega = 847.6\text{ W}$, Total active power delivered = 2609.3 W .
13. (i) $5.5\angle 107.94^\circ \text{ V}_{\text{rms}}$ (ii) Line currents are $9.7\angle -15.16^\circ \text{ A}_{\text{rms}}$, $7\angle -127^\circ \text{ A}_{\text{rms}}$ and $10.85\angle 105.5^\circ \text{ A}_{\text{rms}}$. Load Line voltages are $167.37\angle 20.5^\circ \text{ V}_{\text{rms}}$, $172.2\angle -103.4^\circ \text{ V}_{\text{rms}}$ and $159.8\angle 137^\circ \text{ V}_{\text{rms}}$. Load Phase voltages are $97\angle -15.16^\circ \text{ V}_{\text{rms}}$, $105\angle -127^\circ \text{ V}_{\text{rms}}$ and $86.8\angle 105.5^\circ \text{ V}_{\text{rms}}$. (iii) Active power delivered to $10\ \Omega = 940.9\text{ W}$, Active power delivered to $15\ \Omega = 735\text{ W}$, Active power delivered to $8\ \Omega = 941.8\text{ W}$, Total active power delivered = 2618 W . Neutral wire current towards source neutral = $3.22\angle 45.9^\circ \text{ A}_{\text{rms}}$.
14. (i) Line currents are $11.72\angle -32.51^\circ \text{ A}_{\text{rms}}$, $10.6\angle -147.6^\circ \text{ A}_{\text{rms}}$ and $12.02\angle 94.5^\circ \text{ A}_{\text{rms}}$. (ii) Load Line voltages are $182\angle 34.3^\circ \text{ V}_{\text{rms}}$, $187.4\angle -78.20^\circ \text{ V}_{\text{rms}}$ and $205.3\angle 156.8^\circ \text{ V}_{\text{rms}}$. (iii) Complex power delivered by R-phase = $(1136.6 + j724.4)\text{ VA}$, Complex power delivered by Y-phase = $(965.8 + j743.8)\text{ VA}$, Complex power delivered by B-phase = $(1051.4 + j898)\text{ VA}$. Three-phase active power delivered = 3.154 kW , Three-phase reactive power delivered = 2.366 kVAr . (iv) 3.074 kW , 2.246 kVAr (inductive).
15. (i) Line currents are $20\angle -26.76^\circ \text{ A}_{\text{rms}}$, $16.22\angle -175.73^\circ \text{ A}_{\text{rms}}$ and $10.34\angle 99.24^\circ \text{ A}_{\text{rms}}$. (ii) Load branch currents are $9.87\angle -5.3^\circ \text{ A}_{\text{rms}}$, $6.72\angle -161.5^\circ \text{ A}_{\text{rms}}$ and $11.4\angle 134.8^\circ \text{ A}_{\text{rms}}$. Source branch are $12.6\angle -20^\circ \text{ A}_{\text{rms}}$, $7\angle -128^\circ \text{ A}_{\text{rms}}$ and $7.6\angle 142^\circ \text{ A}_{\text{rms}}$. (iii) Load Line voltages are $98.7\angle -5.3^\circ \text{ V}_{\text{rms}}$, $101.04\angle -113^\circ \text{ V}_{\text{rms}}$ and $117.7\angle 119.9^\circ \text{ V}_{\text{rms}}$. (iv) Complex power delivered by RY source = $(1362.7 + j496)\text{ VA}$, Complex power delivered by YB source = $(732.3 + j238)\text{ VA}$, Complex power delivered by BR source = $(886.7 - j271.1)\text{ VA}$, Total active power delivered by source = 2972.7 W , Total reactive power delivered by source = 462.9 VAr (Inductive). (v) Total active power delivered to load 2731.6 W , Total reactive power delivered to load 179 VAr (Inductive).
16. (i) Load Line voltages are $107.14\angle 30.4^\circ \text{ V}_{\text{rms}}$, $110.44\angle -76^\circ \text{ V}_{\text{rms}}$ and $130.38\angle 156.1^\circ \text{ V}_{\text{rms}}$. Line currents are $21.16\angle -36.5^\circ \text{ A}_{\text{rms}}$, $17.42\angle -149^\circ \text{ A}_{\text{rms}}$ and $21.66\angle 95.5^\circ \text{ A}_{\text{rms}}$. (ii) 3254.3 W , 2440.7 VAr (iii) $3200\text{ W} + 54.04\text{ W} = \sim 3254\text{ W}$; $2400\text{ VAr} + 40.5\text{ VAr} = 2440.5\text{ VAr}$.
17. (i) Load Line voltages are $144.3\angle -13.8^\circ \text{ V}_{\text{rms}}$, $176.6\angle -115.6^\circ \text{ V}_{\text{rms}}$ and $204\angle 108.3^\circ \text{ V}_{\text{rms}}$. Load Phase voltages are $102\angle -48.2^\circ \text{ V}_{\text{rms}}$, $83.3\angle -150^\circ \text{ V}_{\text{rms}}$ and $117.8\angle 88^\circ \text{ V}_{\text{rms}}$. Source Line voltages are $144.3\angle -13.8^\circ \text{ V}_{\text{rms}}$, $176.6\angle -115.6^\circ \text{ V}_{\text{rms}}$ and $204\angle 108.3^\circ \text{ V}_{\text{rms}}$. Source Phase voltages are $100.43\angle -53.8^\circ \text{ V}_{\text{rms}}$, $93.3\angle -150^\circ \text{ V}_{\text{rms}}$ and $112.8\angle 92.3^\circ \text{ V}_{\text{rms}}$. (ii) 2495.6 W ; -1874 VAr .
18. (i) Load Line voltages are $140.5\angle 58.9^\circ \text{ V}_{\text{rms}}$, $172.1\angle -42.9^\circ \text{ V}_{\text{rms}}$ and $198.7\angle -179.1^\circ \text{ V}_{\text{rms}}$. Load Phase voltages are $99.35\angle 24.5^\circ \text{ V}_{\text{rms}}$, $81.13\angle -77.3^\circ \text{ V}_{\text{rms}}$ and $114.7\angle 160.7^\circ \text{ V}_{\text{rms}}$. Source Line voltages are $142.3\angle 60^\circ \text{ V}_{\text{rms}}$, $176.64\angle -41.8^\circ \text{ V}_{\text{rms}}$ and $204\angle -178^\circ \text{ V}_{\text{rms}}$. Source Phase voltages are $111.2\angle 27.7^\circ \text{ V}_{\text{rms}}$, $77.8\angle -70.3^\circ \text{ V}_{\text{rms}}$ and $114.4\angle 157.1^\circ \text{ V}_{\text{rms}}$. (ii) 2465 W ; 1779 VAr .
19. (i) Load Line voltages are $140.5\angle 58.9^\circ \text{ V}_{\text{rms}}$, $172.1\angle -42.9^\circ \text{ V}_{\text{rms}}$ and $198.7\angle -179.1^\circ \text{ V}_{\text{rms}}$. Load Phase voltages are $99.58\angle 21.7^\circ \text{ V}_{\text{rms}}$, $85.9\angle -76.6^\circ \text{ V}_{\text{rms}}$ and $111.3\angle 162.45^\circ \text{ V}_{\text{rms}}$. Source Line voltages are $142.3\angle 60^\circ \text{ V}_{\text{rms}}$, $176.64\angle -41.8^\circ \text{ V}_{\text{rms}}$ and $204\angle -178^\circ \text{ V}_{\text{rms}}$. Source Phase voltages are $102.5\angle 22.6^\circ \text{ V}_{\text{rms}}$, $88.55\angle -75.3^\circ \text{ V}_{\text{rms}}$ and $113.8\angle 163.7^\circ \text{ V}_{\text{rms}}$. (ii) 2470 W ; 1782.8 VAr .
20. Zero sequence component = 0 V , Positive sequence component = $396.5\angle 2.47^\circ \text{ V}_{\text{rms}}$, Negative sequence component = $17.54\angle -77.2^\circ \text{ V}_{\text{rms}}$.
21. $23.36\text{ V}_{\text{rms}}$
22. Zero sequence component = 0 V , Positive sequence component = $325.76\angle -10.23^\circ \text{ V}_{\text{rms}}$, Negative sequence component = $15\angle 17.45^\circ \text{ V}_{\text{rms}}$. Line voltages are $339.1\angle -9.05^\circ \text{ V}_{\text{rms}}$, $325.5\angle -132.9^\circ \text{ V}_{\text{rms}}$ and $313.2\angle 111.24^\circ \text{ V}_{\text{rms}}$.

Chapter 10

1. (i) $i(t) = \begin{cases} 2(1 - e^{-10t})\text{ A}; & 0 \leq t \leq 0.12\text{ s} \\ (4 - 2.6e^{-5(t-0.12)})\text{ A}; & 0.12 < t \end{cases}$
- (ii) $v_L(t) = \begin{cases} 10e^{-10t}\text{ V}; & 0 \leq t \leq 0.12\text{ s} \\ 6.5e^{-5(t-0.12)}\text{ V}; & 0.12 < t \end{cases}$
- (iii) $v(t) = \begin{cases} 10(1 - e^{-10t})\text{ V}; & 0 \leq t \leq 0.12\text{ s} \\ 10 - 6.5e^{-5(t-0.12)}\text{ V}; & 0.12 < t \end{cases}$
- (iv) 0.494 s
2. $i_L(t) = \left(3 - \frac{7}{3}e^{-4t}\right)u(t)\text{ A}$, $v_L(t) = \frac{14}{3}e^{-4t}u(t)\text{ V}$
3. (ii) 69.74 Hz (iii) $(i_L(t))_{\text{av}} = 10.85\text{ A}$, $(i_{s1}(t))_{\text{av}} = 7.58\text{ A}$, $(i_{s2}(t))_{\text{av}} = -3.29\text{ A}$, (iv) 91 W ; 79 W (v) 86.8%
4. (ii) 54.55 Hz
- (iii) $(i_L(t))_{\text{av}} = 10.88\text{ A}$, $(i_{s1}(t))_{\text{av}} = 10.88\text{ A}$, $(i_{s2}(t))_{\text{av}} = 4.94\text{ A}$, (iv) 118.6 W ; 130.6 W (v) 90.8%
5. (ii) 54.55 Hz (iii) $(i_L(t))_{\text{av}} = 10.88\text{ A}$, $(i_{s1}(t))_{\text{av}} = 5.933\text{ A}$, $(i_{s2}(t))_{\text{av}} = 10.88\text{ A}$ (iv) 130.6 W ; 142.4 W (v) 91.7%
6. $i_L(t) = 2e^{-100t}u(t)\text{ A}$, $v_L(t) = -20e^{-100t}u(t)\text{ V}$
7. $\frac{\tau}{T} = 0.1$
8. $k = 15$
9. $k = 3.43$
10. $v(t) = 12.5e^{-100t}u(t)\text{ V}$
11. (i) Zero state response: $i_L(t) = (1 - e^{-100t})u(t)\text{ A}$, $v(t) = 0\text{ V}$
Zero-input response = 0 ,
Total response = $i_L(t) = (1 - e^{-100t})u(t)\text{ A}$, $v(t) = 0\text{ V}$
- (ii) $i_L(t) = 0\text{ A}$, $v(t) = 50u(t)\text{ V}$
12. $k = -0.09\text{ volt-sec}$
13. (i) $R_1 = 40\ \Omega$, $R_2 = 160\ \Omega$, $L = 0.03\text{ H}$
- (ii) $i(t) = 6.25e^{-1000t}u(t)\text{ mA}$
14. 400 W , 400 W
15. (i) $v_0(t) = 10 + 1.264\sin(314t - 82.74^\circ)\text{ V}$
- (ii) 25.3% (iii) 6.4%
16. (iii) 60 V , 1.28 V ; 50 V , 0.5 V ; 63.7 V , 1.7 V
17. $v_0(t) = (-5e^{-500t} - 30e^{-1000t})u(t)\text{ V}$;

$$18. V_1 = V e^{-\left(\frac{T-T_1}{2\tau}\right)} \frac{e^{\frac{T_1}{2\tau}} - e^{-\frac{T_1}{2\tau}}}{e^{\frac{T}{2\tau}} - e^{-\frac{T}{2\tau}}};$$

$$V_2 = V e^{\left(\frac{T-T_1}{2\tau}\right)} \frac{e^{\frac{T_1}{2\tau}} - e^{-\frac{T_1}{2\tau}}}{e^{\frac{T}{2\tau}} - e^{-\frac{T}{2\tau}}};$$

$$\text{For } \frac{\tau}{T} \gg 1,$$

$$V_1 = \left(V \frac{T_1}{T} \right) \left(1 - \frac{T-T_1}{2\tau} \right); V_2 = \left(V \frac{T_1}{T} \right) \left(1 + \frac{T-T_1}{2\tau} \right)$$

Chapter 11

$$1. (i) v_c(t) = \begin{cases} 10(1 - e^{-9.091t}) \text{ V}; 0^+ \leq t \leq 0.12^- \\ (10 - 4.03e^{-18.182(t-0.12)}) \text{ V}; 0.12^+ \leq t \end{cases}$$

$$(ii) i_c(t) = \begin{cases} 2e^{-9.091t} \text{ mA}; 0^+ \leq t \leq 0.12^- \\ 6.5e^{-18.182t} \text{ mA}; 0.12^+ \leq t \end{cases}$$

$$(iii) v(t) = \begin{cases} 10e^{-10t} \text{ V}; 0^+ \leq t \leq 0.12^- \\ 4.03e^{-18.12(t-0.12)} \text{ V}; 0.12^+ \leq t \end{cases}$$

$$2. v_c(t) = (18 - 6e^{-277.8t})u(t) \text{ V}$$

$$i_c(t) = 1e^{-277.8t}u(t) \text{ A}$$

$$3. v_c(t) = 10e^{-t}u(t) \text{ V}$$

$$i_c(t) = 10e^{-t}u(t) \text{ mA}$$

$$4. v(t) = 0.33e^{-6.67t}u(t) \text{ V}$$

$$6. (ii) T \geq 10RC$$

$$7. k = -0.25$$

$$8. v_c(t) = \begin{cases} 10(t - 0.3(1 - e^{-3.33t})) \text{ V}; 0 \leq t \leq 1 \\ 7.107e^{-3.33(t-1)} \text{ V}; 1 < t \end{cases}$$

$$9. 0.6 \text{ ms}, 1 \text{ ms}; 0.5 \text{ ms}, 0.67 \text{ ms}$$

$$10. (a) 0.15 \text{ mW}, 0.03 \text{ mA} \quad (b) 15 \text{ mW}, 3 \text{ mA} \quad (c) 1.35 \text{ mW}, 0.09 \text{ mA} \quad (d) 135 \text{ mW}, 9 \text{ mA}$$

$$11. (i) I_1 = I_0 e^{-\left(\frac{T-T_1}{2\tau}\right)} \frac{e^{\frac{T_1}{2\tau}} - e^{-\frac{T_1}{2\tau}}}{e^{\frac{T}{2\tau}} - e^{-\frac{T}{2\tau}}};$$

$$I_2 = I_0 e^{\left(\frac{T-T_1}{2\tau}\right)} \frac{e^{\frac{T_1}{2\tau}} - e^{-\frac{T_1}{2\tau}}}{e^{\frac{T}{2\tau}} - e^{-\frac{T}{2\tau}}};$$

$$\text{For } \frac{\tau}{T} \gg 1,$$

$$I_1 = \left(I_0 \frac{T_1}{T} \right) \left(1 - \frac{T-T_1}{2\tau} \right); I_2 = \left(I_0 \frac{T_1}{T} \right) \left(1 + \frac{T-T_1}{2\tau} \right)$$

$$(ii) [i_R(t)]_{\text{av}} = dI_0; \left(d = \frac{T_1}{T} \right)$$

$$[v_0(t)]_{\text{av}} = dRI_0$$

$$(iii) \text{Peak-to-peak ripple} = dRI_0(1-d)\frac{T}{\tau},$$

$$\text{ripple factor} = (1-d)\frac{T}{\tau}$$

$$12. \% \text{ tilt} = 50\frac{T}{\tau}\%$$

$$13. (iii) 60 \text{ V}, 1.6 \text{ V}; 50 \text{ V}, 0.625 \text{ V}; 63.7 \text{ V}, 2.1 \text{ V}$$

$$15. v_0(t) = [1.25\sin(\omega t - 51.34^\circ) + 0.31\sin(3\omega t + 104.9^\circ) + 0.126\sin(5\omega t - 80.9^\circ)] \text{ V}$$

$$16. v_0(t) = (-19.41e^{-704.4t} + 64.13e^{-5546t})u(t) \text{ V}$$

$$18. 34.7 \text{ ms}$$

$$19. (i) \frac{d^2 v_0}{dt^2} + \left(\frac{1}{R_2 C} + \frac{R_1}{L} \right) \frac{dv_0}{dt} + \left(1 + \frac{R_1}{R_2} \right) v_0 = \frac{v_s}{LC}; t \geq 0^+$$

$$(ii) \omega_n = 3240 \text{ rad/s}; \xi = 0.2314$$

$$(iii) v_0(t) = [10.03e^{-750t} \cos(3152.42t + 4.5^\circ)]u(t) \text{ V}$$

$$20. \text{Maximum \% overshoot is } 74.6\% \text{ and maximum value of output reaches } 20.8 \text{ V}$$

$$21. \text{Maximum \% overshoot is } 59.5\% \text{ and maximum value of output reaches } 18.7 \text{ V}$$

$$22. (i) L = 0.796 \text{ mH}, C = 32 \mu\text{F} \quad (ii) 0.267 \text{ ms}, 4.78 \text{ ms}, 0.5 \text{ ms}, 72.9\%$$

$$23. (i) L = 2.48 \text{ mH}, C = 2.56 \mu\text{F}, R = 10 \Omega \quad (ii) 52.25 \Omega$$

$$(iii) v_R(t) = 0.25 \times 10^6 te^{-12550t} u(t) \text{ V}$$

$$24. (i) \omega_n = \pi \times 10^4 \text{ rad/s}; \omega_d \approx \pi \times 10^4 \text{ rad/s}$$

$$\xi = 0.02; Q = 25;$$

$$(ii) 250 \text{ V}, 250 \text{ V} \quad (iii) \text{Capacitor voltage: } 250.05 \text{ V at } 4998 \text{ Hz, Inductor voltage: } 250.05 \text{ V at } 5002 \text{ Hz}$$

$$25. (a) v_L(t) = 1.00915 \sin(2 \times 10^4 \pi t + 2.5^\circ) \text{ V}$$

$$v_C(t) = 0.01 \sin(2 \times 10^4 \pi t - 177.5^\circ) \text{ V}$$

$$(b) v_L(t) = 0.01 \sin(2 \times 10^4 \pi t + 177.5^\circ) \text{ V}$$

$$v_C(t) = 1.00915 \sin(2 \times 10^4 \pi t - 2.5^\circ) \text{ V}$$

$$26. v_C(t) = [51.24e^{-250t} \cos(312.25t - 38.67^\circ)]u(t) \text{ V}$$

$$i_L(t) = [1.025e^{-250t} \cos(312.25t + 12.64^\circ)]u(t) \text{ A}$$

$$27. v_C(t) = [10 + 0.455e^{-1916.52t} - 10.455e^{-83.492t}]u(t) \text{ V}$$

$$i_L(t) = [1 + 0.0019e^{-1916.52t} - 1.0019e^{-83.492t}]u(t) \text{ A}$$

$$28. v_C(t) = [16666.67te^{-833.33t}]u(t) \text{ V}$$

$$29. (i) L = 0.08 \text{ mH}, C = 0.8 \mu\text{F}, Q \text{ of inductor} = 99.5$$

$$(ii) 111.11 \Omega$$

$$30. (a) i_C(t) = 2 \cos 1.6 \times 10^4 \pi t \text{ A}$$

$$i_L(t) \approx 0 \text{ A}$$

$$(b) i_C(t) \approx 0 \text{ A}$$

$$i_L(t) = 2 \cos 160 \pi t \text{ A}$$

Chapter 12

$$1. (i) \frac{di_1}{dt} + 50i_1 - 50i_2 = 5v_s(t)$$

$$\frac{di_2}{dt} + 40i_2 - 20i_1 = 0$$

$$(ii) \frac{d^2 v_0}{dt^2} + 90 \frac{dv_0}{dt} + 1000v_0 = 2.5 \frac{dv_s}{dt}; t \geq 0^+, \text{ Natural}$$

$$\text{frequencies are } -13 \text{ nepers/sec and } -77 \text{ nepers/sec}$$

$$(iii) v_0(t) = (107e^{-77t} - 57e^{-13t})u(t) \text{ V}$$

$$(iv) v_0(t) = (6.25e^{-77t} - 6.25e^{-13t})u(t) \text{ V}$$

$$(v) \text{ No}$$

- (vi) $i_1(t) = (0.65e^{-13t} + 0.15e^{-77t} + 0.2)u(t)$ A
2. (i) $\frac{dv_1}{dt} + 2v_1 - v_2 = \frac{dv_s(t)}{dt}$
 $\frac{dv_2}{dt} + v_2 - v_1 = 0$
- (ii) $\frac{d^2v_0}{dt^2} + 3\frac{dv_0}{dt} + v_0 = \frac{dv_s}{dt}; t \geq 0^+$, Natural frequencies are -0.382 nepers/sec and -2.618 nepers/sec.
- (iii) $v_0(t) = (0.362e^{-0.382t} + 0.138e^{-2.618t})u(t)$ V
- (iv) $v_0(t) = (0.98e^{-0.382t} - 1.48e^{-2.618t})u(t)$ V
- (v) $i_{C1}(t) = (4.93e^{-0.382t} + 1.57e^{-2.618t})u(t)$ mA
3. (i) $(L_1, L_2) = (0.19 \text{ H}, 0.04 \text{ H})$ or $(0.021 \text{ H}, 0.378 \text{ H})$
(ii) 0 V
4. (i) $\frac{d^2v_C}{dt^2} + \frac{dv_C}{dt} + 0.5v_C = 0.5v_s; t \geq 0^+$, The natural frequencies are $-0.5 \pm j 0.5$.
(ii) $v_C(t) = [1 - 3.6e^{-0.5t} \cos(0.5t + 56.3^\circ)]u(t)$ V
(iii) $\frac{d^2i_s}{dt^2} + \frac{di_s}{dt} + 0.5i_s = 0.5v_s + \frac{dv_s}{dt}; t \geq 0^+$, The natural frequencies are $-0.5 \pm j 0.5$, $i_{s(0^+)} = 0$ A; $\left. \frac{di_s}{dt} \right|_{0^+} = 1.5$ A/s
5. $v_{c1(0^+)} = -2.5$ V; $\left. \frac{dv_{c1}}{dt} \right|_{0^+} = \frac{5}{8}$ V/s;
 $v_{c2(0^+)} = 2.5$ V; $\left. \frac{dv_{c2}}{dt} \right|_{0^+} = -\frac{5}{8}$ V/s
6. $i_{l(0^+)} = -\frac{20}{3}$ A; $\left. \frac{di_l}{dt} \right|_{0^+} = \frac{100}{3}$ A/s
7. (i) $-0.5698, -0.2151 \pm j1.307$
(ii) $v_o(t) = [1 - 0.9565e^{-0.57t} - 0.4337e^{-0.215t} \cos(1.307t - 84.24^\circ)]u(t)$ V
8. (i) 0, $\pm j\sqrt{2}$ (ii) No (iii) $v_o(t) = 0.5(1 - \cos\sqrt{2}t)u(t)$ V
9. (i) 1.35 V (ii) 0.1015 V (iii) 15 s
10. $[(t-1) - e^{-t}]u(t)$ A
11. (i) $10^{12}te^{-10^6t}u(t)$ V
(ii) $v_o(t) = \{10[u(t) - u(t-2)] - 10e^{-10^6t}[1 + 10^6t]u(t) + 10e^{-10^6(t-0.002)}[1 + 10^6(t-0.002)]u(t-0.002)\}$ V
12. (a) $H(j\omega) = \frac{2}{\sqrt{\omega^2 + 2.25}} \angle \tan^{-1}\left(\frac{\omega}{1.5}\right)$, bandwidth = 0.24 Hz.
 $|H(j\omega)| = \frac{1.061\sqrt{\omega^2 + 6.25}}{\sqrt{(4.25 - \omega^2)^2 + \omega^2}}$
(b) $\angle H(j\omega) = \left\{ \tan^{-1}\left(\frac{\omega}{2.5}\right) - \tan^{-1}\left(\frac{\omega}{4.25 - \omega^2}\right) \right\}$
15. (i) $\frac{d^4v_o}{dt^4} + \frac{d^3v_o}{dt^3} + 3\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = v_s; t \geq 0^+$
(ii) Natural frequencies are $-0.105 \pm j1.5525, 0.395 \pm j0.5067$.
(iii) This part of question is withdrawn since it is difficult to do this part by hand calculation.
16. $\frac{d^2i_x}{dt^2} - \frac{di_x}{dt} + i_x = -(0.5v_s + 0.5\frac{dv_s}{dt} + 0.5\frac{d^2v_s}{dt^2});$
Natural frequencies are $0.5 \pm j 0.866$; Circuit is unstable.
17. (i) Impulse response from v_{s1} location:

$$v_x(t) = (1.2072e^{-3.414t} - 0.2072e^{-0.5858t})u(t) \text{ V}$$

Impulse response from v_{s2} location:

$$v_x(t) = (0.8536e^{-3.414t} + 0.1464e^{-0.5858t})u(t) \text{ V}$$

$$(ii) v_x(t) = 1.414(e^{-0.5858t} - e^{-3.414t})u(t) \text{ V}$$

$$18. (i) \frac{t^2 e^{-t}}{2} u(t) \text{ V} \quad (ii) H(j\omega) = \frac{1}{(1+j\omega)^3}$$

$$19. (i) \text{ Natural frequencies are } -50 \pm j 86.6$$

$$(ii) v_o(t) = [2\delta(t) - 231.2e^{-50t} \cos(86.6t + 150^\circ)]u(t) \text{ V}$$

$$(iii) H(j\omega) = \frac{2(j\omega)^2}{(j\omega)^2 + 100(j\omega) + 100^2}$$

Chapter 13

1. (i) Advance the waveform by $\pi/4$ s.

$$(ii) v(t) = 0.818 \cos(t - \frac{\pi}{4}) - 0.106 \cos(3t - \frac{3\pi}{4}) - 0.0212 \cos(5t - \frac{5\pi}{4}) + 0.015 \cos(7t - \frac{7\pi}{4}) + 0.007 \cos(9t - \frac{9\pi}{4}) + \dots$$

$$2. v(t) = \sum_{n=-\infty}^{\infty} \frac{j16}{\pi^2} \sin \frac{n\pi}{2} \sin^2 \frac{n\pi}{8} e^{jnt}$$

3. (i) Fig. 13.14-3 shows integral of the waveform in Fig. 13.14-2 except for a possible DC component.

$$(ii) v(t) = \sum_{n=-\infty}^{\infty} \frac{8}{\pi} \left(\frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \right) \left(\sin^2 \frac{n\pi}{8} \right) e^{jnt}$$

$$(iii) v(t) = \frac{16}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \left(\frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \right) \left(\sin^2 \frac{n\pi}{8} \right) \cos nt$$

4. (i) The waveform in Fig. 13.14-4 is the integral of waveform in Fig. 13.14-3 except for a possible DC component.

$$(ii) v(t) = -j \sum_{n=-\infty}^{\infty} \frac{8}{\pi} \left(\frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \right) \left(\sin^2 \frac{n\pi}{8} \right) e^{jnt}$$

$$(iii) v(t) = \frac{16}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n} \left(\frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \right) \left(\sin^2 \frac{n\pi}{8} \right) \sin nt$$

$$5. v(t) = \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n} \left(\sin \frac{n\pi}{2} \right) \left(\sin \frac{3n\pi}{8} \right) \left(\cos \frac{n\pi}{8} \right) \sin n\pi t$$

$$6. (ii) \sum_{\substack{n=2 \\ \text{even } n}}^{\infty} \left(\frac{2n}{\pi(n^2-1)} \cos \frac{n\pi}{2} \right) \sin(2n\pi t - \frac{n\pi}{2})$$

$$7. (ii) \frac{2}{\pi} \cos \alpha + \sum_{\substack{k=2 \\ \text{even } k}}^{\infty} \frac{2}{\pi} \left[\frac{\cos(k+1)\alpha}{k+1} - \frac{\cos(k-1)\alpha}{k-1} \right] \cos kt$$

9. The circuit functions as an averaging filter with a gain of T2 for DC content of input signal.

$$10. v_o(t) \approx \frac{40}{\pi^2} \left[\cos(\omega_o t - 64.3^\circ) + \frac{1}{9} \cos(3\omega_o t - 124.1^\circ) + \frac{1}{25} \cos(5\omega_o t - 144.7^\circ) + \frac{1}{49} \cos(7\omega_o t - 154.4^\circ) + \frac{1}{81} \cos(9\omega_o t - 159.9^\circ) \right]$$

where $\omega_o = 2\pi \times 10^3$ rad/sec.

$$11. v_o \approx \begin{bmatrix} -6.84 \cos(\omega_o t + 32.5^\circ) - 0.9 \cos(3\omega_o t + 12^\circ) \\ -0.32 \cos(5\omega_o t + 7.3^\circ) \\ -0.164 \cos(7\omega_o t + 5.2^\circ) \\ -0.1 \cos(9\omega_o t + 4^\circ) + \dots \end{bmatrix}$$

where $\omega_o = 5\pi \times 10^3$ rad/sec.

$$12. v_o \approx \begin{bmatrix} -0.473 \cos(t - 84.3^\circ) + 0.307 \cos(3t - 88.1^\circ) \\ -0.11 \cos(5t - 88.9^\circ) \\ +0.01 \cos(7t - 89.2^\circ) \\ -0.006 \cos(9t - 89.4^\circ) + \dots \end{bmatrix}$$

$$13. \text{(i)} 2 \text{ A}, 12 \text{ V} \text{ (iii)} 4 \text{ A}, 4.8 \text{ W}$$

$$14. v_o(t) = \begin{bmatrix} 10 + 17.5 \cos(2000\pi t) \\ +0.76 \cos(4000\pi t - 86.2^\circ) \end{bmatrix} \text{ V}$$

$$15. P_N = \left\{ 10^6 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{5 \times 10^5}{16(n\pi)^4 + 24(n\pi)^2 + 1} \right\} \text{ W}$$

$$16. \text{(i)} v_o(t) = [312 \sin 100\pi t + 12 \sin 300\pi t - 4 \sin 500\pi t] \text{ V}$$

$$\text{(ii)} 4.05\%$$

$$17. \text{(i)} 10 \text{ A}$$

$$\text{(ii)} i(t) = \left\{ \sum_{n=-\infty}^{\infty} \frac{12.96}{\sqrt{0.096^2 + (300n\pi)^2}} \left(\frac{\sin 0.27\pi n}{0.27\pi n} \right) \times e^{-j0.27n\pi} e^{-\tan^{-1}(3125\pi)} \times e^{j2 \times 10^4 n\pi t} \right\}$$

$$\text{(iii)} \sim 10 \text{ A}, 9.6 \text{ W}$$

$$18. \text{(i)} 10 \text{ A} \text{ (ii)} i(t) \approx 10 \text{ A} \text{ (iii)} 10 \text{ A}, 9.6 \text{ W}, 120 \text{ W} \text{ (iv)} 2.3 \text{ A}$$

$$19. \text{(i)} 228.5 \text{ V} \text{ (ii)} \begin{bmatrix} 3.41 \sin(100\pi t - 20.9^\circ) \\ -0.4 \sin(300\pi t - 16.6^\circ) \\ -0.194 \sin(500\pi t - 21^\circ) \end{bmatrix} \text{ A}$$

$$\text{(iii)} 519 \text{ W}, 555.6 \text{ VA}$$

$$20. v_s(t) = \begin{bmatrix} 100 + (103.75 - j96.21)e^{j40\pi t} \\ + (103.75 + j96.21)e^{-j40\pi t} \\ + (27.7 + j23.4)e^{j120\pi t} \\ + (27.7 - j23.4)e^{-j120\pi t} \end{bmatrix} \text{ V}$$

Chapter 14

$$1. v(t) = [0.5u(t) + u(t-2) - u(t-5) - 0.5u(t-7)] \text{ V}$$

$$V(j\omega) = \left[3 \left(\frac{\sin 1.5\omega}{1.5\omega} \right) + 3.5 \left(\frac{\sin 3.5\omega}{3.5\omega} \right) \right] e^{-j3.5\omega}$$

$$2. \text{(i)} V(j\omega) = \left\{ \left(\frac{500}{\omega^2} \cos 0.003\omega + \frac{1.5}{\omega} \sin 0.003\omega - \frac{500}{\omega^2} \right) + j \left(\frac{1.5}{\omega} \cos 0.003\omega - \frac{500}{\omega^2} \sin 0.003\omega \right) \right\}$$

$$\text{(ii)} 0$$

$$3. V(j\omega) = j \frac{2}{\omega} \left(\cos 0.002\omega - \frac{\sin 0.002\omega}{0.002\omega} \right)$$

$$4. \text{(i)} V_1(j\omega) = \frac{6j\omega}{2+3j\omega}$$

$$\text{(ii)} V_2(j\omega) = \frac{3e^{-2j\omega}}{2-3j\omega}$$

$$\text{(iii)} V_3(j\omega) = \frac{6}{1-6j\omega}$$

$$\text{(iv)} V_4(j\omega) = \frac{3e^{-j3}e^{-5j\omega}}{(2+3j)+3j\omega}$$

$$\text{(v)} V_5(j\omega) = \frac{9(2+3j\omega)}{(2+3j\omega)^2 + (18\pi)^2}$$

$$\text{(vi)} V_6(j\omega) = \frac{3e^{-j0.75\omega}}{8+3j\omega}$$

$$5. v(t) = \frac{1}{2\pi} \left\{ \frac{\sin(\frac{\pi}{2} + t)}{(\frac{\pi}{2} + t)} + \frac{\sin(\frac{\pi}{2} - t)}{(\frac{\pi}{2} - t)} \right\}$$

$$6. \text{(i)} v(t) = \frac{4}{\pi} \left(\frac{1}{t^2 - 2t + 5} \right)$$

$$\text{(ii)} v(t) = \frac{1}{2} \left\{ \frac{\sin \pi(t+2)}{\pi(t+2)} - \frac{\sin \pi(t-2)}{\pi(t-2)} \right\}$$

$$\text{(iii)} v(t) = (0.5e^{-t} - 1.5e^{-3t})u(t)$$

$$\text{(iv)} v(t) = (2e^{-3t} - 5te^{-3t})u(t)$$

$$\text{(v)} v(t) = (e^{-2t} - e^{-t} + e^{-4t})u(t)$$

$$\text{(vi)} v(t) = \frac{e^{-3(t-2)}u(t-2) - e^{3(t+2)}u(-(t+2))}{2}$$

$$\text{(vii)} v(t) = \frac{1}{\sqrt{3}} \left[e^{-0.5(t-3)} \sin \frac{\sqrt{3}}{2} t u(t-3) + e^{0.5(t+3)} \sin \frac{\sqrt{3}}{2} t u(-(t+3)) \right]$$

$$9. 6(e^{-1.5t} - e^{-4t})u(t)$$

$$10. (V_T)^2 \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$$

$$11. V(j\omega) = 1 + 2\cos\omega + 2\cos 2\omega$$

$$12. V(j\omega) = 8 \left(\frac{\sin \omega}{\omega} \right)^2 \cos \omega$$

$$13. \text{(i)} \text{Energy spectral density} = \frac{320000}{\pi} \frac{1}{(1+400\omega^2)^2} \text{ J-sec/rad}$$

$$\text{(iii)} 0.078 \text{ rad/s bandwidth}$$

$$14. \text{(i)} V(j\omega) = 0.001 \left[\frac{e^{-j0.0005(\omega-2\pi \times 10^6)} \frac{\sin 0.0005(\omega-2\pi \times 10^6)}{0.0005(\omega-2\pi \times 10^6)} + e^{-j0.0005(\omega+2\pi \times 10^6)} \frac{\sin 0.0005(\omega+2\pi \times 10^6)}{0.0005(\omega+2\pi \times 10^6)} \right]$$

$$\text{(ii)} \text{Bandwidth} = 1.464 \text{ kHz, Centre frequency} = 1 \text{ MHz}$$

$$15. 0.1272 \text{ J}$$

$$16. \text{(i)} H(j\omega) = \frac{2}{j\omega + 0.2} + \frac{0.809j\omega + 44.49}{(j\omega + 0.05)^2 + 75^2}$$

$$\text{(ii)} 10.008 \text{ V}$$

$$17. v_o(t) = \{e^{-2t}u(t) - e^{-2(t-0.2)}u(t-0.2)\} \text{ V}$$

$$18. v_o(t) = \{10 - 20e^{-50t} \cos 50\sqrt{3}t\}u(t) \text{ V}$$

$$19. v_R(t) = \{2000e^{-4 \times 10^8 t} + 10 \sin 100\pi t\}u(t) \text{ V}$$

$$20. H(j\omega) = \left[\frac{10^9}{(j\omega + 10^4)^2} - \frac{10^5}{(j\omega + 10^4)} \right],$$

$$\text{Impulse response} = (-10^5(1-10^4 t)e^{-10^4 t})u(t) \text{ V.}$$

$$21. \text{(i)} H(j\omega) = \left[\frac{10j\omega}{(j\omega)^2 + 120(j\omega) + 1000} \right]$$

$$\text{(iii)} \text{Impulse response} = 10(1.088e^{-111t} - 0.088e^{-9.01t})u(t) \text{ V,}$$

Normalised energy = 0.417 J

$$22. \left\{ \begin{aligned} &[u(t) - u(t - 20 \mu s)] \\ &-8[e^{-55550t}u(t) - e^{-55550(t-20 \mu s)}u(t-20 \mu s)] \end{aligned} \right\}$$

Chapter 15

1. (a) 1 V, 1.602 V (b) 0 V, 1.602 V
2. $f(t) = \sqrt{2}e^{-t} \sin(t - \frac{\pi}{4})$ A, -1 A
3. 2.666
4. $i(t) = \left[\begin{aligned} &4.21e^{-0.5t} \cos(0.866t + 121.2^\circ) \\ &+ 3.29e^{-0.2t} \cos(t - 48.6^\circ) \end{aligned} \right] u(t)$ A
6. (i) 0.982 V (ii) $v(2^-) = 0.271$ V; $v(2^+) = -1.73$ V;
(iii) 484.13 V (iv) -0.06 V (v) 0 V
(vi) $v(2^-) = 0$ V; $v(2^+) = 2e^{-6}$ V; (vii) -0.0038 V
(viii) -13.75 V
7. (i) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2x; t \geq 0^-$
(ii) $y(t) = (3 - 3e^{-t} + e^{-2t})u(t)$ units.
8. (i) $\frac{d^2i}{dt^2} + \frac{di}{dt} + i = \frac{dv_s}{dt}; t \geq 0$
(ii) $i(t) = e^{-0.5t} \left[\begin{aligned} &(1.27 \cos(2t - 76^\circ) - \\ &0.615 \cos(0.866t - 60^\circ)) \end{aligned} \right] u(t)$ A
9. $i(t) = 1.284e^{-0.5t} \times$
 $\left[\begin{aligned} &(0.97 \cos(0.866t + 169.7^\circ) - \\ &+ 0.99 \cos(2t - 75.6^\circ)) \end{aligned} \right] u(t - 0.5)$ A
11. (i) $e^{-0.5s} \frac{\tanh(0.25s)}{s}$ (ii) $\frac{1}{s(1 + e^{-0.5s})(1 + e^{-s})}$
12. $i_1(t) = \left\{ \begin{aligned} &1.755e^{-0.586t} - 0.1647e^{-3.414t} \\ &+ 0.61 \sin(3t - 75.3^\circ) \end{aligned} \right\} u(t)$
 $i_2(t) = \left\{ \begin{aligned} &0.727e^{-0.586t} + 0.4e^{-3.414t} \\ &+ 0.72 \sin(3t - 83.3^\circ) \end{aligned} \right\} u(t)$
13. $y(t) = 0.25(1 - e^{-2t})u(t) + 0.5tu(t)$
14. $1.5t^2u(t)$
15. $120[e^{-0.005t} - e^{-0.1t}]u(t)$
16. $0.5t^2[u(t) - 2u(t-2) + u(t-4)] +$
 $2t[u(t-2) - u(t-4)]$
17. (i) $\frac{s}{s + \pi^2} [1 - e^{-2s}]$
(ii) $\frac{0.4(1 - e^{-2s})}{s^2 - 0.04} - \frac{0.8215e^{-2s}}{s}$
18. (i) $\frac{12}{(s+1)^4}$ (ii) $\ln \frac{s+1}{s+3}$
19. (i) $\frac{\pi}{2} - \frac{\tan^{-1}(0.5s)}{2}; 4 \left(\frac{\sin 2t}{2t} \right) u(t)$
(ii) $\frac{-8s}{(s^2 + 4)^2}; -2t \sin 2t u(t)$

20. $0.306 \Omega; \{0.97 - 0.99e^{-2.03t} \cos(9.95t - 11.5^\circ)\} u(t)$ V
22. (a) $\frac{V_o(s)}{V_s(s)} = \frac{1}{s^2 + 3s + 1}; Z(s) = \frac{2(s^2 + 3s + 1)}{s(s + 2)}$
(c) $v_o(t) = (1 - 1.171e^{-0.382t} + 0.171e^{-2.618t})u(t)$ V
 $i_s(t) = (0.13821e^{-0.382t} + 0.36181e^{-2.618t})u(t)$ A
(d) Initial value of $v_o(t) = 0$ V, Final value of $v_o(t) = 1$ V,
Initial value of $i_s(t) = 0.5$ A, Final value of $i_s(t) = 0$ A.
23. $e^{-0.5t} \left\{ \begin{aligned} &\cos 0.866t [u(t) - 1.6u(t-1)] \\ &+ 1.03 \sin 0.866t \times u(t-1) \end{aligned} \right\}$
25. (ii) $i_x(t) = \{1.56 - 2.5e^{-0.2151t} \cos(1.307t + 51.4^\circ)\} u(t)$ A
 $v_o(t) = \left\{ \begin{aligned} &2 - 1.913e^{-0.57t} \\ &- 0.8525e^{-0.2152t} \cos(1.307t - 84.1^\circ) \end{aligned} \right\} u(t)$ V
26. (i) 2Ω (ii) $Z(s) = \frac{s^2 + 20s + 66.67}{s^2 + 40s + 200}$
(iii) $i_s(t) = \left\{ \begin{aligned} &0.95 \cos(2t + 19.65^\circ) \\ &+ 0.154e^{-4.23t} - 0.425e^{-15.77t} \end{aligned} \right\} u(t)$ A
27. $v_x(t) = 1.414(e^{-0.586t} - e^{-3.414t})u(t)$ V
28. $v_x(t) = -0.5(e^{-0.586t} - e^{-3.414t})u(t)$ V
29. (a) $H(s) = \frac{0.505}{(s+0.5)(s^2 + 0.2s + 1.01)}$ and impulse response
 $= \{0.5e^{-0.5t} - 0.54e^{-0.1t} \cos(t + 21.8^\circ)\} u(t)$ units
(b) $H(s) = \frac{1}{(s^2 + 1.848s + 1)(s^2 + 0.765s + 1)}$ and impulse
response $= \left\{ \begin{aligned} &2.415e^{-0.924t} \cos(0.3824t - 67.5^\circ) \\ &- 1.311e^{-0.38t} \cos(0.925t - 45.2^\circ) \end{aligned} \right\} u(t)$ units
(c) $H(s) = \frac{5.05}{(s+0.5)(s^2 + 0.2s + 1.01)}$ and impulse response
 $= \{5e^{-0.5t} - 5.4e^{-0.1t} \cos(t + 22.05^\circ)\} u(t)$ units
(d) $H(s) = \frac{0.505(s^2 + 1)}{(s+0.5)(s^2 + 0.2s + 1.01)}$ and impulse response
 $= \{0.5e^{-0.5t} + 0.1076e^{-0.1t} \cos(t + 87.3^\circ)\} u(t)$ units
30. $k < 2$
31. (i) 150 (ii) Poles: -2.92×10^6 nepers/sec, $-0.54 \pm j0.75$,
No zeros
33. $312.5 \sin(4t - 88.16^\circ)$ V
35. (a) $L_p = 22$ mH, $L_s = 1.375$ mH, $M = 5.39$ mH
(b) $Y_p(s) = \frac{1145.8(s + 1563.64)}{s^2 + 39500s + 179166.7}$ S,
 $H(s) = \pm 0.2273 \frac{36730s}{s^2 + 39596s + 179614}$
(c) Poles of transfer function at -4.54 , -39591.5 nepers/sec,
zero of transfer function at $s = 0$
(d) Upper cut-off at 6.3 kHz and lower cut-off at 0.72 Hz.
Bandwidth = 6.3 kHz.
36. $0.2274[e^{-4.54t} - e^{-39591.5t}]u(t)$ V; $\psi_p = 220$ mWb, $\psi_s = 53.9$ mWb,
1.1 J
36. (i) $i_2(t) = -392e^{-1563.6t}u(t)$ A
 $v_o(t) = -784e^{-1563.6t}u(t)$ V
 $v_s(t) = (10 + 3303.7e^{-1563.6t})u(t)$ V

- (ii) Energy dissipated in the switch = 4.36 J; Energy dissipated in the $2\ \Omega$ resistor = 98.27 J

Chapter 16

- $y_{11} = 1.146\ \text{S}$, $y_{12} = -0.625\ \text{S}$, $y_{21} = -0.625\ \text{S}$, $y_{22} = 1.25\ \text{S}$.
- (a) $\begin{bmatrix} 1.125 & -0.125 \\ -0.125 & 1.125 \end{bmatrix}\ \text{S}$ (b) $\begin{bmatrix} 0.625 & -0.125 \\ -0.125 & 1.625 \end{bmatrix}\ \text{S}$
(c) $\begin{bmatrix} 1.125 & -0.625 \\ -0.625 & 1.625 \end{bmatrix}\ \text{S}$
- $\begin{bmatrix} \frac{0.5(s^2+3s+1)}{s+2} & -\frac{0.5(s+1)^2}{s+2} \\ -\frac{0.5(s+1)^2}{s+2} & \frac{0.5(s^2+3s+1)}{s+2} \end{bmatrix}\ \text{S}$
- (i) $\begin{bmatrix} 10.1 & 0.1 \\ 1000.1 & 1.1 \end{bmatrix}\ \text{k}\Omega$ (ii) -13.04
- (i) $\begin{bmatrix} \frac{(10^{-4}+10^{-11}s)}{1.1+10^{-8}s} & -\frac{10^{-11}s}{1.1+10^{-8}s} \\ \frac{(0.1+10^{-11}s)}{1.1+10^{-8}s} & \frac{(1.1\times 10^{-3}-0.97910^{-9}s)}{1.1+10^{-8}s} \end{bmatrix}\ \text{S}$
(ii) $\frac{V_o(s)}{V_s(s)} = 75.76 \left[\frac{1+10^{-12}s}{1-1.5\times 10^{-6}s} \right]$; unstable transfer function, notion of bandwidth is not relevant.
- $[g] = \begin{bmatrix} 900\ \Omega & 1 \\ 299 & 0.011\ \text{S} \end{bmatrix}$; $\frac{v_o}{i_s} = 99\ \Omega$
- (a) $\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}\ \Omega$ (b) $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}\ \Omega$ (c) $\begin{bmatrix} 2.2 & 1.4 \\ 1.4 & 1.8 \end{bmatrix}\ \Omega$
(d) $\begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix}\ \Omega$ (e) $\begin{bmatrix} 2.75 & 0.5 \\ 0.5 & 1 \end{bmatrix}\ \Omega$
- $v_1 = 3.05\ \text{V}$; $v_2 = 0.92\ \text{V}$
- $[ABCD] = \begin{bmatrix} \frac{(10+s)}{0.05s} & \frac{0.2475s^2+7.5s+50}{0.05s} \\ \frac{20}{s}\ \text{S} & \frac{(5s+100)}{s} \end{bmatrix}\ \Omega$
- $[ABCD] = \begin{bmatrix} (3+0.2s) & (25+2s)\ \Omega \\ \frac{(3+0.2s)}{s}\ \text{S} & \frac{(25+2s)}{s} \end{bmatrix}$
- (i) $\frac{v_o}{v_s} = -150.4\ \text{V/V}$ (ii) $2.42\ \text{k}\Omega$ (iii) $-150.4\ v_s$ voltage source in series with $1.955\ \text{k}\Omega$.
- $A(s) = \frac{s^2+4s+1}{s^2+s+1}$
- $\frac{V_o(s)}{V_s(s)} = 4.672 \left[\frac{s+5.5}{s+3.4} \right]$

18. $Z_o = 1.005\ \Omega$; $\gamma = 0.1 + j\frac{\pi}{2}$

- $L = 238.7\ \text{mH}$, $C = 42.44\ \mu\text{F}$ (ii) 2.634 nepers, 22.88 dB; 5.99 nepers, 52 dB (iii) 11.5° phase delay; 60° phase delay.
- (i) $L = 59.7\ \text{mH}$, $C = 10.6\ \mu\text{F}$ (ii) 4.585 nepers, 39.82 dB; 3.75 nepers, 32.55 dB (iii) 83.6° phase lead; 23.1° phase lead.
- $490.1\ \Omega$ in series arms, $121.21\ \Omega$ in shunt arm.
- $2970\ \Omega$ in the series arm and $733.33\ \Omega$ in the shunt arms.
- Bridge with $490.91\ \Omega + 733.33\ \Omega$ in one arm and $733.33\ \Omega + 490.91\ \Omega$ in the other arm.
- $440.7\ \Omega$ series arm followed by $1052.82\ \Omega$ shunt arm and $171.42\ \Omega$ series arm.

Chapter 17

1. (i) No (ii) Yes (iii) Yes

3. $P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$; $A_i^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

4. 0 A in the remaining branches

5. (i) 45 (ii) 24 (iii) 8 (iv) 0 (v) 21 (vi) 8

6. $\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \end{bmatrix}$

7. $v_1 = 0\ \text{V}$, $v_2 = 0\ \text{V}$, $v_3 = 1\ \text{V}$, $v_4 = 2\ \text{V}$, $v_5 = 1\ \text{V}$, $v_6 = 1\ \text{V}$, $v_7 = -2\ \text{V}$, $v_8 = 1\ \text{V}$

8. $v_1 = 0\ \text{V}$, $v_2 = 0\ \text{V}$, $v_4 = 2\ \text{V}$, $v_7 = -2\ \text{V}$

- $i_3 = i_5 = i_6 = i_8 = 0\ \text{A}$

9. $i_1 = 0\ \text{A}$, $i_4 = -2\ \text{A}$, $i_6 = 3\ \text{A}$

10. 15 W

11. 2 A, 1 A

12. $v_x = 14/3\ \text{V}$, $i_x = 5/3\ \text{A}$, $i_y = -2/3\ \text{A}$

13. 26.48 W

14. $B_f = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$

15. $v_1 = 2\ \text{V}$, $v_2 = 0\ \text{V}$, $v_3 = 1\ \text{V}$, $v_4 = -2\ \text{V}$

17. $A = \begin{bmatrix} 1 & 3 & 6 & 7 & 8 & 2 & 4 & 5 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix}$

18. 80 W

19. $B_m = B_f$; $i_1 = -i_5$; $i_2 = i_5 - i_7$; $i_3 = i_7 - i_9$; $i_4 = i_9$; $i_6 = -i_5 + i_7$; $i_8 = i_9 - i_7$

20. They form independent KVL equations. One more equation is needed. The choices are $\{4, 5, 9\}$, $\{7, 8, 5, 4\}$, $\{8, 5, 4, 3\}$, $\{1, 6, 5, 4, 7\}$, $\{1, 2, 3, 4, 5, 6\}$.

21. No, there is no such tree.

$$22. \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{m1} \\ i_{m2} \\ i_{m3} \\ i_{m4} \end{bmatrix}$$

23. $18.21 \angle -159.6^\circ \text{ V}_{\text{rms}}$; $139.8 \angle -13.1^\circ \text{ V}_{\text{rms}}$; $48.1 \angle -141.8^\circ \text{ V}_{\text{rms}}$;

25. -3.75 V

29. (i) Yes (ii) No (iii) Yes (iv) No (v) No

30. (i) $i_1 - i_6 = 0$; $i_1 + i_2 + i_8 - i_7 = 0$; $i_5 + i_9 - i_6 - i_8 = 0$; $i_2 - i_7 + i_9 + i_5 = 0$. (ii) No, they are not independent. (iii) No

$$32. B_f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$33. Q_f = \begin{bmatrix} 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$v_1 = 4 \text{ V}$; $v_2 = -3 \text{ V}$; $v_3 = 2 \text{ V}$.

$$34. Q_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

36. $35 \angle -137.7^\circ \text{ V}_{\text{rms}}$; $82.35 \angle -117.7^\circ \text{ V}_{\text{rms}}$

39. 1.56 V

40. $v_1 = 11.74 \text{ V}$; $v_2 = -0.435 \text{ V}$; $v_3 = 9.565 \text{ V}$

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