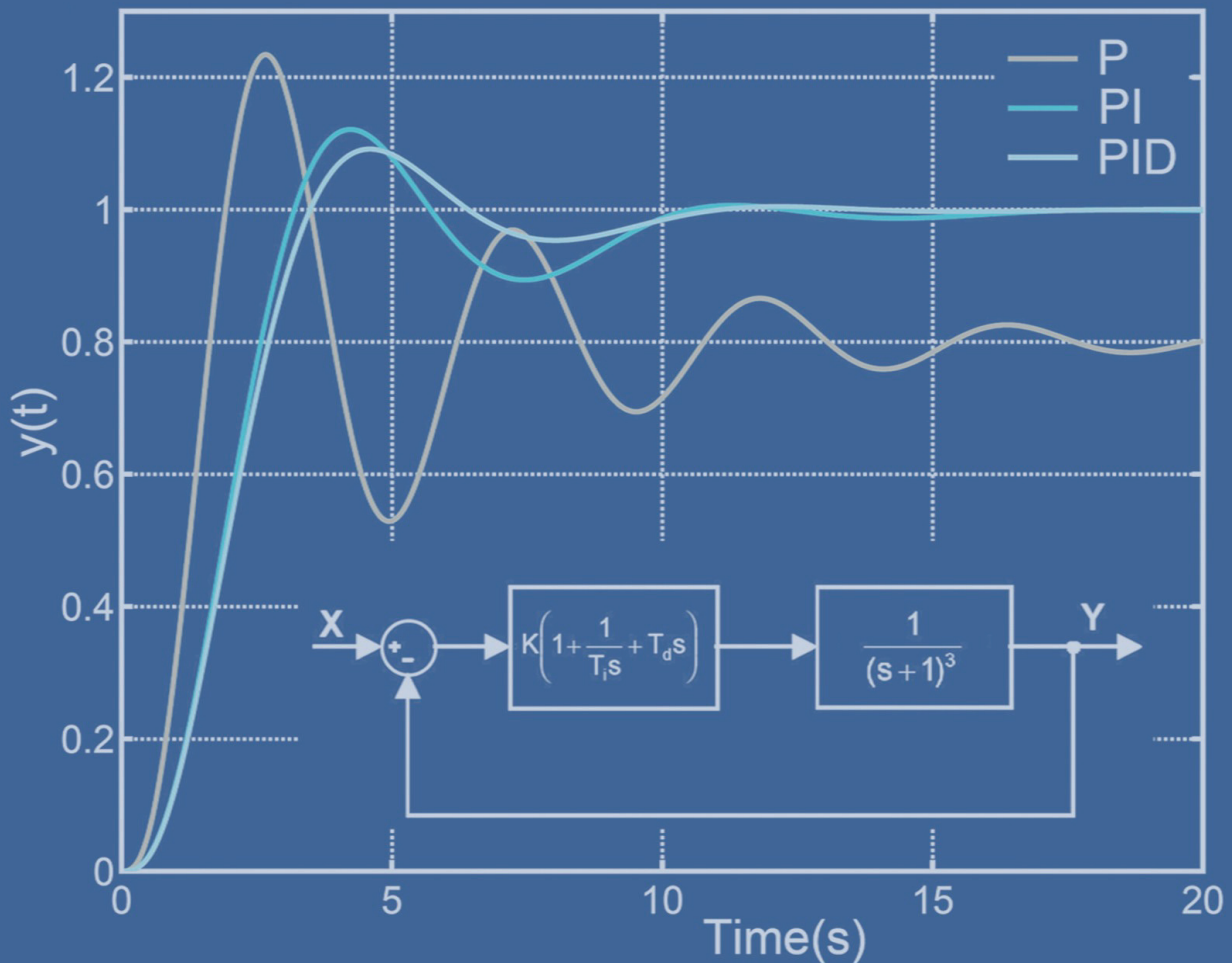
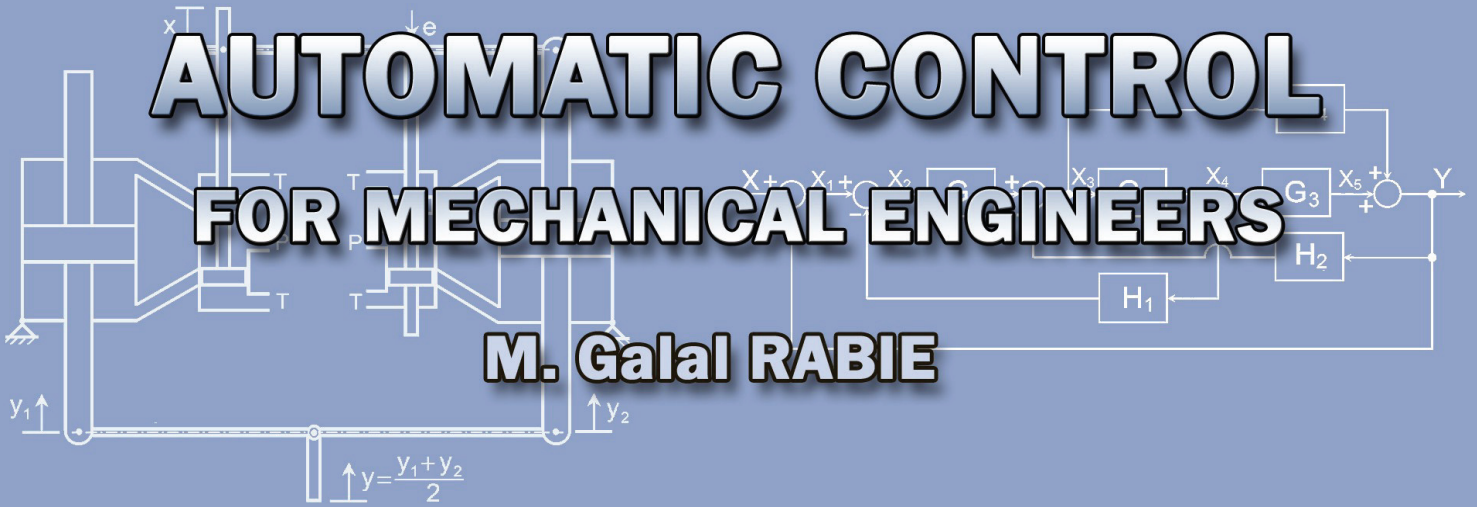


# AUTOMATIC CONTROL FOR MECHANICAL ENGINEERS

M. Galal RABIE



# **AUTOMATIC CONTROL**

## **For Mechanical Engineers**

**M. Galal RABIE**  
**Professor of Mechanical Engineering**

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*To my wife, Fatema Raafat,  
To Rehab, Alaa, Mohammed, Hoda, Malak and Hassan*

## PREFACE

Automatic control plays an important role in the advance of engineering and science. It is of extreme importance in most of the engineering fields; such as the aerospace engineering, chemical engineering, robotic systems, automotive and mobile equipment engineering as well as manufacturing and industrial processes. Automatic control provides the means of understanding the problems of stability and precision of dynamic systems. Actually most engineers must have good understanding of this field.

The majority of textbooks on automatic control are most appropriated for electrical engineers. The main problem in designing and analyzing a control loop for non-electrical systems normally arises when deducing adequate mathematical model for the system. Generally, the components cannot readily be represented by simple discrete ideal elements. The classical approach based on the transfer function and associated techniques of analysis is more easily comprehended and related to practice by the beginners than is the modern control theory. Therefore this text is prepared for the mechanical engineering students. It deals with the basics of linear control theory. The text includes simple examples enabling applicants to understand the problems of dynamic systems accuracy and stability. The text includes examples and exercises that facilitate the comprehension of the control theory, especially for the mechanical engineering students. The text is arranged in ten chapters dealing with the following topics:

1. An introduction giving the basic definitions and methods of system representation, **Chapter 1**.
2. A revision of selected topics from mathematics, **Chapter 2**.
3. Deduction of the transfer functions using mathematical models, block diagrams and signal flow graphs, **Chapters 3, 4 & 5**.
4. Analysis of the transient and frequency responses of the system and how does the response vary with the form of transfer function and the input excitation. The text discusses also how a transfer function can be determined by practical testing of a system, **Chapters 6 & 7**.
5. Analysis of the accuracy and stability of the feedback system, **Chapter 8**.
6. Root locus analysis, **Chapter 9**.
7. Improvement of the system stability by introducing and designing different types of compensators, mainly the P, PI & PID controllers, **Chapter 10**.

I am indebted to my colleague Prof. Dr. Gamal Ahmed El-Sheikh, for his objective comments on the whole text.

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## COVER DESIGN

The author is indebted to the cover designer, Rehab RABIE, MSc.

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# 1 | INTRODUCTION

## 1.1 INTRODUCTION

It is essential for engineers to have an understanding of the nature of the dynamic behavior of systems, as well as the available methods for analyzing and improving their stability and precision. As for mechanical engineers, they should be able to deal with the analysis and design of dynamic systems of different physical nature. A simple electro-hydraulic servo-valve, for example, includes electric, magnetic, hydraulic and mechanical components. Therefore, an increasing proportion of engineers should be acquainted with the modeling of the multi-disciplinary systems as well as the fundamentals of control theory.

## 1.2 SYSTEM DEFINITION

The system can be defined as being that part of the universe in which interest lays. It is a combination of components acting together to perform a specific function. A system may be an aircraft, a jet engine, the engine fuel pump or even the pump controller. A component is a single functioning unit of a system. A single component of a large assembly may be looked at as a system. The system should be defined carefully and separated from the environment by means of an imaginary boundary. It interacts with the environment by means of three types of signals, as shown by in Fig.1.1.

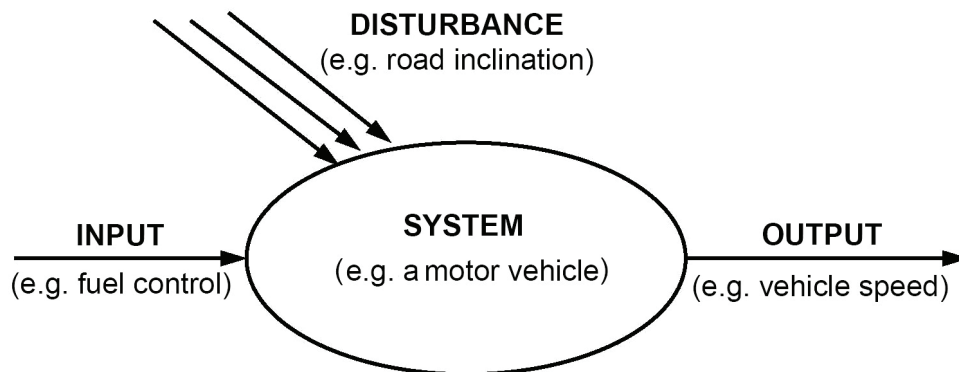


Fig.1.1 Interaction between system and environment

**System inputs** are the controllable signals passing from the environment to the system.

**System outputs** are signals passing from the system to the environment.

**Disturbance (Noise)** is external, random uncontrollable input.

**Linear system:** A system is said to be linear if and only if it obeys the superposition principles. This requires that if the separate application of the independent inputs  $I_1(t)$  and  $I_2(t)$  produces outputs  $O_1(t)$  and  $O_2(t)$  respectively, then the application of an input  $I(t) = I_1(t) + I_2(t)$  produces an output  $O(t) = O_1(t) + O_2(t)$ , Fig.1.2. No real system component is completely linear, but sometimes the range of operation is such that the linearity can be assumed.

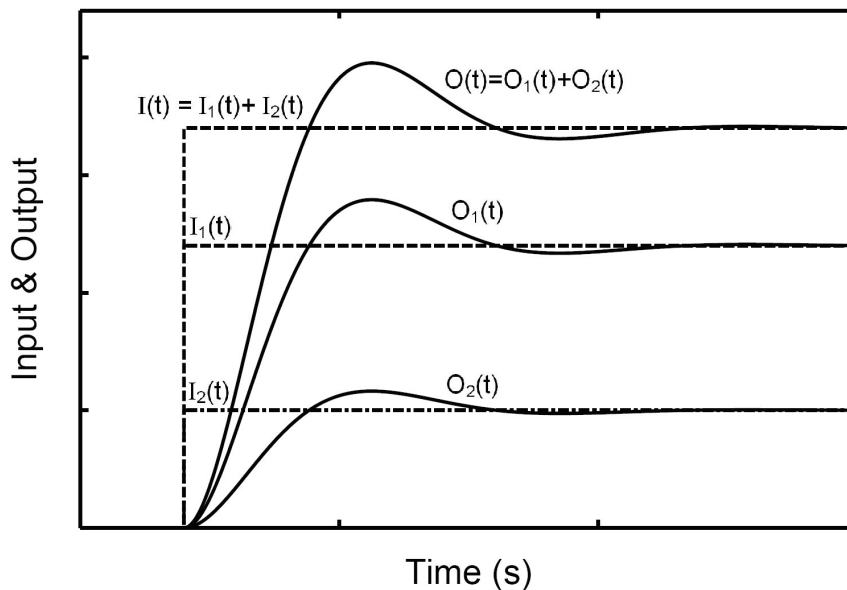


Fig.1.2 Transient response of a linear system

**Continuous and discrete systems:** Continuous-time systems are systems in which the signals involved are continuous in time. These systems may be described by differential equations. If one or more of system variables are changed only at discrete instants of time, the system is a discrete-time and it may be described by difference equations.

**Static system** is a system whose outputs depend on the present value of inputs.

**Dynamic system** is a system whose variables are time dependent and where the output depends on the history of its inputs. Actually all of the systems are dynamic. But, in some operating conditions, a dynamic system can be treated as a static one for simplicity.

## 1.3 SYSTEM CONTROL

The objective of system control is to force the controlled system to produce certain required outputs. It is necessary to adjust / select / design the values of one or more of the inputs to obtain the required output.

**Regulation** is the simplest variety of control. It may be defined as maintaining a specified quantity characterizing the process at a given level or as variation of this quantity in accordance with a specified law (keeping constant engine speed corresponding to a throttle lever position or variation of the speed of a jet engine with flight altitude according to certain law).

**Control** is much more comprehensive concept. It is usually understood as automatic implementation of a group of actions selected from among a set of feasible actions on the basis of available data and intended to maintain or improve the function of a system in keeping with the goal of control. All problems of regulation are essentially simpler cases of control problems.

There exist two classes of control systems; open loop control system and closed loop (feedback) control system.

### 1.3.1 Open Loop Control

On the basis of knowledge about the system and of past experience, a prediction is made of what input should be applied to produce the required output. Thus the objective of an open loop control system is to achieve the required output by utilizing an actuating device to control the process directly without measuring the system output, Fig.1.3. Such control is frequently unsatisfactory because of the effect of unexpected disturbances, which can lead to the deviation of the output from the required value. However, the open loop control can be satisfactory and cost effective for disturbance free applications.

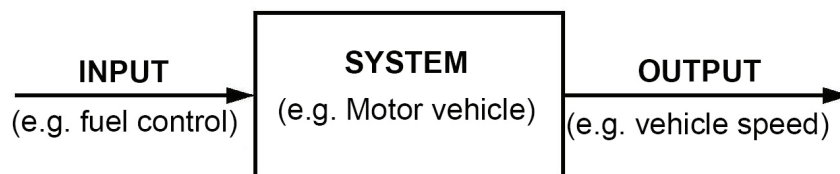


Fig.1.3 Open loop control structure

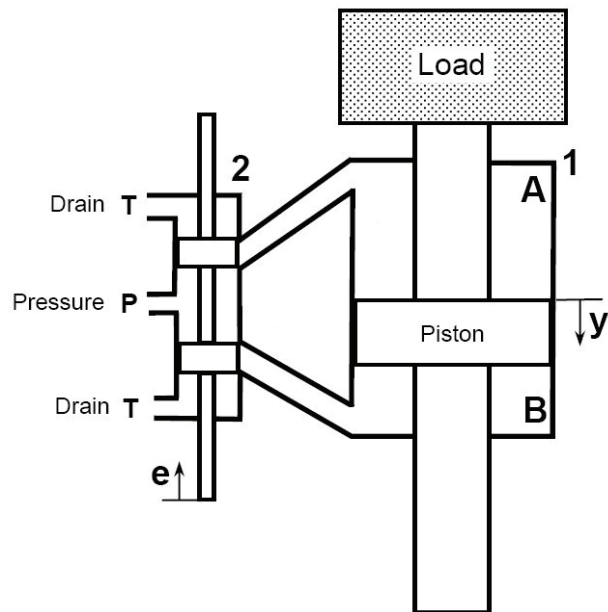


Fig.1.4 Hydraulic cylinder controlled by means of a directional control valve

Figure 1.4 shows an example of an open loop system where a hydraulic cylinder **1** is controlled by means of a directional control valve **2**. When the spool of the valve is displaced by distance **e**, it connects the pressure line to the upper piston chamber **A** and connects the lower piston chamber **B** to the drain line **T**. The throttling area, oil flow rate and piston speed are proportional to the displacement **e**. Therefore, the piston speed can be controlled via the valve displacement **e**. This is a typical example of open loop control system. By adjusting the input  $e(t)$  the system produces a corresponding output, piston speed for example. But, for the same input, if the operating pressure, load or internal leakages change, the system output changes, while keeping the input unchanged.

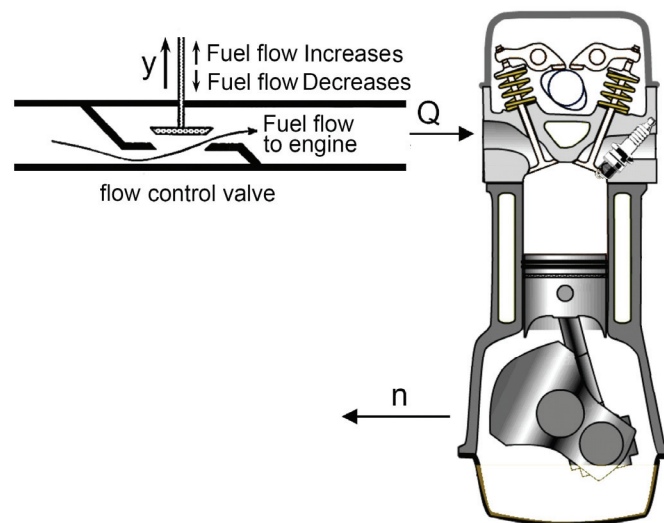


Fig.1.5 Open loop control of a piston engine

Another typical example of open loop system is the simple speed control of piston engine. The engine speed is controlled by means of the fuel throttle valve, Fig.1.5. By displacing the accelerator upwards, the throttle opens and the flow rate of fuel to engine increases. Consequently, the engine speed is increased. But the actual value of output speed  $n$  is not measured and not compared with the required value. If the operating conditions are not changed, the speed  $n$  is kept constant, corresponding to input  $y$ . But the disturbance, engine load for example, affect the system output, even while keeping the input unchanged.

### 1.3.2 Closed Loop (Feedback) Control

The system output is measured and compared with the desired value, Fig.1.6. The system attempts continuously to reduce the difference (error) between the two signals. The measured output value is called **feedback** (FB) signal. Sometimes, the loop is closed through the human being as in the case of motor vehicle control. Generally, the feedback reduces the effect of noise and disturbance on the system performance.

Closing a loop makes the system more accurate, but it can make the system oscillatory or even unstable. Therefore it is necessary to stress more on the analytical approach for system analysis, to reach the required system accuracy and stability and reduce cost and time.

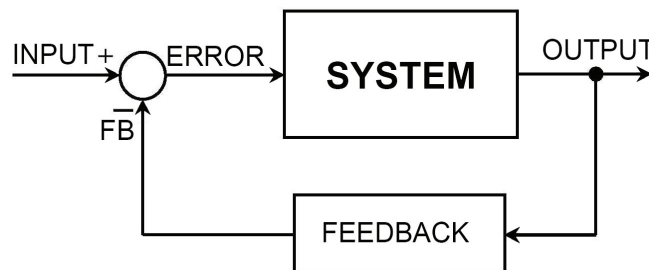


Fig.1.6 Closed loop system control

A functional scheme of typical hydraulic servo actuator is shown in Fig.1.7. It consists of a hydraulic cylinder **1**, directional control valve **2** and feedback linkage **3**. The feedback linkage links the spool displacement  $e$  to the input displacement  $x$  and the output displacement  $y$ . The operation of the servo actuator is explained in the following.

The initial position of the feedback linkage is shown by the centerline connecting points **4** & **5**, Fig. 1.7(a). When an input displacement  $x$  is applied, the feedback linkage first pivots about its connection with the piston rod, point **5**, Fig.1.7(b). The piston is hold temporarily in its position, shown by the dashed line connecting points

**5 & 6.** The spool is thus displaced by distance  $e$ . It allows the pressurized oil to flow to the upper side of cylinder, **A**. The piston starts to move downwards by distance  $y$  which causes the feedback linkage to pivot about the point **6**, Fig.1.7(c). The piston motion continues until the spool returns back to the original, closing position. This position is reached when the piston displacement  $y$  causes a spool displacement  $e$  equal to that caused by the input  $x$ , but opposite in direction. The total displacement of the spool becomes zero. The final position of the feedback linkage is indicated by the line connecting points **6** & **7**.

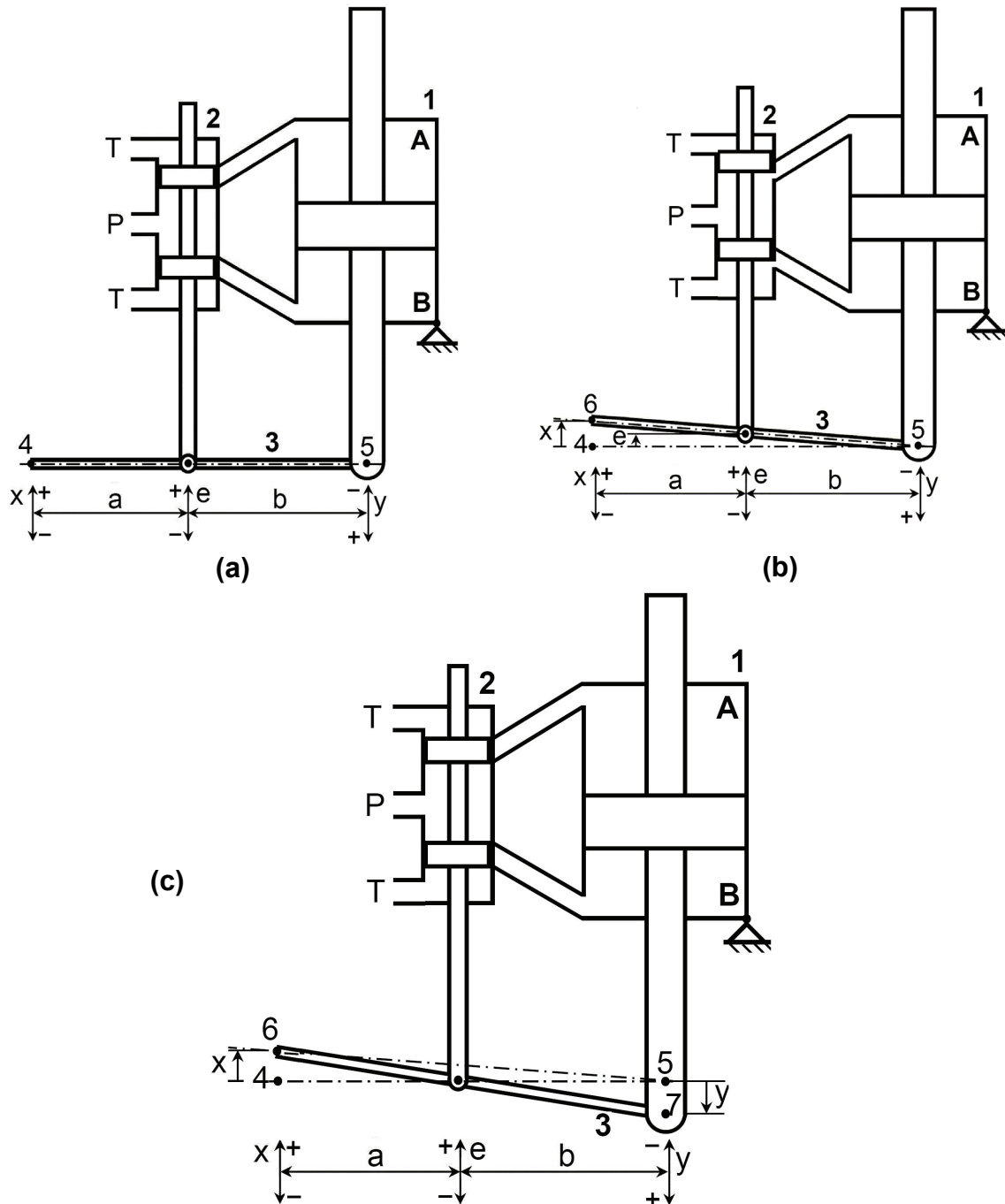


Fig.1.7 Functional scheme of hydraulic servo actuator

Another example of the closed loop control system is the speed control of some internal combustion engines (ICE). Figure 1.8 shows a typical speed control system of a piston engine **6**. The position of the throttle lever sets the desired speed of the engine. The speed control is shown at certain reference operating mode. The positive direction of motion of the system parts is indicated by the arrows.

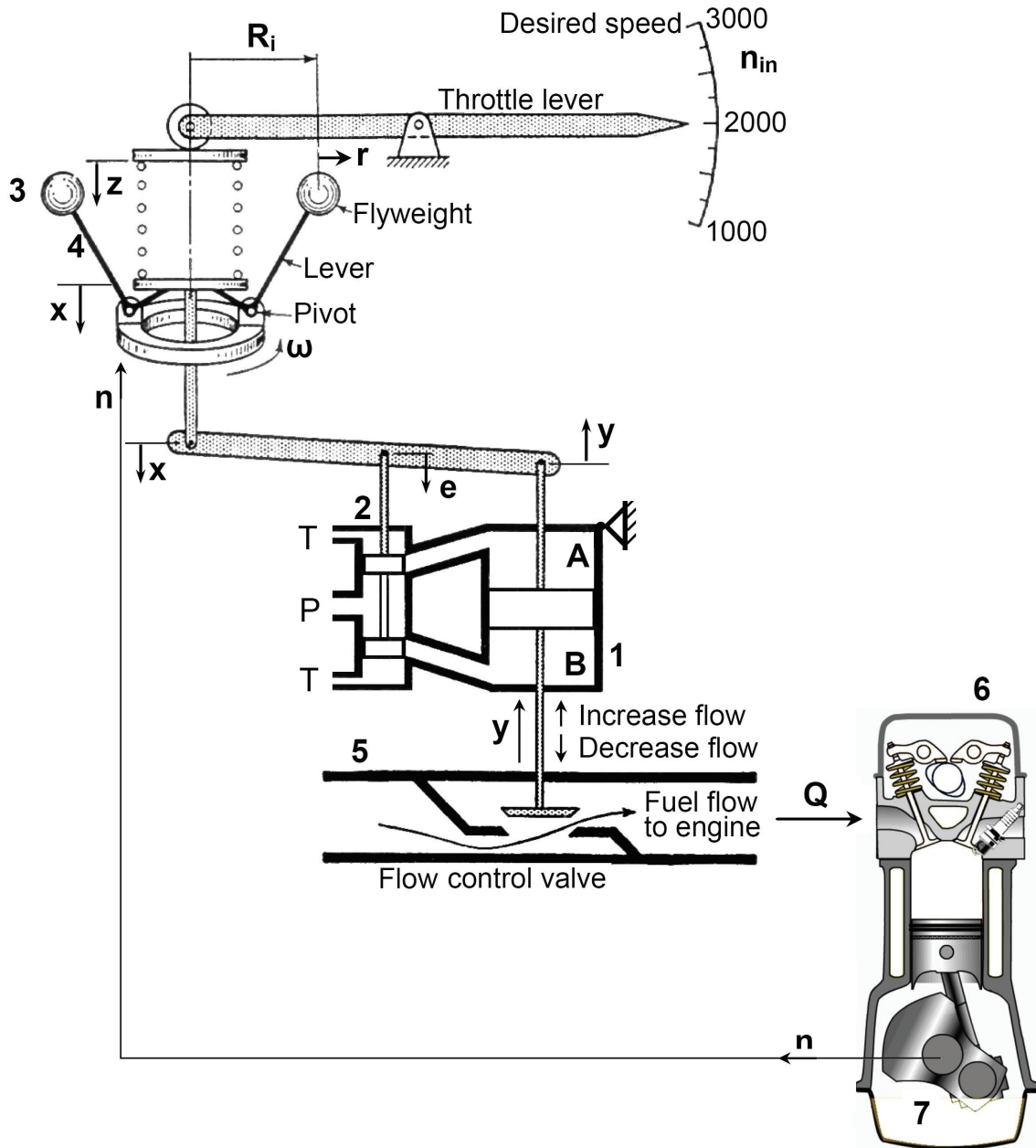


Fig.1.8 Closed loop speed control of ICE.

The center of gravity of the flyweights **3** is at a distance ( $R = R_i + r$ ) from the axis of rotation. The flyweights are geared directly to the output shaft **7**, so that the speed  $\omega$  of the flyweights is proportional to the engine output speed  $n$ . A pivoting lever **4** transmits the centrifugal force from the flyweights to the lower spring seat. The pivot and lever rotate with the flyweights as one unit. If the speed of the engine decreases

below its reference value, the centrifugal force of the flyweights decreases, decreasing the force exerted on the bottom of the spring. This causes the spring seat to move downwards (displacement  $x$ ), which in turn moves the spool downwards (displacement  $e$ ). The pressurized fluid then flows to the lower piston chamber **B**, which causes an increase of the displacement  $y$  and thus opens the fuel flow control valve **5** wider. By supplying more fuel, the speed of the engine will increase until the equilibrium is reached again. As the throttle lever is moved to a higher speed setting, the spring upper-seat moves downwards by distance  $z$ . Consequently, the fuel control valve opens to increase the fuel flow rate. Then, the engine speed is increased.

## 1.4 SYSTEM REPRESENTATION

The analysis of the system performance necessitates the representation of the system in a way visualizing the connection between the system elements and enabling the deduction of the necessary mathematical relations describing its behavior. The following are the most commonly used methods of system representation. These methods are related to each other by the physical laws governing the system behavior.

### 1.4.1 Schematic Diagrams

Figure 1.9 shows a schematic of a quarter-car. It shows how the system components are interconnected, define the system variables and form the basis for an analytical study.

Where:  $f$  = Applied force,  
 $p$  = Damper force,  
 $q$  = Spring force,  
 $m$  = Mass,  
 $x$  = Displacement.  
 $f_r$  = Friction coefficient,  
 $k$  = Spring stiffness.

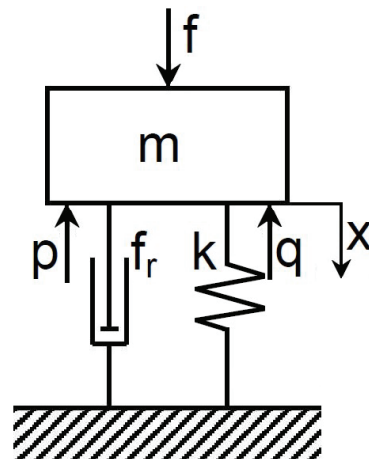


Fig.1.9 Scheme of a quarter-car

### 1.4.2 Mathematical Model

The mathematical model consists of a set of differential and algebraic equations describing the system. By applying physical laws to a specific system, it is possible to develop a mathematical model that describes its dynamics.

In determining a reasonably simple model, one must decide which physical variables and relationships are negligible and which are crucial to the accuracy of the model. Neglecting the effect of some parameters and assuming linear relations instead of nonlinear ones lead to simpler model. But, if these decisions are not justified, the results obtained from this model will not agree with the real system behavior. Moreover, there is no guarantee for the controller to be applicable.

The following are the mathematical modeling procedure:

1. Draw a schematic diagram of the studied system.
2. Deduce the mathematical model by applying the physical laws, considering justified simplifying assumptions.
3. Check if the deduced model is solvable.
4. Verify the validity of the model by comparing the model response with experimental results.

The following equations describe the car suspension system drawn schematically in Fig.1.9.

$$f - q - p = m \frac{d^2x}{dt^2} \quad (1.1)$$

$$p = f_r \frac{dx}{dt} \quad \text{and} \quad q = k x \quad (1.2)$$

### 1.4.3 Transfer Function

The transfer function is written for linear system with, for simplicity, zero initial conditions. It is defined as the ratio of the Laplace transform of the system output to that of the input. The dynamics of car suspension, Fig.1.9, is described by the following Differential equation.

$$m \frac{d^2x}{dt^2} + f_r \frac{dx}{dt} + k x = f \quad (1.3)$$

The system input is the driving force  $f$  and the output is the displacement  $x$ . Then, by applying Laplace transform, for zero initial conditions, the following transfer function can be obtained.

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + f_r s + k} \quad (1.4)$$

### 1.4.4 Block Diagram

The block diagram is a pictorial representation of the system clarifying the inter-relation between its different parts. This diagram can be used for system analysis. In this case, it must be supplied by a quantitative description of the relations between system variables in the form of appropriate mathematical expressions; usually transfer functions. Figure 1.10 shows a block diagram describing the quarter-car illustrated by Fig.1.9.

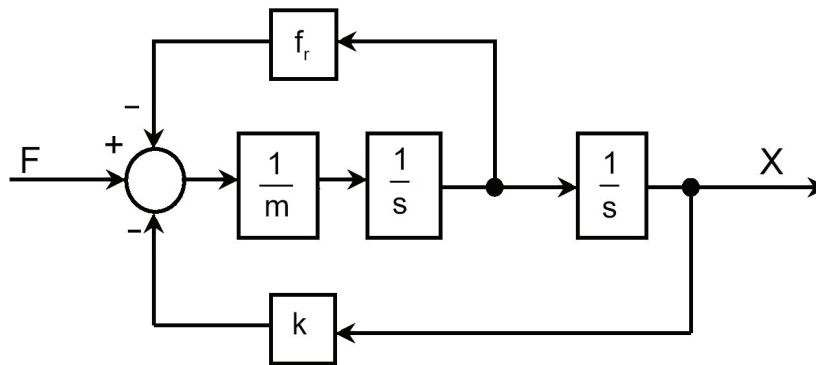


Fig.1.10 Block diagram of a quarter-car

### 1.4.5 Signal Flow Graph

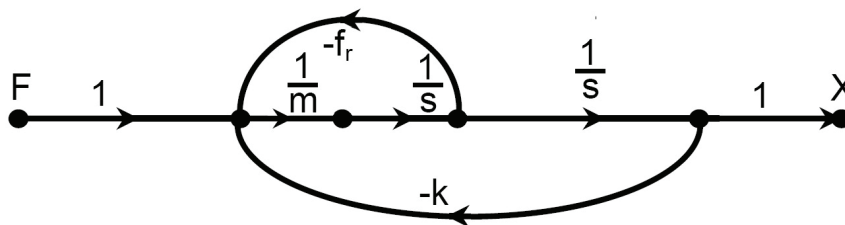


Fig.1.11 Signal flow graph of a quarter-car

The signal flow graph is an alternative pictorial representation of the system. It illustrates the passage of signals through the system and enables to deduce, in a simple way, the overall system transfer function. The quarter-car, illustrated by Fig.1.9, can be represented by the signal flow graph shown in Fig.1.11.

### 1.4.6 State Space Representation

The state space representation is an alternative method of mathematical representation, where the mathematical model includes only first order differential equations. Instead of a single  $n^{\text{th}}$  order differential/difference equation, the problem is

turned to solve n simultaneous differential equations of first order. A state space model of the quarter-car illustrated by Fig.1.9, system is deduced as follows:

$$m \frac{d^2x}{dt^2} + f_r \frac{dx}{dt} + k x = f \quad (1.5)$$

Define,

$$x_1 = \frac{dx}{dt} \quad (1.6)$$

then,

$$m \frac{dx_1}{dt} + f_r x_1 + k x = f \quad (1.7)$$

That is, the second order differential equation is replaced by the following two first order differential equations.

$$\frac{dx_1}{dt} = \frac{1}{m}(f - f_r x_1 - k x) \quad (1.8)$$

$$\frac{dx}{dt} = x_1 \quad (1.9)$$

Or, in a matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{f_r}{m} \end{bmatrix} \begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [f] \quad (1.10)$$

### 1.4.7 Bond Graph

The bond graph is an alternative way of pictorial representation of the system. It describes the transfer, storage and dissipation of energy between the basic elements of the system. It carries both of the physical and mathematical structures of the system. The equations describing the system can be deduced systematically from the bond graph in a form convenient for the computer simulation programs. The state space model of the system can be deduced from the bond graph in a simple way. Figure 1.12 shows a bond graph of the quarter car illustrated by Fig.1.9.

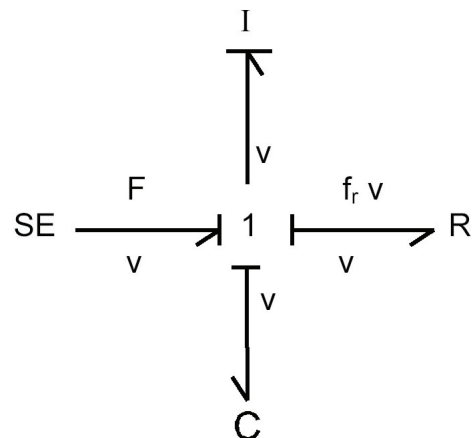


Fig.1.12 Bond graph

where: SE = Source of effort  
F = Force; effort  
v = Speed; flow  
I = Mass (inertia)  
C = Capacitance (1/k)  
R = Resistance (f<sub>r</sub>)

## 1.5 SYSTEM ANALYSIS

When the system input changes, the system output varies depending on the nature of the input signal, the system design and the disturbance signals. In the absence of disturbance signals and for known system structure, the system output depends only on the input signals. The most commonly used system input forcing functions are:

1. Transient disturbances such as step, ramp or impulse functions.
2. Sinusoidal signals.
3. Statistical signals of random nature.
4. Discrete signals.

This text deals with the first two types of input signals for the time and frequency domains analysis.

## 1.6 EXERCISE

1. What is a system? Give some examples of real systems.
2. Define the different types of signals connecting a system with the environment. Give some examples of real systems and define the different types of signals for each system.
3. Discuss briefly the open and closed loop control systems; give some examples of each type.
4. Define the static and dynamic systems giving some examples for each type.
5. Discuss briefly how to deduce the mathematical model of a real system.
6. State the different ways of system representation.

# 2 | MATHEMATICAL TOPICS

## 2.1 INTRODUCTION

One of the problems facing engineers is to deduce the equations describing the operation of components and systems. Once the equations are determined, they can be solved to find the response of the system to various inputs relevant to the way the system will be used.

Linear differential equations can be solved using the formal classical methods. However, engineers are more comfortable with an automated technique such as Laplace transform. This chapter is directed to the revision of selected mathematical topics needed for the analysis of control systems performance.

## 2.2 DIFFERENTIAL EQUATIONS

This part deals with linear differential equations (**DE**) of constant coefficients. The linear differential equation includes neither variables raised to a power different than unity nor a product of variables.

Examples of Linear differential equations:

$$\frac{dx}{dt} + x = 8 \quad (2.1)$$

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 3x = 10 \quad (2.2)$$

Examples of Non-Linear differential equations:

$$\frac{d^2x}{dt^2} + x\frac{dx}{dt} + 4tx = 10 \quad (2.3)$$

$$\left(\frac{d^2x}{dt^2}\right) \times \left(\frac{dx}{dt}\right) + 2x = 8 \quad (2.4)$$

The order of differential equations is the highest time derivative present. For the solution of a differential equation, a number of initial conditions are necessary, equal to the order of the DE.

The solution of a differential equation yields the system response which consists of two parts; steady state response and transient response.

If the variable of the differential equation is  $x$ , the solution can be written as:

$$x = x_t + x_s \quad (2.5)$$

where  $x_t$  = Transient part of the response which is time dependent.

$x_s$  = Steady state, time independent, part of the response.

**Example 2.1** Solve analytically:  $\frac{dx}{dt} + 2x = 8, x(0)=0$

$$\dot{x} + 2x = 8$$

$$\dot{x}_t + \dot{x}_s + 2(x_t + x_s) = 8$$

The right hand side of the differential equation is the input or forcing function. Assuming that, after long time, the transient part of the response has decayed while the steady state response part remains;  $\dot{x}_t = 0$ , then,

$$\dot{x}_s + 2x_s = 8 \quad \text{To be solved for the steady state response,}$$

and  $\dot{x}_t + 2x_t = 0 \quad \text{To be solved for the transient response.}$

Since  $x_s$  is time independent, then,  $\dot{x}_s = 0$  and hence  $x_s = 4$

$$\dot{x}_t / x_t = -2, \quad \text{The integration of this equation yields:}$$

$$\ln(x_t) = -2t + c$$

$$x_t = e^{-2t+c} = Ae^{-2t}$$

Since  $x(t) = x_t + x_s$ ,

then,  $x = 4 + Ae^{-2t}$

The constant, A, can be calculated using the initial conditions;

For  $x(0) = 0$ ,  $A = -4$  and  $x = 4(1 - e^{-2t})$

The variation of  $x_s$ ,  $x_t$  and  $x$  with time is plotted in Fig.2.1.

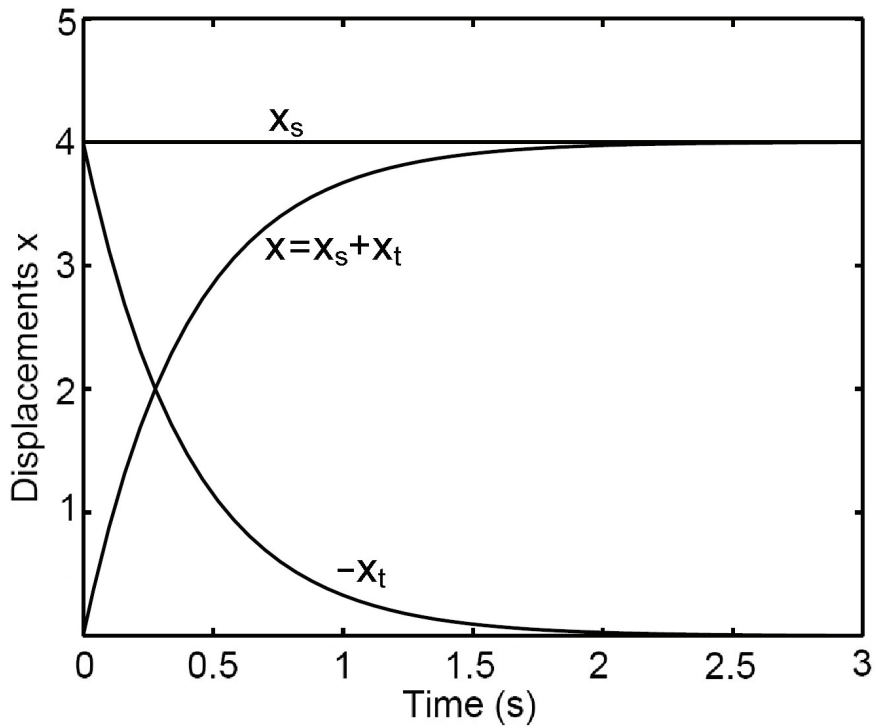


Fig.2.1 Plot of the solution of DE  $\left(\frac{dx}{dt} + 2x = 8\right)$

**Example 2.2** The mass suspension system given in Fig.2.2, can be represented mathematically by the following differential equations:

$$m \frac{d^2x}{dt^2} + f_r \frac{dx}{dt} + kx = f$$

Assuming that the numerical values are such that:

$$\frac{d^2x}{dt^2} + 3 \frac{dx}{dt} + 2x = 6$$

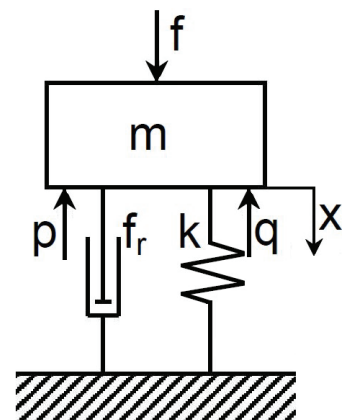


Fig.2.2 Scheme of a quarter-car

and  $x(0) = 1, \frac{dx}{dt}(0) = 0$

To solve this equation and find  $x(t)$ , put  $x(t) = x_t + x_s$

The steady state response  $x_s$  is obtained as follows.

$$\frac{d^2x_t}{dt^2} + 3\frac{dx_t}{dt} + 2x_t + \frac{d^2x_s}{dt^2} + 3\frac{dx_s}{dt} + 2x_s = 6$$

After long time, the transient part of solution decays;  $x_t=0$ . The time derivatives of the steady state part are zeros, then:

$$\frac{dx_s}{dt} = 0, \quad \frac{d^2x_s}{dt^2} = 0 \quad \text{and } x_s = 3$$

The transient response  $x_t$  is obtained by solving:

$$\frac{d^2x_t}{dt^2} + 3\frac{dx_t}{dt} + 2x_t = 0$$

Assuming that the transient response is of the form:  $x_t = Ae^{st}$ , then, by substituting in the above equation, the following equation is obtained;

$$s^2e^{st} + 3se^{st} + 2e^{st} = 0$$

Since  $e^{st} \neq 0$ , then  $s^2 + 3s + 2 = 0$

The last equation is called the characteristic equation. It can be obtained directly from the homogeneous equation by using the following substitution rule.

$$x_t \rightarrow 1, \quad \frac{dx_t}{dt} \rightarrow s, \quad \frac{d^2x_t}{dt^2} \rightarrow s^2, \dots \text{ etc.}$$

The deduced characteristic equation has the following two roots:

$$s = -1 \quad \text{or} \quad s = -2$$

Then,  $x_t = A_1e^{-t} + A_2e^{-2t}$

and 
$$x = x_s + x_t = 3 + A_1 e^{-t} + A_2 e^{-2t}$$

The constants  $A_1$  and  $A_2$  are calculated considering the following initial conditions;

$$x(0) = 1 \text{ and } \frac{dx}{dt}(0) = 0 .$$

$$A_1 + A_2 = -2 \quad \text{and} \quad A_1 + 2 A_2 = 0$$

Then  $A_1 = -4$  and  $A_2 = 2$

$$x = 3 - 4e^{-t} + 2e^{-2t}$$

This method becomes very complicated as the order of the differential equation increases. Therefore, it would be better to use an algebraic method that combines the natural response, forced response and determination of the coefficients in one procedure. The Laplace transform method does just that.

## 2.3 LAPLACE TRANSFORM

When a differential equation expressed in terms of time  $t$  is operated on by Laplace integral, a new equation results, which is expressed in terms of a complex term ( $s$ ). Laplace transform transforms the time dependent function from the time domain to the frequency (or Laplace) domain. The transformed equation is in purely algebraic form.

### 2.3.1 Direct Laplace Transform

The direct Laplace transform is given by the following expression.

$$X(s) = \mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st} dt \quad (2.6)$$

**Example 2.3** Laplace transform of a step function,

$$x(t) = \begin{cases} 0 & \text{for } t = 0 \\ a & \text{for } t > 0 \end{cases}$$

$$X(s) = \int_0^{\infty} a e^{-st} dt = a \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{a}{s}$$

**Example 2.4** Laplace transform of an exponential function,

$$x(t) = k e^{-at} \quad \text{where } a \text{ \& } k \text{ are constants}$$

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt = \int_0^{\infty} k e^{-at} e^{-st} dt = k \int_0^{\infty} e^{-(a+s)t} dt = \frac{k}{s+a}$$

**Example 2.5** Laplace transform of a trigonometric function,

$$x(t) = \cos(\omega t) = (e^{i\omega t} + e^{-i\omega t})/2 \quad \text{where } \omega = \text{constant}$$

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

$$X(s) = \frac{1}{2} \int_0^{\infty} e^{-st} (e^{i\omega t} + e^{-i\omega t}) dt = \frac{1}{2} \left[ \int_0^{\infty} e^{-(s-i\omega)t} dt + \int_0^{\infty} e^{-(s+i\omega)t} dt \right]$$

Or 
$$X(s) = \frac{s}{s^2 + \omega^2}$$

### 2.3.2 Inverse Laplace Transform

The inverse Laplace transform is an integral operator, which transforms from the Laplace domain to the time domain. It is given by the following expression.

$$x(t) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\sigma_1 - iR}^{\sigma_1 + iR} X(s) e^{st} ds \quad (2.7)$$

Actually, all functions in the time domain have direct Laplace transform but some of the functions in the Laplace domain have no inverse Laplace transform.

### 2.3.3 Properties of Laplace Transform

Some of the Laplace transforms properties used in this text are summarized as follows:

$$1. \mathcal{L} [f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s) \quad (2.8)$$

$$2. \mathcal{L} [af(t)] = aF(s) \quad (2.9)$$

$$3. \mathcal{L} [e^{-at}f(t)] = F(s+a) \quad (2.10)$$

$$4. \text{Initial value problem } f(0^+) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \quad (2.11)$$

$$5. \text{Final value problem } f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \quad (2.12)$$

$$6. \mathcal{L} \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0^+) \quad (2.13)$$

$$\mathcal{L} \left[ \frac{d^2f(t)}{dt^2} \right] = s^2F(s) - sf(0^+) - \dot{f}(0^+); \text{ where } \dot{f}(t) = \frac{df(t)}{dt} \quad (2.14)$$

$$\mathcal{L} \left[ \frac{d^nf(t)}{dt^n} \right] = s^nF(s) - s^{n-1}f(0^+) - s^{n-2} \frac{df}{dt}(0^+) - s^{n-3} \frac{d^2f}{dt^2}(0^+) - \dots - \frac{d^{n-1}f}{dt^{n-1}}(0^+) \quad (2.15)$$

$$7. \mathcal{L} \left[ \int f(t)dt \right] = \frac{1}{s} \left[ F(s) + \int f(t)dt \Big|_{0^+} \right] \quad (2.16)$$

**Example 2.6** Find  $\mathcal{L} \left[ \dot{x}(t) \right]$ , if  $x(t) = e^{-2t}$

$$X(s) = \frac{1}{s+2} \quad \text{and} \quad x(0^+) = 1$$

$$\mathcal{L} \left[ \dot{x}(t) \right] = sX(s) - x(0^+) = \frac{s}{s+2} - 1 = \frac{-2}{s+2}$$

**Example 2.7** Find  $x(t)$  after very long time if  $X(s) = \frac{1}{s+2}$

$$x(\infty) = \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) = \lim_{s \rightarrow 0} \frac{s}{s+2} = 0$$

### 2.3.4 Partial Fraction Expansion

**Case 1: Non-repeated real roots of characteristic equation** (i.e. non-repeated real roots of the polynomial in the denominator).

Given  $X(s) = \frac{P(s)}{Q(s)}$ , where  $Q(s) = \prod_{i=1}^m (s + a_i)$  with real distinct  $a_i$  and the degree of  $P(s)$  is less than that of  $Q(s)$  then:

$$X(s) = \sum_{i=1}^m \frac{k_i}{s + a_i} \quad \text{Where } k_i = \lim_{s \rightarrow -a_i} (s + a_i)X(s) \quad (2.17)$$

**Example 2.8** Find  $x(t)$  given  $X(s) = \frac{1}{s(s+2)}$

The inverse Laplace transform of this expression can be obtained after performing a partial fraction expansion as follows.

$$\frac{1}{s(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+2}$$

$$k_1 = \lim_{s \rightarrow 0} sX(s) = 0.5$$

$$k_2 = \lim_{s \rightarrow -2} (s+2)X(s) = -0.5$$

$$X(s) = 0.5 \left( \frac{1}{s} - \frac{1}{s+2} \right)$$

$$x(t) = 0.5(1 - e^{-2t})$$

**Example 2.9** Find  $v(t)$ , given:  $V(s) = \frac{s+5}{s(s+1)(s+2)}$

$$V(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$k_1 = \lim_{s \rightarrow 0} sV(s) = 2.5$$

$$k_2 = \lim_{s \rightarrow -1} (s+1)V(s) = -4$$

$$k_3 = \lim_{s \rightarrow -2} (s+2)V(s) = 1.5$$

$$v(t) = 2.5 - 4e^{-t} + 1.5e^{-2t}$$

**Case 2: Repeated real roots of characteristic equation** (i.e. Repeated real roots of the polynomial of the denominator).

Given  $X(s) = \frac{P(s)}{Q(s)}$ , where  $Q(s) = \prod_{i=1}^m (s+a_i)$  with real  $a_i$  and the degree of  $P(s)$  is less than that of  $Q(s)$ , the root  $(-a_1)$  is repeated with multiplicity  $n$ , then:

$$k_{n-i} = \frac{1}{i!} \lim_{s \rightarrow -a_1} \left[ \frac{d^i}{ds^i} \{ (s+a_1)^n X(s) \} \right]; \text{ where } i = 0, \dots, n-1 \quad (2.18)$$

**Example 2.10** Find the partial fractions of  $Y(s) = \frac{s+4}{(s+1)(s+2)^2}$

$$Y(s) = \frac{s+4}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2}$$

$$k_1 = \lim_{s \rightarrow -1} (s+1)Y(s) = 3$$

$$k_3 = \lim_{s \rightarrow -2} (s+2)^2 Y(s) = -2$$

$$k_2 = \lim_{s \rightarrow -2} \left[ \frac{d}{ds} \{ (s+2)^2 Y(s) \} \right] = \lim_{s \rightarrow -2} \left[ \frac{d}{ds} \left\{ \frac{s+4}{s+1} \right\} \right] = \lim_{s \rightarrow -2} \left[ \frac{(s+1) - (s+4)}{(s+1)^2} \right] = -3$$

$$Y(s) = \frac{3}{s+1} - \frac{3}{s+2} - \frac{2}{(s+2)^2}$$

Or, calculate  $k_1$  and  $k_3$  as shown above then calculate  $k_2$  as follows.

$$\frac{s+4}{(s+1)(s+2)^2} = \frac{3}{s+1} + \frac{k_2}{s+2} + \frac{-2}{(s+2)^2} = \frac{3(s+2)^2 + k_2(s+1)(s+2) - 2(s+1)}{(s+1)(s+2)^2}$$

The value of  $k_2$  is found by equating the coefficients of  $s$  in the nominator of the two equation sides;

$$1 = 12 + 3k_2 - 2, \quad \text{then } k_2 = -3$$

### Case 3: Complex roots of characteristic equation.

**Example 2.11:** Find  $x(t)$  given:  $X(s) = \frac{8}{s(s^2 + 2s + 2)}$

The equation  $s^2 + 2s + 2 = 0$  has two conjugate complex roots. Therefore, by completing the square:

$$X(s) = \frac{8}{s[(s+1)^2 + 1]}$$

Find the inverse Laplace transform,  $x(t)$ , using the Laplace transform tables.

$$x(t) = \frac{k}{\omega^2 + a^2} - \frac{k e^{-at} \sin(\omega t + \phi)}{\omega \sqrt{a^2 + \omega^2}}, \quad \phi = \tan^{-1}(\omega/a)$$

where  $\omega = 1$ ,  $a = 1$  and  $k = 8$

### 2.3.5 Solving Differential Equations Using Laplace Transform

To solve the differential equations using Laplace transform, the following steps are followed;

1. Apply Laplace transform to the differential equation.
2. Solve the resulting algebraic equation to find an expression for the variable in terms of Laplace variable  $s$ ,  $X(s)$ .
3. Expand the result into partial fractions.
4. Find the inverse Laplace transform,  $x(t)$ .

**EXAMPLE 2.12** Solve  $\ddot{x} + 3\dot{x} + 2x = 6$  for  $x(0)=1$  and  $\dot{x}(0) = 0$

Step 1: Find Laplace Transform of the DE

$$\mathcal{L} [\ddot{x}(t)] = s^2 X(s) - sx(0) - \dot{x}(0) = s^2 X(s) - s$$

$$\mathcal{L} [\dot{x}(t)] = sX(s) - x(0) = sX(s) - 1$$

$$\mathcal{L} [6] = \frac{6}{s}$$

$$\{s^2 X(s) - s\} + 3\{sX(s) - 1\} + 2X(s) = \frac{6}{s}$$

$$X(s)(s^2 + 3s + 2) = \frac{6}{s} + 3 + s = \frac{s^2 + 3s + 6}{s}$$

Step 2: 
$$X(s) = \frac{s^2 + 3s + 6}{s[s^2 + 3s + 2]} = \frac{s^2 + 3s + 6}{s(s+1)(s+2)}$$

Step 3: 
$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2} \quad \text{where } k_1 = 3, k_2 = -4 \text{ and } k_3 = 2$$

Step 4: 
$$x(t) = 3 - 4e^{-t} + 2e^{-2t}$$

**Example 2.13** Solve  $\ddot{x} + 2\dot{x} + x = 8$  for  $x(0) = 0$  and  $\dot{x}(0) = 0$

Step 1: 
$$(s^2 + 2s + 1) X(s) = \frac{8}{s}$$

Step 2: 
$$X(s) = \frac{8}{s(s+1)^2}$$

Step 3: 
$$X(s) = \frac{k_1}{s} + \frac{k_2}{(s+1)^2} + \frac{k_3}{s+1} \quad \text{where } k_1 = -k_2 = -k_3 = 8$$

Step 4: 
$$x(t) = 8(1 - te^{-t} - e^{-t})$$

**Example 2.14** Solve  $\ddot{y} + 4y = 8$  for  $y(0) = 0$  and  $\dot{y}(0) = 0$

Step 1: 
$$(s^2 + 4)Y(s) = \frac{8}{s}$$

Step 2: 
$$Y(s) = \frac{8}{s(s^2 + 4)}$$

Step 3: 
$$Y(s) = \frac{k_1}{s} + \frac{k_2s + k_3}{s^2 + 4}, \text{ where } k_1=2, k_2=-2 \text{ \& } k_3=0$$

$$Y(s) = \frac{2}{s} - \frac{2s}{s^2 + 4}$$

Step 4: 
$$y(t) = 2(1 - \cos 2t)$$

## 2.4 COMPLEX VARIABLES

The following are some of the important relations of complex variables, needed for further analysis.

$$z = x + iy = r[\cos(\vartheta) + i \sin(\vartheta)] \quad (2.19)$$

$$z = re^{i\vartheta} \quad (2.20)$$

$$z = |z| \angle \vartheta \quad (2.21)$$

$$z^n = |z|^n \angle n\vartheta \quad (2.22)$$

where

$$r = \sqrt{x^2 + y^2} \quad \& \quad \vartheta = \tan^{-1}(y/x) \quad (2.23)$$

$$\sin \vartheta = y/r \quad \text{and} \quad \cos \vartheta = x/r \quad (2.24)$$

$$e^{i\vartheta} = \cos \vartheta + i \sin \vartheta \quad (2.25)$$

$$e^{-i\vartheta} = \cos \vartheta - i \sin \vartheta \quad (2.26)$$

$$\cos \vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2} \quad (2.27)$$

$$\sin \vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i} \quad (2.28)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots \quad (2.29)$$

## 2.5 LAPLACE TRANSFORM TABLES

The pairs of Laplace transform are given in tables published in textbooks of mathematics. The following are the pairs of Laplace transform of some key functions:

No.	F(s)	f(t)
1.	1	$\delta(t)$
2.	$\frac{k}{s}$	k
3.	$\frac{k(n!)}{s^{n+1}}$	$k t^n$
4.	$\frac{k}{s+a}$	$k e^{-at}$
5.	$\frac{k}{(s+a)^2}$	$k t e^{-at}$
6.	$\frac{k(n!)}{(s+a)^{n+1}}$	$k t^n e^{-at}$
7.	$\frac{k \omega}{s^2 + \omega^2}$	$k \sin \omega t$
8.	$\frac{k s}{s^2 + \omega^2}$	$k \cos \omega t$
9.	$\frac{k \omega}{(s+a)^2 + \omega^2}$	$k e^{-at} \sin \omega t$
10.	$\frac{k(s+a)}{(s+a)^2 + \omega^2}$	$k e^{-at} \cos \omega t$
11.	$\frac{k}{s[(s+a)^2 + \omega^2]}$	$\frac{k}{\omega^2 + a^2} - \frac{k e^{-at} \sin(\omega t + \phi)}{\omega \sqrt{a^2 + \omega^2}}$ , $\phi = \tan^{-1}(\omega/a)$

## 2.6 EXERCISE

1. Find Laplace transform of the following functions:

a)  $x(t) = 1 + 2t$

b)  $x(t) = 4(1 - e^{-2t})$

c)  $x(t) = t^2 + 2 \sin 4t$

d)  $x(t) = -2e^{-3t} \cos 6t$

e)  $x(t) = 2e^{-4t} \sin 4t$

2. Find the partial fraction expansion of the following:

a)  $\frac{3s+1}{s(s+1)}$

b)  $\frac{6s^2+10s+2}{s^3+3s^2+2s}$

c)  $\frac{5s+6}{(s+2)(s+3)}$

d)  $\frac{1}{4s(s+4)}$

3. Find the inverse transforms,  $f(t)$ , of the following:

a)  $F(s) = \frac{6}{s(s+3)}$

b)  $F(s) = \frac{5}{s^2+1}$

c)  $F(s) = \frac{20}{s(s^2+10s+9)}$

d)  $F(s) = \frac{4-2s}{s^3+4s}$

e)  $F(s) = \frac{3s}{s^2+9}$

f)  $F(s) = \frac{s}{s^2+2s+1}$

g)  $F(s) = \frac{s}{s^2+2s+2}$

4. Find  $F(s)$  for the following differential equations for  $f(0) = 2$  and  $\frac{df(t)}{dt}(0) = 1$ .

a)  $3\frac{df(t)}{dt} + 4f(t) = 5$  where  $f(0) = 0$

b)  $2\frac{d^2f(t)}{dt^2} + 3\frac{df(t)}{dt} + 4f(t) = e^{-5t} + 6\cos 7t$

5. Solve the following differential equations using Laplace transform.

a)  $\frac{dx}{dt} + 2x = 8$  where  $x(0) = 0$

b)  $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 6$  where  $y(0) = 0$  &  $\frac{dy}{dt}(0) = 0$

c)  $\frac{d^2y}{dt^2} + 9y = 0$  where  $y(0) = 0$  &  $\frac{dy}{dt}(0) = 3$

d)  $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 10$  where  $x(0) = 0$  &  $\frac{dx}{dt}(0) = 0$

6. Find  $y(t)$  if  $Y(s) = \frac{100}{s\{(s+5)^2 + 25\}}$

7. Find the following Laplace transforms

a)  $\mathcal{L} \left[ \frac{dx}{dt} \right]$  where  $X(s) = \frac{2}{s}$  &  $x(0) = 2$

b)  $\mathcal{L} \left[ \frac{d^2x}{dt^2} \right]$  where  $X(s) = \frac{5}{s+1}$ ,  $x(0) = 5$  and  $\frac{dx}{dt}(0) = -5$

c)  $\mathcal{L} \left[ \int x(t)dt \right]$  where  $X(s) = \frac{3}{s^2 + 9}$  and  $\int x(t)dt = -\frac{1}{3}$  at  $t = 0$

8. The Laplace transform of error function of a servo system is given by.

$$E(s) = \frac{3s^2 + s + 8}{5s^3 + 4s^2 + 2s}$$

a) Find the initial error value  $e(0)$ .

b) Find the final (steady state) value of error  $e(\infty)$ .

9. The given figure shows a mass supported by a dashpot and spring. Give the equation of motion of mass and solve it by using Laplace transform if the exciting force  $f(t)$  is:

a)  $f(t) = a \sin(\omega t)$

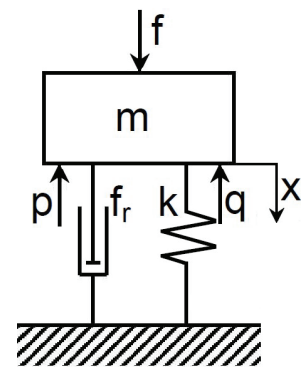
where  $a = 100$  N and  $\omega = 50$  rad/s

b)  $f(t)$  is a step function of 100 N magnitude applied at  $t=0$ .

Given:  $p = 6000 \frac{dx}{dt}$

$q = 5000 x$

$m = 1000$  kg



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# 3 | TRANSFER FUNCTIONS

## 3.1 BASIC DEFINITIONS

The static and dynamic behavior of a linear, single-input single-output system (SISO) can be described by the transfer function. The transfer function (TF) is defined as the Laplace transform of the system output divided by that of its input. It is systematically deducible for linear system of zero initial conditions. If the initial conditions are different from zero, then, for simplicity of mathematical manipulation, assume zero initial conditions by simple transformation of axes and then consider their real values during solution.

Conventionally, the symbol  $G(s)$  is used for the transfer function, but if the element appears in the feedback path, the symbol  $H(s)$  is used.

The linear, single-input single-output, system can be described by a single linear differential equation of zero initial conditions. This differential equation (DE), can be transformed into Laplace domain by replacing the term  $(d/dt)$  by  $(s)$ . The resulting transfer function takes the form:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)} \quad (3.1)$$

$$P(s) = \sum_{i=0}^m b_i s^i \quad \text{and} \quad Q(s) = \sum_{i=0}^n a_i s^i \quad (3.2)$$

For real systems, the order of  $P(s)$  should not exceed that of  $Q(s)$ ;  $n \geq m$ .

**Example 3.1** Consider the linear system, of input  $x(t)$  and output  $y(t)$ , described by the following differential equation of zero initial conditions.

$$a_4 \frac{d^4 y}{dt^4} + a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_2 \frac{d^2 x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x \quad (3.3)$$

The transfer function (TF) can be deduced by applying Laplace transform as follows.

$$(a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0)Y(s) = (b_2s^2 + b_1s + b_0)X(s) \quad (3.4)$$

Then, 
$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_2s^2 + b_1s + b_0}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0} \quad (3.5)$$

**Characteristic equation** is the equation obtained by equating the denominator of the overall transfer function to zero. The roots of characteristic equation are referred to as the poles of TF since they are the values of **s** that cause the transfer function amplitude to be infinity.

$$a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0 \quad (3.6)$$

**Order of the system** is the number of roots of characteristic equation (order of characteristic equation).

**Matrix transfer function:** For a multi-input multi-output system (MIMO), the system inputs and outputs can be related by a transfer matrix, Eq. 3.7. The transfer functions relating the system inputs and outputs can be directly obtained from this matrix, where **X** and **Y** are the input and output vectors respectively.

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \\ G_{31}(s) & G_{32}(s) \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} \quad (3.7)$$

**Example 3.2** In the case of a hydraulic line, Fig.3.1, the inlet pressure  $P_o$  and the volumetric flow rate  $Q_o$  are related to the outlet parameters,  $P_L$  and  $Q_L$  by a transfer matrix:

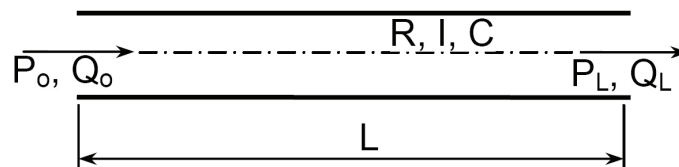


Fig.3.1 Fluid flow in circular transmission line

$$\begin{bmatrix} P_o(s) \\ Q_o(s) \end{bmatrix} = \begin{bmatrix} ICs^2 + RCs + 1 & Is + R \\ Cs & 1 \end{bmatrix} \begin{bmatrix} P_L(s) \\ Q_L(s) \end{bmatrix}$$

where  $R$  = line resistance.

C = line capacitance.

I = line inertia.

$$\text{Then } P_o(s) = (ICs^2 + RCs + 1)P_L + (Is + R)Q_L$$

$$\text{and } Q_o(s) = CsP_L + Q_L$$

The line end can be closed,  $Q_L=0$ , or open to atmosphere,  $P_L=0$ . For closed end line,  $Q_L=0$ , the transfer functions describing the line are:

$$\frac{P_L}{P_o}(s) = \frac{1}{ICs^2 + RCs + 1} \quad \text{and} \quad \frac{P_L}{Q_o} = \frac{1}{Cs}$$

While for open end line,  $P_L=0$ , the line transfer functions are:

$$\frac{Q_L}{P_o}(s) = \frac{1}{Is + R} \quad \text{and} \quad \frac{Q_L}{Q_o} = 1$$

**Gain of transfer function** is the proportionality coefficient relating the input and output in the steady state, i.e. as  $t \rightarrow \infty$  or  $s \rightarrow 0$ .

In order to obtain the transfer function of a system it is necessary to define the boundaries of the system and the relevant signals (variables). The relationships between these variables can be determined by writing down the equations governing the system behavior. The variables, which are not of direct interest, can be eliminated, leaving the relation between the required input and output. If the governing equations are linear, the transfer function can be easily deduced. But in the case of nonlinear relations, a linearization procedure or simplifying assumptions should be considered to obtain linearized equations.

## 3.2 TRANSFER FUNCTION OF SOME BASIC ELEMENTS

### 3.2.1 Proportional Element

The proportional element has a transfer function of the form  $G(s)=K$ , where K is constant. The simple coil spring is a typical proportional element. The simple equation describing the coil spring can be deduced considering the following assumptions:

1. The force is not so large that the spring coils touch each other and then the spring stiffness k is constant.

2. The reduced mass of the spring is negligible.
3. The structural damping of spring material is negligible.

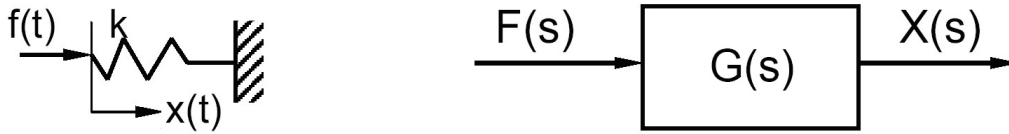


Fig.3.2 Simple coil spring

$$f(t) = kx(t) \quad (3.8)$$

$$F(s) = kX(s) \quad (3.9)$$

$$\boxed{G(s) = \frac{X(s)}{F(s)} = \frac{1}{k} = K} ; \text{ where } K \text{ is a constant} \quad (3.10)$$

### 3.2.2 Integrating Element

The integrating element has a transfer function of the form  $\boxed{G(s)=k/s}$ , where k is constant. The following are some examples of integrating members.

#### 3.2.2.1 Ideal hydraulic cylinder

Figure 3.3 shows a scheme of a hydraulic cylinder. Neglecting the inertia of moving parts, the effect of oil compressibility and wall elasticity, the relation between the system input (flow rate q) and output (displacement x) can be deduced as follows.

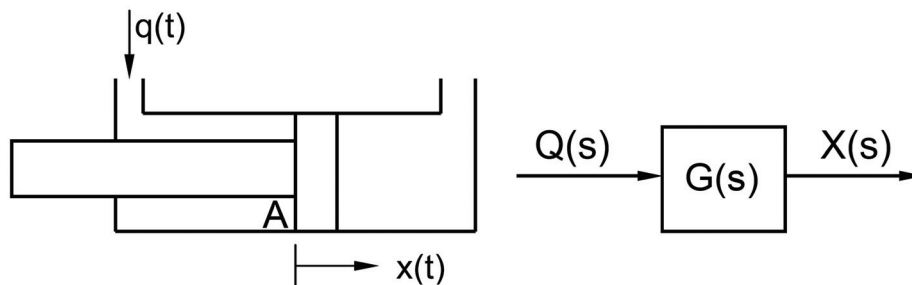


Fig.3.3 Scheme of a hydraulic cylinder

$$q(t) = A \frac{dx}{dt}, \quad (3.11)$$

where A is the piston rod-side area.

$$Q(s) = AsX(s) \quad (3.12)$$

$$G(s) = \frac{X(s)}{Q(s)} = \frac{1}{As} = \frac{k}{s}; \quad k = \frac{1}{A} \quad (3.13)$$

This is the transfer function of an integrating element. The output depends on the time integral of the input, or:

$$x(t) = k \int q(t)dt \quad (3.14)$$

### 3.2.2.2 Valve controlled actuator

Figure 3.4 shows a typical example of open loop system. The hydraulic cylinder **1** is controlled by means of a directional control valve **2**. When the spool of the valve is displaced by distance **e**, it connects the pressure line to the upper piston chamber **A** and connects the drain line to the lower piston chamber **B**. The throttling area and oil flow rate are proportional to displacement **e**. Therefore, for an input **e(t)**, a proportional output piston speed ( $dy/dt$ ) is produced. However, the output changes, for the same input, if the operating pressure, load or internal leakage are changed.

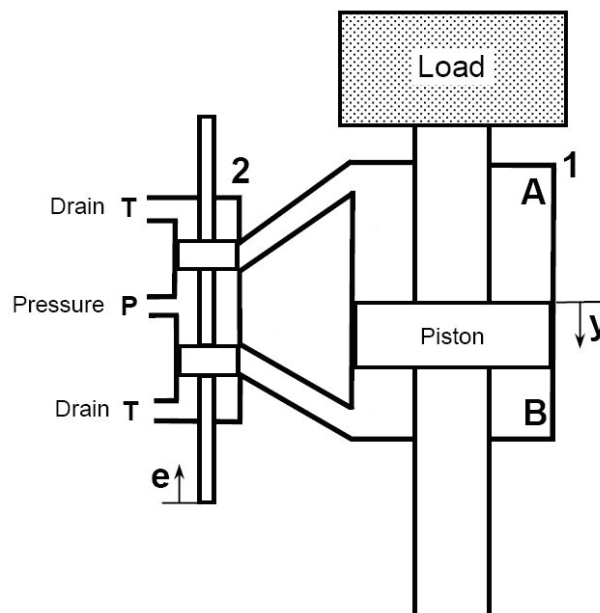


Fig.3.4 Hydraulic cylinder controlled by means of a directional control valve

The transfer function of this system can be deduced considering the following assumptions:

1. No internal leakage and the oil is incompressible.

2. Negligible inertia of moving parts.

3. Constant cylinder load and constant pressure drop across the valve restrictions.

The flow rate through the valve opening is given by:

$$q(t) = C_d A_t(e) \sqrt{2\Delta p / \rho} \quad (3.15)$$

$$A_t(e) = \omega e \quad (3.16)$$

Where  $C_d$  = Discharge coefficient = 0.611 for sharp edged restriction.

$A_t$  = Throttling area. For rectangular slot,  $A_t = \omega e$

$\rho$  = Fluid density

$\omega$  = Width of the opened area ( $\omega$  = constant for rectangular opening).

$\Delta p$  = Pressure difference across the throttle

For constant pressure difference, the flow rate equation becomes:

$$q(t) = k e(t) \quad (3.17)$$

where  $k$  is the proportionality coefficient

The piston speed is given by:

$$q(t) = A \frac{dy}{dt} \text{ or } \frac{dy}{dt} = q(t) / A, \text{ where } A \text{ is the piston area} \quad (3.18)$$

The application of Laplace transform to Eqs. 3.17 and 3.18 yields

$$Q(s) = kE(s) \quad \text{and} \quad sY(s) = \frac{Q(s)}{A} \quad (3.19)$$

Then

$$G(s) = \frac{Y}{E} = \frac{Y}{Q} \times \frac{Q}{E} = \frac{1}{As} \times k \quad \text{or} \quad \boxed{G(s) = \frac{K}{s}} \quad (3.20)$$

where  $K=k/A$ .

$$\text{Actually; } q(t) = A \frac{dy}{dt}, \text{ then: } y = \frac{1}{A} \int q(t) dt = \frac{k}{A} \int e(t) dt = K \int e(t) dt \quad (3.21)$$

Equation 3.21 indicates that the output  $y(t)$  is directly proportional to the time integral of the input  $\left(\int e(t) dt\right)$ . Therefore the shown valve controlled actuator is said to be an integrating element.

### 3.2.3 First Order Element

The first order element has a transfer function of the form:  $G(s) = k_1 / (Ts + 1)$ . It is also called simple lag element. The following are some examples of first order elements.

#### 3.2.3.1 Hydraulic servo actuator

The hydraulic servo actuator shown in Fig.3.5, is actuated by a low power input signal  $x(t)$  and produces a corresponding sufficiently high power output signal  $y(t)$ . The construction and operation of this system are explained in sec.1.3.2.

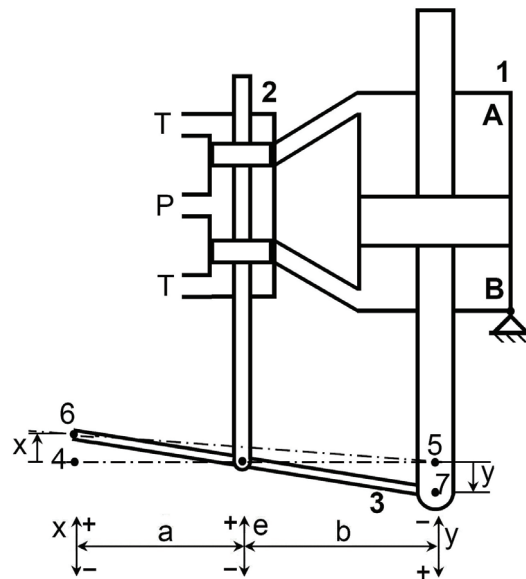


Fig.3.5 Hydraulic servo actuator with mechanical feedback

The following transfer function was deduced considering the discussion of valve controlled hydraulic cylinder, section 3.2.2.2, page 33:

$$G(s) = \frac{Y}{E} = \frac{K}{s} \quad (3.22)$$

The displacement  $e(t)$  is given by:

$$e = \frac{b}{a+b}x - \frac{a}{a+b}y \quad \text{or} \quad e = 0.5(x - y) \quad \text{for } a=b \quad (3.23)$$

The application of Laplace transform to Eq.3.23 yields:

$$E = 0.5(X - Y) \quad (3.24)$$

The hydraulic servo-actuator is described mathematically by Eqs. 3.22 and 3.24. The block diagram shown by Fig. 3.6 was developed using these equations and the system transfer function is:

$$\boxed{\frac{Y}{X}(s) = \frac{0.5K}{s + 0.5K} = \frac{1}{Ts + 1}} \quad (3.25)$$

This is a unity gain first order transfer function of time constant  $T=2/K$ .

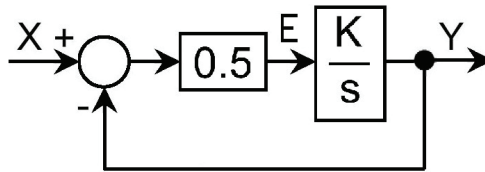


Fig.3.6 Block diagram of the hydraulic servo actuator

The following is the general form of transfer function of first order element, where  $T$  is the time constant and  $k_1$  is the gain

$$\boxed{G(s) = \frac{k_1}{Ts + 1}} \quad (3.26)$$

### 3.2.3.2 Resistance-capacitance network

The basic elements of the RLC circuits are described by the following expressions:

Resistance **R**       $v = Ri$  (3.27)

Inductance **L**       $v = L \frac{di}{dt}$  (3.28)

Capacitance **C**       $i = C \frac{dv}{dt}$  (3.29)

Where:  $v$  = Applied electric potential difference, V.  
 $i$  = Electric current, A

The given R-C circuit, Fig.3.7, can be described by the following equations.

$$i = \frac{v_i(t) - v_o(t)}{R} = C \frac{dv_o}{dt} \quad (3.30)$$

$$v_i(t) = v_o(t) + RC \frac{dv_o(t)}{dt} \quad (3.31)$$

Assuming zero initial conditions, the application of Laplace transform to Eq. 3.31, yields:

$$V_i(s) = V_o(s) + RCsV_o(s) = V_o(s)(1 + RCs) \quad (3.32)$$

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{RCs + 1} = \frac{1}{Ts + 1} \quad (3.33)$$

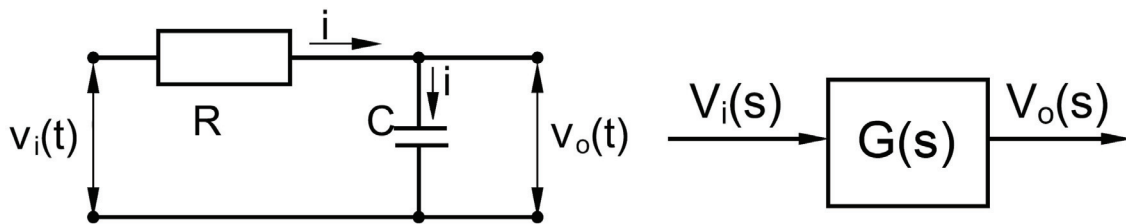


Fig.3.7 Simple RC Circuit

### 3.2.4 Second Order Element

The second order element has a transfer function of the form:  $G(s) = \frac{k}{as^2 + bs + 1}$ .

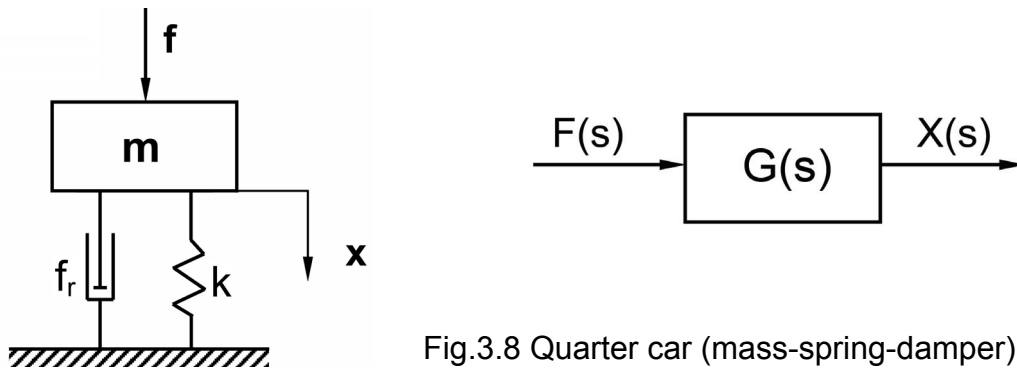


Fig.3.8 Quarter car (mass-spring-damper) system

The quarter car model shown by Fig. 3.8, is a typical example of the second order element. Assuming ideal linear spring and damper and neglecting the spring inertia, the following equation can be deduced.

$$m \frac{d^2x}{dt^2} + f_r \frac{dx}{dt} + kx = f \quad (3.34)$$

Assuming zero initial conditions, then the application of Laplace transform yields:

$$ms^2X(s) + f_r sX(s) + kX(s) = F(s) \quad (3.35)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + f_r s + k} \quad (3.36)$$

The general form of this equation is the second order transfer function, given by:

$$G(s) = \frac{k_1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1} = \frac{k_1 \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3.37)$$

where  $\omega_n = \sqrt{k/m}$  = Natural frequency of the system (the frequency by which the non-damped system oscillates in the absence of external forces).

$$\zeta = \frac{f_r}{2\sqrt{km}} = \text{damping ratio}$$

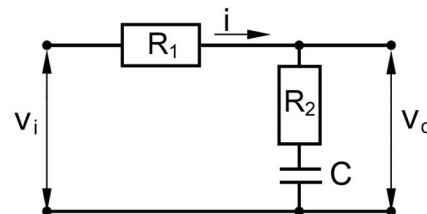
$$k_1 = 1/k = \text{Gain of transfer function}$$

### 3.3 EXERCISE

1. Derive the transfer function

$$G(s) = \frac{V_o}{V_i}(s) \text{ of the shown phase}$$

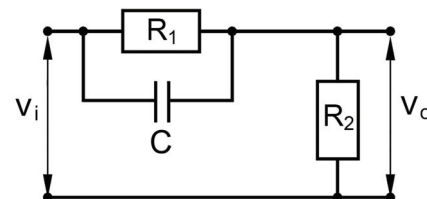
lag compensating RC circuit.



2. Derive the transfer function

$$G(s) = \frac{V_o}{V_i}(s) \text{ of the given phase}$$

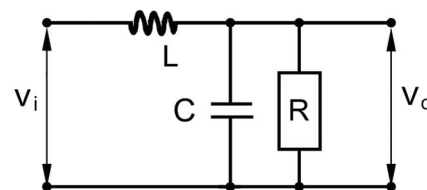
lead compensator RC circuit.



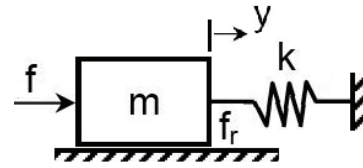
3. Derive the transfer function

$$G(s) = \frac{V_o}{V_i}(s) \text{ of the given RLC}$$

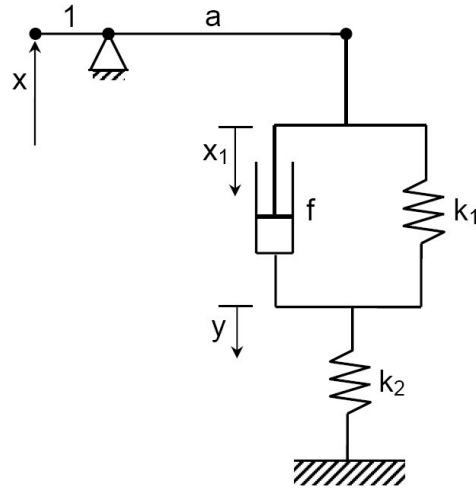
Circuit.



4. Derive the transfer function  $G(s) = \frac{Y}{F}(s)$  of the given system and give the mathematical expressions for the system gain, natural frequency and damping ratio.



5. Shown is a mechanical phase-lag compensator. Its elements are of negligible inertia. Derive the transfer function  $G(s) = \frac{Y}{X}(s)$

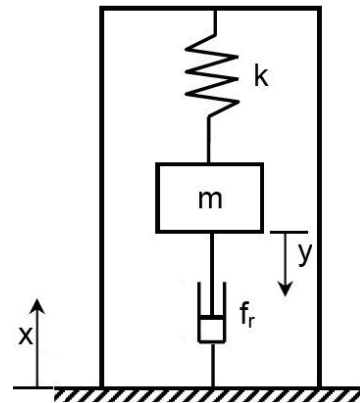


6. A typical accelerometer is shown in the given figure.

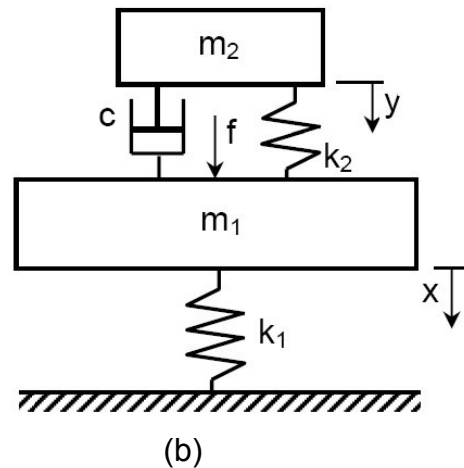
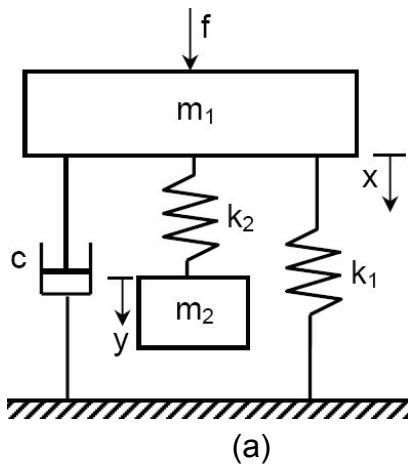
- a) Derive the mathematical model describing its dynamic behavior.  
 b) Derive the transfer function

$$G(s) = \frac{Y}{A}(s), \text{ where } a = d^2x/dt^2$$

- c) Find the accelerometer gain.

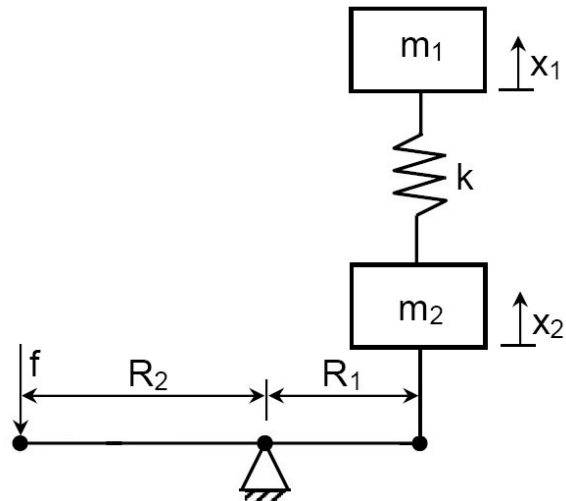


7. Derive the transfer function,  $G(s) = \frac{X(s)}{F(s)}$ , of each of the given systems.



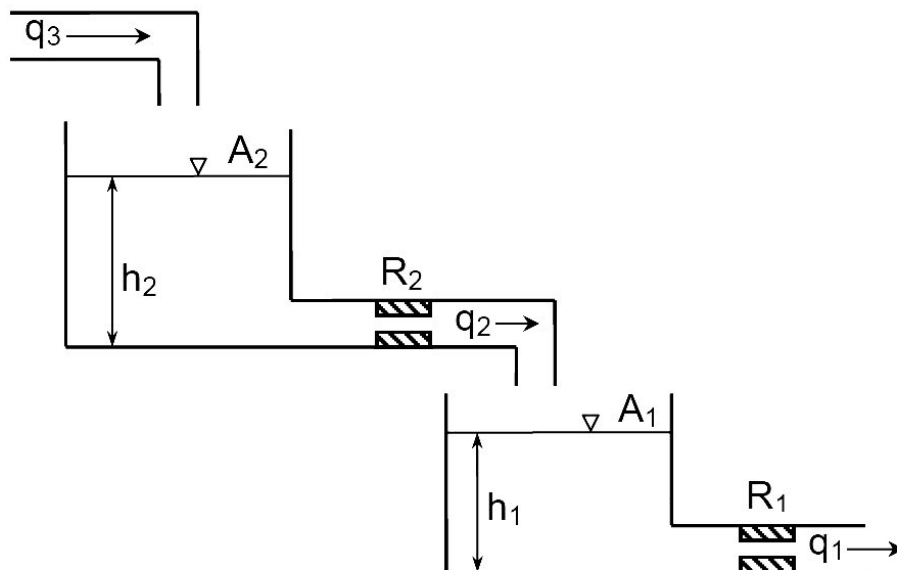
8. Derive the transfer function for the given system.

$$G(s) = X_1(s)/F(s)$$

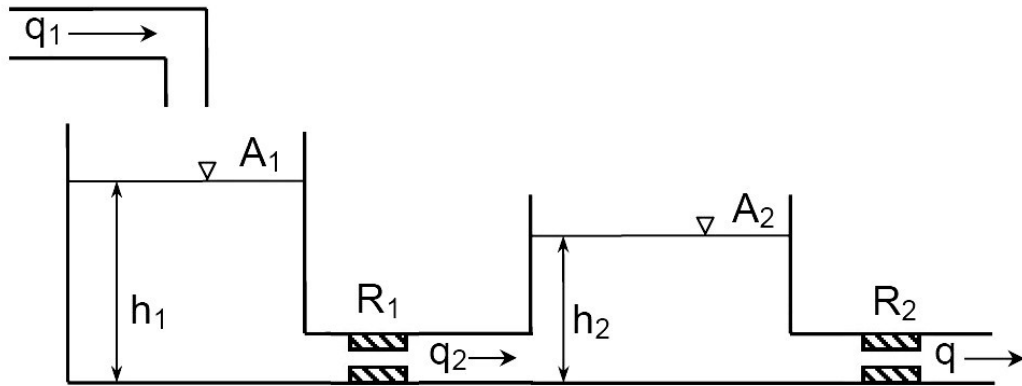


9. A process plant consists of two tanks with cross-sectional areas  $A_1$  and  $A_2$ . The flow rate through the throttle valve is given by:  $q = \Delta h / R$ , where  $\Delta h$  is the head difference across the valves in meters and  $R$  is the throttle valve resistance. Write the equations describing the system operation and derive the following transfer functions:

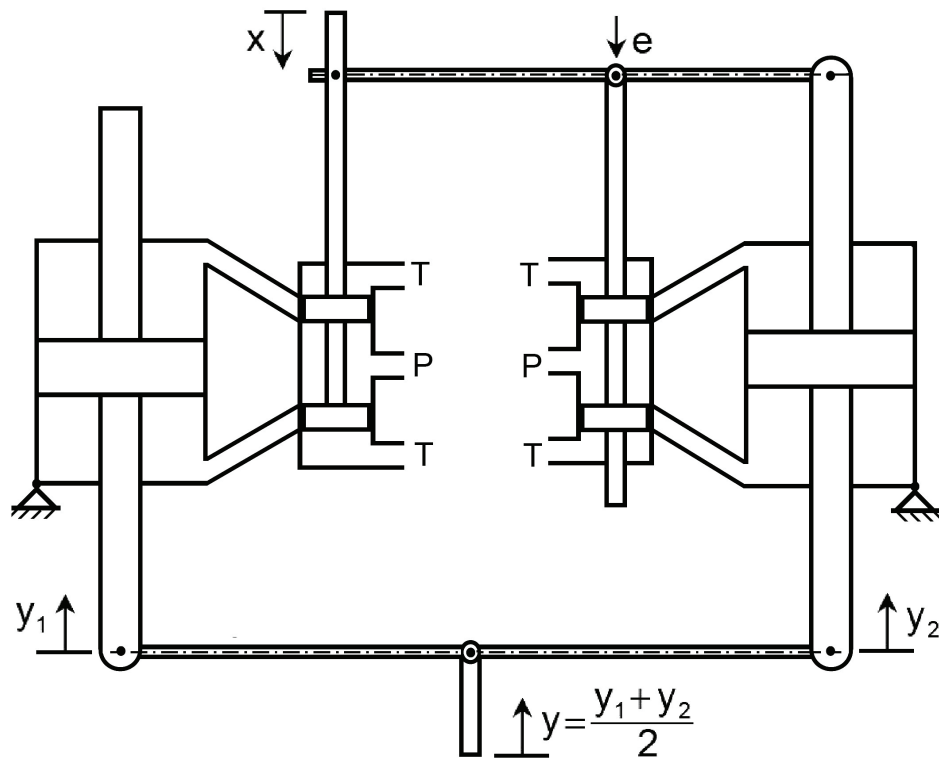
$$G_1(s) = \frac{H_1}{Q_2}, \quad G_2(s) = \frac{Q_2}{Q_3} \quad \text{and} \quad G_3(s) = \frac{H_1}{Q_3}$$



10. The given hydraulic system includes two rigid cylindrical tanks with areas  $A_1$  and  $A_2$ . Write the equations describing the system operation and derive the transfer function  $G(s) = Q(s) / Q_1(s)$ .



11. Derive the transfer function  $G(s) = Y(s)/X(s)$  for the given proportional plus integral control system.



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# 4 | BLOCK DIAGRAMS

## 4.1 INTRODUCTION

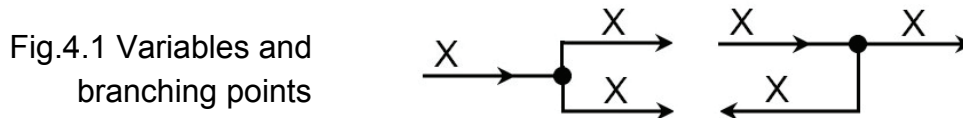
When dealing with the dynamic behavior of a system, one starts usually by the derivation of the equations and the transfer functions describing the behavior of its components. These transfer functions are needed for the analysis of the dynamic behavior of the components. Therefore, when carrying out the integrated system analysis, it is necessary to derive the transfer function of the overall system. This derivation becomes easier when representing the system by a block diagram.

Most of the systems consist of several subsystems interconnected in forward and feedback paths. The algebraic manipulation of the mathematical relations is simplified by the use of a shorthand notation for the transfer function. A popular presentation uses  $G(s)$  with a suitable subscript for the forward path transfer functions and  $H(s)$  for the feedback path transfer functions.

## 4.2 CONVENTIONS FOR BLOCK DIAGRAMS

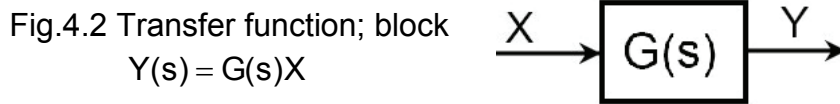
The following symbols are used as basic structural elements of the block diagrams.

**Variable X** is represented by a straight line. An arrow indicates the direction of the signal transmission.

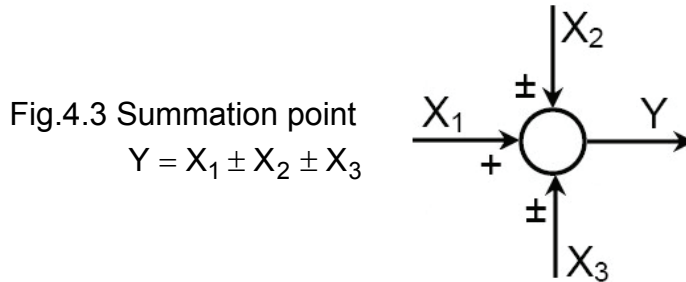


**Branching Points** are indicated by heavy point with single incoming signal and one or more outgoing signals. All of the out-going lines carry the same signal, Fig.4.1.

**Transfer Function** is represented by a rectangle (named a block). The mathematical expression of the transfer function, or its symbol, is written inside the block. The block is connected to the system by means of two lines carrying its input and output signals, Fig.4.2.



**Summation Point** is represented by a small circle with more than one in-going signal and only one out-going signal. The in-going signal sign is indicated by (+) or (-) signs. The value of the out-going signal equals the sum of the in-going signals, each with its assigned sign, Fig.4.3.



### 4.3 DEDUCING SYSTEM TRANSFER FUNCTION

The system transfer function can be deduced from its block diagram by extracting and manipulating the equations relating the system variables. But, in the case of complicated block diagrams, it might be necessary to simplify the diagram by applying the rules of block diagram algebra.

Figure 4.4 shows the block diagram of a typical system with negative feedback. The system transfer function can be deduced from the block diagram as follows:

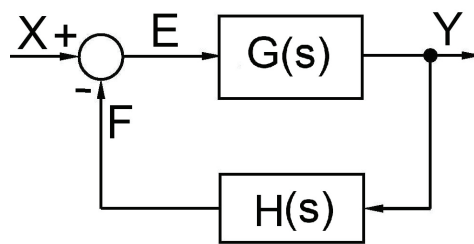


Fig.4.4 Block diagram of negative feedback system

$$Y = GE, \quad F = HY \quad \text{and} \quad E = X - F \quad (4.1)$$

Or

$$Y = G(X - HY) \quad (4.2)$$

Then

$$\frac{Y}{X} = \frac{G}{1 + GH} \quad (4.3)$$

It is important to give the following definitions for the closed loop (feedback) system.

**Closed loop transfer function**

$$G(s) = \frac{Y}{X} = \frac{G}{1+GH} \quad (4.4)$$

**Compensated open loop transfer function**

$$G_o(s) = \frac{F}{E} = GH \quad (4.5)$$

**Actuating signal ratio**

$$\frac{E}{X} = \frac{1}{1+GH} \quad (4.6)$$

**Characteristic equation**

$$1+GH = 0 \quad (4.7)$$

**Principle of superposition:** In the case of multi-input systems, the principle of superposition can be applied. According to this principle, the response  $y(t)$  of a linear system due to simultaneously acting inputs  $x_1(t)$ ,  $x_2(t)$ , ...,  $x_n(t)$ , is equal to the sum of the responses to each input acting alone. If  $y_i(t)$  is the response due to the input  $x_i(t)$ , then,

$$y(t) = \sum y_i(t) \quad (4.8)$$

For Positive feedback, Fig. 4.5, the transfer function is:

$$G(s) = \frac{Y}{X} = \frac{G}{1-GH} \quad (4.9)$$

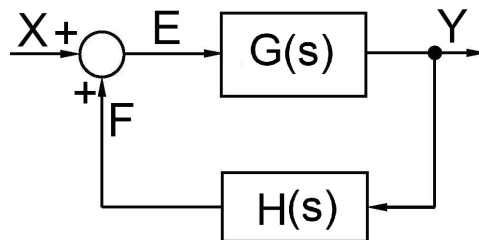


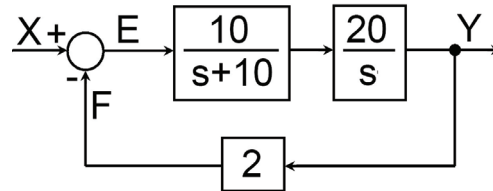
Fig.4.5 Block diagram of positive feedback system

### 4.4 BLOCK DIAGRAM ALGEBRA

The block diagrams of complicated control systems can be simplified by applying simple rules of block diagram algebra, which are summarized in the Table 4.1.

**Example 4.1** shown is the block diagram of a feedback system. Find its transfer function,  $G=Y/X$ .

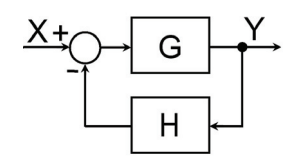
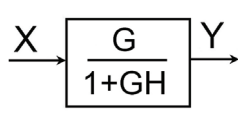
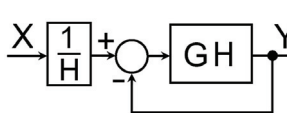
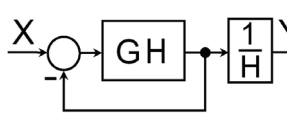
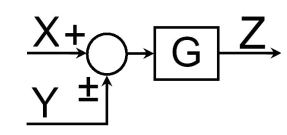
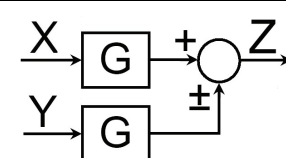
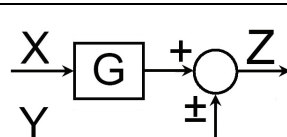
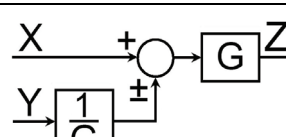
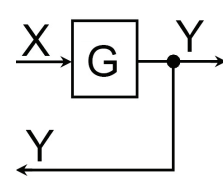
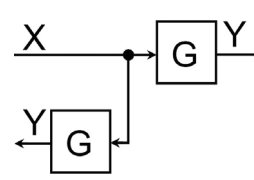
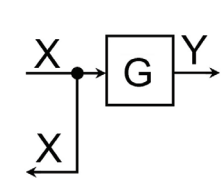
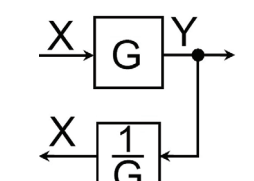
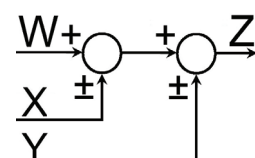
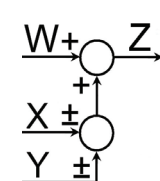
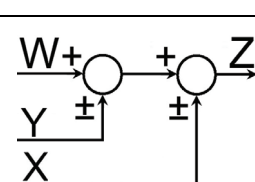
$$\frac{Y}{E} = \frac{200}{s(s+10)}$$



$$G(s) = \frac{Y}{X} = \frac{\frac{200}{s(s+10)}}{1 + \frac{2 \times 200}{s(s+10)}} = \frac{200}{s^2 + 10s + 400} = \frac{0.5}{\left(\frac{s}{20}\right)^2 + \frac{0.5}{20}s + 1}$$

Table 4.1 Rules of Block Diagram Algebra

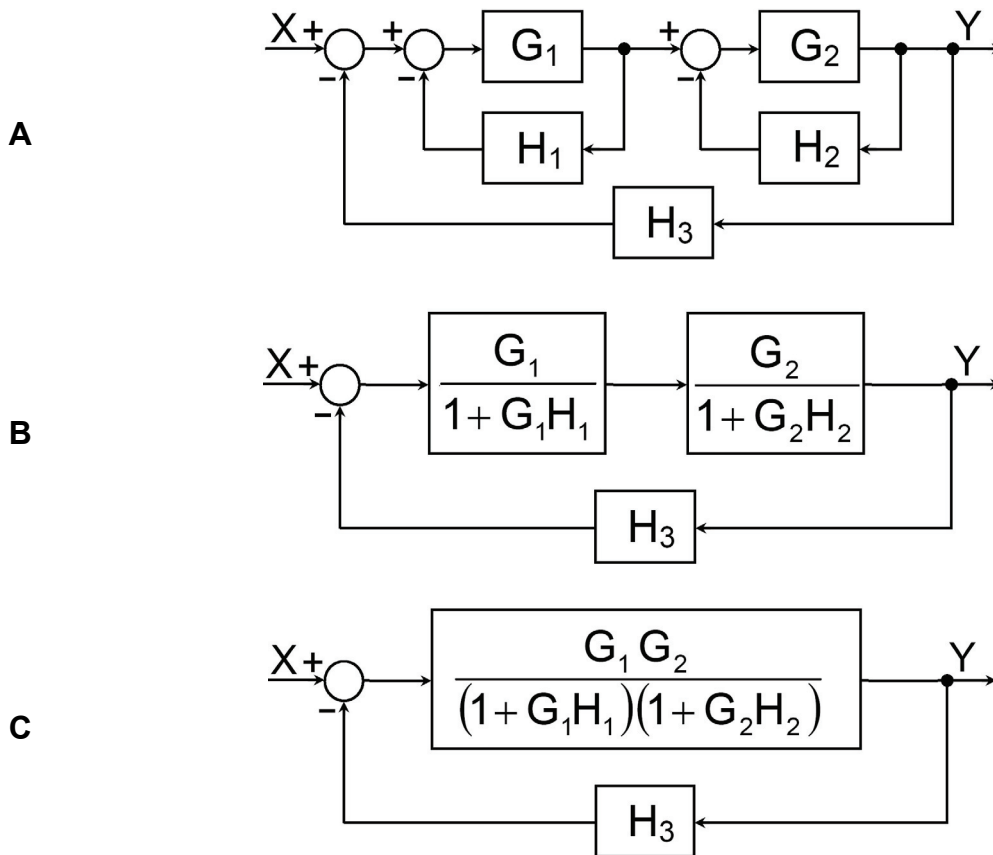
Type	Equation	Block Diagram	Equivalent Block Diagram
1. Series System	$Y = G_1G_2X$		
2. Parallel System	$Y = (G_1 \pm G_2)X$		

Type	Equation	Block Diagram	Equivalent Block Diagram
3. Loop Reduction	$Y = G(X - HY)$ Or $Y = \frac{G}{1+GH} X$		
			
			
4. Displacement of Summation Point	$Z = G(X \pm Y)$		
			
5. Displacement of Feedback point	$Y = GX$		
			
6. Re-arrangement of Summation Points	$Z = W \pm X \pm Y$		
			

Type	Equation	Block Diagram	Equivalent Block Diagram
7. Modification of position of feedback and summation points	$Z = X \pm Y$		

**Example 4.2** Apply the rules of block diagram algebra to deduce the transfer function of the system described by the given block diagram.

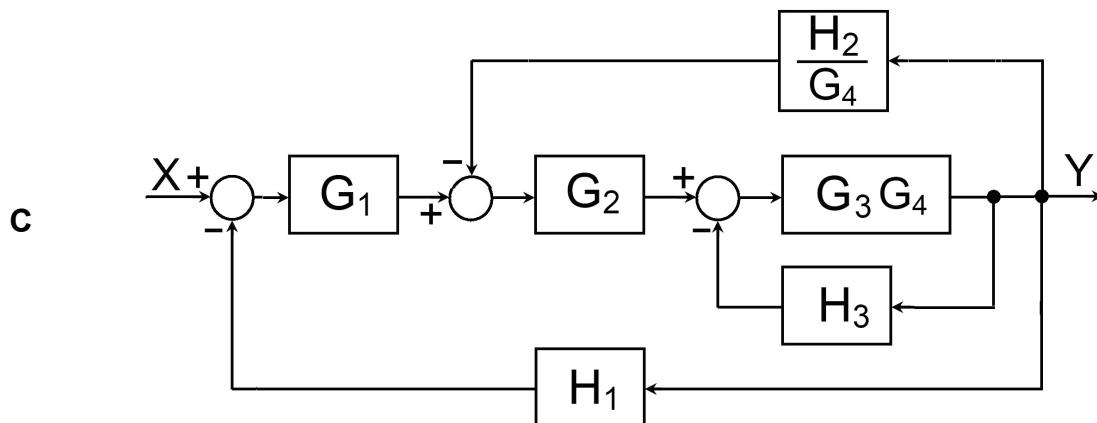
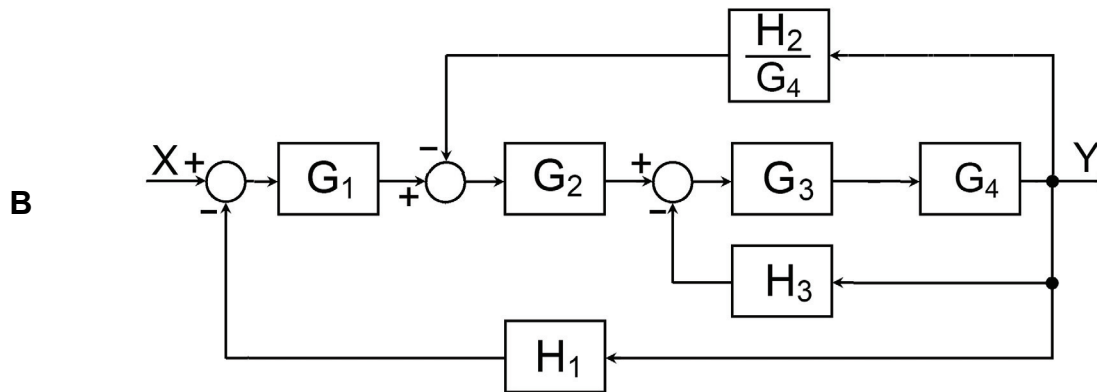
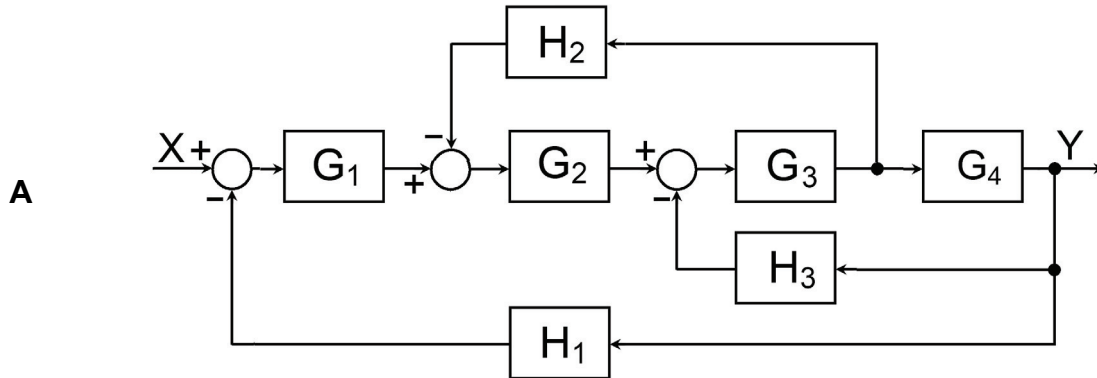
The following are the individual steps of block diagram reduction.

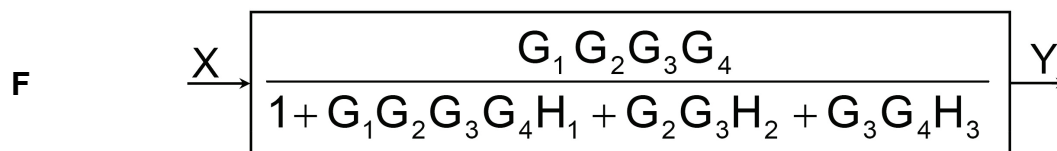
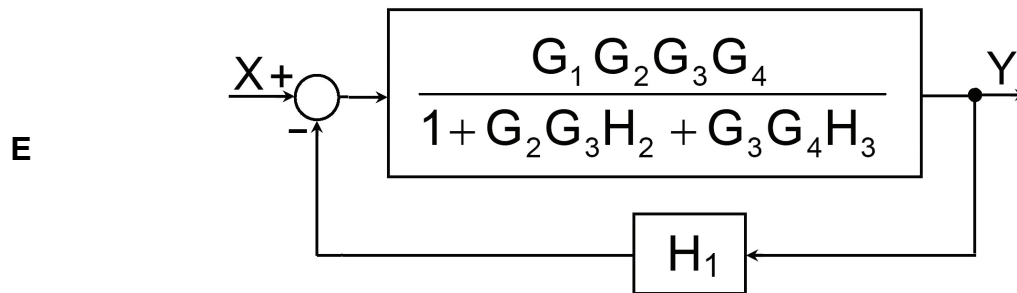
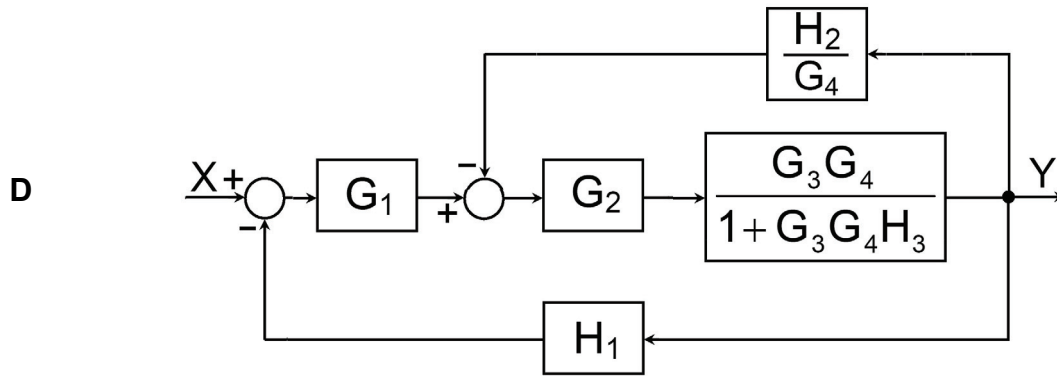


**D**

$$\frac{Y}{X} = \frac{G_1 G_2}{(1 + G_1 H_1)(1 + G_2 H_2) + H_3 G_1 G_2}$$

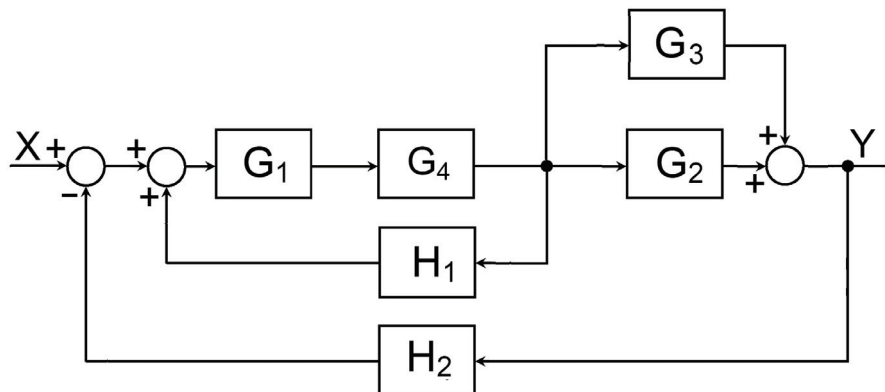
**Example 4.3** Find the system transfer function,  $G(s)=Y(s)/X(s)$ , by successive block diagram reduction.



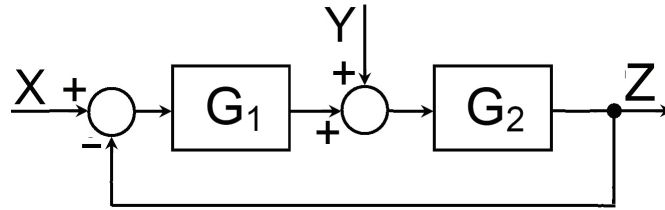


### 4.5 EXERCISE

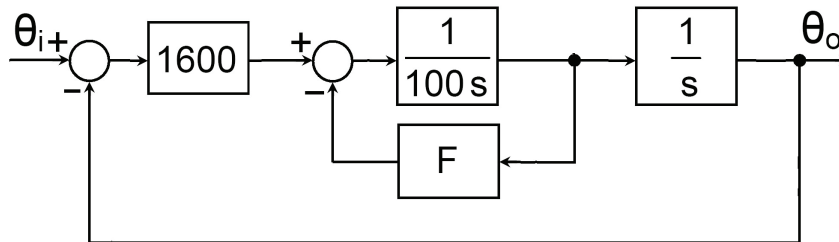
1. Derive the transfer function of the following system.



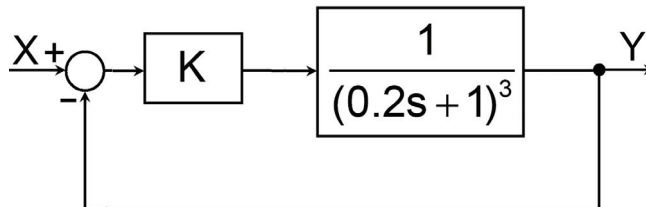
2. A closed-loop control system is subjected to a disturbance  $Y(s)$  as shown in figure. Show by the principle of superposition the effect of input  $X(s)$  and disturbance  $Y(s)$  on the system output  $Z(s)$ .



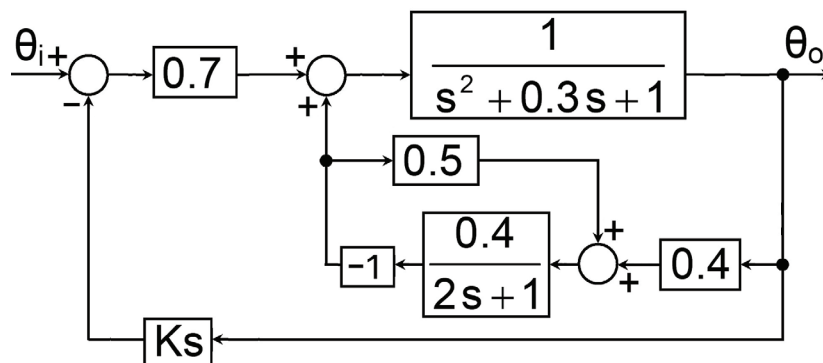
3. Find the transfer function,  $G(s) = \theta_o(s)/\theta_i(s)$ , of the system described by the following block diagram.



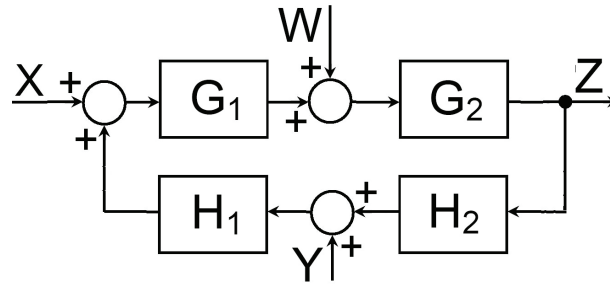
4. Find the transfer function  $G(s)=Y(s)/X(s)$



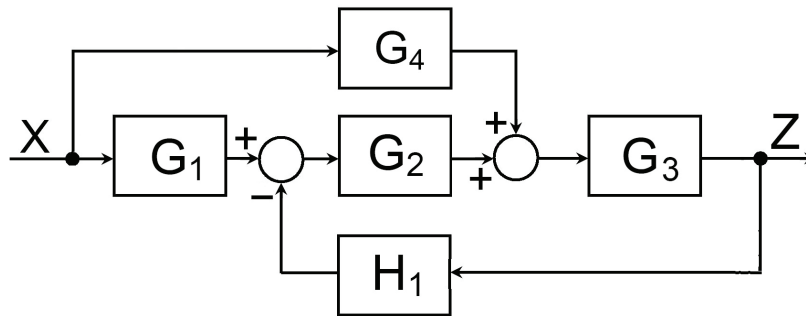
5. Derive the transfer function of the system described by the following block diagram;  $G(s) = \frac{\theta_o}{\theta_i}(s)$ .



6. Find an expression for  $Z(s) = f(W, X, Y)$  for the following system.



7. Redraw the following block diagram to obtain the transfer function  $G(s) = Z/X$ .



# 5 | SIGNAL FLOW GRAPH

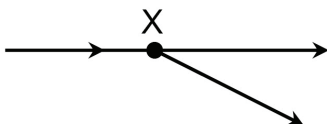
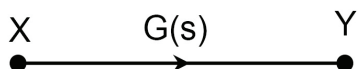
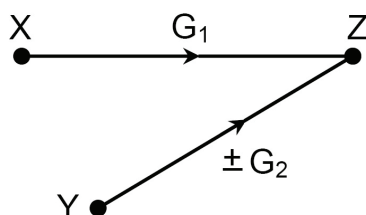
## 5.1 INTRODUCTION

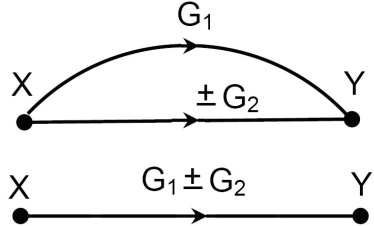
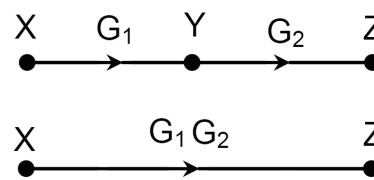
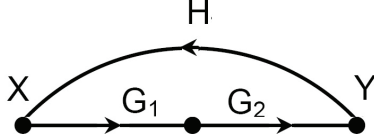
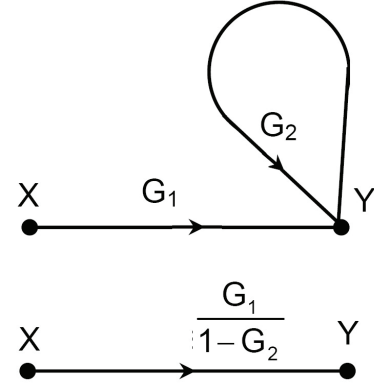
Signal flow graphs are alternatives to block diagrams. They are drawn either using the system of equations or based on the block diagrams. The signal flow graph consists of nodes, which represent the system variables and directed line segments which carry the relation-ship between the variables. Originally, the signal flow graph was introduced by S. J. MASON (1955), for the cause and effect representation of linear systems, described by algebraic equations.

## 5.2 CONVENTIONS FOR SIGNAL FLOW GRAPHS

The signal flow graphs consist of line segments (called Branches) and junction points (called nodes). The, nodes represent the variables, summation points and branching points. They are connected together by the line segments. The branches have associated branch gain and direction. The signal is transmitted through a branch in the direction of the arrow. The conventions for signal flow graphs are summarized in Table 5.1.

Table 5.1 Conventions for signal flow graphs

Element	Convention
1. Variable X	
2. Transfer Function $G(s) = \frac{Y}{X}$ Or $Y = GX$	
3. Summation Point $Z = G_1X \pm G_2Y$	

<p>4. Parallel Connection</p> $Y = G_1X \pm G_2X$ $= (G_1 \pm G_2)X$	
<p>5. Series Connection</p> $Z = G_2Y$ $Y = G_1X$ $Z = G_1G_2X$	
<p>6. Loop</p> <p>With loop gain = <math>G_1G_2H</math></p>	
<p>7. Self Loop</p> $Y = G_1X + G_2Y$ $Y = \frac{G_1}{1-G_2}X$	

**Example 5.1** Figure 5.1 shows a signal flow graph of a system described by five state variables,  $X_1, X_2, X_3, X_4$  and  $X_5$ . This graph was constructed for a system described by the following equations, where  $t_i$  is a transfer function.

$$X_2 = t_{12}X_1 + t_{32}X_3$$

$$X_3 = t_{23}X_2 + t_{43}X_4$$

$$X_4 = t_{34}X_3 + t_{24}X_2 + t_{44}X_4$$

$$X_5 = t_{25}X_2 + t_{45}X_4$$

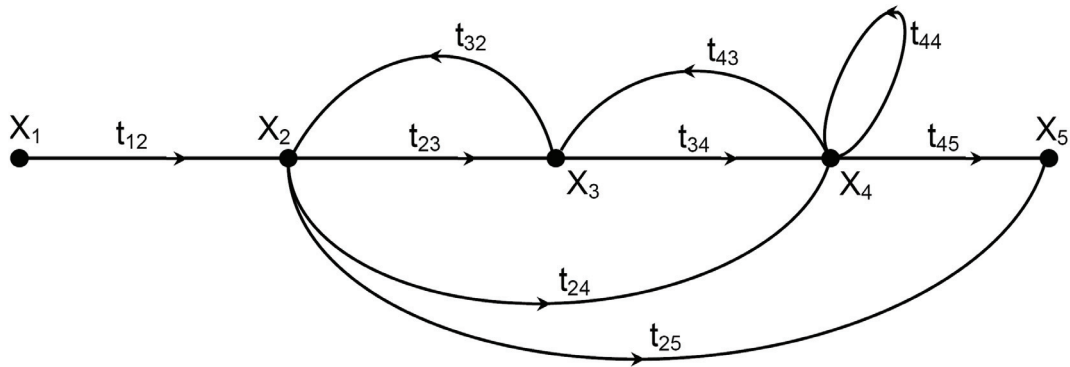


Fig.5.1 Signal flow graph

The transfer function of the system can be obtained from the signal flow graph, in a relatively simple way, by applying Mason's formula. For this purpose, it is important to give the following definitions, applied to the signal flow graph, Fig.5.1.

**Input node (Source):** is a node which has only outgoing branches,  $X_1$ .

**Output node (Sink):** is a node, which has only ingoing branches  $X_5$ .

**Forward path:** is a path from input node to output node along which no node is encountered more than once.

**Feedback loop:** is a path, which originates and terminates at the same node, along which no node is encountered more than once.

**Path gain:** is the product of all gains (transfer functions) along the forward path. The following are the path gains of signal flow graph shown in Fig.5.1.

$$M_1 = t_{12} t_{23} t_{34} t_{45}$$

$$M_2 = t_{12} t_{24} t_{45}$$

$$M_3 = t_{12} t_{25}$$

**Loop gain:** is the product of the gains of branches forming the loop. The following are the loop gains of signal flow graph shown in Fig.5.1.

$$\phi_1 = t_{23} t_{32}$$

$$\phi_2 = t_{34} t_{43}$$

$$\phi_3 = t_{44}$$

$$\phi_4 = t_{24} t_{43} t_{32}$$

### 5.3 MASON'S FORMULA

Given a signal flow graph, the task of solving for input-output relations by algebraic manipulation may be quite tedious. Fortunately, there exists a general gain formula available, known as the Mason's formula. It enables to deduce the system transfer function on the basis of the signal flow graph. This general formula is:

$$G(s) = \frac{\sum_{i=1}^k M_i \Delta_i}{\Delta} \quad (5.1)$$

$$\Delta = 1 - \sum_{i=1}^n \phi_i + (-1)^r \sum_{i=1}^m P_{mr} \quad (5.2)$$

Or

$$\Delta = 1 - \Sigma (\text{all loops gains}) + \Sigma (\text{gain products of all combinations of two mutually non-touching loops}) - \Sigma (\text{gain products of all combinations of three mutually non-touching loops}) + \dots \text{etc.} \quad (5.3)$$

and

$$\Delta_i = \Delta \text{ for all loops that do not touch the } i^{\text{th}} \text{ forward path} \quad (5.4)$$

Where

- $k$  = Number of forward paths.
- $M_i$  = Gain of  $i^{\text{th}}$  forward path.
- $n$  = Number of loops
- $P_{mr}$  = Gain product of  $m$  possible combinations of  $r$  mutually non-touching loops.
- $\phi_i$  = Gain of  $i^{\text{th}}$  loop.

**Example 5.2** Find the transfer function of the system represented by the signal flow graph of Fig.5.1.

1. Forward Paths Gains

$$M_1 = t_{12} t_{23} t_{34} t_{45}$$

$$M_2 = t_{12} t_{24} t_{45}$$

$$M_3 = t_{12} t_{25}$$

## 2. Loops Gains

$$\varphi_1 = t_{23} t_{32}$$

$$\varphi_2 = t_{34} t_{43}$$

$$\varphi_3 = t_{44}$$

$$\varphi_4 = t_{24} t_{43} t_{32}$$

## 3. $\Delta$ and $\Delta_i$

$$\Delta = 1 - (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) + \varphi_1\varphi_3$$

$$\Delta = 1 - (t_{23}t_{32} + t_{34}t_{43} + t_{44} + t_{24}t_{43}t_{32}) + t_{23}t_{32}t_{44}$$

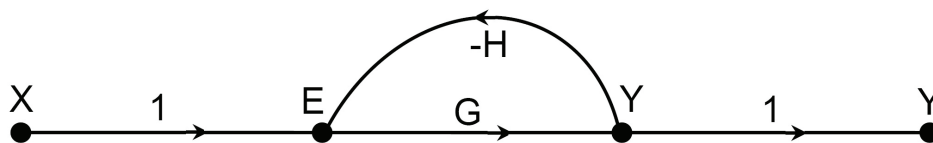
$$\Delta_1 = 1, \quad \Delta_2 = 1 \quad \text{and} \quad \Delta_3 = 1 - (\varphi_2 + \varphi_3) = 1 - t_{34}t_{43} - t_{44}$$

## 4. Transfer Function

$$G(s) = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

$$G(s) = \frac{t_{12}t_{23}t_{34}t_{45} + t_{12}t_{24}t_{45} + t_{12}t_{25}(1 - t_{34}t_{43} - t_{44})}{1 - t_{23}t_{32} - t_{34}t_{43} - t_{44} - t_{24}t_{43}t_{32} + t_{23}t_{32}t_{44}}$$

**Example 5.3** Find the transfer function of the feedback system shown in Fig.5.2.



$$M_1 = G,$$

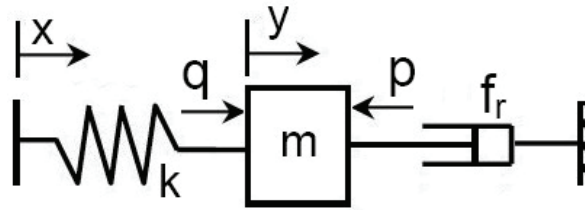
$$\varphi_1 = -GH$$

$$\Delta = 1 - (-GH) = 1 + GH,$$

$$\Delta_1 = 1$$

$$G(s) = \frac{M_1\Delta_1}{\Delta} = \frac{G}{1 + GH}$$

**Example 5.4** Derive a mathematical model for the following system, draw its signal flow graph and derive its transfer function by applying Mason's formula.



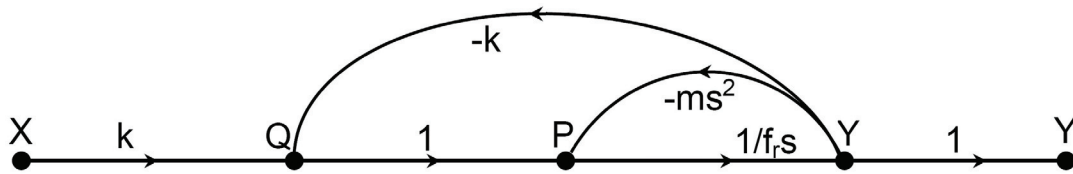
1. Mathematical Model

$$q = k(x - y) \quad \xrightarrow{\text{Applying Laplace Transform}} \quad Q = k(X - Y)$$

$$q - p = m \frac{d^2y}{dt^2} \quad \quad \quad Q - P = ms^2 Y; \quad P = Q - ms^2 Y$$

$$p = f_r \frac{dy}{dt} \quad \quad \quad P = f_r s Y; \quad Y = \frac{P}{f_r s}$$

2. Signal Flow Graph



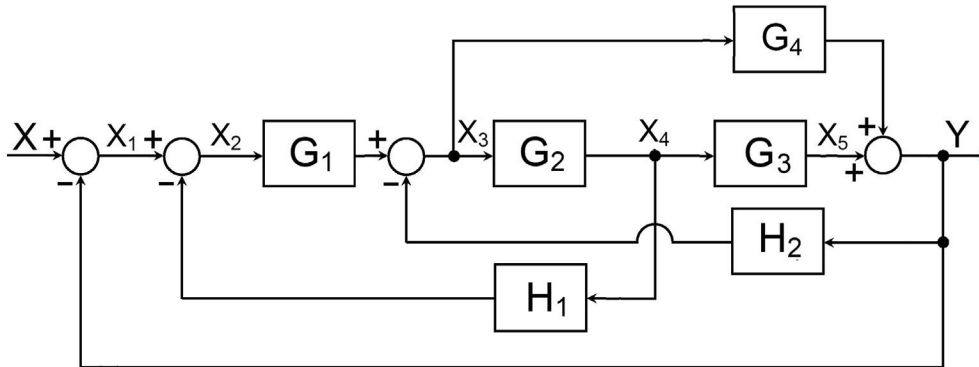
3. Transfer Function

$$M_1 = \frac{k}{f_r s}, \quad \varphi_1 = -\frac{k}{f_r s} \quad \text{and} \quad \varphi_2 = -\frac{ms}{f_r}$$

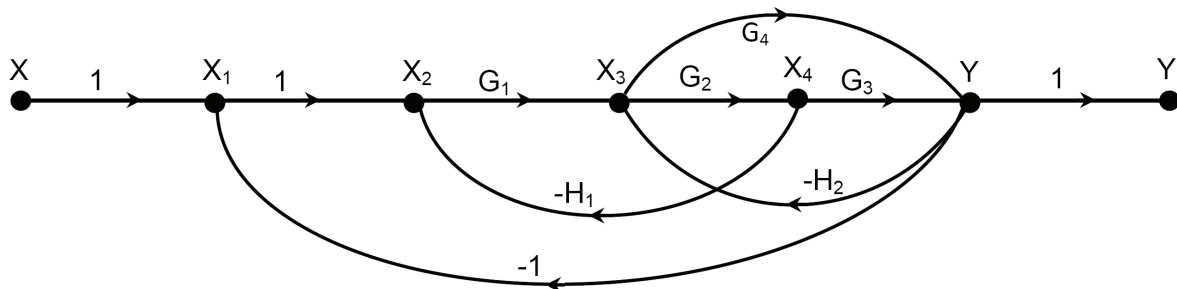
$$\Delta = 1 - \left( -\frac{k}{f_r s} - \frac{ms}{f_r} \right) = 1 + \frac{k + ms^2}{f_r s}, \quad \Delta_1 = 1$$

$$G(s) = \frac{Y}{X} = \frac{M_1 \Delta_1}{\Delta} = \frac{\frac{k}{f_r s} \times 1}{1 + \frac{k + ms^2}{f_r s}} = \frac{k}{ms^2 + f_r s + k}$$

**Example 5.5** Draw the signal flow graph of the system represented by the given block diagram and find its transfer function (apply the conversion guide lines on page 61).



1. Signal Flow Graph



2. Loops and paths gains

$$M_1 = G_1 G_2 G_3$$

$$M_2 = G_1 G_4$$

$$\phi_1 = -G_1 G_2 G_3$$

$$\phi_2 = -H_1 G_1 G_2$$

$$\phi_3 = -H_2 G_2 G_3$$

$$\phi_4 = -H_2 G_4$$

$$\phi_5 = -G_1 G_4$$

3. Application of Mason's formula

$$\Delta = 1 - \phi_1 - \phi_2 - \phi_3 - \phi_4 - \phi_5$$

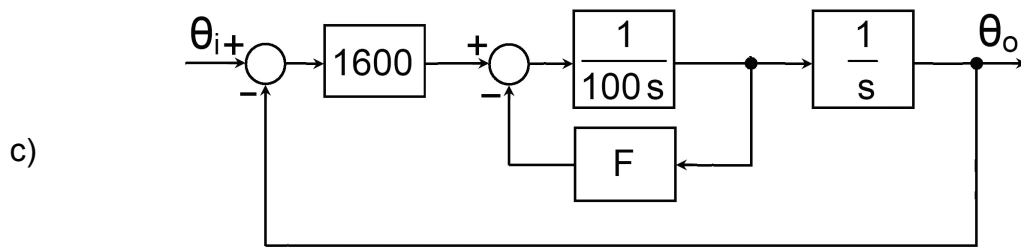
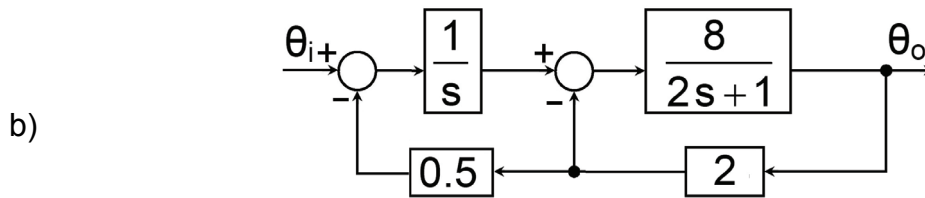
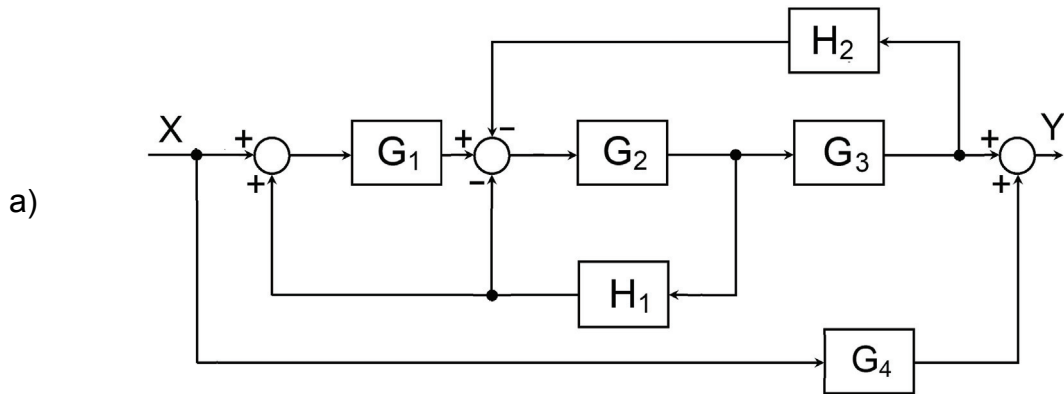
$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

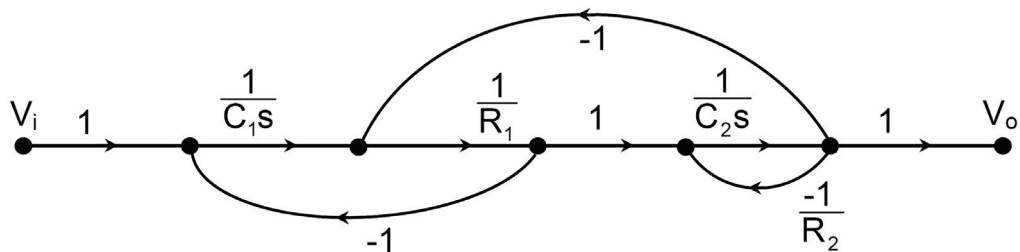
$$G(s) = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 G_3 + H_1 G_1 G_2 + H_2 G_2 G_3 + H_2 G_4 + G_1 G_4}$$

### 5.4 EXERCISE

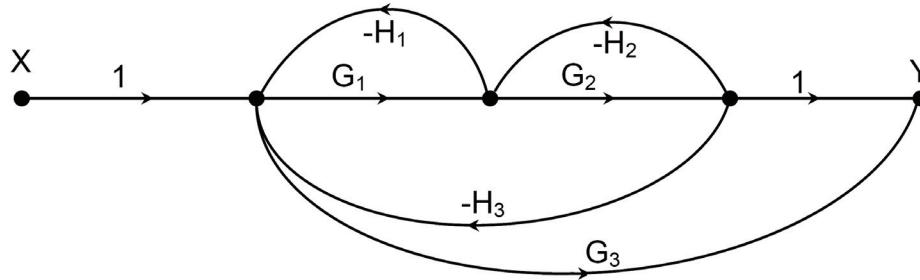
1. Draw the signal flow graph for the systems described by the following block diagrams and find the closed loop transfer function using Mason's formula.



2. Find the closed-loop transfer function  $V_o/V_i$  of the system described by the following signal flow graph.



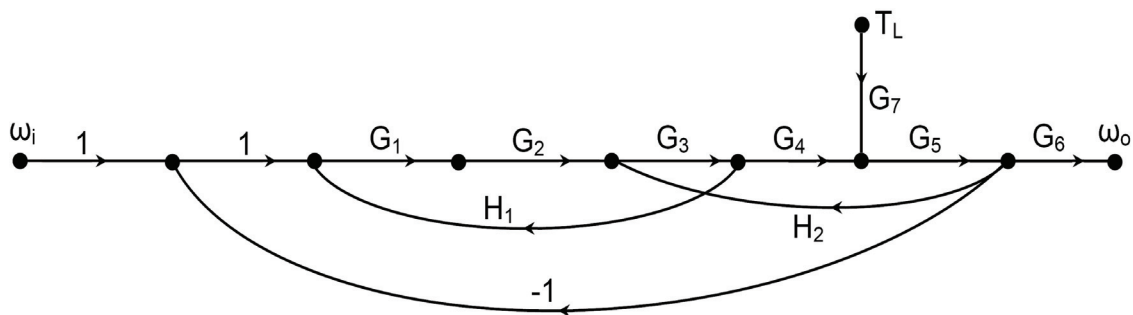
3. Draw a block diagram for the system represented by the following signal flow graph and find its transfer function.



4. Derive the expression relating the output velocity  $\omega_o$  to the load torque  $T_L$  and input signal  $\omega_i$  for the following signal flow graph, given:

$$G_1 = \frac{90}{1+0.06s} \quad G_2 = \frac{2.5}{1+0.19s} \quad G_3 = 25 \quad G_4 = 3.8 \quad G_5 = \frac{1}{1355s}$$

$$G_6 = 0.4 \quad G_7 = -0.4 \quad H_1 = -0.001 \quad H_2 = -4$$



**NB** Guide lines for the conversion of Block diagrams to Signal flow graphs

	Block Diagram Element	Equivalent Signal Flow Graph Element
a)	Block (transfer function)	Branch (transfer function)
b)	Summation point	Node with only one outgoing branch
c)	Branching point	Node with only one ingoing branch
d)	Summation point followed by branching point	Single node (Multi-input multi-output)
e)	Branching point followed by summation point	Two separate nodes. One having single-input multi-output followed by a multi-input single-output node

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# 6 | TIME DOMAIN ANALYSIS

## 6.1 INTRODUCTION

In the transient conditions, the variables describing the dynamic systems change with time. It is of interest to evaluate how do these variables vary with time. In the analysis problem, a reference input signal is applied to the system and the variation of the state parameters with time is observed and analyzed (Time Domain Analysis). The step, ramp and impulse signals are usually used as input signals for the time domain analysis. The step input is most widely used and represents the most possible severe excitations to which the system may be subjected.

### Step Function

$$f(t) = \begin{cases} 0 & \text{for } t = 0 \\ k_2 & \text{for } t > 0 \end{cases} \quad (6.1)$$

$$F(s) = \frac{k_2}{s} \quad (6.2)$$

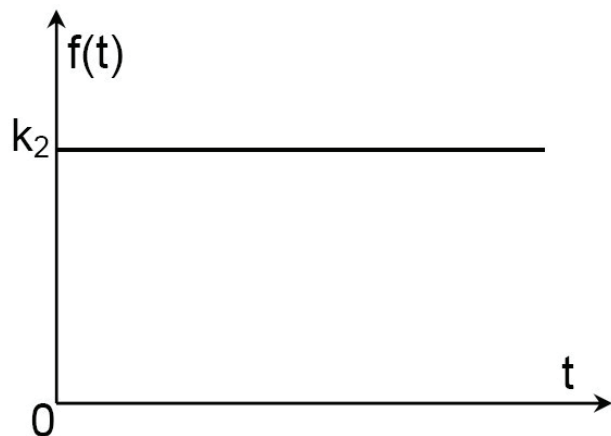


Fig.6.1 Step function

### Ramp Function

$$f(t) = k_2 t \quad (6.3)$$

$$F(s) = \frac{k_2}{s^2} \quad (6.4)$$

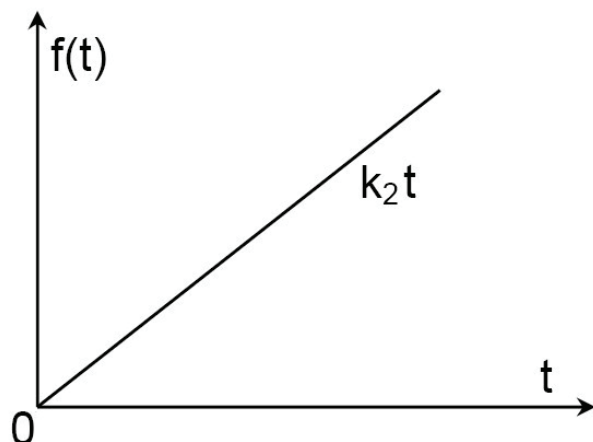


Fig.6.2 Ramp function

**Impulse Function**

$$f(t) = k_2 \delta(t) \quad (6.5)$$

$$F(s) = k_2 \quad (6.6)$$

where  $\delta(t)$  is the Delta Dirac function;  $\int \delta(t) dt = 1$

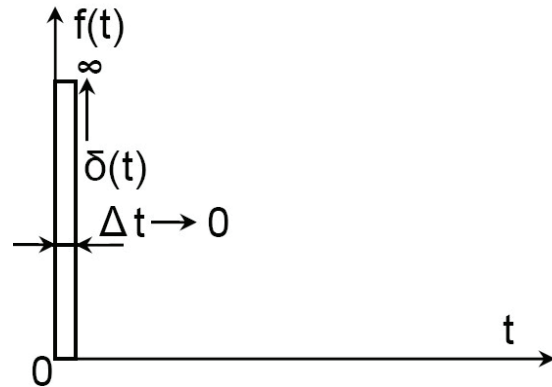


Fig.6.3 Impulse function

The response of the dynamic system to the input signal is divided into two parts: transient part and steady state part. The transient part decays with time. It becomes zero after certain time. The steady state part of response is the part which remains constant after the transient part has decayed. Given the transfer function, the transient response can be calculated according to the following procedure.

1. Find the Laplace transform of the input signal  $X(s)$
2. Find the Laplace transform of the output signal;  $Y(s) = G(s)X(s)$
3. Find the system response by applying the inverse Laplace transform;

$$y(t) = \mathcal{L}^{-1} [Y(s)] \quad (6.7)$$

**Example 6.1** Find the response of the system described by the following transfer function to a step input of magnitude  $k_2$ .

$$G(s) = \frac{Y(s)}{X(s)} = \frac{5}{(s+1)(s+2)(s+3)}$$

The input  $x(t)$  is step function of magnitude  $k_2$ , its Laplace transform is:

$$X(s) = \frac{k_2}{s}$$

$$\text{and } Y(s) = G(s)X(s) = \frac{5 k_2}{s(s+1)(s+2)(s+3)} = k_2 \left\{ \frac{5}{6s} - \frac{2.5}{s+1} + \frac{2.5}{s+2} - \frac{5}{6(s+3)} \right\}$$

$$\text{Then: } y(t) = k_2 \left( \frac{5}{6} - 2.5e^{-t} + 2.5e^{-2t} - \frac{5}{6}e^{-3t} \right).$$

Figure 6.4 shows the step response, evaluated analytically, where the steady state value of  $y$  is  $y_{ss} = 5k_2 / 6$  .

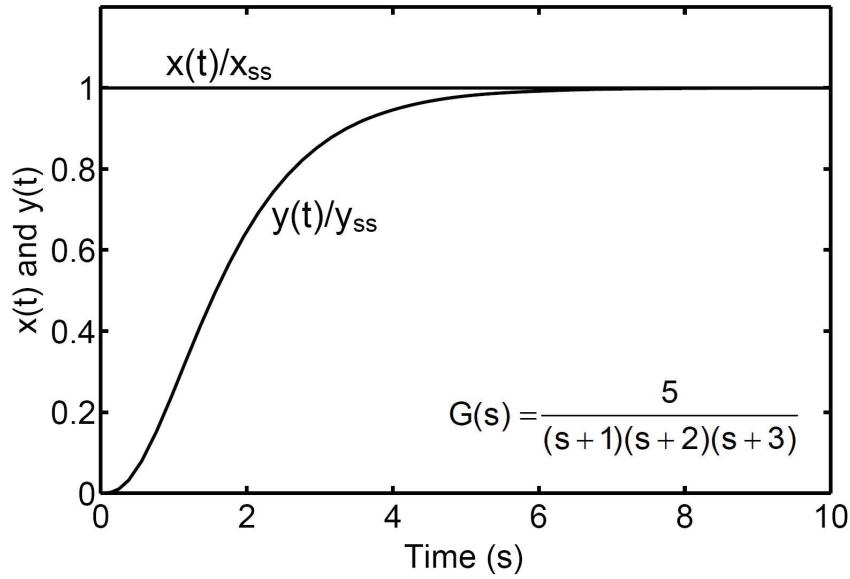


Fig.6.4 Step response evaluated analytically

## 6.2 TIME RESPONSE OF BASIC ELEMENTS

Herein, the responses of the basic elements to step, ramp and impulse input signals are evaluated.

### 6.2.1 Integrating Member

The transfer function of integrating member is:

$$G(s) = \frac{k_1}{s} \tag{6.8}$$

#### 6.2.1.1 Response to step input

$$X(s) = \frac{k_2}{s} \tag{6.9}$$

$$Y(s) = G(s)X(s) = \frac{k}{s^2} \tag{6.10}$$

$$k = k_1 k_2 \tag{6.11}$$

$$y(t) = k t \tag{6.12}$$

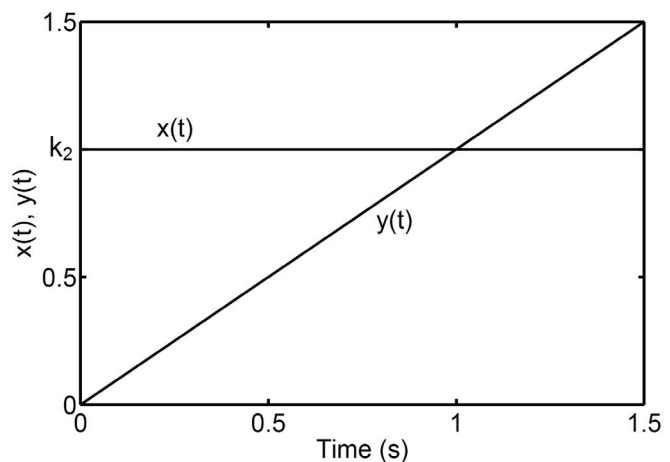


Fig. 6.5 Step response of integrating element

### 6.2.1.2 Response to ramp input

$$X(s) = \frac{k_2}{s^2} \quad (6.13)$$

$$Y(s) = G(s)X(s) = \frac{k}{s^3} \quad (6.14)$$

$$k = k_1 k_2$$

$$y(t) = 0.5k t^2 \quad (6.15)$$

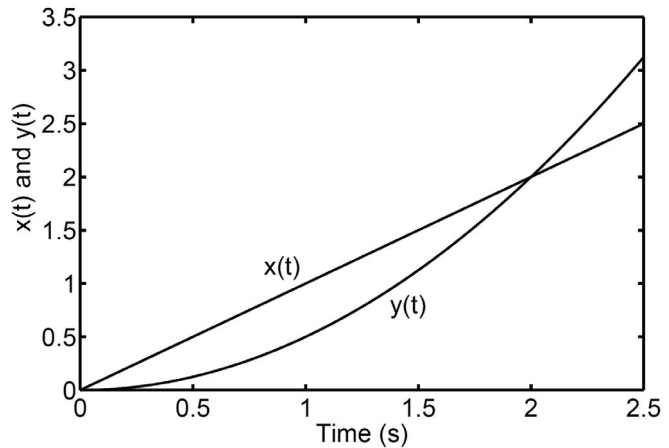


Fig. 6.6 Response of integrating element to ramp input,  $k_1 = k_2 = 1$

### 6.2.1.3 Response to input impulse

$$X(s) = k_2 \quad (6.16)$$

$$Y(s) = G(s)X(s) = \frac{k}{s}, \quad k = k_1 k_2 \quad \text{and} \quad y(t) = k \quad (6.17)$$

## 6.2.2 First Order Element

The first order element has a transfer function of the following form.

$$G(s) = \frac{k_1}{Ts + 1} \quad (6.18)$$

### 6.2.2.1 Step response

$$X(s) = \frac{k_2}{s}, \quad \text{a step input of magnitude } k_2 \quad (6.19)$$

$$Y(s) = G(s)X(s) = \frac{k_1 k_2}{s(Ts + 1)} = \frac{k}{s(Ts + 1)} = k \left( \frac{1}{s} - \frac{T}{Ts + 1} \right) = k \left( \frac{1}{s} - \frac{1}{s + 1/T} \right) \quad (6.20)$$

$$y(t) = k(1 - e^{-t/T}) \quad \text{where } k = k_1 k_2 \quad (6.21)$$

In the case of unity gain system, the unit step response is given by:

$$y(t) = 1 - e^{-t/T} \quad (6.22)$$

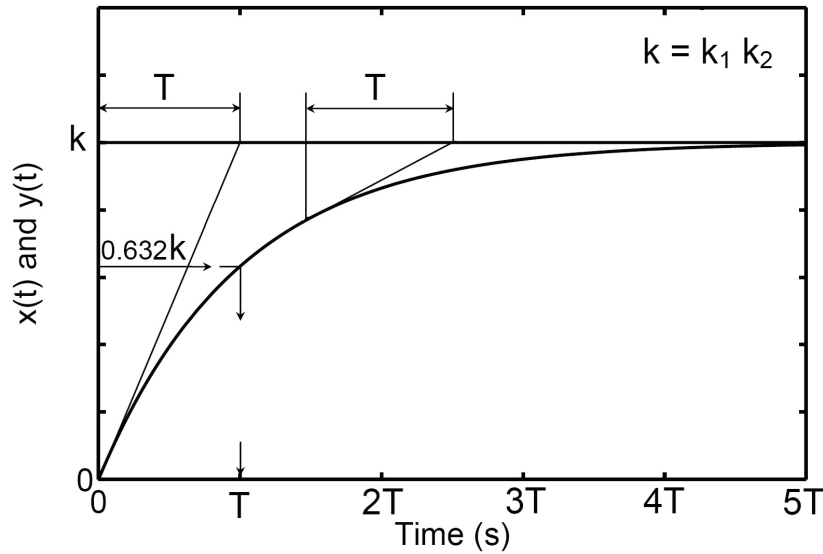


Fig.6.7 Transient response of first order element to step input

The step response of a first order element is plotted in Fig.6.7. The study of this figure shows that:

- i)  $y(T) = 0.632 y_{ss}$
- ii) A tangent drawn at a point on the response curve intersects the  $y_{ss}$  horizontal line  $T$  seconds later.
- iii) For  $t \geq 4T$ ,  $y_{ss} - y(t) \leq 0.02 y_{ss}$
- iv) If the transfer function gain is  $k_1$  and the applied step magnitude is  $k_2$  then the steady state output is  $k = k_1 k_2$ .

### 6.2.2.2 Response to ramp input

$$x(t) = k_2 t, \quad X(s) = \frac{k_2}{s^2} \quad (6.23)$$

$$Y(s) = G(s)X(s) = \frac{k}{s^2(Ts + 1)} = k \left( \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s + 1/T} \right), \quad k = k_1 k_2 \quad (6.24)$$

$$y(t) = k \{ t - T(1 - e^{-t/T}) \} \quad (6.25)$$

The response of a first order element to ramp input is plotted in Fig.6.8. The transient part of response ( $kTe^{-t/T}$ ) decays with time. The steady state response is  $\{k(t - T)\}$ .

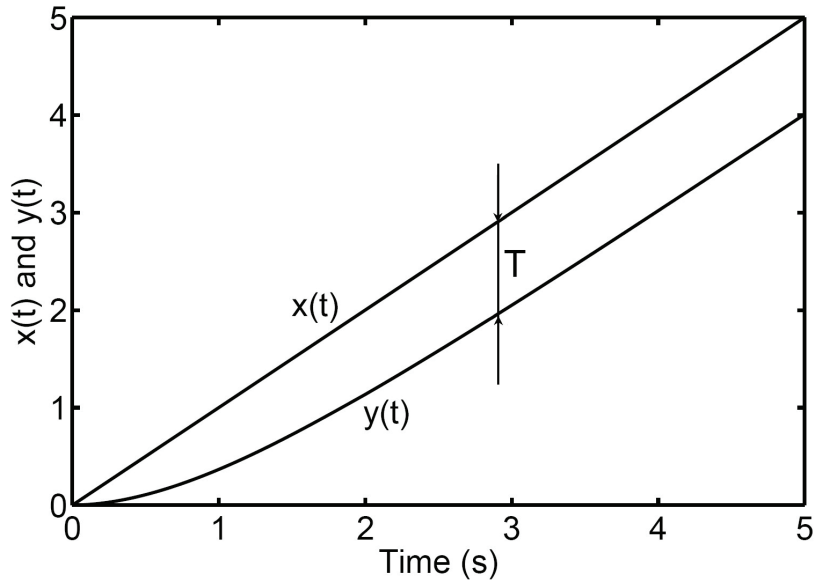


Fig.6.8 Response of first order element to ramp input ,  $k_1=k_2=1$

### 6.2.2.3 Response to input impulse

$$x(t) = k_2 \delta(t), \quad X(s) = k_2 \quad (6.26)$$

$$Y(s) = \frac{k_1 k_2}{Ts + 1} = \frac{k/T}{s + 1/T}, \quad \text{where } k = k_1 k_2 \quad (6.27)$$

$$y(t) = \frac{k}{T} e^{-t/T} \quad (6.28)$$

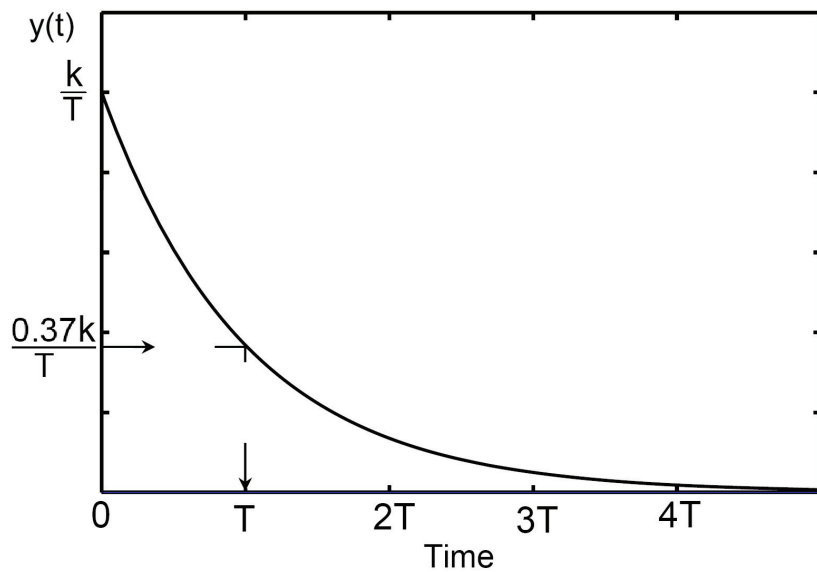


Fig.6.9 Response of first order element to input impulse

The response decays exponentially to a final zero steady state value after a sudden rise to  $(k/T)$  at  $t=0$ , Fig.6.9.

### 6.2.3 Second Order Element

The following is the general form of transfer function of a second order element:

$$G(s) = \frac{k_1 \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k_1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1} \quad (6.29)$$

where  $k_1$  = Gain of transfer function  
 $\omega_n$  = natural frequency, rad/s  
 $\zeta$  = damping ratio

#### 6.2.3.1 Step response of second order element

For a second order system of gain  $k_1$ , subjected to step input of magnitude  $k_2$ , the step response can be calculated as follows:

$$X(s) = \frac{k_2}{s} \quad (6.30)$$

$$Y(s) = G(s)X(s) = \frac{k_1 k_2 \omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = k \left( \frac{B}{s} + \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} \right) \quad (6.31)$$

$$B = \lim_{s \rightarrow 0} \left( \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) = 1 \quad (6.32)$$

**Assignment:** prove that the constants  $A_1$  and  $A_2$  and the roots of the characteristic equation  $p_1$  and  $p_2$  are as given by the following expressions:

$$A_1 = -\frac{1}{2} \left( 1 + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \right), \quad A_2 = -\frac{1}{2} \left( 1 - \frac{\zeta}{\sqrt{\zeta^2 - 1}} \right) \quad (6.33)$$

$$p_1 = -\omega_n (\zeta - \sqrt{\zeta^2 - 1}), \quad p_2 = -\omega_n (\zeta + \sqrt{\zeta^2 - 1}) \quad (6.34)$$

The step response is then given by the following relation.

$$y(t) = k(1 + A_1 e^{p_1 t} + A_2 e^{p_2 t}) \quad (6.35)$$

**Assignment:** Prove that  $y(t)$  is tangent to the horizontal axis at the origin.

Four distinct types of response are possible according to the types of roots,  $p_1$  and  $p_2$  as follows:

1.  $\zeta > 1$  Gives two real negative unequal roots of the characteristic equation. The constants  $A_1$  and  $A_2$  are real. The transient part of response decays exponentially. There are no response overshoot and the response is said to be over-damped.
2.  $\zeta = 1$  Gives two equal real negative roots. The transient part of response decays exponentially. There are no response overshoot and the response is said to be critically damped.
3.  $1 > \zeta > 0$  Gives a pair of complex conjugate roots. The response is under-damped; with overshoot and damped transient oscillations.
4.  $\zeta = 0$  Gives a pair of complex conjugate roots of zero real part and the response is oscillatory.

#### 6.2.3.1.1 Step response of over-damped second order element $\zeta > 1$

The over-damped system is characterized by  $\zeta > 1$ , the roots  $p_1$  and  $p_2$  are real negative and unequal while the coefficients  $A_1$  and  $A_2$  are real. The step response of an over-damped second order system, given by Eq. 6.35, is plotted in Fig.6.10.

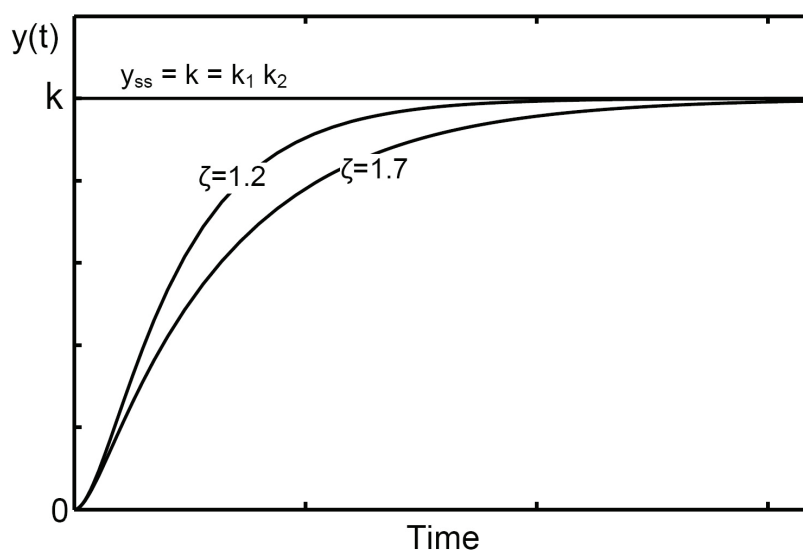


Fig.6.10 Step response of an over-damped second order system.

$$y(t) = k(1 + A_1 e^{p_1 t} + A_2 e^{p_2 t}) \quad (6.36)$$

The increase in the damping ratio results in an increase of the duration of the transient response; the system response slows down.

### 6.2.3.1.2 Step response of critically-damped second order element $\zeta = 1$

An expression for the step response of critically damped second order,  $\zeta = 1$ , is deduced as follows:

$$X(s) = \frac{k_2}{s} \quad (6.37)$$

$$Y(s) = G(s)X(s) = \frac{k_1 k_2 \omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} = \frac{k \omega_n^2}{s(s + \omega_n)^2} = k \left\{ \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2} \right\} \quad (6.38)$$

$$y(t) = k(1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}) = k \{ 1 - (1 + \omega_n t) e^{-\omega_n t} \} \quad (6.39)$$

The step response of a critically damped system is shown in Fig.6.11. The same figure carries the response of an over-damped element.

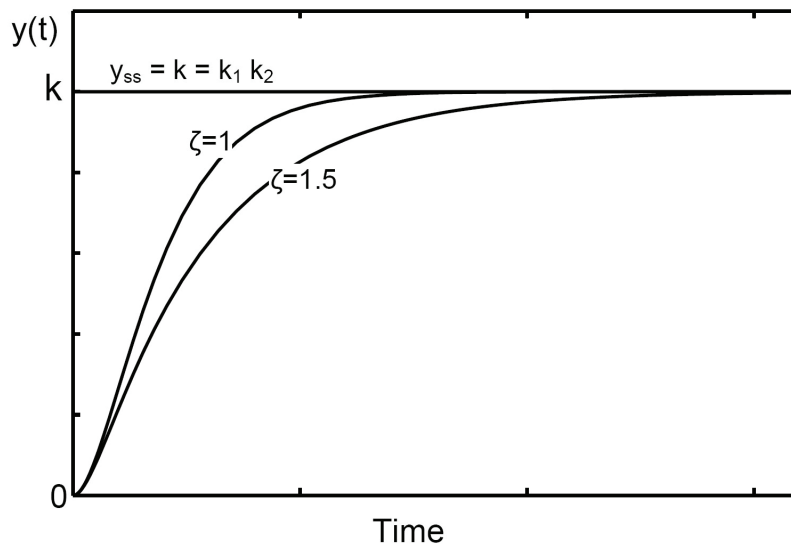


Fig.6.11 Step response of critically damped and over-damped second order systems

### 6.2.3.1.3 Step response of under-damped second order element $\zeta < 1$

In the case of under damped second order system,  $1 > \zeta > 0$ , the characteristic

equation has two complex conjugate roots. An expression for the step response,  $y(t)$ , is deduced as follows:

$$\boxed{A_1 = -\frac{1}{2} \left( 1 - \frac{i\zeta}{\sqrt{1-\zeta^2}} \right)} \quad \text{and} \quad \boxed{A_2 = -\frac{1}{2} \left( 1 + \frac{i\zeta}{\sqrt{1-\zeta^2}} \right)} \quad (6.40)$$

$$\boxed{p_1 = -\omega_n (\zeta - i\sqrt{1-\zeta^2})} \quad \text{and} \quad \boxed{p_2 = -\omega_n (\zeta + i\sqrt{1-\zeta^2})} \quad (6.41)$$

$$\boxed{y(t) = k(1 + A_1 e^{p_1 t} + A_2 e^{p_2 t})} \quad (6.42)$$

$$y(t) = k \left\{ 1 + A_1 e^{(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2})t} + A_2 e^{(-\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2})t} \right\} \quad (6.43)$$

$$y(t) = k \left[ 1 + e^{-\zeta\omega_n t} \left\{ \left( -\frac{1}{2} + \frac{i\zeta}{2\sqrt{1-\zeta^2}} \right) e^{i\omega_n\sqrt{1-\zeta^2}t} + \left( -\frac{1}{2} - \frac{i\zeta}{2\sqrt{1-\zeta^2}} \right) e^{-i\omega_n\sqrt{1-\zeta^2}t} \right\} \right] \quad (6.44)$$

$$= k \left[ 1 + e^{-\zeta\omega_n t} \left\{ -\frac{1}{2} (e^{i\omega_n\sqrt{1-\zeta^2}t} + e^{-i\omega_n\sqrt{1-\zeta^2}t}) + \frac{i\zeta}{2\sqrt{1-\zeta^2}} (e^{i\omega_n\sqrt{1-\zeta^2}t} - e^{-i\omega_n\sqrt{1-\zeta^2}t}) \right\} \right] \quad (6.45)$$

$$\cos \vartheta = \frac{1}{2} (e^{i\vartheta} + e^{-i\vartheta}) \quad \text{and} \quad \sin \vartheta = \frac{1}{2i} (e^{i\vartheta} - e^{-i\vartheta}) \quad (6.46)$$

$$y(t) = k \left[ 1 - e^{-\zeta\omega_n t} \left\{ \cos(\omega_n\sqrt{1-\zeta^2}t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_n\sqrt{1-\zeta^2}t) \right\} \right] \quad (6.47)$$

$$= k \left[ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left\{ \sqrt{1-\zeta^2} \cos(\omega_n\sqrt{1-\zeta^2}t) + \zeta \sin(\omega_n\sqrt{1-\zeta^2}t) \right\} \right] \quad (6.48)$$

$$= k \left[ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left\{ \sin(\varphi) \cos(\omega_n\sqrt{1-\zeta^2}t) + \cos(\varphi) \sin(\omega_n\sqrt{1-\zeta^2}t) \right\} \right] \quad (6.49)$$

Or:

$$\boxed{y(t) = k \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n\sqrt{1-\zeta^2}t + \varphi) \right\}} \quad \text{where} \quad \boxed{\varphi = \cos^{-1}(\zeta)} \quad (6.50)$$

Figure 6.12 shows the different components constituting the step response  $y(t)$  while

Fig.6.13 shows the step response of second order systems with different damping ratios. These figures show that the response of under damped system,  $1 > \zeta > 0$ , overshoots during the transient period. The maximum overshoot  $\sigma$  is one of the parameters characterizing the response of the under damped system.

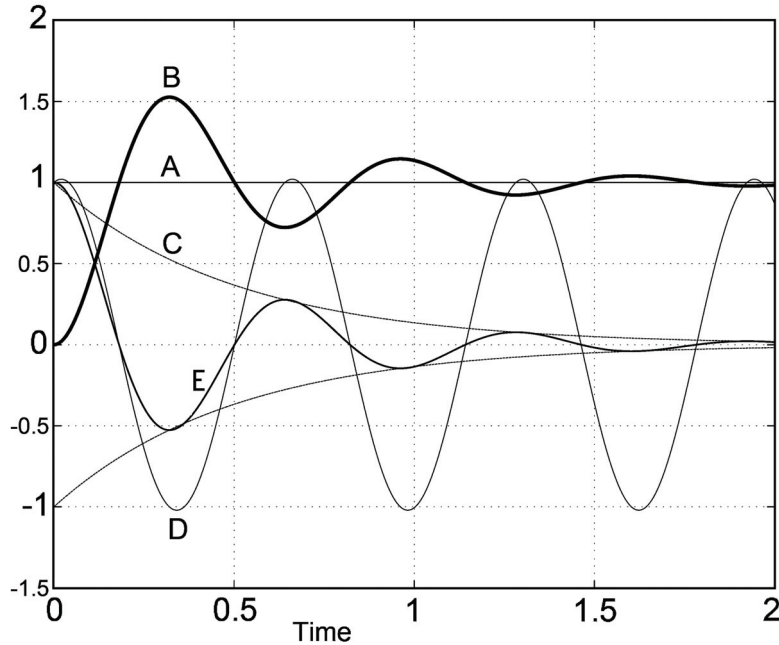


Fig.6.12 Step response of under-damped second order system,  $k_1=k_2=1$

(A)  $y_{ss}$ , (B)  $y(t)$ , (C)  $e^{-\zeta\omega_n t}$ , (D)  $\sin(\omega_n \sqrt{1-\zeta^2} t + \varphi) / \sqrt{1-\zeta^2}$ ,

(E)  $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t + \varphi)$

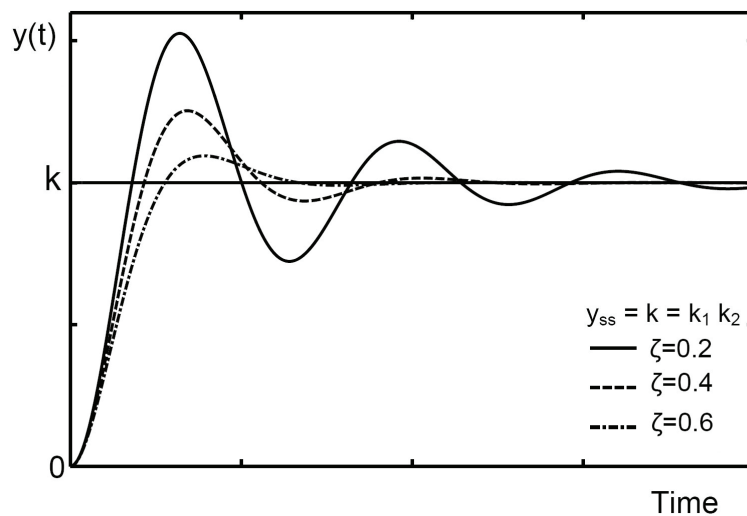


Fig.6.13 Step response of under-damped second order systems of different damping ratios

An expression for the maximum overshoot is deduced as follows.

$$y(t) = k \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t + \varphi) \right\} \quad (6.51)$$

$$\frac{dy}{dt} = k \left\{ \zeta\omega_n \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t + \varphi) - \omega_n e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2} t + \varphi) \right\} \quad (6.52)$$

$$dy/dt=0 \quad \text{at } y_{\max} \text{ and } y_{\min}, \quad (6.53)$$

Or

$$\zeta\omega_n \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t + \varphi) - \omega_n e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2} t + \varphi) = 0 \quad (6.54)$$

$$\tan(\omega_n \sqrt{1-\zeta^2} t + \varphi) = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan(\varphi) \quad (6.55)$$

Then

$$\omega_n \sqrt{1-\zeta^2} t = n\pi \quad \text{where } n = 0, 1, 2, 3, \dots \quad (6.56)$$

$$t = \frac{n\pi}{\omega_n \sqrt{1-\zeta^2}} \quad (6.57)$$

The maximum value of the transient response corresponds to  $n=1$ , then:

$$t = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \quad (6.58)$$

and

$$y_{\max} = k + k e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \quad (6.59)$$

The maximum response overshoot ratio is defined as follows:

$$\sigma = \frac{y_{\max} - y_{ss}}{y_{ss}} = \frac{y_{\max} - k}{k} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \quad \text{and} \quad \zeta = \frac{\sqrt{\{\ln(\sigma)\}^2}}{\sqrt{\{\ln(\sigma)\}^2 + \pi^2}} \quad (6.60)$$

The maximum percentage overshoot is given by the following expression:

$$\sigma = 100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \% \quad (6.61)$$

The transient response oscillates by a frequency different than the natural frequency of the system. The frequency of the transient oscillation of response is called **damped natural frequency**, given by the following expression, see Eq. 6.50.

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (6.62)$$

The duration of one cycle of transient oscillations, also called **the period**, is given by:

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad (6.63)$$

#### 6.2.3.1.4 Step response of un-damped second order element with $\zeta = 0$

$$X(s) = \frac{k_2}{s} \quad (6.64)$$

$$Y(s) = G(s)X(s) = \frac{k \omega_n^2}{s(s^2 + \omega_n^2)} = k \left( \frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \right); \quad \text{where } k = k_1 k_2 \quad (6.65)$$

$$y(t) = k \{1 - \cos(\omega_n t)\} \quad (6.66)$$

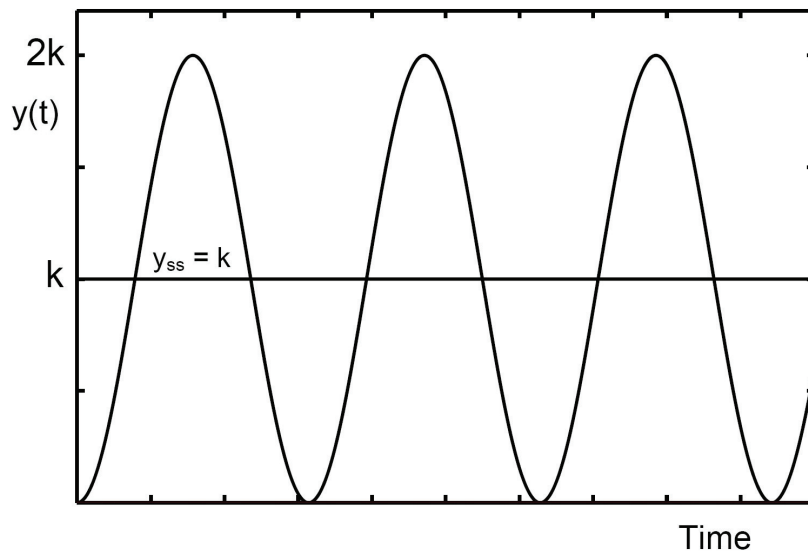


Fig.6.14 Step response of second order element of zero damping

In the case of zero damping, the response is oscillatory. The output  $y(t)$  oscillates harmonically around the steady state value  $y_{ss}$  as shown in Fig.6.14.

If the damping ratio is negative, the system becomes unstable.

Figure 6.15 shows the transient response of second order system to unit step input for different values of damping ratio.

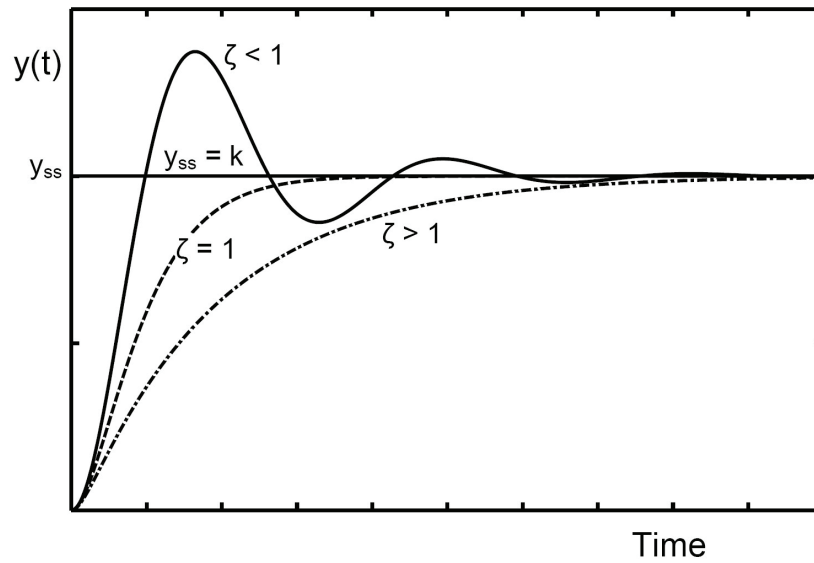


Fig.6.15 Typical step response of over damped, critically damped and under damped second order system

### 6.2.3.2 Response of second order element to ramp input

$$x(t) = k_2 t, \quad \text{and} \quad X(s) = \frac{k_2}{s^2} \quad (6.67)$$

$$Y(s) = G(s)X(s) = \frac{k \omega_n^2}{s^2(s^2 + 2\zeta\omega_n s + \omega_n^2)} = k \left( \frac{B_1}{s^2} + \frac{B_2}{s} + \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} \right); \quad k = k_1 k_2 \quad (6.68)$$

The constants  $A_1$ ,  $A_2$ ,  $B_1$  and  $B_2$  are:

$$B_1 = 1 \quad \text{and} \quad B_2 = -2\zeta/\omega_n \quad (6.69)$$

$$A_1 = \frac{\zeta}{\omega_n} + \frac{2\zeta^2 - 1}{2\omega_n \sqrt{\zeta^2 - 1}} \quad \text{and} \quad A_2 = \frac{\zeta}{\omega_n} - \frac{2\zeta^2 - 1}{2\omega_n \sqrt{\zeta^2 - 1}} \quad (6.70)$$

The expression of the transient response is:

$$y(t) = k \left( t - \frac{2\zeta}{\omega_n} + A_1 e^{p_1 t} + A_2 e^{p_2 t} \right) \quad (6.71)$$

Where:

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \quad \text{and} \quad p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \quad (6.72)$$

The transient response, Fig.6.16, depends on the damping ratio as follows:

1. For  $\zeta > 1$  The response is over damped. In the steady state, the transient part of response decays and  $y(t) = k(t - 2\zeta/\omega_n)$
2. For  $\zeta = 1$  The response is critically damped.
3. For  $0 < \zeta < 1$  The response is of damped oscillatory character with overshoot
4. For  $\zeta \leq 0$  For  $\zeta = 0$ , the response is oscillatory and for  $\zeta < 0$ , the system is unstable, the amplitude of the transient oscillation increase with time.

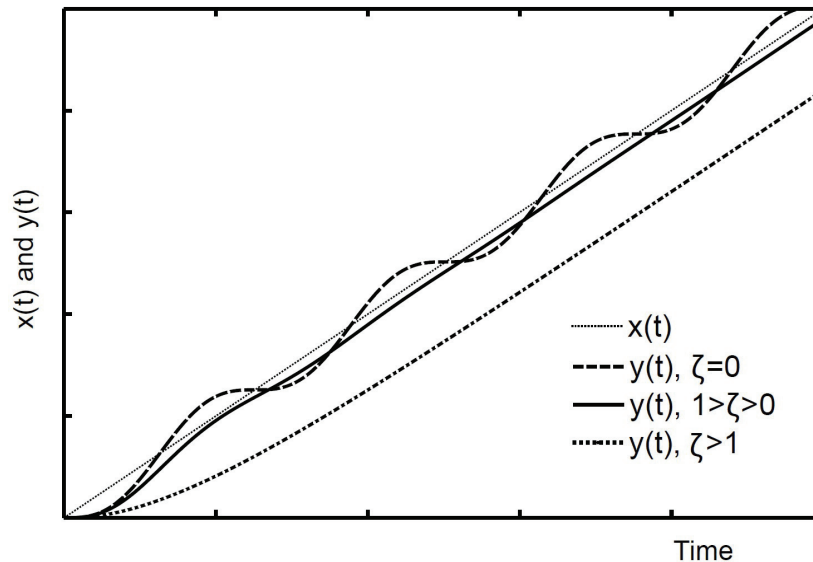


Fig.6.16 Response of the second order system to ramp input

### 6.2.3.3 Response of second order element to input impulse

$$x(t) = k_2\delta(t) \quad \text{and} \quad X(s) = k_2 \quad (6.73)$$

$$Y(s) = G(s)X(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (6.74)$$

$$y(t) = k(A_1 e^{p_1 t} + A_2 e^{p_2 t}) \quad (6.75)$$

Where:

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \quad \text{and} \quad p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \quad (6.76)$$

$$A_1 = -A_2 = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \quad (6.77)$$

The response of a second order element to unit impulse input is plotted in Fig.6.17, for different values of damping ratio. This figure shows that:

1. For  $\zeta > 1$  The response is over damped.
2. For  $\zeta = 1$  The response is critically damped.
3. For  $0 < \zeta < 1$  The response is of damped oscillatory character with overshoot

Moreover, for  $\zeta = 0$ , the response is oscillatory and for  $\zeta < 0$ , the system is unstable, as the amplitude of the transient oscillation increases with time.

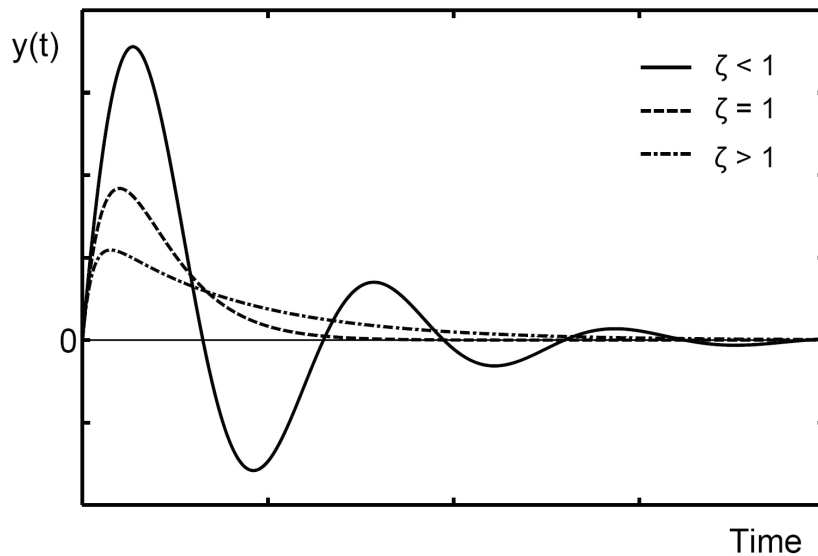


Fig.6.17 Response of the second order system to an input impulse

### 6.2.4 Third and Higher Order Systems

The general form of the transfer function is the following:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)} \quad (6.78)$$

where  $P(s)$  and  $Q(s)$  are polynomials in the Laplace operator ( $s$ ).

The order  $N$  of polynomial  $Q(s)$  should be greater than or equal to that of  $P(s)$ . The polynomial  $Q(s)$  has  $N$  roots;  $p_1, p_2$  and  $p_N$ . Therefore, for a step input,  $X(s) = k_2/s$ , the system response  $y(t)$  can be calculated by the following expression:

$$y(t) = k(1 + A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_N e^{p_N t}) \quad (6.79)$$

Where  $A_1$ ,  $A_2$ , and  $A_N$  are constants, depending on the initial conditions.

The transient part of response consists of the summation of terms of the form  $A_i e^{p_i t}$ . The contribution which each term makes towards the overall response depends on the magnitude and sign of  $A_i$  and on the position of the poles in the complex, R-I, plane.

### 6.2.5 Effect of Root Location

A second order element has a transfer function of the form:

$$G(s) = \frac{k_1 \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (6.80)$$

The roots of the characteristic equation are:

$$p = -\omega_n \left( \zeta \pm \sqrt{\zeta^2 - 1} \right) \quad (6.81)$$

The effect of location of roots, in the R-I plane, on the step response of a second order element, is illustrated by Fig.6.18. The transfer function has either real roots or pairs of complex conjugate roots. The responses, plotted in Fig.6.18, are calculated for unity gain second order system and unit step input function. This figure shows that:

1. If the two roots are located at the origin, the system is simply a double integrating member. For bounded input, step input for example, the output increases continuously with time. Then the system is unstable due to the unbounded output.
2. For roots located on the left hand side of the real (horizontal) axis, the two roots are real negative. The step response has the form:  $y(t) = k(1 + A_1 e^{p_1 t} + A_2 e^{p_2 t})$ . Therefore, if the roots are located on the horizontal axis (negative real roots) the step response is damped and non-oscillatory. The displacement of the roots to the left decreases the settling time.
3. If the roots are located at the vertical axis, zero damping,  $\zeta = 0$ , the system is oscillatory. It has two complex conjugate roots with zero real part. The step response is of the form  $y(t) = k\{1 - \cos(\omega_n t)\}$ . The displacement of the root upwards increases the frequency of oscillations.

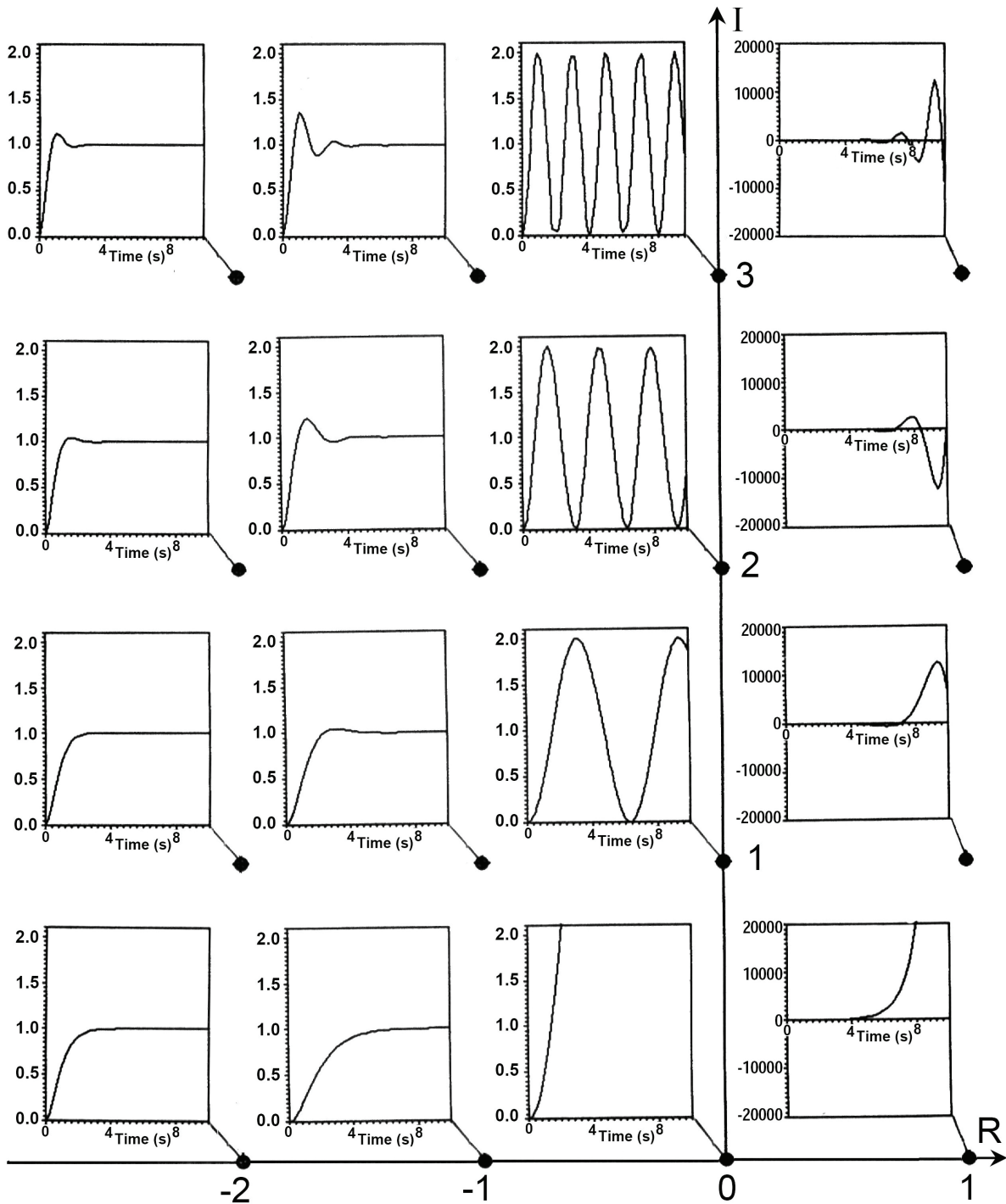


Fig.6.18 Effect of roots location on the step response

4. If the damping ratio is in the range  $1 > \zeta > 0$ , the two roots are complex conjugate with negative real part. They are located to the left of the vertical axis.

The roots are;  $p = -\omega_n (\zeta \pm i\sqrt{1-\zeta^2})$ , where the real part and imaginary parts are:

$$R = -\omega_n \zeta \quad \text{and} \quad I = \pm \omega_n \sqrt{1-\zeta^2}$$

Figure 6.19 shows the location of the root in the R-I plane, where  $\varphi = \cos^{-1}(\zeta)$ . In this case the step response has an overshoot and damped transient oscillations.

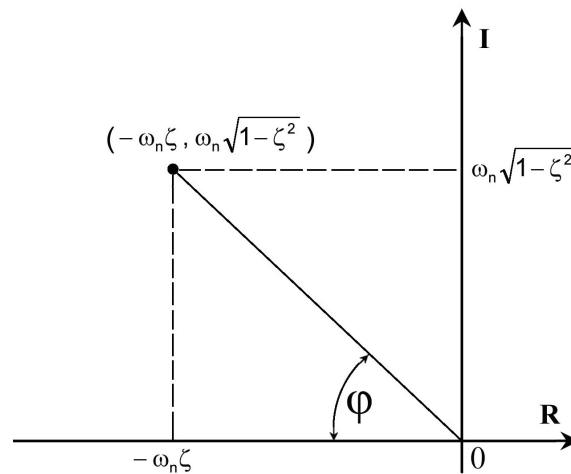


Fig.6.19 Location of root of under damped second order system

The farthest poles from the imaginary axis will have contribution to the transient response, which decay most rapidly. The system response is influenced mostly by the poles closest to the imaginary axis, called the **dominant poles**. Generally, the poles located 5 times far to the left from the vertical axis relative to the dominant roots can be neglected.

The increase of the magnitude of imaginary part of root increases the frequency of transient oscillations.

5. The presence of any root with positive real part, located to the right of vertical axis, increases the magnitude of system output without limit and the system is unstable. The instability means that the system output is unbounded for bounded input.

### 6.3 TRANSIENT RESPONSE CHARACTERISTICS

A first order system can be completely described by specifying the time constant and gain of transfer function. A second order system can be clearly described by specifying its gain, damping ratio and natural frequency. For higher order system, the algebraic equation is not very helpful since the form of the response is not readily apparent and the plot of the response is not satisfactory since a numerical description is required for analysis. Generally, with reference to Fig.6.20, the following parameters are used to describe the step response of a system:

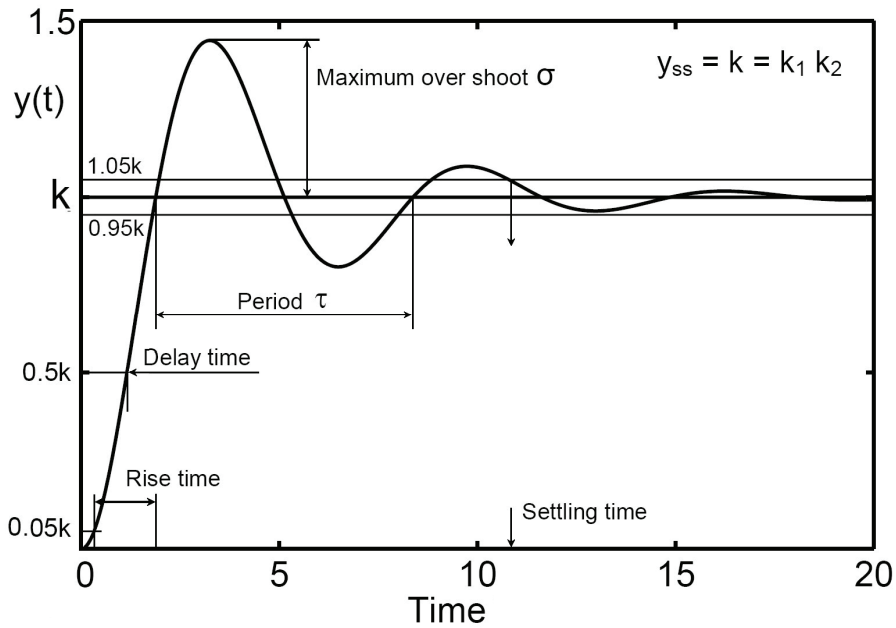


Fig.6.20 Parameters characterizing the time domain response of a control system

**Maximum overshoot** is usually expressed in percentage of the steady state value of the output.

$$\sigma = \frac{y_{\max} - y_{ss}}{y_{ss}} = \frac{y_{\max} - y_{ss}}{y_{ss}} \times 100\% \quad (6.82)$$

For a second order element, the maximum percentage overshoot is:

$$\sigma = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \% \quad (6.83)$$

**Delay time** is the time required for the response to reach 50% of the final steady state value

**Rise time** is the time required for the response to rise from 5% to 95% (or from 10% to 90%) of the step size. This definition avoids the practical difficulty of having to determine the exact start of the transient response.

**Settling time** is the time required for the response to reach an end state within 5% of final, steady state, value. For second order system, the settling time is:

$$t_s \leq \frac{3 - 0.5 \ln(1 - \zeta^2)}{\zeta \omega_n} \quad \text{Or} \quad t_s \cong \frac{3}{\zeta \omega_n} \quad (6.84)$$

Equation 6.84 can be used **only** to estimate the settling time for an under-damped second order element. It cannot be used to calculate the damping ratio or the natural frequency. As for the **first order element**, the following expression can be deduced systematically.

$$\boxed{t_s = 3T} \quad (6.85)$$

**Steady state error** is the steady state deviation between the actual response  $y(t)$  and the required steady state value.

**Period** is the duration of one complete cycle of the transient response oscillations

Sometimes it might be useful to describe the whole transient response by a simple numerical value, called performance index. Some frequently used indices are:

**Integral of absolute error**

$$IAE = \int_0^{\infty} |e(t)| dt; \text{ where } e(t) = y_{ss} - y(t) \quad (6.86)$$

**Integral of error squared**

$$IES = \int_0^{\infty} \{e(t)\}^2 dt \quad (6.87)$$

**Integral of time absolute error**

$$ITAE = \int_0^{\infty} t|e(t)| dt \quad (6.88)$$

The best response is that which gives the minimum value of performance index. For a second order element, the optimum (best) response is obtained at  $\zeta = 0.7$  for the IAE and ITAE. On the other hand, using the IES, the best response is obtained at  $\zeta = 0.5$ . In practice, a damping factor of 0.7 is preferred.

## 6.4 STEP RESPONSE TESTING OF PRACTICAL SYSTEMS

A system of unknown internal structure is called black-box system. The dynamic characteristics of such system can be obtained by practical testing. The results are used to determine a representative / describing mathematical model of the system. The general shape of step response gives an idea of the type of transfer function and its parameters and can also give some indication of how linear the system is. The information about the dynamic performance of a system may be presented in the form of transient response curves (non-parametric model). However, usually, a parametric model is required. It might be necessary to fit a transfer function to the response curve. To do this, it is necessary to select some error criterion to quantify

the goodness of fit and then to adjust the model and its parameters values to minimize the chosen error index.

#### 6.4.1 Response Apparently of First Order

If the step response apparently rises exponentially to the new steady state, it is likely that the system is of first order. The time constant can be found by direct measurement on the step response plot, as shown in Fig.6.7.

Superimposing the curve,  $y(t)=k(1-e^{-t/T})$ , will show qualitatively whether this is a reasonable model, or whether it is necessary to use a higher order model.

The gain of the transfer function ( $k_1$ ) is simply found as:

$$\boxed{k_1 = k / k_2} \quad (6.89)$$

Where  $k$  = Steady state value of response.  
 $k_2$  = Magnitude of input step.

#### 6.4.2 Response Apparently of Under Damped Second Order

If the response is apparently of an under damped second order, Fig.6.20, a describing transfer function can be found by calculating the damping ratio, natural frequency and gain  $k_1$ . The following calculation steps can be followed:

1. In the case of step input of magnitude  $k_2$ , if the steady state value is  $k$  then the gain of transfer function is given by:  $k_1 = k / k_2$
2. Find the period of one oscillation cycle,  $\tau$  from the plotted response and calculate the damped natural frequency,  $\boxed{\omega_d = 2\pi / \tau}$  rad/s.
3. Find the overshoot ratio  $\sigma$  from the plot then calculate the damping ratio.

$$\boxed{\sigma = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}} \quad \text{Or} \quad \boxed{\zeta = \sqrt{\frac{\{\ln(\sigma)\}^2}{\{\ln(\sigma)\}^2 + \pi^2}}} \quad (6.90)$$

4. Calculate the natural frequency

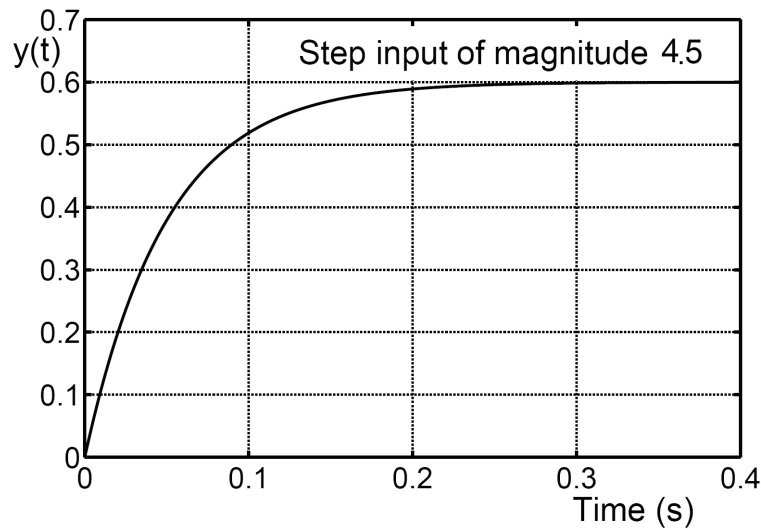
$$\boxed{\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}} \quad (6.91)$$

### 6.5 EXERCISE

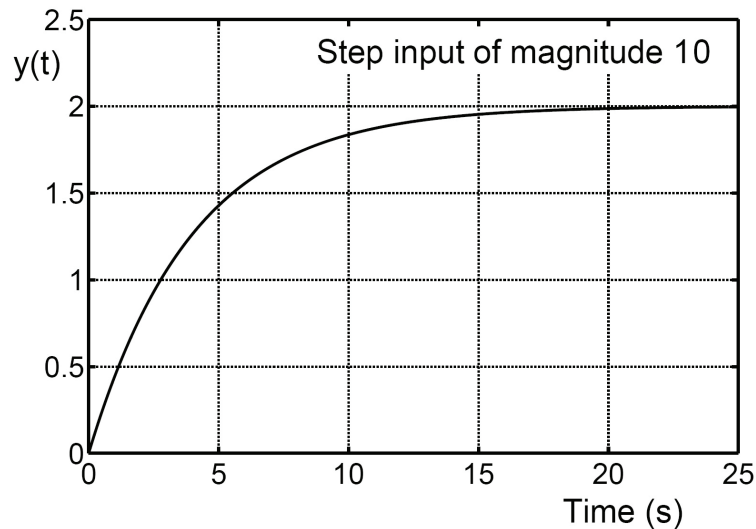
1. Derive an expression for and plot, the transient response of the systems described by the following transfer functions to step input of magnitude 4.

$$G(s) = \frac{0.45}{s+0.5}, \quad G(s) = \frac{0.1}{2s+0.5} \quad \text{and} \quad G(s) = \frac{1}{3s+1}$$

2. Estimate the transfer function of the system whose step response is given in the following figure. Find also, from the graph, the settling time.



3. Estimate the transfer function of the system whose step response is given in the following figure. Find also, from the graph, the settling time, rise time and delay time.



4. A flywheel driven by an electric motor is controlled to follow the movement of a hand wheel. This system is described by the following transfer function:

$$\frac{\theta_o}{\theta_i}(s) = \frac{240}{15s^2 + 60s + 240}$$

Plot the transient response of the system to the input angular displacement  $\vartheta(t)$  where  $\vartheta_i(t) = 0$  at  $t = 0$  and  $\vartheta_i(t) = \pi/3$  at  $t > 0$

5. The following is the characteristic equation of a second order system.

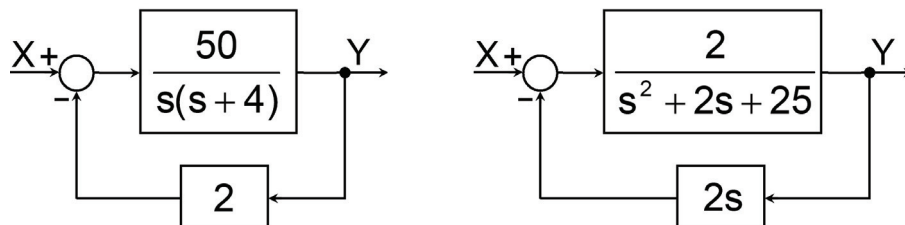
$$s^2 + 0.6s + 9 = 0$$

Calculate: a) The roots of the characteristic equation.

b) The natural frequency, damping ratio and the damped natural frequency.

c) The maximum percentage overshoot and settling time.

6. The following are two block diagrams of two control systems. Find the natural frequency, damping ratio, damped natural frequency, settling time and maximum percentage overshoot for each of them.

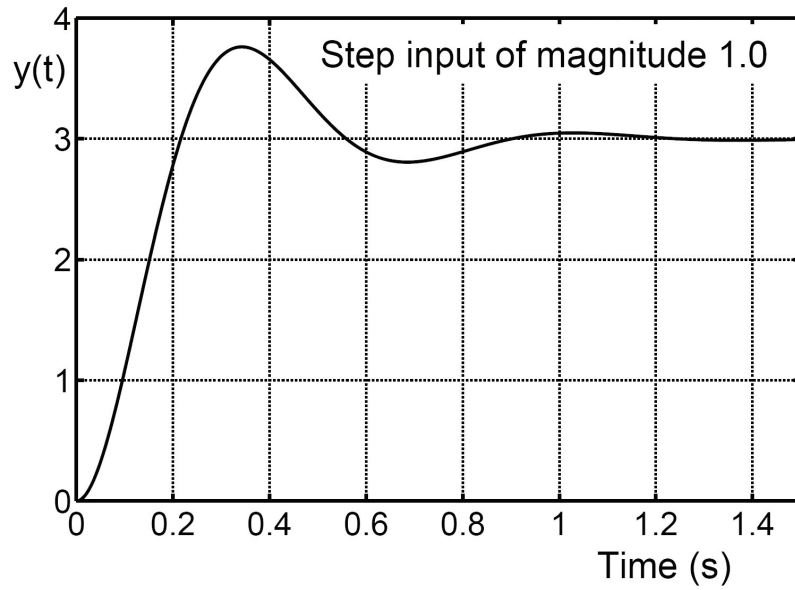


7. Find an expression for the response of the system having the following transfer function to a unit step input.

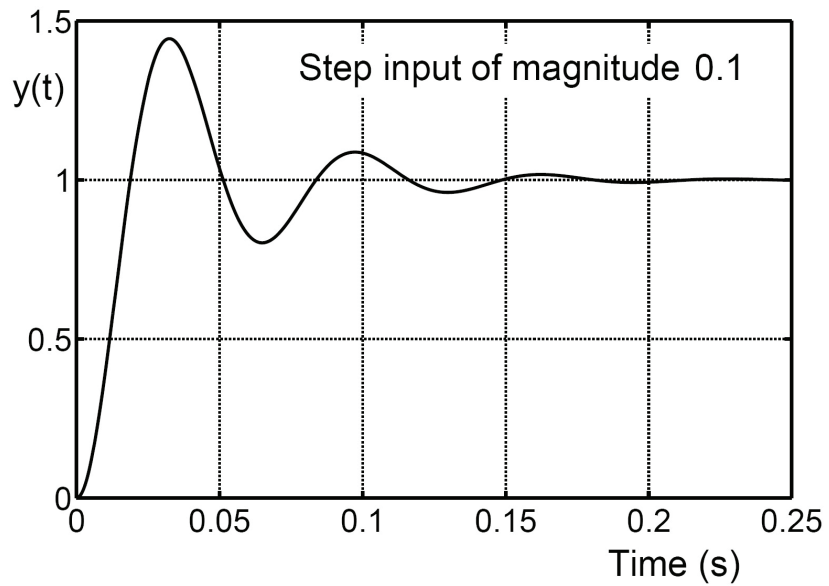
$$G(s) = \frac{s(3 + 5s)}{(s + 1)(s + 2)(s + 3)}$$

8. The following figures show the responses of four different systems to step inputs of different magnitudes.
- Find the settling time, rise time, delay time, period of one oscillation cycle and maximum percentage overshoot.
  - Write the system transfer function.
  - Calculate the transfer function gain, damping ratio and natural frequency of each system.

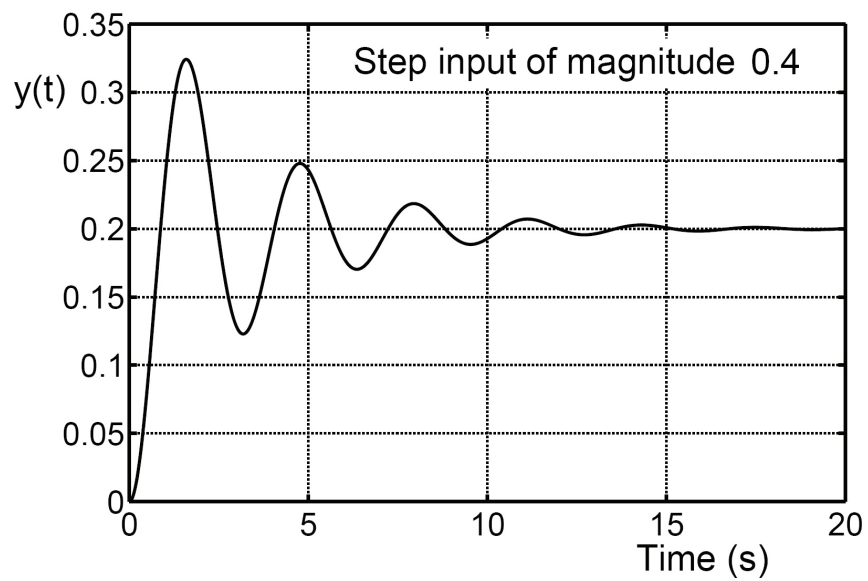
(A)

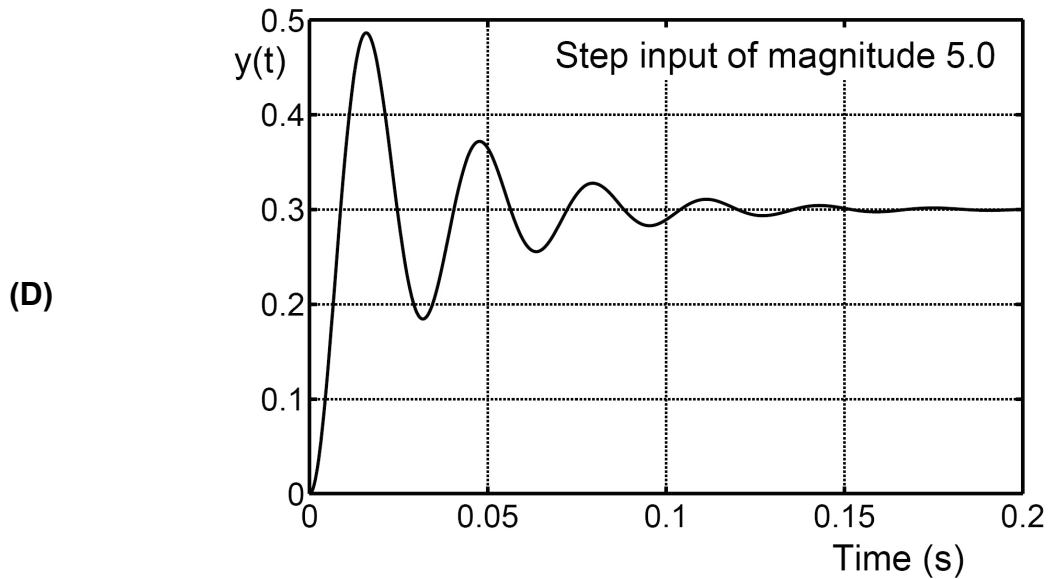


(B)

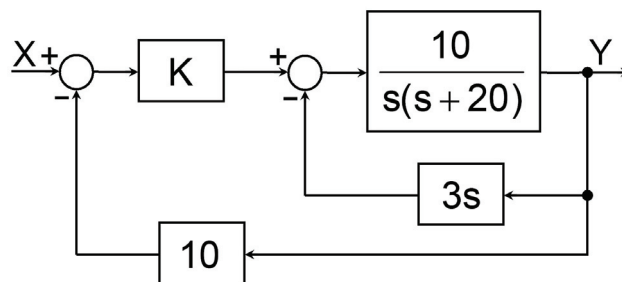


(C)





9. Find the value of constant K for the critical damping of the system described by the following block diagram.



10. The step response of an under damped second order element is given by the following expression.

$$y = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi)$$

- a) Derive an expression for the maximum percentage overshoot.
- b) An electro hydraulic proportional valve is described by the following transfer function:

$$G(s) = \frac{1}{s^2 + 15s + 400}$$

Calculate the natural frequency, damping ratio, gain, damped natural frequency, period, maximum percentage overshoot and settling time.

11. A hydraulic transmission line of 18 m length (L), 0.01 m diameter (D), transmitting oil of density  $\rho = 868 \text{ kg/m}^3$ , oil bulk modulus  $B = 1.6 \text{ GPa}$  and kinematic viscosity  $\nu = 56 \times 10^{-6} \text{ m}^2/\text{s}$ . The line inlet pressure and flow rate are  $p_o$  and  $q_o$  and the outlet parameters are  $p_L$  and  $q_L$ . The following is the transfer matrix relating the inlet and outlet parameters of the line.

$$\begin{bmatrix} P_o(s) \\ Q_o(s) \end{bmatrix} = \begin{bmatrix} ICs^2 + RCs + 1 & Is + R \\ Cs & 1 \end{bmatrix} \begin{bmatrix} P_L(s) \\ Q_L(s) \end{bmatrix}$$

where  $R = \text{line resistance} = \frac{128\mu L}{\pi D^4}$

$$C = \text{line capacitance} = \frac{\pi D^2 L}{4B}$$

$$I = \text{line inertia} = \frac{4\rho L}{\pi D^2}$$

- a) If the line is open at its extremity,  $p_L = 0$ , the relation between  $Q_L(s)$  and  $P_o(s)$  will be given by:

$$\frac{Q_L}{P_o}(s) = \frac{1}{Is + R}$$

- b) The transfer function relating the pressures at both extremities of a closed end line is given by:

$$\frac{P_L}{P_o} = \frac{1}{ICs^2 + RCs + 1}$$

Calculate the parameters of the transfer functions and plot, in scale, the step response of the line in the two cases.

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# 7 | FREQUENCY RESPONSE

## 7.1 INTRODUCTION

The description of system performance in the frequency domain is given in terms of the steady state response to a sinusoidal input; after all initial transients have died out, Fig.7.1. If the system is linear, the steady state output is also sinusoidal and has the same frequency as the input but with a shift in phase and change in magnitude, Fig.7.2. The response is evaluated by the magnitude ratio and the phase shift, defined as follows:

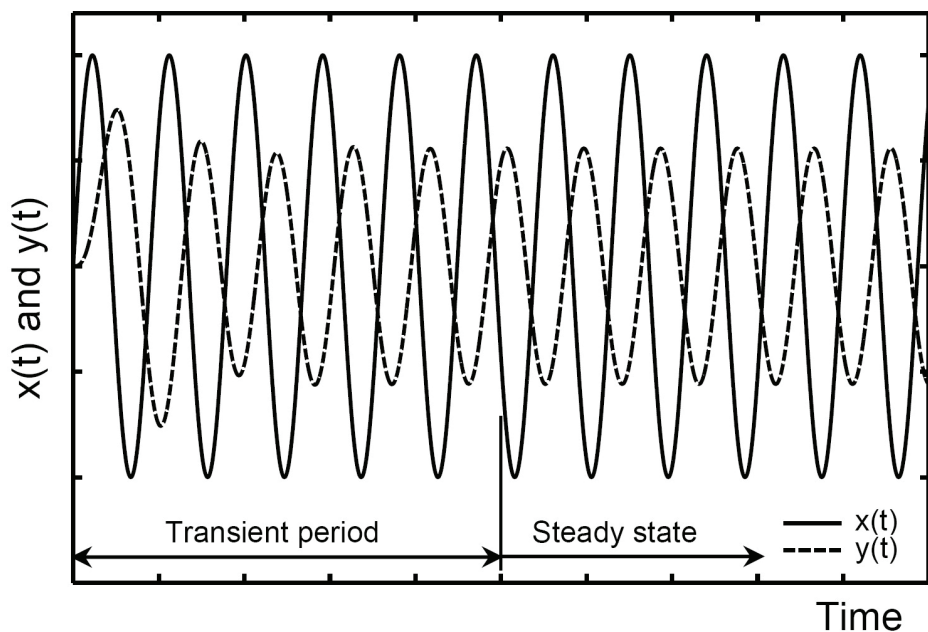


Fig.7.1 Typical linear system response to sinusoidal input

### Input signal

$$x(t) = |x(t)| \sin(\omega t) \quad (7.1)$$

where  $|x(t)|$  = magnitude of input signal.  
 $\omega$  = frequency of input signal (rad/s)

### Steady state output signal

$$y(t) = |y(t)| \sin(\omega t + \phi) \quad (7.2)$$

**AMPLITUDE RATIO** is the ratio of the amplitude of the output sinusoidal wave to that of the input in the steady state conditions (also called the magnitude or gain).

$$\text{Gain} = \frac{|y(t)|}{|x(t)|} \quad (7.3)$$

**PHASE SHIFT** is the shift in phase ( $\phi$ ) between the output sinusoidal wave relative to the input (also called the phase).

**FREQUENCY RESPONSE** Is the variation of amplitude ratio and phase with the frequency of the input signal.

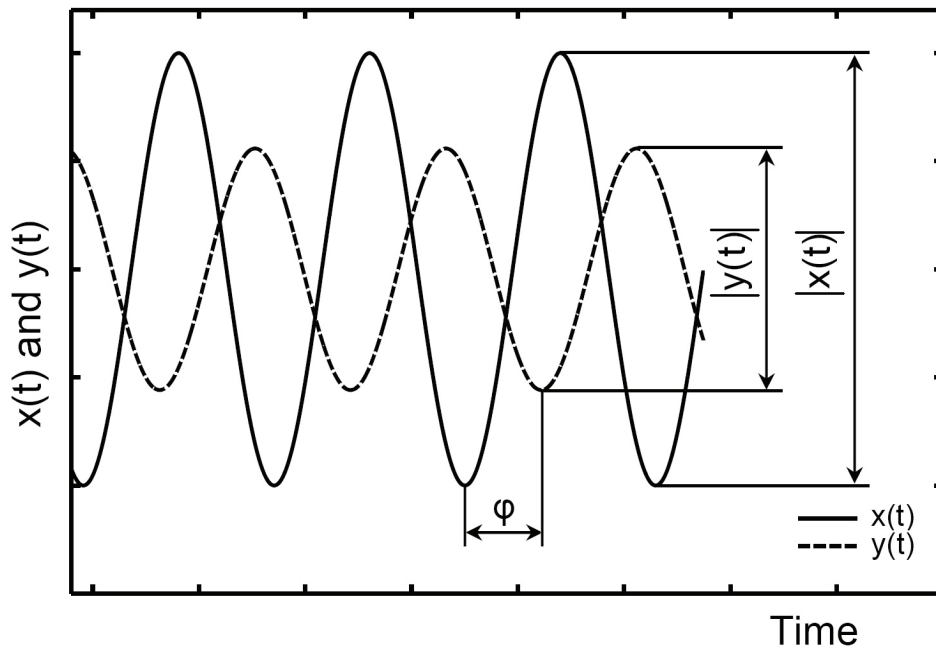


Fig.7.2 Steady state response of linear system to sinusoidal input

In the case of nonlinear systems, the nonlinearity introduces signal components of higher frequencies. The output signal will be distorted from the sinusoidal. It consists of a basic component and other harmonics. In this case, a signal treatment is necessary to separate the basic harmonic, having a frequency equal to that of the input signal. The separated basic harmonic is treated as the system output and is used to calculate the frequency response.

## 7.2 CALCULATION OF THE FREQUENCY RESPONSE

The following is the general form of the transfer function of a linear system:

$$G(s) = \frac{k(s-a_1)(s-a_2)\cdots(s-a_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \quad \text{where } n \geq m \quad (7.4)$$

The frequency response is calculated by applying an input sinusoidal signal;  $x(t) = A \sin(\omega t)$ . The response to this input signal,  $y(t)$ , can be deduced as follows:

$$X(s) = \frac{A\omega}{s^2 + \omega^2} = \frac{A\omega}{(s-i\omega)(s+i\omega)} \quad (7.5)$$

$$Y(s) = G(s)X(s) = \frac{A\omega G(s)}{(s-i\omega)(s+i\omega)} = \frac{kA\omega(s-a_1)(s-a_2)\cdots(s-a_m)}{(s-i\omega)(s+i\omega)(s-p_1)(s-p_2)\cdots(s-p_n)} \quad (7.6)$$

Or

$$Y(s) = \frac{A_1}{(s-i\omega)} + \frac{A_2}{(s+i\omega)} + \frac{B_1}{(s-p_1)} + \frac{B_2}{(s-p_2)} + \cdots + \frac{B_n}{(s-p_n)} \quad (7.7)$$

Then:

$$y(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} + B_1 e^{p_1 t} + B_2 e^{p_2 t} + \cdots + B_n e^{p_n t} \quad (7.8)$$

The first two terms of  $y(t)$  describe the steady state response, while the other terms give the transient part of the response. For a system to be stable, the poles  $p_1, p_2, \dots, p_n$  should be real negative or complex of negative real part. In this case, the terms of the transient response decay as the time increases. Therefore, after all transients have died out, the steady state part of response is:

$$y(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} \quad (7.9)$$

The constants  $A_1$  and  $A_2$  are calculated as follows:

$$A_1 = \lim_{s \rightarrow i\omega} \left( \frac{A\omega G(s)}{s+i\omega} \right) = \frac{A}{2i} G(i\omega) \quad (7.10)$$

$$A_2 = \lim_{s \rightarrow -i\omega} \left( \frac{A\omega G(s)}{s-i\omega} \right) = -\frac{A}{2i} G(-i\omega) \quad (7.11)$$

$$G(i\omega) = R\{G(i\omega)\} + iI\{G(i\omega)\} = |G(i\omega)| e^{i\phi} \quad (7.12)$$

$$G(-i\omega) = |G(i\omega)|e^{-i\varphi} \quad (7.13)$$

where

$$|G(i\omega)| = \sqrt{R^2\{G(i\omega)\} + I^2\{G(i\omega)\}} \quad (7.14)$$

$$\varphi = \tan^{-1} \frac{I\{G(i\omega)\}}{R\{G(i\omega)\}} = \angle G(i\omega) \quad (7.15)$$

$$y(t) = \frac{A}{2i}|G(i\omega)|e^{i\varphi}e^{i\omega t} - \frac{A}{2i}|G(i\omega)|e^{-i\varphi}e^{-i\omega t} \quad (7.16)$$

$$y(t) = \frac{A}{2i}|G(i\omega)|\{e^{i(\varphi+\omega t)} - e^{-i(\varphi+\omega t)}\} \quad (7.17)$$

$$y(t) = A|G(i\omega)|\sin(\omega t + \varphi) \quad (7.18)$$

The magnitude ratio =  $\boxed{g = \frac{|y(t)|}{|x(t)|} = |G(i\omega)|}$  and the phase shift  $\boxed{\varphi = \angle G(i\omega)}$

Then, for linear system, the frequency response is independent of the magnitude of input signal. The gain (magnitude ratio) and phase (phase shift), can be calculated using the transfer function  $G(s)$  according to the following procedure:

1. Find  $G(i\omega)$  by replacing  $(s)$  by  $(i\omega)$  in the expression of  $G(s)$ .
2. Separate the real and imaginary parts of  $G(i\omega)$  by multiplying both the numerator and denominator by the conjugate of the denominator.
3. Find the expressions for phase and gain.
4. Substitute for  $\omega$  to find the gain and phase corresponding to different values of the frequency of input signal.

**Example 7.1** Find the expressions for the gain and phase of the first order element.

$$G(s) = \frac{1}{Ts + 1}$$

$$G(i\omega) = \frac{1}{1+i\omega T} \times \frac{1-i\omega T}{1-i\omega T} = \frac{1}{1+\omega^2 T^2} - i \frac{\omega T}{1+\omega^2 T^2}$$

$$\text{Gain} = |G(i\omega)| = \sqrt{\frac{1}{(1+\omega^2 T^2)^2} + \frac{\omega^2 T^2}{(1+\omega^2 T^2)^2}} = \frac{1}{\sqrt{1+\omega^2 T^2}}$$

$$\text{Phase} = \angle G(i\omega) = \tan^{-1}(-\omega T) = -\tan^{-1}(\omega T)$$

### 7.3 POLAR PLOT (NYQUIST DIAGRAM)

The frequency response information for a system can be displayed on an R-I diagram (Polar plot). The input sinusoidal is considered to be represented by a unit vector lying along the positive real axis. For any frequency, the magnitude and phase of the output can be defined by a corresponding output vector. A phase lag is represented by the rotation of the vector in the clockwise direction.

The Nyquist diagram is a plot showing the variation of the magnitude and phase in polar coordinates. The curve drawn is the locus of the termini of the system output vectors. In this locus, numerical values of  $\omega$  should be indicated against the points.

#### 7.3.1 Polar Plot for First Order Element

The following expressions for gain and phase of first order element were deduced in example 7.1.

$$R\{G(i\omega)\} = \frac{1}{1 + \omega^2 T^2} \quad (7.19)$$

$$I\{G(i\omega)\} = -\frac{\omega T}{1 + \omega^2 T^2} \quad (7.20)$$

$$\text{Gain} = \frac{1}{\sqrt{1 + \omega^2 T^2}} \quad (7.21)$$

$$\text{Phase} = \tan^{-1}(-\omega T) = -\tan^{-1}(\omega T) \quad (7.22)$$

$$I^2 + R^2 = \frac{1}{1 + \omega^2 T^2} = R \quad (7.23)$$

$$I^2 + R^2 - R = 0 \quad (7.24)$$

$$I^2 + \left(R - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \quad (7.25)$$

The frequency response of a first order element is plotted in the polar plot of Fig.7.3. The plot is a semi-circle with center (0.5,0) and radius 0.5. For some input signal frequencies, the corresponding points on the plot are marked by determining the corresponding phase; Phase =  $-\tan^{-1}(\omega T)$ .

For  $\omega = 0$ , the phase = 0 and for  $\omega = 1/T$ , the phase =  $-45^\circ$ . The further increase in the frequency increases the phase lag up to a maximum value of  $90^\circ$ . Thus, the polar plot of a first order element occupies only the first quadrant of the R-I plane.

If the transfer function gain is different from unity, ( $k_1=2$ , for example), the scale of both of I and R axes is multiplied by  $k_1$ . The frequency response plot starts at the point ( $k_1, 0$ ) for  $\omega = 0$ .

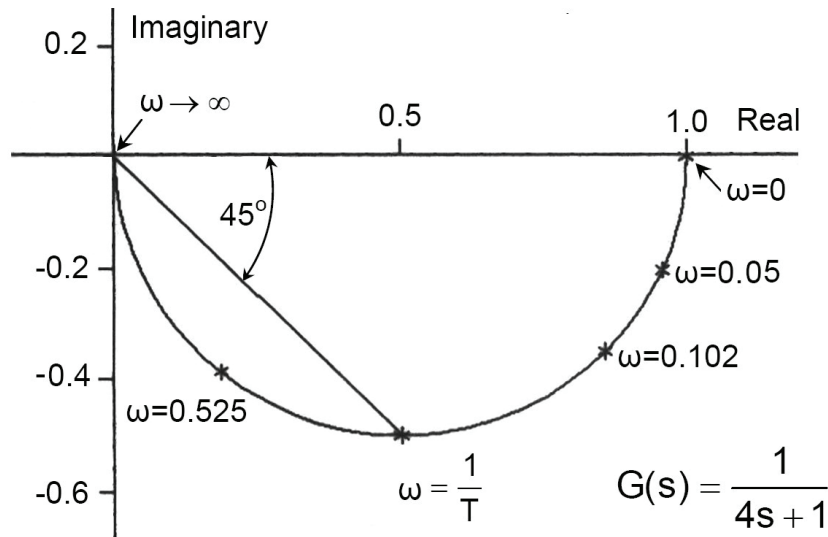


Fig.7.3 Polar (Nyquist) plot of a first order element,  $k_1=1$

### 7.3.2 Polar Plot of Second Order Element

The expressions for the gain and phase of second order element are deduced as follows:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (7.26)$$

$$G(i\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n} \times \frac{\omega_n^2 - \omega^2 - i2\zeta\omega\omega_n}{\omega_n^2 - \omega^2 - i2\zeta\omega\omega_n} \quad (7.27)$$

$$G(i\omega) = \frac{1 - (\omega/\omega_n)^2}{\{1 - (\omega/\omega_n)^2\}^2 + 4\zeta^2\omega^2/\omega_n^2} - i \frac{2\zeta\omega/\omega_n}{\{1 - (\omega/\omega_n)^2\}^2 + 4\zeta^2\omega^2/\omega_n^2} \quad (7.28)$$

$$\text{Gain} = |G(i\omega)| = \frac{1}{\sqrt{\{1 - (\omega/\omega_n)^2\}^2 + 4\zeta^2\omega^2/\omega_n^2}} \quad (7.29)$$

$$\text{Phase} = \tan^{-1} \left\{ \frac{-2\zeta\omega / \omega_n}{1 - (\omega / \omega_n)^2} \right\} = -\tan^{-1} \left\{ \frac{2\zeta\omega / \omega_n}{1 - (\omega / \omega_n)^2} \right\} \quad (7.30)$$

The study of the expressions for gain and phase of second order element shows that:

For  $\omega = 0$     Gain = 1    and    Phase = 0

For  $\omega = \omega_n$     Gain =  $1/2\zeta$     and    Phase =  $-90^\circ$

For  $\omega \rightarrow \infty$     Gain = 0    and    Phase =  $-180^\circ$

The Nyquist plot of a second order element is shown by Fig.7.4. It occupies the first two quadrants of the R-I Plane.

If the transfer function gain is different from unity, ( $k_1$  for example), the scale of both of I and R axes is multiplied by  $k_1$ . The frequency response plot starts at the point ( $k_1, 0$ ) for  $\omega = 0$ .

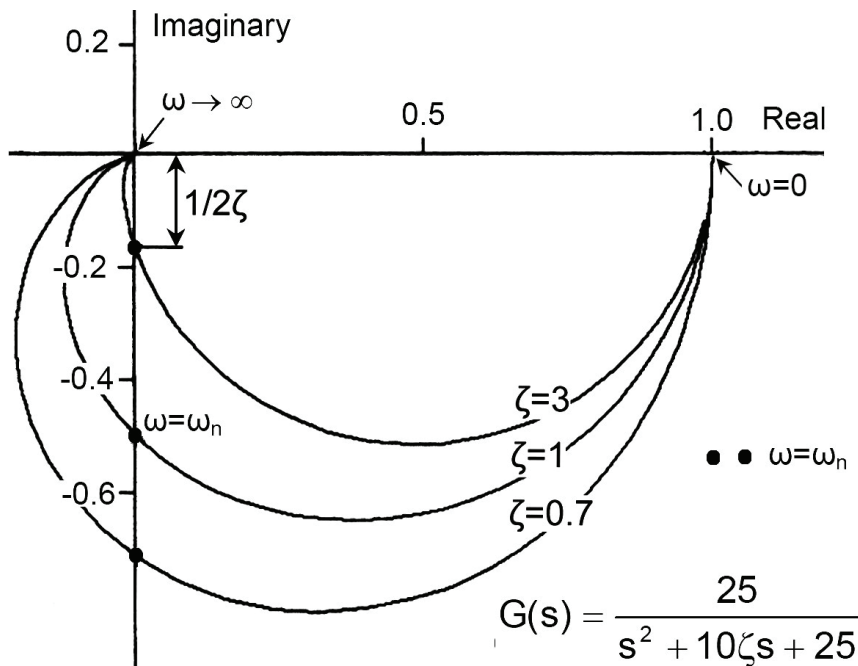


Fig.7.4 Nyquist plot of a second order element,  $k_1=1$

### 7.3.3 Polar Plot of Integrating Member

$$G(s) = \frac{1}{s} \quad (7.31)$$

$$G(i\omega) = \frac{1}{i\omega} = -i\frac{1}{\omega}; I = \frac{-1}{\omega} \text{ \& } R = 0 \quad (7.32)$$

$$G(i\omega) = \frac{1}{i\omega} = -i\frac{1}{\omega}; I = \frac{-1}{\omega} \text{ \& } R = 0 \quad (7.32)$$

$$\text{Gain} = \frac{1}{\omega} \quad (7.33)$$

$$\text{Phase} = \tan^{-1}\left(\frac{I}{R}\right) = -90^\circ \quad (7.34)$$

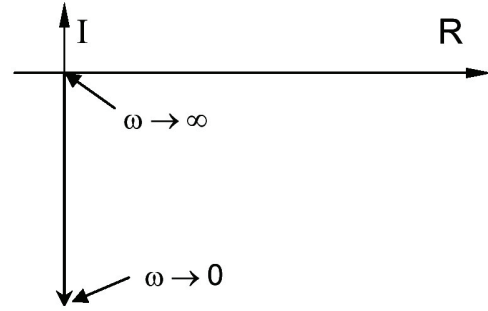


Fig.7.5 Polar plot of integrating member

The whole plot lies on the vertical (imaginary) axis. The phase lag is  $-90^\circ$  for all frequencies. The magnitude is inversely proportional to the input signal frequency  $\omega$ .

### 7.3.4 Polar Plot of Higher Order Elements

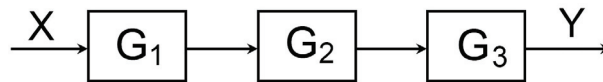


Fig.7.6 Series connected elements

In the case of higher order elements, the frequency response can be calculated from the transfer function according to the procedure discussed in sec. 7.2. However, if the transfer function appears in factorized form, the phase and gain are obtained more easily by thinking of the system as a number of elements connected in series, Fig.7.6. The overall phase for any given input signal frequency is obtained by adding the individual phase components and the overall system gain is obtained by multiplying together the individual gains.

$$G(s) = G_1(s) \times G_2(s) \times G_3(s) \quad (7.35)$$

$$G(i\omega) = G_1(i\omega) \times G_2(i\omega) \times G_3(i\omega) \quad (7.36)$$

$$\text{Phase} = \angle G(i\omega) = \angle G_1(i\omega) + \angle G_2(i\omega) + \angle G_3(i\omega) \quad (7.37)$$

$$\text{Gain} = |G(i\omega)| = |G_1(i\omega)| \times |G_2(i\omega)| \times |G_3(i\omega)| \quad (7.38)$$

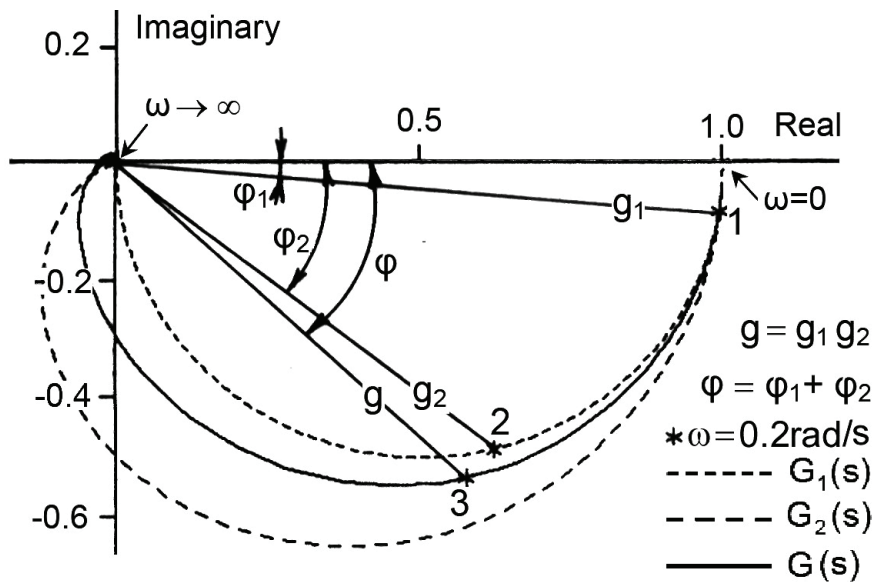
If the polar plots of the individual elements are given (From experimental results for example), the polar plot of the system can be constructed graphically.

**Example 7.2** Draw the Nyquist plot of the system described by the following transfer function:

$$G(s) = \frac{25}{(4s+1)(s^2+10\zeta s+25)} = \frac{1}{4s+1} \times \frac{25}{s^2+10\zeta s+25}$$

Given the Nyquist plot of the first and second order elements, the frequency response of the system described by  $G(s)$  can be calculated point by point as shown by Fig.7.7.

For certain frequency, find the phases  $\phi_1$  &  $\phi_2$  and gains  $g_1$  &  $g_2$  of the second order and first order elements respectively. Then find the resultant gain ( $g = g_1 \times g_2$ ) and resultant phase ( $\phi = \phi_1 + \phi_2$ ). Repeat this process point by point to construct the resultant plot. The resulting polar plot of a third order element occupies three quadrants. Generally, the polar plot of a system of order  $n$  occupies the first  $n$  quadrants of the plane.



$$G_1(s) = \frac{1}{4s+1}, G_2(s) = \frac{25}{s^2+10\zeta s+25} \text{ \& } G(s) = \frac{25}{(s^2+10\zeta s+25)(4s+1)}$$

Fig.7.7 Graphical plotting of the Nyquist diagram of a third order element, given the plots of its basic elements

## 7.4 BODE DIAGRAM

### 7.4.1 Introduction

The Bode diagram is a plot of the frequency response consisting of two different plots; magnitude plot and phase plot, against the frequency of input signal, Fig.7.8.

The Bode diagram is plotted on a semi-logarithmic paper. The frequency is plotted on the horizontal logarithmic scale while the gain and phase are plotted on vertical linear scales. The gain is expressed in decibels (dB), where:

$$\text{Gain} = 20 \log_{10} |G(i\omega)| \quad \text{dB} \quad (7.39)$$

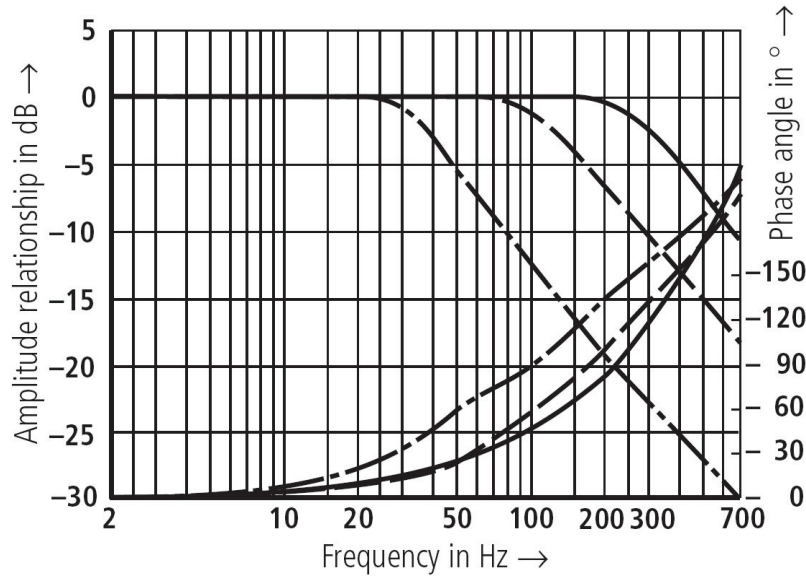


Fig.7.8 Typical Bode plot of a commercial electrohydraulic servovalve

Figure 7.6 shows a system consisting of elements connected in series. The Bode diagram of the system can be obtained if the Bode plots of its components are known, by graphical addition of phase and gain plots.

$$\text{Phase} = \angle G(i\omega) = \angle G_1(i\omega) + \angle G_2(i\omega) + \angle G_3(i\omega) \quad (7.40)$$

$$\text{Gain} = |G(i\omega)| = |G_1(i\omega)| \times |G_2(i\omega)| \times |G_3(i\omega)| \quad (7.41)$$

$$\begin{aligned} \text{Gain (dB)} &= 20 \log_{10} (\text{Gain}) = 20 \log_{10} |G(i\omega)| = \\ &= 20 \log_{10} |G_1(i\omega)| + 20 \log_{10} |G_2(i\omega)| + 20 \log_{10} |G_3(i\omega)| \end{aligned} \quad (7.42)$$

Or

$$\text{Gain (dB)} = \text{Gain}(G_1) + \text{Gain}(G_2) + \text{Gain}(G_3) \quad (7.43)$$

Generally, the transfer function is composed of product combination of the following basic types:

$$K, \quad \frac{1}{s^n}, \quad s^n, \quad \frac{1}{Ts+1}, \quad Ts+1, \quad \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \text{and} \quad \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$$

Therefore, the Bode diagrams of the majority of systems are easily constructed if the Bode plots of these typical elements are known.

### 7.4.2 Bode Plot of Basic Elements

#### 7.4.2.1 Proportional element

$$G(s) = K, \quad I=0 \text{ \& } R=K \quad (7.44)$$

$$\text{Gain} = 20 \log_{10} K \quad \text{dB} \quad (7.45)$$

$$\text{Phase} = \angle G(i\omega) = \tan^{-1}(0) = 0 \quad (7.46)$$

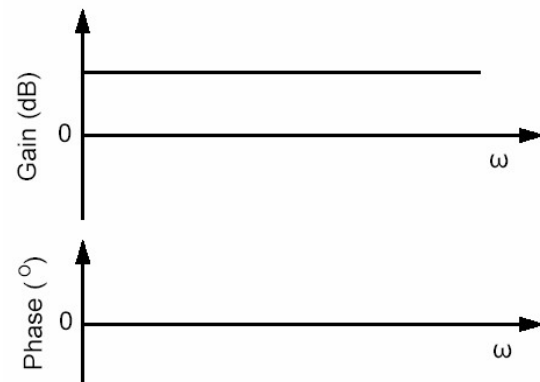


Fig.7.9 Bode plot of proportional element

The constant term (also called proportional term or gain term) presents a constant gain and zero phase as shown in Fig.7.9.

#### 7.4.2.2 Integrating Element

$$G(s) = \frac{1}{s} \quad (7.47)$$

$$G(i\omega) = \frac{1}{i\omega} = -i \frac{1}{\omega}, \quad I = -\frac{1}{\omega} \text{ \& } R=0 \quad (7.48)$$

$$|G(i\omega)| = \frac{1}{\omega} \quad (7.49)$$

$$\text{Gain} = -20 \log(\omega) \quad (7.50)$$

$$\text{Phase} = \tan^{-1}\left(\frac{I}{R}\right) = -90^\circ \quad (7.51)$$

The Bode plot of an integrating member,  $G(s)=1/s$ , is shown in Fig.7.10. The gain plot is a straight line inclined by  $-20 \text{ dB/decade}$ , (one decade is the distance on the horizontal axis along which the frequency is increased by 10 times). The plot is a straight line of slope  $-20 \text{ dB/decade}$  passing by the point  $(1,0)$ . It presents a constant phase of  $-90$  degrees.

**Assignment:** Derive the expressions for the gain and phase of integrating and differentiating members of different powers and discuss their plots given in Fig.7.10.

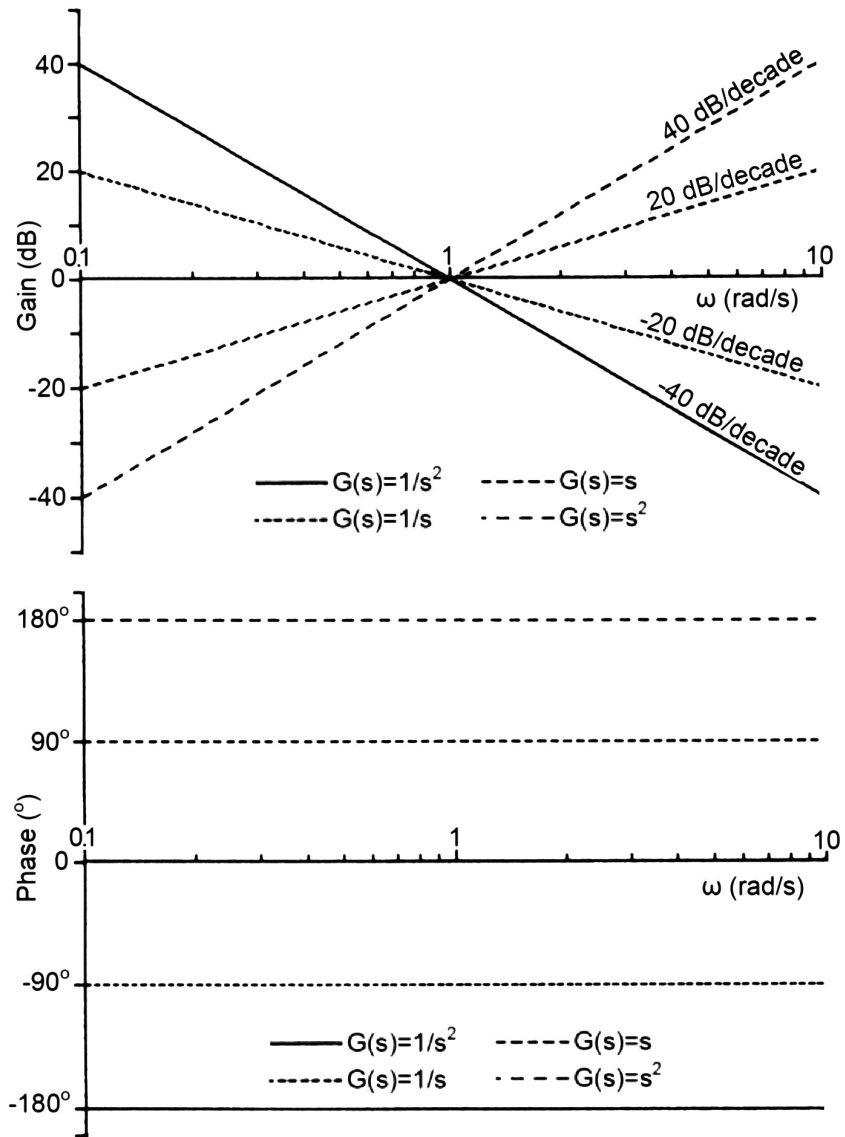


Fig.7.10 Bode plots of Integrating and differentiating members

### 7.4.2.3 First order element (simple lag)

$$G(s) = \frac{1}{Ts + 1} \tag{7.52}$$

$$G(i\omega) = \frac{1}{1+i\omega T} \times \frac{1-i\omega T}{1-i\omega T} = \frac{1}{1+\omega^2 T^2} - i \frac{\omega T}{1+\omega^2 T^2} \tag{7.53}$$

$$\text{Gain} = |G(i\omega)| = \sqrt{\frac{1}{(1+\omega^2 T^2)^2} + \frac{\omega^2 T^2}{(1+\omega^2 T^2)^2}} = \frac{1}{\sqrt{1+\omega^2 T^2}} \tag{7.54}$$

$$\text{Gain} = 20 \log \left( \frac{1}{\sqrt{1 + \omega^2 T^2}} \right) = -20 \log \left( \sqrt{1 + \omega^2 T^2} \right) \quad (7.55)$$

$$\text{Phase} = \angle G(i\omega) = \tan^{-1}(-\omega T) = -\tan^{-1}(\omega T) \quad (7.56)$$

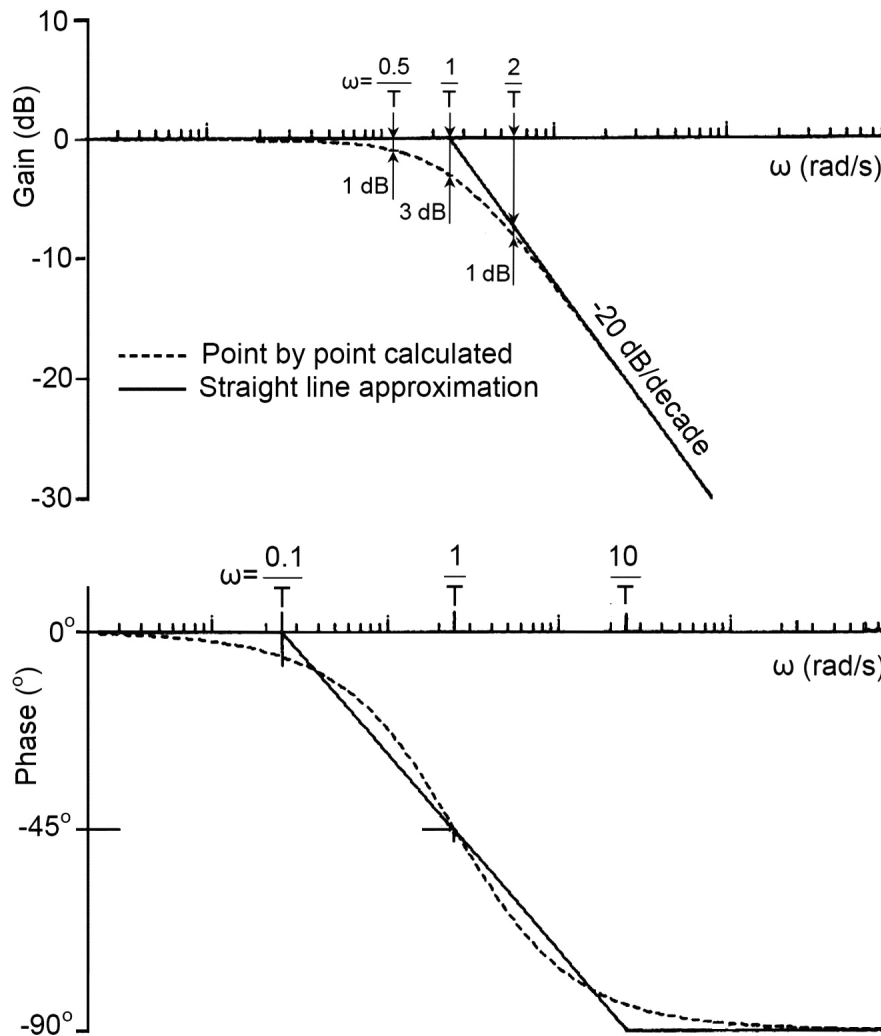


Fig.7.11 Bode plot of first order element (Simple lag)

The gain and phase plots of the first order elements are shown in Fig.7.11. These plots can be simplified by asymptotic approximation lines as follows:

for  $\omega T \ll 1$ ; (Low frequency portion)

$$\text{Gain} = -20 \log(1) = 0 \text{ dB} \quad \text{and} \quad \text{Phase} = 0^\circ$$

The gain plot is a line segment along the horizontal axis.

for  $\omega T \gg 1$ ; (High frequency portion)

$$\text{Gain} = -20 \log\left(\sqrt{\omega^2 T^2}\right) = -20 \log \omega T \quad \text{and} \quad \text{Phase} = -90^\circ$$

The gain plot is a straight line inclined by -20 dB/decade, intersecting the horizontal line at  $\omega = 1/T$ .

These two lines intersect at a corner frequency (Break point)  $\omega_c = 1/T$ . The real plot rounds off the junction. The maximum difference is 3 dB at  $\omega = \omega_c$ , as shown in Fig.7.11. At a frequency of  $\omega = 0.5/T$  and  $\omega = 2/T$ , the difference between the straight-line approximation and real plot is 1 dB. Therefore, for simplicity, the straight-line approximation can be used.

A linear approximation may be also plotted for the phase as shown in Fig.7.11. But, the real phase plot is frequently used.

#### 7.4.2.4 Simple lead element

$$G(s) = 1 + Ts \tag{7.57}$$

**Assignment:** Prove that the gain and phase of the simple lead element are given by the following expressions and plot the Bode diagram; real plot and straight line approximation. Show that the plots are mirror image of the simple phase-lag plots around the zero gain and zero phase lines

$$\text{Gain} = 20 \log|G(i\omega)| = 20 \log\sqrt{1 + \omega^2 T^2} \tag{7.58}$$

$$\text{Phase} = \tan^{-1}(\omega T) \tag{7.59}$$

#### 7.4.2.5 Second order element

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7.60}$$

$$|G(i\omega)| = \frac{1}{\sqrt{\{1 - (\omega/\omega_n)^2\}^2 + 4\zeta^2\omega^2/\omega_n^2}} \tag{7.61}$$

$$\text{Gain} = -20 \log\sqrt{\{1 - (\omega/\omega_n)^2\}^2 + 4\zeta^2\omega^2/\omega_n^2} \quad \text{dB} \tag{7.62}$$

$$\text{Phase} = -\tan^{-1} \left\{ \frac{2\zeta\omega / \omega_n}{1 - (\omega / \omega_n)^2} \right\} \quad (7.63)$$

The Bode plots of a second order element are shown in Fig.7.12, for different values of damping ratio. A straight line approximation of the gain is also plotted.

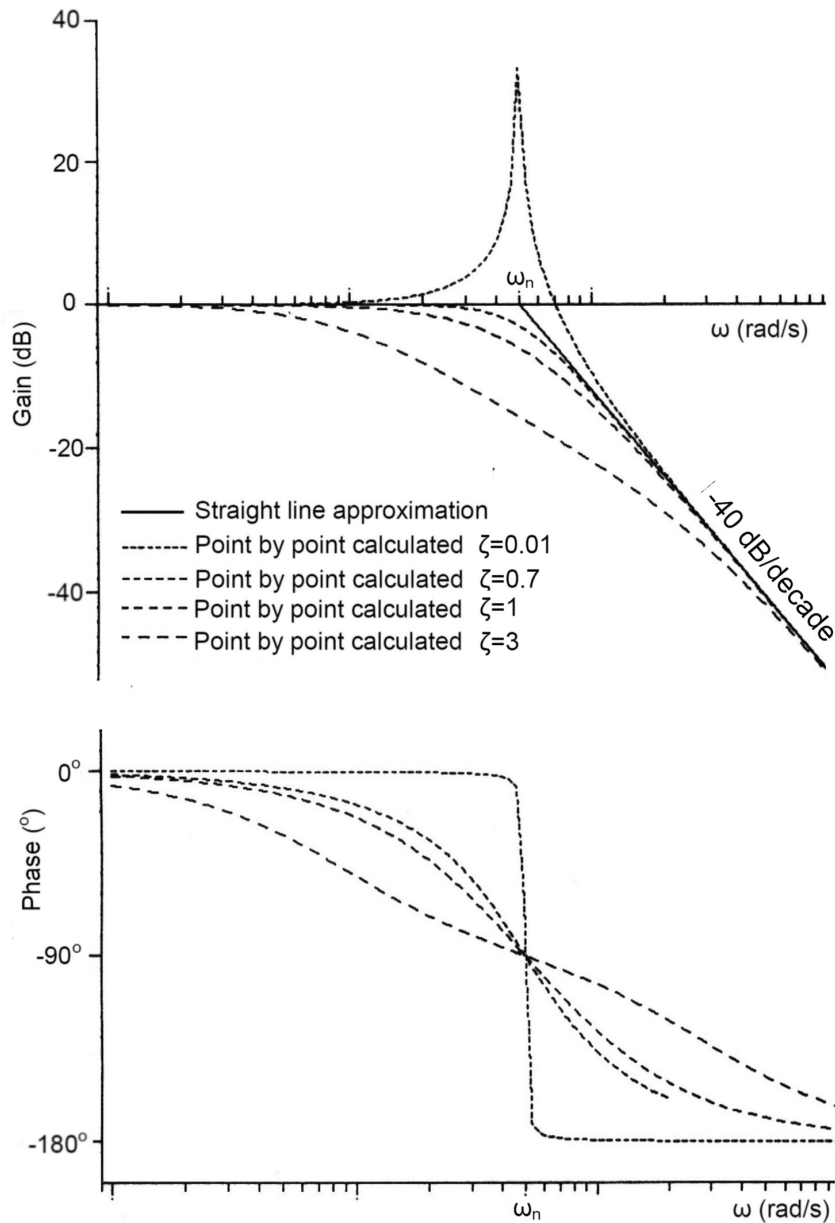


Fig.7.12 Bode plot of second order element

For  $\omega \ll \omega_n$  Gain = 0 dB and Phase =  $0^\circ$

The gain plot is a line segment along the horizontal axis.

For  $\omega \gg \omega_n$  Gain =  $-40 \log(\omega/\omega_n)$  dB and Phase =  $-180^\circ$

The gain plot is a straight line inclined by - 40 dB/decade.

These two straight lines intersect at  $\omega = \omega_n$ ; this is called the corner frequency (Break point). At this frequency the phase =  $-90^\circ$  and the gain depends only on the damping ratio.

$$\text{gain} = -20 \log(2\zeta)$$

There is no convenient straight-line approximation for the phase.

#### 7.4.2.6 Quadratic lead element

$$G(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2) / \omega_n^2 \quad (7.64)$$

**Assignment:** Prove that the gain and phase of the quadratic lead element are given by the following expressions and plot the Bode diagram; real plot and straight line approximation.

$$\text{Gain} = 20 \log \sqrt{\left\{1 - (\omega/\omega_n)^2\right\}^2 + 4\zeta^2 \omega^2 / \omega_n^2} \quad \text{dB} \quad (7.65)$$

$$\text{Phase} = \tan^{-1} \left\{ \frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \right\} \quad (7.66)$$

The magnitude and phase have the same numerical values as those of second order element, but have opposite sign. They are represented on the Bode plot by families of curves which are mirror image of those of the second order element, with respect to the zero gain and zero phase lines.

**Example 7.3** Plot the Bode diagram of the system described by the following transfer function:

$$G(s) = \frac{(5 + 0.5s)(2 + 0.05s)}{(1 + 0.25s)(1 + 0.004s)(1 + 0.004s)}$$

$$G(s) = 10 \times (1 + 0.1s) \times (1 + 0.025s) \times \frac{1}{(1 + 0.25s)} \times \frac{1}{(1 + 0.004s)^2}$$

The Bode plot of the system can be drawn by plotting the frequency response of the individual elements, by straight line approximation and finding out the gain plot of the system by graphical summation as shown in Fig.7.13. The real gain plot of the system and the straight-line approximation are shown in Fig.7.14. The point by point plot is directly calculated by using the gain expression deduced from the transfer function.

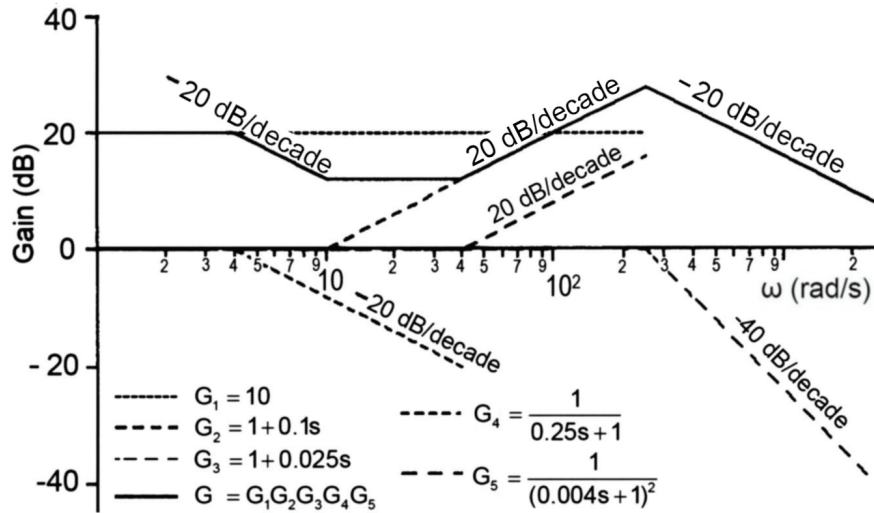


Fig.7.13. Graphical construction of Bode plot

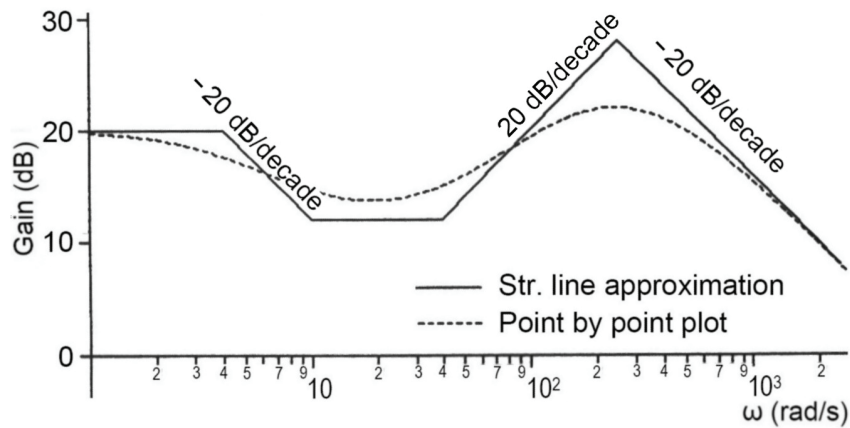
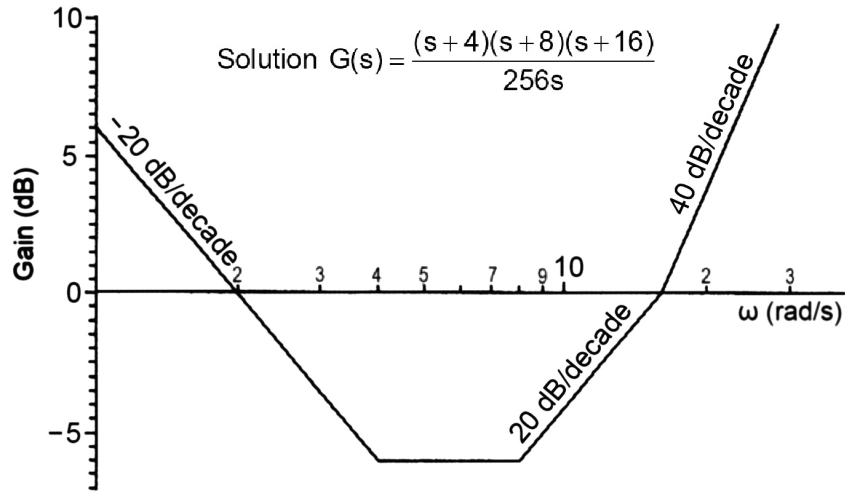


Fig.7.14 Real and straight-line approximation gain plots

**Example 7.4** Find the transfer function of the subsystem whose straight-line approximation of the Bode gain plot is given in Fig.7.15.



### 7.5 NICHOL'S CHART

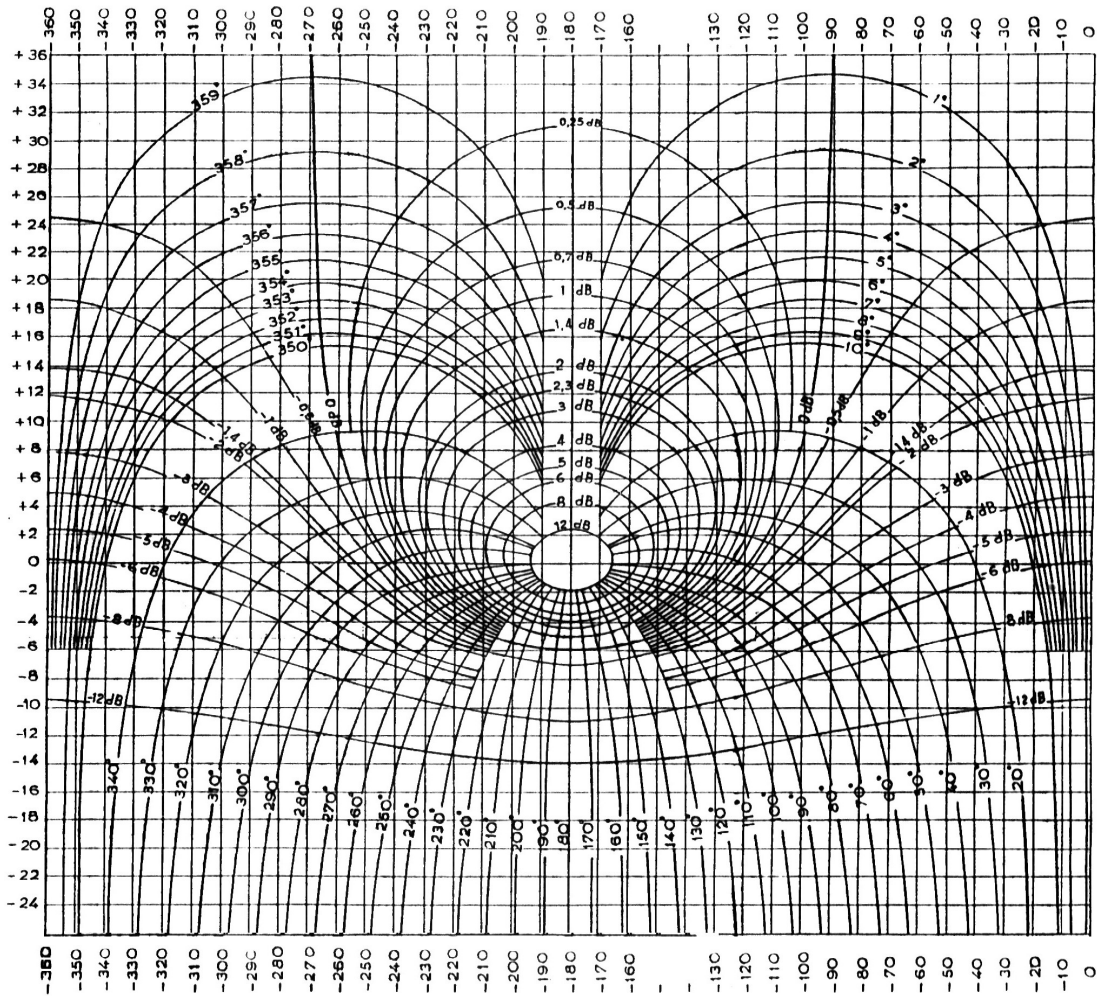


Fig.7.16 Nichol's chart

The Nichol's chart enables to find, in a simple way, the frequency response of the closed loop, if the open loop response is known. The open loop frequency response is plotted on the horizontal and vertical coordinates; the horizontal is for the phase in degrees while the vertical is for the gain in decibels. The values of frequency of the input signal ( $\omega$ ) should be indicated on the plot in a way similar to the polar (Nyquist) plot. The gain of the closed loop response can be read off directly from the intersection of the open loop frequency response plots with the contours encircling the point  $(-180^\circ, 0 \text{ dB})$ . The phase of the closed loop response is found at the intersection with the contours radiated from the point  $(-180^\circ, 0 \text{ dB})$ . The Nichol's chart is of special interest for the analysis of system stability.

## 7.6 EXERCISE

1. Draw the polar plots of the following systems.

$$(a) G(s) = \frac{1}{1+2s}$$

$$(b) G(s) = \frac{1}{s(1+2s)}$$

$$(c) G(s) = \frac{1}{(s^2 + s + 1)}$$

$$(d) G(s) = \frac{1}{s(s^2 + s + 1)}$$

$$(e) G(s) = \frac{1}{(1+2s)(s^2 + s + 1)}$$

$$(f) G(s) = \frac{1}{s(1+0.02s)(1+0.01s)}$$

2. Draw the Bode plot; magnitude only, for the following systems using the straight-line approximations.

$$(a) G(s) = \frac{100s}{s+10}$$

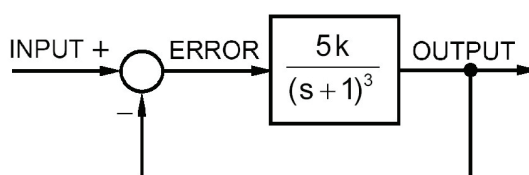
$$(b) G(s) = \frac{1}{s^2 + s + 1}$$

$$(c) G(s) = \frac{2}{(0.5s+1)(0.125s+1)}$$

$$(d) G(s) = \frac{2}{(s+1)(s+2)(s^2 + 3s + 16)}$$

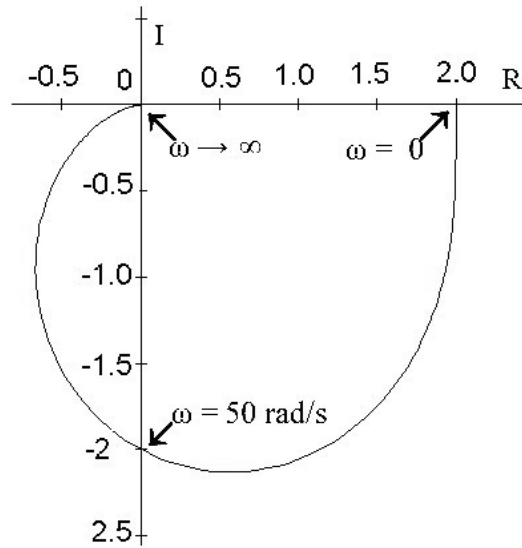
$$(e) G(s) = \frac{s}{100}(s+10)^2$$

3. Plot the Bode and Nyquist diagrams for the compensated open loop of the system described by the following block diagram, assume three different values for  $k < 20$ .

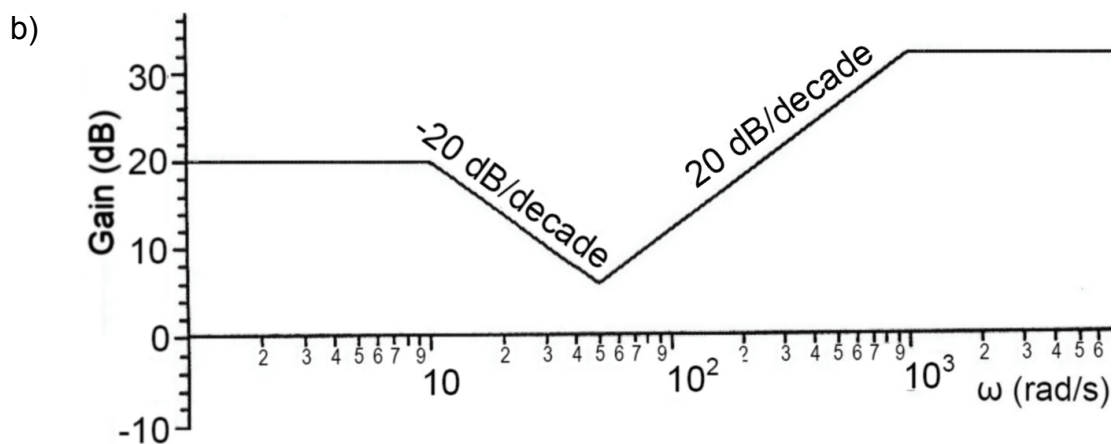
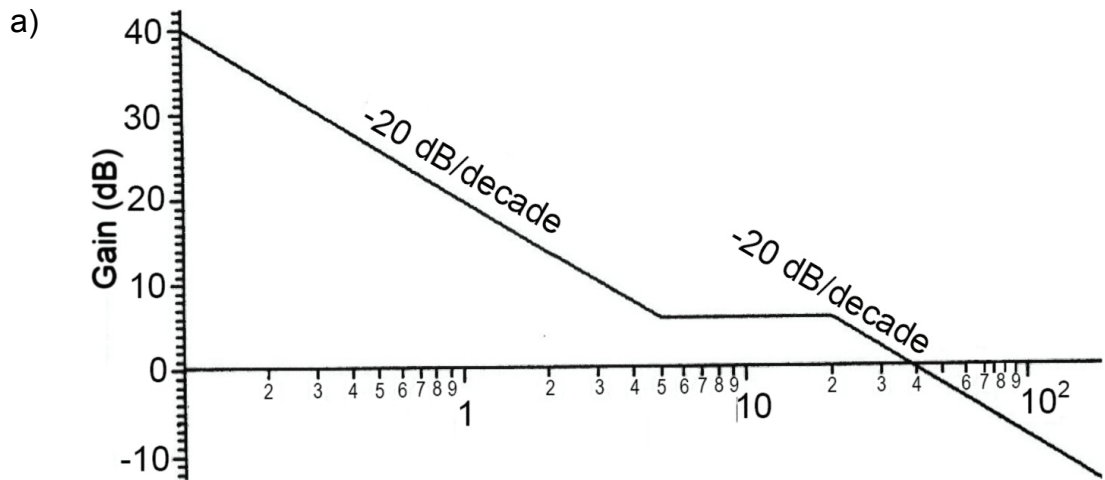


4. Draw the magnitude and phase Bode plots for the system (c) of problem 1.

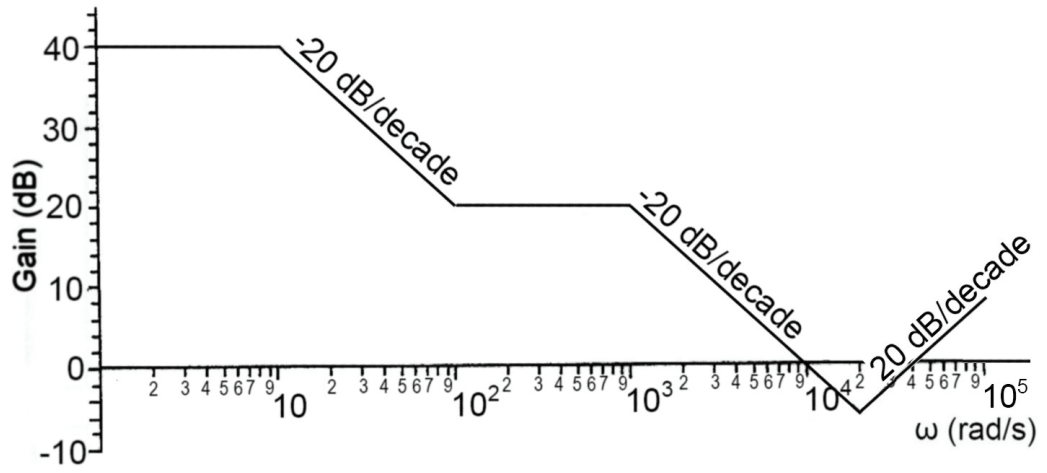
5. Shown is the frequency response plot of a system, write down its transfer function.



6. Find the transfer functions for the systems having the following magnitude Bode plots.



c)



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# 8 | FEEDBACK SYSTEM ACCURACY AND STABILITY

## 8.1 INTRODUCTION

In the case of open loop systems, the system output is not compared with the required value (which is determined by the input signal). The external disturbances, or deterioration with increasing age, may cause changes of the system parameters and its transfer function. Therefore the actual output may deviate far from the required one. Introducing a feedback path allows the actual system output to be compared with the input signal (required output), Fig.8.1. An error signal is generated and forms the actuating signal. An element, of transfer function  $H(s)$ , is inserted in the feedback path. It represents the dynamic characteristics of the transducer which measures the output signal and converts it to the same type as the reference input signal. The main reason for introducing the feedback is to minimize the error between the actual and desired system outputs.

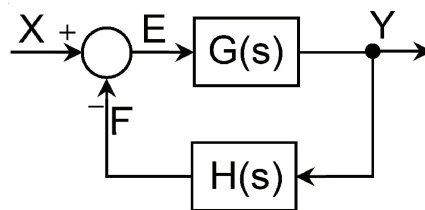


Fig.8.1 Block diagram of feedback control system

The magnitude of the steady state error is a measure of the system accuracy. Closing the loop should reduce or eliminate the steady state error. But the system can be very oscillatory or even unstable.

## 8.2 STEADY STATE ERROR

The steady state error is a measure of the system accuracy. It is the value to which the error signal tends as the transient disturbances die out.

Considering the system shown in Fig.8.1, the error signal is given as follows:

$$E(s) = X(s) - H(s)Y(s) = X(s) - H(s)G(s)E(s) \quad (8.1)$$

$$E(s) = \frac{X(s)}{1 + G(s)H(s)} \quad (8.2)$$

The steady state error  $e_{ss}$  is the limiting value of  $e(t)$ ;

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \quad (8.3)$$

Or,

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sX(s)}{1 + G(s)H(s)} \quad (8.4)$$

Equation 8.4 shows that the steady state error depends on the compensated open loop transfer function,  $G(s)H(s)$  and input signal  $X(s)$ .

### 8.2.1 Steady State Error with Step Input

$$X(s) = \frac{k_2}{s} \quad (8.5)$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sX(s)}{1 + G(s)H(s)} = \lim_{s \rightarrow 0} \frac{k_2}{1 + G(s)H(s)} \quad (8.6)$$

$$e_{ss} = \frac{k_2}{1 + \lim_{s \rightarrow 0} G(s)H(s)} \quad (8.7)$$

Or

$$\boxed{e_{ss} = \frac{k_2}{1 + k_p}}, \text{ where } \boxed{k_p = \lim_{s \rightarrow 0} G(s)H(s)} \quad (8.8)$$

Where  $k_2$  = Magnitude of applied step input.

$k_p$  = Position error coefficient.

It is evident that, in the case of a step input, the steady state error becomes zero only if  $k_p \rightarrow \infty$ . This condition is satisfied if  $G(s)H(s)$  includes an integrating element (a factor  $s^n$  in the denominator,  $n \neq 0$ ).

If the compensated open loop transfer function contains no integral term, the system response has a nonzero steady state error  $e_{ss}$ .

It is more convenient to define the relative steady state error rather than the absolute error. The relative steady state error is defined as:

$$\bar{e}_{ss} = \frac{e_{ss}}{k_2} = \frac{\text{Actual steady state error}}{\text{Required output}} \quad (8.9)$$

Or

$$\bar{e}_{ss} = \frac{1}{1+k_p} \quad (8.10)$$

Then

$$k_p = \frac{k_2 - e_{ss}}{e_{ss}} = \frac{1 - \bar{e}_{ss}}{\bar{e}_{ss}} \quad (8.11)$$

The position error coefficient equals the gain of the compensated open loop transfer function,  $k_p = \lim_{s \rightarrow 0} G(s)H(s)$ . Generally, greater open loop gain results in smaller steady state error.

**Example 8.1** Find the required position error coefficient  $k_p$  if the maximum allowable relative steady state error  $\bar{e}_{ss} \leq 1\%$ .

$$\bar{e}_{ss} \leq 0.01 \quad \text{or} \quad \frac{1}{1+k_p} \leq 0.01 \quad \text{then} \quad k_p \geq 99$$

**Example 8.2** The following system, Fig.8.2, is excited by a step input of magnitude 5, calculate the steady state error then find the required gain of the open loop if the maximum allowable steady state error is less than 2% of the required output.

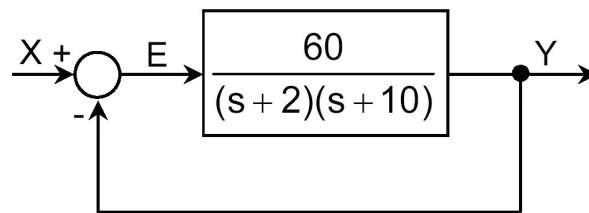


Fig.8.2

$$G(s)H(s) = \frac{60}{(s+2)(s+10)}, \quad k_2 = 5$$

$$k_p = \lim_{s \rightarrow 0} G(s)H(s) = 3$$

$$\bar{e}_{ss} = \frac{1}{1+k_p} = \frac{1}{1+3} = 0.25 = 25\%$$

$$\bar{e}_{ss} = \frac{e_{ss}}{k_2} \quad \text{or} \quad e_{ss} = k_2 \bar{e}_{ss} = 5 \times 0.25 = 1.25$$

For a maximum steady state error of  $0.02k_2$ ,

$$\bar{e}_{ss} \leq 0.02 \quad \text{or} \quad \frac{1}{1+k_p} \leq 0.02 \quad \text{then} \quad k_p \geq 49$$

Therefore, the gain of the open loop should be increased K times by introducing a proportional element (where;  $K \geq 49/3$ ). Figure 8.3 shows the system after adding a gain within the recommended range.

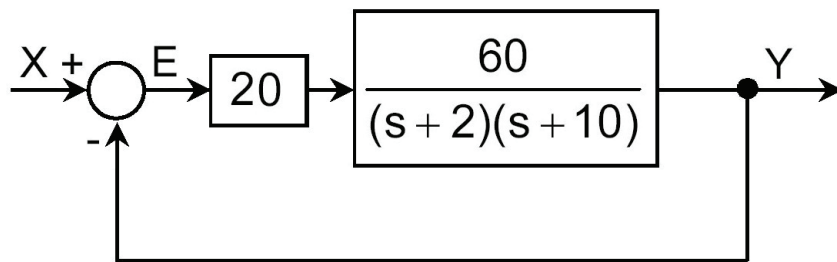


Fig.8.3

### 8.2.2 Steady State Error with Ramp Input

$$X(s) = \frac{k}{s^2} \quad (8.12)$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sX(s)}{1+G(s)H(s)} = \lim_{s \rightarrow 0} \frac{k}{s + sG(s)H(s)} \quad (8.13)$$

$$e_{ss} = \frac{k}{\lim_{s \rightarrow 0} \{sG(s)H(s)\}} \quad (8.14)$$

Or

$$e_{ss} = \frac{k}{k_v}, \quad \text{where} \quad k_v = \lim_{s \rightarrow 0} sG(s)H(s) \quad (8.15)$$

Where  $k$  = Proportionality coefficient of the ramp input.

$k_v$  = Velocity error coefficient;

In the case of ramp input, the steady state error becomes zero, only if  $k_v \rightarrow \infty$ . This condition is satisfied if,  $G(s)H(s)$ , includes a factor  $s^n$  in the denominator, where  $n \geq 2$ .

### 8.3 STABILITY OF FEEDBACK SYSTEMS

#### 8.3.1 Routh-Hurwitz Stability Criterion

Routh-Hurwitz stability criterion judges the stability of a system, given its characteristic equation. It determines whether the system is stable or unstable.

A system is said to be stable if its response to any bounded input is also bounded. For a system to be useful, it should be stable. It has been shown (sec. 6.2.5), that the transient response of a dynamic system is highly affected by the location of the roots of the characteristic equation in the R-I plane. For a system to be stable, the roots should take place in the left half of the s-plane, or, in other words, the roots should be real negative or complex with negative real part and none of them is of zero real part. Therefore, a decision about the system stability can be reached if the locations of the roots of the characteristic equation of the closed loop in the R-I plane are known.

The characteristic equation of the closed loop system, Fig.8.1, is:

$$1 + G(s)H(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0 \quad (8.16)$$

The roots of the first and second order characteristic equations are easily found. However, in the case of higher order characteristic equation, finding out the roots is too complicated. Therefore, around 1880, Routh and Hurwitz developed independently a method for determining whether any root of the characteristic equation has positive real part.

A necessary but not sufficient condition for the polynomial roots to have non zero negative real parts is that all of the coefficients of the characteristic equation should have the same sign and that none of them is zero. If this condition is satisfied, then the necessary and sufficient condition of stability is that the Hurwitz determinants of the polynomial must be positive, where the determinants are given by:

$$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \quad D_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_2 & a_4 \end{vmatrix}, \dots \text{etc.} \quad (8.17)$$

The arithmetic's involved in evaluating these determinants can be avoided by producing the following Routh array:

$$\begin{array}{c|cccc}
 n & a_0 & a_2 & a_4 & a_6 & \dots \\
 n-1 & a_1 & a_3 & a_5 & a_7 & \dots \\
 n-2 & b_1 & b_2 & b_3 & b_4 & \dots \\
 n-3 & c_1 & c_2 & c_3 & \dots & \\
 n-4 & d_1 & d_2 & \dots & & \\
 \dots & \dots & \dots & & & \\
 0 & \dots & & & & 
 \end{array}$$

The first two rows are formed by the coefficients of the characteristic equation, while the subsequent rows are calculated as follows:

$$b_1 = a_2 - \frac{a_0 a_3}{a_1} \quad b_2 = a_4 - \frac{a_0 a_5}{a_1} \quad b_3 = a_6 - \frac{a_0 a_7}{a_1}, \dots \quad (8.18)$$

$$c_1 = a_3 - \frac{a_1 b_2}{b_1} \quad c_2 = a_5 - \frac{a_1 b_3}{b_1} \quad c_3 = a_7 - \frac{a_1 b_4}{b_1}, \dots \quad (8.19)$$

$$d_1 = b_2 - \frac{b_1 c_2}{c_1} \quad d_2 = b_3 - \frac{b_1 c_3}{c_1}, \dots \quad (8.20)$$

The values of the elements of the second column of this array ( $a_0, a_1, b_1, c_1, d_1, \dots$ ) should be positive. Every change of sign means that there exists a root with positive real part.

**Example 8.3** Apply the Routh-Hurwitz stability criterion to check the stability of the system having the following characteristic equation:

$$s^4 + 2s^3 + s^2 + 4s + 2 = 0$$

All of the coefficients are positive and none of them is zero. The Routh array is constructed as follows:

$$\begin{array}{c|ccc}
 4 & 1 & 1 & 2 \\
 3 & 2 & 4 & 0 \\
 \hline
 2 & -1 & 2 & 0 \\
 1 & 8 & 0 & \\
 0 & 2 & & 
 \end{array}$$

There are two sign changes in the second column, which indicates that the characteristic equation has either two real positive roots or a pair of complex-conjugate roots with positive real part. Actually, this characteristic equation has four roots, two real roots and one pair of complex conjugate roots.

$$p_1 = -0.515589, \quad p_2 = -2.1877,$$

$$p_3 = 0.34 + 1.3i \quad \text{and} \quad p_4 = 0.34 - 1.3i$$

Consequently, this system is unstable.

When constructing the Routh array, two kinds of difficulties can occur because of existence of zeros in the second column of the array:

- A zero appears in the second column
- A complete row of zero elements appear

If a zero appears in the first column, the division by zero should be avoided as follows. Replace  $s$  by  $1/\sigma$  in the characteristic equation and apply the Routh-Hurwitz criterion to the new equation in  $\sigma$ . If the polynomial in  $\sigma$  has neither real positive roots nor complex roots with positive real parts then, the polynomial in  $s$  also has neither real positive roots nor complex roots with positive real parts.

**Example 8.4**       $s^5 + s^4 + 4s^3 + 4s^2 + 2s + 1 = 0$

5	1	4	2
4	1	4	1
3	0	1	
2	$\infty$		
1			
0			

By replacing  $s$  by  $1/\sigma$ , then, after arrangement, the following equation results:

$$\sigma^5 + 2\sigma^4 + 4\sigma^3 + 4\sigma^2 + \sigma + 1 = 0$$

5	1	4	1
4	2	4	1
3	2	0.5	
2	3.5	1	
1	-0.07		
0	1		

There are two changes in sign in the first column then the characteristic equation has two roots of positive real parts. Then as  $\sigma$  has a positive value,  $s$  also will have a positive value, since  $s=1/\sigma$ .

If a complete row of zero elements appears, the equation having the row just above that one is to be differentiated. The coefficients of the resulting equation replace the zero values of the row and the construction of the array can be continued.

**Example 8.5**  $s^3 + 10s^2 + 16s + 160 = 0$

$$\begin{array}{c|cc}
 3 & 1 & 16 \\
 2 & 10 & 160 \\
 1 & \theta - 20 & \theta - 0 \\
 0 & 160 & 
 \end{array}
 \xrightarrow{\text{Write the equation}}
 10s^2 + 160
 \xrightarrow{d/ds}
 20s + 0
 \xleftarrow{\text{Replace}}
 \begin{array}{cc}
 20 & 0
 \end{array}$$

The system having this characteristic equation is stable.

**Example 8.6**  $s^6 + 3s^5 + 5s^4 + 12s^3 + 6s^2 + 9s + 1 = 0$

$$\begin{array}{c|cccc}
 6 & 1 & 5 & 6 & 1 \\
 5 & 3 & 12 & 9 & \\
 4 & 1 & 3 & 1 & \\
 3 & 3 & 6 & & \\
 2 & 1 & 1 & & \\
 1 & 3 & & & \\
 0 & 1 & & & 
 \end{array}$$

There exist no sign changes in the second column then the system having this characteristic equation is stable.

### 8.3.2 Nyquist Stability Criterion

Nyquist stability criterion judges the stability of closed loop system on the basis of its compensated open loop frequency response. It decides if the system is stable or not and determines the degree of stability.

#### 8.3.2.1 Stability analysis using Nyquist plot

The Routh-Hurwitz criterion determines if the system is stable or unstable, but it does not give any information about the degree of its stability. The system may be stable, but if the roots of its characteristic equation are too close to the vertical axis of the R-I

plane, the transient oscillations may take too long time to decay, Fig.6.18. On the other hand, the polar plot (Nyquist diagram) of the compensated open loop transfer function,  $G(i\omega)H(i\omega)$ , enables to determine whether any of the roots of the characteristic equation of the closed loop system have positive real part, without actually evaluating them. This plot shows also the degree of system stability and how to improve it. The experimental results of the frequency response evaluation can be directly used without any need to determine the system transfer function.

The Nyquist stability criterion states that **if the compensated open loop of a system is stable then the system with closed loop is also stable provided that the open loop polar plot does not enclose the critical point (-1,0).**

Figure 8.4 shows the case of an unstable system whose open loop polar plot encircles the point (-1, 0) and a stable system whose open loop polar plot does not encircle this critical point.

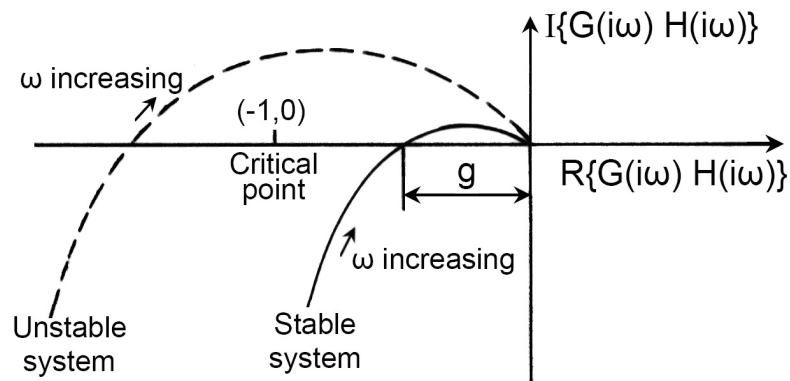


Fig.8.4 Open loop polar plots of stable and unstable systems

When the system is excited by a harmonic signal, the phase shift varies with the frequency of input signal. The polar plot of the compensated open loop intersects with the negative part of the real, horizontal, axis at certain input frequency, called the phase cross-over frequency. At this frequency, the phase lag is  $180^\circ$  and the gain is  $g$ , Fig.8.4. The Nyquist stability criterion can be explained as follows.

If the reference input to the closed loop is a sine wave  $x(t) = A \sin(\omega t)$ , then the signal returning to the error detector will have a different amplitude and phase, Fig.8.5(a). At the phase cross-over frequency, the phase lag is  $180^\circ$  and the magnitude ratio is  $(g)$ ;  $f(t) = ge(t)$ . Then, in the steady state, the returning signal is inverted and added to reference input, Fig. 8.5(b) and the steady state error is given as follows:

$$e(t) = x(t) + ge(t) \quad (8.21)$$

$$e(t) = \frac{x(t)}{1-g} = \frac{A}{1-g} \sin(\omega t) \quad (8.22)$$

There are three different stability cases, depending on the magnitude ratio at the phase cross-over frequency:

- The amplitude of the returning signal is less than that of the input signal,  $g < 1$ . The amplitude of the error signal, converges to a constant value of  $A/(1-g)$ . Consequently, the system output converges to a steady state bounded value and the system is stable, Fig.8.5(c).
- The magnitude ratio at the phase cross-over point is equal to one ( $g = 1$ ). The open loop polar plot intersects the horizontal axis at the critical point  $(-1, 0)$ . The amplitude of the error signal tends to infinity and the system becomes unstable. Moreover, if the input signal is removed, the error signal and system output will continue to oscillate.
- The amplitude of the returning signal is greater than one,  $g > 1$ . The open loop polar plot intersects the horizontal axis at a point  $(-g, 0)$ . The locus encloses the point  $(-1,0)$ . The amplitude of error signal will build up continuously and the system is unstable, Fig.8.5(d). Moreover, if the input signal is removed, the error signal continues to increase or grow up (for  $g > 1$ ).

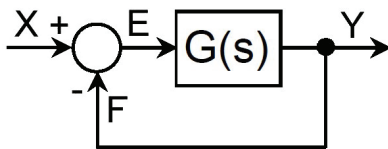


Fig.8.5(a) Typical feedback system

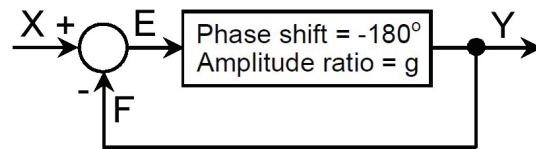


Fig.8.5(b) System excited by sinusoidal input at the phase cross-over point

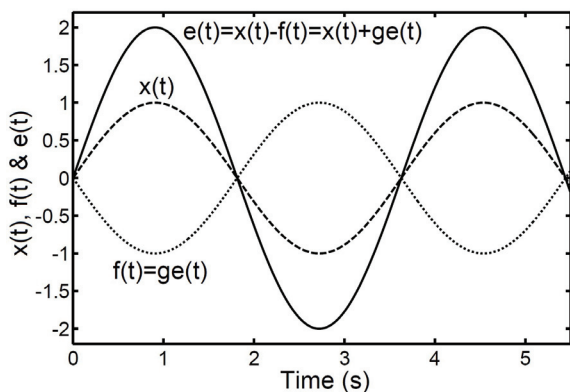


Fig.8.5(c) Steady state input, feedback and error signals for feedback system, excited by harmonic input at the phase cross-over point,  $g < 1$

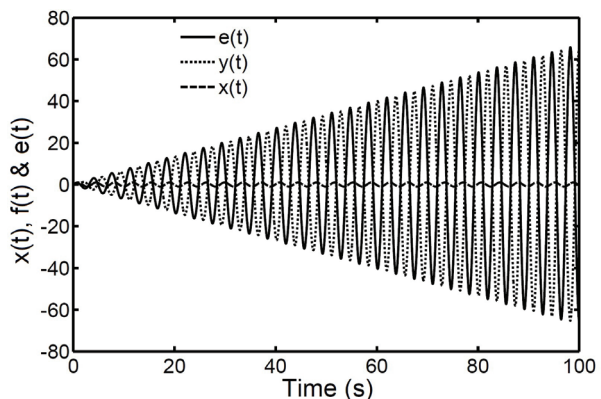
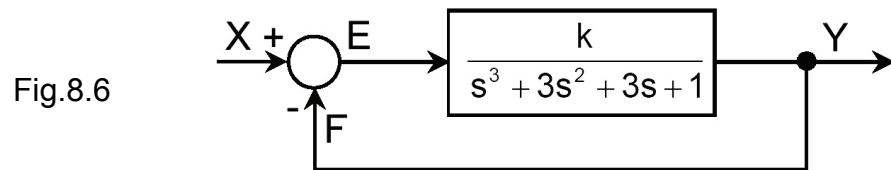


Fig.8.5(d) Steady state input, feedback and error signals for feedback system, excited by harmonic input at the phase cross-over point,  $g > 1$

The magnitude ratio  $g$  depends on the gain  $k$  of the compensated open loop. The effect of open loop gain on the system response is illustrated by example 8.7.

**Example 8.7** Plot the open loop Nyquist diagram, the closed loop response to unit step input and the closed loop Bode diagram of the following system, for open loop gain  $k=1, 5, 8$  &  $10$ . Find the relative steady state error for each case.

Apply Routh-Hurwitz stability criterion to find the limiting value of the open loop gain, for system stability and discuss the obtained results.



The closed loop characteristic equation is:

$$s^3 + 3s^2 + 3s + 1 + k = 0$$

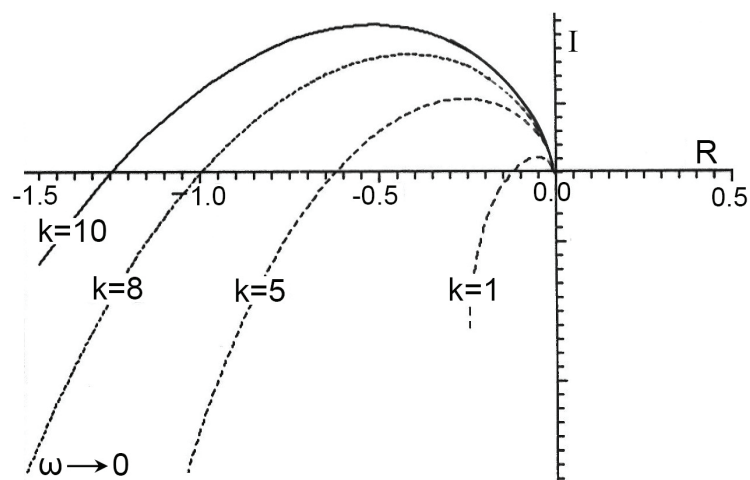
3	1	3
2	3	1+k
1	$3 - \frac{1+k}{3}$	0
0	1+k	

The system is stable if  $3 - \frac{1+k}{3} > 0$  and  $1+k > 0$

Or  $8 > k > -1$

Fig.8.7 Nyquist plot of the open loop of transfer function for different values of gain;

$$GH(s) = \frac{k}{s^3 + 3s^2 + 3s + 1}$$



The relative steady state error is  $\bar{e}_{ss} = \frac{1}{1+k_p}$ , Where  $k_p = \lim_{s \rightarrow 0} G(s)H(s)$

k	$k_p$	$\bar{e}_{ss}$	$\bar{e}_{ss} \%$
1	1	0.5	50
5	5	0.1667	16.67
8	8	0.1111	11.11
10	10	0.0909	9.09

The closed loop gain is  $k/(1+k)$ . Figure 8.7 shows the Nyquist plot of the open loop for different values of open loop gain k. The step response of closed loop to unit step input is plotted in Fig.8.8 and its Bode plot is shown in Fig.8.9.

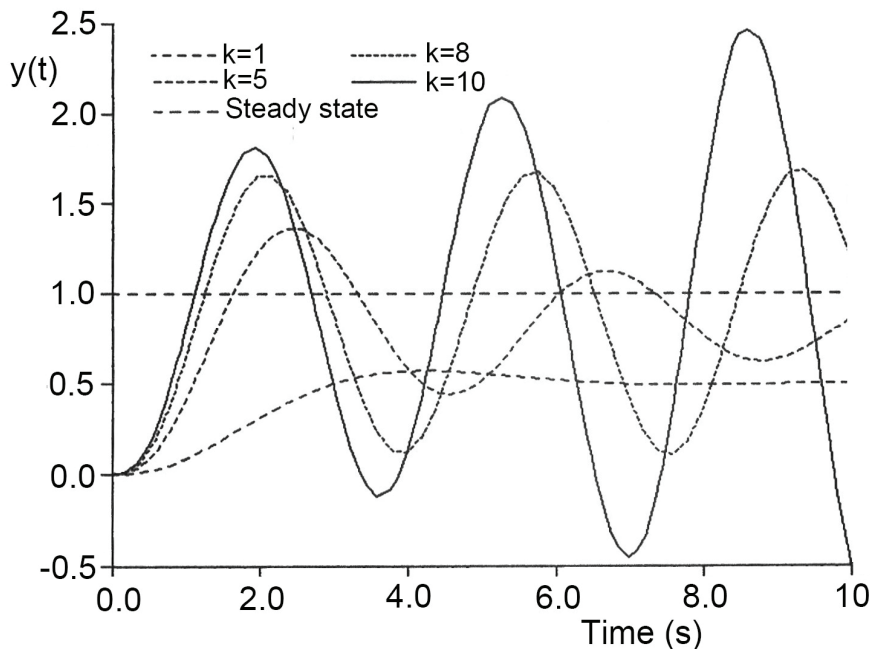


Fig.8.8 Response of the closed loop to unit step input for different values of open loop gain k

The study of Figs. 8.7, 8.8 and 8.9 shows the following:

1. For  $k=1$ , the open loop Nyquist plot intersects the real axis at  $(-0.125,0)$ ; far enough from the critical point. The transient response is stable with considerable steady state error and slight overshoot. The closed loop Bode plot shows low magnitude ratio at resonance frequency.
2. For  $k=5$ , the open loop Nyquist plot intersects the real axis at  $(-0.625,0)$ ; nearer to the critical point. The system is still stable since the locus does not enclose the critical point. The transient response presents considerable transient damped oscillations, overshoot and smaller steady state error. The Bode plot shows higher

magnitude ratio at resonance and the slope of the phase curve becomes steeper. It is evident that this system is less stable than the preceding case.

3. For  $k = 8$ , the open loop Nyquist plot passes through the critical point. The step response is oscillatory. The gain plot of Bode diagram shows that the closed loop magnitude ratio tends to infinity at resonance, which indicates clearly the system instability.
4. For  $k = 10$ , the open loop Nyquist plot encloses the critical point and the system is unstable. The step response becomes unbounded. The closed loop Bode plot shows finite resonance gain and the phase becomes positive for frequencies greater than the resonance frequency.

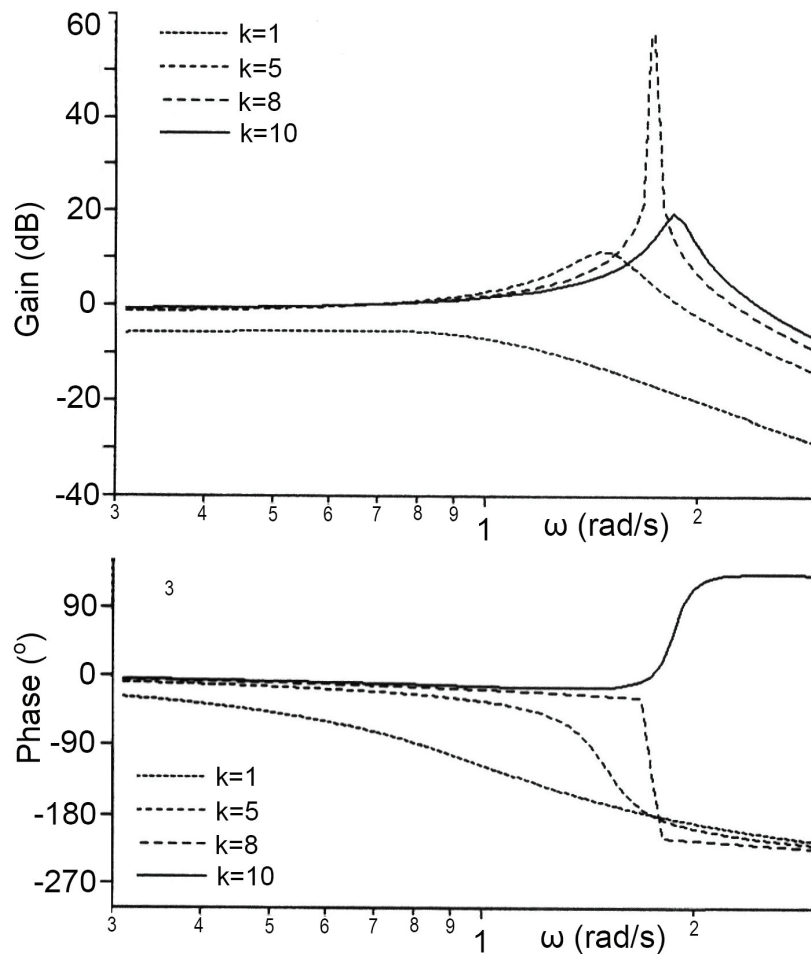


Fig.8.9 Bode diagram of the closed loop for different values of open loop gain.

Generally, it is not sufficient to have a stable system, but the system should have certain **degree of stability, relative stability**. In the time domain, the relative stability can be measured by the maximum percentage overshoot, settling time and damping ratio. In the frequency domain, the relative stability can be evaluated, on the basis of the Nyquist stability criterion, by means of the compensated open loop polar plot;  $G(i\omega)H(i\omega)$ . The closeness of the compensated open loop polar plot to the

critical point gives an indication of the degree of stability. The **gain margin** and **phase margin** are measures of the degree stability, Fig.8.10. These margins are defined as follows:

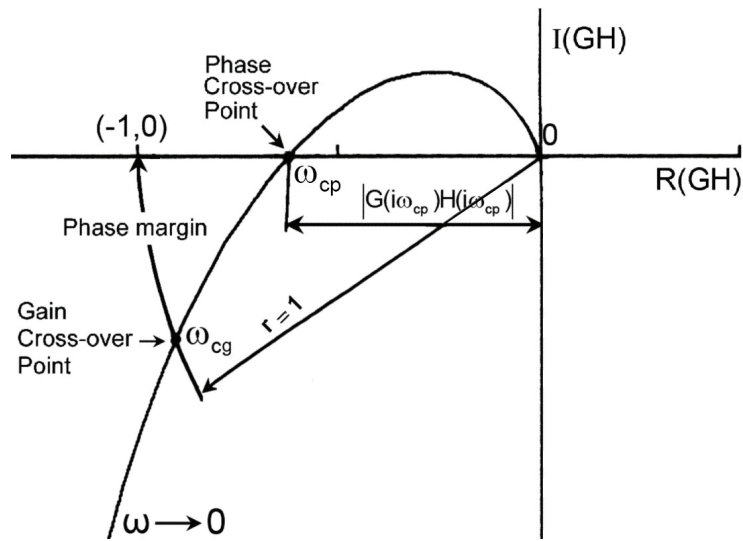


Fig.8.10 Definition of the gain margin and phase margin in the polar coordinates

**Phase cross-over point** is a point at which the Nyquist plot of  $G(i\omega)H(i\omega)$  intersects the negative real axis where  $\angle G(i\omega_{cp})H(i\omega_{cp}) = -180^\circ$ .

**Phase cross-over frequency**,  $\omega_{cp}$  is the frequency at the phase crossover point.

**Gain cross-over point** is a point on the compensated open loop polar plot at which the magnitude of  $G(i\omega)H(i\omega)$  is unity,  $|G(i\omega)H(i\omega)| = 1$ .

**Gain cross-over frequency**,  $\omega_{cg}$  is the frequency at the gain crossover point.

**Gain margin (GM)** is the allowable increase in the open loop gain in decibels (dB) before the closed loop system becomes unstable.

$$\text{Gain Margin (GM)} = \frac{1}{g} = \frac{1}{|G(i\omega_{cp})H(i\omega_{cp})|} \quad (8.23)$$

Or

$$\text{Gain Margin (GM)} = 20 \log \frac{1}{|G(i\omega_{cp})H(i\omega_{cp})|} = -20 \log(|G(i\omega_{cp})H(i\omega_{cp})|) \text{ dB} \quad (8.24)$$

Generally, a minimum recommended gain margin is within 6-12 dB.

The gain margin can be determined analytically as follows:

- Solve  $\angle G(i\omega_{cp})H(i\omega_{cp}) = -180^\circ$  for  $\omega_{cp}$
- Calculate the gain margin  $= -20 \log(|G(i\omega_{cp})H(i\omega_{cp})|)$  dB

**Phase margin** is the angle in degrees through which the compensated open loop polar plot can be rotated about the origin, in the clockwise direction such that the gain crossover point on the locus reaches the critical point (-1,0).

$$\text{Phase Margin (PM)} = \angle G(i\omega_{cg})H(i\omega_{cg}) + 180^\circ \quad (8.25)$$

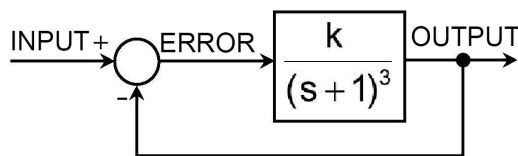
Generally, a minimum recommended phase margin is in the range of  $40^\circ$  to  $60^\circ$ .

The phase margin can be determined analytically as follows:

- Solve  $|G(i\omega_{cg})H(i\omega_{cg})| = 1$  for  $\omega_{cg}$
- Calculate the phase margin  $= \angle G(i\omega_{cg})H(i\omega_{cg}) + 180^\circ$

**Example 8.8** For the system described by the following block diagrams:

- (a) Find the values of constant k for a steady state error  $\bar{e}_{ss} = 0.1$  and for  $\bar{e}_{ss} = 0.05$ .
- (b) Derive the expressions for the open loop gain and phase.
- (c) Calculate the phase cross-over frequency and find the value of constant k if the gain margin is 10 dB
- (d) Calculate the gain cross-over frequency and phase margin for the value of k calculated in (c)
- (e) Give your comment on the calculation results.



$$(a) \quad k_p = \lim_{s \rightarrow 0} G(s)H(s) = \lim_{s \rightarrow 0} \frac{k}{(s+1)^3} = k$$

$$k_p = \frac{1 - \bar{e}_{ss}}{\bar{e}_{ss}}$$

For  $\bar{e}_{ss} = 0.1$ ,  $k_p = 9 = k$     or     $k = 9$

For  $\bar{e}_{ss} = 0.05$ ,  $k_p = 19 = k$  or  $k = 19$

The increase in gain reduces the steady state error. For  $\bar{e}_{ss} = 0.05$ , the open loop gain  $k$  should be:  $k = 19$

$$(b) \quad \text{Gain} = |G(i\omega)| = \frac{k}{(\sqrt{1 + \omega^2})^3}$$

$$\text{Phase} = \angle G(i\omega) = -3 \tan^{-1}(\omega)$$

(c) At the phase cross-over frequency,  $\omega = \omega_{cp}$  the phase  $\phi = -180^\circ$

$$-180^\circ = -3 \tan^{-1}(\omega_{cp}) \quad \text{Or} \quad \omega_{cp} = \sqrt{3} \text{ rad/s}$$

$$\text{Gain} = g = \frac{k}{(\sqrt{1 + \omega_{cp}^2})^3} = \frac{k}{8}$$

For a gain margin of 10 dB;  $GM = 10 = 20 \text{ Log}(1/g)$

Or  $g = 1/\sqrt{10}$ , Then  $k = 8/\sqrt{10} = 2.53$

For gain margin  $GM = 10$  dB, the open loop gain should be  $k = 2.53$

(d) At the gain cross-over frequency,  $\omega = \omega_{cg}$ ,  $g = 1$

$$g = |G(i\omega_{cg})| = \frac{k}{(\sqrt{1 + \omega_{cg}^2})^3} = 1$$

For  $k = 2.53$ ,  $\omega_{cg} = 0.926 \text{ rad/s}$

At this frequency, the phase  $\phi$  is calculated as follows.

$$\phi = -3 \tan^{-1}(\omega_{cg}) = -128.36^\circ$$

The phase margin  $PM = 180^\circ + \angle G(i\omega_{cg})$

Or  $PM = 180^\circ - 128.36^\circ = 51.64^\circ$

- (e) To achieve the required stability level, the open loop gain should be  $k \leq 2.53$ . Meanwhile, to achieve the required precision, the open loop gain should be  $k \geq 19$ . The two demands are in contradiction. They cannot be satisfied simultaneously without implementation of a convenient controller.

### 8.3.2.2 Stability analysis using Bode plot

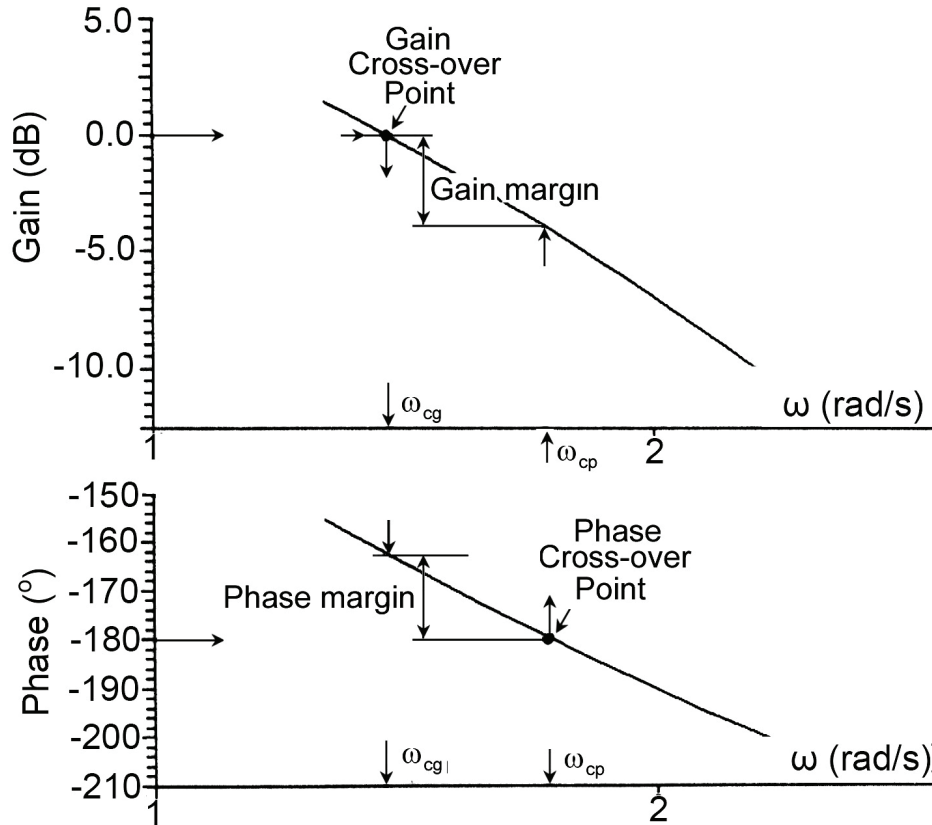


Fig.8.11 Gain and phase margins on the open loop Bode plot

The Bode plot of the open loop frequency response  $G(i\omega)H(i\omega)$  is a powerful tool for the stability analysis, Fig.8.11. It enables to find the phase and gain margins by direct readings on the plot. The phase margin is determined as follows:

1. Find the gain crossover point on the gain plot; the point on the plot corresponding to gain = 0 dB. Then find the gain crossover frequency  $\omega_{cg}$ .
2. Find the corresponding phase, on the phase diagram  $\angle G(i\omega_{cg})H(i\omega_{cg})$ .
3. The phase margin can be directly obtained from the phase diagram as shown in Fig.8.11.

The gain margin can be found from the Bode diagram as follows:

1. Find the phase crossover point on the phase diagram; the point at which the phase equals  $-180^\circ$ . Then find the phase crossover frequency  $\omega_{cp}$ .
2. Find the corresponding gain on the gain plot.
3. The gain margin can be directly found from the gain plot as shown in Fig.8.11.

$$\text{Gain Margin (GM)} = -20 \log |G(i\omega_{cp})H(i\omega_{cp})|$$

### 8.3.2.3 Stability analysis using Nichol's chart

When plotting the compensated open loop frequency response on Nichol's chart, the gain and phase margins can be directly obtained as shown in Fig.8.12.

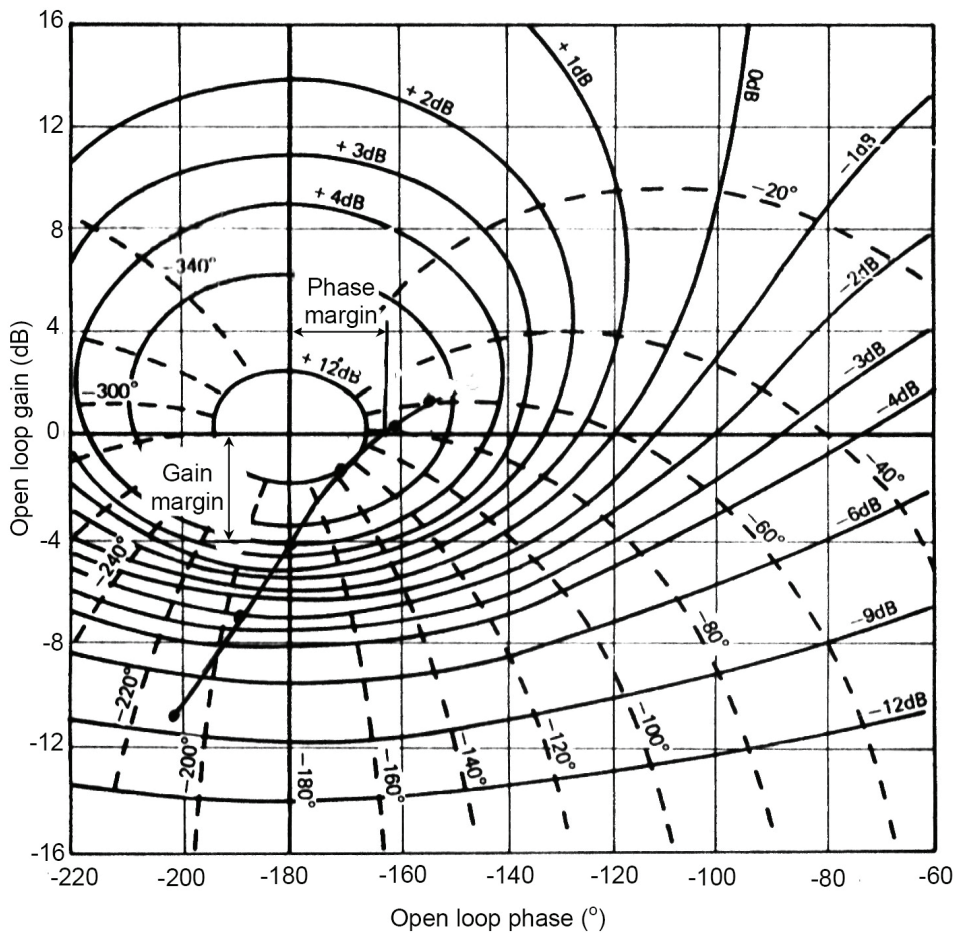


Fig.8.12 Gain and phase margins on the Nichol's chart

**The phase margin** is given by the horizontal distance from the origin ( $-180^\circ, 0$ ) to the intersection point of the open loop plot with the horizontal (zero dB) axis.

**The gain margin** is given by the vertical distance from the origin ( $-180^\circ, 0$ ) to the intersection point of the open loop plot with the vertical ( $-180^\circ$ ) axis.

## 8.4 EXERCISE

1. Consider a system having the following compensated open loop transfer function. Calculate the steady state error if a step input of magnitude 5 is applied.

$$G(s)H(s) = \frac{60}{(s+2)(s+10)}$$

2. A servo system controlling an angular displacement  $\vartheta_o$  is described by the following equations.

$$t_m = 0.004 \left( \vartheta_e + k \frac{d^2 \vartheta_o}{dt^2} \right) \text{ Nm}$$

$$\vartheta_e = \vartheta_i - \vartheta_o$$

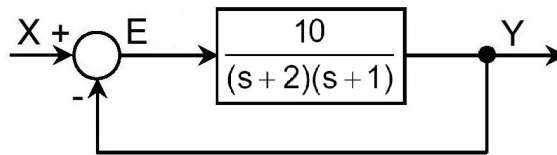
$$J \frac{d^2 \vartheta_o}{dt^2} + f \frac{d\vartheta_o}{dt} = t_m$$

The system has a moment of inertia  $J = 10^{-5} \text{ kgm}^2$ , a friction coefficient  $f = 10^{-4} \text{ Nms/rad}$  and its input is  $\vartheta_i$ .

- (a) Draw the block diagram of this system.
  - (b) Determine the value of  $k$  if the system is critically damped.
  - (c) Find the steady state error if a unit step input rotation angle is applied.
3. The following is the compensated open loop transfer function of a position control servo. Calculate the steady state error if the Laplace transform of the input is  $(10/s)$ .

$$G(s)H(s) = \frac{2000}{(s+2)(s+10)}$$

4. For the system represented by the following block diagram, find the positional error coefficient, the steady state error  $e_{ss}$  and the steady state value of  $y(t)$  if a step input of magnitude  $k_2$  is applied.



5. Applying the Routh-Hurwitz stability criterion, investigate the stability of the systems having the following characteristic equations.

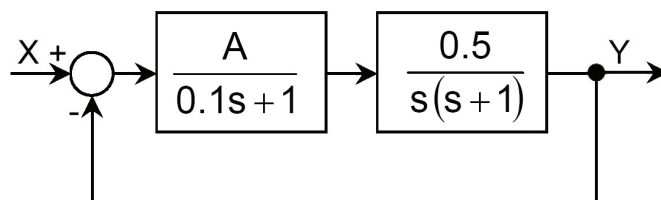
- (a)  $0.1s^3 + s^2 + k = 0$
- (b)  $0.1s^2 + 1.1s + k + 1 = 0$
- (c)  $0.1s^3 + s^2 + s + k = 0$
- (d)  $T_1T_2 s^3 + (T_1 + T_2)s^2 + s + k = 0$
- (e)  $s^4 + 2s^3 + 6s^2 + 8s + k = 0$
- (f)  $s^4 + 20ks^3 + 5s^2 + (10+k)s + 15 = 0$ .

6. The open loop transfer function of control system with unity feed back is given by:

$$G(s) = \frac{k(s+1)}{s(1+Ts)(1+2s)}$$

Find the closed loop transfer function then apply the Routh-Hurwitz stability to find the conditions of system stability.

7. Applying the Routh-Hurwitz criterion to calculate the values of  $A$  for which the control system shown in the following figure is stable.



8. Plot the Bode diagrams for the system described by the given compensated open loop transfer function then determine the phase margin and gain margin.

$$GH(s) = \frac{5}{s(1+0.6s)(1+0.1s)}$$

9. The open loop transfer function of a unity feedback system is

$$G(s) = \frac{4k(1+0.01s)}{s(1+0.1s)(1+0.4s)}$$

Plot the bode diagrams and hence find:

- (a) The value of k for a gain margin of 22 dB.
- (b) The value of k for a phase margin of 45°.

**NB** You can use the CODAS or MATLAB for the plot.

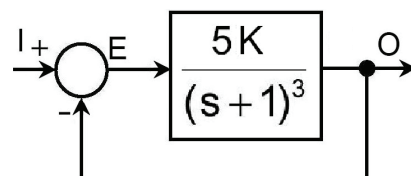
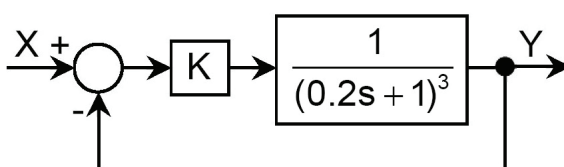
10. The following is the open loop transfer function of a unity feedback system.

$$G(s) = \frac{k}{s(1+0.1s)(1+0.01s)}$$

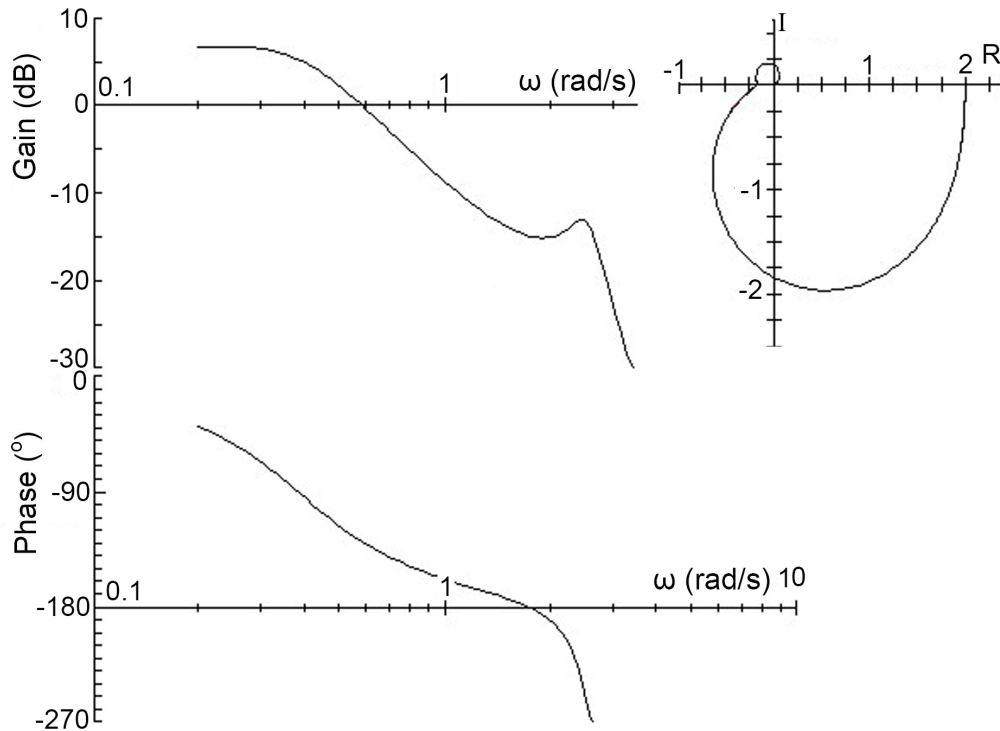
- (a) Find the limiting range of k for stability.
- (b) Check the result by applying Routh-Hurwitz criterion.
- (c) Find the phase margin and gain margin if k is half the limit value.

11. For the shown block diagrams of control systems:

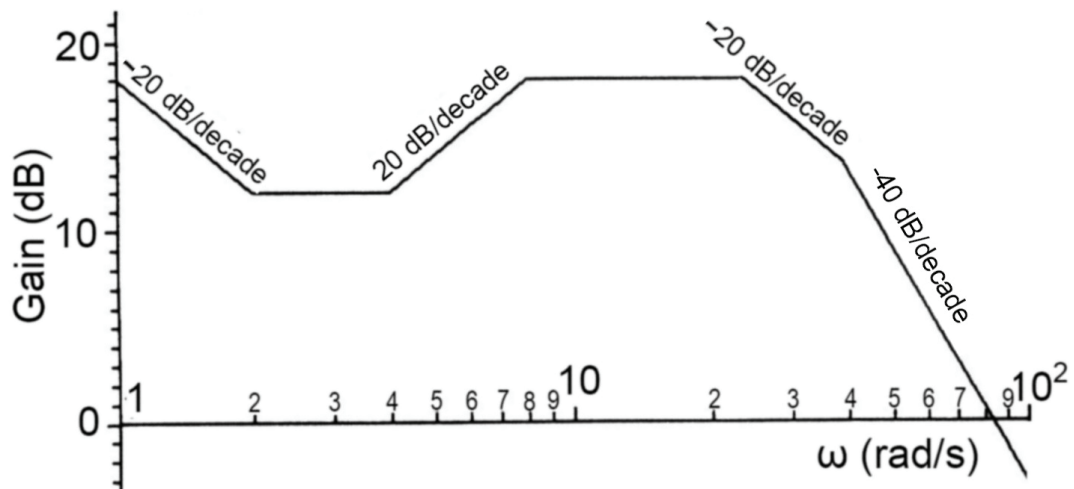
- (a) Find the values of constant K for a steady state error  $\bar{e}_{ss} = 0.1$  and for  $\bar{e}_{ss} = 0.01$ . Comment on the obtained results.
- (b) Give the expressions for the open loop gain and phase of the systems.
- (c) Calculate the phase cross-over frequency and find the value of constant K if the gain margin is 8 dB
- (d) Calculate the gain cross-over frequency and phase margin for the value of K calculated in (c)
- (e) Give your comment on the calculation results.



12. Given the frequency plot of the compensated open loop of a control system. Find the phase and gain cross-over frequencies, the gain and phase margins and the gain of the closed loop transfer function if  $H(s)=1$ .



13. The Bode diagram of the compensated open loop of a unity feedback control system is given in the following figure. Deduce the forward path transfer function for the system and sketch the phase plot of the frequency response. Hence comment on the stability and the transient and steady state performance of the system when operating in the closed-loop mode.



14. Sketch the Nyquist diagrams and check the stability of systems with the following open loop transfer functions.

(a)  $G(s)H(s) = \frac{1}{s(s+1)}$

(b)  $G(s)H(s) = \frac{1}{s^5(s+1)}$

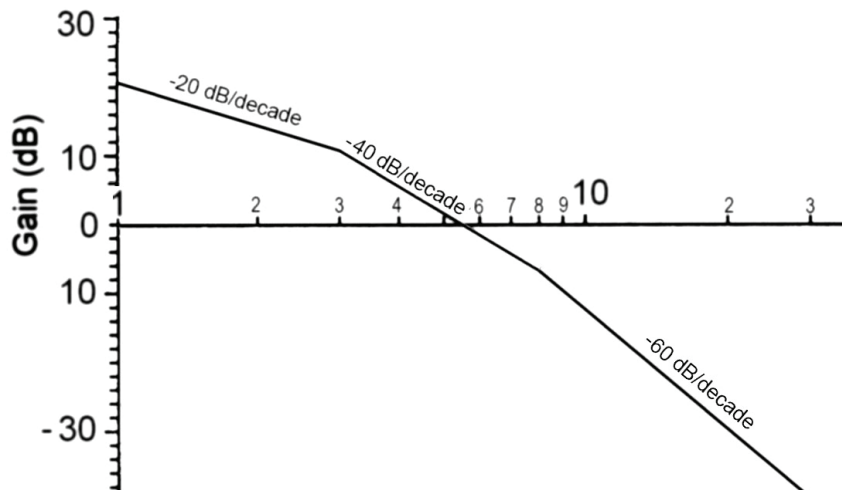
15. Plot the open loop frequency response of the following system and find the gain margin and phase margin.

$$G(s)H(s) = \frac{10}{s(1+0.2s)(1+0.02s)}$$

16. The following figure shows the asymptotic gain Bode plot of a third order system having a transfer function of the form:

$$G(s) = \frac{k}{s(1+T_1s)(1+T_2s)}$$

- (a) Find the numerical values of the constants  $k$ ,  $T_1$  and  $T_2$ .
- (b) Plot the phase diagram.
- (c) Find graphically the gain and phase margins.
- (d) Find analytically the gain margin.



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# 9 | ROOT LOCUS ANALYSIS

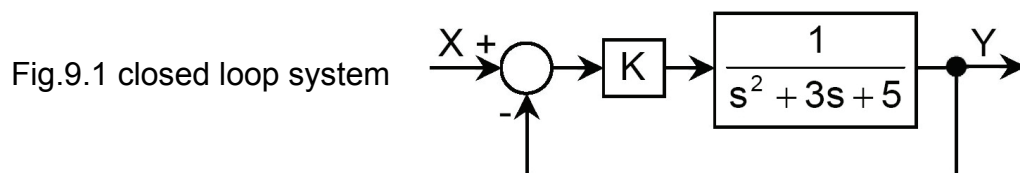
## 9.1 INTRODUCTION

The discussions of the transient response of a linear system, chapter 6, showed that the system stability depends on the location of the roots of the characteristic equation in the R-I plane. The root locus is a plot of the location of roots of the characteristic equation of the closed loop system.

The process of constructing the root locus includes:

- Finding out the characteristic equation of the closed loop.
- Derivation of expressions for the characteristic equations, considering the proportional gain K, Fig.9.1.
- Plotting the roots locations on the R-I plane for different values of gain K;  $0 \leq K < \infty$ . The plot indicates the loci of the different roots, for varying values of K as shown in Figs 9.2 and 9.6.

This process is illustrated by plotting the root locus of the system described by Fig.9.1.



The characteristic equation of the closed loop system is:  $s^2 + 3s + 5 + K = 0$

This equation has the following two roots:  $s = -1.5 \pm \sqrt{2.25 - (5 + K)}$

For  $K=0$ , the two roots are  $s = -1.5 \pm i\sqrt{2.75}$

The roots location in the R-I plane is plotted in Fig.9.2 for different values of K in the range  $0 \leq K < \infty$ . This plot is called root locus. The response of the closed loop system to a unit step input is given in Fig.9.3. The study of Figs 9.2 and 9.3 shows that:

- The closed loop is always stable for all values of K.

- The increase in K increases the frequency of the transient oscillations, decreases the damping ratio and decreases the steady state error.

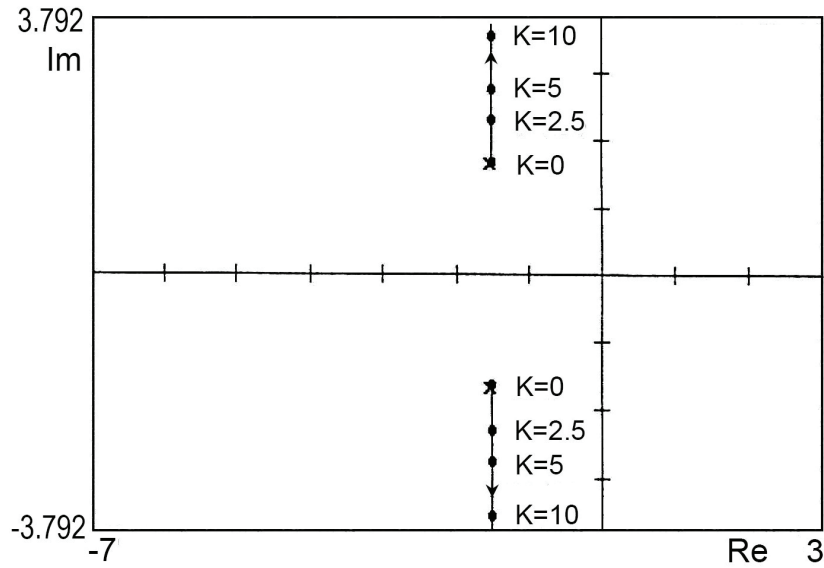


Fig.9.2 Root locus of  $s^2 + 3s + 5 + K = 0$

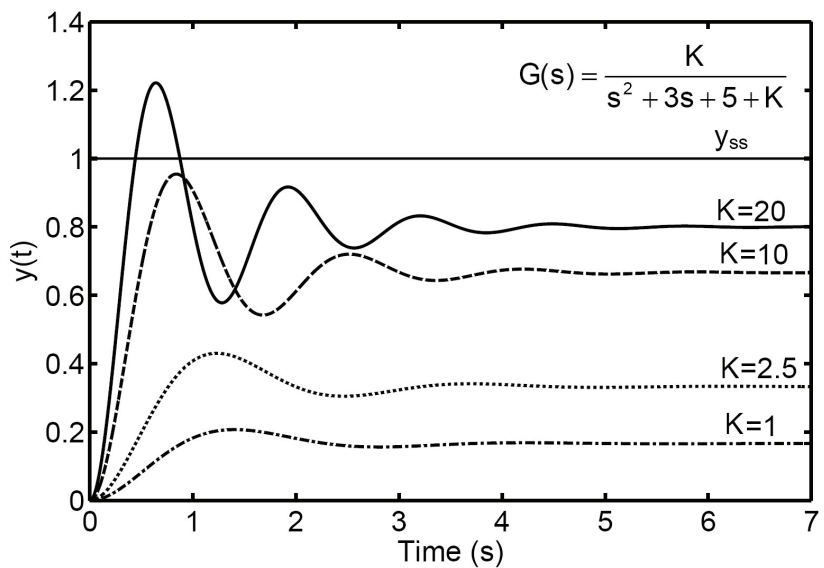


Fig.9.3 Unit step response of closed loop system

## 9.2 INTERPRETATION OF ROOT LOCUS

The complete root loci are symmetrical with respect to the real axis of the R-I plane and the number of branches of the root loci is equal to the order of the characteristic equation. The construction and analysis of root loci gives good support to the system analysis as well as to the synthesis and design of closed loop systems. There are several available computer programs for roots evaluation and for the plot of the root loci such as the MATLAB and CODAS. This section presents several examples of

closed loop control systems with different types of controllers. Their root loci are plotted and analyzed. The block diagrams of these systems are of the form given in Fig.9.4.

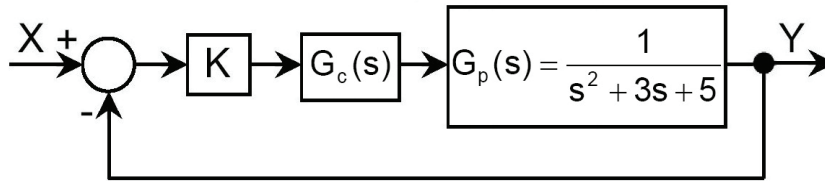


Fig.9.4 Block diagram of a typical closed loop system

The closed loop transfer function and characteristic equation are given by the following expressions:

$$G(s) = \frac{KG_c(s)G_p(s)}{1+KG_c(s)G_p(s)} \quad (9.1)$$

The characteristic equation is:

$$\boxed{1+KG_c(s)G_p(s) = 0} \quad (9.2)$$

The case of proportional controller,  $G_c = 1$ , is demonstrated by example 9.1. The following examples show the root loci of the system with different types of controllers and the effect of these controllers on the closed loop transient response.

It is important for a control engineer to build up knowledge that enables him to relate root position to the transient behavior of the system. Moreover, it is necessary to know the effect of altering the position of poles and zeros or introducing new poles or zeros on a root locus plot and hence on the system dynamic behavior.

Generally, the addition of a pole (located in the left half of the R-I plane) to the transfer function pushes the root loci towards the right half-plane and reduces the system relative stability. This is illustrated by example 9.1 and 9.2. Example 9.1 shows the effect of introducing a pole at the origin and example 9.2 shows the effect of introducing a pole at  $s = -1$ . The effect of adding a pole on the transient response is shown in Figs. 9.5 to 9.10. The study of these figures shows that the addition of a pole slows down the response and reduces the system stability or even makes it unstable.

**Example 9.1**  $G_c = \frac{1}{s}$  (Simple integrator, introducing a pole at  $s=0$ )

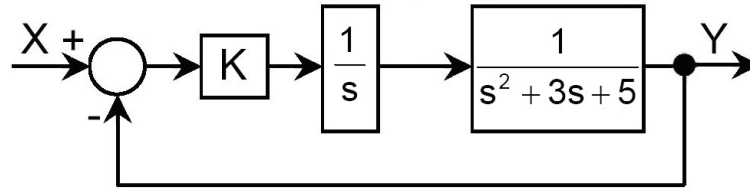


Fig.9.5 Introducing a pole at  $s=0$

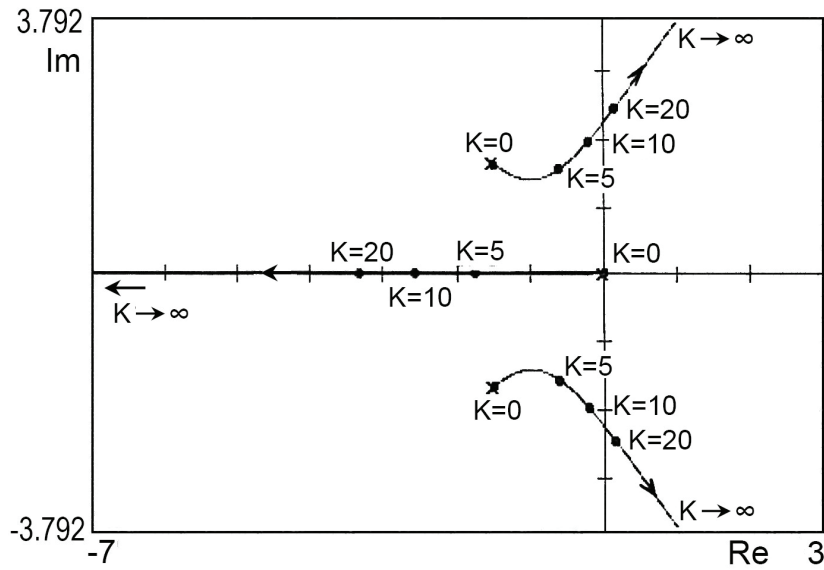


Fig.9.6 Root locus of  $s(s^2 + 3s + 5) + K = 0$

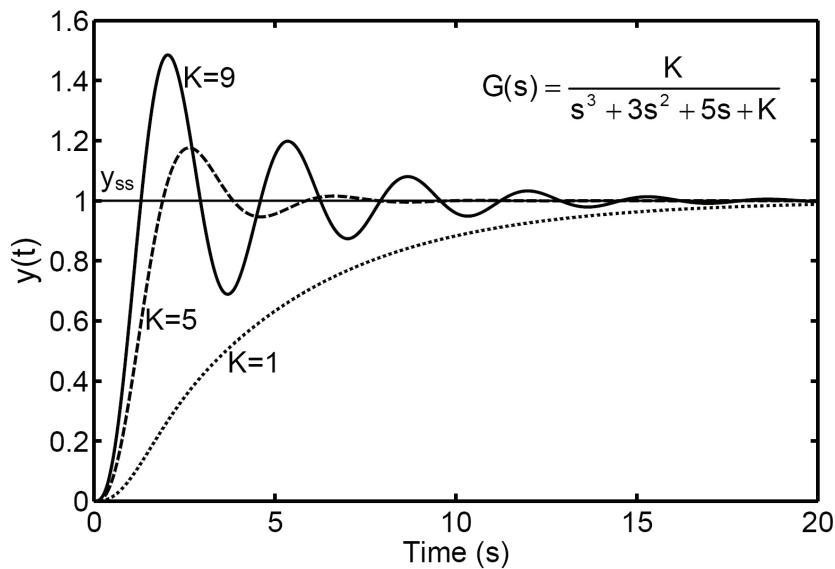


Fig.9.7 Unit step response of closed loop system

**Example 9.2**  $G(s) = \frac{1}{s+1}$  (Simple lag; introducing a pole at  $s = -1$ )

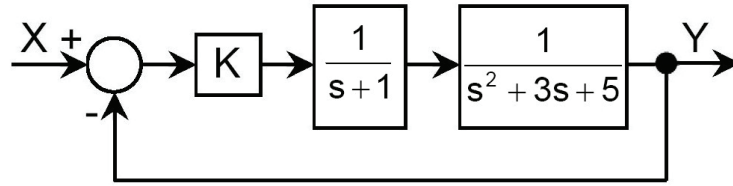


Fig.9.8 Introducing a pole at  $s = -1$

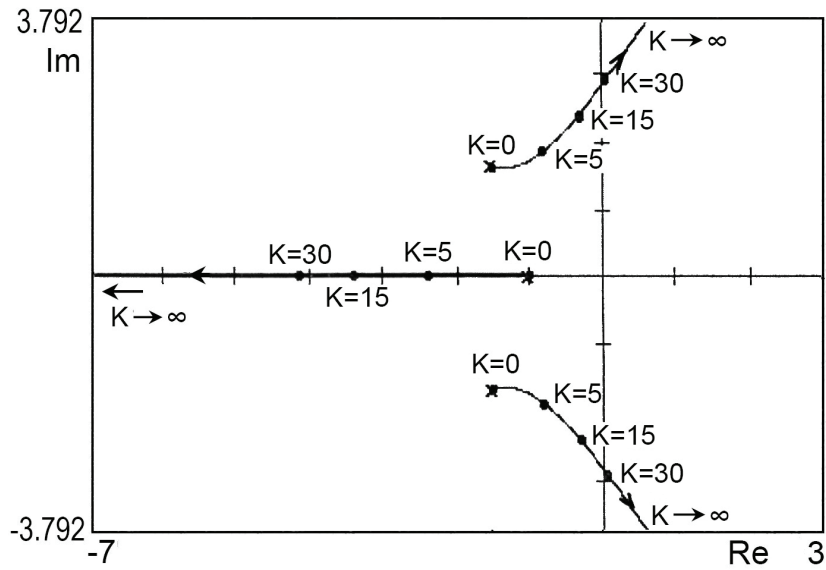


Fig.9.9 Root locus of  $(s+1)(s^2 + 3s + 5) + K = 0$

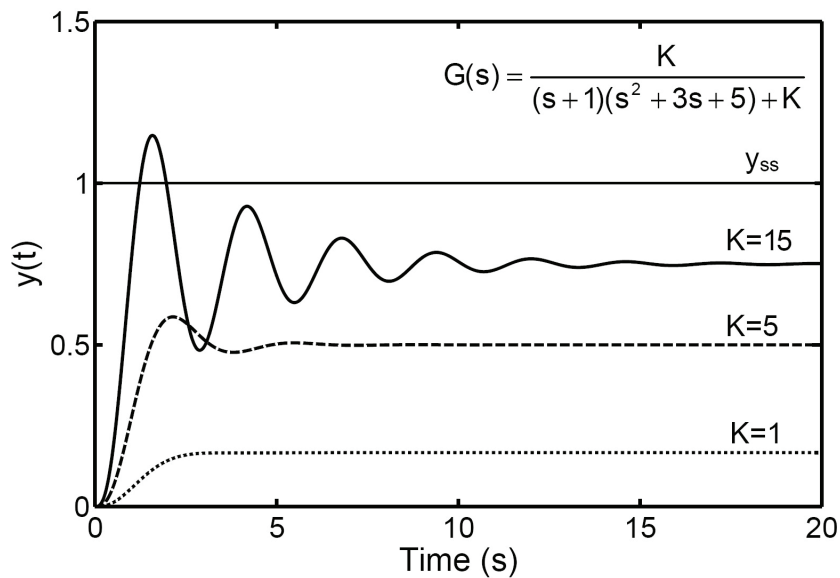


Fig.9.10 Unit step response of closed loop system

**Example 9.3**  $G_c = (s + 2)$  (Proportional plus differential controller PD)

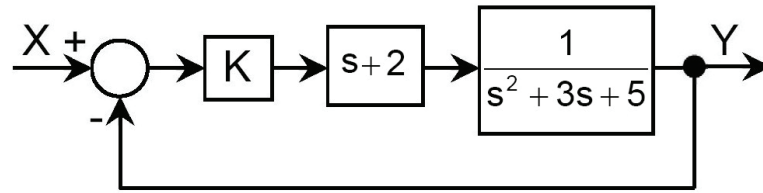


Fig.9.11 Introducing a PD controller

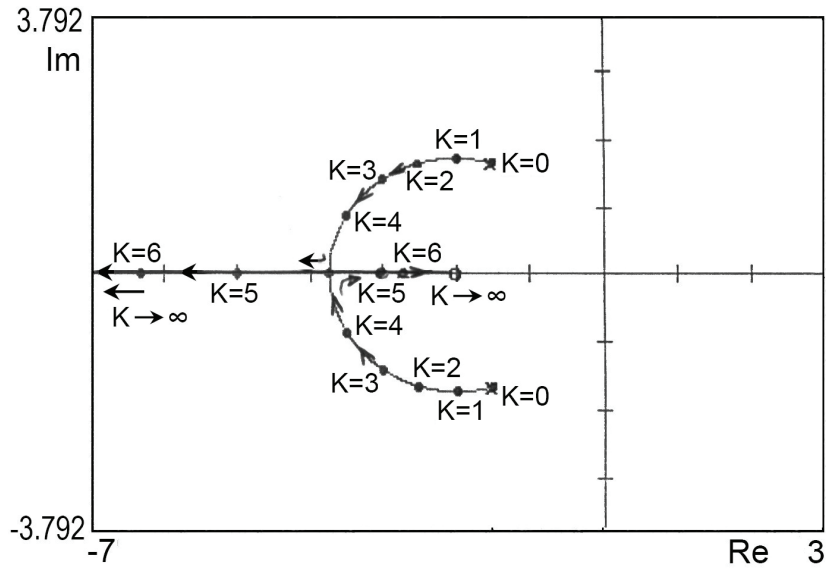


Fig.9.12 Root locus of  $s^2 + 3s + 5 + K(s + 2) = 0$

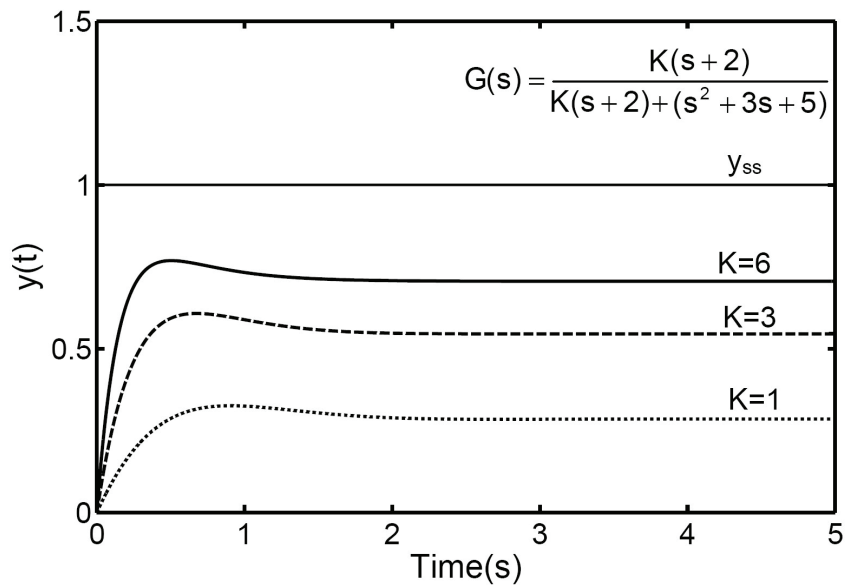


Fig.9.13 Unit step response of closed loop system

**Example 9.4**  $G_c = \frac{s+2}{s} = \left(1 + \frac{2}{s}\right)$  (Proportional plus integral controller, PI)

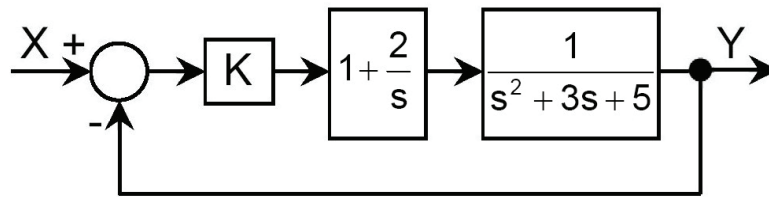


Fig.9.14 Introducing a PI controller

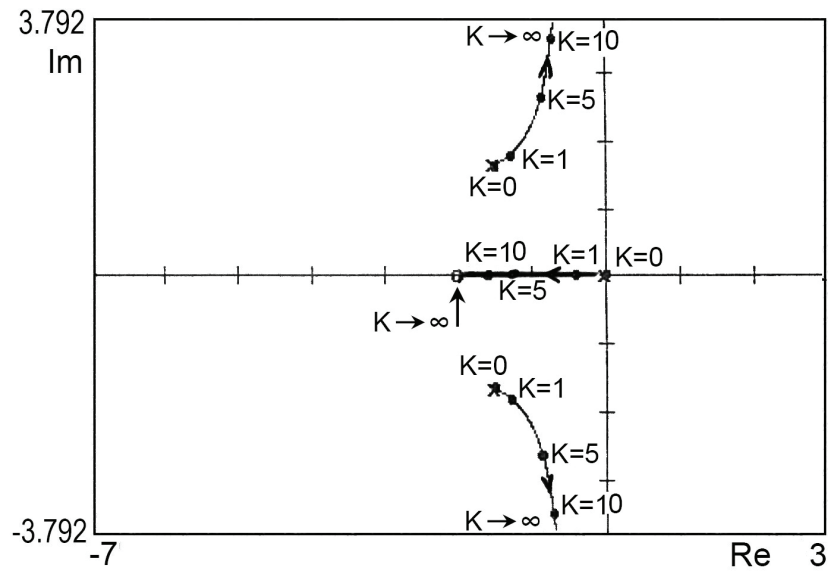


Fig.9.15 Root locus of  $s(s^2 + 3s + 5) + K(s + 2)$

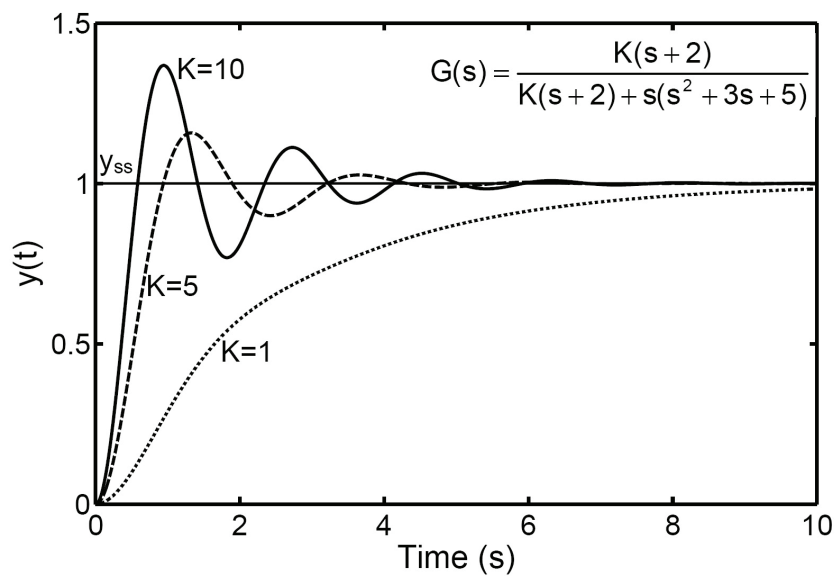


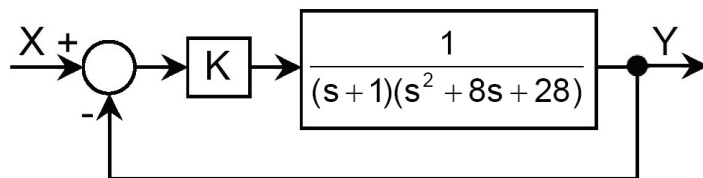
Fig.9.16 Unit step response of closed loop system

The effect of adding a zero to the open loop function is illustrated by example 9.4, where a zero (at  $s = -2$ ) is added. Generally, the addition of a left-half plane zeros to the open loop transfer function has the effect of moving and bending the root loci toward the left-half of R-l plane, Figs. 9.11 thru 9.13, which tends to make the system more stable and of faster response.

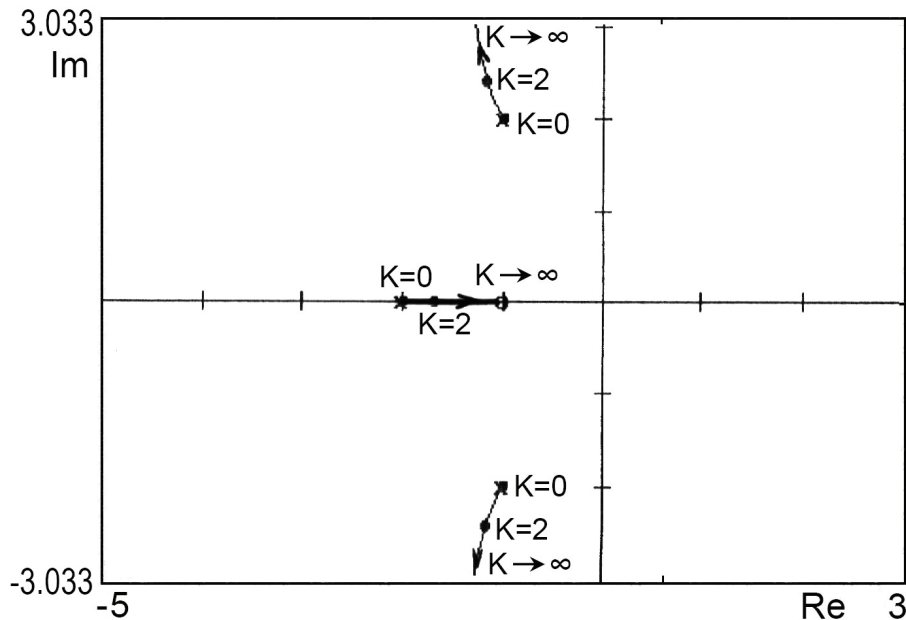
Finally, the effect of implementation of a proportional plus integral controller is explained by example 9.4 and Figs. 9.14 to 9.16.

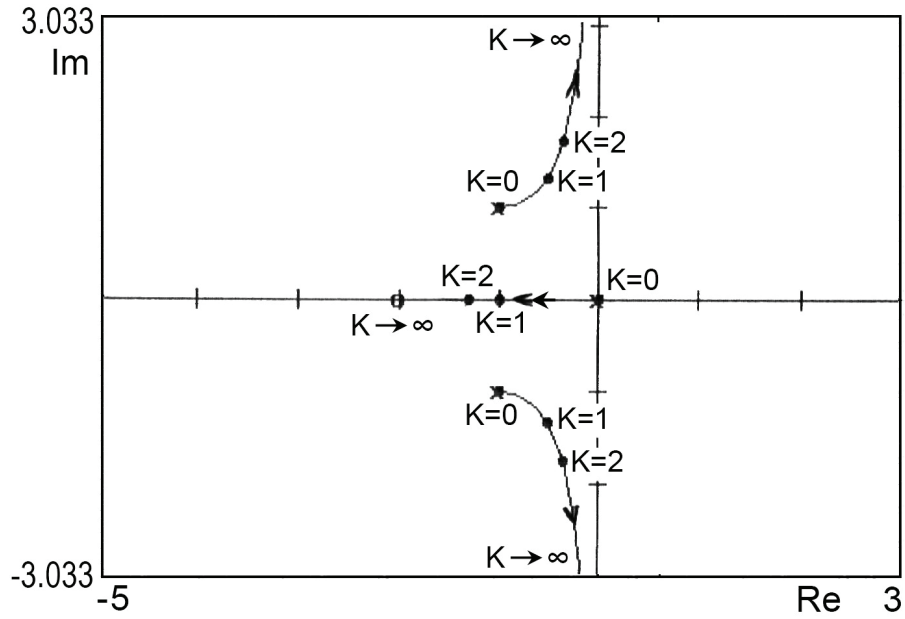
### 9.3. EXERCISE

1. Plot the root locus of the following system and sketch the response of the closed loop to unit step input for  $K=1$  &  $K=100$ .

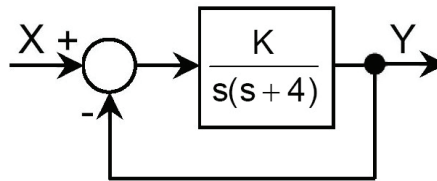


2. Find the transfer function of the systems whose closed loop root loci are plotted in the following figures. Sketch the closed loop response for three different values of gain  $K$ .





3. Plot the root locus for the given system and find the location of roots and value of  $k$  for a damping ratio  $\zeta = 0.5$



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# 10 | COMPENSATION OF CONTROL SYSTEMS

## 10.1 INTRODUCTION

The control system should satisfy simultaneously the performance specifications of limited steady state error and relative stability. The specification of steady state error requires certain minimum value of open loop gain  $K_1$ . To insure the recommended relative stability, the phase and gain margins should be greater than the pre-determined minimum values. This defines certain maximum value of the open loop gain  $K_2$ . If  $K_1 > K_2$  then these two requirements are in contradiction. The specifications of steady state error and relative stability cannot be both satisfied unless some form of compensation is introduced.

where  $K_1$  = Minimum compensated open loop gain recommended to satisfy the system accuracy specifications.  
 $K_2$  = Maximum compensated open loop gain recommended to satisfy the relative stability specifications.

**Example 10.1** Find the open loop gain  $K$  needed to produce a steady state position error  $< 5\%$ . Find the value required to give a gain margin of 6 dB and a phase margin of  $27^\circ$ .

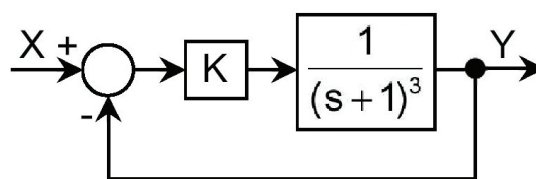


Fig.10.1

When applying a step input of magnitude  $k_2$ , the steady state error  $e_{ss}$  is given as follows.

$$e_{ss} = \frac{k_2}{1+k_p} \quad \text{or} \quad \bar{e}_{ss} = \frac{1}{1+k_p}$$

Where  $k_2$  = Magnitude of the applied step input.

$$k_p = \text{Position error coefficient; } k_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$\bar{e}_{ss} = \text{relative steady state error} = e_{ss} / k_2$$

$$k_p = \lim_{s \rightarrow 0} \frac{K}{(s+1)^3} = K$$

$$\text{for } \bar{e}_{ss} < 0.05, \quad \frac{1}{1+k_p} < 0.05 \quad \text{and} \quad K > 19$$

Then, to reach the required precision, K should be greater than or equal 19 ( $K \geq 19$ ). However, the Nyquist plot of the open loop frequency response, Fig.10.2 shows that the system is unstable for  $K \geq 19$ .

The value of K needed to satisfy the relative stability requirements is  $K=4$ ; for a gain margin = 6 dB and a phase margin =  $27^\circ$ . Naturally, decreasing the value of K below this value improves the relative stability. Then, in order to satisfy the stability requirements, K should be smaller than or equal 4 ( $K \leq 4$ ).

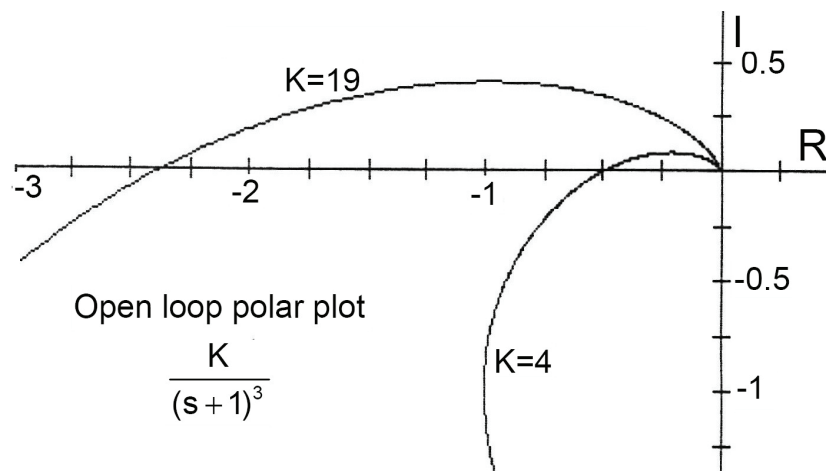


Fig.10.2 Effect of open loop gain K on the system stability

The requirements of the steady state error are satisfied if  $K > 19$ . But the requirements of the relative stability are satisfied only if  $K < 4$ . Then the two requirements are in contradiction. Therefore, the addition of compensator is required to reshape the open loop frequency response so that the low frequency gain is high enough, in addition, the plot avoids the critical point  $(-1,0)$  insuring the required gain and phase margins.

The Nyquist plot can be reshaped to satisfy these requirements by one of the following methods:

1. Start with gain  $K=19$  (to satisfy the precision requirement) and introduce a phase lead at high frequencies in order to attain the required phase and gain margins (phase lead compensation)
2. Start with gain  $K=4$  (Satisfying the stability requirement) and introduce a phase lag at low frequencies to meet the steady state error requirements (phase lag compensation).
3. Start with a gain between the two limits;  $4 < K < 19$ , then introduce phase lag at low frequencies and phase lead at high frequencies (lag-lead compensator).

The compensator is frequently included as an additional dynamic element in the system. The compensating element can be connected in the system in different ways; series, parallel or feed-forward connection

### Series Compensation

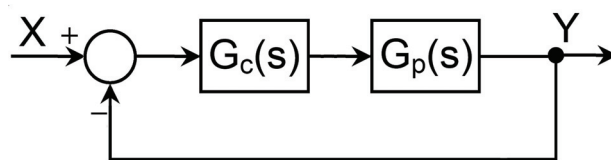


Fig.10.3 Series connection of compensator

The compensated open loop transfer function is given by:  $G_c(s)G_p(s)$

### Parallel, Feedback, Compensation

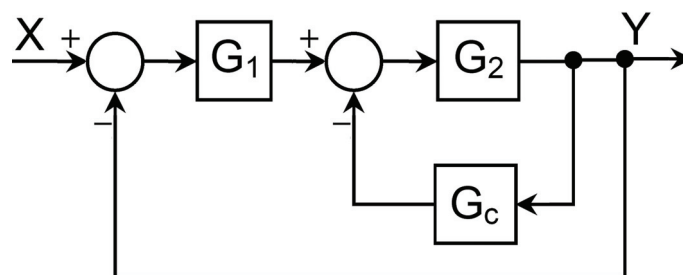


Fig.10.4 Parallel connection of compensator

The compensated open loop transfer function is given by:  $\frac{G_1G_2}{1+G_2G_c}$

Formally, the parallel compensator can be always selected to obtain the same results as in series compensation.

## Feed Forward Compensation

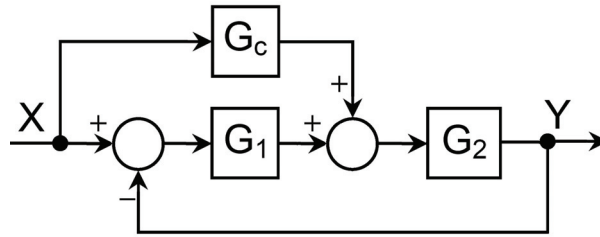


Fig.10.5 Connection of feed forward compensator

The compensating unit is included in a supplementary feed forward loop; along which the disturbance signal is introduced in the system.

## 10.2 PHASE LEAD COMPENSATOR

The transfer function of the phase lead compensator is of the following form:

$$G_c = \frac{1}{a} \frac{1+aTs}{1+Ts}; \quad a > 1 \quad (10.1)$$

The gain can be kept unity by connecting a proportional element (gain = a), in series with the compensator. The compensator transfer function becomes:

$$\boxed{G_c = \frac{1+aTs}{1+Ts}} \quad (10.2)$$

The frequency response of a typical phase lead element is given in Fig.10.6. This figure shows that the phase lead element presents a phase lead within certain range of frequency associated with permanent gain increase at high frequency range. The phase lead can be calculated as follows.

$$\varphi = \tan^{-1} \frac{\omega T(a-1)}{1+aT^2\omega^2} \quad (10.3)$$

The maximum phase shift is at  $\omega = \omega_m$ , where  $\omega_m$  is given by:

$$\log(\omega_m) = \frac{1}{2} \left\{ \log\left(\frac{1}{aT}\right) + \log\left(\frac{1}{T}\right) \right\} \quad (10.4)$$

Or

$$\omega_m = \frac{1}{\sqrt{aT}} \quad (10.5)$$

$$\phi_{\max} = \phi(\omega_m) = \tan^{-1}\left(\frac{a-1}{2\sqrt{a}}\right) = \sin^{-1}\left(\frac{a-1}{a+1}\right) \quad (10.6)$$

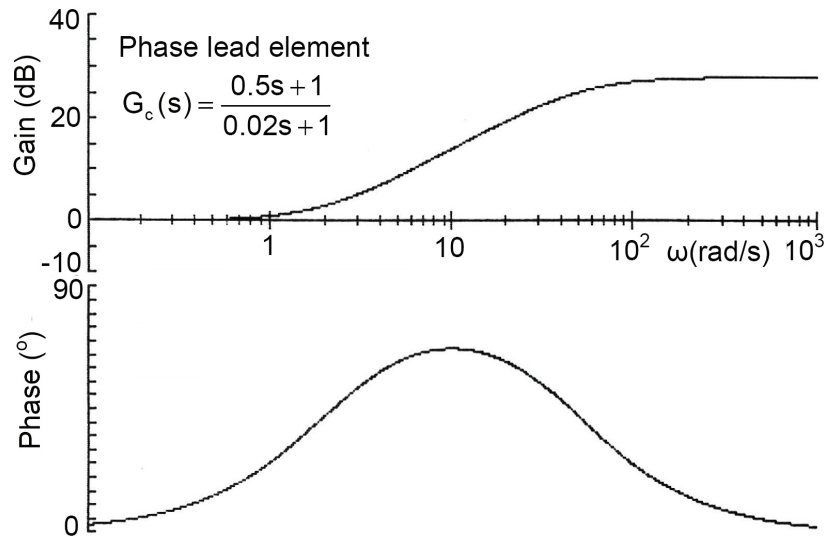


Fig.10.6 Bode plot of a phase lead element,  $T=0.02$  sec &  $a=25$

For a phase lead element having  $T = 0.02$  sec and  $a = 25$ , the frequency response parameters are calculated as follows.

$$a = 25, \quad T = 0.02 \text{ sec.}$$

$$\text{Corner frequencies } \omega_1 = \frac{1}{aT} = 2 \text{ (1/sec)} \quad \text{and} \quad \omega_2 = \frac{1}{T} = 50 \text{ (1/sec)}$$

$$\omega_m = 10 \text{ (1/sec)}, \quad \phi_{\max} = \sin^{-1} \frac{25-1}{25+1} = 67.38^\circ \quad \text{and} \quad \phi\left(\frac{1}{T}\right) = \phi\left(\frac{1}{aT}\right) = 42.7^\circ$$

**Example 10.2** Plot the Nyquist and Bode plots of the open loop of the following system with and without compensation and discuss the effect of introducing a phase lead compensator having the following transfer function:

$$G_c = \frac{0.5s + 1}{0.02s + 1}$$

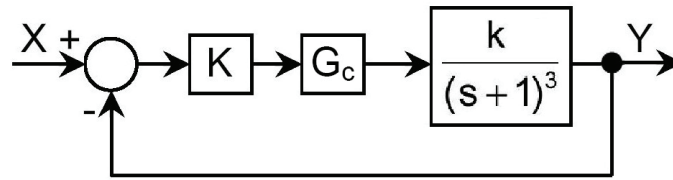
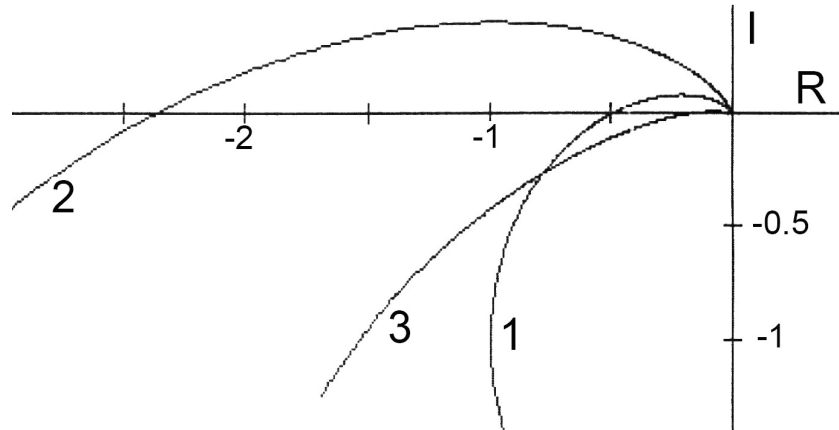


Fig.10.7



- (1)  $K=4$ , Stable with  $GM=6$  dB and  $PM=27^\circ$  (2)  $K=19$ , Unstable without compensator  
 (3)  $K=19$ , Stable with phase lead compensator,  $GM=15$  dB and  $PM=22^\circ$

Fig.10.8 Nyquist plot of the non-compensated open loop system and of the system with series connected phase lead compensator.

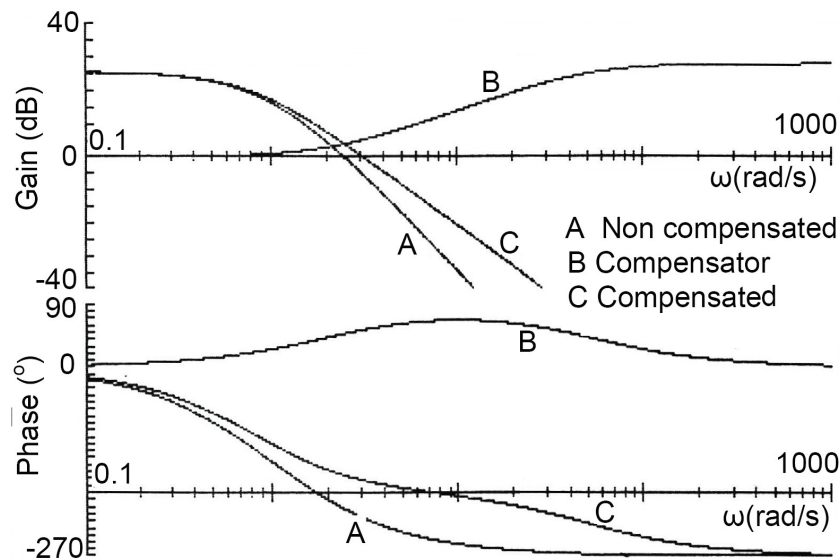


Fig.10.9 Bode plot of the non-compensated open loop system and system with series connected phase lead compensator.

The open loop frequency response plots of the system with  $K=19$ , with and without compensation are given in Figs. 10.8 and 10.9. The study of these figures shows the following:

- The non-compensated system is unstable.
- The compensator introduces significant phase lead in the range of frequency  $\omega = 0.5$  to 200 rad/sec. The maximum phase lead is at  $\omega = 10$  rad/sec.
- The system with compensator is stable with a phase margin of  $22^\circ$  and a gain margin of 15 dB.

**Assignment:** Figure 10.10 shows a mechanical phase-lag compensator. Its elements are of negligible inertia. Derive the transfer function  $G(s) = Y/X$  and show that it is a phase lead element with

$$T = \frac{f}{k_1 + k_2} \text{ \& } a = \frac{k_1 + k_2}{k_1} .$$

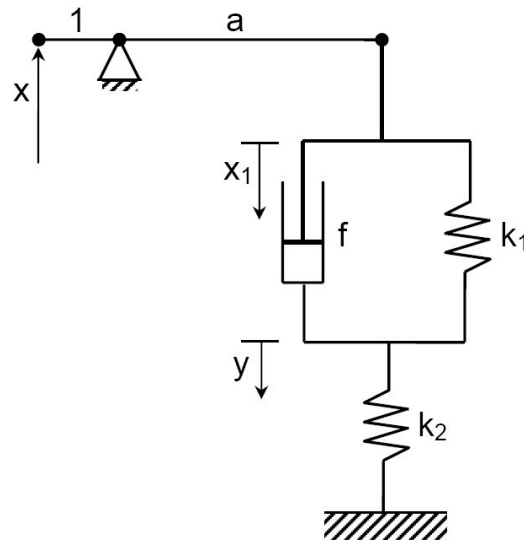


Fig.10.10 Mechanical phase-lag compensator

### 10.3 PHASE LAG COMPENSATOR

The phase lag element has a transfer function of the following form:

$$G_c = \frac{1 + Ts}{1 + aTs}; \quad a > 1 \quad (10.7.)$$

The Bode plot of a typical phase lag element is shown in Fig.10.11. This phase lag element presents a phase lag associated with gain reduction (negative gain in dB). It permits the gain of the original system to be increased without affecting the stability. The time constant  $T$  can be chosen such that the phase margin of the non-compensated system is not much changed.

The phase lag compensator is used when the phase margin is acceptable but the gain is too low to satisfy the steady state error requirements.

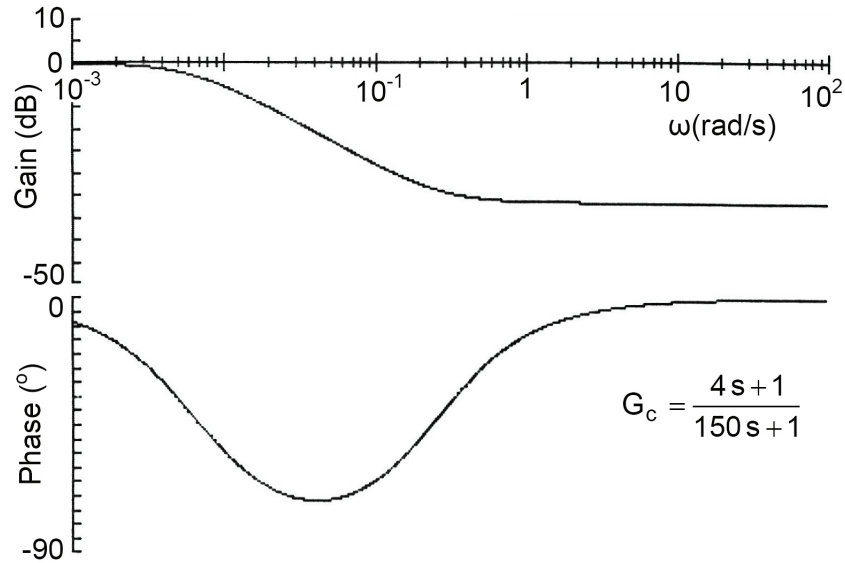


Fig.10.11 Bode plot of a phase lag element,  $T=4$  s &  $a=37.5$

**Example 10.3** Plot the Bode diagram of the open loop of the following system with and without compensation and discuss the effect of introducing a phase lag compensator having the following transfer function:

$$G_c = \frac{4s + 1}{150s + 1}$$

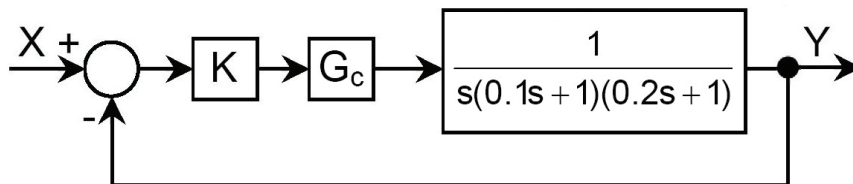


Fig.10.12

The open loop frequency plots of the system, for  $K= 100$ , with and without compensation are given in Fig.10.13. The study of this figure shows the following:

- The non-compensated system is unstable.
- The compensator introduced significant phase lag in the range of frequency  $\omega = 0.001$  to  $1$  rad/sec. The maximum phase lag is at  $\omega = 0.041$  rad/sec.
- The system with compensator is stable with a phase margin of  $45^\circ$  and a gain margin of  $14.3$  dB.

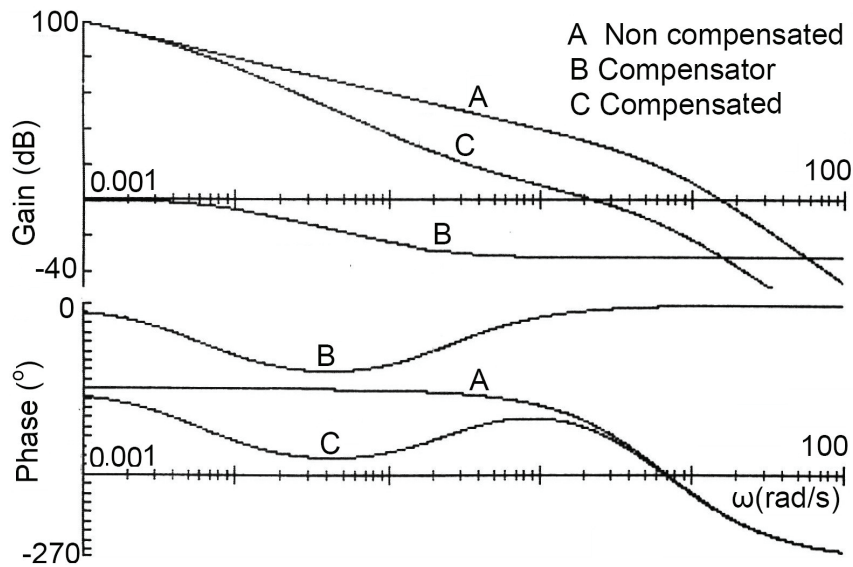


Fig.10.13 Bode plot of the non-compensated open loop system and system with series connected phase lag compensator

### 10.4 LAG-LEAD COMPENSATOR

The lag-lead element has a transfer function of the following form:

$$G_c = \frac{1+bT_2s}{1+T_2s} \times \frac{1+aT_1s}{1+T_1s}, \quad \text{where} \quad a > 1, b < 1 \text{ and } T_1 < T_2 \quad (10.8)$$

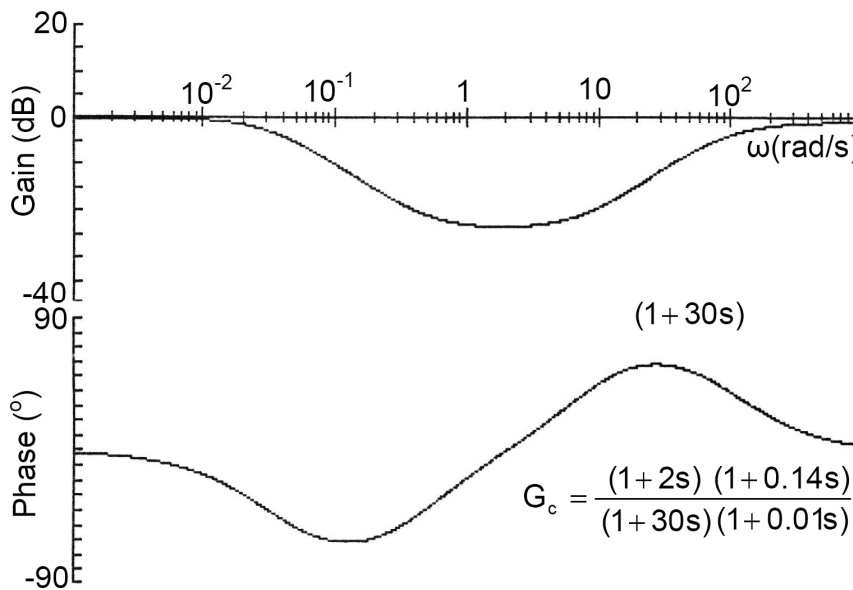


Fig.10.14 Bode plot of a lag lead element,  $T_1 = 0.01$  sec,  $T_2 = 30.5$  sec,  $a=14$  and  $b=0.0667$

The Bode plot of the frequency response of a typical lag-lead element is shown in Fig.10.14. The phase lag-lead compensator combines the advantages of the phase lead and phase lag compensators.

The lag-lead compensator can be thought of as a separate phase lag compensator and a phase lead compensator. Lag compensation permits an increase in gain to reduce steady state error while a phase lead compensator permits an increase in response time without loss of stability.

**Example 10.4** Plot the Bode plot of the open loop of the following system with and without compensation and discuss the effect of introducing a lag-lead compensator having the following transfer function:

$$G_c = \frac{1+2s}{1+30.6s} \times \frac{1+0.4s}{1+0.01s}$$

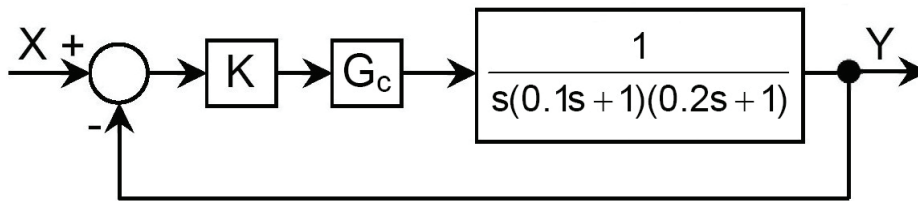


Fig.10.15

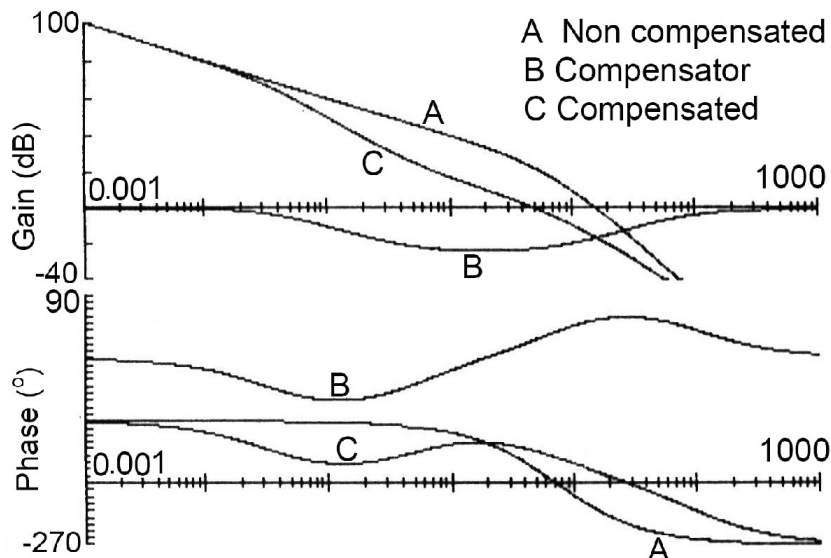


Fig.10.16 Bode plot of the non-compensated system and system with series connected lag lead compensator.

The open loop Bode plots of the system, for  $K= 100$ , with and without compensation are given in Fig.10.16. The study of this figure shows the following:

- The non-compensated system is unstable.
- The compensator introduced significant phase lag in the range of frequency  $\omega = 0.004$  to  $1.5$  rad/sec and significant phase lead in the range  $\omega = 2.5$  to  $850$  rad/sec. The maximum phase lag is  $60.4^\circ$  at  $\omega = 0.123$  rad/sec. The maximum phase lead is  $59^\circ$  at  $\omega = 27.8$  rad/sec. The compensator introduces significant gain attenuation. The maximum gain attenuation is  $23.11$  dB at  $\omega = 2$  rad/sec.
- The system with compensator is stable with a phase margin of  $45^\circ$  and a gain margin of  $24$  dB.

## 10.5 P, PI AND PID CONTROLLERS

The proportional integral derivative controller (PID) is the most common form of feedback. It was an essential element of early governors. Today, more than 95% of the control loops are of PID or PI type. The PID controllers are found in all areas where control is used. They have survived many changes in technology, from mechanics and pneumatics to microprocessors via electronic tubes, transistors and integrated circuits. The microprocessor has had a dramatic influence on the PID controller. Practically all the PID controllers made today are based on microprocessors. This has given opportunities to provide additional features like automatic tuning, gain scheduling and continuous or online adaptation.

The PID algorithm is described by:

$$u(t) = K \left\{ e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right\} \quad (10.9)$$

The error signal  $e(t)$  is the difference between the instantaneous values of the input signal,  $x(t)$  and the feedback signal  $f(t)$ ; as illustrated by Fig.10.17.

$$e(t) = x(t) - f(t) \quad (10.10)$$

The control signal is the sum of three terms:

- **Proportional-term**; proportional to the error,  $P(t) = Ke(t)$ ,
- **Integral-term**; proportional to the integral of the error,  $I(t) = K \frac{1}{T_i} \int_0^t e(\tau) d\tau$ ,
- **Derivative-term**; proportional to the derivative of the error,  $D(t) = KT_d \frac{de(t)}{dt}$ .

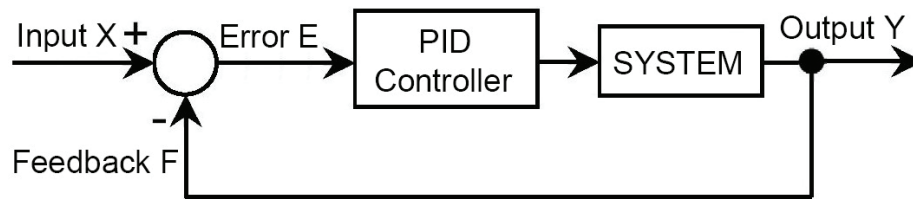
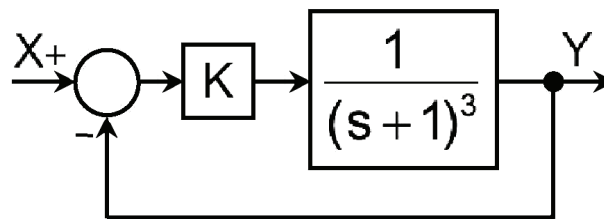


Fig.10.17 Connection of the PID controller in the feedback loop

The controller parameters are the proportional gain  $K$ , the integral time  $T_i$  and the derivative time  $T_d$ . The most well-known methods for estimating and tuning the PID parameters are those developed by Ziegler and Nichols. They have had a major influence on the practice of the PID control for more than half a century. The estimation methods are based on characterization of process dynamics by a few parameters and simple equations for the controller parameters. The PID controllers can be designed, according to Ziegler-Nichols rule as explained by the following example. The process of design of a PID controller is illustrated by replacing the proportional controller, Fig.10.18, by the PID controller, Fig.10.17.

Fig.10.18 Feedback system with proportional controller of gain  $K$ 

The Implementation of the PID controller is carried out according to the following four steps:

- a) Find the limiting open loop gain for the stability.
- b) Design P, PI and PID controller
- c) Implementation of the P, PI and PID controllers
- d) Tuning of the controllers parameters

#### a) Find the limiting open loop gain for the stability

- Connect the system as shown in Fig.10.18 then apply a step input (with gain  $K=1$ , for example).
- Calculate the step response then change the proportional gain  $K$  until continuous oscillations are observed. The resulting step responses are shown in Fig.10.19. For proportional gain  $K = 1$ , the calculated step response shows that the system response has considerable steady state error ( $\overline{e_{ss}} = 50\%$ ).

For  $K=8$ , the transient response converged to sustained oscillations. The limiting value of gain  $K$ , which makes the system marginally stable (the response is oscillatory), is called the ultimate gain  $K_L$ . The duration of one complete cycle is the ultimate period  $\tau$ . For the studied system,  $K_L=8$  and  $\tau=3.63$  s.

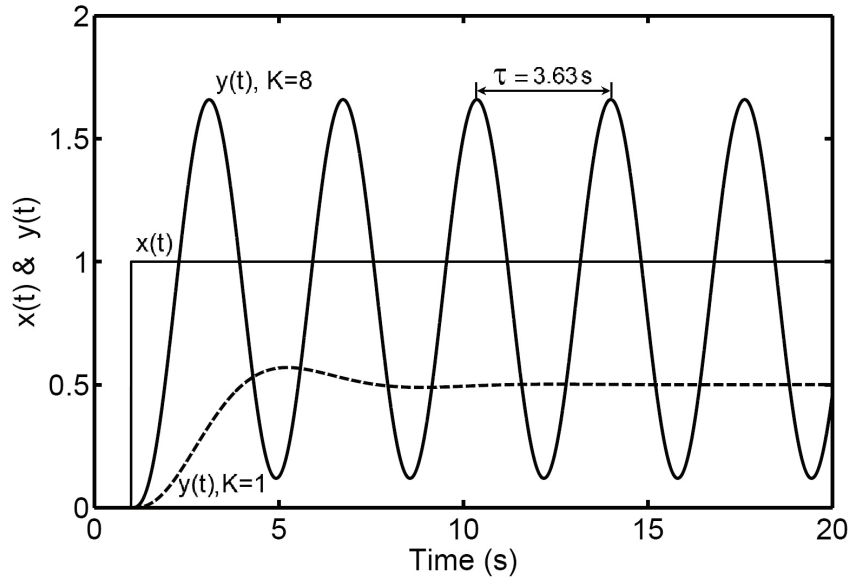


Fig.10.19 Step response of the closed loop system for open loop gain  $K=1$  and for the limiting gain  $K_L=8$

### b) Design P, PI and PID controller

The transfer functions of the proportional **P**, proportional integral **PI** and proportional integral derivative **PID** controllers are the following:

**P-Controller**

$$D(s) = K = 0.5K_L \quad (10.11)$$

**PI-Controller**

$$D(s) = K \left( 1 + \frac{1}{T_i s} \right) \quad (10.12)$$

**PID-Controller**

$$D(s) = K \left( 1 + \frac{1}{T_i s} + T_d s \right) \quad (10.13)$$

The first estimate of the PID controller parameters is calculated by applying Ziegler and Nichols rule, table 10.1. The calculation results are given in table 10.2.

Table 10.1 Summary of formulae used to calculate a first estimate of the P, PI and PID controllers, according to Ziegler-Nichols rules.

Controller	Symbol	Gain	$T_i$ [s]	$T_d$ [s]
Proportional	P	$K = 0.5 K_L$	-	-
Proportional Integral	PI	$K = 0.45 K_L$	$0.8 \tau$	-
Proportional Integral Derivative	PID	$K = 0.6 K_L$	$0.5 \tau$	$0.125 \tau$

Table 10.2 First estimate and tuned parameters of the P, PI and PID controllers according Ziegler-Nichols rules.

Controller	First estimate			Tuned parameters		
	K	$T_i$ [s]	$T_d$ [s]	K	$T_i$ [s]	$T_d$ [s]
<b>P</b>	4	-	-	4	-	-
<b>PI</b>	3.6	2.9	-	1.5	3.03	-
<b>PID</b>	4.8	1.815	0.454	1.3	2.5	0.08

**c) Implementation of the P, PI and PID controllers**

Connect the controllers as shown in Fig.10.20 and calculate the step response. The resulting response is shown in Fig.10.21. This figure shows that the P-controller response has great steady state error. The PI and PID controllers stabilized the system, with no steady state error. The controller setting according to Ziegler-Nichols rule improved the closed loop system stability and precision. But a final tuning of the controller parameters must be done iteratively until satisfactory response is obtained.

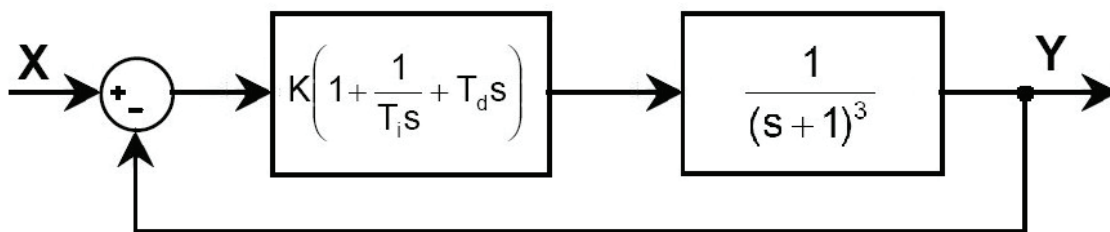


Fig.10.20 Closed loop system equipped with PID controller.

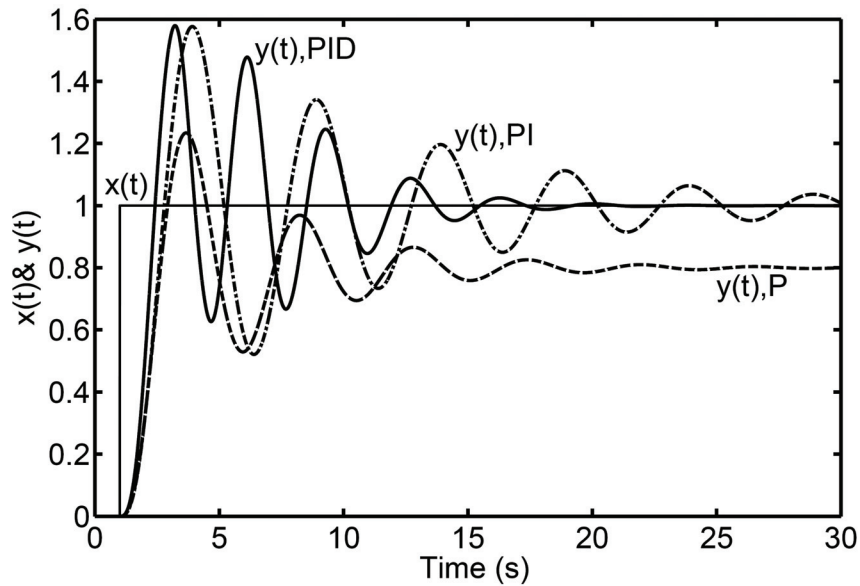


Fig.10.21 Step response of the closed loop system, with P, PI and PID controllers.

**d) Tuning of the controllers parameters**

The tuning of the gain of the proportional controller does not improve the system behavior, due to the contradiction between the stability and precision requirements. The tuning of the PI and PID controllers improved radically the system stability. Figure 10.22 shows that the responses of the system with PI and PID controllers converge rapidly to the required steady state value, without steady state error.

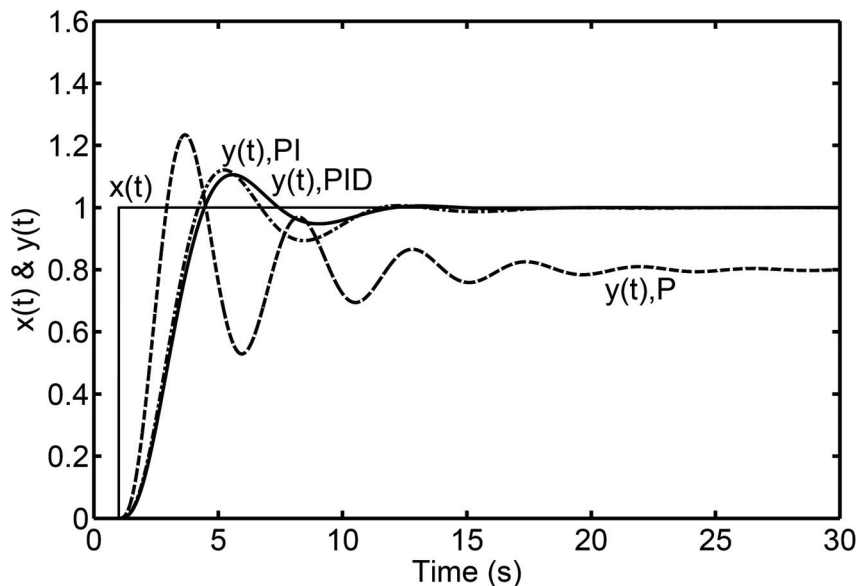


Fig.10.22 Step response of the closed loop system, equipped with P, PI and PID controllers, with tuned parameters.

## 10.6 EXERCISE

1. Consider the system described by the following open loop transfer function:

$$G(s)H(s) = \frac{K}{s(s+1)(1+0.125s)}$$

- Find the limiting value of K for system stability.
- Design PI and PID controllers for this system, tune the parameters and calculate the step response controlled by the PI and PID controllers.

2. Consider the system described by the following open loop transfer function:

$$G(s)H(s) = \frac{K}{s(0.5s+1)^2}$$

- Plot the Bode diagrams for K=1 and K=10 and discuss the system stability.
- A series phase lag compensator having the following transfer function is introduced:

$$G_c(s) = \frac{1+10s}{1+100s}$$

Plot the Bode plot of the compensated system and discuss the results for K=1 and K=10.

3. Consider the system described by the following open loop transfer function:

$$G(s)H(s) = \frac{K}{s(0.5s+1)^2}$$

- Find the limiting value of K for system stability.
- Design a PI and a PID controller for this system, tune the parameters and calculate the step response of the system controlled by the PI and PID controllers.

4. Consider the system described by the following open loop transfer function:

$$G(s)H(s) = \frac{K}{s(0.2s+1)(0.3s+1)(0.5s+1)}$$

- Find the limiting value of K for system stability.
- Design a PI and a PID controller for this system, tune the parameters and calculate the step response of the system controlled by the PI and PID controllers.

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