Mechanical Vibrations

Thammaiah Gowda
Jagadeesha T
D V Girish

Mechanical Vibrations

ABOUT THE AUTHORS

Thammaiah Gowda is presently working as Professor and Head in the Department of Mechanical and Industrial Production Engineering at Adichunchanagiri Institute of Technology, Chickmagalur, Karnataka. Dr Gowda obtained both his BE (Mechanical) and postgraduation (PG) (Engineering Management) degrees from SJCE, Mysore (University of Mysore), ME(Machine Design) from UVCE, Bangalore (Bangalore University) and doctoral degree in the area of SCF from UBDT College of Engineering, Davangere (Kuvempu University). He has more than thirty years of teaching experience in Mechanical Vibration, FEM, Tribology and Bearing Design, Mechanics of Materials, Design of Machine Elements, Kinematics of Machines, Dynamics of Machinery, Automatic Control Engineering, CAD/CAM, Theory of Elasticity and Plasticity to both graduate and undergraduate students. He has guided several PhD and MSc students. He has published several research papers in international and national journals and conferences. He is also a member of Institution of Engineers (Fellow, IE) and ISTE (Life Member, ISTE).

Jagadeesha T is currently working as Assistant Professor in the Department of Mechanical and Production Engineering at National Institute of Technology (NIT), Calicut, Kerala. He earned his BE (Mechanical) degree from NIT Surathkal, ME(Machine Tool Design) degree from PSG college of Technology, Master of Science(Manufacturing and Automation) from National University of Singapore and Master of Science (AMNS) from Singapore MIT alliance. He has 20 years of experience in industry, teaching, academic research, consultation and doing projects with reputed organizations. He has worked with TATA Engineering Locomotive Company (India), TVS Suzuki (India), IBM Pvt. Ltd (Singapore), ASM (Singapore), and Applied Materials (Singapore and US), APP Systems and Services (Singapore) ST Microelectronics (Singapore), Chartered Semiconductor Manufacturing (Singapore) and Sitronics (Singapore). His articles have appeared in various publications and journals. He has developed workbooks on Fluid Power Control, Mechanical Vibration, Machine Design, Machining Science and IC Engines. He is a member of several professional bodies in India and abroad and is a Certified Professional Engineer (Australia). His innovative research on BPSG CVD process won best IFIT award at ST Microelectronics, Singapore. He won more than thirty quality suggestion awards at TELCO and Best Employee award three times at Singapore. He teaches Solid Mechanics, Mechanics of Machinery, Dynamics of Machinery, Design of Machine Elements, Fluid Power and Control at NIT Calicut.

D V Girish recently retired from the position of Vice Principal of Malnad Engineering College, Hasan, Karnataka. He has four decades of teaching experience. At Hasan, he taught Machine Design, Theory of Machines, Advanced Kinematics, Mechanical Vibration and Automatic Control Engineering. He obtained his PhD from IIT Madras, having received his master's degree from IIT Kanpur and BE degree from Malnad Engineering College. He has published several research papers in national and international journals and presented a good number of technical papers in conferences. He has also guided several PhD and MSc students. He headed the department for more than 12 years and the institution has seen several improvements under his leadership. He served several times as Chairperson of Board of Examiners, Board of Studies and Senate Member at VTU.

Mechanical Vibrations

Thammaiah Gowda
Professor and Head
Department of Mechanical and Industrial Production Engineering
Adichunchanagiri Institute of Technology
Chickmagalur, Karnataka

Jagadeesha T
Assistant Professor
Department of Mechanical and Production Engineering
National Institute of Technology (NIT)
Calicut, Kerala

D V Girish
Retired Vice Principal
Malnad Engineering College
Hasan, Karnataka

Tata McGraw Hill Education Private Limited
NEW DELHI

McGraw-Hill Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

Tata McGraw Hill

Published by Tata McGraw Hill Education Private Limited,

7 West Patel Nagar, New Delhi 110 008.

Mechanical Vibrations

ISBN (13): 978-1-25-900617-3

ISBN (10): 1-25-900617-4

Vice President and Managing Director—McGraw Hill Education: *Ajay Shukla*
Head—Higher Education Publishing and Marketing: *Vibha Mahajan*

Publishing Manager—SEM & Tech Ed.: *Shalini Jha*
Senior Editorial Researcher: *Harsha Singh*
Executive—Editorial Services: *Sohini Mukherjee*
Senior Production Manager: *Satinder S Baveja*
Proof Reader: *Yukti Sharma*

Marketing Manager—Higher Education: *Vijay Sarathi*
Sr. Product Specialist – SEM & Tech Ed.: *Tina Jajoriya*
Graphic Designer—Cover: *Meenu Raghav*

General Manager—Production: *Rajender P Ghansela*
Production Manager: *Reji Kumar*

Typeset at Bharati Composers, D-6/159, Sector-VI, Rohini, Delhi 110 085, and printed at

The McGraw-Hill Companies

RESPECTFULLY DEDICATED TO

Padmabhushana Jagadguru
Sri Sri Sri Dr Balagangadharanatha Mahaswamiji
Adichunchanagiri Maha Samsthana Math, Karnataka

AND

Babuji Maharaj
Founder President, Shri Ramchandra Mission
Shahjahanpur, Uttar Pradesh

CONTENTS

PREFACE

Target Audience

Mechanical Vibrations plays an important role in engineering and is considered to be one of the most fundamental applied subjects in the engineering discipline. This book is designed primarily for the use of degree-level students of mechanical engineering as well as students who are preparing for AMIE and various other competitive examinations. It will also benefit postgraduate students to some extent as it contains advanced topics like random and transient vibrations. The book will also prove helpful to practicing design engineers in their day-to-day work and thus be a welcome addition to design offices and provide useful addition to their libraries.

Prerequisites for this Course

We have endeavoured to present the subject in a simple and rational way. In preparation of this book, we have taken advantage of the vast experience gained in the course of our work during the past thirty years. It has been our aim to show the basic principles underlying mechanical vibration by means of typical examples. Empirical formulae have been used only when it is not practical to use mathematical analysis. According to our experience, a sound knowledge of mechanics of materials is very essential to take up the study of mechanical vibrations and design for vibration. It is expected that the students using this book have completed a course in applied mathematics.

Rationale for Writing this Book

The main objective of writing this book has been to give a clear understanding of the concepts underlying Mechanical Vibrations. We have endeavoured to teach the subject on a scientific basis, to maintain the physical perceptions in the various derivations and to give short comprehending solutions to a variety of complex problems. The parameters kept in mind while writing the book are coverage of contents to suit the syllabi of various Indian universities, prerequisite knowledge of the users of this book, lucidity of writing, clarity of thoughts and diagrams, and a variety of solved and unsolved numerical problems including problems from competitive examinations.

This book is specifically concerned with the fundamentals of vibration analysis for mechanical engineers, structural engineers, mining engineers, production engineers, maintenance engineers, etc.

Vibration is omnipresent, ranging from the fluttering of aeroplane wings to sweet musical tunes of flutes to the beats of modern-day percussion instruments. The subject of vibration has close interface with other subjects like finite element analysis, instrumentation and rotor dynamics. We have presented random vibration and transient vibration in this book. In future editions, nonlinear vibrations, random vibration and transient vibration will be discussed in great detail. We also wish to add a chapter on rotor dynamics in the next edition.

Despite the importance and relevance of the subject, it is observed that the subject has not been given its justified importance in the undergraduate engineering course curriculum of Indian technical institutes. In most cases, the subject has been camouflaged in the broader outlines of theory of machines or has been set aside for advanced reading in the postgraduate section. We feel that the subject should be a separate one in the undergraduate level, where fundamentals of physics of vibrations should be taught with great care and with sufficient mathematical exposure.

A significant amount of advances emerged over the last thirty years or so. It would be an impossible task to attempt to cover all this material in a textbook aimed at providing the reader with a fundamental basis for vibration analysis. This book is, therefore, only concerned with some of the more important fundamental considerations required for a systematic approach to mechanical vibration, the main emphasis being the practical approach.

This book serves as a preparatory textbook to take up challenging research work in instrumentation to measure vibrations. This book will also serve as reference to practicing engineers.

Chapter Organization

This textbook embodies eleven self-contained chapters, each of which is summarized here. SI units have been adopted throughout the textbook.

- *Chapter 1* introduces the basic terms and parameters related to mechanical vibration. Also, analytical and numerical methods for harmonic analysis have been added.
- *Chapter 2* introduces the reader to the free vibration of an undamped single-degree-freedom system and some practical applications. Students should focus on this chapter as understanding later concepts depends on understanding this chapter well.
- *Chapter 3* introduces us to free vibration of a damped single-degree-freedom system. The primary aspects of this chapter are to determine the damped natural frequency and rate of decay of vibrations.
- *Chapter 4* introduces us to the study of forced vibrations in single degree of freedom. In this chapter, various forms of damping are also discussed.

- *Chapter 5* deals with various instruments to measure the vibrations. Critical speed of shaft without and with damping has been added.
- In *Chapter 6*, two-degree-freedom systems are discussed in great detail. It also contains nonharmonic behaviour of two masses and their solutions.
- As the number of degrees of freedom increases, it becomes very tedious to solve the equations of motion and to determine the natural frequencies and mode shapes. The natural frequencies and mode shapes can be determined easily and quickly with the help of numerical methods. *Chapter 7* deals with many such multi-degree-freedom systems using exact analysis.
- *Chapter 8* complements Chapter 7 and is about the interaction of multi-degree-freedom system using numerical methods.
- *Chapter 9* introduces the vibration of continuous or distributed system such as beams, rods, cables, plates, etc. A continuous system is equivalent to an infinite element of masses concentrated at different points and, hence, it is an infinite-degree-freedom system.
- In *Chapter 10*, transient vibration is taken up which is the vibration caused by an impulsively acting force or moment on a system.
- *Chapter 11* finally introduces random vibration in brief.

Each chapter begins with the fundamental concepts and related theory and takes the reader to logical conclusion of the physical meaning of the systems at the end. We have endeavoured to maintain continuity of thoughts and concepts between chapters. Each chapter has a concise and comprehensive treatment of topics with strong emphasis on fundamental concepts. A number of theoretical questions and unsolved exercises are given for practice to widen the horizon of comprehension of the topic.

Apart from all these, each chapter contains *Objective-type Questions* to prepare the student in a better way to face the challenges of competitive examination. Chapters 9 to 11 involve specialist topics more suited to postgraduate courses.

The material has been arranged in an order beneficial to most students and teachers. The text contains many illustrative examples and a number of problems to be worked out by the students. Important tables are included to enhance the scope of the subject.

Salient Features

- Complete and comprehensive coverage of Mechanical Vibrations with presentation of basic theory in simple and readily understandable form
- Balanced presentation of mathematical and concept approaches
- Separate chapters on Exact Analysis and Numerical Methods of Multi-Degree-Freedom Systems
- Dedicated discussion on Single-Degree-Freedom Systems: Undamped Free, Damped Free and Forced Vibration
- Comprehensive coverage of Vibrations of Continuous Systems, Transient and Random Vibrations

- Introduction at the beginning of each chapter sums up the aim and contents of the chapter
- Simple diagrams given for easy visualisation of the explanations
- Large number of solved problems and unsolved problems picked up from examinations of various Indian technical institutes and universities
- Each chapter contains typical objective-type questions useful for students appearing in competitive examinations
- Pedagogy:
  - Solved Examples : 180
  - Numerical Problems : 137
  - Review Questions : 115
  - Multiple Choice Questions : 120

How to use the Book

- *For Instructors*—The first six chapters constitute a reasonable introduction to mechanical vibration, and they could be a satisfactory basis for a one-semester course. The book is laid out allowing one week for each chapter through 3, and two weeks each for the chapters 3 to 6. Chapters 3 to 6 bring the student through free and forced vibration, forced and undamped vibration and, into a discussion of systems with two degrees of freedom. The rest of the book can be used at the postgraduate level.

  We have found that five problems a week is an ample assignment. Certain problems can be assigned to students as take-home assignments to solve using C or MATLAB.
- *For Students*—Mathematics is intimately concerned with the study of vibration. In order to study the characteristics of vibration of any system, you have to resort to understanding the physical meaning and modelling of the system and write the characteristic differential equation of motion, which essentially represents the dynamic behaviour of the system. To solve vibration problems by numerical methods, knowledge of matrix algebra is essential. Knowledge of complex numbers, trigonometry and Fourier series is also very much essential to derive full benefit from this textbook.
- *Secondary readers* are postgraduate students. If the secondary readers are exposed to fundamentals of vibration in their undergraduate study, they can start from Chapter 6 on measurement of vibration. They will also find chapters on random vibration and transient vibration very useful.

Web Supplements

There are a number of supplementary resources available on *http://www.mhhe.com/gowda/mv1*, and updated from time to time to support this book.

For Instructors:

- Solution Manual
- PPTs

For Students:

- 1 Sample Chapter
- 2 Model Question Papers
- Web links for further reading

Acknowledgements

This book has been nearly a decade in the making. During that period, a multitude of friends, clients and associates have provided us with support, helped us solidify our ideas regarding mechanical-vibration theory and many of our students have indirectly assisted us by asking thought-provoking questions, thus inspiring us to seek knowledge.

We would like to thank Dr C K Subbaraya, Principal; Mr C NChandra Shakeriah, Registrar; Mr S Chandre Gowda, Ex-Registrar; and Akshatha S Gowda, Assistant Professor, AIT Chickmagalur, for their help and support. We would like place on record our deep sense of gratitude to Mr H N Suresh, System Analyst, for his incalculable help in preparing this textbook—without him, this book could not have been done.

Special acknowledgements are due to our teachers and families, especially our parents for encouraging us to pursue an academic career. Dr Gowda wishes to thank his wife, M S Leelavathy, for enduring the very long hours that we had to work during the gestation period of this textbook, his young sons, Ullas and Uttam, and his brother Paneesha Gowda, who has rendered so much help.

Jagadeesha T would like specifically to thank his wonderful sister Prabhavathi T for her endless sacrifices, patience and motherly love. Acknowledgements are due to several of his colleagues at National Institute of Technology, Calicut. These include Dr G R C Reddy, Director, NIT Goa, (Former Director of NIT Calicut), a dynamic administrator, who provided the most ideal learning atmosphere at NIT Calicut; and Dr R Sridharan, Professor and Head of Mechanical Engineering Department, without whose presence, encouragement and comfort, this textbook would have remained merely a good intention. Dr R Sridharan possesses all the qualities most people could only hope to have.

It takes a team of many people and lots of hard work to create a quality textbook. Thanks most certainly to the publishing team at Tata McGraw Hill. Many thanks are due to Harsha Singh, Senior Editorial Researcher, who set the tone for excellence and who provided the vision and leadership to create such a quality product. We would also like to put on record our immense appreciation for Yukti Sharma, Proof Reader, who worked long hours to improve our prose and produce this text from the first page of the manuscript to the final, bound product; and to Sohini Mukherjee, Executive—Editorial Services, whose hard work and dedication can never be repaid.

We would like to express our gratitude to the following reviewers who helped us form our ideas concerning mechanical vibration. We are indebted to them for reviewing our original manuscript and unconsciously implanting in us the drive for perfection and confidence that all enabled us to create this book.

S P Harsha	Indian Institute of Technology (IIT) Roorkee, Uttarakhand
K S Tripathi	Institute of Technology—Banaras Hindu University (IT-BHU), Varanasi, Uttar Pradesh
S C Mohanty	National Institute of Technology (NIT) Rourkela, Orissa
L A Kumaraswamidhas	National Institute of Technology (NIT) Tiruchirapalli, Tamil Nadu
Jitendra Bhaskar	Harcourt Butler Technological Institute (HBTI) Kanpur, Uttar Pradesh
Manoj Kr. Barai	Future Institute of Engineering and Management, Kolkata
H D Desai	Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat, Gujarat
L Bhaskara Rao	School of Mechanical and Building Sciences VIT University, Chennai
Suresh Kumar Jyothula	Jawaharlal Nehru Technological University (JNTU-H) College of Engineering, Hyderabad

Feedback

We express our gratitude towards the publishers who have extended their cooperation in bringing out this book in a short period.

Although great care has been taken in correcting proofs and checking numerical examples, errors may be present and further suggestions to improve upon remain. The authors will be highly grateful to the readers for any corrections. Please send your feedback to *jagdishsg@nitc.ac.in*

Thammaiah Gowda
Jagadeesha T
D V Girish

Publisher's Note

Do you have any feedback? We look forward to receive your views and suggestions for improvement. The same can be sent to *tmh.mechfeedback@gmail.com,* mentioning the title and author's name in the subject line.

FUNDAMENTALS OF VIBRATIONS

1

1.1 INTRODUCTION

Vibration is a ubiquitous phenomenon. It may occur due to earthquakes or tremors, resonance effects on bridges, taking-off or landing of aeroplanes, starting of an automobile engine, burning of fuel for space-launch vehicles and operations of machines with moving, rotating or reciprocating parts operated without suitable foundations.

Speech is caused by vibrations of the vocal chord, tongue and lips. Vision requires the vibration of eyelids, walking requires oscillation of legs, breathing requires vibration of lungs and life requires vibration of the heart.

One can observe small vibration dampers on high-voltage transmission lines near towers or rubber pads separating, home appliances. Stones on which rails are laid are padded with wooden sleepers to decrease the impact of vibration due to train motion.

Is the presence of vibrations desirable or undesirable?

It has many desirable effects in the following applications:

1. Musical instruments like flute, harmonium, tabor (tabla), violin, *veena*, etc.
2. Agitators used in concrete setting.
3. Horns and sirens, etc.

It has many undesirable effects in the following applications: machine tools, automobiles, bridges, dams, buildings as it produces noise or sound, excessive stress, unbalanced forces in rotating and reciprocating parts, etc.

Vibration analysis has a vital role to play in society for improving living conditions and ensuring smooth functioning of industrial equipments, minimising the losses due to unwanted vibrations. This can be made possible only through a clear understanding and proper study of vibrations.

Systems possessing mass and elasticity are capable of having relative motion.

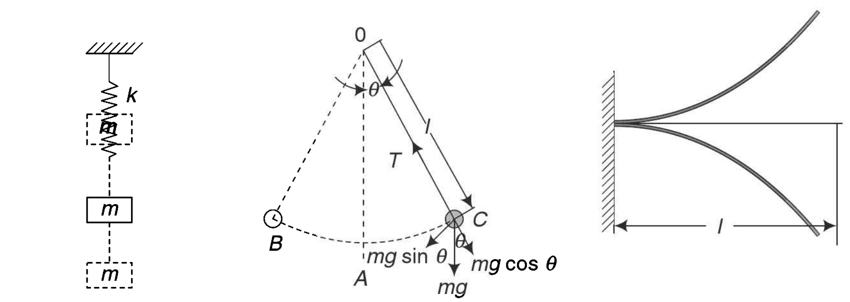
DEFINITIONS

- **If the motion of such a system repeats itself in a given interval of time, it is known as vibration.**
- **Any motion which repeats itself after an interval of time is known as vibration (periodic motion).**

- **The to-and-fro motion of a body about a mean position is known as vibration.**

Example Classical spring mass system, simple pendulum, cantilever beam, as shown in Fig. 1.1.

The study of vibrations is very important in engineering. In general, vibration is a wasted form of energy and unwanted in many cases. Vibration generates unwanted noise, breaking parts of machinery, causes dry friction between matching surfaces, develops unbalanced forces and excessive stress in machine parts. Because of large vibrations, proper readings of (accuracy) instruments cannot be taken and also it is too dangerous for human beings. Vibration can be minimised by removing the causes of vibration, resting the machinery on proper types of isolators, and using dampers, shock absorbers, springs, rubber, etc.



(a) Spring mass system (b) Simple pendulum (c) Centilever beam

Fig. 1.1 *Objects undergoing vibration*

1.1 AIM OF VIBRATION ANALYSIS

The main consideration of vibration analysis is to find the natural frequency of vibration, which is one of the characteristic frequencies of vibration of a body when it is under free vibration. We need to see that the structure is excited by frequencies far away from the natural frequencies, if necessary, to limit the amplitude of vibration. If the excitation frequency is very near the natural frequency, the amplitude of vibration will be excessively large which readily leads to failure due to resonance.

1.3 LIMITS OF VIBRATION ANALYSIS

Generally, elastic analysis of vibration problems are considered and hence the first limit on displacements, strains, and loads is that the maximum stress developed should not exceed the proportionality limit stress. Hence, the displacements are to be small. Similarly, if the displacements are not small (finite or large), the equations involved become nonlinear and solutions become complicated. To avoid such difficulties and to readily obtain simpler solutions, the restrictions are again that the displacements be small.

1.4 IMPORTANT DEFINITIONS IN VIBRATION ANALYSIS

1. ***Inertia*** The property of a body (either in rest or in motion) by which it continues to be in its present state unless acted upon by an external force.
2. ***Displacement (x)*** The change in position of an object in a particular direction by application of an external force.
3. ***Disturbance*** Any action which destroys the static equilibrium of a vibrating system may be called a disturbance to that system.
4. ***Restoring force*** The displaced body does not stay in the new position, because of the restoring force which is provided either by gravity or by elasticity or by both.
5. ***Undamped vibration*** If vibrations take place in the absence of a damping force then the vibration is known as undamped vibration.
6. ***Damping*** It is the resistance offered to the motion of the vibratory body.
7. ***Damped vibration*** If energy is lost or dissipated during oscillation, the vibration is known as damped vibration.
8. ***Periodic motion*** Motion which repeats itself after equal intervals of time.

Periodic motion can be represented as shown in Fig. 1.2. Vibration is periodic and any of its properties like displacement 'x' can be conveniently represented by a periodic function varying with time. Such functions are the trigonometric functions sin θ, cos θ, tan θ. The functions sin θ or cos θ are very much convenient in vibration analysis. The expression for the displacement 'x' measured from the static equilibrium position in a to-and-fro motion can be generated by considering a vector $\vec{X}$ of modulus 'X' revolving at a constant angular velocity 'ω' and considering its resolved component about the direction of motion. Hence, $x = X \cos\theta$ about the horizontal axis, and $x = X \sin\theta$ about the vertical axis. In a vector revolving with angular velocity 'ω', the angle 'θ' is given by $\theta = \omega t$.

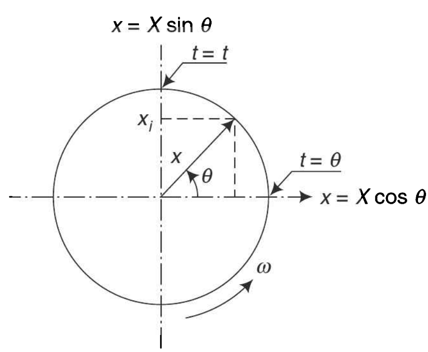


Fig. 1.2 *Representation of periodic motion*

If 't' is measured as zero when 'θ' is zero, $\omega t = 2\pi f t$ or $\omega = 2\pi f$,

where f = Linear frequency. Hence, $x = X\cos\theta$ or $x = X\cos\omega t$,

$x = X\sin\theta$ or $x = X\sin\omega t$ or $x = X\sin(\omega t + \phi)$. If '$t$' is measured arbitrarily, the angle 'ϕ' is known as phase angle. The simplest type of periodic motion is harmonic motion or Simple Harmonic Motion (SHM).

Example Sine wave and cosine wave are shown in Fig. 1.3.

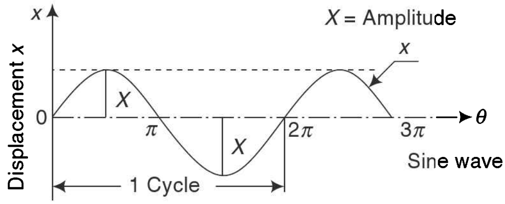


Fig. 1.3 *Sine wave*

9. ***Amplitude (X)*** The maximum displacement from the mean position of the vibrating body, (X) is shown in Fig. 1.3.
10. ***Cycle*** It is the motion completed during one time period.
11. ***Time period or periodic time (τ)*** It is the time required for a particle to complete one cycle, i.e. $s = v \times t$, $2\pi = \omega \times \tau$, or $\tau = \frac{2\pi}{\omega}$ seconds, where s = distance travelled = 2π, v = velocity = ω and t = time in seconds as shown in Fig. 1.4.

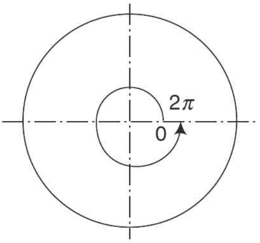


Fig. 1.4 *Representation of periodic time*

12. ***Frequency (f)*** It is the number of cycles per unit time or seconds. Frequency is the reciprocal of time period,

i.e. $$f = \frac{1}{\tau} = \frac{1}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi} \text{ cps or Hz.}$$

Or $$\omega = 2\pi f \text{ rad/s}$$

13. ***Natural frequency (ω_n)*** It is the frequency of free vibration without damping denoted by 'ω_n'.
14. ***Fundamental mode of vibration*** It is the mode having lowest natural frequency of a vibrating system.
15. ***Damped natural frequency (ω_d)*** It is the frequency of the system having free vibration with friction denoted by 'ω_d'.
16. ***Resonance*** When the frequency of the external excitation (force) equals the natural frequency of a system, the amplitude of vibration will increase without bound and is governed only by the amount of damping present in the system. At resonance the amplitude of vibration becomes too large.
17. ***Phase difference (ϕ)*** It is the angle between two rotating vectors representing simple harmonic motion of the same frequency. Let the first vector be $x_1 = X_1 \sin(\omega t)$ and second vector $x_2 = X_2 \sin(\omega t + \phi)$; then the term '$\phi$' is called phase difference.

1.4 THE BASIC COMPONENTS OF A MECHANICAL VIBRATING SYSTEM

The main components of vibrating systems are mass(m) damper(c) and spring(k).

These components are represented in Fig. 1.5.

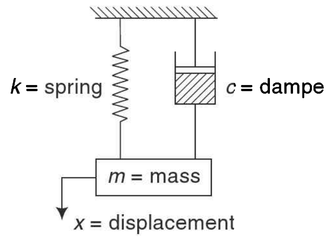


Fig. 1.5 *Basic components of a vibrating system*

The mass of the vibrating system provides inertia force, the damper provides the resisting force and the spring provides the restoring force. The constant coefficients m, c and k represent the system parameters. These parameters are known as mechanical components of the system. They are generally passive elements. The active element of the mechanical system is the exciting forces. They will have their associated mechanical impedances.

Mass, or inertia element (m)

The mass or inertia element is assumed to be rigid body. During vibration, the velocity of mass changes. Hence, kinetic energy can be gained or lost. The work done on the mass will be stored in the form of kinetic energy $= \frac{1}{2} mv^2 = \frac{1}{2} m\dot{x}^2$ (Translation system) or $\frac{1}{2} J.\dot{\theta}^2$ (pure rotational system)

In practical cases, for simple analysis, we replace several masses by a single equivalent mass.

Case (i) Translation masses connected by a rigid bar

Let the masses 'm'$_1$, 'm'$_2$, 'm'$_3$ be attached to the rigid bar at locations (1), (2), (3) respectively as shown in Fig. 1.6(a). The equivalent mass 'm_{eq}' be assumed to be located at 'l_l' as shown in Fig 1.6(c) and FBD is as shown in Fig. 1.6(b).

Let the displacement of masses 1, 2 and 3 be x_1, x_2 and x_3 respectively, and hence the velocities will be x_1, x_2 and x_3 respectively.

Let the velocity of masses m_2 and m_3 (i.e. x_2 and x_3 respectively) be expressed in terms of velocity of mass m_1 (i.e. x_1).

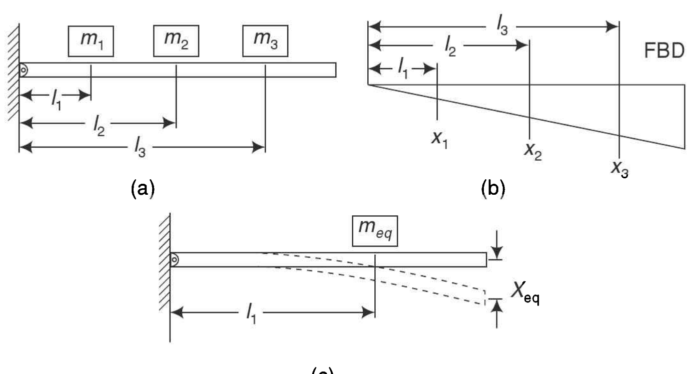


Fig. 1.6 *Combination of masses*

From similar triangular properties, we can write

$$\frac{l_1}{x_1}=\frac{l_2}{x_2}=\frac{l_3}{x_3},\ x_2=\left(\frac{l_2}{l_1}\right)x_1,\ \dot{x}_2=\left(\frac{l_2}{l_1}\right)\dot{x}_1.\ \text{Similarly, } x_3=\left(\frac{l_3}{l_1}\right)x_1,\ \dot{x}_3=\left(\frac{l_3}{l_1}\right)\dot{x}_2$$

And the velocity for equivalent mass $x_{eq} = \dot{x}_1$

The kinetic energy of equivalent mass is equal to the sum of kinetic energies of individual masses.

$$(\text{KE})_{eq} = (\text{KE})_1 + (\text{KE})_2 + (\text{KE})_3 = \frac{1}{2} m_{eq}\,\dot{x}_{eq}^2 = \frac{1}{2} m_1\,\dot{x}_1^2 + \frac{1}{2} m_2\,\dot{x}_2^2 + \frac{1}{2} m_3\,\dot{x}_3^2$$

$$= m_{eq}\,\dot{x}_1^2 = m_1\,\dot{x}_1^2 + m_2\left(\frac{l_2}{l_1}\right)^2\dot{x}_1^2 + m_3\left(\frac{l_3}{l_1}\right)^2\dot{x}_1^2,\ \text{or } m_{eq} = m_1 + m_2\left(\frac{l_2}{l_1}\right)^2 + m_3\left(\frac{l_3}{l_1}\right)^2$$

The equivalent mass can be placed anywhere on the rigid bar, but the magnitude changes everywhere.

Case (ii) Translational and rotational masses coupled together

Let a mass having translational velocity 'x_1' be coupled to another mass having mass moment of inertia 'J' having rotational velocity 'θ' as in a rack.

The pinion arrangement is shown in Fig. 1.6(d). These two masses can be combined as either a single equivalent translational mass 'm_{eq}' or a single equivalent rotational mass 'J_{eq}.'

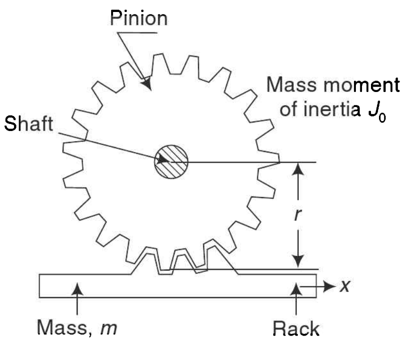


Fig. 1.6 (d) *Translational and rotational masses*

Equivalent translational mass; kinetic energy of two masses

For mass 1, $\quad \text{KE} = \frac{1}{2} m.\dot{x}^2$

For mass 2, $\quad \text{KE} = \frac{1}{2} J_0.\dot{\theta}^2$

$\therefore \quad \text{KE} = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_0\,\dot{\theta}^2$

Kinetic energy of equivalent mass $= \frac{1}{2} m_{eq}\,x_{eq}^2$, since $x_{eq} = x$

Kinetic energy of equivalent mass is equal to the sum of KE of individual masses.

$$\therefore \quad \frac{1}{2} m_{eq}\,\dot{x}_{eq}^2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_0\,\dot{\theta}^2 \text{ or } m_{eq}\dot{x} = m\dot{x}^2 + J_0\,\dot{\theta}^2$$

From the geometry of the figure, $x = R\dot{\theta}$,

On differentiating, $\quad x = R\dot{\theta}$

$$\therefore \quad m_{eq}\,\dot{x}^2 + m\dot{x}^2 + J_0\,\frac{\dot{x}^2}{R^2},\ m_{eq} + m + \frac{J_0}{R^2}$$

This is the translational equivalent mass.

Case (iii) Equivalent rotational mass

KE of equivalent mass = $(\text{KE})_1 + (\text{KE})_2, \frac{1}{2} J_{eq}\, \dot{\theta}_{eq}^2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_0\, \dot{\theta}^2$

$$x = R\theta,\ x = R\dot{\theta}, \text{ and } \dot{\theta}_{eq} = \theta,\ J_{eq}\, \dot{\theta}^2 = mR^2\, \dot{\theta}^2 + J_0\, \dot{\theta}^2$$

1.3 SPRING ELEMENT (*K*)

Whenever there is relative motion between the two ends of a spring, a force called *spring force* or *restoring force* is developed. The spring force 'F' is directly proportional to the amount of deformation, i.e. $F \,\alpha\, x$ or $F = kx$

where k = Stiffness of spring or spring constant

The spring stiffness 'k' in the spring force required to cause a unit deformation of the spring $k = F/x$ N/mm. (See Sec. 2.4, Chapter-2).

1.3 DAMPER

Damping is the resistance offered to the motion of a vibrating body. Damping occurs as a result of system vibrations in a fluid. Hydraulic dashpots and shock absorbers are systems which can be represented by viscous damping. Most mechanical systems themselves have damping which is quiet complex. They can be represented by an equivalent viscous damping. In viscous damping, the damping resistance is proportional to the relative velocity between piston and cylinder ($c\dot{x}$) where 'c' is the damping coefficient. The importance of viscous damping is that it affords an easy analysis of the damping element (c).

The vibrational energy is generally converted into heat or sound. Hence, the displacement during vibration gradually reduces. The mechanism by which vibrational energy is gradually converted into heat or sound is known as damping, as shown in Fig. 1.7. A damper is assumed to have neither mass nor elasticity. To determine causes of damping in a practical system, damping is modelled as one or more of the following types:

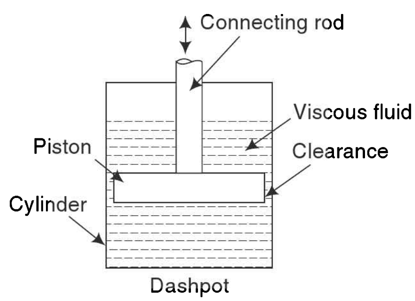


Fig. 1.7 *Damper*

1. Viscous damping
2. Coulomb damping or dry friction damping
3. Structural damping or solid or hysterisis or material damping
4. Slip or Interfacial damping
5. Proportional damping. (See Sec. 3.3.1, Chapter-3).

1.3 TYPES OF VIBRATIONS

Vibration in a system can be classified as follows.

1. *Free vibration, or natural vibration (ω_n)*

When no external force acts on a body after giving it an initial displacement then the body is said to be under free or natural vibration (no external force and no damping) as shown in Fig. 1.8.

Based on movement of the particles of the body, the free vibrations are also classified into three important types as follows.

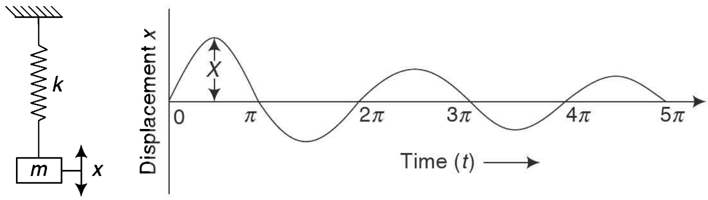


Fig. 1.8 *Free vibration*

- **(a) Rectilinear, or longitudinal vibration** When the particles of a body or shaft moves parallel to the axis of the body or shaft then the vibrations are known as longitudinal vibration [Fig. 1.9(a)].
- **(b) Lateral, or transverse vibration** When the particles of a body or system move approximately perpendicular to the axis of the body or system then the vibrations are known as lateral or transverse vibration [Fig. 1.9(b)].
- **(c) Torsional vibration** When the particles of the body or system move in a circular about its axis of the body or system then the vibration is known as torsional vibration [Fig. 1.9(c)].

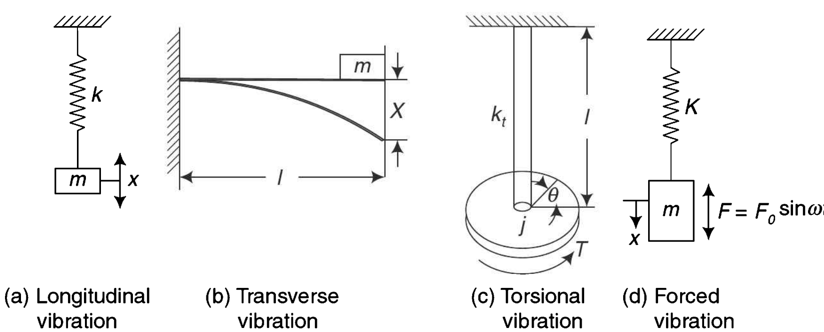


Fig. 1.9 *Types of vibrations*

2. *Forced vibration*

When the body vibrates under the influence of an external force ($F = F_o \sin \omega t$ or $F = F_o \cos \omega t$) then the body is said to be under forced vibrations (damping and external force are present) as shown in Fig. 1.9(d).

Table 1.1 *Difference between rectilinear and torsional vibrations*

Rectilinear system Fig. 1.9(a)	*Torsional system Fig. 1.9(c)*
1. Mass (m) kg	1. Mass MI (J) kg – m^2
2. Stiffness (k) N/m	2. Torsional stiffness (k_t) N-m/rad
3. Damping coefficient (c) N-m/s	3. Torsional damping coefficient (c_t) N-m/rad/s
4. Inertia force ($m\ddot{x}$)	4. Inertia torque ($T\ddot{\theta}$)
5. Restoring force (kx)	5. Restoring torque ($k_t\theta$)
6. Damping force ($c\dot{x}$)	6. Damping torque ($c_t\dot{\theta}$)

Table 1.2 *Difference between free and forced vibrations*

Free vibration Fig. 1.9(a)	*Forced vibration Fig. 1.9(d)*
1. Free vibration takes place without the application of external force.	1. Forced vibration takes place with the application of external force.
2. The frequency with which the system is vibrating is called natural frequency.	2. The frequency with which the system is vibrating is called forced frequency.
3. The natural frequency is a property of a system and it is constant for a particular system.	3. There can be any number of forced frequencies for a particular system.

3. Damped vibration When there is a reduction in the amplitude over every cycle of vibration of a vibrating body, it is said to be under damped or transient vibration (damping force is present and no external force) as shown in Fig. 1.10.

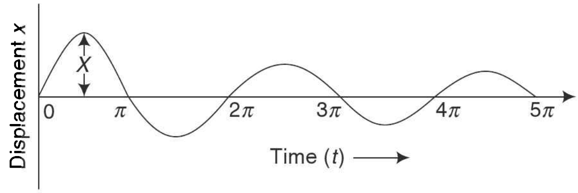


Fig. 1.10 *Damped vibration*

Note: The inertia force and elastic force are always present in all the cases.

4. Linear vibration If all the basic components of a vibratory system behave linearly, the resulting vibration is known as linear vibration. The differential equations that govern a linear vibratory system are linear. If the vibration is linear the principles of superposition hold and mathematical techniques of analysis are well developed.

Example Spring-mass and damper system

5. Nonlinear vibration If any of the basic components of a vibratory system behave nonlinearly, the resulting vibration is known as nonlinear vibration. The differential equations that govern a nonlinear vibratory system are nonlinear. If the vibration is nonlinear the principles of superposition don't hold good.

6. Deterministic vibration If the magnitude of excitation on a vibratory system is known at any given time, the resulting vibration is known as deterministic vibration.

7. Nondeterministic vibration, or random vibration If the magnitude of excitation acting on a vibratory system at any given time can't be predicted, the resulting vibration is known as nondeterministic vibration or random vibration.

Example Road roughness, wind velocity and ground motion during earthquakes.

1.3 DEGREES OF FREEDOM

The number of independent coordinates in a system required to describe the motion of a system at any instant is called degrees of freedom.

A body with a certain mass and elastically supported can undergo a maximum of three translations in the coordinate directions, i.e. three mutual directions (x, y and z) and three rotations about the coordinate directions(ω_x, ω_y and ω_z). Hence, it can have six independent motions in general. Therefore, six coordinates are required to describe the position of a body as it moves in general. Hence, **'the number of independent coordinates in a system to describe the motion of a system at any instant is called its degrees of freedom',** i.e. how many mass or masses will be there in a system? Each mass will have its own natural frequency (ω_n). Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom as shown in Fig. 1.11(h).

Example

(a) Single-degree-freedom system One independent coordinate is required to describe the motion of the system and it will have one natural frequency, as in Fig. 1.11(a), (b).

(b) Two degree-freedom system Two independent coordinates, in a system are required to describe the motion and it will have two natural frequencies as in Fig. 1.11(c), (d).

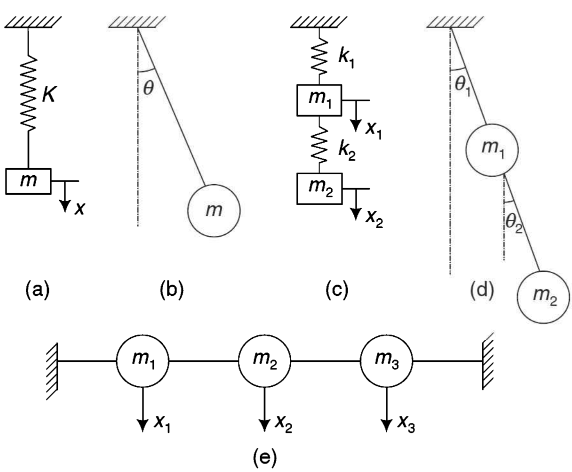


Fig. 1.11 *Degrees of freedom*

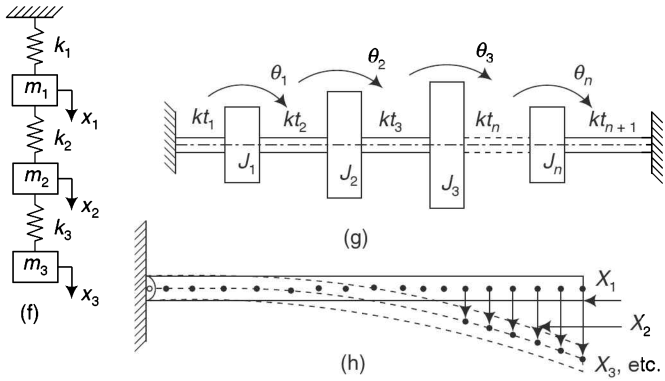


Fig. 1.11 *Contd.*

(c) Discrete or multidegree-freedom system An 'n' degree-freedom system will have 'n' natural frequencies as in Fig. 1.11(e), (f). (g).

(d) Distributed mass or continuous-degree freedom system In general, the number of degrees of a freedom system can be stated as the number of mass or masses in the system and number of possible types of motion of each mass or masses, as in Fig. 1.11(h).

1.10 HARMONIC MOTION AND ITS PROPERTIES

Periodic motions may be harmonic or nonharmonic. In fact, a harmonic motion is the simplest form of periodic motion. The motion represented by the circular functions, sine or cosine, is a harmonic motion. **All harmonic motions are periodic, but not all periodic motions are harmonic.**

It has been found by Fourier that any periodic motion may be converted into harmonic motions in the form of a series consisting of sine and cosines having frequencies which are multiples of the frequency of a given motion. The harmonic motion having the same frequency as that of the given periodic motion is called the **fundamental harmonic**. The harmonic motion having a frequency twice that of the fundamental is called the '**second harmonic**', and so on.

Let $x = X \sin \omega t$ represent a harmonic motion as shown graphically in Fig. 1.12 in which the motion (x) is obtained by the projection of a rotating vector 'X' on the vertical diameter. If projection on the horizontal diameter is considered, the motion becomes $x_i = X \cos \omega t$, which is also a harmonic as it moves around a circle with constant angular speed 'ω' rad/s as shown in Fig. 1.12.

1.10.1 Graphical Representation of Harmonic Motion

Since the circular function repeats itself in '2π' radians, a cycle is complete when

$$\omega \tau = 2\pi, \text{ or } \tau = \frac{2\pi}{\omega} \quad \ldots 1.1\text{a}, \quad \text{where } \tau = \text{Time period in seconds}$$

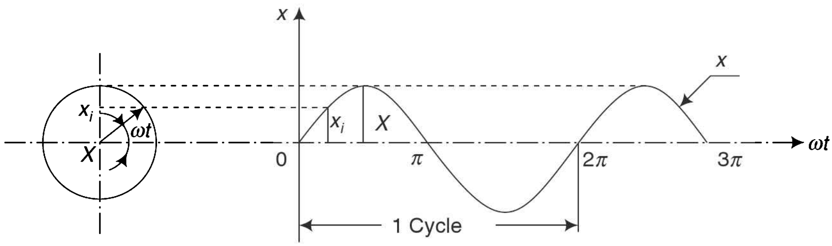


Fig. 1.12 *Graphical representation of harmonic motion*

Therefore, by definition, $$f = \frac{1}{\tau} = \frac{\omega}{2\pi} \qquad ...1.1b$$

The velocity and acceleration at any time are given by,

$$\text{Velocity} = \frac{dx}{dt} = \dot{x} = \omega X \cos \omega t = \omega X \sin(\omega t + \pi/2) \qquad ...1.2$$

$$\text{Acceleration} = \frac{d^2x}{dt^2} = \ddot{x} = -\omega^2 X \sin \omega t = \omega^2 X \sin(\omega t + \pi) \qquad ...1.3$$

Equations 1.1 and 1.2 show that in harmonic motion, the velocity and acceleration are also harmonics having the same frequency as that of the displacement(x) but leading the displacement by 90° ($\pi/2$) and 180° (π) respectively. Also the magnitudes of velocity and acceleration are respectively represented by ω and ω^2 times that of the displacement magnitude. This is shown graphically in Fig. 1.12.

1.10.2 Displacement (x), Velocity ($\dot{x}$) and Acceleration ($\ddot{x}$) in Harmonic Motion

Equation 1.3 can also be written as

$$\frac{d^2x}{dt^2} = -\omega^2 X \sin \omega t = -\omega^2 x \qquad ...1.4$$

i.e. acceleration $\ddot{x} = -\omega^2 x$ or $\ddot{x} + \omega^2 x = 0$

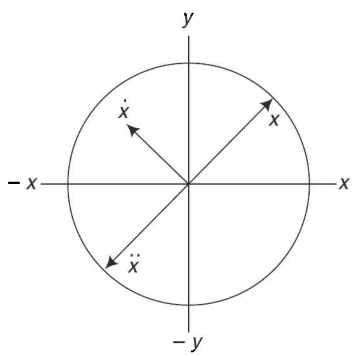


Fig. 1.13 *Relative positions of displacement, velocity and acceleration in harmonic motion*

Therefore, it can be said that in harmonic motion, acceleration is proportional to displacement and is always directed to the origin [as –ve sign appears in the above equation 1.4]. This property of a harmonic motion is used sometimes to define it.

Equation 1.4 can be expressed in a most general form:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ or } \ddot{x} + \omega^2 x = 0 \qquad ...1.5$$

which is called the 'differential equation of motion'. Equation 1.5 has been derived by assuming a harmonic motion. Conversely, it can be said that for a system governing Eq. 1.5, the motion of the system must be harmonic.

1.10.3 Examples of Simple Harmonic Motion

1. ***Linear*** Piston, slider-crank mechanism, liquid in U-tube, spring-mass system, etc.
2. ***Angular***
   (a) The to-and-fro motion of a simple pendulum is simple harmonic for small amplitudes.
   (b) When a rigid body capable of rotating freely about a horizontal axis through it is displaced through on a small angle and let go, the body executes simple harmonic motion.
   (c) The oscillations of a magnet suspended in a magnetic field are simple harmonic.
   (d) When a wire fixed at the top end and loaded at the lower end is given a small twist at lower end and released, the torsional oscillations are found to be simple harmonic.

1.11 SIMPLE HARMONIC MOTION (SHM)

Consider the simplest equation of a vibrational motion as $x = X \sin \omega t$ ...1.6

as shown in Fig. 1.14, where 'x' is the displacement at time 't' seconds and X = Amplitude, ω = Angular velocity in rad/s.

The velocity at time 't' seconds $= \frac{dx}{dt} = \dot{x} = X\omega \cos \omega t$ ...1.7

The acceleration at time 't' seconds $= \frac{d^2x}{dt^2} = \ddot{x} = -X\omega^2 \sin \omega t$ or ...1.8

$$\ddot{x} = -\omega^2 x$$

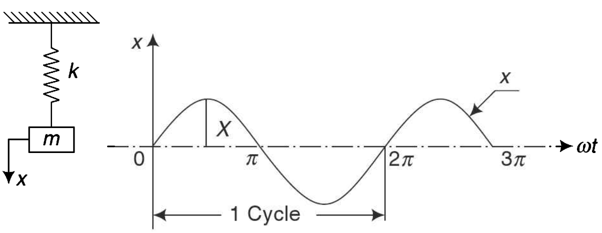


Fig. 1.14 *Linear vibration of spring-mass system with a graph*

$$\therefore \ddot{x} = -\omega^2 X \sin \omega t, \quad \ddot{x} = -\omega^2 x, \quad \ddot{x} + \omega^2 x = 0 \qquad \text{...1.9}$$

For above example, in case of simple-spring mass system

$$m\ddot{x} + kx = 0 \quad \text{or } \ddot{x} + \frac{k}{m} x = 0, \quad \omega^2 = \frac{k}{m} \quad \therefore \omega = \sqrt{\frac{k}{m}} \text{ rad/s} \qquad \text{...1.10}$$

Such a vibration where the acceleration is directly proportional to the displacement and is directed towards the mean position (–ve sign gives the direction towards the mean position) is called simple harmonic motion (SHM).

$x = X \cos \omega t$ is another example.

1.11.1 Solution of Differential Equation of a Body Executing Simple Harmonic Motion

For a particle executing simple harmonic motion, the differential equation of motion is given by $\ddot{x} + \omega^2 x = 0$ ...1.11

The solution of differential equation is given by,

$$x = A \sin \omega t + B \cos \omega t \qquad \text{...1.12}$$

where 'A' and 'B' are arbitrary constants which are to be determined from initial conditions.

Taking the initial conditions, $x = x_0$, at $t = 0$ ...(i) and $\dot{x} = \dot{x}_0$, at $t = 0$ ...(ii)

Substituting the values of Eq. (i) in Eq. 1.12, we get

$$x_0 = A \sin(\omega \times 0) + B \cos(\omega \times 0), \; x_0 = B$$

Differentiating Eq. 1.12 w. r. t. time 't',

$$\dot{x} = A\omega \cos \omega t - B\omega \sin \omega t \qquad \text{...1.13}$$

Substituting the values of Eq. (ii) in Eq. 1.13, we get

$$\dot{x}_0 = A\omega \cos(\omega \times 0) - B\omega \sin(\omega \times 0), \; x_0 = A\omega, \; A = \frac{x_0}{\omega}$$

Substituting the values of 'A' and 'B' in Eq. 1.12, we get

$$x = \frac{x_0}{\omega} \sin \omega t + x_0 \cos \omega t$$

This is the complete solution of the differential equation

$$x = A \sin \omega t + B \cos \omega t$$

where $\quad A = X \cos \phi$ ...(iii), $\quad B = X \sin \phi$ ...(iv)

Squaring Eqs. (iii) and (iv) and adding, we get

$$A^2 + B^2 = X^2 (\cos^2\phi + \sin^2\phi)$$

$$A^2 + B^2 = X^2 \quad \therefore X = \sqrt{A^2 + B^2}$$

The solution of the differential equation $\ddot{x} + \omega^2 x = 0$ can also be given by

$$x = X \sin(\omega t + \phi) \qquad \text{...1.14}$$

where 'X' and 'ϕ' are two arbitrary constants which are to be determined by imposing initial conditions.

1.11.2 Alternate Method to Determine the Complete Solution of Simple Harmonic Motions

We have, acceleration $\ddot{x} = -\omega^2 x$

i.e. $\ddot{x} + \omega^2 x = 0$ ...1.15

The general solution of Eq. 1.11 can also be written as

$$x = X\sin(\omega t + \phi) \quad ...1.16$$

Here 'X' and 'ϕ' are the two arbitrary constants to be determined from the initial conditions.

The equation 1.16 can be expressed as $x = X\sin\omega t\cos\phi + X\cos\omega t\sin\phi$

$$x = \sin\omega t\,(X\cos\phi) + \cos\omega t\,(X\sin\phi),\ x = X\sin\omega t\cos\phi + X\cos\omega t\sin\phi$$

$$x = A\sin\omega t + B\cos\omega t$$

where $A = X\cos\phi$ ...(i) $B = X\sin\phi$ ...(ii)

Squaring Eqs. (i) and (ii) and adding, we get

$$A^2 + B^2 = X^2(\cos^2\phi + \sin^2\phi)$$

$$A^2 + B^2 = X^2 \quad \therefore X = \sqrt{A^2 + B^2}$$

Dividing Eq. (ii) by Eq. (i), $\tan\phi = \frac{B}{A} \quad \therefore \phi = \tan^{-1}\left(\frac{B}{A}\right)$

1.11.3 Addition of two Simple Harmonic Motions

There are two methods for addition of two simple harmonic motions.

1. Analytical method 2. Graphical method

1. Analytical method When we add two harmonic motions of same frequencies, the resulting motion is also a harmonic motion.

Consider two harmonic motions of amplitudes 'X_1' and 'X_2' having same frequencies and phase difference 'ϕ' and are given by

$$x_1 = X_1\sin\omega t,\ x_2 = X_2\sin(\omega t + \phi)$$

The addition of these two SHMs is given as

$$x = x_1 + x_2 = X_1\sin\omega t + X_2\sin(\omega t + \phi)$$

Expand the term, $X_2\sin(\omega t + \phi)$

$$= X_2\sin\omega t\cos\phi + X_2\cos\omega t\sin\phi$$

$$x = X_1\sin\omega t + X_2\sin\omega t\cos\phi + X_2\cos\omega t\sin\phi$$

$$x = \sin \omega t(X_1 + X_2 \cos \phi) + X_2 \cos \omega t \sin\phi \qquad ...1.17$$

Substituting $X_1 + X_2 \cos \phi = X \cos \theta$ ...(i)

$X_2 \sin \phi = X \sin \theta$ ...(ii) in Eq. 1.17, we get

$\therefore \quad x = X \sin \omega t \cos \theta + X \cos \omega t \sin \theta \quad \therefore\ x = X \sin (\omega t + \theta)$

Hence, the resultant is also a simple harmonic motion of amplitude 'X' and phase angle θ.

2. Graphical method, or vectors method The graphical representation of the two vectors and resultant vector is represented as shown in Fig. 1.15.

The value of 'X' is given by

$$X \sin \theta = X_2 \sin \phi, \quad X \cos \theta = X_1 + X_2 \cos \phi \qquad ...1.18$$

Squaring and adding the equation 1.18, we have

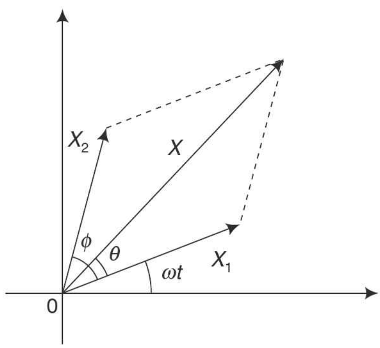


Fig. 1.15 *Graphical representation of the two vectors in simple harmonic motions*

$$X^2 (\sin^2 \phi + \cos^2 \phi) = (X_2 \sin \phi)^2 + (X_1 + X_2 \cos \phi)^2$$

$$X = \sqrt{(X_1 + X_2 \cos \phi)^2 + (X_2 \sin \phi)^2}$$

$$X = \sqrt{X_1^2 + X_2^2 \cos^2 \phi + X_2^2 \sin^2 \phi + 2\, X_1 X_2 \cos \phi}, X = \sqrt{X_1^2 + X_2^2 + 2X_1 X_2 \cos \phi}$$

The resultant phase difference is given by $\tan \theta = \dfrac{X_2 \sin \phi}{X_1 + X_2 \cos \phi}$

$$\theta = \tan^{-1}\left(\frac{X_2 \sin \phi}{X_1 + X_2 \cos \phi}\right)$$

1.11.4 Difference between Periodic Motion and Harmonic Motion

A periodic motion is that which repeats itself at equal intervals of time called the **time period**. Periodic motion may be harmonic or nonharmonic.

In fact, a harmonic motion is the simplest form of periodic motion. The motion represented by the circular functions, sine or cosine, is a harmonic motion.

In harmonic motion, the particle under consideration moves such that its acceleration is always proportional to its displacement and is always directed towards a fixed point.

All harmonic motion is periodic, but not all periodic motion is harmonic.

1.11 THE BEATS PHENOMENON

The addition of two harmonic motions of the same frequency is a harmonic motion. But when the frequencies are different, the resultant motion is nonharmonic. A special case occurs when the frequencies of the two motions to be added together are very near to each other.

Let us consider two harmonic motions given by $x_1 = a \sin \omega_1 t$ and $x_2 = b \sin \omega_2 t$, as shown in Fig. 1.16(a).

When two harmonic motions, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as beats. This is not a simple harmonic motion but similar.

Let us consider two waves of the same amplitude 'X' and slightly different frequencies as shown in Fig. 1.16(b) given by $x_1 = X \cos \omega t$ and $x_2 = X \cos (\omega + \delta)t$ where 'δ' is a small quantity. The addition of these two motions gives

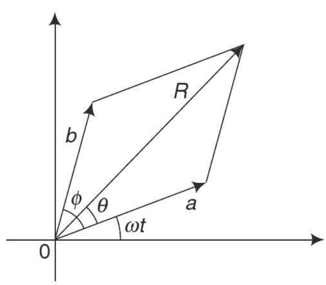
$$x = x_1 + x_2 = X[\cos \omega t + \cos (\omega + \delta)\, t]$$

$$\because \quad \cos A + \cos B = 2 \cos\left(A + \frac{B}{2}\right) \cos\left(A - \frac{B}{2}\right)$$

$$x = 2X\left[\cos\left(\frac{\omega + \omega + \delta}{2}\right) t \cos\left(\frac{\delta}{2}\right)t\right] = 2X \cos\left(\frac{\delta t}{2}\right) \cos\left(\omega + \frac{\delta}{2}\right)t$$

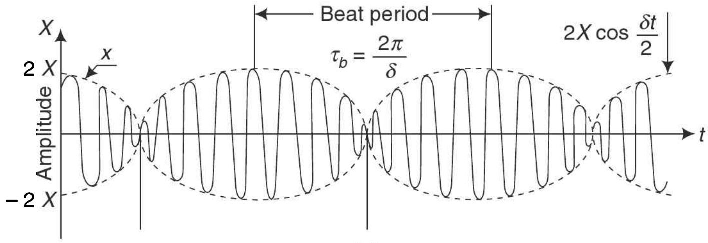
The equation is shown graphically in Fig. 1.16(a).

The resulting motion 'X' represents cosine wave with frequency $\left(\omega + \dfrac{\delta}{2}\right) \approx \omega$ and with varying amplitude $2X \cos \dfrac{\delta t}{2}$.



(a)

Fig. 1.16 *Beats phenomenon*



(b)

Fig. 1.16 *Contd.*

The frequency 'δ' at which the amplitude builds up and dies down between '0' and '2*X*' is known as **beat frequency** .

Beat phenomenon is found in machines, structures and electric powerhouses. In machines, the beat phenomenon occurs when the frequency is close to the natural frequency of the system.

EXAMPLE 1.1

Add the following harmonic motions analytically and check the solution graphically.

$$x_1 = 3 \sin(\omega t + 30°), x_2 = 4 \cos(\omega t + 10°).$$

Solution Let 'x' be the resultant motion of x_1 and x_2 $\therefore x = x_1 + x_2$

$$x = 3 \sin(\omega t + 30°) + 4 \cos(\omega t + 10°)$$

In the given harmonic motion, the frequency is same (i.e. ωt) for both x_1 and x_2
Therefore, the resultant motion can also be written as $x = A \sin(\omega t + \theta)$

$$\therefore \quad A \sin(\omega t + \theta) = 3 \sin(\omega t + 30°) + 4 \cos(\omega t + 10°)$$

Expanding the equation, we have

$$A \sin \omega t \cos\theta + A \cos \omega t \sin\theta = 3 \sin \omega t \cos 30° + 3 \cos \omega t \sin 30° + 4 \cos \omega t \cos 10° - 4 \sin \omega t \sin 10°$$

Rearranging above terms,

$$\sin \omega t (A \cos\theta) + \cos \omega t (A \sin\theta) = 1.904 \sin \omega t + 5.44 \cos \omega t$$

By equating the corresponding coefficients of $\cos \omega t$ and $\sin \omega t$ on both the sides,

$$A \cos\theta = 1.904 \text{ ...(a)} \quad A \sin\theta = 5.44 \text{ ...(b)}$$

Now squaring and adding the equations (a) and (b), we have

$$A^2 \cos^2\theta + A^2 \sin^2\theta = (1.904)^2 + (5.44)^2$$

$$A^2(\cos^2\theta + \sin^2\theta) = (1.904)^2 + (5.44)^2$$

$$A^2 = 33.208 \quad \therefore A = 5.763$$

Dividing Eq. (b) by Eq. (a), we have

$$\tan\theta = \frac{5.44}{1.904} \quad \therefore\ \theta = 70.712°$$

$$\therefore \quad x = 5.763 \sin(\omega t + 70.712°)$$

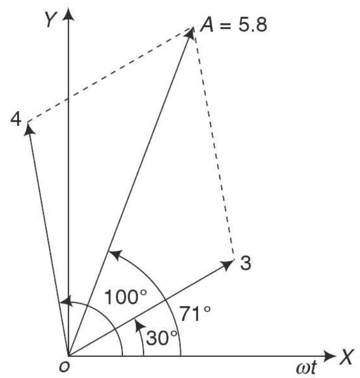


Fig. p-1.1 *Graphical representation of the two vectors in simple harmonic motions*

Graphical method is as shown in Fig. p-1.1, $x_1 = 3\sin(\omega t + 30°)$

$x_2 = 4\cos(\omega t + 10°) = 4\sin(\omega t + 10 + 90°) = 4\sin(\omega t + 100°)$

$\therefore$ $x = 5.8\sin(\omega t + 71°)$ which agrees closely with the analytical results.

EXAMPLE 1.2

A harmonic motion is given by $x(t) = 10\sin(30t - \pi/3)$mm, where '$t$' is in seconds and phase angle in radians. Determine (i) frequency and period of motion, and (ii) maximum displacement, velocity and acceleration.

Solution The given harmonic motion is $x(t) = 10\sin(30t - \pi/3)$ ...(a)

Let the harmonic motions be in the form $x = A\sin(\omega t - \phi)$ ...(b)

where 't' is in seconds, A = Maximum displacement in mm, ω = Frequency in rad/s and ϕ = Phase angle in radians

Comparing the equations (a) and (b), A = 10 mm, ω = 30 rad/s, $\phi = \pi/3 = 60°$

For the maximum velocity, differentiate Eq. (b) w.r.t. time 't'

$$x = A\sin(\omega t - \phi) \quad ...(c), \dot{x} = \omega A = 30 \times 10 = 300 \text{ mm/s}^2$$

And for the maximum acceleration, again differentiate Eq. (b) w.r.t. time 't'

$$\ddot{x} = -\omega^2 A = (-30)^2 \times 10 = 900 \text{ mm/s}^2$$

Time period $$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{30} = 0.21 \text{ second}$$

EXAMPLE 1.3

A body is subjected to two harmonic motions: $x_1 = 15 \sin\left(\omega t + \frac{\pi}{6}\right), x_2 = 8 \cos\left(\omega t + \frac{\pi}{3}\right)$. What harmonic motion should be given to the body to bring it to equilibrium?

Solution Let us consider $x_3 = A \sin(\omega t + \phi)$ be the extra harmonic motion given to bring it to equilibrium position, i.e. $x_1 + x_2 + x_3 = 0$

$x_1 = 15 \sin\left(\omega t + \frac{\pi}{6}\right)$ or $15 \sin(\omega t + 30°)$, $x_2 = 8 \cos\left(\omega t + \frac{\pi}{3}\right)$ or $8 \cos(\omega t + 60°)$ and

$x_3 = A \sin(\omega t + \phi)$

Now, $15 \sin(\omega t + 30°) + 8 \cos(\omega t + 60°) + A \sin(\omega t + \phi) = 0$

Expanding the above terms:

$15 \sin \omega t \cos 30° + 15 \cos \omega t \sin 30° + 8 (\cos \omega t \cos 60° - \sin \omega t \sin 60°) + A(\sin \omega t \cos \phi + \cos\phi \sin \omega t) = 0$

$15 \sin \omega t\left(\frac{\sqrt{3}}{2}\right) + 15 \cos \omega t\left(\frac{1}{2}\right) + 8\left(\cos \omega t\left(\frac{1}{2}\right) - \sin \omega t\left(\frac{\sqrt{3}}{2}\right)\right) + A(\sin \omega t \cos \phi + \cos \omega t \sin \phi) = 0.7 \sin \omega t \times \frac{\sqrt{3}}{2} + \frac{23}{2} \cos \omega t + A \sin \omega t \cos \phi + A \cos \omega t \sin \phi = 0$

$$\sin \omega t\left(\frac{7\sqrt{3}}{2} + A \cos \phi\right) + \cos \omega t\left(A \sin \phi + \frac{23}{2}\right) = 0$$

$$\sin \omega t (A \cos \phi + 6.06) + \cos \omega t (A \sin \phi + 11.5) = 0$$

$$A \cos \phi = -6.06 \quad \text{...(a)}, \quad A \sin \phi = -11.5 \quad \text{...(b)}$$

Dividing Eq. (b) by Eq. (a), we have

$$\frac{A \sin \phi}{A \cos \phi} = \frac{-11.5}{-6.06} = \tan \phi = 1.894 = 62°\ 17' \text{ or } 180° + 62° = 242°.$$

Now squaring and adding the Eqs. (a) and (b), we have

$$A^2 (\cos^2 \phi + \sin^2 \phi) = (6.06)^2 + (11.5)^2, A^2 = 169 \quad \therefore A = \sqrt{169} = 13.$$

$\therefore$ the equilibrium of harmonic motion $x_3 = 13 \sin(\omega t + 242° \cdot 17')$.

EXAMPLE 1.4

A harmonic motion is given by the equation $x = 5 \sin(4t + \phi)$. Find its two components: one that leads it by 30° and the other that lags it by 80°.

Solution Let x_1 and x_2 be the required components.

Resolve the vectors along x_1 and a direction at right angles to x_1, as shown in Fig. p-1.4.

$$x_1 - x_2 \sin 20° = 5 \cos 80° \quad \text{...(a)}, \quad x_2 \cos 20° = 5 \cos 10°$$

$$\therefore \quad x_2 = \frac{5 \times 0.9848}{0.9397} = 5.24.$$

From Eq. (a), $x_1 = 5.24 \sin 20° + 5 \cos 80°$

$= 1.79 + 0.87 = 2.66,\ x_1 = 2.66$

$x_1 = 2.66 \sin (4t + \phi - 80°),\ x_2 = 6.24 \sin (4t + \phi + 30°)$

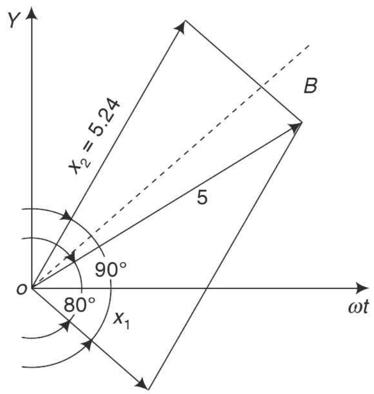


Fig. p-1.4 *Graphical representation of the two vectors in simple harmonic motions*

EXAMPLE 1.5

Find the amplitude of the sum of the two harmonic motions

$x_1 = 3 \cos (2t + 1°)$; $x_2 = 4 \cos (2t + 1.5°)$.

Solution Let $x = X \sin (\omega t + \theta)$ be the resultant of the two harmonic motions.

$$\therefore \qquad x = x_1 + x_2,\ x = 3 \cos (2t + 1°) + 4 \cos (2t + 1.5°)$$

Expanding the above terms,

$$x = 3 \cos 2t \cdot \cos 1° - 3 \sin 2t \cdot \sin 1° + 4 \cos 2t \cdot \cos 1.5° - 4 \sin 2t \cdot \sin 1.5°$$

Rearranging above terms,

$$x = 3 \cos 2t\,(3 \cos 1° - 4 \cos 1.5°) - \sin 2t\,(3 \sin 1° + 4 \sin 1.5°)$$

$$x = \cos 2t\,(x \sin \theta) - \sin 2t\,(x \cos \theta)$$

Equating the coefficient of $\sin \omega t$ and $\cos \omega t = 0$

$$x \sin \theta = 3 \cos 1° + 4 \cos 1.5° \qquad \text{...(a)}$$

$$x \cos \theta = 3 \sin 1° + 4 \sin 1.5° \qquad \text{...(b)}$$

$$x \sin \theta = 6.998 \qquad \text{...(c)}$$

$$x \cos \theta = 0.157 \qquad \text{...(d)}$$

Squaring and adding Eqs. (c) and (d),

$$x^2 (\cos^2 \theta + \sin^2 \theta) = 6.998^2 + 0.157^2,\ x^2 = 48.99,\ x = 6.99$$

Dividing Eq. (c) by Eq. (d), $\dfrac{x \sin \theta}{x \cos \theta} = \dfrac{6.998}{0.157} = \tan \theta = \dfrac{6.998}{0.157} = \therefore\ \theta = 44.58°,$

$$\theta = 88°\ 71'\ x = 6.99 \sin (\omega t + 88°)$$

EXAMPLE 1.6

A particle is under the influence of two harmonic motions, y_1 = 0.03 sin (14t + 68.8°) and y_2 = 0.02 sin (10t + 59.6°). Determine the resulting amplitude and phase angle.

Solution Let the resulting amplitude is

$$X^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2)$$

$$= 0.03^2 + 0.02^2 + 2 \times 0.03 \times 0.02 \times \cos(68.8 - 59.6°), X = 0.05$$

The phase angle of the resulting motion

$$\tan\theta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2} = \frac{0.03 \sin 68.8° + 0.02 \sin 59.6°}{0.03 \cos 68.8° + 0.02 \cos 59.6°} = 65.12°$$

EXAMPLE 1.7

Show that the resultant motion of the three harmonic motions given below is zero.

$$\mathbf{x_1 = A \sin \omega t, x_2 = A \sin(\omega t + 2\pi/3), x_3 = A \sin(\omega t + 4\pi/3)}$$

Solution $x_1 = A \sin \omega t$, $x_2 = A \sin(\omega t + 2\pi/3)$, or $x_2 = A \sin(\omega t + 120°)$

$x_3 = A \sin(\omega t + 4\pi/3)$, or $x_3 = A \sin(\omega t + 240°)$

The resultant harmonic motion is given by $x = x_1 + x_2 + x_3$

$$x = A \sin \omega t + A \sin(\omega t + 120°) + A \sin(\omega t + 240°)$$

Expanding the above terms,

$$= A \sin \omega t + A \sin \omega t \cos 120° + A \cos \omega t \sin 120°$$

$$+ A \sin \omega t \cos 240° + A \cos \omega t \sin 240$$

$$x = A \sin \omega t - 0.5\, A \sin \omega t + 0.866\, A \cos \omega t$$

$$-0.5\, A \sin \omega t - 0.866\, A \cos \omega t = 0$$

Hence, the resultant motion of the three harmonic motions is zero.

EXAMPLE 1.8

Split up the harmonic motion x = 20 sin (ωt + π/6) into two harmonic motions, one having a phase angle of zero and other having a phase angle of 50°.

Solution $x = 20 \sin(\omega t + \pi/6) = 20 \sin(\omega t + 30°)$

Let $x_1 = X_1 \sin(\omega t + 0) = X_1 \sin \omega t$ and $x_2 = X_2 \sin(\omega t + 50°)$

$$x = x_1 + x_2$$

$$20 \sin(\omega t + 30°) = X_1 \sin \omega t + X_2 \sin(\omega t + 50°)$$

Expanding the above terms and equating the coefficients of sin ωt and cos ωt, we get

$$20 \sin \omega t \cos 30° + 20 \cos \omega t \sin 30° = X_1 \sin \omega t + X_2 \sin \omega t \cos 50° + X_2 \cos \omega t \sin 50°$$

$$\therefore \quad X_1 = 8.93, X_2 = 13.055$$

$$x_1 = 8.93 \sin \omega t, x_2 = 13.055 \sin (\omega t + 50°).$$

EXAMPLE 1.9

The displacement of a vibrating body is given by 5 sin(31.415*t* + π/4).

Draw the variation of displacement for one cycle of vibration and also determine the displacement of the body after 0.11 second.

Solution The given equation of motion, $x = 5 \sin (31.415t + \pi/4)$

The general solution is $x = X \sin (\omega t + \phi)$. Here, $\omega = 31.415$ rad/s.

Also, $\omega = 2\pi f$ or $31.415 = 2\pi f$ $\therefore$ $\omega = 5$ cyc/s.

Also the time period $\tau = 1/f = 1/5 = 0.2$ s.

By equation of motion, $x = 5 \sin (31.415t + \pi/4)$

At	$t = 0$,	$x = 3.53$.	$t = 0.025$,	$x = 5$.	$t = 0.05$,	$x = 3.53$.
	$t = 0.075$,	$x = 0$.	$t = 0.1$,	$x = -3.53$	$t = 0.125$,	$x = -5$.
	$t = 0.15$,	$x = -3.53$.	$t = 0.175$,	$x = 0$.	$t = 0.2$,	$x = 3.53$,

Displacement of the body after 0.11 second, $x = 5 \sin (31.415t + \pi/4)$, $x = -4.45$.

Displacement versus time graph for one complete cycle of vibration is shown in Fig. p-1.9.

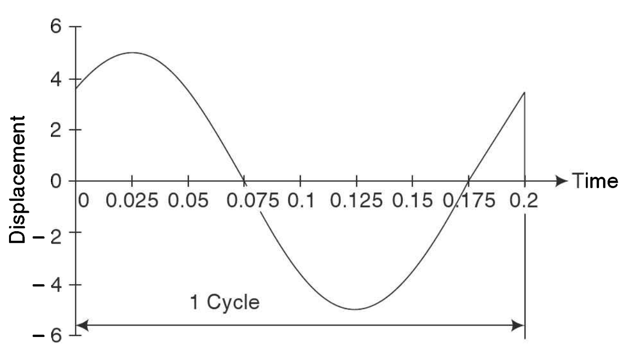


Fig. p-1.9 *Displacement versus time graph*

Note: If the function is cosine then the curve starts from the negative sign below the mean line.

1.13 HARMONIC ANALYSIS—FOURIER'S SERIES

In actual experiments on vibrating systems, a plot is made of the vibration the system is undergoing. Such a plot is generally periodic. This periodic function needs to be analysed for knowing the characteristics of the vibration under study. Analysing this function in terms of its harmonics is called 'harmonic analysis.'

The French mathematician and physicist. Jacques Fourier (1768–1830) was the first to use a certain series in his work, now called the Fourier series. Fourier has shown that any periodic function can be represented by a series of sines and cosines. It is this work of Fourier which helps in analysing the experimentally obtained vibrations and plots analytically.

If $x(t)$ is a periodic function with time period 'τ' then it is represented by the Fourier series as

$$x(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\,\omega t + a_3 \cos 3\,\omega t + \dots\dots$$

$$b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \dots\dots$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t] \qquad \dots 1.19$$

where $\omega = 2\pi/\tau$ is the fundamental or first harmonic frequency.

The term $(a_1 \cos \omega t + b_1 \sin \omega t)$ is the fundamental or first harmonic.

The term $(a_2 \cos 2\omega t + b_2 \sin 2\omega t)$ is the second harmonic, and so on.

The various coefficients $a_1\, a_2 \dots b_1\, b_2 \dots$ of the individual sine waves and the constant a_0 is called the Fourier constant and can be determined analytically or numerically when $x(t)$ is given.

Note: To write $\frac{a_0}{2}$ instead of 'a_0' is a conventional device to be able to get more symmetric formulae for the coefficient.

1. *To determine a_0* Integrate both sides of Eq. 1.19 over any interval of time $\tau = 2\pi/\omega$. Now according to the above formula, all the integrals on the right-hand side (RHS) of the equation are **zero** except the one containing **'a_0,'**

i.e. $$\int_0^{\tau} x(t)\,dt = \int_0^{\tau} (a_0)\,dt, = (a_0)t\Big]_0^{\tau}, \int_0^{\tau} x(t)\,dt = a_0.\tau$$

Since $\tau = 2\pi/\omega$,

$$\int_0^{\tau} x(t)\,dt = a_0 \frac{2\pi}{\omega} \quad \therefore\ a_0 = \frac{\omega}{2\pi}\int_0^{2\frac{\pi}{\omega}} x(t)\,dt \qquad \dots 1.20$$

2. *To determine a_n* Multiply both sides of Eq. 1.19 by '$\cos n\omega t$' and integrate over any interval of time $\tau = \frac{2\pi}{\omega}$. Then all the integrals on the right-hand side (RHS) are **zero** except the one containing **'a_n,'**

i.e. $$a_n = \int_0^{\frac{2\pi}{\omega}} x(t) \cos n\,\omega t \cdot dt = \int_0^{\frac{2\pi}{\omega}} a_n \cos^2 (n\,\omega t)\cdot dt = \int_0^{\frac{2\pi}{\omega}} a_n \left[\frac{1 + \cos 2\,n\omega t}{2n\omega}\right] dt$$

$$= \frac{a_n}{2}\left[t + \frac{\sin 2n\omega t}{2n\omega}\right]_0^{2\pi/\omega} = a_n \frac{\pi}{\omega} \text{ or } a_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\cdot \cos (n\omega t)\,dt \qquad \dots 1.21$$

3. To determine b_n Multiply both the sides of Eq. 1.19 by 'sin $n\omega t$' and integrate over any interval of time $\tau = \frac{2\pi}{\omega}$. Then all the integrals on the right-hand side (RHS) are **zero** except the one containing 'b_n',

i.e. $$b_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\cdot \sin(n\omega t)\, dt \qquad ...1.22$$

Substituting the values of a_0, a_n and b_n in Eq. 1.19, we get

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos n\omega t + b_n \sin n\,\omega t] \qquad ...1.23$$

Note: In case of even function if $x(t) = x(-t)$, 'a_0' and 'a_n' will be present and $b_n = 0$, i.e. symmetric.

If $x(t) = -x(-t)$, only 'b_n' will be present and $a_0 = a_n = 0$, is called the **odd function**, i.e. unsymmetric.

EXAMPLE 1.10

Determine the Fourier series for the curve as shown in Fig. p-1.10(a).

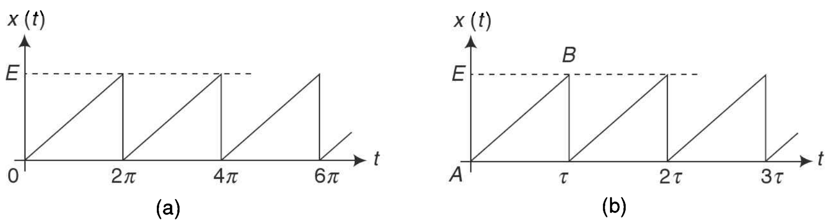


Fig. p-1.10 *Sawtooth curve*

Solution The time taken by the vector to rotate through 2π radians is called time period (τ). So the above motion can be represented as shown in Fig. p-1.10(b).

$$\tau = \frac{2\pi}{\omega} \quad \text{Or } \omega = \frac{2\pi}{\tau}$$

The equation for straight line 'AB' is given by $y = mx + c$, $\left(\frac{E}{\tau}\right)t + 0$

$0 \le x \le \tau$, $c = 0$ since it starts from origin.

The Fourier series is represented by the equation for the above motion as

$$x(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\,\omega t + ...$$
$$b_1 \sin \omega t + b_2 \sin 2\,\omega t + ...$$
$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega t + b_n \sin n\,\omega t) \qquad \text{(I)}$$

To determine a_0 The equation for $a_0 = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\, dt$ or $a_0 = \int_0^{\frac{2\pi}{\omega}} \left(\frac{E}{\tau}\right)t.\, dt$

$$= \frac{\frac{2\pi}{\tau}}{2\pi} \int_0^{\tau} \left(\frac{E}{\tau}\right) t.\, dt = \frac{1}{\tau} \cdot \frac{E}{t} \int_0^{\tau} t.\, dt = \frac{E}{\tau^2}\left[\frac{t^2}{2}\right]_0^{\tau} \quad \therefore\ a_0 = \frac{E}{\tau^2}\left[\frac{\tau^2}{2}\right] = \frac{E}{2} \qquad \text{...(a)}$$

Note: To write $\frac{a_0}{2}$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficient.

To determine a_n The equation for $a_n = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) \cos n\omega t\, dt.$

Substituting the values of 'ω' and $x(t)$,

$$a_n = \frac{2\pi}{\tau\pi} \int_0^{\frac{2\pi}{2\pi\tau}} \left(\frac{Et}{\tau}\right) \cos n\omega t\, dt \text{ or } a_n = \frac{2}{\tau} \int_0^{\tau} \frac{E}{\tau} t \cos n\omega t\, dt$$

$$= \frac{2}{\tau}\frac{E}{\tau}\left[t\frac{\sin n\,\omega\tau}{n\omega} + \frac{\cos n\,\omega t}{(n\omega)^2}\right]_0^{\tau}$$

$$= \frac{2E}{\tau^2}\left[\left(\frac{\tau \sin n\,\omega\tau}{n\omega} + \frac{\cos n\omega\tau}{(n\omega)^2}\right) - \left(0 + \frac{1}{(n\omega)^2}\right)\right]$$

$$= \frac{2E}{\tau^2}\left[\left(\frac{\tau \sin n\frac{2\pi}{\tau}\tau}{n\frac{2\pi}{\tau}} + \frac{\cos n\frac{2\pi}{\tau}\tau}{(n\omega)^2}\right) - \frac{1}{(n\omega)^2}\right]$$

$$\frac{2E}{\tau^2}\left[\frac{1}{(n\omega)^2} - \frac{1}{(n\omega)^2}\right] = 0$$

$$\therefore \qquad a_n = 0 \qquad \text{...(b)}$$

To determine b_n The equation for $b_n = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) \sin n\omega t.\, dt.$

Substituting the values of 'ω' and $x(t)$,

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{E}{2\pi}\right) t \sin \omega\, nt\, dt$$

$$= \frac{1}{\pi}\left(\frac{E}{2\pi}\left\{t - \frac{\cos \omega nt}{\omega n} + 1 - \frac{\sin \omega nt}{\omega^2 n^2}\right\}_0^{2\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{E}{2\pi}\left\{\frac{2\pi(-1)}{n} - 0 + 0\right\}\right) = -\frac{E}{n\pi}$$

Fourier series representation of the series is $x(t) = \frac{E}{4} + \sum_{n-1}^{\infty} -\frac{E}{n\pi} \sin n\omega t$

$$= \frac{E}{4} + \left(\frac{-E}{n}\right) \sum_{n-1}^{\infty} \sin\left(\frac{2\pi}{\tau}\right) nt$$

$$\therefore \qquad b_n = 0 \qquad \text{...(c)}$$

For even function, $b_n = 0$, and for odd function, $a_0 = 0$ and $a_n = 0$

Substituting the values of a_0, a_n and b_n in Eq. (I), the Fourier representation of harmonic motion of the given curve is

$$x(t) = \frac{E}{2} - \left(\frac{E}{\pi}\right) \sum_{1}^{\infty} \frac{1}{n} \sin n\omega t$$

EXAMPLE 1.11

Determine the Fourier series for the curve shown in Fig. p-1.11(a).

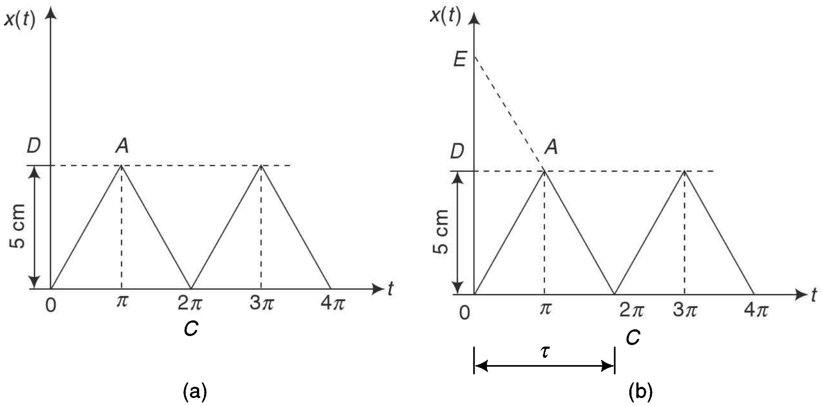


Fig. p-1.11 *Representation of periodic motion of a triangular curve*

Solution The time period 'τ' is (OC) one complete cycle $\tau = \dfrac{2\pi}{\omega}$ second or the frequency $\omega = \dfrac{2\pi}{\tau}$ rad/s.

The equation for the straight line 'OA', $y = mx + c$, $x(t) = (5/\pi)\, t + 0$, $c = 0$

Since start from origin, Fig. p-1.11(a), $x(t) = \left(\dfrac{5}{\pi}\right) t \; 0 \le x \le \pi$

The equation for the straight line 'AC', $y = (-5/\pi)t + (5 + 5)$ is shown in Fig. p-1.11(b), twice the slope, i.e. $(EA + AC) = (5 + 5)$

$x(t) = (-5/\pi)\, t + 10$, slope is –ve, since it is measured in clockwise direction from adjacent side (base).

$$x(t) = \left(\frac{-5}{\pi}\right) t + 10,\ \pi \le x \le 2\pi.$$

The Fourier series is represented by the following equation for the above motion:

$$x(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\,\omega t + \ldots$$

$$b_1 \sin \omega t + b_2 \sin 2\,\omega t + \ldots \quad \text{or}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,\omega t + b_n \sin n\omega t) \qquad \ldots\text{(I)}$$

To determine a_0 The equation for $a_0 = \dfrac{\omega}{2\pi} \displaystyle\int_0^{\tau} x(t)\, dt$

$$a_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{\omega}{2\pi}\left\{\int_0^{\pi} x(t)\,dt + \int_{\pi}^{2\pi} x(t)\cdot dt\right\}$$

$$= \frac{\omega}{2\pi}\left\{\int_0^{\pi}\left(\frac{5}{\pi}\right)t.dt + \int_{\pi}^{2\pi}\left(\frac{-5}{\pi}\right)t.\,dt\right\},\quad a_n = \frac{\omega}{2\pi}\left[\frac{5}{\pi}\left\{\frac{t^2}{2}\right\}_0^{\pi} - \frac{5}{\pi}\left\{\frac{t^2}{2}\right\}_{\pi}^{2\pi} + [10t]_{\pi}^{2\pi}\right]$$

$$= \frac{\omega}{2\pi}\left[\frac{5}{\pi}\left\{\frac{\pi^2}{2}\right\} - \frac{5}{\pi}\left\{\frac{4\pi^2-\pi^2}{2}\right\} + 10(2\pi-\pi)\right] = \frac{\omega}{2\pi}\left[\frac{5\pi}{2} - \frac{15\pi}{2} + 10\pi\right]$$

$$= \frac{1}{2\pi\times 2}[5\pi - 15\pi + 20\pi] = \frac{10\pi}{2\pi\times 2},\ a_0 = \frac{5}{2} = 2.5 \quad ...(a)$$

To determine a_n The equation for

$$a_n = \frac{\omega}{\pi}\int_0^{\tau} x(t)\cos n\omega t.\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos n\,\omega t.\,dt$$

$$= \frac{\omega}{\pi}\left[\int_0^{\pi}\left(\frac{5}{\pi}\right)t\cos n\,\omega t + \int_{\pi}^{2\pi}\left\{-\frac{5}{\pi}t + 10\right\}\cos n\,\omega t\right]$$

$$= \frac{\omega}{\pi}\left[\frac{5}{\pi}\left\{t\frac{\sin nt}{n} + \frac{1.\cos nt}{n^2}\right\}_0^{\pi} + \frac{-5}{\pi}\left\{t\frac{\sin nt}{n} + \cos\frac{nt}{n^2}\right\}_0^{2\pi} + 10\left\{\frac{\sin nt}{n}\right\}_0^{2\pi}\right]$$

$$= \frac{\omega}{\pi}\left[\frac{5}{\pi}\left\{\pi\cos\frac{\pi.\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right\} - \frac{5}{\pi}\left\{0 + \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2}\right\}\right]$$

$$= \frac{\omega}{\pi}\left[\frac{5}{\pi}\left\{\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right\} - \frac{5}{\pi}\left\{\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right\}\right]$$

$$= \frac{\omega}{\pi}\left[\frac{5}{\pi}\frac{\cos n\pi}{n^2} - \frac{5}{\pi n^2} - \frac{5}{\pi n^2} + \frac{5}{n}\frac{\cos n\pi}{n^2}\right] = \frac{1}{\pi}\left[\frac{5}{\pi n^2} - \frac{5}{\pi n^2} - \frac{5}{\pi n^2} - \frac{5}{\pi n^2}\right]$$

$$\therefore\quad a_n = -\frac{20}{\pi n^2}\ ...(b)\ \text{if}\quad n = \text{odd}$$

$a_n = 0$, if n is even.

To determine b_n The equation for

$$b_n = \frac{\omega}{\pi}\int_0^{\tau} x(t)\sin n\omega t.\,dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\sin n\omega t.\,dt$$

$$= \left[\frac{\omega}{\pi}\int_0^{\pi}\left(\frac{5}{\pi}\right)t\cos n\omega t + \left(-\frac{5}{\pi}t + 10\right)\sin n\omega t.\,dt\right]$$

$$= \frac{\omega}{\pi}\left[\frac{5}{\pi} - \left\{-t\frac{\cos nt}{n} + \frac{\sin nt}{n^2}\right\}_0^{\pi} + \frac{-5}{\pi}\left\{t + \frac{\cos nt}{n} + \frac{\sin nt}{n^2}\right\}_0^{2\pi} - \left\{10\frac{\cos nt}{n}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{\omega}{\pi}\left[\frac{5}{\pi}\left\{-\pi\frac{\cos n\pi}{n}\right\} - \frac{5}{\pi}\left\{-\frac{2n\pi}{n} - 0 - \pi\frac{\cos n\pi}{n^2}\right\} - 10\left\{\frac{1}{n} - \frac{\cos nt}{n}\right\}_{\pi}^{2\pi}\right]$$

If n is odd,

$$b_n = \frac{\omega}{\pi}\left[\frac{5}{\pi}\left(\frac{\pi}{n}\right) - \frac{5}{\pi}\left(-\frac{2\pi}{n} - \frac{\pi}{n}\right) - 10\left(\frac{1}{n} + \frac{1}{n}\right)\right], = \frac{1}{\pi}\left[\frac{5}{n} + \frac{10}{n} - \frac{5}{n} - \frac{10}{n} - \frac{10}{n}\right] = 0$$

If n is even,

$$b_n = \left[-\frac{5}{n} + \frac{10\pi}{n} - \frac{5\pi}{n} - \frac{10}{n} + \frac{10}{n}\right] = 0 \qquad ...(c)$$

Substituting the values of a_0, a_n and b_n in Eq. (a), the Fourier representation of harmonic motion of a given curve is

$$x(t) = \frac{5}{2} + \sum_{n-1}^{\infty} \frac{-20}{\pi^2 n^2} \cos n\,\omega t \text{ if '}n\text{' is odd, and } x(t) = \frac{5}{2} \text{ if '}n\text{' is even.}$$

EXAMPLE 1.12

Find the Fourier series for the curve as shown in Fig. p-1.12(a).

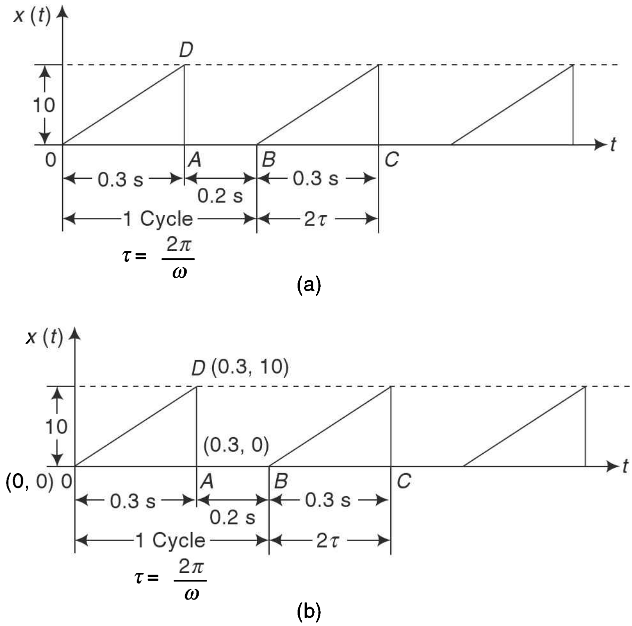


Fig. p-1.12 *Representation of periodic motion*

Solution In Fig. p.1.12(a), one complete cycle =

$$\tau = OA + AB = OB,\ \tau = 0.3 + 0.2 = 0.5 \text{ s}$$

But $\omega = \frac{2\pi}{\tau}$ or $\omega = \frac{2\pi}{0.5} = 4\pi$ rad/s.

[In Fig. 1.12(b), OD is a straight line. The equation of a straight line is given by

$$\left(\frac{y - y_1}{x - x_1}\right) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \left(\frac{x(t) - 0}{t - 0}\right) = \left(\frac{10 - 0}{0.3 - 0}\right)$$

$$\therefore \quad x(t) = \left(\frac{100t}{3}\right) [0 \le x \le 0.3, x(t) = 0 \;\; 0.3 \le x \le 0.5]$$

The straight line '*OD*' is given by the equation $y = mx + c$, $x(t) = (10/0.3)\, t + 0$, $c = 0$, since it starts from origin.

$$x(t) = \frac{10t}{0.3} = \frac{100t}{3} \quad \text{when } 0 \le x \le 0.3$$

$$x(t) = \frac{0}{0.2} = 0 \;\; 0.3 \le x \le 0.5$$

The Fourier series is represented by the equation for the above motion as

$$x(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\,\omega t + \cdots$$
$$b_1 \sin \omega t + b_2 \sin 2\,\omega t + \cdots$$

or $$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega t + b_n \sin n\,\omega t) \quad \text{...(I)}$$

To determine a_0 The equation for $a_0 = \dfrac{\omega}{2\pi} \displaystyle\int_0^{\tau} x(t)\, dt$

Put $\omega = 4\pi$ rad/s, $x(t) = \dfrac{100t}{3}$ and limits $0 \le x \le 0.3$

$$a_0 = \frac{\omega}{2\pi}\int_0^{0.3} x(t).dt = \frac{4\pi}{2\pi}\int_0^{0.3} \left(\frac{100t}{3}\right).dt = \frac{4\pi}{2\pi}\left(\frac{100}{3}\right)\left(\frac{t^2}{2}\right)_0^{0.3}$$

$$a_0 = \frac{4\pi}{2\pi}\left(\frac{100}{3}\right)\left[\frac{0.3^2}{2}\right] = 2\left(\frac{100}{3}\right)\left(\frac{0.09}{2}\right) = 9/3$$

$$\therefore \quad a_0 = 3 = 3/2 = 1.5 \quad \text{...(a)}$$

To write $a_0/2$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

To determine a_n The equation for $a_n = \dfrac{\omega}{\pi} \displaystyle\int_0^{2\pi/\omega} x(t) \cos n\,\omega t.\, dt$

$$a_n = \frac{4\pi}{\pi}\int_0^{0.3} \left(\frac{100t}{3}\right) \cos(4\pi n t)\, dt = \frac{400}{3}\left[\frac{t \sin 4\pi\, nt}{4\pi n} + \frac{\cos 4\pi\, nt}{(4\pi n)^2}\right]_0^{0.3}$$

$$= \frac{400}{3}\left[\frac{0.3 \sin 4\,\pi n \times 0.3}{4\pi n} + \frac{\cos 4\pi n \times 0.3}{(4\pi n)^2} - 0 - \frac{1}{(4\pi n)^2}\right]$$

$$a_n = \left[\frac{10 \sin \times 1.2\pi n}{\pi n} + 25\,\frac{\cos 2\,\pi n}{(\pi^2 n^2)} - \frac{25}{3(4\pi n)^2}\right]$$

$$\therefore \quad a_n = \frac{10}{\pi n}\sin(1.2\,\pi n) + \frac{8.33}{\pi^2 n^2}\{\cos(1.2\,\pi n) - 1\} \quad \text{...(b)}$$

To determine b_n The equation for $b_n = \dfrac{\omega}{\pi} \displaystyle\int_0^{2\pi/\omega} x(t) \sin n\omega t.\, dt$

$$= \frac{4\pi}{\pi}\int_0^{0.3} \left(\frac{1000t}{3}\right) \sin(4\pi nt)dt = \frac{400}{3}\left[\frac{-t\cos 4\pi nt}{4\pi n} + \frac{\sin 4\pi nt}{(4\pi n)^2}\right]_0^{0.3}$$

$$= \frac{400}{3}\left[-0.3\frac{\cos(4\pi n \times 0.3)}{4\pi n} + \frac{\sin(4\pi n \times 0.3)}{(4\pi n)^2}\right],$$

$$= -\frac{10}{\pi n}\cos(1.2\,\pi n) + \frac{8.33}{\pi^2 n^2}\sin(1.2\pi n) \qquad \text{(c)}$$

Substituting the values of a_0, a_n and b_n in Eq. (I), the Fourier representation of harmonic motion of a given curve is

$$\therefore \quad x(t) = \frac{3}{2} + \sum_{n=1}^{\infty}\left\{\frac{10}{\pi n}\sin(1.2\pi n) + \frac{8.33}{\pi^2 n^2}(\cos 1.2\,\pi\, n - 1)\right\}\cos 4\pi nt$$

$$+ \sum_{n=1}^{\infty}\left\{-\frac{10}{\pi n}\cos 1.2\pi n + \frac{8.22}{\pi^2 n^2}\sin 1.2\,\pi n\right\}\sin 4\,\pi nt$$

EXAMPLE 1.13

Determine the Fourier series for the curve shown in Fig. p-1.13(a).

Solution The one complete cycle, $(OC + CE) = 2\pi$ is shown in Fig. p.1.13(b).

$\therefore$ time period or periodic time $\tau = 2\pi$, The frequency

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2\pi} = 1 \text{ rad/s}$$

The straight line 'OB' (forward direction +ve) is given by the equation $y = mx + c$

$x(t) = (A/\pi/2)\,t + 0 = (2A/\pi)t.$ $\qquad c = 0$, since it starts from origin.

$$x(t) = \left(\frac{+2A}{\pi}\right)t,\ 0 \le x \le \pi/2$$

The straight line 'BC' (reverse direction –ve) is given by the equation

$y = mx + c$, $x(t) = (-A/\pi/2)\,t + (A + A)$, c = twice the slope $(FB + BC) = (A + A)$ as shown in Fig. P.1.13(b). $x(t) = \left(\frac{-2A}{\pi}\right)t + (2A) \quad = \left(\frac{-2A}{\pi}\right)t + 2A,\ \pi/2 \le x \le \pi.$

The Fourier series is represented by the following equation for the above motion.

$$x(t) = a_0 + a_1\cos\omega t + a_2\cos 2\,\omega t + \cdots$$

$$b_1 \sin\omega t + b_2 \sin 2\,\omega t + \cdots \quad \text{or}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega t + b_n \sin n\,\omega t) \qquad \text{...(I)}$$

To determine a_0 The equation for $a_0 = \dfrac{\omega}{2\pi}\displaystyle\int_0^{\tau} x(t)\,dt$

$$x(t) = \frac{\omega}{2\pi}\int_0^{\frac{\pi}{2}} x(t)\,dt + \int_{\pi/2}^{\pi} x(t)\,dt$$

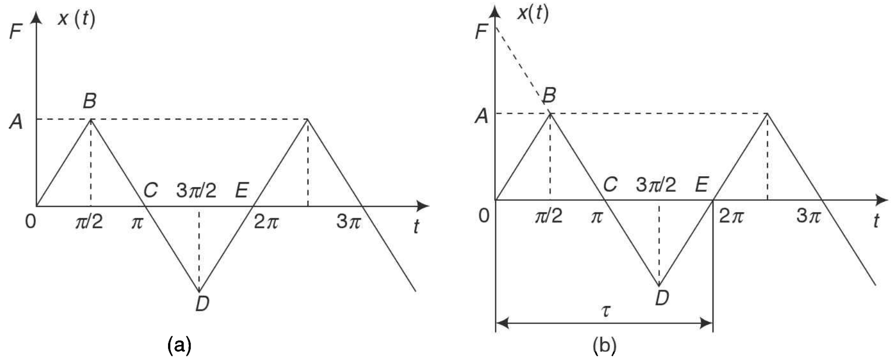


Fig. p-1.13 *Representation of periodic motion*

Substituting the values of 'ω' and $x(t)$,

$$a_0 = \frac{\omega}{2\pi}\left[\frac{2A}{\pi}\left\{\frac{t^2}{2}\right\}_0^{\pi/2} + \frac{-2A}{\pi}\left(\frac{t^2}{2}\right)_{\pi/2}^{\pi} + \{2A(t)\}_{\pi/2}^{\pi}\right]$$

$$= \frac{\omega}{2\pi}\left[\frac{2A}{\pi}\times\frac{\pi^2}{8} + \frac{-2A}{p}\left\{\frac{\pi^2}{2} - \frac{\pi^2}{8}\right\} + 2A\left(\pi - \frac{\pi}{2}\right)\right]$$

$$= \frac{\omega}{2\pi}\left[\frac{4\pi}{4} - \frac{2A}{\pi}\left\{\frac{3\pi^2}{8}\right\} + 2A\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{2\pi}\left[\frac{4\pi}{4} - \frac{3A\pi}{4} + A\pi\right] = \frac{1}{\pi}\left[\frac{A\pi - 3A\pi + 4A\pi}{4}\right] = \frac{1}{\pi}\left[\frac{2A\pi}{4}\right]$$

$$a_0 = \frac{A}{2} \qquad \text{...(a)}$$

To determine a_n The equation for $a_n = \frac{w}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\cos n\omega t.\,dt$

Substituting the values of 'ω'and $x(t)$,

$$a_n = \frac{\omega}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(\frac{2A}{\pi}\right)t\cos n(2)\,t\,dt + \int_{\frac{\pi}{2}}^{\pi}\left\{-\left(\frac{2A}{\pi}\right)t + 2A\right\}\cos 2nt.\,dt\right]$$

$$= \frac{\omega}{\pi}\left[\frac{2A}{\pi}\left\{t.\frac{\sin 2nt}{2n} + \frac{\cos 2nt}{4n^2}\right\}_0^{\frac{\pi}{2}} + \frac{-2A}{\pi}\left\{t.\frac{\sin 2nt}{2n} + \frac{\cos 2nt}{4n^2}\right\}_{\frac{\pi}{2}}^{\pi} + 2A\left\{\frac{\sin 2nt}{2n}\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$a_n = \frac{\omega}{\pi}\left[\frac{2A}{\pi}\left\{\frac{\cos n\pi}{4n^2} - \frac{1}{4n^2}\right\} - \frac{2A}{\pi}\left\{\frac{1}{4n^2} - \frac{\cos n\pi}{4n^2}\right\}\right]$$

$$a_n = \frac{\omega}{\pi}\left[\frac{A\cos n\pi}{2n^2\pi} - \frac{A}{2\pi n^2} - \frac{A}{2\pi n^2} + \frac{A\cos n\pi}{2\pi n^2}\right] = \frac{\omega}{\pi}\left[\frac{-A}{2\pi n^2}(4)\right]$$

$$= \frac{-2 \times A \times 4}{2\pi^2 n^2} = \frac{-4A}{n^2 \pi^2} \text{ ...(b), where } n = \text{odd. If '}n\text{' is even, } a_n = 0.$$

To determine b_n The equation for $b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t). \sin n\, \omega t.\, dt.$

Substituting the values of 'ω' and $x(t)$,

$$b_n = \frac{\omega}{\pi}\left[\int_0^{\frac{\pi}{2}} \left(\frac{2A}{\pi}\right) t.\, \sin n\, \omega t + \int_{\frac{\pi}{2}}^{\pi} \left(\frac{-2A}{\pi}\right) t + 2A \sin n\omega t.\, dt\right]$$

$$= \frac{\omega}{\pi}\left[\frac{2A}{\pi}\left\{t - \frac{\cos n2t}{2n} + \frac{\sin 2nt}{4n^2}\right\}_0^{\pi/\omega}\right.$$

$$\left. + \frac{-2A}{\pi}\left\{\frac{-t\cos 2nt}{2n} + \frac{\sin nt}{4n^2}\right\}_{\pi/2}^{\pi} + 2A\left\{\frac{-\cos nt}{2n}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{\omega}{\pi}\left[\frac{2A}{\pi}\left\{\frac{-\pi}{2} \cdot \frac{\cos n\pi}{2n}\right\} - \frac{2A}{\pi}\left\{\frac{-\pi}{2} \times \frac{1}{2n} + \frac{\pi}{2} \cdot \frac{\cos n\pi}{2n}\right\} + 2A\left\{-\frac{1}{2n} + \frac{\cos n\pi}{2n}\right\}\right]$$

$$b_n = \frac{\omega}{\pi}\left\{\frac{2A}{\pi} - \frac{A\cos n\pi}{2n} + \frac{A}{n} - \frac{A\cos n\pi}{2n} - \frac{A}{n} + A\frac{\cos n\pi}{n}\right\}, bn = 0 \qquad \text{...(c)}$$

Substituting the values of a_0, a_n and b_n in Eq. (I), the Fourier representation of harmonic motion of a given curve is

$$x(t) = \frac{A}{4} + \sum_{n=1}^{\infty} \left(\frac{-4A}{\pi^2 n^2}\right) \cos n\omega t \text{ for '}n\text{' = odd, and } x(t) = \frac{A}{4} \text{ for '}n\text{' is even.}$$

1.14 WORK DONE BY HARMONIC MOTION

The work done by a harmonic force on a harmonic motion of the same frequency is a practical requirement in vibration study.

Let a harmonic force $F = F_0 \sin \omega t$ be acting on a vibrating body having SHM

$$x = X \sin(\omega t - \phi)$$

The work done by the harmonic force during a small displacement dx is $F.\, dx$

or
$$F. \frac{dx}{dt}. dt \qquad \text{...1.24}$$

∴ for work done in one complete cycle (τ), integrating the Eq. 1.14 between limits '0' to 'τ' we have,

$$\text{Work done} = \int_0^{\tau} F \cdot dx = \int_0^{t} F \cdot \frac{dx}{dt}\, dt \qquad \text{...1.25}$$

Substituting the values of 'F' and 'x' in Eq. 1.25,

$$\text{Work done} = \int_0^{\tau} \left[F_0 \sin \omega t \frac{d}{dt}[X \sin(\omega t - \phi)]\right] dt$$

$$= \int_0^{\tau} [F_0 \sin \omega t \cdot \omega X \cos(\omega t - \phi)] dt$$

$$= F_0 X \omega \int_0^{\tau} \{\sin \omega t \cdot \cos(\omega t - \phi)\} dt$$

$$\text{Work done} = F_0 X\omega \int_0^{\tau} \{\sin \omega t \cdot \cos \omega t \cos \phi + \sin \omega t \cdot \sin \omega t \cdot \sin \phi\} dt$$

$$= F_0 X\omega \int_0^{\tau} \{\sin \omega t \cdot \cos \omega t \cdot \cos \phi + \sin^2 \omega t \cdot \sin \phi\} dt$$

$$= F_0 X\omega \int_0^{\tau} \left\{\frac{\sin 2\,\omega t}{2} \cdot \cos \phi + \frac{(1 - \cos 2\omega t)}{2} \sin \phi\right\} dt$$

$$= F_0 X\omega \int_0^{\tau} \{\sin \omega t \cdot \cos \omega t \cdot \cos \phi + \sin^2 \omega t. + \sin \phi\} dt$$

$$= F_0 X\omega \int_0^{\tau} \left\{\frac{\sin 2\,\omega t}{2} \cdot \cos \phi + \frac{(1 - \cos 2\omega \cdot t)}{2} \sin \phi\right\} dt$$

$$\text{Work done} = F_0\, \omega X \left\{\frac{\cos \phi}{2}\left(\frac{-\cos 2\,\omega t}{2\omega}\right) + \left(\frac{\sin \phi}{2}\right)\left(\frac{t - \sin 2\omega t}{2\omega}\right)\right\}_0^{\tau}$$

$$= F_0 X \omega \left[\frac{\cos \phi}{4\omega}\{-\cos 2\,\omega t + 1\} + \frac{\sin \phi}{2}\left\{\tau - \frac{\sin 2\,\omega t}{2\omega} - 0\right\}\right]$$

$$\omega = \pi X F_0 \sin \phi \qquad \text{...1.26}$$

If $\phi = 0$, in Eq. 1.26 then work done will be zero. It means force and displacement should not be in-phase to get the work done.

$$= F_0 X \omega x \left[\frac{\cos \phi}{4\omega}\left\{-\cos 2\omega \frac{2\pi}{\omega} + 1\right\}\right] + \frac{\sin \phi}{2}\left[\frac{2\pi}{\omega} - \sin 2\omega \cdot \frac{2\pi}{\omega} - 0\right]$$

$$= F_0 X\omega \left[\frac{\cos \phi}{4\omega}\{-1 + 1\}\right] + \frac{\sin \phi}{2}\left[\frac{2\pi}{\omega} - 0\right], = F_0 X\omega \left[0 + \frac{\sin \phi}{2} - \frac{2\pi}{\omega}\right]$$

Work done = $F_0 X \sin \phi$, work done = $\pi F_0 X \sin \phi$.

Thus, it is only that component of the force in phase with the velocity ($F_0 \sin \phi$) which does the work. The component in phase with displacement does no work in harmonic motion.

The maximum work ($\pi F_0 X$) is done when $\phi = 90°$, that is when the force is leading the displacement by an angle of $\pi/2$.

If the force is lagging the displacement then work is done on the force by the system.

If it is required to find the work done during a fraction of a cycle then actual integration should be carried out within the given time limits.

1.15 REPRESENTING HARMONIC MOTION IN COMPLEX FORM

Consider two lines $X^1 OX$, $Y^1 OY$ at right angles to each other. Let all the real numbers be represented by points on the line $X^1 OX$ called the real axis, positive real numbers being along 'OX' and negative ones along OX^1. Let the point 'L' on 'OX' represent the real number 'x' shown in Fig. 1.17(a). Since the multiplication of a real number by 'i' is equivalent to the rotation of its direction through a right angle, therefore, let all the imaginary numbers be represented by points on the line YOY^1 called the imaginary axis M, the positive ones along 'OY' and negative along 'OY^1. Let the point 'M' on the 'OY' represent the imaginary number 'iy'. Complete the rectangle $OLPM$. Then the point whose Cartesian coordinates are (x, y) uniquely represents the complex number $z = x + iy$ on the complex plane z.

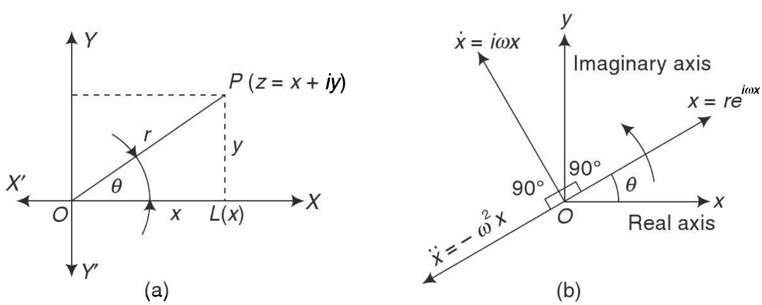


Fig. 1.17 *Representing harmonic motion in complex form*

If (r, θ) be the polar coordinates of 'P' then 'r' is the modulus of 'z' and 'θ' is its amplitude. Let 'v' represent complex number $v = x + iy$, where $i = \sqrt{-1}$ and 'x' and 'y' denote the real and imaginary components of 'z' respectively as shown in Fig. 1.17(a).

Let $v = \sqrt{x^2 + y^2}$ be the modulus of complex number and it is also equal to the magnitude of the vector. If the vector makes an angle 'θ' with the x-axis then $\theta = \tan^{-1} (y/x)$. It is also the angle that the vector makes with the x-axis. Therefore, the vector can also be written as $v = r(\cos\theta + i\sin\theta) = re^{i\theta}$ by Euler's formula.

For a particle if 'r' is the amplitude and 'ω' is its circular frequency then,

Displacement $\quad x = r(\cos\omega t + \mathrm{i}\sin\omega t) = re^{i\omega t}$ ...1.27

Differentiating Eq. 1.27 *w.r.t.* time 't',

Velocity $\quad x = \omega r(-\sin\omega t + i\cos\omega t) = i\,\omega r(\cos\omega t + i\sin\omega t)$ ...1.28

$\therefore \quad i\omega r\, e^{i\omega t} = i\,\omega x$ known as velocity vector

Differentiating Eq. 1.27 w.r.t. time 't' once again,

Acceleration $x = i\omega^2 r(-\sin \omega t + i \cos \omega t) = i^2 \omega^2 r (\cos \omega t + i \sin \omega t) = i^2 \omega^2 r e^{j\omega t}$ $= -\omega^2 x$ ...(1.29) known as acceleration vector.

From Fig. 1.17 (b), it is seen that velocity vector leads the displacement vector by 90° and the acceleration vector leads the velocity vector by 90°. It can be seen that the multiplication of a complex number by 'i' is equivalent to rotation of the corresponding vector by 90°.

EXAMPLE. 1.14

Represent the following complex numbers in exponential form.

(i) $-2 + i4$ (ii) $-2 - i4$ (iii) $-3 + j4$ (iv) $3 + j4$

Solution

(i) $r = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

$$\theta = \tan^{-1}\frac{4}{-2} = -63.43° = -1.107 \text{ rad}, \theta = \tan^{-1}\frac{4}{-2} = 2\sqrt{5}\, e^{i1.107}$$

(ii) $r = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

$$\theta = \tan^{-1}\frac{4}{-2} = -63.43° = -1.107 \text{ rad}, -2 - i4 = 2\sqrt{5}\, e^{i1.107}$$

(iii) Let $V = -3 + j4, = r\cos\phi + jr\sin\phi = r(\cos\phi + j\sin\phi) = re^{j\theta}$

$$-3 = r\cos\theta \ldots\text{(a)} \quad 4 = r\sin\theta \ldots\text{(b)}$$

Squaring and adding Eq. (a) and Eq. (b),

$$(-3)^2 + 16 = r^2\cos^2\theta + r^2\sin^2\phi, 25 = r^2, r = 5$$

Dividing Eq. (b) by Eq. (a),

$$\frac{r\sin\theta}{r\cos\theta} = \frac{4}{-3}, \tan\theta = \frac{-4}{3}, \theta = \tan^{-1}\left(\frac{-4}{3}\right) = -0.927 \quad \therefore V = 5\, e^{-j0.927}$$

(iv) Let $V = 3 + j4$

Put $3 = r\cos\theta$ ...(a) $4 = r\sin\theta$ ...(b)

$$V = r\cos\theta + j\, r\sin\theta, = r(\cos\theta + j\sin\theta), V = r\, e^{j\theta}$$

From Eq. (a) and (b), $3 = r\cos\theta$, $4 = r\sin\theta$.

Squaring and adding the above equations,

$$9 + 16 = r^2\cos^2\theta + r^2\sin^2\theta, 25 = r^2(\cos^2\theta + \sin^2\theta)$$

$$r^2 = 25 \text{ because } \cos^2\theta + \sin^2\theta = 1, r = 5$$

Dividing Eq. (b) by Eq. (a), $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{4}{3}, \tan\theta = \dfrac{4}{3}, \theta = \tan^{-1}\left(\dfrac{4}{3}\right) = 0.927$

$$\therefore \quad V = 5\, e^{j0.927}$$

EXAMPLE 1.15

Represent the following complex numbers in rectangular form.

(i) $5\,e^{10-1}$ (ii) $17\,e^{-j3.74}$

Solution (i) $V = 5\,e^{10-1}$, 0.1 being the angle in radians.

$= 5[\cos 0.1 + j \sin 0.1] = 5\,[0.925 + j\,0.099]$ or $V = 4.97 + j\,0.49$.

(ii) $V = 17\,e^{-j3.74}$, 3.74 being the angle in radians.

$= 17[\cos 3.74° - j \sin 3.74°] = 17\,[(-0.820) - j\,(-4.559)]$ or $V = -14.08 + j\,9.50$

REVIEW QUESTIONS

(1) What is vibration? Explain clearly the different types of vibrations: linear vibration, nonlinear vibration, deterministic vibration, nondeterministic vibration or random vibration.

(2) Define natural frequency. Why is it important to determine the natural frequency of a vibrating system?

(3) Define the following terms: free, undamped, damped, and forced vibration; resonance; phase difference; periodic motion; time period; amplitude and degrees of freedom.

(4) Explain clearly the rectilinear or longitudinal vibration, lateral or transverse vibrations and torsional vibrations.

(5) Distinguish between the
   (i) Free vibration and forced vibration
   (ii) Undamped vibrations and damped vibrations
   (iii) Rectilinear system and torsional system
   (iv) Deterministic vibration and nondeterministic vibration.

(6) What is harmonic analysis? Suggest two methods for finding the time derivative of a harmonic motion.

(7) Explain simple harmonic motion and beat phenomenon.

(8) Distinguish between periodic motion and harmonic motion.

(9) Explain work done by harmonic force on a harmonic motion.

(10) Write notes on (i) Causes of vibration (ii) Effects of vibration.

(11) Explain the various elements of a vibratory system. Compare the system parameters of a rectilinear and torsional system.

PROBLEMS FOR PRACTICE

(1) Add the following harmonic motions analytically and check the solution graphically.

$$x_1 = 15 \sin\left(\omega t + \frac{\pi}{6}\right),\ x_2 = 8\cos\left(\omega t + \frac{\pi}{3}\right).$$

Ans. $x = 12 \sin(\omega t + \pi/3)$.

(2) Add the following harmonic motions analytically and check the solution graphically. $x_1 = 2\cos(\omega t + 0.5)$, $x_2 = 5\sin(\omega t + 1.0)$.

Ans. $x = 6.2\sin(\omega t + 1.28°)$, $x = 6.2\sin(\omega t + 73.6°)$.

(3) Add the following harmonic motions analytically and check the solution graphically. $x_1 = 2\sin(\omega t + 60°)$, $x_2 = -3\cos(\omega t + 120°)$ and express the sum in the form $x = X\sin(\omega t + \phi)$. Check the solution graphically.

Ans. $x = 4.84\sin(\omega t + 42°)$, $x = 6.2\sin(\omega t + 73.6°)$.

(4) A body is subjected to harmonic motions $x_1 = 10\sin(\omega t + 30°)$ and $x_2 = 5\cos(\omega t + 60°)$. What harmonic motion should be given to the body to bring it to equilibrium?

Ans. $x = 8.66\sin(\omega t + 240°)$.

(5) Split up the harmonic motion $x = 8\cos(\omega t + \pi/4)$ into two harmonic motions, one of them having a phase angle of zero and the other having a phase angle of 60°.

Ans. $3.77\cos\omega t$, $6.53\cos(\omega t + \pi/3)$.

(6) The equation of motion of a single-degree-freedom undamped system is given by $x = 5\sin(2\pi t + \pi/4)$, where '$x$' is in cm and '$t$' is in seconds. Draw the variation of displacement for one cycle of vibration and also determine the displacement after a lapse of 0.1 second.

(7) A body describes simultaneously two motions: $x_1 = 3\sin 40t$, $x_2 = 4\sin 41t$. What is the maximum and minimum amplitude of combined solution and what is the beat frequency?

Ans. 7, 1 and $1/2\pi$.

(8) A harmonic displacement is given by $x(t) = 6\sin(20t + \pi/3)$ mm, where t is in seconds and phase angle in radians. Determine (i) frequency and period of motion, and (ii) the maximum displacement, velocity and acceleration.

Ans. (i) 20 rad/s, 0.314 s (ii) 120 m/s, 2400 mm^2/s.

(9) A harmonic displacement is given by $x = 12.5\sin(15\pi t + 60°)$ where t is in seconds. x is measured in cm and phase angle in radians. Determine (i) frequency and period of motion, (ii) the maximum displacement, velocity and acceleration, and (iii) the displacement, velocity and acceleration at $t = 0.2$ s and $t = 0.25$ s.

Ans. (i) Frequency $f = 7.5$ cps, period $\tau = 0.133$.

(ii) Displacement $X = 1.25$ cm, velocity = 0.6 m/s and acceleration = 27.8 m/s^2.

(iii) At $t = 0.2$ $X = 1.08$ cm, velocity = 0.3 m/s and acceleration = 24.05 m/s^2
At $t = 0.25$ $X = -1.21$ cm, velocity = – 0.1525 m/s and acceleration = 26.8 m/s^2.

(10) An instrument has a natural frequency of 8 Hz. It can withstand a maximum acceleration of 10 m/s^2. Determine (i) frequency, (ii) amplitude, and (iii) maximum velocity.

Ans. (i) 50.3 rad/s. (ii) 0.0032 m. (iii) 0.15 m/s.

(11) Show that the Fourier series expansion for the function $x(t)$ defined in the finite interval $-\pi \le t \le \pi$ by $x(t) = 0$ $[-\pi \le t \le 0]$, $x(t) = \sin t$ $[0 \le t \le \pi]$ is given by

$$X(t) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos 2n\pi}{4n^2 - 1}\right) + \frac{1}{2}\sin t$$

(12) Develop the Fourier series for the curve shown in Fig. p.p.-1.12.

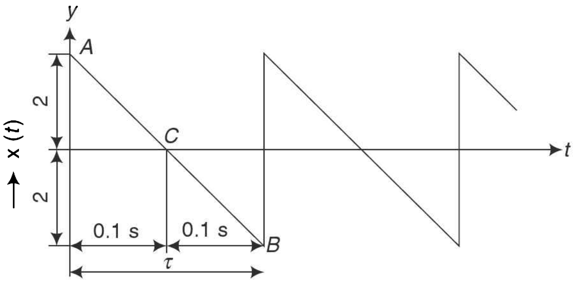


Fig. p. p-1.12

Ans. The Fourier representation of harmonic motion of a given curve is

$$x(t) = \sum_{n=1}^{\infty} \frac{4}{\pi n} \sin 10\,\pi nt = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin 10\pi nt.$$

(13) Determine the Fourier series for the part of the sine wave curve as shown in Fig. p.p-1.13.

Ans. $x(t) = \frac{2E_0}{\pi} - \frac{4E_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}\right)\cos 100\,n\pi t.$

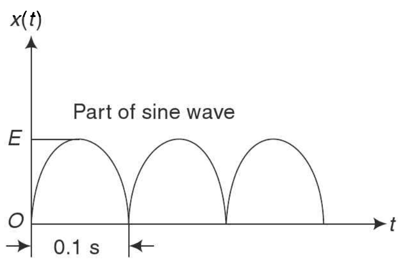


Fig. p.p 1.13

(14) In a certain periodic motion, the period time $\tau = 0.1$ s is given by

$$(x) = 200t\ 0 \le t \le 0.05,\ (x) = -200t + 20, \quad 0.05 \le t \le 0.1$$

Determine the Fourier series.

Ans. $x(t) = 5 - \frac{40}{\pi^2}\left[\cos 20\pi t - \left(\frac{40}{9\pi^2}\right)\cos 60\,\omega t\right]$ as $\omega = 20\pi$.

(15) Represent the following complex numbers in exponential form.

(i) $3 - j4$ (ii) $-3 - j4$

Ans. (i) $5e^{-J\,0.026}$, (or $5e^{J5.358}$) (ii) $5\,e^{J4.068}$.

OBJECTIVE-TYPE QUESTIONS

(1) The interval of time taken by a vibrating body to complete a cycle is called
- (a) frequency (b) amplitude
- (c) period
- (d) none of the above

(2) Resonance is a phenomenon when the frequency of the external exciting force is
- (a) twice the natural frequency of the system
- (b) half the natural frequency of the system
- (c) same as the natural frequency of the system
- (d) none of the above

(3) Periodic motion is a phenomenon of
- (a) the to-and-fro motion of a body about a mean position
- (b) motion which repeats itself after equal interval of time
- (c) the interval of time taken by a vibrating body
- (d) none of the above

(4) The harmonic motion is
- (a) sine wave (b) cos wave
- (c) sine wave. cos wave. tan wave
- (d) the simplest type of periodic motion

(5) Beats phenomenon is
- (a) this is not a simple harmonic motions but similar
- (b) two harmonic motions with frequencies closed to one another
- (c) circular function repeating itself in '2π' radians
- (d) none of the above

(6) When there is a reduction in the amplitude over every cycle of vibration of a vibrating body, it is said to be
- (a) damped vibration
- (b) free vibration
- (c) forced vibration
- (d) undamped vibration

(7) When the particles of a body or system move approximately perpendicular to the axis of the body then the vibration is said to be
- (a) longitudinal vibrations
- (b) undamped vibrations
- (c) torsional vibrations
- (d) lateral or transverse vibrations

(8) If all the basic components of a vibratory system behave linearly, the resulting vibration is known as
- (a) nonlinear vibration
- (b) deterministic vibration
- (c) linear vibration
- (d) random vibration

(9) In analysis of Fourier's series, 'a_0' and 'a_n' will be present and $b_n = 0$ if
- (a) in case of even function if $x(t) = x(-t)$
- (b) in case of even function if $x(t) = -x(-t)$
- (c) if $x(t) = -x(-t)$ and if $x(t) = x(-t)$
- (d) none of the above

(10) A body is subjected to two harmonic motions $x_1 = 15 \sin(\omega t + \pi/6)$ superposed on a body. Which harmonic motion should bring the body to equilibrium?
- (a) $x = 8.66 \sin(\omega t + 240°)$
- (b) $x = -8.66 \sin(\omega t + 240°)$
- (c) $x = 20.2 \sin(\omega t + 220°)$
- (d) $x = -20.2 \sin(\omega t + 220°)$

Answers

(1) c	(2) c	(3) b	(4) d	(5) a	(6) a
(7) d	(8) c	(9) a	(10) d		

UNDAMPED FREE VIBRATION OF SINGLE-DEGREE-OF-FREEDOM SYSTEMS

2

1.14 INTRODUCTION

When an elastic system is disturbed from its equilibrium by an impressed force, the system starts vibrating. If the impressed force remains absent during the resulting vibration, the system is said to have free vibration.

The system in such a case vibrates at its natural frequency. In course of time, the vibration dies down due to energy dissipation by the motion itself. The natural frequency of a system and rate of decay of its motion are the two interesting aspects of vibration study. If damping of the system is negligible (only spring mass) then only the determination of natural frequency remains to be done.

Note: In case of undamped free vibrations, there will be two forces acting on a system, i.e. inertia force and spring force (elastic forces).

In solving vibration problems using equilibrium method, the steps are involved as follows.

1. Study the physical system.
2. Take a displaced position of the mass or masses.
3. Assume a direction (+ve or –ve) for the displacement.
4. Draw the free-body diagram (See Sec. **2.12.** Note for FBD) indicating the forces acting in various directions.
5. Apply Newton's second law of motion, i.e. $\Sigma F = ma$.

1.14 ANALYSIS OF VIBRATION PROBLEMS

The analysis of vibration problem involves the following steps.

1. Formulation of the physical problem into mathematical model.
2. Writing down the equation of motion. This could be done by the first principles of dynamics. Both equilibrium methods and energy methods can be employed. The choice depends upon the convenience of the resulting analysis.
3. Writing down the solutions for the equations of motion and determining the constants occurring in the solution by using the initial conditions.

2.3 METHODS OF ANALYSIS

The methods of analysis of vibration problems are as follows.

(1) Equilibrium method (2) Energy method (3) Rayleigh's method

1. Equilibrium method

The equilibrium method is once again subdivided into two methods:

(a) Newton's second law of motion (b) D'Alembert's principle

(a) Applying Newton's second law of motion The equation of motion is just another form of Newton's second law of motion. By Newton's second law of motion,

$$\text{Sum of forces} = \text{Mass} \times \text{Acceleration.}$$

The algebraic sum of forces is equal to the mass into acceleration, i.e. $\Sigma F = ma$.

Or $\Sigma F = m\ddot{x}$ (Restoring force) = (Inertia force) and $\Sigma M = I_o\,\ddot{\theta}$, i.e. the algebraic sum of moments is equal to the mass moment of inertia into angular acceleration.

(b) Applying D'Alembert's principle D'Alembert's principle states that "a body which is not in static equilibrium by virtue of some displacement can be brought to static equilibrium by introducing on it the inertia force which is equal to mass times the acceleration of the body and acts through the centre of gravity (CG) of the body but in the opposite direction to the acceleration" (displacement).

In this method for equilibrium, the algebraic sum of forces and moments acting on it must be equated to zero. The forces are **Inertia force, the spring force (restoring force) damping force and the external force,**

i.e. $$\Sigma F = 0 \text{ and } \Sigma M = 0.$$

2. Energy method

The energy method states that the sum of the kinetic energy and potential energy is constant in a conservative system,

i.e., $$\text{PE} + \text{KE} = \text{Constant}$$

Differentiating both sides with respect to time 't', i.e. $\frac{d}{dt}[\text{PE} + \text{KE}] = \frac{d}{dt}(\text{constant})$

The resulting equation is the equation of motion.

Energy may be defined as capacity to do work. It exists in many forms, i.e. mechanical, electrical, chemical, heat, light, etc.

Though there are many types of **mechanical energy**, the following two types are important: (a) Potential energy (b) Kinetic energy.

(a) Potential Energy (PE) It is the energy possessed by a body, for doing work, by virtue of its position, for example,

(i) A body raised to some height above the ground level possesses some potential energy, (PE) because it can do some work by falling on the earth's surface.

(ii) Compressed air also possesses potential energy, because it can do some work in expanding to the volume it would occupy at atmospheric pressure. A compressed spring also possesses potential energy because it can do some work in recovering its original shape.

Now consider a body of mass (m) raised through a height (h) above the datum level. We know that work done in raising the body = Weight × Distance = mgh

This work (mgh) is stored in the body as potential energy. A little consideration will show that the body while coming down to its original level, is capable of doing work equal to mgh.

(b) Kinetic Energy (KE) It is the energy possessed by a body, for doing work, by virtue of its mass and velocity of motion. Now consider a body which has been brought to rest by a uniform retardation due to the applied force.

Let

m = Mass of the body

u = Initial velocity of the body

F = Force applied on the body to bring it to rest

a = Constant retardation

s = Distance travelled by the body before coming to rest

Since the body is brought to rest, therefore, its final velocity,

$v = 0$ and work done, W = Force × Distance = $F \times s$ ...2.1

Now substituting value of $F = m \times a$ in Eq. 2.1,

$$W = m\,a\,s \quad \text{...2.2}$$

$$v^2 = u^2 - 2\,as \quad \cdots \text{(minus sign due to retardation)}$$

$$2\,as = u^2 \text{.......... } (v = 0, \text{ initial velocity) or } as = \frac{u^2}{2}$$

Now substituting the value of (as) in Eq. 2.2 and replacing work done with kinetic energy,

$$\text{KE} = \frac{mu^2}{2}.$$

Note: In most of cases, the initial velocity is taken as 'v' (instead of u). Therefore, kinetic energy, KE = $\frac{mv^2}{2}$.

3. Rayleigh's method In this method, "the maximum kinetic energy at the mean position is equal to the maximum potential energy at the extreme position and which is equal to the total energy of the system", i.e. $(\text{KE})_{max}$ at mean position = $(\text{PE})_{max}$ at extreme position = Total energy of the system.

The resulting equation is the equation of motion which will readily yield the natural frequency of the system.

Let us consider that motion of a vibrating body to be simple harmonic motion of

$$x = X \sin \omega_n t$$

The velocity at time 't' seconds, $\frac{dx}{dt} = \dot{x} = X\,\omega_n \cos \omega_n\, t$ and maximum amplitude, $x_{max} = X$.

The maximum kinetic energy at the mean position,

$$\text{KE} = \frac{1}{2} mv^2 = \frac{1}{2} m\dot{x}_{\max}^2 = \frac{1}{2} m\omega_n^2 X^2$$

The maximum potential energy at the extreme position, PE $= \frac{1}{2} kX^2$

As we know that $(\text{KE})_{\max} = (\text{PE})_{\max}$

$$\frac{1}{2} m\omega_n^2 X^2 = \frac{1}{2} kX^2 \therefore \omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

2.3 SPRING ELEMENT

Whenever there is relative motion between the two ends of a spring, a force is developed called, **spring force** or **restoring force.** The spring force 'F' is directly proportional to the amount of deformation shown in Fig. 2.1,

i.e. $F \alpha x$ or $F = kx$.

where k = Stiffness of spring constant in N/m or N/mm.

The spring stiffness 'k' in the spring force is received to cause a unit deformation of the spring, i.e. $k = F/x$.

Work done in deforming a spring is equal to the strain or potential energy in the spring.

Energy stored, i.e. strain energy (SE) = PE

$$\text{PE} = 1/2\ Fx, \text{ SE or PE} = 1/2\ kxx = 1/2\ kx^2$$

Spring stiffness, $k = \frac{\text{Force}}{\text{Deflection}} = \frac{F}{x} \quad \therefore F = kx.$

$\therefore$ Potential energy = Area shaded = $1/2 \times kx \times x$, PE $= 1/2\ kx^2$.

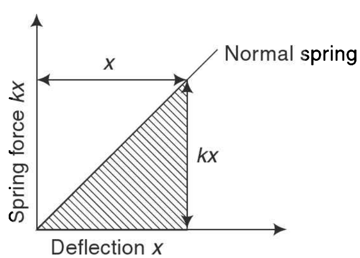


Fig. 2.1 *Strain energy in a spring*

2.3 SPRINGS IN SERIES

Let us consider a spring in series as shown in Fig. 2.2(a). We know that the spring stiffness k = Force/Unit deflection, i.e. $k = F/x$.

But force F = Mass × Acceleration

Let 'x' be the displacement given to mass 'm'.

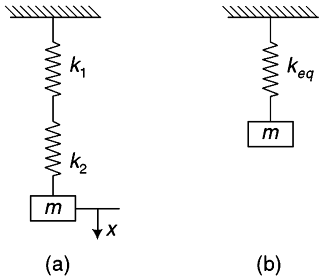


Fig. 2.2 *Springs-mass system in series*

Then the spring 'k_1' will move by 'x_1' distance, and spring 'k_2' will move by 'x_2' distance.

$$\therefore \quad k_1 = \frac{mg}{x_1} \quad \text{and} \quad k_2 = \frac{mg}{x_2} \quad \therefore x_1 = \frac{mg}{k_1} \text{ and } x_2 = \frac{mg}{k_2}.$$

But total displacement, $x = x_1 + x_2$, but $k_{eq} = \frac{mg}{x} \quad \therefore x = \frac{mg}{k_{eq}}$.

$$\therefore \quad \frac{mg}{k_{eq}} = \frac{mg}{k_1} + \frac{mg}{k_2}, \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \text{ or } k_{eq} = \frac{k_1 k_2}{k_1 + k_2}.$$

Then the given spring mass system will be reduced to as shown in Fig. 2.2(b). In general, when the springs are in series, the equivalent spring stiffness is given by the equation as follows.

$$\therefore \quad \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \ldots\ldots\ldots\ldots \text{ if } k_1 = k_2 = k_3 = k_4 = k,$$

in this case, $k_{eq} = k/4$.

2.3 SPRINGS IN PARALLEL

Let us consider a spring in parallel, as shown in Fig. 2.3(a). Let 'x' be the displacement given to mass 'm'. Then both the springs will move by 'x' distance.

$\therefore$ Total weight = Weight shared by spring k_1 + Weight shared by spring k_2

$\therefore$ $mg = k_1 x + k_2 x$, but for equivalent system, $k_{eq} = mg/x \therefore mg = k_{eq}\, x$

$$\therefore \quad k_{eq}\, x = k_1 x + k_2 x \text{ or } k_{eq} = k_1 + k_2.$$

When the springs are in parallel, the equivalent springs stiffness is reduced to as shown in Fig. 2.3(b).

$\therefore k_{eq} = k_1 + k_2$

In general, when the springs are in parallel, the equivalent spring stiffness is given by the equation as follows:

$k_{eq} = k_1 + k_2 + k_3 + k_4 + \ldots$, if $k_1 = k_2 = k_3 = k_4 = k$ in this case, $k_{eq} = 4k$.

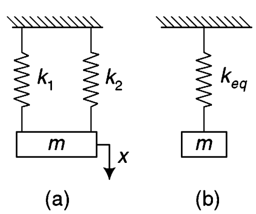


Fig. 2.3 *Spring-mass system in parallel*

2.3 INCLINED SPRING

Let us consider a spring inclined at an angle 'α' as shown in Fig. 2.4.

By applying a load 'P' in the direction of 'x', the axial component of 'P' in the direction of the spring, which is inclined at an angle 'α' to x-direction, is $F = P \cos \alpha$.

Stretching of the spring $\delta = \frac{F}{k} = \frac{P \cos \alpha}{k}$.

Component of this deflection in the x-direction is

$$\delta_x = \frac{P \cos \alpha}{k} \cdot \cos \alpha = \frac{P \cos^2 \alpha}{k}.$$

Equivalent spring constant is

$$k_e = \frac{P}{\delta_x} = \frac{Pk}{P \cos^2 \alpha} = k \sec^2 \alpha.$$

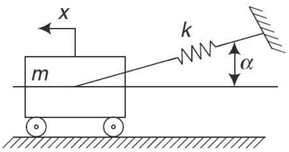


Fig. 2.4 *Inclined spring-mass system*

2.8 LAWS OF A SIMPLE PENDULUM

The following laws of a pendulum are important from the subject point of view:

1. *Law of isochronism's* It states "the time period (τ) of a simple pendulum does not depend on its amplitude of vibrations and remains the same provided the angular amplitude (θ) does not exceed 4°."

2. *Law of mass* It states, "the time period (τ) of a simple pendulum does not depend upon the mass of the body suspended at the free end of the string."

3. *Law of length* It states, "the time period (τ) of a simple pendulum is proportional to the square root of the length ($\sqrt{l}$) where l is the length of the string."

4. *Law of gravity* It states, "the time period (τ) of a simple pendulum is inversely proportional to $\sqrt{g}$, where 'g' is the acceleration due to gravity."

Note: The above laws of a simple pendulum are true from the equation of the time period, i.e. $\tau = 2\pi \sqrt{\frac{l}{g}}$.

2.3 NEWTON'S LAWS OF MOTION

The following are the three Newton's laws of motion

1. Newton's first law of motion states, **"every body continues in its state of rest or of uniform motion, in a straight line, unless it is acted upon by some external force"**.
2. Newton's second law of motion states, **"the rate of change of momentum is directly proportional to the impressed force, and takes place in the same direction in which the force acts"**, i.e. $\Sigma F = ma$.
3. Newton's third law of motion states, **"to every action, there is always an equal and opposite reaction."**

2.8 NEWTON'S LAWS OF MOTION OF ROTATION

The following are the three Newton's laws' of motion of rotation:

1. Newton's first law of motion of rotation states, **"every body continues in its state of rest or of uniform motion of rotation about an axis, unless it is acted upon by some external torque".**
2. Newton's second law of motion of rotation states that **"the rate of change of angular momentum of a body is directly proportional to the impressed torque, and takes place in the same direction in which the torque acts"** $\Sigma T = I_0\ddot{\theta}$.
3. Newton's third law of motion of rotation states, **"to every torque, there is always an equal and opposite torque."**

2.8 LAW OF CONSERVATION OF ENERGY

It states that **"the energy can neither be created nor destroyed, though it can be transformed from one form into any of the forms, in which the energy can exist."**

1. In an electrical heater, the electrical energy is converted into heat energy.
2. In an electrical bulb, the electrical energy is converted into light energy.
3. In a dynamo, the mechanical energy is converted into electrical energy.

2.12 DERIVATION OF DIFFERENTIAL EQUATION OF A CLASSICAL SPRING-MASS SYSTEM AND ITS NATURAL FREQUENCY

Now let us consider a classical spring mass system. The system of Fig. 2.5(a) is in equilibrium, due to its static equilibrium condition because the mass and gravitational force (mg) will act vertically in the downward direction at the centre of mass, whereas the same amount of static force ($k\delta$) (where 'δ' is static deflection and 'k', the spring force or spring stiffness) will act vertically in the upward direction as shown in Fig. 2.5(b), i.e. $mg = k\delta$ and corresponding free-body diagram (FBD) is Fig. 2.5(c). Now give linear displacement(x) to the mass 'm' assuming the **light spring** in

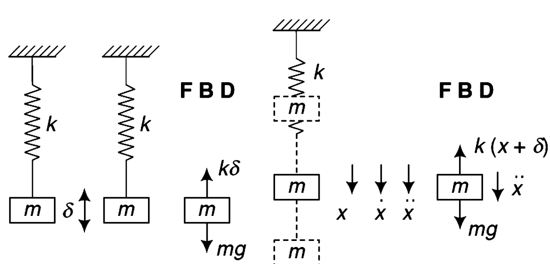


Fig. 2.5 *Undamped classical spring-mass system with free-body diagram*

Fig. 2.5(d) (either to the upward or downward direction). Afterwards, write down the free-body diagram (FBD) of the mass or masses.

The free-body diagram indicates [Fig. 2.5(e)] what are the force or forces, moment or moments acting in various directions in a displaced position, taking the equilibrium position as reference point. Now apply any one method for deriving the differential equation of motion.

Now, in this case applying **Newton's method,** we solve the above problem.

By Newton's method, $\Sigma F = ma$, (restoring force) = (inertia force).

From FBD of Fig. 2.5(c), $k(x + \delta) - mg = -m\ddot{x}$, or $kx + k\delta - mg = -m\ddot{x}$.

But from free-body diagram of Fig. 2.5(c),

$$k\delta - mg = 0 \quad \therefore \; kx = -m\ddot{x} \text{ or } m\ddot{x} + kx = 0$$

This is the differential equation of motion for a spring-mass undamped system.

$\therefore \quad m\ddot{x} + kx = 0$, divided by '$m$' throughout. $\quad \therefore \; \ddot{x} + \frac{k}{m}x = 0$ ...2.3

Comparing Eq. 2.3 with the SHM, $\omega^2 = \frac{k}{m}$

Since the vibrations are natural,

$\therefore \qquad \omega = \omega_n$

$\therefore \quad$ natural angular acceleration, $\omega_n^2 = \frac{k}{m}$ or $\omega_n = \sqrt{\frac{k}{m}}$ rad/s.

But from SHM, $x = X \sin \omega t$, differentiating w.r.t. time 't' twice.

$\therefore \qquad \dot{x} = X\omega \cos \omega t, \qquad \ddot{x} = -\omega^2 X \sin \omega t, \ddot{x} + \omega^2 x = 0$ ...2.4

Since $x = X \sin \omega t$, comparing Eqs. 2.3 and 2.4 to Eq. 1.5 (* See Sec.10.2, Chapter 1), $\omega^2 = \frac{k}{m}$ or $\omega_n = \sqrt{\frac{k}{m}}$

$\therefore$ natural angular acceleration, $\omega_n^2 = \frac{k}{m}$ or $\omega_n = \sqrt{\frac{k}{m}}$ rad/s.

Natural frequency $f_n = \frac{\omega_n}{2\pi}$, $\quad \therefore \; f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ Hz or cps.

The time period $\tau = \frac{1}{f_n}$ or $\tau = 2\pi\sqrt{\frac{m}{k}}$ s.

Note: We have $mg = k\delta$ or $\frac{k}{m} = \frac{g}{\delta}$. $\quad \therefore \; \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta}}$ rad /s

Note: A free-body diagram (FBD) is a sketch or a drawing of a portion or a part or a component isolated from the total system or mechanism. The forces or moments acting on the portion are also indicated. The portion taken for free-body diagram (FBD) should also be in equilibrium if the system is also in equilibrium.

2.13 SOLUTION OF DIFFERENTIAL EQUATION DERIVED FOR A CLASSICAL SPRING-MASS SYSTEM

Consider the simplest equation of a vibrational motion as $x = X \sin \omega t$ ...2.5

as shown in Fig. 2.6, where 'x' is the displacement at time 't' s,

and X = Amplitude, ω = Angular velocity in rad/s.

The velocity at time 't' seconds, $\frac{dx}{dt} = \dot{x} = X\,\omega \cos \omega t$ ...2.6

The acceleration at time 't' seconds, $\frac{d\dot{x}}{dt} = \ddot{x} = -X\,\omega^2 \sin \omega t$ ...2.7

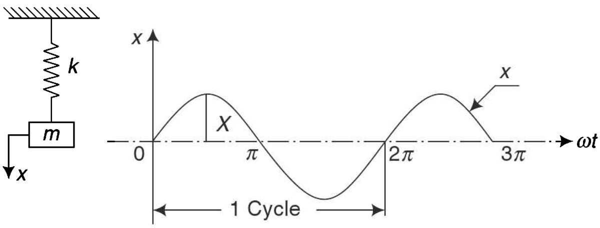


Fig. 2.6. *Linear vibration of undamped spring-mass system with a graph*

$\therefore \quad \ddot{x} = -\,\omega^2 X \sin \omega t = -\,\omega^2 x$, since $x = X \sin \omega t$, i.e. $\ddot{x} + \omega^2 x = 0$ ...2.8

For the above example in Fig. 2.6,

$$m\ddot{x} + kx = 0 \text{ or } \ddot{x} + \frac{k}{m}x = 0,\ \omega^2 = \frac{k}{m} \text{ or let } \omega_n^2 = \frac{k}{m}\ \therefore \omega_n = \sqrt{\frac{k}{m}} \text{ rad/s} \quad ...2.9$$

Such a vibration where the acceleration is directly proportional to the displacement and is directed towards the mean position (–ve sign gives the direction towards the mean position) is called a simple harmonic motion (SHM).

$x = X \cos \omega t$ is another example.

2.13.1 Solution of Differential Equation of a Body Executing Simple Harmonic Motion

For a particle executing simple harmonic motion, the differential equation of motion is given by

$$\ddot{x} + \omega^2 x = 0 \quad ...2.10$$

The solution of the differential equation is given by

$$x = A \sin \omega t + B \cos \omega t \quad ...2.11$$

where 'A' and 'B' are arbitrary constants which are to be determined from initial conditions.

Taking the initial conditions $x = x_0$ at $t = 0$ ...(a), $\dot{x} = \dot{x}_0$ at $t = 0$ ...(b)

Substituting the values of Eq. (a) in Eq. 2.11, we get

$$x_0 = A \sin(\omega \times 0) + B \cos(\omega \times 0),\ x_0 = B$$

Differentiating Eq. 2.11 w.r.t. time 't',

$$\dot{x} = A\,\omega \cos \omega t - B\,\omega \sin \omega t \qquad \text{...2.12}$$

Substituting the values of Eq. (b) in Eq. 2.12, we get

$$\dot{x}_0 = A\,\omega \cos(\omega \times 0) - B\,\omega \sin(\omega \times 0),\ x_0 = A\omega,\ A = \frac{x_0}{\omega}$$

Substituting the values of 'A' and 'B' in Eq. 2.11, we get

$x = \frac{\dot{x}_0}{\omega} \sin \omega t + x_0 \cos \omega t$. This is the complete solution of the differential equation

$$x = A \sin \omega t + B \cos \omega t$$

where, $A = X \cos \phi$ ...(a) $B = X \sin \phi$ ...(b)

Squaring Eqs. (a) and (b) and adding, we get

$$A^2 + B^2 = X^2 (\cos^2 \phi + \sin^2 \phi)$$

$$A^2 + B^2 = X^2 \quad \therefore X = \sqrt{A^2 + B^2}$$

The solution of the differential equation $\ddot{x} + \omega^2 x = 0$ can also be given by

$$x = X \sin(\omega t + \phi) \qquad \text{...2.13}$$

where 'X' and 'ϕ' are two arbitrary constants which are to be determined by imposing initial conditions.

2.13.2 Alternate Method to Determine the Complete Solution of Simple Harmonic Motions

We have, acceleration $\ddot{x} = -\omega^2 x,$

i.e. $\ddot{x} + \omega^2 x = 0$ ...2.14

The general solution of Eq. 2.11 can also be written as

$$x = X \sin(\omega t + \phi) \qquad \text{...2.15}$$

Here 'X' and 'ϕ' are the two arbitrary constants to be determined from the initial conditions.

The equation 2.15 can be expressed as $x = X \sin \omega t \cos \phi + X \cos \omega t \sin \phi$

$$x = \sin \omega t\,(X \cos \phi) + \cos \omega t\,(X \sin \phi),\ x = X \sin \omega t \cos \phi + X \cos \omega t \sin \phi$$

$$x = A \sin \omega t + B \cos \omega t$$

where, $A = X \cos \phi$ ...(i) $B = X \sin \phi$ ...(ii)

Squaring Eqs. (i) and (ii) and adding, we get

$$A^2 + B^2 = X^2 (\cos^2\phi + \sin^2\phi)$$

$$A^2 + B^2 = X^2 \quad \therefore X = \sqrt{A^2 + B^2}.$$

Dividing Eq. (ii) by Eq. (i), $\tan \phi = \frac{B}{A} \quad \therefore \phi = \tan^{-1}\left(\frac{B}{A}\right).$

EXAMPLE 2.1

Determine the differential equation of a classical spring-mass system and its natural frequency by using (i) D'Alembert's principle, (ii) energy method, and (iii) Rayleigh's method as shown in Fig. p-2.1(a).

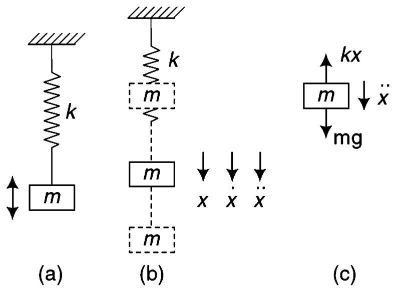


Fig. p-2.1 *Classical spring-mass system with free-body diagram*

Solution

(i) D'Alembert's Principle Fig. p-2.1(a).

Neglecting the initial tension in the spring after putting the mass, the FBD of the system as shown in Fig. p-2.1.1.

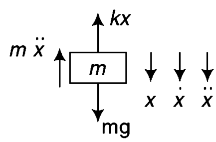


Fig. p-2.1.1 *FBD of Fig. p-2.1*

$\therefore$ spring force $= kx$, Inertia force $= m\ddot{x}$

$\therefore$ $m\ddot{x} + kx = 0$. This is the equation of motion.

$\therefore$ $m\ddot{x} + kx = 0$.

Divided by 'm' throughout,

$$\ddot{x} + \frac{k}{m}x = 0.$$

Comparing with the SHM,

$$\omega_n^2 = \frac{k}{m} \quad \text{or } \omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

$$f_n = \text{Natural frequency}, f_n = \frac{\omega_n}{2\pi} \text{ Hz}, \; f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz or cps.}$$

(ii) Energy method

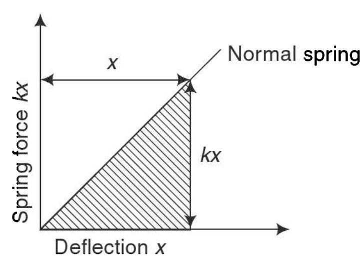


Fig. p–2.1.2 *Strain energy in spring*

Spring stiffness $k = \dfrac{\text{Force}}{\text{Deflection}} = \dfrac{F}{x}$

$\therefore$ $F = kx$

$\therefore$ Potential energy $=$ Area shaded $= 1/2 \times kx \times x$

PE $= 1/2\, kx^2$

By energy method in Fig. p-2.1.2,

PE + KE = Constant.

$\therefore$ PE for the given system $= \dfrac{1}{2}kx^2$

KE for the given system $= \dfrac{1}{2}mv^2 = \dfrac{1}{2}m\dot{x}^2$

$\therefore \frac{1}{2}kx^2 + \frac{1}{2}m\,\dot{x}^2 = \text{Constant}$

Differentiating both sides w.r.t. time 't',

$$\frac{d}{dt}\left[\frac{1}{2}kx^2 + \frac{1}{2}m\,\dot{x}^2\right] = \frac{d}{dt}(\text{Constant}), \frac{1}{2}2k\,x\,\dot{x} + \frac{1}{2}2m\,\dot{x}\,\ddot{x} = 0$$

$$kx\,\dot{x} + m\,\ddot{x}\,\dot{x} = 0, \quad [m\,\ddot{x} + kx]\,\dot{x} = 0, \dot{x} \neq 0$$

$$\therefore \quad m\,\ddot{x} + kx = 0$$

This is the equation of motion. Divided by 'm' throughout, $\ddot{x} + \frac{k}{m}x = 0$

Comparing with the SHM, $\omega_n = \sqrt{\frac{k}{m}}$ rad/s

Natural frequency $f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ cps or Hz

(iii) Rayleigh's method

By Rayleigh's method (Fig. p-2.1.3),

$$(\text{KE})_{\max} = (\text{PE})_{\max} = \text{Total energy of the system}$$

$$\text{PE}_{\max} = \frac{1}{2}kx^2 \quad \ldots \text{(a)}$$

$$\text{KE}_{\max} = \frac{1}{2}m(\dot{x})^2_{\max}$$

By SHM, $x = x \sin \omega t$,

$$\dot{x} = \omega\,x \cos \omega t, \; At\; \omega t = 0, \pi, 2\pi, \ldots$$

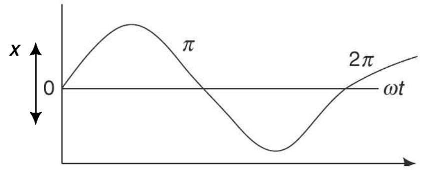


Fig. p-2.1.3 *Sine wave*

$\dot{x}$ will be maximum $\therefore \dot{x}_{\max} = \omega x$

$$\therefore \quad (\text{KE})_{\max} = \frac{1}{2}m(\omega x)^2 \quad \ldots\text{(b)}$$

From Eqs. (a) and (b), $\frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$

$$\therefore \quad k = m\,\omega^2 \quad \therefore \omega^2 = k/m, \text{ or } \omega_n = \sqrt{\frac{k}{m}} \text{ rad/s.}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi}$

$$\therefore \quad f_n = \frac{1}{2\pi}\sqrt{k/m} \text{ cps or Hz.}$$

EXAMPLE 2.2

Find the natural frequency of a spring-mass system considering the weight of the spring as shown in Fig. p-2.2.

Solution Let us consider a spring-mass system as shown in Fig. p-2.2(a). Let '*m*' be the mass of the spring. Let '*k*' be the spring stiffness and '*l*' be the length of the spring.

Consider an elemental length '*dx*' at a distance '*X*' from the fixed end and let '*W*' be the weight of the spring/unit length. Let 'δ' be the static deflection of the small elemental length '*dx*' and '*y*' be the static deflection of the entire length of the spring '*l*' as shown in FBD Fig. p-2.2(b).

∴ mass of the small elemental length '*dx*' of the spring = $W/g.\ dx$.

From similar triangles in Fig. p-2.2(b),

$$\frac{\delta}{y} = \frac{x}{l} \quad \therefore\ \delta = \frac{x}{l} y$$

Total KE = KE of mass + KE of spring.

To find KE of spring

We have KE of an elemental length *dx* can be written as

$$\text{KE}_{dx} = \frac{1}{2}\frac{w}{g}.\ dx.\delta^2$$

$$\text{KE}_{dx} = \frac{1}{2}\frac{w}{g}\left(\frac{x}{l}\right)^2 y^2\ dx \qquad \ldots(a)$$

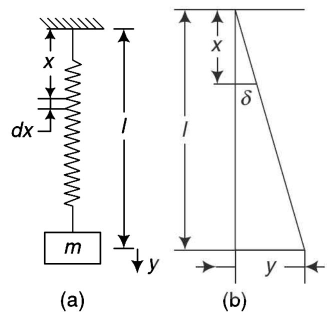


Fig. p-2.2 *Spring-mass system with effect of mass*

is the KE of the small elemental length '*dx*'.

∴ KE of the entire length '*l*' of the spring is given by integrating Eq. (a) between the limits 0 to *l*.

$$\therefore \quad \text{KE of spring} = \int_0^l \frac{1}{2}\frac{w}{g}\left(\frac{x}{l}\right)^2 \dot{y}^2\ dx = \frac{1}{2}\frac{w}{g}\frac{x}{l^2}\dot{y}^2\int_0^l x^2\ dx = \frac{1}{2}\frac{w}{gl^2}\dot{y}^2\left[\frac{x^3}{3}\right]_0^l$$

$$\therefore \quad = \frac{1}{2}\frac{w}{gl^2}\dot{y}^2\left[\frac{l^3}{3} - 0\right]$$

$$\therefore \quad \text{KE of spring} = \frac{1}{2}\left[\frac{wl}{3g}\right]\dot{y}^2,\ PE = \frac{1}{2}ky^2.$$

By energy method

$$\therefore \quad \text{Total KE} = \frac{1}{2}\left[\frac{wl}{3g}\right]\dot{y}^2 + \frac{1}{2}m\,\dot{y}^2, \quad \frac{d}{dt}[KE + PE] = 0$$

$$\text{KE} = \frac{1}{2}\left[m + \frac{wl}{3g}\right]\dot{y}^2 \ \therefore\ \frac{d}{dt}\left[\frac{1}{2}\left\{m + \frac{wl}{3g}\right\}\dot{y}^2 + \frac{1}{2}ky^2\right] = 0$$

$$\left(m + \frac{wl}{3g}\right)\dot{y}\,\ddot{y} + ky\,\dot{y} = 0$$

$$\left(m + \frac{wl}{3g}\right)\ddot{y} + ky = 0$$

Since w = Weight of the spring/unit length, wl = Weight of the spring.

Let $w_s = wl \therefore \left(m + \frac{wl}{3g}\right)\ddot{y} + ky = 0 \therefore \ddot{y} + \left[\frac{3kg}{3mg + w_s}\right]y = 0$

This is in the form of SHM.

$$\therefore \qquad \omega_n^2 = \frac{3kg}{3mg + w_s} \text{ rad/s}$$

$$\therefore \qquad \omega_n = \sqrt{\frac{3kg}{3mg + w_s}} \text{ rad/s}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi}$

$$\therefore \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{3kg}{3mg + w_s}} \text{ Hz or cps.}$$

EXAMPLE 2.3

Determine the natural frequency of a simple pendulum, as shown in Fig. p-2.3(a) if the mass of the rod is not negligible by using (i) energy method, and (ii) Newton's method.

Solution Let 'm' be the mass of the bob and 'm_r' be the mass of the rod.

(i) *By Energy Method*

We know that PE + KE = Constant, $\frac{d}{dt}$[PE + KE] = 0.

For small angular displacement of the pendulum,

$$\text{KE} = \frac{1}{2}mv^2 \text{ or KE} = \frac{1}{2}I_0\,\dot{\theta}^2 \therefore \text{KE} = \text{KE of mass} + \text{KE of rod}$$

$$\text{KE} = \frac{1}{2}ml^2\,\dot{\theta}^2 + \frac{1}{2}\left[\frac{1}{3}m_rl^2\right]\dot{\theta}^2 \therefore \text{KE} = \left[\frac{3ml^2 + m_rl^2}{6}\right]\dot{\theta}^2$$

$\text{KE} = \left[\frac{3ml^2 + m_rl^2}{6}\right]\dot{\theta}^2$, PE = PE of mass + PE of rod = $mgh + m_rgh^1$, where $h^1 = \frac{1}{2} - \frac{1}{2}\cos\theta$

$$\therefore \quad \text{PE} = mgl\,(1 - \cos\theta) + \frac{1}{2}m_rgl\,(1 - \cos\theta) \therefore \text{PE} = \left[mgl + \frac{1}{2}m_rgl\right](1 - \cos\theta)$$

$$\therefore \quad \frac{d}{dt}\left[\left(mgl + \frac{1}{2}m_rgl\right)(1 - \cos\theta) + \left\{\frac{3ml^2 + m_rl^2}{6}\right\}\dot{\theta}^2\right] = 0$$

$$\left(mgl + \frac{1}{2}m_rgl\right)(0 + \sin\theta \times \dot{\theta}) + \left[\frac{3ml^2 + m_rl^2}{6}\right]2\,\dot{\theta}\,\ddot{\theta} = 0$$

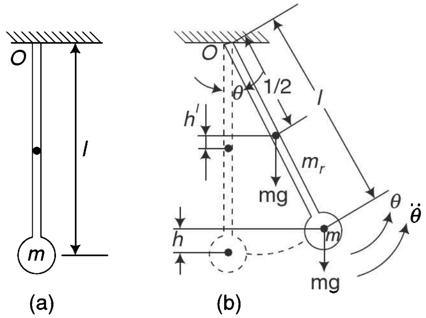


Fig. p-2.3 *Simple pendulum*

$$\left(mg + \frac{1}{2} m_r g\right) l \sin\theta \times \dot{\theta} + \left[ml^2 + \frac{1}{3} m_r l^2\right]\dot{\theta}\,\ddot{\theta} = 0$$

For small angles of 'θ', $\sin\theta \approx \tan\theta = \theta$

$$\therefore \quad \left(mg + \frac{1}{2} m_r g\right) l\,\theta\,\dot{\theta} + \left(m + \frac{1}{3} m_r\right) l^2\,\dot{\theta}\,\ddot{\theta} = 0$$

$$l\,\dot{\theta}\left[\left(m + \frac{1}{3} m_r\right) l\,\ddot{\theta} + \left(m + \frac{1}{2} m_r\right) g\,\theta\right] = 0,\ l\,\dot{\theta} \neq 0$$

$$\therefore \quad \left(m + \frac{1}{3} m_r\right) l\,\ddot{\theta} + \left(m + \frac{1}{2} m_r\right) g\theta = 0, \text{ and } \ddot{\theta} + \left[\frac{\left(m + \frac{1}{2} m_r\right) g}{\left(m + \frac{1}{3} m_r\right) l}\right]\theta = 0$$

$$\therefore \quad \omega_n = \sqrt{\frac{g\left(m + \frac{1}{2} m_r\right)}{l\left(m + \frac{1}{3} m_r\right)}} \text{ rad/s}$$

Natural frequency $f_n = \dfrac{\omega_n}{2\pi}$, $f_n = \dfrac{1}{2}\sqrt{\dfrac{\left(gm + \frac{1}{2} m_r\right)}{l\left(m + \frac{1}{3} m_r\right)}}$ Hz or cps

(ii) ***By Newton's method*** We know that $\Sigma F = ma$, and $\Sigma M_0 = I_0\,\ddot{\theta}$.

For the given system, $\Sigma M_0 = I_0\,\ddot{\theta}$

where $I_0 = MI$ of mass about 'O' (bob) + MI of rod about 'O'

$$\therefore \quad I_0 = Ml^2 + \frac{1}{3} m_r l^2,\ I_0 = \left[m + \frac{1}{3} m_r\right] l^2,$$

$$\therefore \quad -mg\,L \sin\theta\, m_r\, g\left(\frac{l}{2}\right) \sin\theta = \left[m + \frac{1}{3} m_r\right] l^2\,\ddot{\theta}$$

$$\therefore \quad \left[m + \frac{1}{3} m_r\right] l^2\,\ddot{\theta} + mgl \sin\theta + \frac{1}{2} mrgl \sin\theta = 0$$

For small angles of 'θ', $\sin\theta \approx \theta$ $\quad \therefore \left[m + \frac{1}{3}m_r\right]l^2\ddot{\theta} + \left[m + \frac{1}{2}m_r\right]gl\theta = 0$

$$\therefore \qquad l\left[\left(m + \frac{1}{3}m_r\right)l\ddot{\theta} + \left(m + \frac{1}{2}m_r\right)g\theta\right] = 0,\ l \neq 0$$

$$\therefore \left(m + \frac{1}{3}m_r\right)l\ddot{\theta} + \left(m + \frac{1}{2}m_r\right)g\theta = 0 \ \therefore\ \ddot{\theta} = -\left[\frac{\left(m + \frac{1}{2}m_r\right)g}{\left(m + \frac{1}{3}m_r\right)l}\right]\theta$$

$$\therefore \qquad \omega_n = \sqrt{\frac{g\left(m + \frac{1}{2}m_n\right)}{l\left(m + \frac{1}{3}m_r\right)}}\ \text{rad/s.}$$

Natural frequency, $f_n = \frac{\omega_n}{2\pi}$ $\quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{g\left(m + \frac{1}{2}m_r\right)}{l\left(m + \frac{1}{3}m_r\right)}}$ Hz or cps

EXAMPLE 2.4

Let us consider a simple pendulum as shown in Fig. p-2.4(a). At any instant, if the mass 'm' is displaced through an angle 'θ', then find the frequency of oscillation and time period.

(i) Newton's method

$\Sigma M = I_0\ddot{\theta}$, i.e. algebraic sum of external moment about 'O' in the direction of angular acceleration = Mass moment inertia of the system about 'O' × angular acceleration.

$\therefore\ -mgl\sin\theta = I_0\ddot{\theta}$, but $I_0 = ml^2$

$\therefore\ -mgl\sin\theta = ml^2\ddot{\theta}$, but for small angles of displacement, $\sin\theta \approx \tan\theta \approx \theta$.

$\therefore\ ml^2\ddot{\theta} + mgl\theta \quad \therefore\ \ddot{\theta} + \frac{g}{l}\theta = 0$

This is the equation of motion.

But for SHM, $\theta = A\sin\omega t$, $\dot{\theta} = A\omega\cos\omega t$, $\ddot{\theta} = -\omega^2 A\sin\omega t$

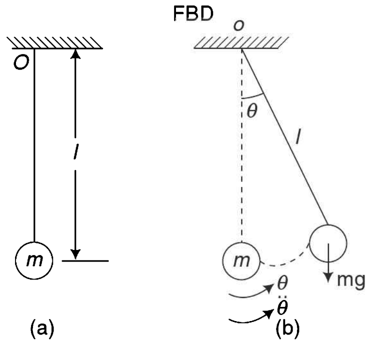


Fig. p-2.4 *Simple pendulum*

$$\therefore \qquad \ddot{\theta} = -\omega^2\theta,\ \ddot{\theta} + \omega^2\theta = 0\ \therefore\ \omega_n^2 = g/l,\ \omega^n = \sqrt{\frac{g}{l}}\ \text{rad/s}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi}$ $\quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ Hz, $\quad$ Time period $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{g/l}$ s

(ii) D'Alembert's principle

By this, $\Sigma F = 0$ or $\Sigma M = 0$, i.e. algebraic sum of forces or moment equal to zero.

$\therefore$ moment of inertia = $I_0 \ddot{\theta}$

Gravitational force due to mass = $-mgl\sin\theta$

$\therefore$ for $\Sigma M = 0$, $-I_0\ddot{\theta} - mg\,l\sin\theta = 0$ in Fig. p-2.4.1(a)

Since $\sin\theta \approx \tan\theta \approx \theta$ and $I_0 = ml^2$,

$\therefore$ $ml^2\ddot{\theta} + mg\,l\theta = 0$, $\ddot{\theta} + \left(\frac{g}{l}\right)\theta = 0$. Comparing with the SHM, $\omega_n = \sqrt{\frac{g}{l}}$ rad/s.

Natural frequency $f_n = \frac{\omega_n}{2\pi}$ $\therefore$ $f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ cps. Time period $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}$ s

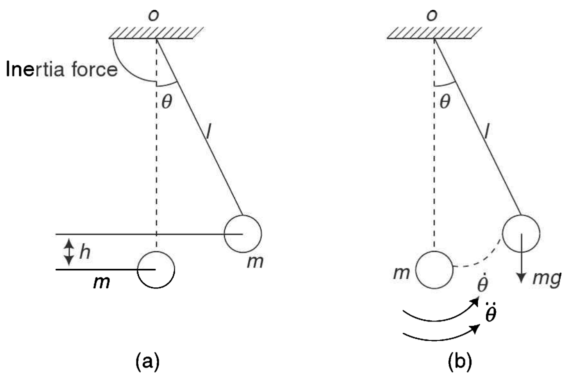


Fig. p-2.4.1 *Simple pendulum*

(iii) Energy method

By energy method, KE + PE = Constant.

$$\frac{d}{dt}(\text{KE} + \text{PE}) = 0$$

$$\therefore \quad \text{KE} = \frac{1}{2}mv^2 \text{ or KE} = \frac{1}{2}I_0\dot{\theta}^2 \quad \therefore \text{ KE} = \frac{1}{2}I_0\dot{\theta}^2$$

PE $= mgh$ in Fig. p-2.4.1(b) where $h = 1 - l\cos\theta$

$$\therefore \quad \text{PE} = mg\,(1 - l\cos\theta) \;\therefore\; \frac{d}{dt}\left[\frac{1}{2}I_0\dot{\theta}^2 + mg(1 - l\cos\theta)\right] = 0$$

$$\left[\frac{1}{2}I_0\, 2\dot{\theta}\ddot{\theta} + mg(0 + l\sin\theta)\,\dot{\theta}\right] = 0,\; I_0\ddot{\theta}\dot{\theta} + mgl\sin\dot{\theta} = 0, \quad (I_0\ddot{\theta} + mgl\sin)\dot{\theta} = 0$$

Since $\dot{\theta} \neq 0$,

$$\therefore \quad I_0\ddot{\theta} + mgl\sin\theta = 0$$

For small angles of θ, $\sin\theta \approx \tan\theta \approx \theta$, and $I_0 = ml^2$

$$\therefore \quad ml^2\ddot{\theta} + mgl\,\theta = 0,\; \ddot{\theta} + \left(\frac{g}{l}\right)\theta = 0$$

Comparing with the SHM,

$$\omega_n = \sqrt{\frac{g}{l}} \text{ rad/s}, \quad \text{time period } \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g} \text{ s}$$

Natural frequency $f_n = \omega_n/2\pi \quad \therefore \quad \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ Hz.

EXAMPLE 2.5

A compound pendulum is a rigid body of mass '*m*' pivoted at '*O*'. The point of the pivot is at a distance '*d*' from the centre of gravity '*G*' as shown in Fig. p-2.5(a) and it is free to rotate about its axis '*O*'. Determine the frequency of oscillation of this pendulum.

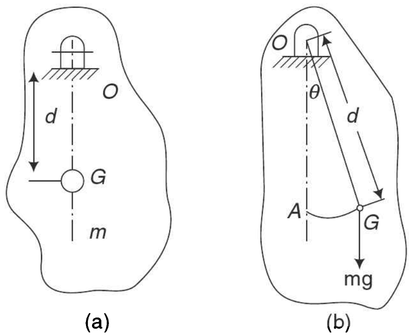


Fig. p-2.5 *Compound pendulum*

Solution For small angular displacement 'θ' of the compound pendulum in Fig. p-2.5(a), the FBD is as shown in Fig. p-2.5(b).

Now apply Newton's second law of motion, $I_0\ddot{\theta} = \Sigma T$ or $I_0\ddot{\theta} = -mgd \sin\theta$

For small angular displacement, $\sin\theta \approx \theta$

$$\therefore \qquad I_o\ddot{\theta} + mgd\,\theta = 0, \ddot{\theta} + \frac{mgd\,\theta}{I_0}$$

Comparing this equation with the SHM, $\omega = \omega_n = \sqrt{\frac{mgd}{I_0}}$ rad/s

$$\therefore \quad \text{natural frequency } f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgd}{I_0}} \text{ Hz}$$

$$\therefore \quad \text{time period } \tau = \frac{1}{f_n} = 2\pi\sqrt{\frac{I_0}{mgd}} \text{ s}$$

But for simple pendulum, $\tau = 2\pi\sqrt{\frac{l}{g}}$ s

EXAMPLE 2.6

The tension '*T*' of the string as shown in Fig. p-2.6(a) can be assumed constant for small displacements. Determine the natural frequency of the vertical vibration of the string and also show that the period of vibration is greatest when *a* = *b*.

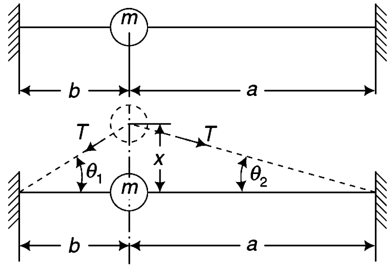


Fig. p- 2.6 *Tension string with mass*

Solution ***Assumptions:*** (i) Weight of the mass is not a factor as the system is in equilibrium. (ii) Tension in the inextensible strings does not change appreciably for small displacement of mass.

From FBD, Fig. p-.2.6(b), the restoring force is $T \sin \theta_1 + T \sin \theta_2$.

$= T\left(\frac{x}{a} + \frac{x}{b}\right) = Tx\left(\frac{1}{a} + \frac{1}{b}\right)$, if θ_1 and θ_2 are very small.

$\therefore$ the equation of motion is $m\ddot{x} = -Tx\left(\frac{1}{a} + \frac{1}{b}\right) \quad \therefore \ddot{x} + \frac{T}{m}\left(\frac{1}{a} + \frac{1}{b}\right). x = 0$

Comparing this equation with the SHM,

$$\omega^2 = \frac{T}{m}\left(\frac{1}{a} + \frac{1}{b}\right) \quad \therefore \omega_n = \sqrt{\frac{T(a+b)}{m\,ab}} \text{ rad/s}$$

$$\therefore \quad f_n = \frac{1}{2\pi}\sqrt{\frac{T(a+b)}{m\,ab}} \text{ rad/s}, \quad \text{time period } \tau = \frac{1}{f} = 2\pi\sqrt{\frac{m\,ab}{T(a+b)}} \text{s.}$$

The quantity $\frac{ab}{a+b}$ is maximum only when $a = b$ as $(a + b)$ is constant [if 'm' and 'T' are fixed], therefore time period (τ) is maximum when $a = b$.

EXAMPLE 2.7

A uniform plate of sides 'l' and mass 'm' is suspended from the midpoint of one of the sides as shown in Fig. p-2.7(a). Find out its frequency of vibration.

Solution At any instant if the square plate is displaced by an angle 'θ' then 'mg' will act vertically in the downward direction at the centre of the square plate, i.e. 1/2 of the length as shown in FBD of Fig. p-2.7(b).

Now apply newton's second law of motion,

$$I_0\,\ddot{\theta} = -\text{Restoring torque} = -mg\left(\frac{l}{2}\right)\sin\theta$$

If 'θ' is very small, $\sin\theta = \theta$

$$\therefore \quad I_0\,\ddot{\theta} = -\frac{mgl\theta}{2}$$

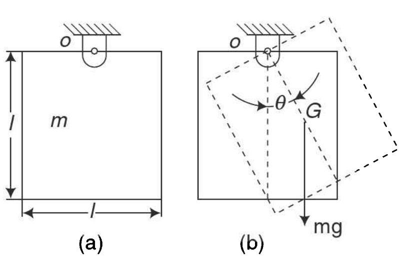


Fig. p-2.7 *Square plate*

Moment of inertia of square plate about centre of gravity (CG) 'G' is $I_G = \frac{ml^2}{6}$.

But $$I_0 = I_G + m\left(\frac{l}{2}\right)^2, \frac{ml^2}{6} + \frac{ml^2}{4} = \frac{2ml^2 + 3ml^2}{12} = \frac{5}{12}ml^2$$

$\therefore$ $$I_0\,\ddot{\theta} = \text{Restoring torque}, \frac{5}{12}ml^2\,\ddot{\theta} = -\frac{mgl\theta}{2}$$

$$\frac{5}{12}ml^2\,\ddot{\theta} + \frac{mgl\,\theta}{2} = 0, \frac{5}{6}l\,\ddot{\theta} + g\theta = 0, \ddot{\theta} + \frac{6g}{5l}\theta = 0, \omega_n^2 = \frac{6g}{5l}, \omega_n = \sqrt{\frac{6g}{5l}}\text{ rad/s}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi}$, or $f_n = \frac{1}{2\pi}\sqrt{\frac{6g}{5l}}$ Hz or cps

EXAMPLE 2.8

If a semicircular isotropic disc of radius 'r' and mass 'm' is pivoted freely about its centre as shown in Fig. p-2.8(a), determine the natural frequency of oscillation for small displacement.

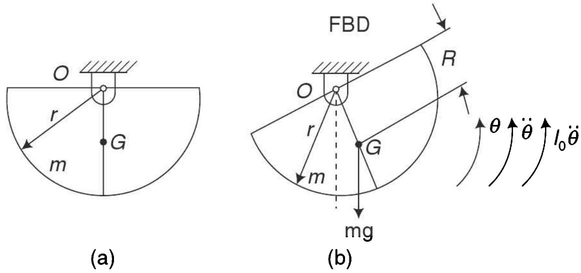


Fig. p-2.8 *Semicircular plate*

Solution Let 'θ' be the small angular displacement given to the homogeneous disc in counter clockwise direction, the FBD is as shown in Fig. p-2.8(b). The restoring torque is due to the tangential component of the weight of the disc acting vertically downward direction at centre of gravity 'G' equal to the at a distance $R = \frac{4r}{3\pi}$

Applying Newton's second law of motion, $\Sigma M_0 = I_o\ddot{\theta} = mg\,R\sin\theta$

If 'θ' is vary small, $\sin\theta = \theta$ and

$$R = \frac{4r}{3\pi}, I_0\,\ddot{\theta} = -mg\left(\frac{4r}{3\pi}\right)\theta, I_0\,\ddot{\theta} + mg\left(\frac{4r}{3\pi}\right)\theta = 0, I_o = \frac{1}{2}mr^2, \frac{mr^2}{2}\ddot{\theta} + mg\left(\frac{4r}{3\pi}\right)\theta = 0,$$

$$\frac{mr^2}{2}\ddot{\theta} + mg\frac{4r}{3\pi}\dot{\theta} = 0, \frac{r}{2}\ddot{\theta} + g\frac{4}{3\pi}\theta = 0$$

$$\therefore\ \ddot{\theta} = -\left[\frac{\frac{4g}{3\pi}}{r/2}\right]\theta, \ddot{\theta} = \frac{-8g}{3\pi r}\theta\quad \omega_n = \sqrt{\frac{8g}{3r\pi}}\text{ rad/s}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi}$ $\quad\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{8g}{3\pi r}}$ Hz or cps

EXAMPLE 2.9

Determine the equivalent stiffness for the system as shown in Fig. p-2.9(a).

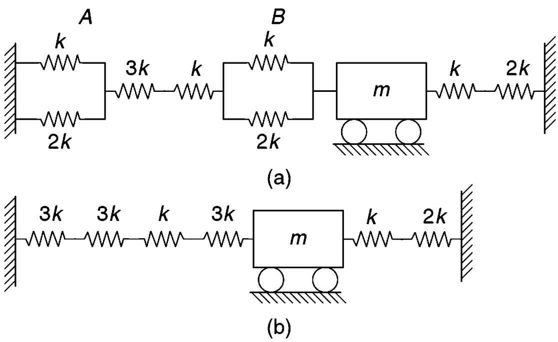


Fig. p-2.9 *Springs in series and parallel with mass*

Solution The springs '*A*' and '*B*' are in parallel and it will be reduced to Fig. p-2.9(b). In Fig. p-2.9(b), the LHS of mass of the springs are in series and also the RHS of the mass of springs are in series. The equivalent spring stiffness on LHS of mass, as follows.

$$\frac{1}{k_{eq\,L}} = \frac{1}{3k} + \frac{1}{3k} + \frac{1}{k} + \frac{1}{3k} \quad \frac{1}{k_{eq\,L}} = \frac{1+1+3+1}{3k} = \frac{6}{3k} = \frac{2}{k}.$$

$k_{eq\,L} = \frac{k}{2}$ is the equivalent spring stiffness on LHS of mass '*m*'.

Similarly, the equivalent spring stiffness on RHS of the mass is also

$$\frac{1}{k_{eq\,R}} = \frac{1}{k} + \frac{1}{2k}, = \frac{3}{2k}, k_{eq\,R} = \frac{2k}{3}$$

The above system now reduces to k_{eqL} and k_{eqR} which are in series; hence the total equivalent stiffness of the above spring system

$$k_{eq} = k_{eq\,L} + k_{eq\,R} = \frac{k}{2} + \frac{2k}{3}, = \frac{3k+4k}{6}, k_{eq} = \frac{7k}{6} k_{eq} = 1.18 \text{ N/m}.$$

EXAMPLE 2.10

Find the stiffness of the system at the point of force of application. The mass 1-2 is free to have rectilinear and angular motion as shown in Fig. p-2.10(a).

Solution Force 'F_0' at '*O*' is equivalent to force $F_0 l_2/(l_1 + l_2)$ and $F_0 l_1/(l_1 + l_2)$ at points '1' and '2' respectively. These forces give corresponding deflections at the two ends in Fig. p-2.10(b) as $\delta_1 = \frac{F_0 l_2}{(l_1 + l_2)k_1}$, $\delta_2 = \frac{F_0 l_1}{(l_1 + l_2)k_2}$.

From these, the deflection at point '*O*' can be obtained as $\delta = \frac{F_0\left(k_1 l_1^2 + k_2 l_2^2\right)}{(l_1 + l_2)^2\, k_1 k_2}$

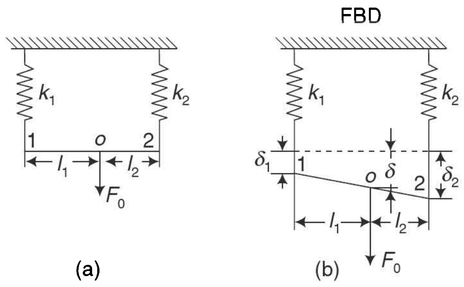


Fig. p-2.10 *Massless parallel spring*

Hence the stiffness at the point '*O*' is $k_0 = \dfrac{F_0}{\delta} = \dfrac{(l_1 + l_2)^2 k_1 k_2}{k_1 l_1^2 + k_2 l_2^2}$.

It may be noted that a point mass 'm_0' attached at the point '*O*' will give a natural frequency of the system as $\omega_n = \sqrt{\dfrac{k_0}{m_0}}$

This is dependent upon the position of the mass m_0.

EXAMPLE 2.11

Determine the equivalent stiffness of the system as shown in Fig. p-2.11(a).

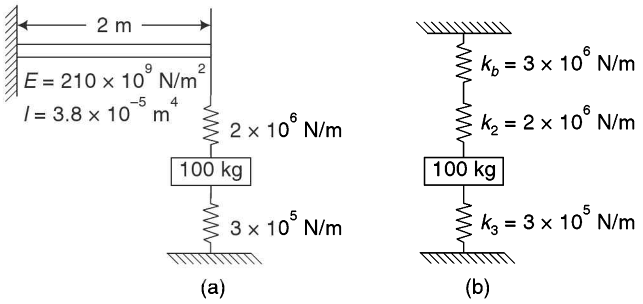


Fig. p-2.11 *Cantilever beam and attached spring mass at the end*

Solution Let 'δ' be the static deflection of the cantilever beam at the free end due to the force '*F*'

$$\therefore \qquad \delta = \frac{Fl^3}{3EI}, \text{ we know that } k = \frac{F}{\delta} \text{ or } \delta = \frac{F}{k}.$$

The stiffness of the cantilever beam is given by the equation $k = \dfrac{3EI}{l^3}$ N/mm.

$$k_b = \frac{3EI}{l^3} = \frac{3 \times 210 \times 10^9 \times 3.8 \times 10^{-5}}{2^3} = 2.9925 \times 10^6 \text{ N/m} \approx 3 \times 10^6 \text{ N/m}$$

The system now reduces to Fig. p- 2.11(b), that is,

$$\frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_2}, \quad \frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_3}$$

$$\frac{1}{k_{eq}} = \frac{1}{3 \times 10^6} + \frac{1}{2 \times 10^6} = \frac{2+3}{6 \times 10^6}, \; k_{eq} = \frac{6 \times 10^6}{5} \text{ N/m}$$

$$\therefore \; k_{eq} = \frac{6}{5} \times 10^6 + \frac{3}{1} \times 10^5, \; k_{eq} = \frac{6 \times 10^6 + 3 \times 10^5}{5}, \; k_{eq} = \frac{7.5 \times 10^6}{5}, k_{eq} = 1.5 \times 10^6 \text{ N/m.}$$

EXAMPLE 2.12

A simply supported beam has a concentrated load acting on the midspan as shown in Fig. p-2.12(a). If the mass of the beam is negligible compared to the mass acting, determine the natural frequency of the system.

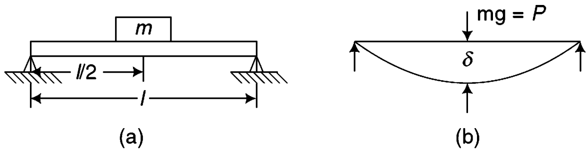


Fig. p-2.12 *Simply supported beam with load at midpoint*

Solution Due to the mass at the centre of the span, let a force 'P' be acting at the same point.

For a simply supported beam with a concentrated load 'P' acting at the centre, as shown in Fig. p-2.12(b), the deflection (δ) of the beam is given by

$\delta = \dfrac{Pl^3}{48EI}$ where l = Length of the beam, E = Young's modulus of the material of the beam, I = Moment of inertia of the section of the beam.

From the definition of spring stiffness, $k = \dfrac{P}{\delta} = \dfrac{P}{\left(\dfrac{Pl^3}{48EI}\right)} = \dfrac{48EI}{l^3}$ N/m.

The system now reduces to a simple spring-mass system and hence the natural frequency is given by

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s.}$$

EXAMPLE 2.13

Determine the natural frequency for the system as shown in Fig. p-2.13(a).

Solution Let 'θ' be the small angular displacement given to the mass 'm'; then the FBD of the system is as shown in Fig. p-2.13(b)

Applying D'Alembert's principle, taking moment $\Sigma Mo = 0$, we have

$$\therefore \quad I_o \ddot{\theta} = -ka\,\theta\, a \text{ or } I_o\ddot{\theta} + ka\,\theta\, a = 0$$

But $I_o = ml^2, ml^2\ddot{\theta} + ka^2\theta = 0, \ddot{\theta} + \frac{ka^2}{ml^2}\theta, \omega_n = \sqrt{\frac{ka^2}{ml^2}}$ rad/s

Natural frequency, $f_n = \frac{1}{2\pi}\sqrt{\frac{ka^2}{ml^2}}$ cps or Hz.

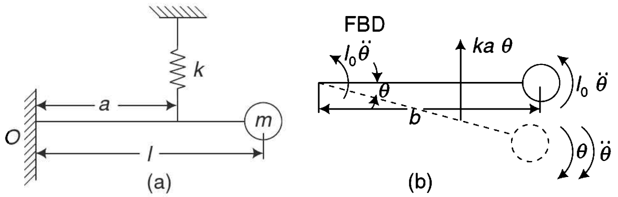


Fig. p-2.13 *System for Example 2.13*

EXAMPLE 2.14

A rigid weightless rod is restrained to oscillate in a vertical plane as shown in Fig. p-2.14(a). Determine the natural frequency of the mass '*m*'.

Solution Let 'θ' be the small angular displacement given to the mass '*m*'; then the FBD of the system is as shown in Fig. p-2.14(b).

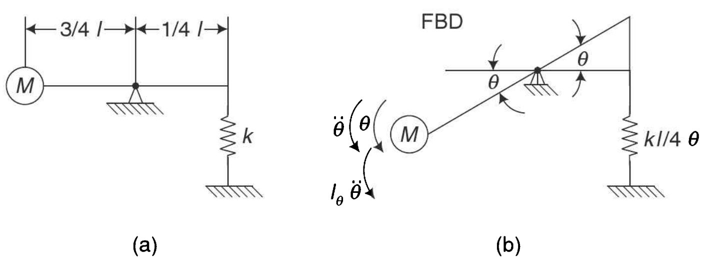


Fig. p-2.14 *Rigid weightless rod*

Applying Newton's second law of motion,

$$\Sigma Mo = I_o\ddot{\theta} \quad \therefore \frac{-kl}{4}\theta.\frac{1}{4} = I_0\ddot{\theta} \quad \therefore I_0\ddot{\theta} + \frac{kl^2}{16}\theta = 0$$

where $I_0 = M \times \left(\frac{3}{4}l\right)^2 = M \times \frac{9}{16}l^2 \quad \therefore I_0 = \frac{9Ml^2}{16} \quad \therefore \frac{9kl^2}{16}\ddot{\theta} + \frac{kl^2}{16}\theta = 0,$

$9M\ddot{\theta} + k\theta = 0$. This is the equation of motion.

$$\therefore \quad \ddot{\theta} + \frac{k}{9M}\theta = 0, \quad \omega_n = \sqrt{\frac{k}{9M}} \text{ rad/s.}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi} \qquad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k}{9M}}$ Hz or cps.

EXAMPLE 2.15

A stiff weightless horizontal bar of length '*L*' is pivoted at one end and carries a mass '*m*' at its other end as shown in Fig. p-2.15. It is held by an inextensible string of length '*l*'. If the mass '*m*' is vibrated in a plane perpendicular to that of the paper, find the natural frequency of the system.

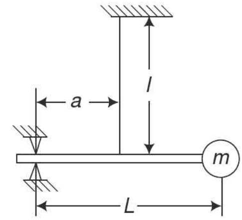


Fig. p-2.15 *Stiff weightless horizontal bar*

Solution In the static equilibrium position, the tension '*T*' in the string is given by

$$mg\,L = Ta \text{ or } T = \frac{mgL}{a}$$

Consider an angular displacement 'θ' of the mass in a plane perpendicular to that of the paper.

Let it result in angular displacement of 'α' in the string.

Then $\quad a\theta = l\alpha \quad \therefore\ \alpha = \dfrac{a\theta}{1}$

The equation of motion is

$I_0\,\ddot{\theta} = -(Ta \sin\alpha) = -Ta\,\alpha$; if α is very small, $\sin\alpha = \alpha$

But $I_0 = mL^2$ and $T = \dfrac{mgL}{a}$ $\quad \therefore\ mL^2\,\ddot{\theta} = -Ta\,\dfrac{a\theta}{l},\ \dfrac{\ddot{\theta} + Ta}{mL^2}\,\dfrac{a}{1}\,\theta = 0$

$$\therefore\ \ddot{\theta} + \frac{Ta}{m\,L^2}\,\frac{a}{1}\,\theta = 0 \quad \therefore\ \omega_n^{\ 2} = \frac{Ta}{lmL^2} = \frac{mgL}{mL^2}\cdot\frac{a}{l} = \frac{ag}{Ll}\ \text{rad/s} \quad \therefore\ f_n = \frac{1}{2\pi}\sqrt{\frac{ag}{Ll}}\ \text{cps.}$$

EXAMPLE 2.16

Determine the equation of motion for the following system as shown in Fig. p-2.16(a) and find the natural frequency of the system.

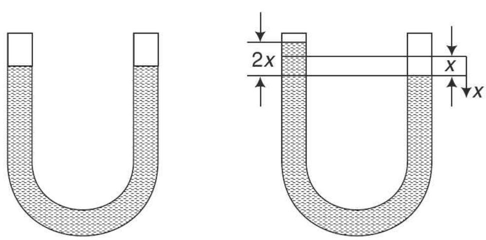


Fig. p-2.16 *U-tube manometer*

Solution Let '*l*' be the total length of water present in the U-tube manometer. Let '*x*' be the small displacement given to the liquid in the manometer in the downward direction as shown in Fig. p-2.16(b).

We know that density of the liquid $(\rho) = \dfrac{\text{Weight}}{\text{Volume}}$

$$= \frac{\text{Mass} \times \text{Acceleration due to gravity}}{\text{Area} \times \text{Length}}$$

$\therefore \qquad \rho = \dfrac{m \times g}{A \times l} \qquad \therefore$ mass of liquid 'm' $= \dfrac{\rho A l}{g}$.

(i) Energy method

$\dfrac{d}{dt}$(KE + PE) = 0, where KE $= \dfrac{1}{2} m v^2 = \dfrac{1}{2}\left(\dfrac{\rho A l}{g}\right) \cdot \dot{x}^2$

Length = l because the complete water in the U-tube is moved for a displacement 'x'.

KE $= \dfrac{1}{2}\left(\dfrac{\rho A l}{g}\right)\dot{x}^2$ ρ = Specific density, A = Area, Volume (v) = $A \times l$.

PE = Weight of the water displaced from mean position × Length (displacement)

$\therefore$ PE = $mg\,h \quad \therefore$ PE = $mg\,x$, PE $= \dfrac{\rho A x}{g} g \times x \quad \therefore$ PE $= \rho A x^2$.

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\left(\frac{\rho A l}{g}\right)\dot{x}^2 + \rho A x^2\right] = 0, \frac{1}{2}\left(\frac{\rho A l}{g}\right) x\,\dot{x}\,\ddot{x} + \rho A 2x.\,\dot{x} = 0, \rho A \dot{x}\left[\frac{l}{g}\ddot{x} + 2x\right]$$

$$= \rho A \dot{x} \neq 0$$

$\therefore \dfrac{l}{g}\ddot{x} + 2x = 0$. This is the equation of motion $\ddot{x} + \left(\dfrac{2g}{l}\right)x = 0$

Comparing with the SHM, we have

$\omega_n = \sqrt{\dfrac{2g}{l}}$ rad/s, natural frequency $f_n = \dfrac{\omega_n}{2\pi} \quad \therefore f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{2g}{l}}$ Hz.

(ii) Newton's method From Newton's second law of motion,

$\Sigma F = ma$, i.e. Restoring force = Inertia force.

$$m\ddot{x} = \Sigma F,\ m\ddot{x} = -(\rho A 2x)g, \text{ or } (\rho A l)\ddot{x} = -(\rho A 2x)g,\ (\rho A l)\ddot{x} + (\rho A 2g)x = 0$$

$$l\ddot{x} + 2gx = 0, \quad \text{or } \ddot{x} + \frac{2g}{l}x = 0,\ \omega^2 = \frac{2g}{l}$$

Natural frequency $\omega = \omega_n = \sqrt{\dfrac{2g}{l}}$ rad/s.

EXAMPLE 2.17

A solid wooden cylinder of radius '*r*' is partially immersed in a bath of distilled water as shown in Fig. p-2.17. The cylinder is slightly depressed and released. Find the natural frequency of oscillation of the cylinder if it stays upright all the time. What will the frequency be if salt water of specific gravity 1.2 is used instead of distilled water?

Solution Let 'x' be the displacement of the cylinder and 'ρ' be the specific gravity of water.

We know that density (ρ) $= \dfrac{\text{Weight}}{\text{Volume}} = \dfrac{\text{Mass} \times \text{Gravity}}{\text{Area} \times \text{Length}}$.

$\therefore$ Mass = $\dfrac{\text{Density} \times \text{Area} \times \text{Length}}{\text{Acceleration due to gravity}}$, mass of the cylinder, $m = \left[\dfrac{\rho \times \pi r^2 \times h}{g}\right]$

Let 's' be the specific gravity of wood.

$$m = \frac{\rho \pi r^2 h}{g} s$$

(i) Energy method

$$\frac{d}{dt}(\text{KE} + \text{PE}) = 0, \text{ KE} = \frac{1}{2} mv^2 = \frac{1}{2}\left[\frac{\rho\pi r^2 hs}{g}\right]\dot{x}^2$$

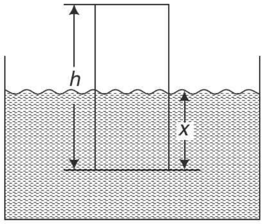


Fig. p-2.17 *Water bath with solid wooden cylinder*

PE = Weight of water displaced × Displacement

$$\text{PE} = \frac{\rho\pi r^2 x}{g} \times gx, \text{ P.E.} = \rho\pi r^2 x^2$$

$$\therefore \quad \frac{d}{dt}\left[\left(\frac{1}{2}\frac{\rho\pi r^2 hs}{g}\right)\dot{x}^2 + (\rho\pi r^2)x^2\right] = 0$$

$$\frac{\rho\pi r^2 hs}{g}\dot{x}\ddot{x} + 2\rho\pi r^2 x\,\dot{x} = 0, \frac{hs}{g}\ddot{x} + 2x = 0, \quad \therefore \ddot{x} + \frac{2g}{hs}x = 0.$$

Comparing this equation with the SHM,

$$\omega_n = \sqrt{\frac{2g}{hs}} \text{ rad/s. Natural frequency } f_n = \frac{1}{2\pi}\sqrt{\frac{2g}{hs}} \text{ Hz or cps.}$$

(ii) Newton's method

$\Sigma F = ma = m\ddot{x}$, mass of the wooden cylinder along with the acceleration will contribute the disturbing force.

Weight of the water displaced will give the restoring force.

$$\therefore \quad m\ddot{x} + \Sigma F = 0 \therefore \frac{\rho\pi r^2 hs}{g}\ddot{x} + 2\rho\pi r^2 x = 0, \frac{hs}{g}\ddot{x} + 2x = 0, \quad \ddot{x} + \frac{2gx}{hs} = 0$$

or $\omega_n = \sqrt{\dfrac{2g}{hs}}$ rad/s, $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{2g}{hs}}$ Hz, we know that specific gravity of distilled water = 1. If the salt water is used having specific gravity of 1.2 then $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{1.2g}{hs}}$ Hz

Therefore, the frequency is increased.

EXAMPLE 2.18

Determine the natural frequency of vibration of the torsion pendulum as shown in Fig. p-2.18 with following dimensions. Length of the shaft l = 100 cm, diameter of the rod d = 50 mm, diameter of the rotor D = 0.3 m, mass of the rotor m = 3 kg, modulus of rigidity $G = 8.4 \times 10^{10}$ N/m².

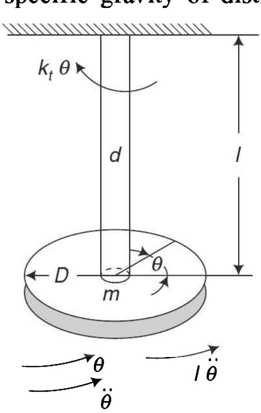


Fig. p-2.18 *Torsional pendulum*

Solution $l = 100$ cm $= 1000$ mm, $d = 50$ mm, $D = 0.3$ m, $G = 8.4 \times 10^{10}$ N/m^2, $m = 3$ kg

We know from torsional equation $\frac{T}{I_P} = \frac{G\theta}{l}, \frac{T}{\theta} = \frac{GI_P}{l}$

$\therefore$ torsional spring stiffness $k_t = \frac{T}{\theta} = \frac{GI_P}{l}$, where I_P = Polar moment of inertia, $= \frac{\pi d^4}{32}$

$\therefore k_t = \frac{\pi d^4}{32l} \times G \quad \therefore k_t = \frac{\pi \times (0.05)^4 \times 8.4 \times 10^{10}}{32 \times 1}$, $k_t = 51541.75$ N-m/ rad,

Applying Newton's second law, $k_t\theta = -I\ddot{\theta} \quad \therefore I\ddot{\theta} + k_t\vartheta = 0$ ...(a)

This is the equation of motion,

where I = Mass moment of inertia of the rotor about its centre

$$I = \frac{mr^2}{2} \text{ where } r = \text{radius of gyration}$$

$$\therefore \quad I = \frac{m \times \left(\frac{D}{2}\right)^2}{2}, I = \frac{3 \times (0.15)^2}{2}, I = 0.03375 \text{ kg-m}^2$$

From Eq. (a), $\ddot{\theta} = \frac{-kt\theta}{I} \quad \therefore \omega_n = \sqrt{\frac{kt}{I}}$ rad/s, $f_n = \frac{1}{2\pi}\sqrt{\frac{kt}{I}}$.

$$\therefore \quad f_n = \frac{1}{2\pi}\sqrt{\frac{51541.75}{0.03375}} = 196.68 \text{ Hz}$$

EXAMPLE 2.19

Determine the natural frequency for the following system as shown in Fig. p-2.19(a).

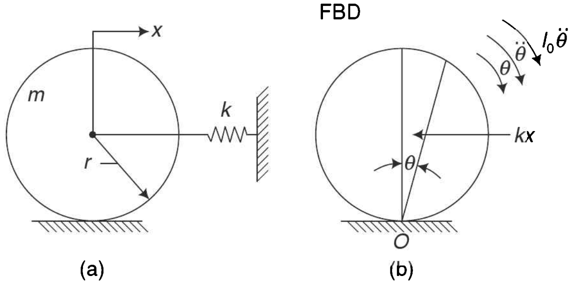


Fig. p-2.19 *Circular cylinder*

Solution Let 'θ' be the angular displacement for the corresponding linear distance 'x' as shown in Fig. p-2.19(b) of FBD.

$$\therefore \quad x = r\theta, \quad \dot{x} = r\dot{\theta}, \quad \ddot{x} = r\ddot{\theta}$$

Applying Newton's second law of motion,

$$\Sigma M_o = I_o\ddot{\theta} \;\therefore\; kx.r = -I_0\ddot{\theta} \quad \therefore\; I_0\ddot{\theta} + kr\theta.r = 0$$

But $\quad I_o = 1/2\, mr^2 + mr^2 = 3/2mr^2 \; \frac{3}{2} mr^2\,\ddot{\theta} + k\theta r^2 = 0$

This is the equation of motion.

$$\therefore \quad \ddot{\theta} = \frac{-2k}{3m}\theta,\ \omega_n = \sqrt{\frac{2k}{3m}}\ \text{rad/s}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$ Hz or cps.

Energy method

$$\text{KE} = \frac{1}{2}I_0\dot{\theta}^2 = \frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot{\theta}^2,\ \text{KE} = \frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot{\theta}^2$$

$$\text{PE} = \frac{1}{2}kx^2,\ \text{PE} = \frac{1}{2}kr^2\theta^2 \;\therefore\; \frac{d}{dt}[\text{KE} + \text{PE}] = 0$$

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\left(\frac{3}{2}m\right)r^2\dot{\theta}^2 + \frac{1}{2}kr^2\dot{\theta}^2\right] = 0,\quad \left(\frac{3}{2}m\right)r^2\dot{\theta}\,\ddot{\theta} + kr^2\theta\,\dot{\theta} = 0$$

$$\frac{3}{2}m\,\ddot{\theta} + k\theta = 0 \;\therefore\; \ddot{\theta} = \frac{-2k}{3m}\theta,\ \omega_n = \sqrt{\frac{2k}{3m}}\ \text{rad/s.}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$ Hz or cps.

EXAMPLE 2.20

A circular cylinder of mass '*M*' and radius '*R*' is connected by a spring of stiffness '*k*' as shown in Fig. p-2.20(a). If it is free to roll on the rough surface which is horizontal without slipping, find its natural frequency. What will happen to frequency if the system is placed on an inclined plane as shown in Fig. p-2.20(b)?

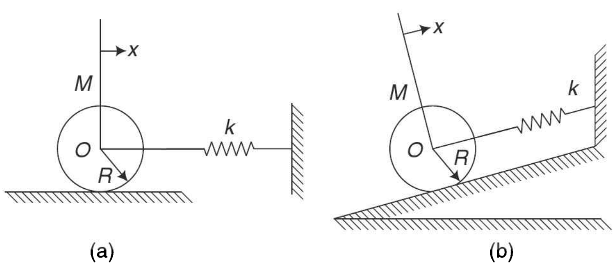


Fig. p-2.20 *Circular cylinder in horizontal and inclined plane*

Solution Let 'θ' be the angular displacement for the corresponding linear distance '*x*' as shown in Fig. p-2.20(b) of FBD. For the cylinder, applying Newton's second law of motion,

$$M\ddot{x} = -kx + F \text{ where '}F\text{' is the frictional force}$$

But $I_0\,\ddot{\theta} = -FR$ or $\left(\frac{1}{2}MR^2\right)\left(\frac{\ddot{x}}{R}\right) = -FR \quad [\because x = R\theta],\ \theta = x/R,\ \ddot{\theta} = \frac{\ddot{x}}{R},\ F = -\frac{1}{2}M\ddot{x}$

$$\therefore \qquad M\ddot{x} + kx + \frac{1}{2}M\ddot{x} = 0, \text{ or } \ddot{x} + \left(\frac{2k}{3m}\right)x = 0, \ \omega_n = \sqrt{\frac{2k}{3m}} \text{ rad/s.}$$

Natural frequency $f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$ cps.

If the cylinder is placed on an inclined plane as shown in Fig. p-2.20(b) then also, the free-body diagram of the cylinder remains unchanged and as such there will not be any change in the natural frequency of the system.

EXAMPLE 2.21

Determine the natural frequency of oscillations for the homogeneous cylinder as shown in Fig. p-2.21(a) by using energy method.

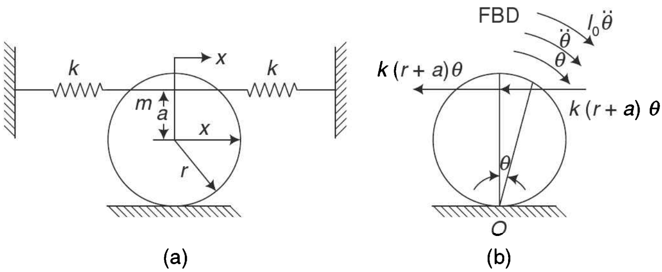


Fig. p-2.21 *Homogeneous solid cylinder restrained by two springs*

Solution Let 'θ' be the angular displacement for the corresponding linear distance 'x' as shown in Fig. p-2.21(b) of FBD.

Energy method

$$\text{KE} + \text{PE} = \text{Constant } \frac{d}{dt}(\text{KE} + \text{PE}) = 0. \quad \therefore \quad \text{KE} = \frac{1}{2}I_0\,\dot{\theta}^2$$

But $$I_0 = \frac{3}{2}mr^2, \ \text{KE} = \frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot{\theta}^2$$

$$\text{PE} = \frac{1}{2}2k(r+a)^2\theta^2, \quad \text{PE} = \frac{1}{2}\left[2k(r+a)^2\right]\theta^2$$

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot{\theta}^2 + \frac{1}{2}\left[2k(r+a)^2\theta^2\right]\right] = 0, \quad \frac{3}{2}mr^2\,\ddot{\theta} + 2k(r+a)^2\,\dot{\theta} = 0$$

$$\therefore \quad \ddot{\theta} = \frac{-4k(r+a)^2}{3mr^2}\theta, \ \omega_n = \sqrt{\frac{4k(r+a)^2}{3mr^2}} \quad \therefore \omega_n = \sqrt{\frac{4k(r+a)^2}{3mr^2}} \text{ rad/s.}$$

Natural frequency $f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{4k(r+a)^2}{3mr^2}}$ Hz or cps.

EXAMPLE 2.22

A homogeneous cylinder of mass 'm' is suspended by a spring 'k' N/m and an inextensible chord as shown in Fig. p-2.22(a). Determine the natural frequency of the cylinder.

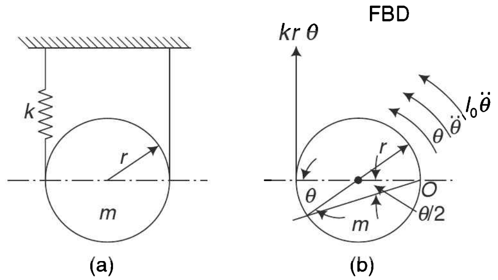


Fig. p-2.22 *Homogeneous solid cylinder suspended by a spring*

Solution Let 'θ' be the small angular displacement given to the pulley in counter-clockwise direction. Then the FBD is as shown in Fig. p-2.22(b). Point 'O' is the fixed one.

Applying Newton's second law of motion,

$$\Sigma M_o = I_o\ddot{\theta} \quad \therefore -2kr.2r = I_o\ddot{\theta},$$

But $$I_O = 1/2\, mr^2 + mr^2 = 3/2mr^2, \quad \left(\frac{3}{2}m\right) r^2\,\ddot{\theta} + 4kr^2\theta = 0$$

$$\therefore \quad \ddot{\theta} = -\frac{4k}{\frac{3m}{2}}\theta \quad \therefore \ddot{\theta} = -\frac{8k}{3m}\theta \quad \therefore \omega_n = \sqrt{\frac{8k}{3m}} \text{ rad/s}$$

$$\therefore \text{Natural frequency } f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{8k}{3m}} \text{ Hz or cps.}$$

Energy method $\text{KE} = \frac{1}{2} I_0\dot{\theta}^2$, but $I_0 = \left(\frac{3}{2} mr^2\right)$ $\text{KE} = \frac{1}{2}\left[\frac{3}{2}mr^2\right]\dot{\theta}^2$ $\text{PE} = \frac{1}{2} k(2r\theta)^2$

$$\text{PE} = \frac{1}{2}[4kr^2\theta^2] \quad \therefore \frac{d}{dt}[\text{KE} + \text{PE}] = 0 \quad \frac{d}{dt}\left[\frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot{\theta}^2 + \frac{1}{2}4kr^2\theta^2\right] = 0$$

$$\frac{3}{2}m\ddot{\theta} + 4k\theta = 0 \therefore \ddot{\theta} = \frac{-4k}{\frac{3m}{2}}\theta \quad \ddot{\theta} = -\frac{8k}{3m}\theta$$

$$\therefore \omega_n = \sqrt{\frac{8k}{3m}} \text{ rad/s}$$

$$\therefore \text{Natural frequency } f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{8k}{3m}} \text{ Hz or cps}$$

EXAMPLE 2.23

The cylinder of mass '*m*' and radius '*r*' rolls without slipping on a circular surface of radius '*R*' as shown in Fig. p-2.23(a). Determine the frequency of oscillation when the cylinder is displaced slightly from its equilibrium position. Use energy method.

Solution At any instant for the small angular displacement 'θ' to the cylinder of mass 'm' as shown in Fig. p-2.23(a). Then the FBD is as shown in Fig. p-2.23(b).

Energy method

$$(\text{KE} + \text{PE})_{\text{system}} = \text{Constant}, \frac{d}{dt}[\text{KE} + \text{PE}] = 0, \text{KE}_{\text{system}} = (\text{KE})_{\text{Rot}} + (\text{KE})_{\text{Translation}}$$

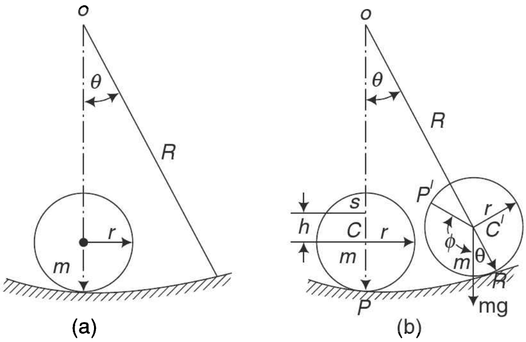


Fig. p-2.23 *Cylinder rolls without slipping on a circular surface*

When the rolling cylinder is at the mean position, the point 'P^1' will coincide with 'P'

$\therefore$ Arc PR = Arc RP^1 $\quad \therefore R\theta = r\phi$

For small angles of θ, $\sin \approx \theta$ $\quad \therefore C^1S = (R-r)\theta \quad \therefore \theta C^1 = (R-r)$

Translation velocity = $(R-r)\,\dot{\theta}$, Rotational velocity = $(\dot{\phi} - \dot{\theta})$

$$\therefore \quad \text{KE}_{\text{system}} = \frac{1}{2}m[(R-r)^2\dot{\theta}^2] + \frac{1}{2}I_o = \frac{1}{2}\text{mr}^2(\dot{\phi}-\dot{\theta})^2$$

$$\text{KE} = \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}\frac{mr^2}{2}(\dot{\phi}-\dot{\theta})^2.$$

$$\text{KE} = mgh,$$

$$\text{PE} = mg\,(R-r)\,[1-\cos\theta]$$

$$\because \quad h = OC - OS, \quad h = (R-r) - (R-r)\cos\theta, \quad h = (R-r)\,[1-\cos\theta]$$

$$\therefore \quad \text{PE} = mg(R-r)\,[1-\cos\theta]$$

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}\frac{mr^2}{2}(\dot{\phi}-\dot{\theta})^2 + mg(R-r)(1-\cos\theta)\right] = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}\frac{mr^2}{2}\left(\frac{R}{r}\dot{\theta} - \dot{\theta}\right)^2 + mg(R-r)(1-\cos\theta)\right] = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{4}mr^2\left(\frac{R}{r}-1\right)^2\dot{\theta}^2 + mg(R-r)(1-\cos\theta)\right] = 0$$

$$m(R-r)^2\dot{\theta}\ddot{\theta} + \frac{mr^2}{2}\frac{(R-r)^2}{r^2}\dot{\theta}\ddot{\theta} + mg(R-r)(0+\sin\theta)\,\dot{\theta} = 0$$

$$\left[m(R-r)^2 + \frac{m}{2}(R-r)^2\right]\ddot{\theta} + mg(R-r)\theta = 0$$

$$\frac{3}{2}m(R-r)^2\ddot{\theta} + mg(R-r)\theta = 0,\ \frac{3}{2}(R-r)\ddot{\theta} + g\theta = 0$$

This is the equation of motion.

$$\ddot{\theta} = \frac{-2g}{3(R-r)}\theta,\ \omega_n = \sqrt{\frac{2g}{3(R-r)}}\ \text{rad/s.}$$

Natural frequency $f_n = \dfrac{\omega_n}{2\pi}$ $\quad \therefore f_n = \dfrac{1}{2}\sqrt{\dfrac{2g}{3(R-r)}}$ Hz or cps.

EXAMPLE 2.24

Determine the equivalent mass (m_{eq}) and equivalent spring stiffness (k_{eq}) for the system as shown in Fig. p-2.24 where '*x*' the downward displacement of the block measured from the system. Equilibrium position is used as the generalised coordinate.

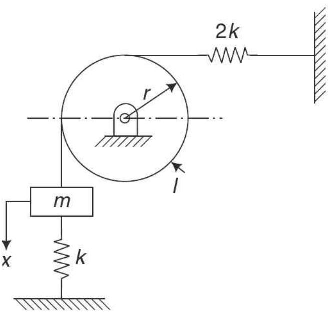


Fig. p-2.24 *Pulley with attached spring-mass*

Solution The equivalent spring stiffness of 'k' and '$2k$' $k_{eq} = 3k$.

The mass 'm' and pulley of mass moment of inertia is 'I'

$\therefore$ the equivalent mass

$$m_{eq} = m + \frac{I}{r^2},\ \text{but } I = \frac{1}{2}Mr^2$$

where M = Mass of the pulley. $\quad \therefore m_{eq} = m + \dfrac{I}{r^2} = m + \dfrac{Mr^2}{2r}. \quad \therefore m_{eq} = m + \dfrac{M}{2}$

EXAMPLE 2.25

Determine the natural frequency for the following system as shown in Fig. p-2.25(a).

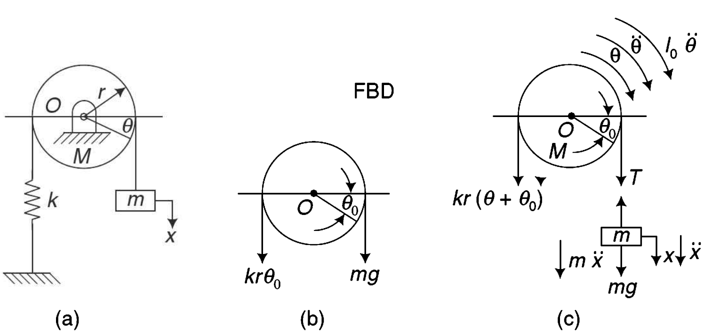


Fig. p-2.25 *Pulley with attached spring-mass*

Solution Let 'θ_0' be the initial static displacement. Let 'x' be the small linear displacement given to the mass 'm' and 'θ' be the angular displacement of the pulley, for the displacement 'x' as shown in Fig. p-2.25(b) and mass 'm' and pulley 'M' as shown in Fig. p-2.25(c).

(i) Newton's method Newton's second law of motion for mass 'm', $\Sigma F = m\ddot{x}$.

$$\therefore \quad mg - T = m\ddot{x} \quad \therefore -T = m\ddot{x} - mg \quad \text{or} \quad T = mg - m\ddot{x}$$

For pulley M, $\Sigma Mo = I_o\ddot{\theta}$ $\therefore kr(\theta + \theta_o).\, r - Tr = -I_o\ddot{\theta}$ in Fig. p-2.25(c).

$$kr^2\,\theta + kr^2\,\theta_o - (mg - m\ddot{x})\,r = -I_o\ddot{\theta}$$

But at equilibrium, $mgr = kr^2\theta_o$ $\therefore kr^2\theta + mr\ddot{x} = -I_0\ddot{\theta}$, $I_0\ddot{\theta} + mr^2\ddot{\theta} + kr^2\theta = 0$.

$$\therefore \quad \left[\frac{Mr^2}{2} + mr^2\right]\ddot{\theta} + kr^2\theta = 0 \quad \text{since } I_0 = \frac{Mr^2}{2}.$$

$$\left[\frac{M}{2} + m\right]\ddot{\theta} + k\theta = 0$$

This is the equation of motion.

$$\therefore \quad \ddot{\theta} = -\left[\frac{k}{\frac{m}{2} + m}\right]\theta \quad \therefore \omega_n = \sqrt{\frac{k}{\left(\frac{m}{2} + m\right)}}\ \text{rad/s} \quad \therefore \omega_n = \sqrt{\frac{2k}{(M + 2m)}}\ \text{rad/s}$$

$$\text{Natural frequency } f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k}{\left(\frac{M}{2} + m\right)}} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{(M + 2m)}}\ \text{Hz or cps}$$

(ii) Energy method KE of the system = KE of mass 'm' + KE of the pulley 'M'

$$\frac{1}{2}mx\dot{x}^2 + \frac{1}{2}I_0\dot{\theta}^2 = \frac{1}{2}mr^2\,\dot{\theta}^2 + \frac{1}{2}\frac{Mr^2}{2}\dot{\theta}^2,\ \text{KE} = \frac{1}{2}\left[m + \frac{M}{2}\right]r^2\,\dot{\theta}^2$$

$$\text{PE} = \frac{1}{2}kx^2,\ \text{PE} = \frac{1}{2}kr^2\theta^2 \quad \therefore \frac{d}{dt}[\text{KE} + \text{PE}] = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}\left[\frac{M}{2} + m\right]r^2\,\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2\right] = 0$$

$$\left[\frac{M}{2} + m\right]r^2\,\dot{\theta}\,\ddot{\theta} + kr^2\,\dot{\theta}\ \theta = 0, \left[\frac{M}{2} + m\right]\ddot{\theta} + k\theta = 0$$

This is the equation of motion,

$$\therefore \quad \ddot{\theta} = -\left[\frac{k}{\frac{M}{2} + m}\right]\theta \quad \therefore \omega_n = \sqrt{\frac{k}{\left(\frac{M}{2} + m\right)}}\ \text{rad/s} \quad \therefore \sqrt{\frac{2k}{(M + 2m)}}$$

$$\text{Natural frequency, } f_n = \frac{\omega_n}{2\pi} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k}{\left(\frac{M}{2} + m\right)}} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{(M + 2m)}}\ \text{Hz or cps.}$$

EXAMPLE 2.26

Determine the equivalent torsional stiffness of the system as shown in Fig. p-2.26.

Solution $G_1 = G_2 = G_3 = 40 \times 10^9$ N /m^2

(1) $r_1 = 5$ cm

(2) $r_2 = 8$ cm

(3) $r_3 = 4$ cm

$l_1 = 60$ mm, $l_2 = 80$ mm, $l_3 = 50$ mm

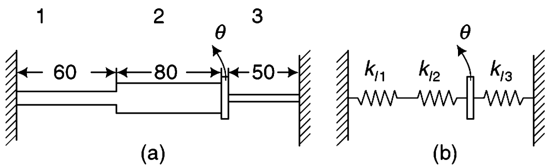


Fig. p-2.26 *Equivalent torsional system*

Let 'k_{t1}', 'k_{t2}' and 'k_{t3}' be the torsional stiffnesses of springs 1, 2 and 3 respectively.

$$k_{t1} = \frac{GI_{P1}}{l_1} = \frac{40 \times 10^9 \times \dfrac{\pi \times 0.1^4}{32}}{0.6} = 0.6545 \times 10^6 \text{ N-m/rad}$$

$$k_{t2} = \frac{40 \times 10^9 \times \dfrac{\pi \times 0.16^4}{32}}{0.8} = 3.217 \times 10^6 \text{ N-m/rad}$$

$$k_{t3} = \frac{40 \times 10^9 \times \dfrac{\pi \times 0.08^4}{32}}{0.5} = 3.217 \times 10^6 \text{ N-m/rad}$$

The system now reduces to Fig. p-2.26(b). The springs 'k_1' 'k_2' and 'k_3' are in series.

$$\therefore \quad \frac{1}{k_{\text{eq}_\text{L}}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}, \frac{1}{k_{\text{eqL}}} = \frac{1}{0.6545 \times 10^6} + \frac{1}{3.217 \times 10^6},$$

$$+ \frac{1}{3.217 \times 10^6}, \frac{1}{k_{\text{eqL}}} = \frac{3.217 + 0.6545}{2.1 \times 10^6}, k_{t\,\text{eqL}} = 0.542 \times 10^6 \text{ Nm/rad.}$$

$$k_{t\,\text{eq}} = k_{t\,\text{eqL}} + k_{t3} = 0.542 \times 10^6 + 3.217 \times 10^6, k_{t\,\text{eq}} = 0.864 \times 10^6 \text{ Nm/rad.}$$

REVIEW QUESTIONS

(1) Explain (i) Newton's method, (ii) Energy method, (iii) Rayleigh's method, and (iv) D'Alembert's principle to determine the natural frequency of a system.

(2) Find the natural frequency of a spring-mass system considering the weight of the spring.

(3) Determine the natural frequency of a simple pendulum, if the mass of the rod is not negligible

(i) By Energy method (ii) By Newton's method

(4) Determine the natural frequency of vibrations of a compound pendulum which is relatively pivoted at a distance 'd' from its centre of mass, taking the mass as 'm'.

PROBLEMS FOR PRACTICE

(1) Calculate the length of a simple pendulum whose time period of oscillation is 2 s. $g = 9.81$ m/s^2.

Ans. 1 m.

(2) Find the natural frequency of the compound pendulum shown in Fig. p.p-2.2.

Ans. $\omega_n = \sqrt{\dfrac{mgr}{I_G + mr^2}}$ rad/s. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{mgr}{I_G + mr^2}}$ Hz.

(3) Find the equivalent spring stiffness and natural frequency for the following system as shown in Fig. p.p-2.3.

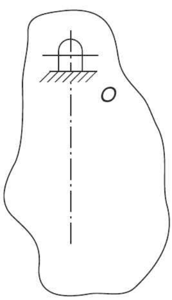


Fig. p.p-2.2

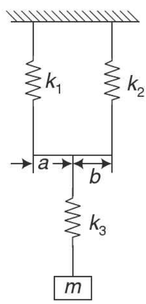


Fig. p.p-2.3

Ans. $k_{eq} = \dfrac{k_3 k_{e1}}{k_3 + k_{e1}}$ $\omega_n = \sqrt{\dfrac{k_{eq}}{m}}$ rad/s.

(4) Determine the natural frequency for the system shown in Fig. p.p-2.4. The bar is horizontal, so that the gravity has no influence.

Ans. $\omega_n = \sqrt{\dfrac{5k}{6m}}$ rad/s.

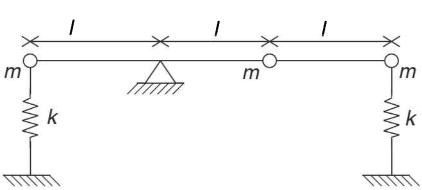


Fig. p.p-2.4

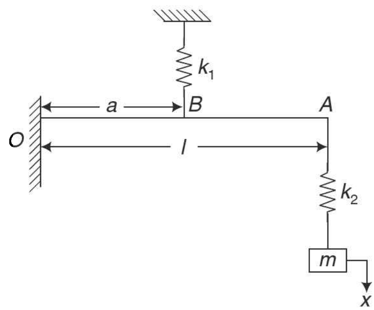


Fig. p.p-2.5

(5) Determine the natural frequency for the system shown in Fig. p.p.2.5, assuming that the bar should be weightless and rigid.

Ans. $\omega_n = \sqrt{\dfrac{k_1 k_2 a^2}{m(k_1 a^2 + k_2 l^2)}}$ rad/s.

(6) If the mass of the pulleys shown in Fig. p.p.2.6 is small and the chord is inextensible, show that the natural frequency of vibration of the system is given by

$$\omega_n = \sqrt{\frac{k_1 k_2}{4m(k_1 + k_2)}} \text{ rad/s.}$$

(7) Determine the natural frequency for the system shown in Fig. p.p-2.7. Neglecting the mass of the pulley, m = 15 kg, $k_1 = 8 \times 10^3$N/m, $k_2 = 6 \times 10^3$N/m.

Ans. $\omega_n = \sqrt{\dfrac{k_{eq}}{m}}$ rad/s. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{4k_1 k_2}{m(4k_1 + k_2)}}$ Hz. f_n = 2.9 Hz.

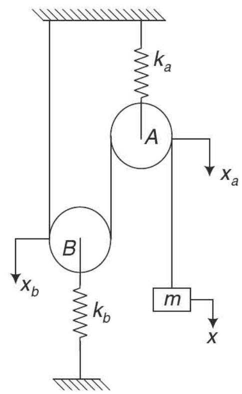


Fig. p.p-2.6

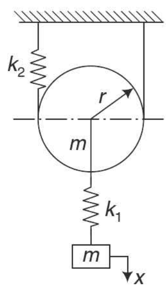


Fig. p.p-2.7

(8) A wheel and axle assembly of *MI* is inclined from the vertical by an angle 'θ' shown in Fig. p.p-2.8. Determine the frequency of oscillation due to a small unbalance weight 'w' at a distance 'd' from the axle.

Ans. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{wd \sin\theta}{1}}$ Hz.

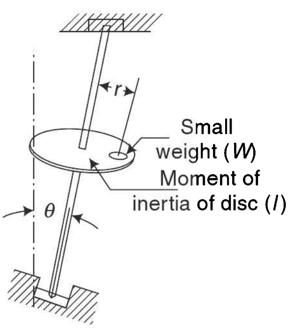


Fig. p.p-2.8

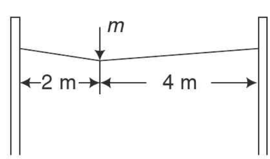


Fig. p.p-2.9

(9) An acrobat weighing 50 kg walks on a tightrope as shown in Fig. p.p-2.9. If the natural frequency of the vibration in the given position in the vertical direction is 10 cycles/min, determine the tension in the rope.

(10) The end of a shaft with a heavy disc of moment of inertia 'I' are built in as shown in Fig. p.p-2.10. Determine the natural frequency of torsional vibration of the disc.

Ans. $\omega_n = \sqrt{\dfrac{\pi D^2 G(l_1 + l_2)}{32 l_1 l_2 I}}$ rad/s.

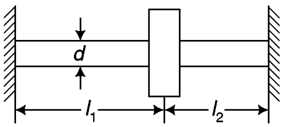


Fig. p.p-2.10

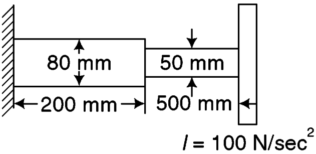


Fig. p.p-2.11

(11) Determine the natural frequency for torsional vibration of the system as shown in Fig. p.p-2.11.

Ans. $\omega_n = \sqrt{\dfrac{k_t}{I}}$ rad/s.

(12) A laboratory has got a set-up as shown in Fig. p.p. 2.12 meant for finding out the modulus of rigidity 'G' of a thin wire carrying a cylindrical rod at its end, the other being fixed. By giving torsional vibration, the time period 'τ' is noted down. Deduce an expression for 'G'.

Ans. $G = \dfrac{8\pi}{3\tau^2} \cdot \dfrac{MI}{d^4}(4L^2 + 3D^2)$.

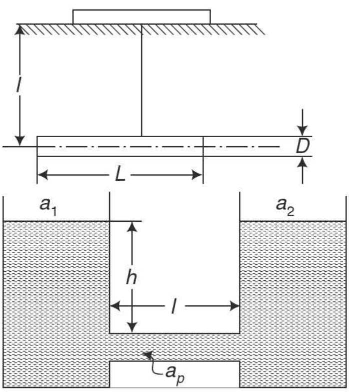


Fig. p.p-2.12

(13) A pipe of length 'l' is the length between two tanks and cross-sectional area 'a_p' connects two tanks of cross-sectional area a_1 and a_2 as shown in Fig. p.p-2.13. Show that the period of motion for small oscillations of the fluid between two tanks is given by $\tau = \sqrt{\dfrac{ha_p(a_1 + a_2) + la_1a_2}{ga_p(a_1 + a_2)}}$, where '$h$' is the height of the fluid in the tanks above the level of the connecting pipe.

(14) Determine the natural frequency of the system as shown in Fig. p.p-2.14 where MI of the mass 'm' about 'O' is I_o.

Ans. $\omega_n = \sqrt{\dfrac{ka^2 - mgb}{I_0}}$ rad/s. $f_n = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{ka^2 - mgb}{I_0}}$ cps or Hz.

(15) Determine the natural frequency for the following system as shown in Fig. p.p-2.15. Assume the chord is an inextensible one, if mass 'm' is displaced slightly and released.

Ans. $\omega_n = \sqrt{\dfrac{k}{\left(\dfrac{3}{2}M + 4m\right)}}$ rad/s. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{\left(\dfrac{3}{2}M + 4m\right)}}$ Hz.

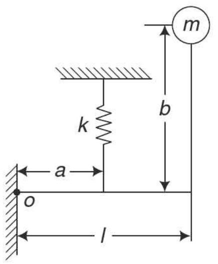


Fig. p.p-2.14

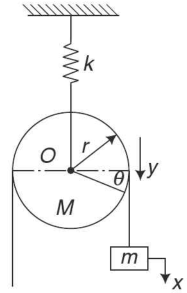


Fig. p.p-2.15

(16) Determine the natural frequency of oscillations for the homogeneous cylinder shown in Fig. p.p-2.16 by Newton's method.

Ans. $\omega_n = \sqrt{\dfrac{4k(r+a)^2}{3mr^2}}$ rad/s. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{4k(r+a)^2}{3mr^2}}$ Hz or cps.

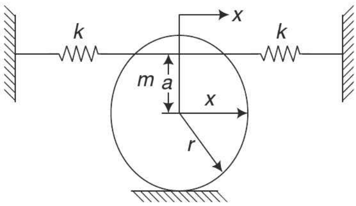


Fig. p.p-2.16

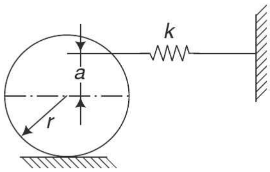


Fig. p.p-2.17

(17) A circular disc of 5 kg mass and 100 mm radius is held by a spring of 200 N/m constant at a distance of 50 mm from the centre and rolls on a smooth horizontal plane as shown in Fig. p.p- 2.17. Find the frequency of the system by energy method.

Ans. $\omega_n = \sqrt{\dfrac{k(r+a)^2}{mr^2+J}}$ rad/s. $\omega_n = 7.75$ rad/s, where $J = \dfrac{mr^2}{2}$.

(18) A circular disc of 5 kg mass and 100 mm radius is held by a spring stiffness of 20000 N/mm at its centre. The disc rolls without slipping on a smooth horizontal surface. Determine the frequency of oscillation.

Ans. 5.2 rad/s.

(19) A circular cylinder of 0.5 kg mass and 100 mm radius rolls without slipping on a cylindrical concave surface of 2 m radius. Determine the frequency of oscillation.

Ans. 0.3 Hz.

(20) Find the natural frequency for the system as shown in Fig. p.p- 2.20 by (i) energy method, and (ii) Newton's method.

Ans. $\omega_n = \sqrt{\dfrac{2(k_1+k_2)}{m}}$ rad/s, $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{2(k_1+k_2)}{m}}$ Hz.

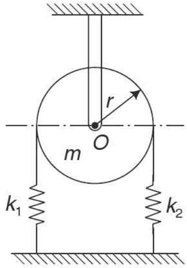


Fig. p.p-2.20

OBJECTIVE-TYPE QUESTIONS

1. In a spring-mass system, the static deflection caused by mass 'm' the spring of stiffness 'k' is 'δ'. The natural frequency of free longitudinal vibration of the system will be given by

   (a) $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{mg}}$ Hz

   (b) $f_n = \frac{1}{2\pi}\sqrt{\frac{mg}{\delta}}$ Hz

   (c) $f_n = \frac{1}{2\pi}\sqrt{\frac{m}{k}}$ Hz

   (d) $f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ Hz

2. In a spring-mass system, the static deflection caused by mass 'm' of the spring of stiffness 'k' is 'δ'. The time period of free longitudinal vibration of the system is given by

   (a) $\tau = 2\pi\sqrt{\frac{m}{k}}$ s

   (b) $\tau = 2\pi\sqrt{\frac{\delta}{g}}$ s

   (c) $\tau = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ s

   (d) both (a) and (b)

3. Two springs of stiffness k_1 and k_2 are connected in series and mass 'm' is attached to it The natural frequency of the longitudinal vibration will be given by

   (a) $f_n = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)}{mk_1k_2}}$ Hz

   (b) $f_n = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m(k_1 + k_2)}}$ Hz

   (c) $f_n = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)}{2m}}$ Hz

   (d) $f_n = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)}{m}}$ Hz

4. The natural frequency of a system is a function of

   (a) the spring-mass system

   (b) mass of the system

   (c) stiffness of the spring

   (d) none of the above

5. In the spring-mass system, if the mass of the system is doubled with spring stiffness halved, the natural frequency of vibration

   (a) remains unchanged

   (b) is doubled

   (c) is halved

   (d) is quadrupled

6. In the spring-mass system the mass 'm' and spring stiffness 'k' is taken to very high altitude, the natural frequency of vibrations

   (a) increases

   (b) remains unchanged

   (c) decreases

   (d) may increase or decrease depending upon the value of the spring-mass system

7. The equation of motion for free-vibrations of spring-mass system without damping is given by

   (a) $\ddot{x} + \frac{c\dot{x}}{m} + \frac{kx}{m} = 0$

   (b) $\ddot{x} + \frac{c\dot{x}}{m} + \frac{k}{m}x = F_0 \sin \omega t$

   (c) $\ddot{x} + \frac{k}{m}x = 0$

   (d) $\ddot{x} + \frac{k}{m}x = F_0 + \cos \omega t$

8. The natural frequency of free torsional vibration of a shaft is given by

   $f_n = 2\pi\sqrt{\frac{kt}{I}}$ $\quad f_n = \frac{1}{2\pi}\sqrt{\frac{kt}{I}}$

   $f_n = \frac{1}{2\pi}\sqrt{k_t I}$ $\quad f_n = 2\pi\sqrt{k_t I}$

9. A mass of 'M' is attached to a spring whose upper end is fixed. The mass and stiffness of the spring are 'm' and 'k' respectively. The natural frequency of the spring-mass system would be

(a) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m+M}}$ Hz

(b) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{3k}{m+3M}}$ Hz

(c) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{2k}{m+M}}$ cps

(d) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m+2M}}$ cps

10. In a spring-mass system of mass '*m*' and stiffness '*k*', the ends of the spring are surely fixed and the mass is attached to intermediate point of the spring. The natural frequency of longitudinal vibration of the system
   (a) is maximum when the mass is attached to the midpoint of the spring
   (b) decrease as the distance from the bottom end where mass is attached decreases
   (c) decrease as the distance from the top end where mass is attached decreases
   (d) is minimum when the mass is attached to the midpoint of the spring

11. In a cantilever beam with load acting at end point, the spring's stiffness is given by
   (a) $k_{eq} = \dfrac{3EI}{l^3}$ N/mm
   (b) $k_{eq} = \dfrac{48EI}{l^3}$ N/mm
   (c) $k_{eq} = \dfrac{l^3}{3EI}$ N/mm
   (d) $k_{eq} = \dfrac{l^3}{48EI}$ N/mm

12. The degrees of freedom of a simple pendulum is given by
   (a) 2 (b) 6
   (c) 0 (d) 1

13. Equivalent spring stiffness of a helical spring under axial load is given by
   (a) $k_{eq} = \dfrac{\pi EDd}{4l}$ N/mm
   (b) $k_{eq} = \dfrac{GD^4}{8nD^3}$ N/mm
   (c) $k_{eq} = \dfrac{EA}{I}$ N/mm
   (d) $k_{eq} = \dfrac{EA}{4Il^2}$ N/mm

14. The static deflection of the cantilever beam when the mass acting at one end is given by the equation
   (a) $\delta = \dfrac{Fl^3}{192EI}$ (b) $\delta = \dfrac{Fl^3}{3EI}$
   (c) $\delta_{max} = \dfrac{Fl^3}{48EI}$ (d) none of these

15. The equivalent mass (M) attached at end of spring of mass '*m*' is given by $m_{eq} = M + m/3$
   (a) $m_{eq} = M + 0\ 23\ m$
   (b) $m_{eq} = 3M + \dfrac{m}{3}$
   (c) $m_{eq} = M + \dfrac{m}{3}$
   (d) $m_{eq} = 5M + \dfrac{m}{3}$

Answers

(1) d	(2) d	(3) b	(4) a	(5) c	(6) b
(7) c	(8) b	(9) b	(10) d	(11) a	(12) a
(13) b	(14) b	(15) c			

FREE-VIBRATION OF SINGLE-DEGREE-FREEDOM SYSTEM WITH DAMPING

3

3.1 INTRODUCTION

Damping is the resistance offered to the motion of a vibrating body. The resistance may be applied by a liquid or solid, internally or externally. All physical systems are associated with one or the other types of damping. In some cases, the amount of damping may be small and in some other cases it may be large. When vibrations take place in the presence of a damping, the amplitude of vibration gradually becomes small, and finally it ceases. The rate at which the amplitude decays depends on the type and amount of damping in the system. The primary aspects of damped free vibrations are (i) damped natural frequency and (ii) rate of decay of vibrations.

In case of free vibration with viscous damping, when a system has got viscous damping then (damping force) comes into consideration, and the general steps for analysing such systems remain same. The steps are the following:

1. Take a displaced position of the mass system.
2. Draw the free-body diagram (FBD) of the system—it indicates the forces acting in various direction.
3. Apply Newton's second law of motion for the equilibrium of the free body.

3.1 METHODS OF VIBRATION DAMPING

Damping is associated with energy dissipation. A system having positive damping is called a '*dissipative*' *system*'. In such systems, the amplitude of vibration decreases with respect to time '*t*' as shown in Fig. 3.1(a). Whereas in a 'regenerative' system having a negative damping, the amplitude of vibration increases with respect to time '*t*' as shown in Fig. 3.1(b).

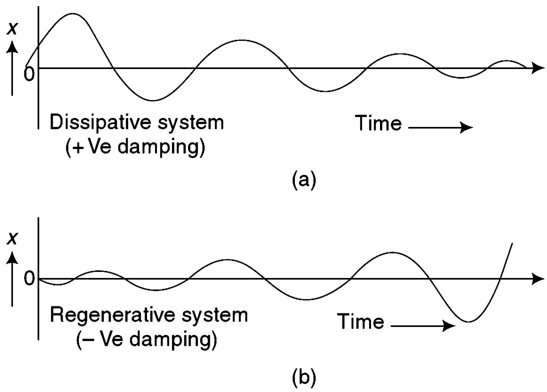


Fig. 3.1 *Dissipative and regenerative damping system*

3.3 COMPONENTS OF A VIBRATING SYSTEM

The main components of a vibrating system are (i) mass (m), (ii) spring (k), and (iii) damper (c).

These components are represented as shown in Fig. 3.2(a) and three forces comes into consideration. They are

1. The mass of the vibrating system provides inertia force ($m\ddot{x}$) (See Sec.1.5, Chapter 1)
2. Spring provides the restoring force (kx) (See Sec. 2.4, Chapter 2)
3. Damper provides the resisting force ($c\dot{x}$) (See Sec. 1.7, Chapter 1)

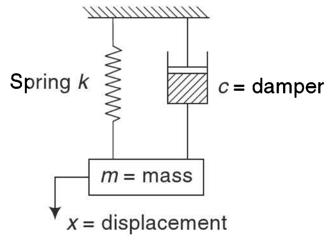


Fig. 3.2(a) *Components of a vibrating system*

3.3.1 Different types of Damping

1. Viscous damping
2. Coulomb damping or dry friction damping
3. Structural damping or solid or hysterisis or material damping
4. Slip or interfacial damping
5. Proportional damping

1. Viscous damping

This is the most commonly used damping mechanism in vibration analyses. When a mechanical system vibrates in a fluid medium such as water, oil, gas at moderate speeds, it experiences a resisting force called viscous damping. The resisting force 'F' is proportional to the velocity 'v' or $F = -cv$ where 'c' is a constant and 'v' is the velocity, i.e. $v = \dot{x}$; therefore, $F = -c\,\dot{x}$, –ve sign indicates the resisting force is opposite to the direction of motion. In this case, the amount of energy depends on many factors such as shape and size of vibrating body, viscosity of fluid, velocity of vibrating body, frequency of vibration, etc. Viscous damping is very common. It also occurs in fluid film around a journal in a bearing, fluid flow through an orifice, fluid flowing on sliding surfaces, moving instruments immersed in oil and in many other instruments. There will be two important types of viscous dampers commonly used as fluid dashpot, and eddy current damping. Now we will discuss these two types of viscous dampers as follows:

(a) Fluid dashpot Fluid dashpot is one of the most important components of a vibrating systems. It consists of a piston–cylinder arrangement and the piston moves to and fro or in reciprocating motion full of viscous fluid as shown in Fig. 3.2(b).

When the piston moves in a reciprocating motion, forces are experienced due to the

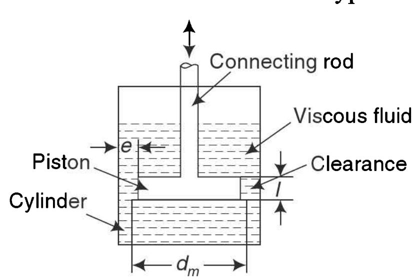


Fig. 3.2(b) *Fluid dashpot*

drag of fluid, the pressure of the fluid that flows through on either side of the clearance of the piston and the damping resistance due the pressure difference on the two sides of the piston. This pressure difference is based on the restriction to the fluid flow due to the piston movement.

Out of these three types of damping, it can be observed that the pressure of the fluid that flows through on either side of the clearance of the piston and cylinder is very small; therefore, the first two components of the damping are ignored and the total resulting damping is fully based on the third component of the damping coefficient 'c' and is given by an approximate expression as follows:

$$c = \frac{12\mu}{\pi} \cdot \frac{a_p l}{d_m e^3} \qquad \ldots 3.1$$

where c = Viscous damping coefficient

e = Clearance between cylinder and the piston

a_p = Cross-sectional area of the flat side of the piston

l = Length of the piston

d_m = Mean diameter of the cylinder and piston

All these parameters are as shown in Fig. 3.2.

Some assumptions are made while deriving Eq. 3.1. The piston-rod diameter is small as compared to the piston diameter. A perfect fluid is used in the damping medium, the laminar flow being expected in the clearance between the cylinder and the piston. The cylinder and the piston being concentric, orifice effect on sharp edges may be ignorable.

(b) Eddy-current damping Eddy-current damping is based on the principle of generation of eddy currents which provide the damping. Let us consider a nonferrous conducting rectangular plate or rod being moved in a direction normal to the lines of magnetic flux which is produced by a permanent magnet as shown in Fig. 3.3. Then as the plate or rod moves, current is induced in the plate and this current is proportional to the velocity of the plate with an assumption that the magnetic flux and the dimension of the plate remains constant. This current is in the form of an eddy current that sets up a magnetic field in a direction opposing the original magnetic field that causes them. This provides a resistance to the motion of the plate in the magnetic field. The resisting force produced by this flux field from eddy currents is also proportional to the velocity of the plate. This is a mechanical type of viscous damping. Mechanical type of viscous damping is used in vibrometers and also some other vibration control systems, etc. The magnetic field can be set up by means of a permanent magnet or a coil

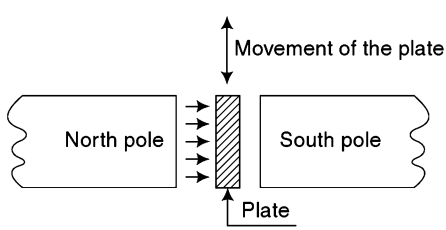


Fig. 3.3 *Eddy-current damping*

wound on a magnetic core by flowing current through it and the flow direction of the magnetic flux as shown in Fig. 3.3.

3.3.2 Dampers are in Series and Parallel

1. Dampers are in series Let us consider dampers c_1, c_1 c_3...are in series as shown in Fig. 3.4(a). When the dampers are in series, the equivalent damping coefficient (c_{eq}) is given by

$$\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$$

Then the equivalent damping (c_{eq}) system will be reduced to $c_{eq} = \dfrac{c_2c_3 + c_1c_3 + c_1c_2}{c_1c_2c_3}$

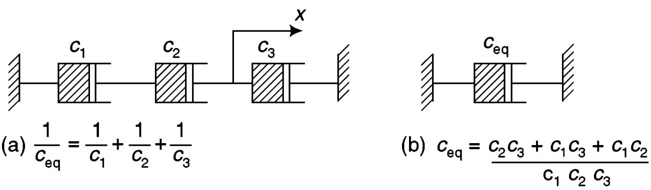


Fig. 3.4 *Dampers are in series*

This equivalent damping system can be as shown in Fig. 3.4(b).

In general, when the dampers are in series the equivalent damping system is given as follows.

$$\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4} + \dots$$

2. Dampers are in parallel Let us consider dampers c_1, c_2 c_3... are in parallel as shown in Fig. 3.4(c). When the dampers are in parallel, the equivalent damping coefficient (c_{eq}) is given by $c_{eq} = c_1 + c_2 + c_3$.

Then the equivalent damping (c_{eq}) system will be reduced to $c_{eq} = c_1 + c_2 + c_3$. This equivalent damping system can be as shown in Fig. 3.4(d).

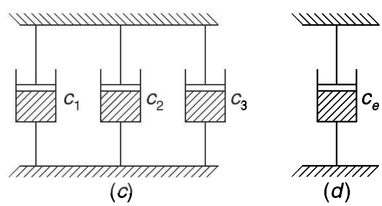


Fig. 3.4 *Dampers are in parallel*

In general, when the dampers are in parallel, the equivalent damping system is given as follows.

$$c_{eq} = c_1 + c_2 + c_3 + c_4 + \cdots$$

3. Dampers are in series and parallel Let us consider dampers c_1, c_2 ... are in parallel and c_3 is in series as shown in Fig. 3.4(e). When the dampers c_1, c_2 are in parallel, the equivalent damper (c_{eq1}) is given by $c_{eq1} = c_1 + c_2$. Then the dampers will be reduced to as shown in Fig. 3.4(f).

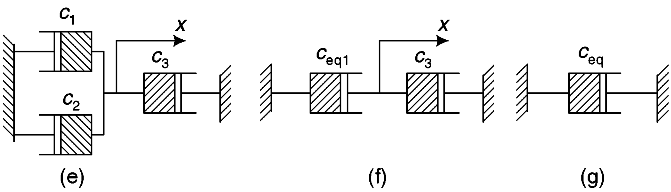


Fig. 3.4 *Dampers are in series and parallel*

In Fig. 3.4(f), c_{eq1} and c_3 are in series. Then equivalent damper (c_{eq2}) is given by

$$\frac{1}{c_{eq2}} = \frac{1}{c_{eq1}} + \frac{1}{c_3}, \frac{1}{c_{eq2}} = \left[\frac{c_3 + c_{eq1}}{c_{eq1}c_3}\right], \quad \frac{1}{c_{eq2}} = \frac{c_1 + c_2 + c_3}{c_3(c_1 + c_2)}$$

Then the dampers will be reduced to as shown in Fig. 3.4(g).

3.3.3 Dampers and Springs are in Parallel and Series

1. Springs and dampers are in parallel Let us consider springs k_1, k_2, k_3 … masses m_1, m_2, m_3 …..are in series and dampers c_1, c_2 c_3 …masses m_1, m_2, m_3 are in series, springs k_1, k_2, k_3 …masses m_1, m_2, m_3 and dampers c_1, c_2 c_3 …masses m_1, m_2, m_3 are in parallel as shown in Fig. 3.5(a).

Then the equivalent spring–mass and the equivalent damper–mass system will be reduced to as shown in Fig. 3.5(c) 3.5(d) respectively.

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots + \frac{1}{k_n}, \quad \frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots + \frac{1}{c_n}$$

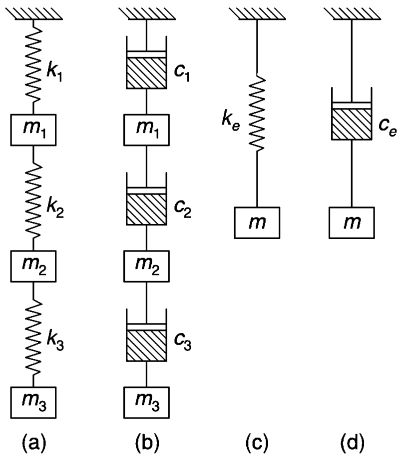
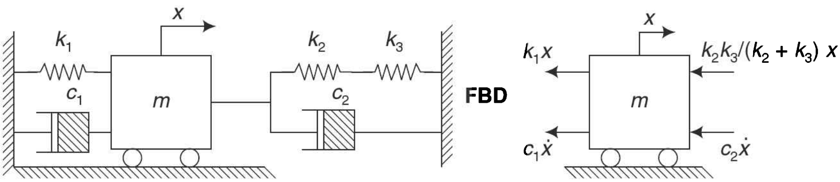


Fig. 3.5 *Springs and dampers are in parallel*

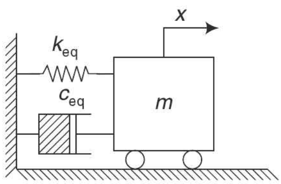
2. Springs and dampers are in series

Let us consider springs k_1, k_2, k_3 …are in series and dampers c_1 and c_2 are in series, as shown in Fig. 3.5(c). Then the equivalent springs are in series is given by $k_{eq} = k_1 + \dfrac{k_2 k_3}{k_2 + k_3}$ and the equivalent damper when they are in series is given by $c_{eq} = c_1 + c_2$. Then the free-body diagram of the system is as shown in Fig. 3.5(d). Then the equivalent system is as shown in Fig. 3.5(e).



(c) Spring and dampers are in series and parallel

(d) Free-body diagram (FBD)



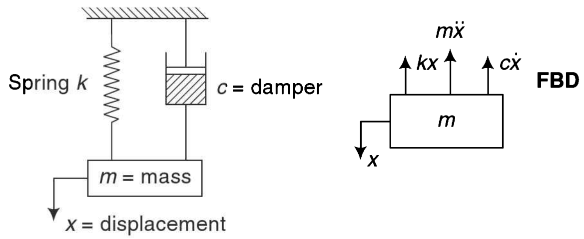
(e)

Fig. 3.5 *(Contd.)* *Simplified spring-mass and dampers system*

3.4 SOLUTIONS OF FREE VIBRATION WITH DAMPING (VISCOUS DAMPING) IN A SINGLE-DEGREE-FREEDOM SYSTEM

Let us consider a classical spring-mass-dashpot system as shown in Fig. 3.6(a). Let 'm' be the mass of the body, 'k' be the spring stiffness of the spring and 'c' be the damping coefficient. The damping resistance at any instant is equal to $c\,\dot{x}$ where $\dot{x}$ is the relative velocity between the piston and the cylinder of the dashpot. For viscous damped system, the damped force F_d is proportional to the velocity v, i.e.

$F_d \alpha\, v$, $F_d = cv$ or $F_d = c\dot{x}$. The springs and dashpot are in parallel.



(a) (b)

Fig. 3.6 *Spring-mass-dashpot system with free-body diagram*

Let at any instant the system be displaced through a distance 'x' from the equilibrium (mean) position as shown in Fig. 3.6(a). Then different forces are acting on the system as shown in free-body diagram (FBD) of Fig. 3.6(b). The spring force 'kx' acting in the upward direction and the damping force $c\,\dot{x}$ acting in the upward direction.

Note: In case of free vibrations with damping, there will be three forces acting on a system, i.e., spring force, damping force and inertia force (See Eq. 3.2.).

Now apply Newton's second law of motion,

$\Sigma F = m.a,\ m\ddot{x} = -c\,\dot{x} - kx$

or $m\ddot{x} + c\dot{x} + kx = 0$ ...3.2

Equation 3.2 is called the *governing equation of motion*.

This is the fundamental homogeneous differential equation of motion of a single-degree-freedom-system having damped vibration of order two. The solution of this differential equation of motion is as follows. Assuming that the solution is

$$x = X_0 e^{st} \quad ...3.3$$

where 'X_0' is the amplitude, 'e' is the base of the natural logarithms, 's' is the constant to be determined and 't' is the time in seconds. Differentiating Eq. 3.3 with respect to time 't' twice, we have

$$\therefore \qquad \frac{dx}{dt} = \dot{x} = sX_0 \cdot e^{st},\ \frac{d\dot{x}}{dt} = \ddot{x} = s^2 X_0 \cdot e^{st}$$

Substituting these values in Eq. 3.2,

$m\left(s^2 X_0 e^{st}\right) + c\left(sX_0 e^{st}\right) + kX_0 e^{st} = 0,\ \left(ms^2 + cs + k\right)X_0 e^{st} = 0$ as $X_0 \cdot e^{st} \neq 0$, $\left(ms^2 + cs + k\right) = 0$ is the characteristic equation. The above equation is quadratic and hence the two values of 's' can be found as $s_{1,2} = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$ ...3.4

$$s_{1,2} = \frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{4mk}{4m^2}}$$

$$s_1, s_2 = \frac{-c}{2m} \pm \sqrt{\left[\frac{c}{2m}\right]^2 - \frac{k}{m}} \quad ...3.5$$

The general solution for the motion of mass 'm' is therefore given by

$$x = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

For critical damping in Eq. 3.4, the quantity under the radical sign should be zero, i.e.

$$c^2 - 4mk = 0, \text{ or } c = c_c = 2\sqrt{mk}$$

or from Eq. 3.5 the quantity under the radical sign should be zero, i.e.

$$\left[\frac{c_c}{2m}\right]^2 - \frac{k}{m} = 0,\ \left[\frac{c_c}{2m}\right]^2 = \frac{k}{m},\ \left[\frac{c}{2m}\right]^2 = \omega_n^2,\ c_c = 2m\omega_n \text{ or } c_c = 2\sqrt{km}$$

Note: Spring-mass dashpot system can also be written in several ways, based on the different applications as shown in Fig. 3.7(a), (c), (e) and (g) and respective free-body diagram (FBD) in Fig. 3.7(b), (d), (f) and(h).

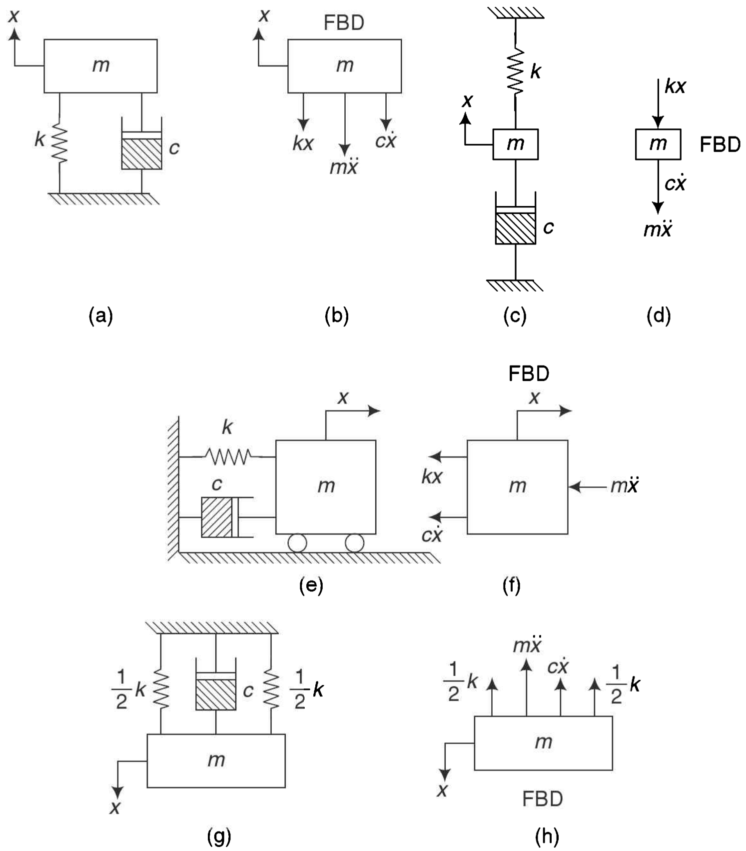


Fig. 3.7 *Spring-mass dashpot system with free-body diagram (FBD)*

3.5 GENERALISED DIFFERENTIAL EQUATION OF THE SYSTEM

We know the equation $m\ddot{x} + c\dot{x} + kx = 0$ ...3.6

Divide Eq. 3.6 by 'm' throughout,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \text{...3.7}$$

since $\omega_n = \sqrt{\frac{k}{m}}.$

The term $\frac{c}{m}$ is multiply and divided by 'c_c'

$$\frac{c}{m} = \frac{c_c \cdot c}{c_c m} = \frac{\xi 2\sqrt{mk}}{m}, \text{ where } \frac{c}{c_c} = \xi \text{ (damping ratio)}$$

$$c_c = 2\sqrt{mk}$$

$$= \frac{2\xi\sqrt{m}\,\sqrt{k}}{\sqrt{m}\,\sqrt{m}} = \frac{2\xi\sqrt{k}}{\sqrt{m}} = 2\xi\omega_n \quad \therefore \frac{c}{m} = 2\xi\omega_n$$

Substituting these values in Eq. 3.6, we have

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = 0 \qquad ...3.8$$

The solution of this equation is given by,

assuming that $x = Ae^{st}$ ...3.9

Differentiating the Eq. 3.9 with respect time twice,

$$\dot{x} = Ase^{st}, \ddot{x} = As^2e^{st}$$

Substituting these values in Eq. 3.8, we get

$\therefore As^2e^{st} + 2\xi\omega_n Ase^{st} + \omega_n^2 Ae^{st} = 0, (s^2 + 2\xi\,\omega_n s + \omega_n^2)\,Ae^{st} = 0$, as $A.e^{st} \neq 0$.

$$s^2 + 2\xi\,\omega_n^2 s + \omega_n^2 = 0 \qquad ...3.10$$

is the auxiliary equation.

The above equation is quadratic and hence the two values can be found out as follows.

$s_{1,2} = -2\xi\omega_n \pm \frac{\sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$, or it can also be written in terms of damping factor 'ξ'

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \qquad ...3.11$$

$\therefore x = e^{s_1 t}$ and $x = e^{s_2 t}$ are the solutions of Eq. 3.6.

The general solution can be written as

$$x = A_1e^{s_1 t} + A_2\,e^{s_2}t \qquad ...3.12$$

where 'A_1' and 'A_2'are arbitrary constants to be evaluated from initial conditions.

Substituting the values of 's_1 and 's_2' in Eq. 3.12,

$$x = A_1e^{\left[-2\xi\omega_n + \omega_n\sqrt{\xi^2} - 1\right]t} + A_2\,e^{\left[-2\xi\omega_n - \omega_n\sqrt{\xi^2} - 1\right]t} \qquad ...3.13$$

we get

$$x = A_1e^{\left[\frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right]t} + A_2e^{\left[\frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right]t} \qquad ...3.14$$

$$x = e^{\left[\frac{-ct}{2m}\right]}\left[A_1e^{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\,t} + A_2e^{-\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\,t}\right] \qquad ...3.15$$

The first term in Eq. 3.15 $e^{\frac{-ct}{2m}}$ is simply an exponential decaying function of time. The behaviour of the terms within the parenthesis depends on whether the numerical value within the square root is positive, zero or negative.

1. When the damping term $\left(\frac{c}{2m}\right)^2$ is larger than $\frac{k}{m}$, the exponents of the equation are real numbers and no oscillations are possible. Hence, we refer this case as over-damped condition.

2. When the damping term $\left(\frac{c}{2m}\right)^2$ is less than $\frac{k}{m}$, the exponents of the above equation becomes imaginary numbers, i.e. $e^{\pm i\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}}$.

This is similar to $e^{i\theta} = \sin\theta + i\cos\theta$

$$\therefore \qquad e^{\pm i\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t} = \sin\left\{\left[\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\right]t\right\} + i\cos\left\{\left[\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\right]t\right\}$$

Hence, the terms of the equation within the parenthesis are oscillatory.

Hence, this condition is referred as an underdamped conditions.

3. The limiting case between the oscillatory and non-oscillatory motion is got when $\frac{k}{m} = \left(\frac{c}{2m}\right)^2$ and this results in the value under the square root to zero. This case is referred as the *critically damped condition*.

Mathematically, critical damping means that value of 'c' which makes the radical in the expression of $s_{1,2}\left[\frac{c_c}{2m}\right]^2 - \frac{k}{m}$ or $\frac{k}{m} = \omega_n^2$, $c_c = 2m\omega_n = 2\sqrt{km}$

The ratio of $\left(\frac{k}{m}\right)$ to $\left[\frac{c}{2m}\right]^2$ indicates the degree of dampness provided in the system and its square root is known as damping factor 'ξ', i.e. $\xi = \sqrt{\frac{\left(\frac{c}{2m}\right)^2}{\frac{k}{m}}} = \frac{c}{2\sqrt{k/m}}$.

3.6 DAMPING RATIO (ξ)

It is defined as the ratio of actual damping coefficient(c) to the critical damping coefficient (c_c),

i.e. Damping ratio $\xi = \frac{\text{Actual damping coefficient}}{\text{Critical damping coefficient}} = \frac{c}{c_c}$, where '$c$' is actual damping coefficient, 'c_c' is critical damping coefficient and $\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n}$ or $c = 2m\xi\omega_n$.

The damping factor is the measure of the relative amount of damping in the existing system with that necessary for the critical damped system. The damping coefficient is a constant depending upon the mass and stiffness of the system and is independent of the actual amount of damping.

There will be four cases for damping ratio 'ξ' which are as follows:

Case (i) If $c = 0$ or $\xi = 0$, the system is called no damping (undamped system).

Case (ii) If $c < c_c$ or $\xi < 1$, the system is called an underdamped system.

Case (iii) If $c = c_c$ or $\xi = 1$, the system is called a critical damped system.

Case (iv) If $c > c_c$ or $\xi > 1$, the system is called an overdamped system.

Case (i) No damping (undamped system $\xi = 0$) The roots of the characteristics equation (3.11) are $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

Now $\xi = 0$

The roots of the equation are from Eq. 3.11, $s_{1,2} = \pm\,\omega_n\sqrt{-1}$ $s_{1,2} = \pm J\omega_n$

$\therefore$ the solution is $A_1e^{s_1t} + A_2e^{s_2t}$, $x = A_1e^{J\omega_nt} + A_2e^{-J\omega_nt}$

$$x = A\sin\omega_nt + B\cos\omega_nt$$

To find constant 'A_1' and 'A_2' using initial conditions.

At $t = 0$, $x(0) = x_0$ $\therefore$ $B = x_0$, $\dot{x}(0) = 0$, $\dot{x} = A\omega_n\cos\omega_nt - B\omega_n\sin\omega_nt$, $0 = A.1.\omega_n$ $\therefore A = 0$.

Case (ii) Underdamping system ($\xi < 1$) The roots of the characteristic equation 3.11 are $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

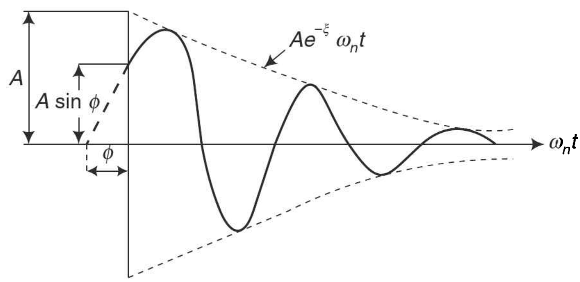


Fig. 3.8 *No damping*

Now for underdamping system $\xi < 1$, the roots of the equation are from Eq. 3.6, $\xi^2 - 1$ will be negative.

$$\therefore\quad \sqrt{\xi^2 - 1} = \sqrt{(-1)(1-\xi^2)}\ \sqrt{(-1)}\ \sqrt{(1-\xi^2)},$$

The radical becomes imaginary and can be written as

$$j\sqrt{1-\xi^2},\ s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}\quad [\text{where } j = \sqrt{(-1)}]$$

$s_{1,2} = -\,\xi\omega_n \pm j\omega_d$, where $\omega_d = \omega_n(\sqrt{1-\xi^2})$ damped natural frequency.

The general solution is given by

$$x = A_1e^{s_1t} + A_2e^{s_2t},\quad \text{i.e. } x = A_1e^{\left(-\xi + j\sqrt{1-\xi^2}\right)\omega_nt} + A_2e^{\left(-\xi + j\sqrt{1-\xi^2}\right)\omega_nt}$$

The above solution can also be written as

$$x = e^{-\xi\omega_nt} - \left(A_1e^{\left(j\sqrt{1-\xi^2}\right)\omega_nt} + A_2e^{\left(-J\sqrt{1-\xi^2}\right)\omega_nt}\right)$$

$$x = e^{-\xi\omega_nt} - \left(c_1\sin\left[\left(\sqrt{1-\xi^2}\right)\omega_nt\right] + c_2\cos\left[\left(\sqrt{1-\xi^2}\right)\omega_nt\right]\right)$$

i.e. 'c_1' and 'c_2' are arbitrary constants to be determined from initial conditions.

Due to the presence of sine and cosine terms in the above equation, the motion of the system is oscillatory with a damped natural frequency (ω_d)

The solution can also be represented as

$$x = Xe^{-\xi\omega_n t}\left[\sin\left\{\left(\sqrt{1-\xi^2}\right)\omega_n t\right\} + \phi\right]$$

It is noted that the above equation is periodic whose amplitude decreases with time. This is the general nature of damped free oscillations. The characteristics of such underdamped system of motion are (i) motion is periodic, (ii) amplitude decreases with time, and (iii) the frequency and hence the time period of oscillations depend on the damping constant (c or ξ). Higher the damping constant, smaller will be the frequency and larger will be the time period as shown in Fig. 3.8.

Case (iii) Critical damping system (ξ = 1)

The roots of the characteristics Eq. 3.11 are $s_{1,\,2} = -\,\xi\omega_n \pm \omega_n\sqrt{\omega\xi^2 - 1}$

Now since $\xi = 1$, the two values of 's' are $s_{1,2} = -\,\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

Since the 'ξ' value is one, critical damping $\left(\sqrt{\xi^2 - 1}\right)$ term tends to zero.

Hence $s_1, s_2 = -\,\omega_n$

The general form of solution for such a case of repeated roots is given by,

$$x = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$x = A_1 e^{-\xi\omega_n t} + A_2 e^{-\xi\omega_n t},\ x = e^{-\xi\omega_n t}(A_1 + A_2)$$

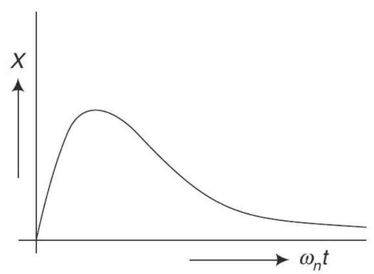


Fig. 3.9 *Critical damping*

where 'A_1' and 'A_2' are arbitrary constants which are evaluated from the initial conditions. As the differential equation is a two-degree equation, the resulting solution is inadequate and therefore not general. However, the assumption $s = te^{-\omega_n t}$ satisfies the equation and results in a general expression for 'x' as follows: $x = e^{-\xi\omega_n t}[A_1 + A_2 t]$. This is shown in Fig. 3.9 for $A_1 = 0$. From Fig. 3.9, it is observed that the motion is periodic.

Importance of critical damping The main importance of the critical damping condition is that it forms a method of measuring the relative amount of damping in a particular system. It is uncommon in practice since it is difficult to maintain the exact relationship required of the passive elements to produce it. Physically, critical damping (c_c) means that amount of damping which will make the vibrating system stop during the least possible time.

The damping factor (ξ) for the critical case will carry the subscript c; and the ratio of the actual amount of damping 'c' to the critical amount of damping 'c_c' is a measure on a dimensionless basis of the relative amount of damping in the system. This ratio will be designated as 'ξ'. With critical damping $(c_c/2m)^2 = k/m$, and the term under the radical of equations 3.11 equals zero so that

$$s_1 = s_2 = s = -\,c_c/2m.$$

Case (iv) Overdamping system ($\xi > 1$) The general solution is given by, $x = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

$x = A_1 e^{[-\xi + \sqrt{\xi^2 - 1}]\omega_n t} + A_2 e^{[-\xi - \sqrt{\xi^2 - 1}]\omega_n t}$ where 'A_1' and 'A_2' are arbitrary constants to be determined from the initial conditions. Since the power of 'e' is negative in both terms of the above equation, both decreases exponentially with the time 't' and therefore 'x' decreases as 't' increases, and it becomes zero as 't' tends to ∞. The system comes to equilibrium position nearly in an exponential manner. Theoretically, the system will take infinite time to come to the equilibrium position once it is displaced from its original position as shown in Fig. 3.10.

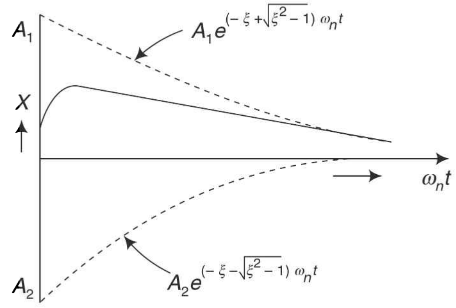


Fig. 3.10 *Overdamping*

Note: In critical damping and overdamping cases, there is no oscillations, since they are aperiodic.

3.7 LOGARITHMIC DECREMENT(δ)

It is an yardstick to measure the amount of damping present in an underdamping case only. (i.e. $\xi < 1$). Also, it is one of the easy ways to determine the amount of damping present in a system. It is 'defined as the natural logarithmic ratio of any two successive amplitudes on the same side of the mean line.' The ratio of any two successive amplitudes is always constant in a free vibration and also it is one of the practical ways of finding out the damping of a system and is denoted by 'δ'.

The main importance of logarithmic decrement is the vibration plot is obtained experimentally and after measuring amplitudes, damping is determined using standard equations.

In Fig. 3.11 at the point of tendency, the amplitudes are approximately given by $x = Xe^{-\xi\omega_n t}$

By definition, $\delta = \ln\left(\frac{X_0}{X_1}\right) = \ln\left(\frac{X_1}{X_2}\right) = \ln\left(\frac{X_2}{X_3}\right) = \cdots \delta = \ln\left(\frac{X_{n-1}}{X_n}\right)$

$$x = Xe^{-\xi\omega_n t} \qquad ...3.16$$

when $t = t^1$, $x = X_0$. Therefore, from Eq. 3.16, $X_0 = (Xe^{-\xi\omega_n t^1})$,

when $t = t^1 + \tau$, $x = X_1$. Therefore, from Eq. 3.16, $X_1 = Xe^{-\xi\omega_n}(t^1 + \tau)$

where τ = Periodic time $= \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1 - \xi^2}}$

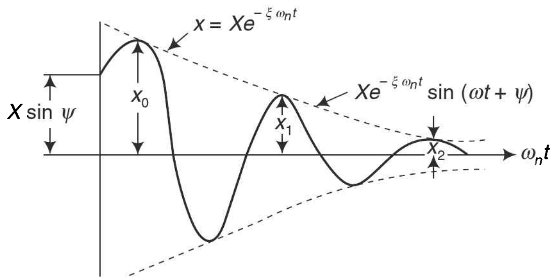


Fig. 3.11 *Logarithmic decrement*

By definition, $\delta = \ln\left(\frac{X_0}{X_1}\right)$

Substituting the values of X_0 and X_1, we have

$$\delta = \ln\left(\frac{Xe^{-\xi\omega_n t^1}}{Xe^{-\xi\omega_n (t^1+\tau)}}\right) = \ln\left(\frac{Xe^{-\xi\omega_n t^1}}{Xe^{-\xi\omega_n t^1}\, e^{-\xi\omega_n \tau}}\right) = \ln\left(\frac{1}{e^{-\xi\omega_n\tau}}\right) = \ln\left(e^{\xi\omega_n\tau}\right),\ \delta = \xi\omega_n\tau \ln e$$

$$\delta = \xi\omega_n\tau \quad \text{since as } \ln e = 1\ \tau = \frac{2\pi}{\omega_d},\quad \delta = \frac{2\pi\xi\omega_n}{\omega_n\sqrt{1-\xi^2}} \quad \therefore\ \delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad \text{...3.17}$$

[$\delta \approx 2\pi\xi$ if $\xi << 1$ (underdamping case)]

Also from the equation, $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$, i.e. $\sqrt{1-\xi^2} = \frac{2\pi\xi}{\delta}$

Squaring both sides,

$$1-\xi^2 = \frac{4\pi^2\xi^2}{\delta^2},\ \delta^2 - \delta^2\xi^2 = 4\pi^2\xi^2,\ \delta^2 = 4\pi^2\xi^2 + \delta^2\xi^2,\ \delta^2 = \xi^2[4\pi + \delta^2]$$

$$\therefore \qquad \xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \qquad \text{...3.18}$$

From these equations, we can determine the 'ξ' value.

If the system takes place in 'n' number of cycles then by definition of logarithmic decrement, $\delta = \ln\left(\frac{X_0}{X_1}\right) = \ln\left(\frac{X_1}{X_2}\right) = \ldots = \ln\left(\frac{X_{n-1}}{X_n}\right)$.

$$\delta = \ln\left(\frac{X_0}{X_1}\right)$$

Taking log on both sides,

$$\left(\frac{X_0}{X_1}\right) = e^{\delta},\ \left(\frac{X_1}{X_2}\right) = e^{\delta},\ \left(\frac{X_{n-1}}{X_n}\right) = e^{\delta} \text{ or } \left(\frac{X_0}{X_n}\right) = \left(\frac{X_0}{X_1}\right)\times\left(\frac{X_1}{X_2}\right)\times\cdots\times\left(\frac{X_{n-1}}{X_n}\right)$$

$$= e^{\delta} \times e^{\delta} \times e^{\delta} \times \cdots \times e^{\delta} \text{ `}n\text{' times} \quad \therefore \frac{X_o}{X_n} = e^{n\delta}$$

Taking log on both sides,

$$\therefore \ln_e\left(\frac{X_0}{X_n}\right) = n\delta \ln e, \ \ln\left(\frac{x_0}{x_n}\right) = n\delta \times 1, \text{ since as } \ln e = 1, \ \delta = \frac{1}{n}\ln\left(\frac{X_0}{X_n}\right) \qquad \text{...3.19}$$

EXAMPLE 3.1

A mass of 0.907 kg is attached to the end of a spring with a stiffness of 7 N/cm. Determine the critical damping coefficient.

Solution $m = 0.907$ kg, $k = 7$ N/cm = 700 N/m.

The critical damping coefficient is given by the equation

$$c_c = 2\sqrt{mk} = 2\sqrt{0.907 \times 700} = 50.394 \text{ N-s/m.}$$

EXAMPLE 3.2

A damped vibration record of a spring-mass-dashpot system shows the following data: Amplitude on second cycle = 12 mm, Amplitude on third cycle = 10.5 mm, Spring constant k = 7840 N/m, Mass of the spring = 2 kg.

Determine the damping constant assuming it to be viscous damping.

Solution $x_2 = 12$ mm, $x_3 = 10.5$ mm, $k = 7840$ N/m = 7.84 N/mm, $m = 2$ kg, $c = ?$

The logarithmic decrement is given by the equation

$$\delta = \log_e\left(\frac{x_2}{x_3}\right) = \log_e\left(\frac{12}{10.5}\right), \ \delta = 0.1335$$

Damping factor is given by the equation

$$\delta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.1335}{\sqrt{4\pi^2 + 0.1335^2}} \quad \therefore \xi = 0.0212.$$

Critical damping coefficient is given by the equation

$$c_c = 2\sqrt{mk} = 2\sqrt{2 \times 7.84}, \ c_c = 7.92 \text{ N-s/m.}$$

or $$c_c = 250.43 \text{ N-s/m}$$

Damping constant $c = \xi \times c_c = 0.0212 \times 7.92 = 0.1678$ or 0.0212×250.43,

$$c = 5.309 \text{ N-s/m.}$$

EXAMPLE 3.3

A mass of 1 kg is to be supported by a spring having a stiffness of 9800 N/m. The damping coefficient is 5.9 N-s/m. Determine the frequency of the system. Find the logarithmic decrement and also the amplitude of the 3 cycles if the initial displacement is 0.3 cm.

Solution $m = 1$ kg, $k = 9800$ N/m, $c = 5.9$ N-s/m, $x_0 = 0.3$ cm $= 0.003$ m.

The natural frequency is given by the equation $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9800}{1}}$, $\omega_n = 98.99$ rad/s.

The frequency is given by the equation $f = \frac{\omega_n}{2\pi} = \frac{98.995}{2\pi} = 15.76$ cyc/s or Hz.

The damping coefficient is given by the equation $c_c = 2\sqrt{mk} = 197.99$ N-s/m.

The damping factor is given by the equation $\xi = \frac{c}{c_c} = \frac{5.9}{197.99} = 0.0298$.

The logarithmic decrement is given by the equation

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.0298}{\sqrt{1-0.298^2}} = 0.187.$$

We have $\left(\frac{x_0}{x_1}\right) = e^{\delta}$, $x_1 = \frac{x_0}{e^{\delta}} = \frac{0.003}{e^{0.187}} = 0.0025$ m

$$x_2 = \frac{x_1}{e^{\delta}} = \frac{0.0025}{e^{0.187}} = 0.0021 \text{ m}$$

$$x_3 = \frac{x_2}{e^{\delta}} = \frac{0.0021}{e^{0.187}} = 0.00171 \text{ m.}$$

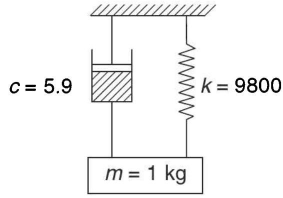


Fig. p-3.3 *Spring-mass damper*

EXAMPLE 3.4

A mass of 20 kg is suspended from a spring of stiffness 39240N/m. A dashpot is fitted and it is found that the amplitude of vibration diminished from its initial value of 25 mm to 6.25 mm in two complete oscillations. Find the resistance offered by the dashpot at a velocity of 0.3 m/s and the frequency of damped vibration. Compare this value with frequency of free vibration.

Solution $m = 20$ kg, $k = 39240$ N/m, $x_0 = 25$ mm, $x_2 = 6.25$ mm, $n = 2$,

velocity $(\dot{x})$ $v = 0.3$ m/s.

We know that $\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_1}\right) = \frac{1}{2}\log_e\left(\frac{25}{6.25}\right) = 0.69$

Also the logarithmic decrement is given by the equation, $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$

or $\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{0.69}{\sqrt{4\pi^2+(0.69)^2}}$ $\therefore\ \xi = 0.11 < 1$ (Underdamping)

But damping factor

$$\xi = \frac{c}{c_c} \quad c = c_c\xi = 2\sqrt{mk},\ \xi = 2 \times \sqrt{\frac{20}{981} \times 39240 \times 0.11},\ c = 194.9 \text{ N-s/m}$$

Force of resistance $F = c\dot{x} = 194.9 \times 0.30$, $F = 58.46$ N or $F = 5.96$ kg

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_n = \sqrt{\frac{39240}{20}} = 44.29 \text{ rad/s}$$

The frequency of damped vibration

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 44.29\sqrt{1-(0.11)^2}, \ \omega_d = 44.02 \text{ rad/s}$$

$\dfrac{\omega_d}{\omega_n} = \sqrt{1-\xi^2} = \sqrt{1-(0.11)^2} = 0.994$ has to be less than one always (underdamping).

EXAMPLE 3.5

A machine of 20 kg mass is mounted on a spring and dashpot as shown in Fig. p-3.5. The total spring stiffness is 10 N/mm and the total damping is 0.15 N/mm/s. If the system is initially at rest and a velocity of 100 mm/s is imparted to the mass, then determine

(i) displacement and velocity of mass as a function of time, and (ii) displacement and velocity at time equal to one second.

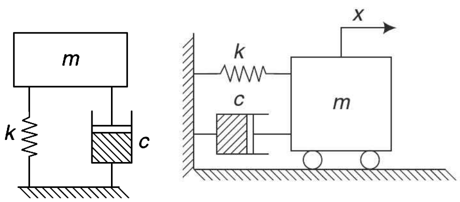


Fig. p-3.5 *Spring-mass damper system*

Solution $m = 20$ kg, $k = 10$ N/mm = 10,000 N/m, $c = 0.15$ N/mm/s = 150 N-s/m, $v = 100$ mm/s = 0.1 m/s

Motion of a system with initial displacement and an initial velocity.

(i) Displacement and velocity of mass as a function of time

Undamped natural frequency of the system is given by the equation

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s} = \sqrt{\frac{10000}{20}} = 22.36 \text{ rad/s}$$

Damping factor $\xi = \dfrac{c}{c_c} = \dfrac{c}{2m\omega_n} = \dfrac{150}{2 \times 20 \times 22.36} = 0.1677$

Damped frequency $\omega_d = \omega_n\sqrt{1-\xi^2} = 22.36\sqrt{1-0.1677^2} = 22.043$ rad/s

Now $\xi\omega_n = 0.1677 \times 22.36 = 3.75$

Displacement $x = e^{-\xi\omega_n t}[A_1 \cos \omega_d t + A_2 \sin \omega_d t]$

Displacement $x = e^{-3.75t}[A_1 \cos 22.043t + A_2 \sin 22.043t]$

Velocity $\dot{x} = -3.75\, e^{-3.75t}[A_1 \cos 22.043t + A_2 \sin 22.043t]$
$+ 22.043 e^{-3.75t}[-A_1 \sin 22.043t + A_2 \cos 22.043t]$

When $t = 0; x = x_{(0)} = 0, \quad 0 = A_1 + 0, \quad \therefore A_1 = 0$

When $t = 0, \dot{x} = \dot{x}_{(0)} = 0.1$ m/s, i.e. $0.1 = 0 + 22.043[0 + A_2]$,

$$A_2 = \frac{0.1}{22.043} = 4.5366 \times 10^{-3}$$

Substituting the values of 'A'$_1$ and 'A'$_2$ in displacement and velocity equation,

Displacement $x = 4.5366 \times 10^{-3}\, e^{-3.75t} \sin 22.043t$

Velocity
$$\dot{x} = -3.75 \times e^{-3.75t} \times 4.5366 \times 10^{-3} \times \sin 22.043t$$
$$+ 22.043e^{-3.75t} \times 4.5366 \times 10^{-3} \times \cos 22.043t$$
$$\dot{x} = e^{-3.75t}\,[0.1 \cos 22.043t - 0.017 \sin 22.043t]$$

(ii) Displacement and velocity at time equal to one second, i.e. $t = 1$

Displacement $x = 4.5366 \times 10^{-3} \times e^{-3.75 \times 1} \times \sin(22.043 \times 1) = 4 \times 10^{-5}$ m

Velocity $\dot{x} = e^{-3.75 \times 1}[0.1 \cos(22.043 \times 1) - 0.017 \sin(22.043 \times 1)]$

$= 2.03 \times 10^{-3}$ m/s.

EXAMPLE 3.6

In a simple spring-mass-damper system, the mass is 20 kg, spring constant is 10 kg/cm and the damping coefficient has a value of 0.15 kg-s/cm and the system is initially at rest. When a velocity of 10 cm/s is given to it, determine the subsequent displacement and velocity of the mass.

Solution $m = 20$ kg, $k = 10 \times 9.81 \times 100 = 9810$ N/m,

$c = 0.15$ kg-s/cm $= 0.15 \times 9.81 \times 100 = 147.15$ N-s/m,

$v = 10$ cm/s $= 10/100 = 0.1$ m/s

Initial conditions: (i) When $t = 0, x = 0$ (ii) When $t = 0, \dot{x} = 0.1$ m/s.

We have
$$c_c = 2\sqrt{mk} = 2\sqrt{\frac{20}{981} \times 9810} = 885.88 \text{ N-s/m.}$$

But
$$\xi = \frac{c}{c_c} = \frac{0.15}{0.9} = 0.16 < 1 \text{ (underdamping system)}$$

Hence,
$$x = e^{-\xi\omega_n t}\,[A \sin \omega_d t + B \cos \omega_d t] \qquad \text{...(a)}$$

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s} = \sqrt{\frac{9810}{20}} = 22.15 \text{ rad/s}$$

Damped natural frequency, $\omega_d = \omega_n\sqrt{1 - \xi^2} = 22.15\sqrt{1 - (0.16)^2}$, $\omega_d = 21.86$ rad/s

$$\therefore \qquad x = e^{-0.16 \times 22.15t}\,[A \sin 21.86t + B \cos 21.86t]$$

Using initial conditions, (i) at $t = 0$ $x = 0$ $\quad \therefore 0 = e^{-0.16 \times 22.15 \times 0}[A \sin 21.86 \times 0 + B \cos 21.86 \times 0]$

$$0 = e^{-0}[A \sin 0° + B \cos 0°] \quad \therefore B = 0$$

$$\dot{x} = e^{-3.51}[A \times 21.86 + \cos 21.86t] + A \sin 21.86t + (-3.513)e^{-3.513t}$$

Again using initial condition, (ii) at $t = 0$, $\dot{x} = 0.1$ m/s

$$0.1 = e^{-0}[A \times 21.86 + 0] \quad \therefore = \frac{0.1}{21.80}, A = 0.0046$$

Substituting the values of $A = 0.0046$, $B = 0$ in Eq. (a).

Hence $\quad x = 0.0046\, e^{-3.513t} \times \sin 21.86\, t$

What will be the displacement after $t = 1$ second?

$$x = e^{-0.166 \times 22.147 \times 1}[0.00457 \sin 21.839°] = 0.0017 \approx 1 \text{ rad} \approx \pi \approx 180°$$

EXAMPLE 3.7

The mass of the spring-mass-dashpot system is given an initial velocity (from the equilibrium position) of $A\ \omega_n$ where 'ω'$_n$ is the undamped natural frequency of the system. Find the equation of motion for the system when (i) $\xi = 0.2$, (ii) $\xi = 1$, and (iii) $\xi = 2$.

Solution *(i) When $\xi = 0.2 < 1$ is an underdamping system.*

For underdamping system, $x = Ae^{-0.2\omega_n t} \sin(\sqrt{0.96}\ \omega_n t + \phi)$, $t = 0$, $x = x_0 = A \sin \phi$

$$\dot{x} = -0.2\omega_n Ae^{-0.2\omega_n t} \sin(\sqrt{0.96}\ \omega_n t + \phi) + Ae^{-0.2\omega_n t}\sqrt{0.96}\ \omega_n \cos(\sqrt{0.96}\ \omega_n t + \phi).$$

$$t = 0,\ \dot{x} = -0.2\omega_n A \sin \phi + \sqrt{0.96}\ \omega_n A \cos \phi$$

$$\therefore \tan \phi = \sqrt{\frac{0.96}{0.2}},\ \phi = 78^0\, 28^1,\ \sin \phi = \sqrt{0.92}$$

$$\frac{A}{x_0} = \frac{1}{\sin \phi} = 1.02,\ A = 1.02\, x_0,\ \frac{x}{x_0} = 1.02e^{-0.2\omega_n t} \sin(\sqrt{0.96}\ \omega_n t + 78^0\, 28^1).$$

(ii) When $\xi = 1$, it is a critical damping system.

For a critical damping system,

$$x = (A + Bt)\, e^{-\omega_n t},\ t = 0\ x_0 = A,\ \dot{x} = -\omega_n(A + Bt)\, e^{-\omega_n t} + Be^{-\omega_n t},\ t = 0\ \dot{x} = 0$$

$$= -\omega_n A + B$$

$$\therefore \quad A = x_0,\ B = \omega_0\, x_0,\ \frac{x}{x_0} = (1 + \omega_n t)e^{-\omega_n t}$$

Motion is still not periodic but decreasing exponentially.

(iii) When $\xi = 2$, $\quad \xi > 1$ is an overdamping system.

For overdamping system, $x = Ae^{(-2+\sqrt{3})\omega_n t} + Be^{(-2-\sqrt{3})\omega_n t}$,

Apply Boundary Conditions at $t = 0$, $x = x_0 = A + B \therefore A + B = x_o$

$$\dot{x} = (-2 + \sqrt{3})\omega_n Ae^{(-2\times\sqrt{3})} + (-2 + \sqrt{3})\omega_n Be^{(-2-\sqrt{3})\omega_n t}$$

$$t = 0\ \ \dot{x} = 0 = (-2 + \sqrt{3})\omega_n A + (-2 - \sqrt{3})\omega_n B$$

or $$A = \left(\frac{2+\sqrt{3}}{-2+\sqrt{3}}\right)B,\ A = x_0 - B\ \therefore\ 0 = -x_0 + B\left(1 + \frac{2+\sqrt{3}}{-2+\sqrt{3}}\right) = -x_0 + \frac{2\sqrt{3}}{-2+\sqrt{3}}B.$$

$$B = \frac{x_0(-2+\sqrt{3})}{2\sqrt{3}} = -0.077,\ A = \left(\frac{2+\sqrt{3}}{-2+\sqrt{3}}\right)\left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right) = x_0 = 1.077\ x_0.$$

Motion is not periodic but decreasing exponentially.

EXAMPLE 3.8

A vibrating system consists of a mass of 50 kg, a spring of 30 kN/m stiffness of a damper. The damping provided is only 20% of the critical value.

Determine (i) the damping factor, (ii) the critical damping coefficient, (iii) the natural frequency of damped vibrations, (iv) logarithmic decrement, and (v) the ratio of two consecutive amplitude.

Solution $m = 50$ kg , $k = 30$ kN/m = 30000 N/m, $c = 0.2c_c$.

(i) The damping factor $\xi = \frac{c}{c_c} = \frac{0.2c_c}{c_c} = 0.2 < 1$ (underdamping)

(ii) The critical damping coefficient, $c_c = 2\sqrt{mk} = 2\sqrt{50 \times 30000} = 2450$ N-s/m = 2.4 N-s/mm

(iii) The natural frequency of damped vibrations,

$$\omega_d = \omega_n\sqrt{1-\xi^2}.\ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000}{50}} = 24.5 \text{ rad/s}\ \therefore\ \omega_d = 24.5\sqrt{1-0.2^2} = 24 \text{ rad/s}$$

(iv) Logarithmic decrement, $\delta = \dfrac{2\pi\xi}{\sqrt{1-\xi^2}} = \dfrac{2\pi \times 0.2}{\sqrt{1-0.2^2}} = 1.28$

(v) Also the ratio of two consecutive amplitude,

$$\delta = \log_e\left(\frac{x_n}{x_{n+1}}\right), \left(\frac{x_n}{x_{n+1}}\right) = e^{\delta}\quad \therefore \left(\frac{x_n}{x_{n+1}}\right) = e^{1.28}, \left(\frac{x_n}{x_{n+1}}\right) = 3.6$$

EXAMPLE 3.9

The disc of a torsional pendulum has a momentum of inertia of 600 kg-cm^2 and immersed in a viscous fluid. The brass shaft attached to it is of 10 cm diameter and 40 cm long. When the pendulum is vibrating, the observed amplitudes on the same side of the rest portion for successive cycles are 9°, 6° and 4°. Determine

(i) logarithmic decrement 'δ', (ii) damping torque at unit velocity, and (iii) the periodic time of vibrations. Assume for the brass shaft $G = 4.4 \times 10^{10}$ N/m^2. What would the frequency be if the disc is removed from the viscous fluid.

Solution (i) The logarithmic decrement 'δ' in case of torsional system is given by the equation

$$\delta = \log_e\left(\frac{\theta_0}{\theta_1}\right) = \log_e\left(\frac{\theta_1}{\theta_2}\right) = \log_e\left(\frac{\theta_2}{\theta_3}\right) = \log_e\left(\frac{9}{6}\right) = \log_e\left(\frac{6}{4}\right) = \log_e(1.5) \approx 0.405$$

Also, the logarithmic decrement

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \text{ or } \xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}},\ \xi = \frac{0.405}{\sqrt{4\pi + 0.405}} = 0.0645$$

(ii) Now damping torque at unit velocity is the torsional damping coefficient 'c' of the system given by

$$\xi = c/c_c \text{ or } c = \xi\, c_c,\ c_c = 2\sqrt{mk} \quad \text{or } 2\sqrt{k_t I}, \quad \therefore\ c = 2.\xi\sqrt{k_t \cdot I}$$

$$d = 10 \text{ cm} = 0.1 \text{ m},\ l = 40 \text{ cm} = 0.4 \text{ m},\ I = 600 \text{ kg-cm}^2 = 0.6 \text{ kg m}^2.$$

$\therefore$ torsional stiffness of the shaft is given by the equation.

$$k_t = \frac{G.I_P}{l} = \frac{G}{l}\cdot\frac{\pi}{32}d^4 \quad \therefore\ T/I_P = G\theta/l, \quad \text{or } T/\theta = k_t = GI_P/l$$

$$k_t = \frac{4.41\times10^{10}}{0.4} = \frac{\pi}{32}(0.1)^4 = 1.08\times10^6 \text{ N-m/rad.}$$

$$\therefore \qquad c = 0.0645\times2\times\sqrt{1.08\times10^6\times0.06}\ \ c = 32.8 \text{ N-m/rad.}$$

This is the damping torque at unit velocity.

(iii) The periodic time of vibration is given by the equation $\tau = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_d}$

$$\tau = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}},\ \omega_n = \sqrt{\frac{k_t}{I}} = \sqrt{\frac{1.08\times10^6}{0.06}} = 4240 \text{ rad/s,}$$

$$\tau = \frac{2\pi}{4240\sqrt{1-0.0645}} = 0.0015 \text{ s.}$$

The frequency when the disc is removed from the viscous fluid is the natural frequency of the system and is given by $\omega_n = 2\pi f_n$ or $f_n = \frac{\omega_n}{2\pi} = \frac{4240}{2\pi} = 675$ Hz or cps.

EXAMPLE 3.10

In a single-degree damped vibrating system, a suspended mass of 8 kg makes 30 oscillations in 18 seconds. The amplitude decreases to 0.25 of the initial value after 5 oscillations. Determine (i) the stiffness of the springs, (ii) the logarithmic decrement, (iii) the damping factor, and (iv) the damping coefficient.

Solution $m = 8$ kg, Number of oscillations is 30 in 18 s

$\therefore$ time taken for one oscillation = 30/18 = 1.67 Hz

We know that $\omega_n = 2\pi\times f = 2\pi\times1.67 = 10.47$ rad/s.

(i) The stiffness of the springs, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{8}} = 10.47$

$\therefore\ k = 8.77$ N/m or 0.877 N/mm.

We know that $\left(\frac{x_0}{x_5}\right) = \frac{x_0}{x_1} = \frac{x_1}{x_2}\times\frac{x_2}{x_3}\times\frac{x_3}{x_4}\times\frac{x_4}{x_5}$, $\quad \frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \left(\frac{x_0}{x_5}\right)^5$

$$\therefore \qquad \left(\frac{x_0}{x_1}\right) = \left(\frac{x_0}{x_5}\right)^{1/5} = \left(\frac{1}{0.25}\right)^{1/5} = 1.32 \quad \therefore\ \delta = \log_e\left(\frac{x_0}{x_5}\right).$$

(ii) The logarithmic decrement, $\delta = \log_e(1.32)$, $\delta = 0.278$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}},\quad 0.278 = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad \sqrt{1-\xi^2} = 22.6\xi,\ 1-\xi^2 = 510.82\xi^2.$$

(iii) The damping factor $\xi = 0.0442 < 1$ (underdamping).

(iv) The damping coefficient $c = 2m\,\omega_n\,\xi = 2 \times 8 \times 10.47 \times 0.0442 = 7.4$ N-s/m.

3.8 ENERGY DISSIPATED PER CYCLE IS PROPORTIONAL TO THE FREQUENCY AND SQUARE OF AMPLITUDE IN CASE OF VISCOUS DAMPING

We have the damping force in case of viscous damping is given by the equation $F = -c\,\dot{x}$.

Since this viscous damping force always resists the motion, this will be positive, when velocity ($\dot{x}$) is negative and this will be negative when velocity ($\dot{x}$) is positive. This is shown in Fig. 3.12. Therefore, the energy dissipated (ED) per cycle is given by

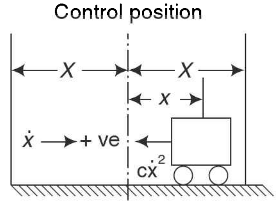


Fig. 3.12 *Energy dissipated in viscous damping*

$$\text{Energy dissipated} = 2\int_{-X}^{X} c\dot{x}\,dx \qquad \ldots 3.20$$

Assuming that the motion is periodic and $x = X\sin\omega t$ and $\dot{x} = \frac{dx}{dt} = -\omega X\cos\omega t$ or $dx = -\omega X\cos\omega t\,dt$

Substituting these values in Eq. 3.20,

we get, $ED = 2\int_{-X}^{X} c(-\omega X\cos\omega t)(-\omega X\cos\omega t)dt.$

$$= 2 \times c\omega X^2 \int_{\pi}^{2\pi} \cos^2\omega t\, d(\omega t) = c\omega X^2\left[(\omega t) + \frac{\sin 2(\omega t)}{2}\right]_{\pi}^{2\pi} = c\omega X^2[\pi] = c\,\omega\pi X^2$$

$\therefore$ energy dissipated/cycle $\propto \omega$, where c, π, X^2, are constant and also energy dissipated/cycle $\propto X^2$.

Energy dissipated by a damping force proportional to the square of the velocity From Fig. 3.12 and discussion, $F = c(\dot{x})^2$

Here, energy dissipated/cycle $= 2\int_{-x}^{x} c(\omega^2 X^2\cos^2\omega t)(\omega X\cos\omega t)dt$

$$= 2\int_{\pi}^{2\pi} c\omega^3 X^3(\cos^3\omega t)\frac{d(\omega t)}{\omega} = -2c\omega^2 X^3 \times 2\int_{\pi}^{3\pi/2} c\omega^2(\omega t)\,d(\omega t)$$

$$= -4c\omega^2 X^3\left[\frac{3}{4}\sin(\omega t) + \frac{1}{12}\sin 3(\omega t)\right]_{\pi}^{3\pi/2} = -4c\omega^2 X^3\left[-\frac{3}{4} + \frac{1}{12}\right] = \frac{8}{3}c\omega^2 X^3$$

3.9 COULOMB DAMPING, OR DRY FRICTION DAMPING

In this type of damping, the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating bodies.

It is caused by friction between the surfaces that are dry or having insufficient lubrication. as shown in Fig. 3.13(a) and FBD of Fig. 3.13(b). When a body slides on a dry surface, the force of resistance between the surfaces or the frictional force is proportional to the normal load. This damping is called Coulomb damping and the relation for force is (F), $F \alpha R_N$ or $F = \mu R_N$ Sgn. ($\dot{x}$) where R_N = Normal load, μ = Coefficient of kinetic friction in Fig. 3.13(c) as normal load verses friction. It is effective only in the final stage of the damped free vibration when the other types of damping are ignorable.

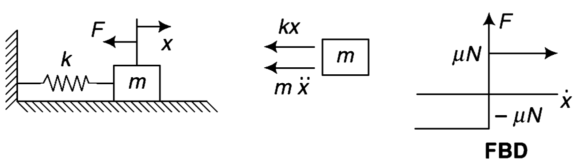


Fig. 3.13 *Coulomb damping*

1. Frequency of damped oscillations Let us consider spring-mass system, capable of sliding on a dry surface as shown in Fig. 3.14(a) and 'μ' is the coefficient of dry friction between the two surfaces. Now we will discuss the following three states of a body in case of a Coulomb damping as follows.

When a mass is at rest, the spring is unstretched and no friction force 'F' acts on the body or either moves to leftwards direction or moves towards right direction are the three possible states of a body.

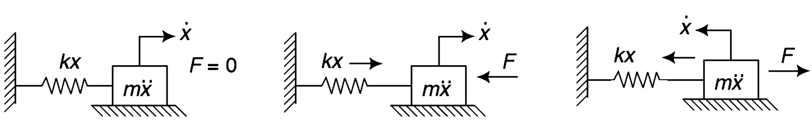


Fig. 3.14 *Coulomb damping*

1. When a mass is at equilibrium or in rest position as shown in Fig. 3.14(a), the equation of a motion is given as, $m\ddot{x} + kx = 0$.
2. When a mass moves towards right direction as shown in Fig. 3.14(b), the equation of a motion is given as, $m\ddot{x} + kx + F = 0$.

3. When a mass moves towards leftwards direction as shown in Fig. 3.14(c), the equation of a motion is given as, $m\ddot{x} + kx - F = 0$.

In the above cases, let us consider that the mass moves towards the leftwards direction and the equation can be rewritten as $m\ddot{x} + kx = F$ and divided by 'm' throughout.

$$\ddot{x} + \frac{k}{m}x = \frac{F}{k} \qquad ...3.21$$

The solution of Eq. 3.21 can be written as

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t + \frac{F}{k} \qquad ...3.22$$

As we know that $\omega_n = \sqrt{\frac{k}{m}}$ rad/s.

Now applying boundary condition $x = x_0$ at $t = 0$ and $\dot{x} = 0$ at $t = 0$, we get

$A = \left(x_0 - \frac{F}{k}\right)$ and $B = 0$. Substitute the values of 'A' and 'B' in Eq. 3.22, we get

$$x = \left(x_0 - \frac{F}{k}\right)\cos\sqrt{\frac{k}{m}}t + \frac{F}{k} \qquad ...3.23$$

The equation 3.23 is the solution for only half the cycle. When time $t = \frac{\pi}{\omega}$, half the cycle is complete as shown in Fig. 3.23(c). In this half cycle, the displacement can be determined by Eq. 3.23.

By substituting $t = \frac{\pi}{\omega}$ and $\cos\pi = -1$, we get

$$x = \left(x_0 - \frac{F}{k}\right)\cos\pi + \frac{F}{k}, \quad x = -\left(x_0 - \frac{F}{k}\right) + \frac{F}{k}, \quad x = -\left(x_0 - \frac{2F}{k}\right) \qquad ...3.24$$

The equation (3.24) is the amplitude for left extreme position of the mass and it clearly shows that the initial displacement 'x_0' is reduced by an amount $\frac{2F}{k}$. On the other hand, in the next half cycle when a mass moves towards, the right extreme position of the initial displacement 'x_0' is reduced by an amount $\frac{2F}{k}$. Now we understand in one complete cycle, the amplitude reduced by the total amount of $\frac{4F}{k}$.

This can be represented by means of a graph as shown in Fig. 3.15. Now it is clearly understood in case of Coulomb damping that the natural frequency of the system remains unchanged.

2. Rate of decay of oscillations Referring Fig. 3.15 also shows the rate of amplitude decay of vibration with Coulomb damping. The rate of amplitude decay can be explained briefly as follows. It can be seen from the figure, when a body moves from point '1' to point '3' through point '2', the motion is a part of a simple harmonic motion about point '2'. Suppose when we consider the motion in equilibrium position, i.e., $x = 0$, it means moving from '1' to '3' the body has lost an amount

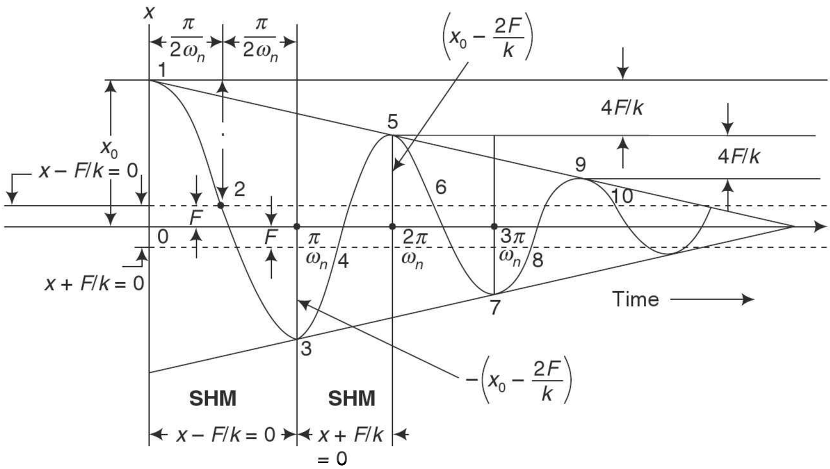


Fig. 3.15 *Amplitude decay in Coulomb damping*

of amplitude equal to $\frac{2F}{k}$. This is the half of the simple harmonic motion or half a cycle. Similarly, there is another loss of amplitude of an amount equal to $\frac{2F}{k}$ about the equilibrium position in the next half of the cycle in Fig. 3.16. Thus the amount of amplitude loss in one complete cycle (Fig. 1-2-3-4-5) equals $\frac{4F}{k}$.

This is same as that of the above equation 3.24 in the energy point of view.

Difference between viscous damping and Coulomb damping is shown in Table 3.1

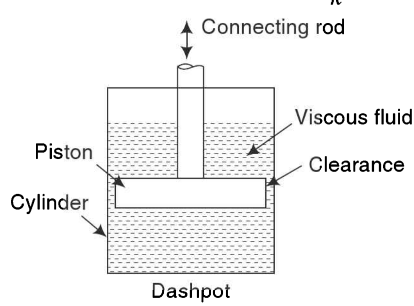


Fig. 3.16 *Viscous damping*

Table 3.1 *Main difference between Viscous Damping and Coulomb damping*

Viscous damping (Fig. 3.16)	*Coulomb damping (Fig. 3.13)*
In case of viscous damping the damping, resistance is directly proportional to the relative velocity.	In case of Coulomb damping, the damping resistance is constant for the entire velocity range.
Viscous damping occurs as a result of fluid friction.	Coulomb damping occurs when two parts work against each other which are dried and unlubricated.
In case of viscous damping, the ratio of two consecutive amplitudes is constant and the envelop of maximum of displacement time plot is an exponential curve.	In case of coulomb damping, the difference between any two consecutive amplitudes is same and the envelope of the maximum of the displacement time plot is a straight line.
In case of viscous damping, the body once disturbed from the equilibrium position finally comes to rest in equilibrium position.	In case of coulomb damping, the body may finally come to rest in equilibrium position.

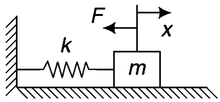


Fig. 3.17 *Coulomb damping*

3.10 STRUCTURAL DAMPING, OR SOLID DAMPING, OR HYSTERISIS DAMPING OR MATERIAL DAMPING

Hysterisis damping is an important type of damping since it occurs in all vibrating systems subjected to elastic restoring forces. This type of damping is due to the internal friction of the molecules of elastic materials as shown in Fig. 3.18. Stress–strain graphs are different types of damping for loading and unloading.

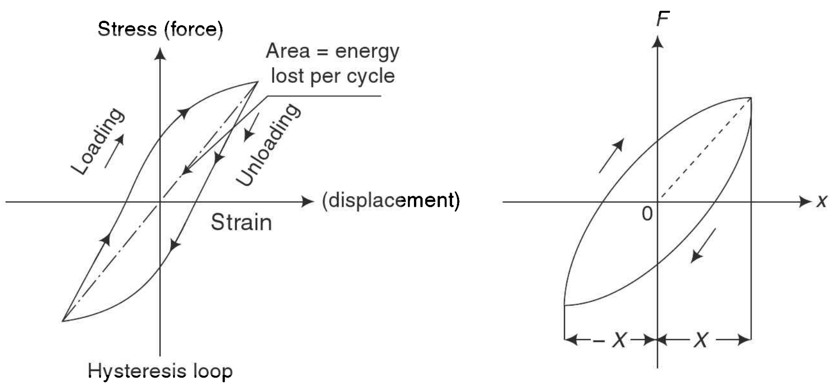


Fig. 3.18 *Structural damping*

When such a material is subjected cyclic reversal of loading, a hysterisis loop appears on the stress–strain plot as shown in Fig. 3.18(a). The area of the thin loop is indicating the energy dissipated per unit volume/cycle. This means that more work is done on the system while straining it than what is recovered during its relaxation. This type of damping is called *hysterisis damping*. The size of the loop depends upon the material of the vibrating body, frequency and amount of dynamic stress.

Thus the enclosed area in the hysterisis loop is the energy loss per cycle loading.

This can be expressed as $E = \int F dx$ ...3.25

From experiment, it was observed that the energy loss per cycle is proportional to the stiffness of the material and the square of the displacement amplitude 'X'. Also, it is independent of frequency. Therefore, the energy loss is given by the expression

$$E = \pi \lambda k X^2 \qquad ...3.26$$

where $k\lambda$ = Shape, size and property of the material

λ = Property of the material

X = Amplitude of vibration

In most of the cases, point '4' is very small and motion is nearly simple harmonic as that of the equation (3.26) may be used to obtain an equivalent damping coefficient.

Assuming that the motion is simple harmonic as

$$x = X\sin(\omega_n t + \varphi) \quad ...3.27$$

We know that the force exerted by the viscous damping is $c\dot{x} = cX\,\omega_n \cos(\omega_n t + \varphi)$

$$\therefore \text{ energy loss per cycle } E = \int_0^{\frac{2\pi}{\omega}} c\,\dot{x}^2\,dt = \int_0^{\frac{2\pi}{\omega}} c\omega_n^2\,X^2\cos^2(\omega_n t + \varphi)dt = \pi\omega_n cX^2 \quad ...3.28$$

By equating equations (3.26) and (3.28), the equivalent damping coefficient is

$$c_{\text{eq}} = \frac{\lambda k}{\omega_n} = \lambda\sqrt{km} \quad ...3.29$$

Since $\omega_n = \sqrt{\frac{k}{m}}$, the value of structural damping coefficient 'λ' may be determined experimentally by means of using logarithmic decrement same as that of viscous damping.

The energy equation for half cycle between any two successive amplitudes is indicated in Fig. 3.19 in points '2' and '4' of the curve.

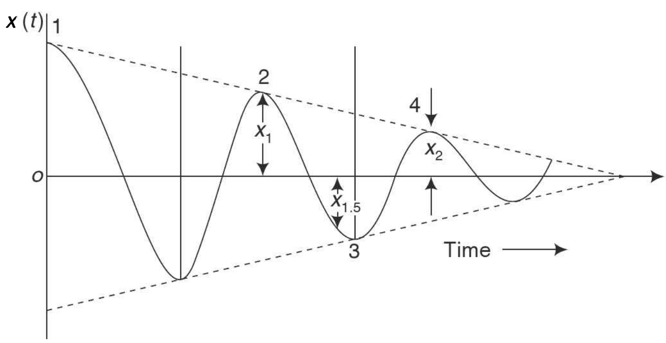


Fig.3.19 *Energy equation for half cycle*

$$\frac{kX_1^2}{2} - \frac{\pi\lambda kX_1^2}{4} - \frac{\pi\lambda kX_{1.5}^2}{4} = \frac{kX_{1.5}^2}{2} \quad \text{i.e.} = \frac{X_1^2}{X_{1.5}^2} = \frac{1 + \frac{\pi\lambda}{2}}{1 - \frac{\pi\lambda}{2}} \quad ...3.30$$

This is for half cycle.

$$\text{Similarly, for next half cycle } \frac{X_{1.5}^2}{X_2^2} = \frac{1 + \frac{\pi\lambda}{2}}{1 - \frac{\pi\lambda}{2}} \quad ...3.31$$

$$\therefore \text{ ratio of successive amplitude } \frac{X_1}{X_2} = \frac{1 + \frac{\pi\lambda}{2}}{1 - \frac{\pi\lambda}{2}} \quad ...3.32$$

But the value of 'λ' is very small in some materials, so the equation (3.32) can be written as

$$\frac{X_1}{X_2} = 1 + \frac{\pi\lambda}{2} \qquad ...3.33$$

By definition of logarithmic decrement,

$$\delta = \ln\frac{X_1}{X_2} = \ln\left(1 + \frac{\pi\lambda}{2}\right) = \pi\lambda. \qquad ...3.34$$

3.11 EQUIVALENT VISCOUS DAMPING

In the same manner let us discuss the equivalent viscous damping.

The equivalent damping factor $\xi_{eq} = \frac{c_{eq}}{c}$

Substituting the values of $c_{eq} = \frac{\lambda k}{\omega_n}$ and $c = 2m\ \omega_n$, we get

$$\frac{\pi\lambda}{\omega_n\, 2m\omega_n} = \frac{\pi\lambda}{2m\omega_n^2}, \quad \xi_{eq} = \frac{\lambda}{2} \qquad ...3.35$$

Also we know that

$$\xi_{eq} = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}, \quad \frac{\lambda}{2} = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \quad \therefore \lambda = \frac{2\delta}{\sqrt{(2\pi)^2 + \delta^2}} \qquad ...3.36$$

The amplitude decay is found to be exponential in nature. When force (F) is plotted versus displacement (x) then a closed loop is formed as shown in Fig. 3.18(b). The area of the loop indicates the energy dissipated by the damper in one complete cycle of motion. This can given by the equation $E = \pi\lambda k X^2$.

3.11 SLIP DAMPING, OR INTERFACIAL DAMPING

Slip damping, or interfacial damping occur where energy is dissipated due to rubbing of parts, backlash, etc. Consider a cantilever bar 'x' on which another bar 'y' is placed. The two bars are pressed together by means of a number of C-clamps to provide a pressure 'p' between the two contact surfaces of bars 'x' and 'y' as shown in Fig. 3.20(a).

Now when the system vibrates in the vertical plane as shown, notice the continuous slip between the two surfaces shown exaggerated in Fig. 3.20(b).

The energy dissipated per cycle based up on the coefficient of friction, the pressure between the two plates and by the means of amplitude is shown in Fig. 3.20(c). At less pressure, there is larger slip but no energy is dissipated in friction because of less pressure and therefore there is no energy loss.

At very high pressure, there is essentially no slip and hence no energy loss. There is an optimum value of pressure for which the energy dissipated is maximum. This is different for different amplitudes of vibration. Larger the energy dissipation, larger is the effective damping in the system.

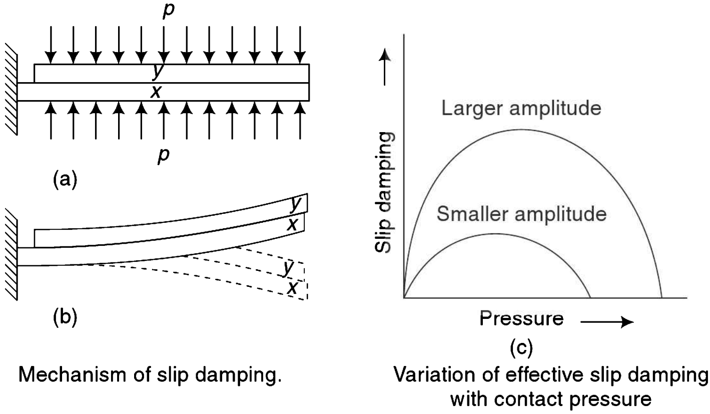


Fig. 3.20 *Slip damping*

EXAMPLE 3.11

Write down the differential equation of motion for the following system shown in Fig. p-3.11(a) and find an expression for (i) critical damping coefficient, and (ii) damped natural frequency.

Solution For an angular downward displacement 'θ' for the mass 'm', the FBD is shown in Fig. p-3.11(b). The force in the spring is $k\,(4a \times \theta)$ and the dashpot is $c(3a\dot{\theta})$

From Newton's second law of motion, $I_0\ddot{\theta} = \Sigma T$

$$m(2a)^2\,\ddot{\theta} = -c(3a \times \dot{\theta})3a - k(4a \times \theta)4a$$

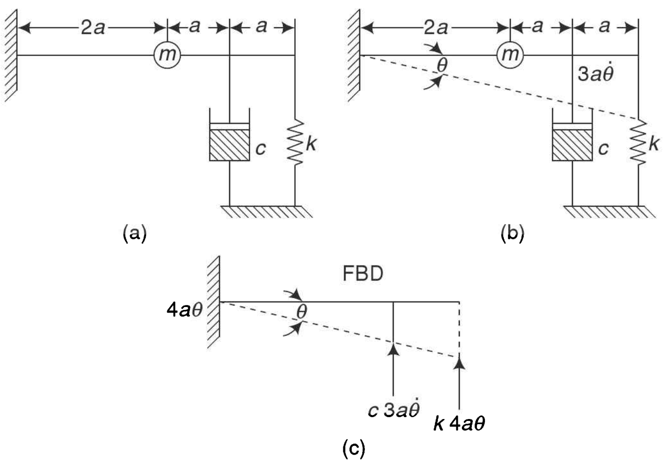


Fig. p-3.11 *Spring-mass damper*

$$4m\,a^2\ddot{\theta} + 9ca^2\dot{\theta} + 16ka^2\theta = 0 \text{ or } \quad \ddot{\theta} + \frac{9}{4}\frac{c}{m}\dot{\theta} + 4\frac{k}{m}\theta = 0 \qquad ...(a)$$

$$\omega_n = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}} \text{ rad/s}$$

The solution of the Eq. (a) is assuming that $\theta = e^{st}$

Differentiating with respect to time.

$$\theta = e^{st}, \dot{\theta} = se^{st}, \ddot{\theta} = s^2 e^{st}$$

Substituting these values in Eq. (a),

$$s^2 e^{st} + \left(\frac{9c}{4m}\right) se^{st} + \left(\frac{4k}{m}\right) e^{st} = 0 \quad e^{st} \neq 0$$

$$\therefore \quad s^2 + \left(\frac{9c}{4m}\right) s + \left(\frac{4k}{m}\right) = 0 \qquad \text{...(b)}$$

$$\therefore \quad s_{1,2} = \frac{\dfrac{-9c}{4m} \pm \sqrt{\left(\dfrac{9c}{4m}\right)^2 - \left(\dfrac{16k}{m}\right)}}{2}$$

$$\therefore \quad s_{1,2} = \frac{-9c}{8m} \pm \sqrt{\frac{1}{4}\left(\frac{9c}{4m}\right)^2 - \left(\frac{16k}{4m}\right)}$$

$$\therefore \quad s_{1,2} = \frac{-9c}{8m} \pm \sqrt{\left(\frac{81c}{64m}\right) - \left(\frac{4k}{m}\right)}$$

The critical damping coefficient is given by (the radical sign must be zero)

$$\left(\frac{9\,c}{8\,m}\right)^2 = \frac{4\,k}{m}, \; c_c^2 = \left(\frac{256\;km^2}{81m}\right), \; c_c = \left(\frac{16}{9}\right)\sqrt{km}, \; \xi = \frac{c}{c_c} = \frac{c}{\dfrac{16}{9}\sqrt{km}}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}, \; \omega_d = \sqrt{\frac{4\,k}{m}}\sqrt{1 - \frac{81c^2}{256\;km}}, \; \omega_d = \sqrt{\frac{4\,k}{m} - \frac{81c^2\,k}{64\;km}},$$

$$\omega_d = \sqrt{\frac{4\,k}{m} - \frac{81c^2}{64\;m}} \text{ rad/s.}$$

EXAMPLE 3.12

The simple pendulum is pivoted at the point '*O*' as shown in Fig. p-3.12(a). If the mass of the rod is small, determine the damped natural frequency of the pendulum for small oscillations.

Solution For an angular displacement 'θ' for the weight 'mg', the FBD is shown in Fig. p-3.12(b).

The equation of motion is given by

$$I_0\ddot{\theta} = -c(b \times \dot{\theta})b - k(a \times \theta)a - mgl\sin\theta \text{ if '}\theta\text{' is small } \sin\theta = \theta$$

$$I_0\ddot{\theta} = -c(b \times \dot{\theta})b - k(a \times \theta)a - mgl\,\theta, \text{ since } I_o = \text{ml}^2 \text{ then,}$$

$$ml^2\ddot{\theta} = -cb^2\,\dot{\theta} - ka^2\theta - mgl\,\theta, \quad ml^2\ddot{\theta} = -cb^2\,\dot{\theta} - (ka^2 + mgl)\theta$$

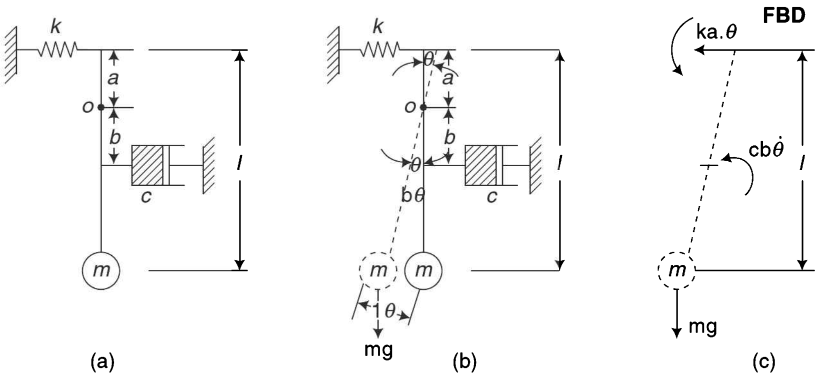


Fig. p-3.12 *Spring-mass damper*

$$ml^2\ddot{\theta} + cb^2\dot{\theta} + (ka^2 + mgl)\theta = 0 \quad \therefore \ddot{\theta} + \frac{cb^2}{ml^2}\dot{\theta} + \left(\frac{ka^2 + mgl}{ml^2}\right)\theta = 0$$

The solution of this equation is $s_{1,2} = \dfrac{-cb^2}{2\,ml^2} \pm \sqrt{\left(\dfrac{cb}{2\,ml^2}\right)^2 - \left(\dfrac{ka^2 + mgl}{ml^2}\right)}$

The damped natural frequency ω_d = Radical with negative sign

$$\omega_d = \sqrt{\frac{ka^2 + mgl}{ml^2} - \left(\frac{cb^2}{2ml^2}\right)^2} \text{ rad/s.}$$

EXAMPLE 3.13

Write the differential equation of motion for the system shown in Fig. p-3.13(a). Determine the natural frequency of damped vibration. Neglecting the mass of the rod holding the spring-mass and dashpot.

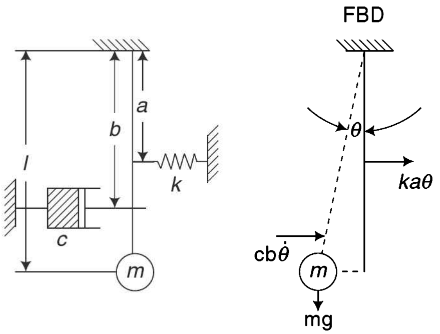


Fig. p-3.13 *Spring-mass damper*

Solution For an angular displacement 'θ'of the weight 'mg', the FBD as shown in Fig. p-3.13(b)

$$\Sigma M = 0,\ I_0\ddot{\theta} = -\Sigma T,\ I_0\ddot{\theta} = -k(a \times \theta)a - c(b \times \dot{\theta})b - mgl \sin\theta$$

$$I_0\ddot{\theta} + k(a \times \theta)\,a + c(b \times \dot{\theta})b + mgl \sin\theta = 0$$

If 'θ' is very small, $\sin\theta = \theta$, and $I_0 = ml^2$

$$\therefore \quad ml^2\ddot{\theta} + cb^2\dot{\theta} + ka^2\theta + mgl\theta = 0, \; ml^2\ddot{\theta} + cb^2\dot{\theta} + (ka^2 + mgl)\theta = 0$$

Divided by ml^2 throughout, we have

$$\therefore \quad \ddot{\theta} + \left(\frac{cb^2}{ml^2}\right)\dot{\theta} + \left(\frac{ka^2}{ml^2} + \frac{mg}{ml}\right)\theta = 0$$

Undamped natural frequency,

$$\omega_n = \sqrt{\left(\frac{ka^2}{ml^2} + \frac{mg}{ml}\right)} \text{rad/s}$$

Damped natural frequency of vibration

$$\omega_d = \omega_d\sqrt{1-\xi^2} = \sqrt{\left(\frac{ka^2}{ml^2} + \frac{mg}{ml}\right)}\sqrt{1-\xi^2} \text{ rad/s.}$$

EXAMPLE 3.14

A system as shown in Fig. p-3.14 consists of a massless spring of scale 'k' and massless piston in a dashpot of damping coefficient 'c'. The piston is displaced a distance 'x_0' and released. For this ideal case, derive an expression for the motion of the piston. Discuss the result.

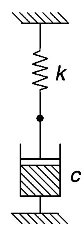


Fig. p-3.14 *Spring-massless damper*

Solution When mass 'm' is negligibly small, there remains only a spring and a damper system, i.e the point of interest is to know the motion of the massless damper piston.

The differential equation is $m\ddot{x} + c\dot{x} + kx = 0$

When a mass 'm' is negligibly small, i.e. $m \approx 0$, the above equation becomes

$c\dot{x} + kx = 0$ or $c\dfrac{dx}{dt} + kx$, $c\dfrac{dx}{dt} = -kx$ divided by 'kx' throughout $\dfrac{c}{k}\dfrac{dx}{x} = -dt$.

Integrating with respect to time 't', we get $t = \dfrac{-c}{k}(\log x + \text{constant})$

As $at\ t = 0, x = x_0$ $\quad\therefore$ constant $= -\log x_0$ $\quad\therefore\ t = \dfrac{-c}{k}\log\dfrac{x}{x_0}$, $\quad$ or $x = x_0 e^{\frac{-kt}{c}}$.

EXAMPLE 3.15

A gun barrel weighing 600 kg has a recoil spring of 30000 kg/m stiffness. If the barrel recoils 1.2 m on firing, determine (i) the initial velocity of the barrel, (ii) the critical damping coefficient of a dashpot engaged at the end of the recoil stroke, and (iii) the time required for the barrel to return to a position 5 cm from its position.

Solution $m = 600$ kg, $k = 30000$ kg/m $\dot{x} = 1.2$ m

If 'v' is the initial recoil velocity then,

(i) $\frac{1}{2}mv^2 = \frac{1}{2}k\dot{x}^2$, $\frac{1}{2} \times \frac{600}{981} \times v^2 = \frac{1}{2} 30000 \times 1.2^2$

$\therefore$ recoil velocity $v = \dot{x} = 27.35$ m/s.

(ii) $c_c = 2\sqrt{mk} = 2\sqrt{\frac{30000}{100} \times \frac{600}{981}} = 27.05$ kg-s/m.

(iii) The general expression for displacement in a critically damped case is given by

$$x = (A_1 + A_2 t)e^{-\omega_n t}, \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{30000 \times \frac{981}{600}}, \quad \omega_n = 22.2 \text{ rad/s}$$

At the instant the dashpot is engaged, the boundary conditions are $x = 1.2$ m $= 120$ cm at $t = 0$, $\dot{x} = 0$ *at* $t = 0$.

The solution under such condition is given by

$$x = 120(1 + \omega_n t)e^{-\omega_n t}, \text{ here } x = 5 \text{ cm} \quad \therefore 5 = 120(1 + 22.2t)e^{-22.2t}$$

Or $$e^{22.2t} = 24 + 532.8\,t,\ 22.2t = \log_e 24,\ t = 0.143 \text{ s.}$$

EXAMPLE 3.16

The barrel of large guns on firing recoil against a spring. At the end of recoil, dampers are engaged to bring the barrel to its original position ready for next firing within the minimum possible time. The barrel weighs 800 kg and the recoil distance is 1.5 m. The gun shots are weighing 15 kg at 1.6 km/s. Find out the proper spring constant and damping coefficient of the dashpot.

Solution If 'v' is the initial velocity with which the barrel recoils then from the principles of conservation of momentum.

$$15 \times 1.6 \times 100 = 800 \times v \quad \therefore v = 30 \text{ m/s.}$$

During the recoil, the damper is inactive and therefore the initial kinetic energy of the barrel must get absorbed in the spring as elastic energy, i.e. $\frac{1}{2}mv^2 = \frac{1}{2}k\dot{x}^2$

$\frac{1}{2} \times \frac{800}{9.81} \times 30^2 = \frac{1}{2}k(1.5)^2$

$\therefore$ spring constant $k = 32600$ kg/m or 326 kg/cm.

We know that damping coefficient of the dashpot $c_c = 2m\,\omega_n = 2\sqrt{mk} = 2\sqrt{\frac{800}{981} \times 326}$ $= 32.6$ kg-s/cm.

EXAMPLE 3.17

Between the solid mass of 10 kg and the floor are kept two isolators, natural rubber and felt in series. The natural rubber slab has a stiffness of 3000 N/m and an equivalent damping coefficient of 100 N-s/m. The felt slab has a stiffness of 12000 N/m and an equivalent viscous damping coefficient of 330 N-s/m. Determine the undamped and the damped natural frequencies of the system in vertical direction. Neglect the mass of the isolators.

Solution $m = 10$ kg, Stiffness of natural rubber $k_r = 3000$ N/m and Stiffness of felt slab $k_f = 12000$ N/m.

The springs 'k' and '$2k$' are in series.

Let k_{eq} = Equivalent stiffness of rubber and felt

$$\therefore \frac{1}{k_{eq}} = \frac{1}{k_r} + \frac{1}{k_f} = \frac{1}{3000} + \frac{1}{12000}\ k_{eq} = 2400 \text{ N/m.}$$

Equivalent damping coefficient of rubber

$$c_r = 100 \text{ N-s/m.}$$

Equivalent viscous damping coefficient of felt

$$c_f = 330 \text{ N-s/m.}$$

Let c_{eq} = Equivalent damping coefficient of rubber and felt.

This can be shown in Fig. p-3.17.

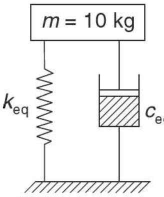


Fig. p-3.17 *Spring-mass damper*

$$\therefore \quad \frac{1}{c_{eq}} = \frac{1}{c_r} + \frac{1}{c_f} = \frac{1}{100} + \frac{1}{330}\ c_{eq} = 77 \text{ Ns/m.}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{2400}{10}} = 15.5 \text{ rad/s,} \quad f_n = \frac{\omega_n}{2\pi} = \frac{15.5}{2\pi} = 2.47 \text{ Hz}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2},\ \xi = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{2\sqrt{k_{eq}m}}$$

$$= \frac{77}{2\sqrt{2400 \times 10}} = 0.3 \text{ HZ. } \omega_d = 15.5\sqrt{1-(0.3)^2} = 15 \text{ rad/s,}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{15}{2\pi} = 2.4 \text{ Hz.}$$

EXMAPLE 3.18

For the horizontal frictionless surface, the spring-mass-damper system is shown in Fig. p-3.18(a). Determine (i) undamped natural frequency, (ii) damped natural frequency, (iii) logarithmic decrement. (iv) If the mass is initially at rest and is given a velocity of 0.1 m/s then calculate the amplitude of vibration after 5 oscillations.

Solution $k = 10000$ N/m, $m = 20$ kg, $c = 150$ N-s/m.

(i) $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{20}} = 22.36 \text{ rad/s.}$$

(ii) $$\omega_d = \omega_n\left(\sqrt{1-\xi^2}\right).$$

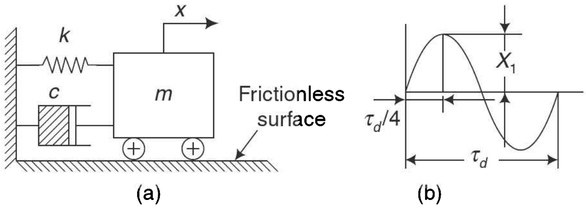


Fig. p-3.18 *Spring-mass damper are in horizontal position*

$$\xi = \frac{c}{c_c},\ c_c = 2\,m\omega_n = 2(20)\,(22.36) = 894.42 \text{ N.s/m}$$

$\therefore$ $$\xi = \frac{150}{894.42} = 0.1671 < 1 \text{ (underdamped)}$$

$\therefore$ $$\omega_d = 22.045 \text{ rad/s}$$

(iii) $$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi(0.167)}{0.9859} = 1.0643.$$

Initial conditions: (iv) $\dot{x} = 10$ cm/s *at* $t = 0$, $x = 0$ *at* $t = 0$

Response of the system:

$$x(t) = e^{-\xi\omega_n t}\,(A\cos\omega_d t + B\sin\omega_d t),\ \xi\omega_n = (0.167)\times(22.36) = 3.734$$

$\therefore$ $$x(t) = e^{-3.734t}\,(A\cos 22.045t + B\sin 22.045t)$$

$$\dot{x}(t) = 3.734\,e^{-3.734t}\,(A\cos 22.045t + B\sin 22.045t) + e^{-3.734t}\,(-A\sin 22.045t + B\cos 22.045t)$$

From the equation, $A = 0$

From the equation,

$$10 = -3.734(0+0) + 22.045\,(0+B),\ 22.045\,B = 10 \quad \therefore B = 0.4536.$$

$$\omega_d = \frac{2\pi}{\tau_d};\ \tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{22.045}, = 0.285 \text{ s} \quad \therefore \text{ time for one oscillation} = 0.285 \text{ s.}$$

From Fig. 3.18(b), x_1 occurs at $\frac{\tau_d}{4}, \frac{\tau_d}{4} = \frac{0.285}{4} = 0.07125.$

$$\therefore X_1 = e^{-3.734\,(0.07125)}\,0.4536\,[\sin(22.045\times 0.07125)] = (0.7664)\,(0.4536)$$
$$= 0.3476 \text{ cm}$$

In $\frac{x_1}{x_6} = 5\delta$, $\therefore \frac{x_1}{x_6} = e^{5\delta}$ $\therefore x_6 = \frac{x_1}{e^{5\delta}} = \frac{0.3476}{e^{5\times(1.0643)}} = 1.69\times 10^{-3}$ cm $= 1.69\times 10^{-5}$ m.

EXAMPLE 3.19

A weight of 300 N is resting on 2 springs of 3000 N/m stiffness each and a dash-pot of damping coefficient 150 N-s/m as shown in Fig. p- 3.19. If an initial velocity of 10 cm/s is given to the mass at its equilibrium position, what will be the displacement from the equilibrium position at the end of first second and derive the formula used.

Solution $m = \frac{w}{g} = \frac{300}{9.81} = 30.58$ kg, $k_{eq} = k_1 + k_2 = 6000$ N/m, $v_0 = 10$ cm/s *at* $t = 0$,

$x_0 = 0$ *at* $t = 0$, $c = 150$ N-s/m.

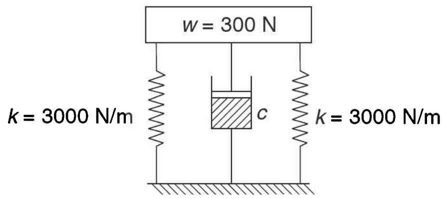


Fig. p-3.19 *Two spring-mass-damper*

To determine whether the system is overdamped, critically damped or underdamped, we determine the value of 'ξ'.

$$c_c = 2\sqrt{mk_{eq}}, = 2\sqrt{30.58 \times 6000} = 856.7 \text{ N-s/m}, \ \xi = \frac{c}{c_c} = \frac{150}{856.7} = 0.175 < 1$$

Hence, the system is underdamped; the general solution of an underdamped system is given by

$$x = e^{-\xi\omega_n t}\left\{A \sin\left(\omega_n\sqrt{1-\xi^2 t}\right) + B\cos\left(\omega_n\sqrt{1-\xi^2 t}\right)\right\}$$

When 'A' and 'B' are arbitrary constants and it is to be evaluated from the initial condition as given below.

$$x_0 = 0 \ at\ t = 0, v_0 = 0.1 \text{ m/s } at\ t = 0, 0 = 1\ [A(0) + B(1)] \ \ B = 0$$

To apply the second initial condition, differentiate the general solution w.r.t. time.

$$\frac{dx}{dt} = e^{-\xi\omega_n t}\sqrt{1-\xi^2}\left[A\cos\omega_n\left(\sqrt{1-\xi^2}\,\omega_n t\right) + A\sin\sqrt{1-\xi^2}\,\omega_n t\left(-\xi\omega_n e^{-\xi\omega_n t}\sqrt{(1-\xi)t}\right)\right]$$

$$-\,e^{-\xi\omega_n t}\sqrt{(1-\xi^2)}\,\omega_n\left[B\sin\left(\omega_n\sqrt{1-\xi^2}\,\omega_n t\right)\right] + B\cos\sqrt{1-\xi^2}\,\omega_n t(-\xi\omega_n e^{-\xi\omega_n t})$$

$$0.1 = 1\lfloor A\,\omega_n\sqrt{1-\xi^2}(1) + 0\rfloor + [0+0] - \xi\omega_n, \quad 0.1 = 1\lfloor A\,\omega_n\sqrt{1-\xi^2}(1) + 0]\rfloor,$$

$$A = \frac{0.1}{\omega_n\sqrt{1-\xi^2}} = \frac{0.1}{14\sqrt{1-(0.175)^2}} = 0.0075,$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 14\sqrt{1-(0.175)^2} = 13.78 \text{ rad/s}.$$

Substituting the values of arbitrary constant 'A' and 'B' in a general solution, we get complete solution as $x = e^{-(0.175)14t}\,[0.0075\sin(13.78\,t)]$

Substituting $t = 1$ s, $x_t = 1 = e^{-2.45}\,[0.0075 \sin(13.78)], = 5.86 \times 10^{-4}$ m,

$$x_t = 1 = 0.586 \text{ mm}.$$

EXAMPLE 3.20

An automobile can be modeled as a mass placed on 4 shock absorbers, each consisting of a spring and a damper such that each spring is equally loaded. Determine the stiffness, damping constant of each shock absorber, so that the natural frequency is 2 Hz and the system is critically damped. The mass of the vehicle is 200 kg.

Solution $m = 200$ kg. $f_n = 2$ Hz, $k = ?$, $c = ?$

Let 'k' be the spring stiffness of each spring and 'c' be the damping coefficient of each dashpot, $k_{eq} = 4k$. Similarly the equivalent damping coefficient $c_{eq} = 4c$.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}, \quad 2 = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{200}}, \quad (2)^2 = \frac{1}{4\pi^2} \times \frac{k_{eq}}{200}, \quad k_{eq} = 31582.73 \text{ N/m.}$$

$$\therefore \qquad k = \frac{k_{eq}}{4} = \frac{31582.73}{4} = 7895.68 \text{ N/m}$$

For the system

$$c = c_{eq} = 2m\omega_n \quad \omega_n = 2\pi f_n \quad \therefore\ c_{eq} = 2 \times 200 \times 2 \times \pi \times 2 = 5026.55 \text{ N-s/m.}$$

$\therefore$ the damping coefficient of each dashpot, $c = \frac{c_{eq}}{4} = 12.56.63$ N-s/m. See Fig. p-3.20(a),(b) and (c).

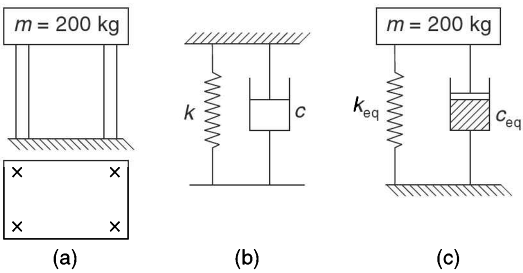


Fig. p-3.20 *Automobile with shock absorber*

REVIEW QUESTIONS

(1) Explain different types of damping with a sketch.

(2) Set up differential equation for a spring-mass-damper system and obtain the complete solution for the underdamped condition.

(3) Discuss the response of (i) underdamped, (ii) critically damped, and (iii) overdamped systems using respective response equations and curves.

Show that the mass of an overdamped system will never pass through the equilibrium position if it is given (i) an initial displacement only, (ii) an initial velocity only.

(4) Explain the difference between Viscous damping and Coulomb damping.

Discuss the various types of damping conditions.

(5) Develop an expression for logarithmic decrement 'δ' in terms of displacements 'x_1'and 'x_{n+1}'which are 'n' cycles apart on a plot of displacement versus time. **or**

(6) Define logarithmic decrement and show that logarithmic decrement 'δ' is given by

$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right)$, where x_0 = Amplitude of the zeroth oscillation,

x_n = Amplitude of the n^{th} oscillation, n = Number of oscillation

(7) For free vibrations of an underdamped spring-mass-damper system, show that logarithmic decrement $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$, where '$\xi$' is the damping ratio, and hence from there show that

$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$.

(8) Explain (i) damped natural frequency (ω_d), (ii) logarithmic decrement 'δ', (iii) critical damping coefficient (c_c) and (iv) Importance of critical damping.

(9) Set up the differential equation for a spring-mass-damper system and obtain the complete solution for the underdamped condition.

(10) Derive an expression for oscillatory motion of a spring-mass-damper system given the initial conditions as $x = x_0$ at $t = 0$ and $\dot{x} = 0$ at $t = 0$.

(11) For a viscous damper, show that energy dissipated/cycle is proportional to square of the amplitude of the harmonic motion.

(12) Define logarithmic decrement and show that logarithmic decrement 'δ' is given by $\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right)$, where x_0 = Amplitude of the zeroth oscillation,

x_n = Amplitude of the n^{th} oscillation, n = Number of oscillation.

(13) Prove from the first principles that with viscous damping the amplitudes of successive oscillations are geometrical progressions in case of free vibration.

PROBLEMS FOR PRACTICE

(1) An underdamped shock absorber is to be designed for an automobile. It is required that the initial amplitude be reduced to $1/16^{th}$ in one complete cycle. The mass of the automobile is 200 kg and damped period of vibration is 1.0 s. Determine the necessary stiffness and damping constants of the shock absorber.

Ans. δ = 2.77, ξ – 0.404, ω_n = 6.86 rad/s, c_c = 2744 N-s/m, c = 1107.75 N-s/m, k = 9411.92 N/m.

(2) Set up the differential equation of motion for the system shown in Fig. p.p-3.2 and determine (i) natural frequency of the system, (ii) critical damping coefficient, (iii) damping ratio, and (iv) damped natural frequency.

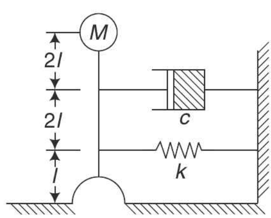


Fig. p.p-3.2

(3) A spring-mass-damper system consists of a spring of 343 N/m stiffness. The mass is 3.43 kg. The mass is displaced 20 mm beyond the equilibrium position and released. Find the equation of motion for the system if the damping coefficient of the dashpot is 13.72 N-s/m.

(4) A spring-mass-damper system is having a mass of 10 kg and a spring of such stiffness which causes a static deflection of 5 mm. The amplitude of vibration reduces to 1/4 the initial value in 10 oscillations. Determine (i) logarithmic decrement, (ii) actual damping present in the system and (iii) damped natural frequency.

(5) A machine weighs 18 kg and is supported on springs and dashpot. The total stiffness of the springs is 12 N/mm and damping is 0.2 N/mm/s. The system is initially at rest

and a velocity of 120 mm/s is imparted to the mass. Determine (i) the displacement and velocity of mass as a function of time. (ii) The displacement and velocity after 0.4 s.

Ans. Displacement $x = 4.76e^{-5.55t}$ $x \sin 25.2t$, velocity $= e^{-5.55t}[120 \cos 25.2t - 26.4 \sin 25.2t]$, Displacement = 0.0905 mm, velocity = 12.3 mm.

(6) In a single-degree-damped vibrating system, a suspended mass of 18 kg makes 15 oscillations in 13 seconds. The amplitude decreases to 0.25 of the initial value after 5 oscillations.

Determine (i) stiffness of the spring, (ii) logarithmic decrement, (iii) damping factor, and (iv) damping coefficient.

Ans. $\delta = 0.2772$, $\xi = 0.044$, $k = 947.96$ N/m, $c = 11.52$ N-s/m.

(7) A vibrating system is defined by the following parameters: $m = 3$ kg, $k = 100$ N/m, $c = 3$N-s/m.

Determine (i) damping factor, (ii) natural frequency of damped vibration, (iii) logarithmic decrement, (iv) ratio of successive amplitudes, and (v) the number of cycles after which the original amplitude is reduced to 20%.

Ans. $\xi = 0.0866$, $\omega_n = 5.77$ rad/s, $\omega_d = 5.75$ rad/s, $\delta = 0.55$, $x_1/x_2 = 1.73$, $n = 2.96$ cycles.

(8) A damped vibration record of a spring-mass dashpot system shows the following dates:

Amplitude at end of 2nd cycle = 9 mm. Amplitude at end of 3rd cycle = 6 mm. Amplitude at end of 4th cycle = 4 mm. Stiffness of the spring is 80 N/cm, weight = 20 N. Determine (i) logarithmic decrement, (ii) damping factor at unit velocity, and (iii) periodic time of vibration.

Ans. $\delta = 0.405$, $\xi = 0.0644$, $c = 16.44$ N-s/m, $\tau_d = 0.1$ s.

(9) The amplitude of a spring-mass-damper system is observed to decrease to 25% of the initial value after five consecutive cycles of motion. Determine damping coefficient 'c' of the system if the stiffness of the spring $k = 2000$ N/m and mass 'm' = 4.5 kg.

(10) The following data are given for a spring-mass-damper system. $m = 5$ kg, $k = 15000$ N/m, $c = 117$ N-s/m. Determine (i) undamped natural frequency, (ii) critical damping coefficient, (iii) damping factor, (iv) damped natural frequency vibration, and (v) logarithmic decrement.

(11) In a single-degree-damped vibrating system, a suspended mass of 20 kg makes 20 oscillations in 10 seconds. The amplitude decreases to 0.25 of the initial value after 5 seconds. Determine

(i) stiffness of the spring, and (ii) logarithmic decrement.

(12) A body of 5 kg, mass, stiffness of the spring 1960 N/m and a 1.96 N-s/m damping coefficient is a velocity of 1 m/s. Determine the ratio of amplitude reduced after 5 cycles.

Ans. $x_0/x_5 = 1.4$.

OBJECTIVE-TYPE QUESTIONS

(1) The energy dissipated per cycle depends upon the coefficient of friction in case of

(a) viscous damping
(b) Coulomb damping
(c) structural damping
(d) slip damping

(2) The damping force is constant in magnitude but opposite in direction to that of the motion of vibrating bodies in case of

(a) viscous damping
(b) Coulomb damping
(c) structural damping
(d) slip damping

(3) The governing equation of motion of free vibration of the spring-mass system with viscous damping is given by the equation

(a) $\ddot{x} + \frac{c\dot{x}}{m} + \frac{kx}{m} = F_0 \sin \omega t$

(b) $\ddot{x} ++ \frac{kx}{m} = 0$

(c) $\ddot{x} + \frac{c\dot{x}}{m} + \frac{kx}{m} = 0$

(d) $\ddot{x} ++ \frac{kx}{m} = F_0 \cos \omega t$

(4) In case of critical damping, the damping factor 'ξ' value is

(a) $\xi = 0$ (b) $\xi < 1$
(c) $\xi = 1$ (d) $\xi > 1$

(5) The general nature of damped free oscillations whose amplitude decreases with time is

(a) no damping system
(b) underdamped system
(c) critical damped system
(d) overdamped system

(6) The natural logarithmic of ratio of any two successive amplitudes on the same side of the mean line is known as

(a) damping factor
(b) critical damping coefficient
(c) logarithmic decrement
(d) overdamping system

(7) In case of underdamped system, if x_1 and x_2 are the successive amplitudes on the same side of the mean position, then logarithmic decrement is given by

(a) $\log \frac{x_2}{x_1}$ (b) $\log \frac{x_1}{x_2}$

(c) $\log_e \frac{x_2}{x_1}$ (d) $\log_e \frac{x_1}{x_2}$.

(8) In a single-degree-damped vibrating system a suspended mass of 1 kg, stiffness of the spring 100 N/mm, damping coefficient 2 N-s/m, the amplitude after 3 cycles in terms of initial amplitude is

(a) $\frac{x_1}{x_3} = 6.6$ (b) $\frac{x_1}{x_2} = 6.6$

(c) $\frac{x_2}{x_3} = 6.6$ (d) $\frac{x_0}{x_3} = 6.6$

(9) The damped natural frequency in case of underdamped system is given by the equation

(a) $\frac{\omega_d}{\omega_n} = 1 - \xi^2$

(b) $\omega_d = \omega_n\sqrt{1 - \xi^2}$

(c) $\omega_d = \omega_n\sqrt{1 - \xi}$

(d) $\omega_d = \sqrt{1 - \xi^2}$

(10) The characteristics of underdamped system of motion are

(a) amplitude increases with time
(b) amplitude decreases with time
(c) amplitude is constant with time
(d) none of the above

(11) In damped free-vibration system,

(a) the spring force vector acts in the direction opposite the displacement
(b) the damping force vector acts in the direction opposite the velocity
(c) the inertia force vector acts in the direction opposite the acceleration
(d) all the above statements are true

Answers

(1) d	(2) b	(3) c	(4) c	(5) b	(6) c
(7) d	(8) d	(9) b	(10) b	(11) d	

FORCED VIBRATION OF SINGLE-DEGREE-FREEDOM SYSTEM

4

4.1 INTRODUCTION

In free vibration, the oscillation dies down in course of time due to energy dissipation by damping. However, if there be an external source of energy, vibration can be maintained at constant amplitude or the rate of dying down may be slowed down.

Vibrations that take place under the excitation of external forces are called forced vibrations. When a system is subjected to harmonic excitation, it is forced to vibrate at the same frequency as that of excitation. When any of the excitation frequency coincides with one of the natural frequencies of a system, resonance occurs under this condition during which large amplitudes of vibrations are observed. To avoid resonance, external frequency (operating speed) may be changed or the properties of the system may be altered to change the natural frequency. In some cases, sufficient amount of damping may be provided to avoid larger amplitudes during resonance. Thus the problems of forced vibration are very important in mechanical design, e.g. internal combustion engines, air compressors, machine tools, turbines, rotating and reciprocating systems, automobiles, electrical motors, etc.

In general, the system when acted upon by external forces has a transient state but very soon it becomes steady and the system vibrates with the frequency of the external sources. Systems having forced vibrations are very common in engineering. In case of forced damped vibration there will be an additional impressed force involved in engineering practice. They are

(i) periodic or harmonic forcing function,

(ii) impulsive or shock type of forcing function, and

(iii) random type of forcing function.

The periodic functions are further classified as

(a) harmonic, and (b) nonharmonic.

Impulsive or shock type of forcing function is common and they lead to transient vibration, e.g. dropping or sudden application of weights.

Random type of forcing function occurs in earthquake, blasts or explosions, excitation, acoustics, etc.

4.2 FORCED VIBRATION DUE TO HARMONIC EXCITATION

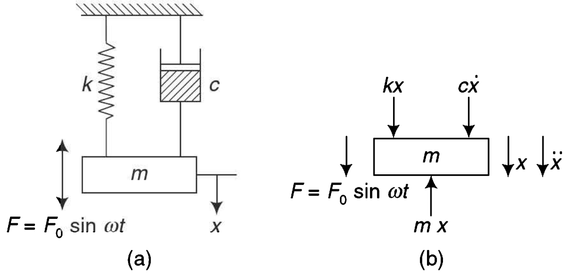


Fig. 4.1 *Spring-mass-dashpot system under forced vibration*

Let us consider a classical spring-mass-dashpot system excited by a sinusoidal forcing function $F = F_0 \sin \omega t$ as shown in Fig. 4.1(a) where 'F'$_0$ is the amplitude and 'ω' is the angular frequency. Let 'k' be the spring stiffness of the spring, 'm' be the mass of the body and 'c' be the damping coefficient. Let at any instant the system be displaced through a distance 'x' from the equilibrium position as shown in Fig. 4.1(a). The body has at the instant a velocity '$\dot{x}$' at the instant in the upward direction, i.e. the direction of positive of 'x' where the external force ($F = F_0 \sin \omega t$) is acting on the system. The forces acting are as shown in the free-body diagram Fig. 4.1(b).

The damping resistance at any instant is equal to $c\dot{x}$.

Note: In case of forced vibrations, there will be four forces acting on a system, i.e. spring force, damping force, inertia force and impressed force or external force. (See Eq. 4.4).

Now applying Newton's second law of motion to mass 'm', i.e. $\Sigma F = m\ddot{x}$

$$-kx - c\dot{x} + F = m\,\ddot{x}, \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad \text{...4.1}$$

This is a linear differential equation of motion, which is a second-order nonhomogeneous differential equation of a single-degree freedom system having free vibration with damping. The complete solution of this equation has two components, viz. complimentary function x_c (CF) and particular integral or function x_p (PI),

i.e. $$x = x_c + x_p$$

1. *Complimentary function 'x_c'* (See Sec. 3.6, Chapter 3)

This can be obtained by equating the left-hand side of Eq. 4.1 to zero. That means there is no forcing function ($F = F_0 \sin \omega t$) on the system. This is also called *transient response* because it will eventually die out.

The resulting equation is $m\ddot{x} + c\dot{x} + kx = 0$.

This equation is a linear, fundamental homogeneous second-order differential equation of motion of a single-degree-of-freedom system having free vibration with damping.

$\therefore \qquad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0.$ Here $\frac{c}{m} = \frac{c}{c_c} \cdot \frac{c_c}{m} = \xi \cdot \frac{2m\omega_n}{m} = 2\xi\omega_n, \frac{k}{m} = \omega_n^2$

where ξ = Damping ratio $\xi = \frac{c}{c_c}\cdot$ Using these values in above equation, we have

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \text{...4.2}$$

Let $\qquad x = Ae^{st}, \dot{x} = Ase^{st} = sx, \ \ddot{x} = A\,s^2e^{st} = s^2x$

Using these values in Eq. 4.2,

$$s^2x + 2\xi\omega_n s\,x + \omega_n x = 0, (s^2 + 2\,\xi\omega_n s + \omega_n^2)x = 0, \text{ as } x \neq 0$$

$$\therefore \qquad s^2x + 2\xi\omega_n + \omega_n^2 = 0$$

This is a quadratic equation of 's' and there will be two roots for 's'

$$\therefore \qquad s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

$$\therefore \qquad s_{1,2} = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2}$$

$$\therefore \qquad s_{1,2} = \omega_n\left(-\xi \pm \sqrt{\xi^2 - 1}\right) \qquad \text{...4.3}$$

When $\xi < 1$, (underdamping) in Eq. 4.3, the solution is

$$x = A_1e^{s_1t} + A_2e^{s_2t} \text{ or } x = e^{-\xi\omega_n t}(c_1 \cos\omega_d t + c_2 \sin\omega_d t)$$

$$s_{1,2} = -\xi\omega_n \pm i\,\omega_d \text{ where } \omega_d = \omega_n\sqrt{1 - \xi^2} \text{ or } x = Ye^{-\xi\omega_n t}\sin(\omega_d t + \psi)$$

where $\qquad Y = \sqrt{c_1^2 + c_2^2} \qquad \psi = \tan^{-1}\left(\frac{c_2}{c_1}\right)$

when $\xi = 1$ (critical damping), the solution is $x = (c_1 + c_2t)e^{-\omega_n t}$,

When $\xi > 1$, (overdamping), the solution is

$$x = A_1e^{s_1t} + A_2e^{s_2t}, s_{1,2} = \omega_n(-\xi \pm \sqrt{\xi^2 - 1}).$$

2. Particular integral or solution 'x_p' (steady–state component) Let $x = X\sin(\omega t - \phi)$, be the particular solution (because the forcing function is a sinusoidal, the particular integral should also be sinusoidal) where 'X' is the constant amplitude of vibration of the system and 'ϕ' is the angle (phase difference) by which the displacement vector lags the force vector and 'ω' is the angular frequency.

$$x = X\sin(\omega t - \phi). \text{ Differentiating with respect to time '}t\text{' twice,}$$

$$\dot{x} = \omega X\cos(\omega t - \phi) = \omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right)$$

$$\ddot{x} = -\omega^2X\sin(\omega t - \phi)$$

Substituting these values in Eq. 4.1, we get

$$m[-\omega^2X\sin(\omega t - \phi)] + c\left[\omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right)\right] + kX\sin(\omega t - \phi) = F_0\sin\omega t$$

$$mX\omega^2\sin(\omega t - \phi + \pi)] + cX\omega\sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX\sin(\omega t - \phi) - F_0\sin\omega t = 0 \qquad \text{...4.4}$$

Inertia force + Damping force + Spring force – Impressed force = 0

From the above equation we absorbed that

- The term $mX\,\omega^2 \sin(\omega t - \phi + \pi)$ is the inertia force
- The term $cX\,\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right)$ is the damping force
- The term $kX \sin(\omega t - \phi)$ is the spring force
- The term $F_0 \sin \omega t$ is harmonic excitation force (impressed Force)

These forces can be vectorially represented as follows:

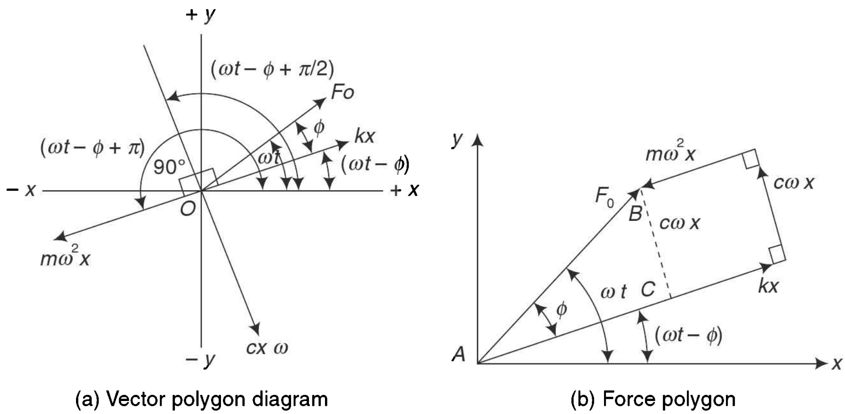


Fig. 4.2 *Vector representation of forces on the system having forced vibration*

From vector diagram Fig. 4.2(a), we can observe that

(a) Spring force is always opposite to the displacement

(b) Damping force lags the displacement by 90°

(c) Inertia force is out of phase with the displacement (180°)

In Fig. 4.2(b), force polygon from the right–angled triangle ABC

$$AB^2 = BC^2 + AC^2, \text{ where } AB = F_0, \quad BC = c\,\omega X, AC = (kX - m\omega^2 X)$$

$$F_0^2 = (c\,X\,\omega)^2 + (kX - mX\omega^2)^2, \quad F_0^2 = X^2[(k - m\omega^2)^2 + (c\omega)^2]$$

$$X^2 = \frac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}$$

The steady state amplitude $X = \sqrt{\dfrac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}}$

$$X = \frac{F_0}{\sqrt{k^2\left(1 - \frac{m}{k}\omega^2\right)^2 + k^2\left(\frac{c}{k}\omega\right)^2}} = \frac{F_0}{k\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}$$

Dividing the right–hand side numerator and denominator by 'k', we get

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}} \quad \text{... 4.5}$$

The static deflection 'X_{st}' due to the harmonic force is given by

$$X_{st} = \frac{F_0}{k}, \text{ and } \frac{m}{k} = \frac{1}{\omega_n^2}, \frac{c}{k} = \frac{c}{c_c} \times \frac{c_c}{k} = \xi \times \frac{2\sqrt{mk}}{k} = \xi \times \frac{2\sqrt{m}}{\sqrt{k}} = 2\xi\sqrt{\frac{m}{k}} = \frac{2\xi}{\omega_n}, \frac{c}{k} = \frac{2\xi}{\omega_n}$$

Using these values in Eq. 4.5,

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = 0 \quad \text{... (Eq. 4.2)}$$

$$X = \frac{X_{st}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \quad \frac{X}{X_{st}} = \frac{1}{\sqrt{(1 - r^2) + (2\xi r)^2}}$$

Let $r = \frac{\omega}{\omega_n}$ where 'r' is the frequency ratio.

Once again from force diagram Fig. 4.2(b) in the right–angled triangle ABC,

$$\tan\phi = \frac{BC}{AC} = \frac{cX\omega}{kX - mX\omega^2}$$

$$\tan\phi = \frac{c\omega}{k - m\omega^2} = \frac{c\omega}{k\left(1 - \frac{m}{k\omega^2}\right)}$$

$$= \frac{\left[\frac{c}{k}\right]\omega}{1 - \frac{m}{k}\omega^2,} = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$\therefore \quad \tan\phi = \frac{2\xi r}{(1 - r^2)} \text{ or } \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right] \quad \text{... 4.6}$$

Considering the underdamped case ($\xi < 1$), the complete solution is given by

$$x = x_c + x_p$$

$$x = X_1 e^{-\xi\omega_n t} \sin(\omega_d t + \psi) + X \sin(\omega t - \phi)$$

where $$X = \frac{X_{st}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}, \quad \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right]$$

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \quad \text{where } X_{st} = \frac{F_0}{k}.$$

3. *Procedure to draw vector diagram and force polygon*

$$mX\omega^2 \sin(\omega t - \phi + \pi) + cX\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX \sin(\omega t - \phi) - F_0 \sin\omega t = 0$$
.... (See Eq. 4.4)

(a) Vector diagram Draw the ordinate '–xox' and '–yoy' as shown in Fig. 4.2(a). From point 'O', draw an inclined line 'F_0' at an angle of 'sin ωt' to a suitable scale indicating impressed force ($F_0 \sin\omega t = 0$) from reference line 'Ox'. Similarly, from the same point 'O' draw an inclined line kX at an angle of sin $(\omega t - \phi)$ with suitable scale indicating spring force [kX sin ($\omega t - \phi$)]. On an inclined line [sin ($\omega t - \phi$)], measure an angle of 90° ((π/2) in the clockwise direction. With a suitable scale draw a line equal to ($c\omega X$) indicating damping force [$cX\,\omega$ sin ($\omega t - \phi + 90°$)] as shown in Fig. 4.2(a). A spring force (kX) is always opposite to the displacement, i.e. $m\omega^2 X$ at an angle of ($\omega t - \phi + \pi$) from a suitable scale indicating inertia force [$mX\,\omega^2$ sin ($\omega t - \phi + \pi$)]. See Eq. 4.4.

(b) Force polygon Draw the ordinates 'Ax' and 'Ay' as shown in Fig. 4.2(b). From point 'A', draw a parallel line 'AB' parallel to 'OF_0' in Fig. 4.2(a) with suitable scale. Again from point 'A', draw a parallel line 'AD' in Fig. 4.2(a) parallel to $O - kx$ with a suitable scale. From point 'D', draw a parallel line 'DE' parallel to '$O - CX\omega$' in Fig. 4.2(a) with a suitable scale. From point 'E', draw a parallel line 'EF_0' in Fig. 4.2(a) parallel to the '$O - m\omega^2 X$', with a suitable scale as shown in Fig. 4.2(b).

4.1 MAGNIFICATION FACTOR (MF)

Magnification factor (MF) is defined as the ratio of steady–state amplitude to the zero frequency deflection (static deflection due to harmonic force),

i.e. Magnification factor, $\text{MF} = \frac{X}{X_{st}}, \frac{X}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ... 4.7

The phase lag, $\phi = \tan^{-1}\left(\frac{2\xi r}{1-r}\right)$ ... 4.8

where $r = \frac{\omega}{\omega_n}$

Dimensionless plots of magnification factor (MF) versus frequency ratio 'r' and phase lag 'ϕ' versus frequency ratio 'r' for different values of damping factor 'ξ'.

Magnification factor and phase lag 'ϕ' are the functions of 'ξ' and frequency ratio $\frac{\omega}{\omega_n} = r$.

Case (i) When $r = 0$, (at zero frequency) in Eq. 4.7, Eq. 4.8 becomes

$$\frac{X}{X_{st}} = \frac{1}{(\sqrt{1-0})^2 + 0} = 1 \text{ or } X = X_{st}$$

which is the definition of zero frequency deflection and

$$\phi = \tan^{-1}(0),\ \phi = 0.$$

∴ the amplitude ratio or magnification factor is independent of the damping ratio 'ξ'.

Case (ii) When $r = 1$ (resonance, i.e. $\omega = \omega_n$) in Eq. 4.7, Eq. 4.8 becomes

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{0 + (2\xi)^2}} = \frac{1}{2\xi} \text{ and } \phi = \tan^{-1}\left(\frac{2\xi r}{-r}\right) = \tan^{-1}(\infty) = 90°.$$

The amplitude ratio or magnification factor depends on damping ratio 'ξ' at resonance. As the value of 'ξ' decreases, magnification factor increases and the converse is true, but at $\xi = 0$, i.e. for undamped system, the value of magnification factor $= \infty$.

Case (iii) When $r >> 1$, and $r^2 >>> 1$

$$\therefore \qquad \frac{1}{r^2} <<< 1 \text{ or } \frac{1}{r^2} \approx 0 \text{ and } \frac{1}{r} \approx 0$$

$\therefore$ equations 4.7 and 4.8 become

$$\therefore \qquad \frac{X}{X_{st}} = \frac{1}{\sqrt{r^2\left(\frac{1}{r^2} - 1\right) + (2\xi)^2}} \text{ and } \phi = \tan^{1}\left[\frac{2\xi r}{r^2\left(\frac{1}{r^2} - 1\right)}\right]$$

$$\therefore \qquad \frac{X}{X_{st}} = \frac{1}{r\sqrt{\left(\frac{1}{r^2} - 1\right) + (2\xi)^2}} \text{ and } \phi = \tan^{-1}\left[\frac{2\xi r}{-r^2}\right],$$

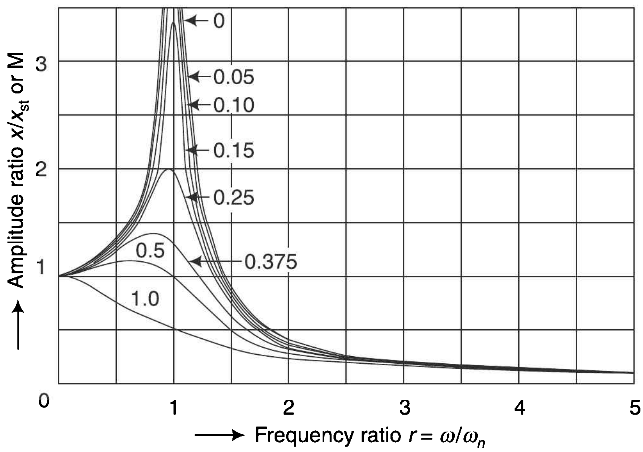
$$\text{MF} = \frac{X}{X_{st}} \approx 0 \text{ and } \phi = \tan^{-1}[-0] = 180°$$

The dimensionless plots of **magnification factor** (MF) versus **frequency ratio(r)** and **phase lag (ϕ)** versus **frequency ratio(r)** for different values of damping factor are shown in Fig. 4.3(a). These curves reveal a lot of interesting and useful information regarding the behaviour of the system to sinusoidal excitation.

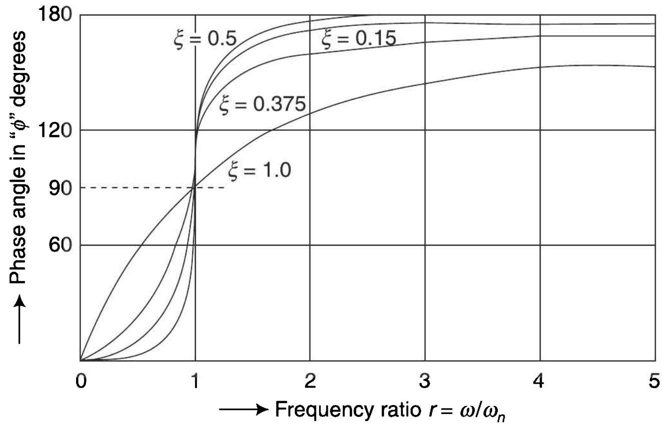
Curves of Fig. 4.3(a) are also known as *frequency-response curves,* since they give the response of the system to various frequencies. It is seen from these curves that the response of a particular system at any particular frequency is lower for higher value of damping and lies below those for lower values of damping. At zero frequency the magnification factor is unity and is independent of the damping, i.e. $X = X_{st}$, which itself is the definition of zero frequency deflection. At very high frequency, the magnification factor tends to zero or the amplitude of vibration becomes very small. At resonance ($\omega = \omega_n$), the amplitude of vibration becomes excessive for small damping and decreases with increase in damping. For zero damping at resonance, the amplitude is infinite theoretically. Practically, however the system may go into destruction much before that or the amplitude may be limited because of other factors.

The phase angle also varies from zero at low frequencies to 180° at very high frequencies. It is 90° at resonance and is independent of damping. Over a small frequency range containing the resonance point, the variation of phase angle is more abrupt for lower values of damping than for higher values. The more abrupt the change in phase angle about resonance, sharper is the peak in the frequency response curve. For zero damping, the phase lag suddenly changes from zero to 180° at resonance. The corresponding zero damping frequency-response curve is also infinitely sharp at resonance.

Let us now study the phenomenon of Fig. 4.3(b) by means of the vector diagram and gain some more insight into what is happening in the system. With reference vector diagram at very low frequency (ω is very small), the inertia term '$m\omega^2 x$' becomes



(a) Magnification factor versus frequency ratio for different amount of damping.



(b) Frequency ratio versus phase angle

Fig. 4.3

negligibly small and damping term '$c\omega X$' is also small. This gives rise to small value of 'ϕ' as shown in Fig. 4.3(b).

The impressed force 'F_0' is almost equal and opposite to the spring force 'kx' under these conditions. Thus, for very low frequencies, the phase angle tends to zero and the impressed force wholly balances the spring force. With increase in the frequency, the damping force vector ($c\omega X$) grows larger. Angle 'ϕ' has also to increase so that component of 'F_0' perpendicular to x-direction may balance the increasing damping force. The inertia force vector grows much more rapidly with increase in frequency because of the factor 'ω^2' in its expression. If we continue to increase the frequency, a time comes when the spring force and inertia force vectors are equal and opposite as shown in the figure and this condition is $kx = m\omega^2 x$ or $\omega = \sqrt{\frac{k}{m}} = \omega_n$. This is the resonance condition of the system and the vector diagram becomes rectangular. The impressed force completely balances the damping force and $\phi = 90°$.

$\therefore$ $c\omega X = F_0$ or amplitude at resonance is $Xr = \dfrac{F_0}{c\omega} = \dfrac{X_{st}}{2\xi\left(\dfrac{\omega}{\omega_n}\right)} = \dfrac{X_{st}}{2\xi}$, since at resonance $\omega = \omega_n$.

$\therefore$ $\dfrac{X_r}{X_{st}} = \dfrac{1}{2\xi}$ at very high frequency, the inertia vector becomes very large and the damping force, the spring force vectors are negligibly small. Angle 'ϕ' tends to 180° and the impressed force is wholly utilised to balance the inertia force as shown in Fig. 4.3(b).

EXAMPLE 4.1

Show that the peak amplitude takes place at a frequency ratio

(r)

$$= \frac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2}$$

Solution The amplitude ratio is given by the equation

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}, \text{ where } r = \frac{\omega}{\omega_n}$$

For 'X' to be maximum 'X_{st}' should be minimum, i.e. denominator of right-hand side should be minimum. For minimum $\dfrac{d}{dr}$ [denominator] = 0,

$\dfrac{d}{dr}\left[\sqrt{(1 - r^2)^2 + (2\xi r)^2}\right] = 0.$

$$2(1 - r^2)\,(0 - 2r) + (2\xi r)\,2\xi = 0,\ (r^2 - 1)\,r + 2\xi^2 r = 0$$

$$r^2 - 1 + 2\xi^2 = 0,\ r^2 = 1 - 2\xi^2,\ r = \sqrt{1 - \xi^2}$$

$\therefore$ frequency ratio $r = \dfrac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2}$

$\therefore$ the peak amplitude will occr at this frequency ratio.

EXAMPLE 4.2

A spring-mass system is excited by a force F_0 sin ωt. At resonance, the amplitude of vibration was found to be 1.2 cm while at frequency 0.8 times the resonant frequency, amplitude was measured to be 0.8 cm. Estimate the damping ratio of the system.

Solution Given condition: At resonance,

$$r = 1,\ \frac{\omega}{\omega_n} = 1, \text{ or } \omega = \omega_n\ X_1 = 1.2 \text{ cm} = 0.012 \text{ m}$$

At $\omega = 0.8\ \omega_n$,

$X_2 = 0.8$ cm = 0.008 m. The magnification factor is given by the equation

$$M = \frac{X}{X_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...(a)}$$

where $r = \dfrac{\omega}{\omega_n}$ $\therefore$ at $r = 1$, $x = 1.2$

Substituting these values in Eq. (a), we get

$$\frac{1.2}{X_{st}} = \frac{1}{\sqrt{(1-1)^2 + (2\xi)^2}} = \frac{1}{2\xi} \qquad \text{... (b)}$$

At $\quad r = 0.8, x = 0.8$

Substituting these values in Eq. (b) we get

$$\frac{0.8}{X_{st}} = \frac{1}{\sqrt{(1-0.8^2)^2 + (2\xi \times 0.8)^2}}, \quad \frac{0.8}{X_{st}} = \frac{1}{\sqrt{0.13 + 2.56\xi^2}} \qquad \text{... (c)}$$

Dividing Eq. (b) by Eq. (c), we have

$$\frac{1.2}{0.8} = \frac{\sqrt{0.13 + 2.56\xi^2}}{\sqrt{(2\xi)^2}} \quad (1.5)^2 = \frac{0.13 + 2.56\xi^2}{4\xi^2}, \quad 9\xi^2 = 0.13 + 2.56\xi^2$$

$$6.44\xi^2 = 0.13, \quad \xi^2 = 0.02$$

$\therefore$ damping ratio $\xi = 0.142. < 1$, (underdamping).

EXAMPLE 4.3

A machine part weighing 5 kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic force of 36 N results in 15 mm resonant amplitude with a period of 0.32 s.

Solution $\quad m = 5$ kg, $F_0 = 36$ N, $X = 15$ mm $= 0.015$ m. $\tau_p = 0.32$ s.

At resonance, $\omega = \omega_n$, i.e. $r = 1$ (given condition)

The amplitude ratio is given by $\dfrac{X}{X_{st}} = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ... (a)

where $\quad r = \dfrac{\omega}{\omega_n}$ and $X_{st} = \dfrac{F_0}{k}$

$\therefore \quad$ at $r = 1$, in Eq. (a)

we have $\quad \dfrac{X}{X_{st}} = \dfrac{1}{2\xi}, \quad \xi = \dfrac{X_{st}}{2X} = \dfrac{F_0}{2kX}, \quad \xi = \dfrac{36}{2.k.0.015}$ ... (b)

Given $\quad \tau_p = 0.32 = \dfrac{1}{f_n} = \dfrac{2\pi}{\omega_n}$, since $\omega_n = 2\pi/\tau$, $\omega_n = \dfrac{2\pi}{0.32}, \quad \sqrt{\dfrac{k}{m}} = \dfrac{2\pi}{0.32}$.

$$\therefore \quad \frac{k}{m} = \left(\frac{2\pi}{0.32}\right)^2, k = 5\left(\frac{2\pi}{0.32}\right)^2, k = 1927.66 \text{ N/m}$$

Using these values in Eq. (b), we get damping ratio

$$\xi = \frac{36}{2 \times 1927.66 \times 0.015}, \xi = 0.62.$$

But $\xi = \dfrac{c}{c_c} = \dfrac{c}{2\sqrt{mk}} \quad \therefore c = \xi \times 2\sqrt{mk}.$

Damping coefficient,

$$c = 2 \times 0.62 \times \sqrt{5 \times 1927.66}, \quad c = 121.74 \text{ N-s/m}.$$

EXAMPLE 4.4

A mass of 10 kg is suspended by a spring having a stiffness of 10000 N/m. The viscous damping causes the amplitude to decrease to one–tenth of the initial value in four complete oscillations. If a periodic force of 150cos50*t* is applied to the mass in vertical direction, find the amplitude of the forced vibrations. What is its value at resonance?

Solution m = 10 kg, k = 10000 N/m, F = 150 cos 50t, F_0 = 150 N, ω = 50 rad/s

$x_4 = \dfrac{1}{10} x_0, \dfrac{x_0}{x_4} = 10$, amplitude ratio is given by

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad \text{...(a)}$$

where $r = \dfrac{\omega}{\omega_n}$, for the given system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{10}} = 31.62 \text{ rad/s}$$

$\therefore$ frequency ratio $r = \dfrac{\omega}{\omega_n} = \dfrac{50}{31.62}$, $r = 1.58$.

Logarithmic decrement (δ) is given by $\delta = \dfrac{1}{n} l_n \left(\dfrac{x_0}{x_n}\right)$.

For four cycles, $n = 4 \quad \therefore \delta = \dfrac{1}{4} l_n \left(\dfrac{x_0}{x_4}\right) = \dfrac{1}{4} l_n 10 \quad \delta = 0.576.$

But $\delta = \dfrac{2\pi\xi}{\sqrt{1-\xi^2}}$, or $\xi = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \dfrac{0.576}{\sqrt{4\pi^2 + (0.576)^2}}$

Damping ratio $\xi = 0.091 < 1$

For zero frequency deflection, $X_{st} = \dfrac{F_0}{k}$, $X_{st} = \dfrac{150}{10000}$, $X_{st} = 0.015$ m, $r = 1.58$.

Using all these values in Eq. (a), we get

$$X = \frac{0.015}{\sqrt{[1-(1.58)^2]^2 + (2 \times 0.091 \times 1.58)^2}}, X = 0.0098 \text{ m} = 9.8 \text{ mm}.$$

Amplitude of the forced vibration, X = 9.8 mm

At resonance, $\omega = \omega_n$, i.e. $r = 1$.

Using these value in Eq. (a), we get

$$\frac{X}{X_{st}} = \frac{1}{2\xi}, \quad X = \frac{X_{st}}{2\xi}, X = \frac{0.015}{2 \times 0.091} = 0.08242 \text{ m}, X = 82.42 \text{ mm}.$$

EXAMPLE 4.5

A periodic torque having a maximum value of 0.588 N-m at a frequency corresponding to 4 rad/s is impressed upon a flywheel suspended from a wire. The wheel has a moment of inertia of 0.12 kg-m^2 and the wire has a stiffness of 1.176 N-m/ rad. The viscous dashpot applies a damping couple of 0.39 N-m at an angular velocity of 1 rad/s. Calculate

(i) Maximum angular displacement from the rest position

(ii) The maximum couple applied to the dashpot

(iii) The angle by which the angular displacement lags the torque

Solution $T_0 = 0.588$ N-m, $\omega = 4$ rad/s, $J = 0.12$ kg-m^2, $kt = 1.176$ N-m/ rad,

$T = T_0 \sin \omega t$, $T = 0.588 \sin \omega t$

Damping torque $T_d = 0.382$ N-m at 1 rad/s, $F_d = c\,\dot{x}$ $c = F_d/\dot{x}$, $c_t = T_d/\dot{x}$

$$c_t = T_d/\theta = 0.392/1 = 0.392 \text{ N-m-s/ rad.}$$

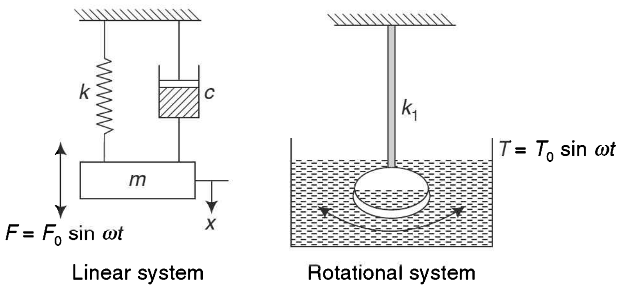


Fig. p-4.5 *Flywheel system*

The amplitude is given by equation for linear system $X = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$,

For torsional system $\theta = \dfrac{T_0}{\sqrt{(k_t - J\omega^2)^2 + (c_t\omega)^2}}$

$$\theta = \frac{0.588}{\sqrt{(1.176 - 0.12 \times 4^2)^2 + (0.392 \times 4)^2}}, \ \theta = 0.34 \text{ rad}$$

The maximum damping couple of the dashpot = $c\omega\theta$

= 0.382 × 4 × 0.34 =

0.53 N-m.

Phase angle at resonance is given by $\phi = \tan^{-1}\left(\dfrac{c_t\omega}{k_t - J\omega^2}\right)$

$$\phi = \tan^{-1}\frac{0.392 \times 4}{(1.76 - 0.12 \times 4^2)}, \quad \text{or} \quad \phi = 115.4°.$$

EXAMPLE 4.6

The damped natural frequency of a system as obtained from a free-vibration test is 10 Hz. During the forced vibration test with constant vibrating force on the same system, the maximum amplitude is found to be at 9.5 Hz. Find the damping factor for the system and the natural frequency of the system.

Solution $f_d = 10$ Hz, $f = 9.5$ Hz, $\omega_d = 2\pi f_d$

But $\omega_d = \omega_n\sqrt{1 - \xi^2}$

$$\omega_n = \frac{2\pi \times 10}{\sqrt{1 - \xi^2}} \quad ...(a) \quad f = \omega/2\pi \quad \therefore \omega = 2\pi \times 9.5 \quad ...(b)$$

Dividing Eq. (b) by Eq. (a), $\frac{\omega}{\omega_n} = \frac{2\pi \times 9.5}{2\pi \times 10} \times \sqrt{1 - \xi^2}, \frac{\omega}{\omega_n} = 0.95\sqrt{1 - \xi^2}$ ...(c)

The amplitude ratio is given by $\frac{X}{X_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$ ...(d)

Amplitude 'X' will be maximum when 'X_{st}' is minimum or denominator of RHS is minimum.

For maximum $\frac{d}{dr} = \left[\sqrt{(1 - r^2)^2 + (2\xi r)^2}\right] = 0, \quad 2(1 - r^2)(-2r) + 2(2\xi r)(2\xi) = 0$

$$(r^2 - 1) + 2\xi^2 = 0, \quad r^2 = 1 - 2\xi^2, r = \sqrt{1 - 2\xi^2}, \quad \text{or } \frac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2} \quad ...(e)$$

From Eq. (c) and Eq. (e), the LHS are equal; therefore RHS are

$$0.95\sqrt{1 - \xi^2} = \sqrt{1 - 2\xi^2}, 0.903\,(1 - \xi^2) = 1 - 2\xi^2, 0.903 - 0.903\xi^2 - 1 + 2\xi^2 = 0$$

$$1.098\xi^2 = 0.098, \quad \xi^2 = \frac{0.098}{1.098}$$

Damping factor $\xi = 0.30 < 1$, (underdamping).

From Eq. (a), the natural frequency,

$$\omega_n = \frac{2\pi \times 10}{\sqrt{1 - (0.3)}}, \omega_n = 65.87 \text{ rad/s}, \quad f_n = \frac{\omega_n}{2\pi} = \frac{65.87}{2\pi}, \quad f_n = 10.48 \text{ Hz.}$$

EXAMPLE 4.7

A weight of 60 N suspended by a spring of stiffness 1.2 k N/m is forced to vibrate by a harmonic force of 10 N. Assuming viscous damping of 0.086 k N-s/m, Determine

(i) The resonant frequency

(ii) Amplitude at resonance

(iii) Phase angle at resonance

(iv) Frequency corresponding to peak amplitude

(v) Peak amplitude

Solution $w = 60$ N, $m = 60/9.81$ kg, $k = 1200$ N/m, $c = 86$ N-s/m, $F_0 = 10$ N.

(i) At resonance, $\omega = \omega_n$,

$$\therefore \text{resonance frequency} = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200 \times 9.81}{60}} \quad \therefore \omega_n = 14.01 \text{ rad/s.}$$

(ii) Amplitude ratio is given by $\frac{X}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ...(a)

At resonance $r = 1$ or $\omega = \omega_n$ $\therefore \frac{X}{X_{st}} = \frac{1}{2\xi}$

To find damping ratio 'ξ'

Damping ratio $\xi = \frac{c}{c_c}, \frac{c}{2mw_n}, \xi = \frac{86}{2 \times \frac{60}{9.81} \times 14.01}, \xi = 0.5$

$$\frac{X}{X_{st}} = \frac{1}{2 \times 0.5} = 1, X = X_{st}, \text{ where } X_{st} = \frac{F_0}{k} = \frac{10}{1200} = 8.33 \times 10^{-3}\text{mm.}$$

$\therefore$ amplitude at resonance, $X = 8.33 \times 10^{-3}$ m, $X = 8.33$ mm.

(iii) The phase angle is given by $\phi = \tan^{-1}\left[\frac{2\xi r}{1-r^2}\right]$

At resonance $r = 1$.

$\therefore$ $\phi = \tan^{-1} \infty, \phi = 90°$.

(iv) Peak amplitude will occur at $r = \sqrt{1-2\xi^2}$ (given)

$$\frac{\omega_p}{\omega_n} = \sqrt{1-2\xi^2}, \omega_p = \omega_n\sqrt{1-2\xi^2}, 14.01\sqrt{1-2 \times 0.5^2}.$$

Frequency at peak amplitude, $\omega_p = 9.91$ rad/s.

(v) To find peak amplitude, $r = \sqrt{1-2\xi^2}, r = \sqrt{1-2 \times 0.5^2}, \quad r = 0.71$

Substituting these values in Eq. (a), we get

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{(1-0.71^2)^2 + (2 \times 0.5 \times 0.71)^2}}, X = \frac{8.33 \times 10^{-3}}{\sqrt{(1-0.71^2)^2 + (2 \times 0.5 \times 0.71)^2}},$$

$$X = 9.62 \times 10^{-3} \text{ m}, X = 9.62 \text{ mm}.$$

EXAMPLE 4.8

A mass attached to a spring of 580 N/m stiffness has a viscous damping device. When the mass was displaced and released, the period of vibrations was found to be 1.8 s and the ratio of consutive amplitudes was 4.2:1. Determine the amplitude and phase angle of vibrations when a force $F = 20 \cos 3t$ acts on the system.

Solution

$$k = 580 \text{ N/m}, t_p = 1.8\text{s}, \frac{x_0}{x_i} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = 4.2, \quad F = 20 \cos 3t, F_0 = 20 \text{ and } \omega = 3.$$

$$\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} = \frac{2\pi}{t_p\sqrt{1-\xi^2}} = \frac{2\pi}{1.8\sqrt{1-0.2236^2}}, \delta = \ln\left(\frac{x_0}{x_1}\right) = \ln(42), \delta = 1.435.$$

$\delta = \dfrac{2\pi\xi}{\sqrt{1-\xi^2}}$, $(1.435)^2 = \dfrac{4\pi\xi^2}{1-\xi^2}$, $\xi = 0.2236$, $\omega_n = 3.58$ rad/s, $r = \dfrac{\omega}{\omega_n} = \dfrac{3}{3.58} = 0.84$

The amplitude of vibration is given by $\dfrac{X}{X_{st}} = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{(1-3.58^2)^2 + (2 \times 2236 \times 3.58)^2}}, X_{st} = F_0/k = 20/580$$

$\therefore X = 0.072$ m = 7.2 mm

Phase angle is given by $\phi = \tan^{-1}\left(\dfrac{2\xi r}{1-r^2}\right) = \tan^{-1}\left(\dfrac{2 \times 0.2236 \times 3.58}{1-3.58^2}\right)$, $\phi = 51.53°$

4.4 ROTATING AND RECIPROCATING UNBALANCE

Unbalance in reciprocating machinery is a common source of vibration. Consider a machine of mass '*m*' mounted on a foundation of stiffness '*k*' and damping coefficient '*c*' as shown in Fig. 4.4(a) The unbalance is represented by the reciprocating mass '*m*' having cranked small rotation '*e*' and connecting rod length '*l*'. Let '*ω*' be the angular velocity of the crank. Let '*x*' be the displacement of non reciprocating mass '*M-m*' at any instant of time '*t*'. The displacement of reciprocating mass '*m*' from static equilibrium position is given by $x + e \sin \omega + \dfrac{e}{l} \sin 2\omega t + \ldots$

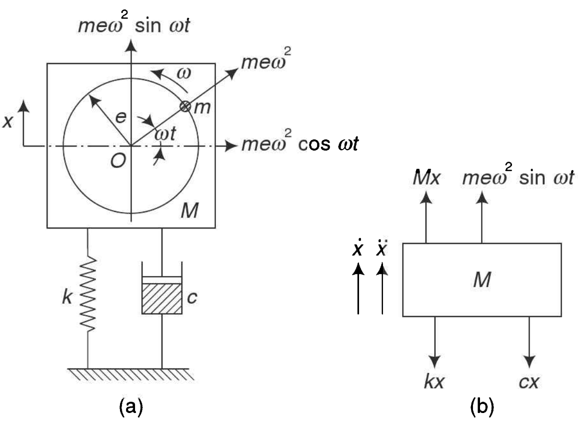


Fig. 4.4 *Rotating unbalance*

Applying D'Alembert's principle or Newton's second law of motion to mass '*M*'

$$\Sigma F = M\ddot{x} \quad \therefore c\dot{x} + kx - me\omega^2 \sin \omega t = -M\ddot{x} \quad M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t \quad \ldots 4.9$$

or let $me\omega^2 = F_0$

$$\therefore \quad M\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \quad \ldots 4.10$$

This is a second-order nonhomogeneous differential equation of motion, whose solution is given by

$$x = x_c + x_p$$

where x_c = Complementary function (transient response)

x_p = Particular integral (steady-state response already discussed in Article 4.2, case 'ii')

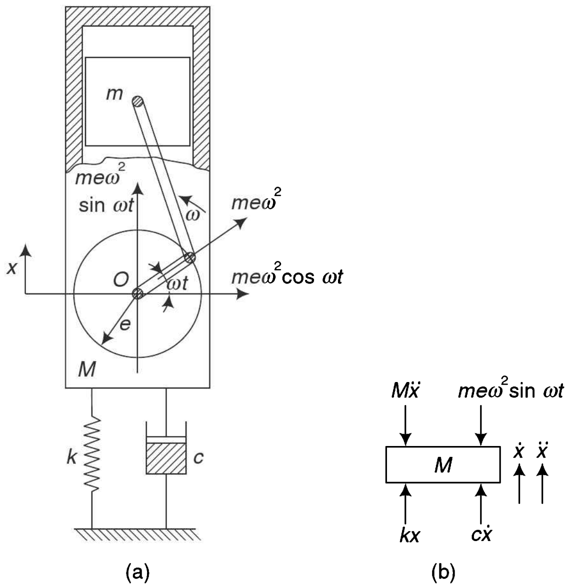


Fig. 4.5 *Reciprocating unbalance*

Considering the steady-state response or to find the particular integral 'x_p',

let $\qquad x_p = x = X\sin(\omega t - \phi)$.

where 'X' is the amplitude of vibration of the system, 'ϕ' is the angle by which the displacement vector lags the force vector, and 'ω' is the angular frequency in rad/s.

$$\dot{x} = \omega X\cos(\omega t - \phi) = \omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right) \ddot{x} = -\omega^2 X\sin(\omega t - \phi).$$

Substituting these value in Eq. (4.9),

$$M[-\omega^2 X\sin(\omega t - \phi)] + c\left[\omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right)\right] + kX\sin(\omega t - \phi) = F_0\sin\omega t$$

Rearranging the above term we have,

$$M\dot{\omega}^2 X\sin(\omega t - \phi + \pi) + cX\omega\sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX\sin(\omega t - \phi) - F_0\sin\omega t = 0$$

From the above equation, we absorbed that

Inertia force + Damping force + Spring force – Impressed force = 0

The term $mX\omega^2\sin(\omega t - \phi + \pi)$ is the inertia force.

The term $cX\omega\sin(\omega t - \phi + \pi/2)$ is the damping force.

The term $kX\sin(\omega t - \phi)$ is spring force.

The term $F_0\sin\omega t$ is harmonic excitation force (impressed force).

These forces can be vectorially represented as follows.

From the force polygon, $\overrightarrow{OB} = F_0$, $\overrightarrow{AB} = c\omega X$, $\overrightarrow{OA} = kX - M\omega^2 X$

In force polygon, from right-angled triangle OAB,

$$OB^2 = OA^2 + AB^2 \quad \therefore F_0{}^2 = (kX - M\omega^2 X)^2 + (c\omega X)^2$$

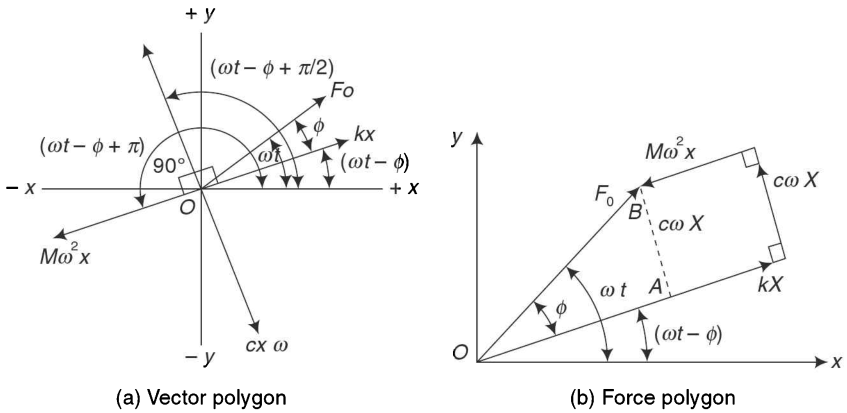


Fig. 4.6 *Force and vector polygon*

$$F_0{}^2 = X^2 [(k - M\omega^2)^2 + (c\omega)^2]\ X^2 = \frac{F_0^2}{(k - M\omega^2)^2 + (c\omega)^2},\ X = \frac{F_0}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}},$$

But $F_0 = me\omega^2$.

$$\therefore \quad X = \frac{me\omega^2}{\sqrt{k^2\left[\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2\right]}},\ X = \frac{me\omega^2}{k\sqrt{\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}$$

Dividing the right-hand side by both numerator and denominator by 'k', we get

$$X = \frac{\frac{me\omega^2}{k}}{\sqrt{\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}$$

We have $\frac{me}{k}$

Dividing and multiplying by 'M', we have $\frac{m}{M} \cdot \frac{M}{k} \cdot e$

But $\frac{k}{M} = \omega_n^2$ or $\frac{M}{k} = \frac{1}{\omega_n^2}$. Divide and multiply by c_c

$$\therefore \frac{me}{k} = \frac{m}{M} \cdot \frac{e}{\omega_n^2}\ \frac{me}{k} = \frac{m}{M} \cdot \frac{e}{\omega_n^2} \times \frac{c_c}{c_c}\ \frac{M}{k} = \frac{1}{\omega_n^2} \text{ and } \frac{c}{k} = \frac{c}{c_c} \cdot \frac{2M\omega_n}{k} = \xi . \frac{2\omega_n}{\omega_n^2} = \frac{2\xi}{\omega_n}$$

$$\therefore X = \frac{\frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}. \quad \text{Let } r = \frac{\omega}{\omega_n} \quad \therefore \frac{MX}{me} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad ...4.11$$

And $\tan\phi = \dfrac{AB}{OA} = \dfrac{cx\omega}{kx - Mx\omega^2} = \dfrac{c\omega}{k - M\omega^2} = \dfrac{\dfrac{c}{k\omega}}{1 - \dfrac{M}{k}\omega^2} = \dfrac{2\xi r}{1 - r^2}$

$$\therefore \quad \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right] \qquad \text{...4.12}$$

Discussion on $\dfrac{MX}{me}$ Versus $\dfrac{\omega}{\omega_n}$.

Case (i) When $r = 0$ in Eq. 4.11

$\dfrac{Mx}{me} = 0$, which is independent of the damping ratio 'ξ'.

Case (ii) When $r = 1$ (Resonance, i.e. $\omega = \omega_n$) in Eq. 4.11

$\dfrac{Mx}{me} = \dfrac{1}{\sqrt{0 + (2\xi)^2}}$, $\dfrac{Mx}{me} = \dfrac{1}{2\xi}$, which depends on the damping ratio 'ξ', at $\xi = 0$, $\dfrac{Mx}{me} = \infty$.

i.e. at resonance and at damping ratio = 0, $\dfrac{Mx}{me} = \infty$.

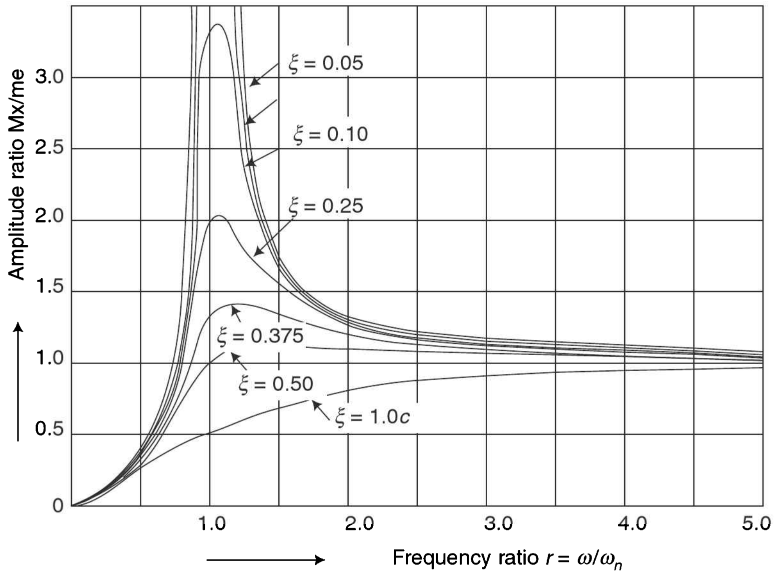


FIG. 4.7 *Amplitude ratio versus frequency ratio*

Case (iii) When $r >> 1$ ($\omega >> \omega_n$), i.e. $r^2 >>> 1$

$$\therefore \quad \frac{1}{r^2} <<< 1, \quad \text{i.e. } \frac{1}{r^2} \approx 0 \text{ from Eq. 4.11}$$

$$\therefore \quad \frac{Mx}{me} = \frac{r^2}{\sqrt{r^4\left(\frac{1}{r^2} - 1\right)^2 + r^4\left(\frac{2\xi}{r^2}\right)^2}}, \quad \frac{Mx}{me} = \frac{r^2}{r^2\sqrt{\left(\frac{1}{r^2} - 1\right)^2 + \left(\frac{2\xi}{r^2}\right)^2}} \quad \therefore \frac{Mx}{me} \approx 1$$

The phase angle 'ϕ' versus frequency ratio 'r' plot is similar to the 'ϕ' versus frequency ratio 'r' plot of magnification factor.

Table 4.1 *Differences between rotating and reciprocating unbalance*

Rotating Unbalance. (Figure 4.4)	*Reciprocating Unbalance. (Figure 4.5)*
In case of rotating unbalance, mass of the rotating element is to be considered.	In case of reciprocating unbalance, reciprocating element is to be considered.
In case of rotating unbalance, the eccentricity '*e*' which is the distance between the centre of rotation and centre of gravity is considered.	In case of reciprocating unbalance, the eccentricity '*e*' is the crank radius and crank radius, *e* = crank length/2.

EXAMPLE 4.9

A single cylinder vertical petrol engine of 300 kg total mass is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine of 20 kg mass and moves through a vertical stroke of 0.15 m with simple harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the velocity and amounts to 1500 N-s/m. Considering the steady state of vibration is reached, determine

(i) The amplitude of forced vibrations when the driving shaft of the engine rotates at 480 rev/min

(ii) The speed of the driving shaft at which resonance will occur

Solution $M = 300$ kg, $m = 20$ kg, $\delta = 2$ mm $= 2 \times 10^{-3}$ m,

$c = 1500$ N-s/m,

Stroke (s) = 0.15 m, eccentricity $e = \frac{s}{2} = \frac{0.15}{2} = 0.075$ m

(i) For reciprocating unbalance is given by

$$\frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad \text{...(a)} \qquad \text{where } r = \frac{\omega}{\omega_n}$$

We know that under static deflection due to weight,

$$k\delta = Mg, \frac{k}{M} = \frac{g}{\delta} \quad \therefore \omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{2 \times 10^{-3}}},$$

Natural frequency $\omega_n = 70.04$ rad/s

$$\text{Forcing frequency } \omega = \frac{2\pi N}{60} \text{ rad/s}, \ \omega = \frac{2\pi \times 480}{60}, \ \omega = 16\pi \text{ rad/s}$$

$$\therefore \quad \text{frequency ratio, } r = \frac{\omega}{\omega_n} = \frac{16\pi}{70.04}, \quad r = 0.72$$

$$\text{Damping ratio} \quad \xi = \frac{c}{c_c} = \frac{c}{2M\omega_n}$$

$$= \frac{1500}{2 \times 300 \times 70.04}, \ \xi = 0.036 < 1$$

Using all these values in Eq. (a), we get

$$\therefore\ X = \frac{mer^2}{M\sqrt{(1-r^2)^2 + (2\xi r)^2}},\ X = \frac{20 \times 0.075 \times (0.72)^2}{300\sqrt{(1-(0.72)^2)^2 + (2 \times 0.036 \times 0.72)^2}},$$

$$X = 5.35 \times 10^{-3}\ \text{m}$$

$$X = 5.35\ \text{mm}$$

(ii) At resonance, $\omega = \omega_n$

$$\therefore\qquad \frac{2\pi N}{60} = \omega_n,\ N = \frac{60\omega_n}{2\pi} = \frac{60 \times 70.04}{2\pi}$$

$\therefore\ N = 668.83$ rev/min, speed of the shaft at which resonance occurs.

EXAMPLE 4.10

A vertical single-stage air compressor having a mass of 500 kg is mounted on spring having stiffness of 1.96 × 10^5 N/m and a dashpot with damping factor of 0.2. The reciprocating unbalanced mass is 20 kg. The stroke is 0.2 m. Determine (i) the dynamic amplitude of vertical motion, and (ii) the phase difference between the motion and exiting force if the compressor is operated at 200 rev/min.

Solution $k = 1.96 \times 10^5$ N/m, $M = 500$ kg, $\xi = 0.2$, $m = 20$ kg, stroke = 0.2 m.

$$\therefore\quad e = \frac{\text{Stroke}}{2} = \frac{0.2}{2} = 0.1\text{m},\ N = 200\ \text{rev/min}.$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{1.96 \times 10^5}{500}} = 19.8\ \text{rad/s},\ \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 200}{60} = 20.93\ \text{rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{20.93}{19.8} = 1.06$$

The dynamic amplitude is given by

$$\frac{X}{\frac{me}{M}} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}},\quad \frac{X}{\frac{20 \times 0.1}{500}} = \frac{1.06^2}{\sqrt{(1-1.06^2)^2 + (2 \times 0.2 \times 1.06)^2}}$$

$$\therefore\qquad X = 0.01\ \text{m}$$

Phase angle is given by

$$\phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 \times 0.2 \times 1.06}{1-1.06^2}\right)\quad \therefore\ \phi = 105.9°.$$

EXAMPLE 4.11

A single-cylinder engine has an out-of-balance force of 500 N at an engine speed of 300 rev/min. The complete mass of the engine is 200 kg and it is carried on a set of springs of total stiffness 30000 N/m. Find the amplitude of the steady motion of the mass and the maximum oscillating force transmitted to the foundation. If a viscous damper is provided in between the mass and the foundation, the damping force is 10 N at a velocity of 0.01 m/s. Find the amplitude of forced damped oscillation of the mass and its angle of lag with disturbing force.

Solution $me\omega^2 = 500$ N, $M = 200$ kg, $N = 300$ rev/min, $k = 30000$ N/m,

Assuming $\xi = 0$ (damping is not given) for rotating unbalance the equation is given by

(i) $\dfrac{MX}{me} = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ...(a) where $r = \dfrac{\omega}{\omega_n}$

Since speed is given

$$\omega = \frac{2\pi N}{60} \quad \omega = \frac{2\pi \times 300}{60} = 10\pi \text{ rad/s} \quad \therefore\ me = \frac{500}{\omega^2} = \frac{500}{10\pi}, \quad me = 0.507.$$

The natural frequency is given by $\omega_n = \sqrt{\dfrac{k}{M}} = \sqrt{\dfrac{30000}{200}}$ rad/s $\therefore\ \omega_n = 12.25$ rad/s.

$\therefore$ frequency ratio $r = \dfrac{\omega}{\omega_n} = \dfrac{10\pi}{12.25}$ $\therefore\ r = 2.57$ and $\xi = 0$.

Using these values in Eq. (a), we get $\dfrac{200X}{0.507} = \dfrac{(2.57)^2}{\sqrt{[1-(2.57)^2]^2 + 0}}$,

$$X = \frac{(2.57)^2 \times 0.507}{200(1-(2.57)^2)}, X = -2.987 \times 10^{-3} \text{ mm}$$

Neglecting the –ve sign, amplitude $X = 2.987$ mm.

The transmissibility ratio is given by $TR = \dfrac{F_t}{F_0} = \dfrac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ...(b)

$$F_t = \frac{F_0\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}, \text{ where } F_0 = me\omega^2.$$

$$F_t = \frac{500\sqrt{1+0}}{\sqrt{[1-(2.57)^2]^2 + 0}}, \ F_t = \frac{500}{\sqrt{(1-(2.57)^2)}} = -89.21 \text{ N.}$$

$\therefore$ force transmitted to the foundation $F_t = -89.21$ N

Neglect negative sign, $F_t = 89.21$ N.

(ii) If viscous damper is provided then viscous force $F = c\dot{x}$,

Given $F = 10$ N $\dot{x} = 0.01$ m/s.

$\therefore$ $c = \dfrac{10}{0.01} = 1000$ N-s/m.

Damping coefficient $c = 1000$ N-s/m

Damping ratio $\xi = \dfrac{c}{c_c} = \dfrac{c}{2\sqrt{Mk}}$, $\xi = \dfrac{1000}{2\sqrt{200 \times 30000}}$, $\xi = 0.2$

From Eq. (a),

$$MX = \frac{mer^2}{M\sqrt{(1-r^2)^2+(2\xi r)^2}},\ X = \frac{0.507(2.57)^2}{200\sqrt{(1-2.57^2)^2+(2\times 0.2\times 2.57)^2}},\ X = 0.00294$$

Amplitude of forced viscous damping oscillation, $X = 2.94$ mm, and the phase lag is given by

$$\phi = \tan^{-1}\left[\frac{2\xi r}{1-r^2}\right],\ \phi = \tan^{-1}\left[\frac{2\times 0.2\times 2.57}{1-2.57^2}\right],\ \phi = -10°23^1 \text{ or, } \phi = 180° - 10°23^1,$$

$\phi = 169°37^1$.

EXAMPLE 4.12

A 75 kg machine is mounted on springs of stiffness $k = 11.76 \times 10^5$ N /m with damping of $\xi = 0.2$. A 2 kg piston within the machine has reciprocating motion with a stroke of 0.08 m and a speed of 3000 rev/min. Assuming the motion of the piston is harmonic motion, determine the amplitude of vibration of the machine.

Solution $M = 75$ kg, $k = 11.76 \times 10^5$ N /m, $\xi = 0.2$, $m = 2$ kg, stroke = 0.08 m,

$N = 3000$ rev/min, $e = \frac{\text{Stroke}}{2} = \frac{0.08}{2} = 0.04\text{m}$, $\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 3000}{60} = 314.43$ rad/s.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{11.76\times 10^5}{75}} = 125.21 \text{ rad/s}$$

The transmissibility ratio is given by

$$T_R = \frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}},\ T_R = \frac{2.5^2}{\sqrt{(1-2.5^2)^2+(2\times 0.2\times 2.5)^2}},\ T_R = 1.46.$$

The amplitude of vibration $T_R = \frac{MX}{me} = X = \frac{1.46\times 2\times 0.04}{75}$, $X = 1.24 \times 10^{-2}$ m .

EXAMPLE 4.13

A 40 kg fan has a rotating unbalance of 0.1 kg-m magnitude. The fan is mounted on the beam as shown in Fig. p-4.13. The beam has been specially treated to add viscous damping. As the speed of the fan is varied, it is noted that its maximum steady-state amplitude is 20.3 mm. What is the fan's steady-state amplitude when it operates at 1000 rev/min? Take $E = 200 \times 10^9$ N/m^2,

$I = 1.3 \times 10^{-4}$ m^4, $m_0e = 0.1$ kg-m, $\omega = 1000$ rev/min.

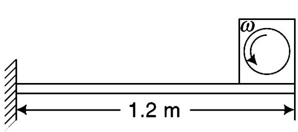


Fig. p-4.13 *Rotating fan*

Solution The maximum values of A is $A_{max} = \dfrac{mX_{max}}{m_0 e} = \dfrac{(40)(0.0203)}{0.1} = 8.12$.

The damping ratio is determined using equation

$$A_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}},\ 8.12 = \frac{1}{2\xi\sqrt{1-\xi^2}},\ \xi = 0.062.$$

The beam stiffness is given by $k = \dfrac{3EI}{L^3} = \dfrac{3(200 \times 10^9)\,(1.3 \times 10^{-6})}{(1.2)^3}$

$$= 4.51 \times 10^5 \text{ N/m}$$

and the system's natural frequency is $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4.51 \times 10^5}{40}} = 106.2$ rad/s

The frequency ratio is $r = \dfrac{\omega}{\omega_n} = \dfrac{(1000)(2\pi)\left(\dfrac{1}{60}\right)}{106.2} = 0.986$

The steady-state amplitude is calculated by $X = \dfrac{m_0 e}{m} A(0.986, 0.0617)$

$$= \frac{0.1}{40} \frac{(0.986)^2}{\sqrt{[1-(0.986)^2]^2 + [2(0.0617)(0.86)]^2}},\ X = 19.48 \text{ mm}.$$

4.5 FORCED VIBRATION DUE TO EXCITATION OF THE SUPPORT MOTION

In most of locomotives and vehicles, the wheels are mounted on a base or support for the systems. These wheels can move vertically up and down on the surface of the base or support on the surface during the moving of the vehicle. In the motion of these vehicles or body of the wheels and base or support, there is a relative motion between the support motion, relative to the wheels and the wheels are having motion relative to the road surface.

In case of support motion, the amplitude of the motion depends upon the speed of the vehicle and the nature of the road surface. The vibration measuring instruments are designed on the support motion approach. In a vibratory system where the support is put to excitation, (i) absolute motion, and (ii) relative motion become most important. Such systems are supported to have a spring-mass-damper system of a single degree of freedom with a moving support or base as shown in Fig. 4.8.

1. Absolute motion (motion transmissibility) Absolute motion of a mass means its motion with respect to the coordinate system attached to the earth as shown in Fig. 4.8(a). The absolute displacement of support is $y = Y \sin \omega t$[sinusoidal motion Fig. 4.8(b)] and the absolute displacement of the mass 'm' from its equilibrium position is 'x'. The displacement of the mass 'm' relative to the support is 'z'.

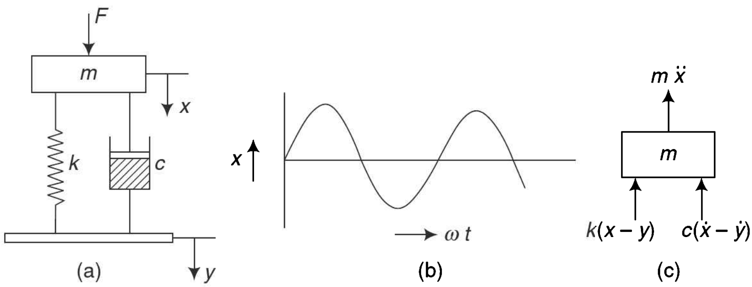


Fig. 4.8 *Support motion*

The net elongation of the spring is $(x - y)$ and the relative motion between the two ends of the damper is $(\dot{x} - \dot{y})$. Then $z = (x - y)$ and $\dot{z} = (\dot{x} - \dot{y})$.

Let us consider a spring-mass-damper system subjected to the support motion as shown in Fig. 4.8(a). Due to the support motion, let 'x' be the absolute motion of the system at any instant of time and respective FBD is as shown in Fig. 4.8(c).

Now apply Newton's second law of motion to mass 'm' $\Sigma F = m\ddot{x}$

$$m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y),\ m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad \text{... 4.13}$$

Since $\qquad y = Y\sin\omega t,\ \dot{y} = \omega Y\cos\omega t$

Substituting these values in Eq. 4.13, we get

$$m\ddot{x} + c\dot{x} + kx = c\omega Y\cos\omega t + kY\sin\omega t,\ \ m\ddot{x} + c\dot{x} + kx = Y(c\omega\cos\omega t + k\sin\omega t)$$

Multiplying and dividing by right-hand side by $\sqrt{k^2 + (c\omega)^2}$ and simplifying, we get

$$= Y\sqrt{k^2 + (c\omega)^2}\left[\frac{k}{\sqrt{k^2 + (c\omega)^2}}\sin\omega t + \frac{c\omega}{\sqrt{k^2 + (c\omega)^2}}\cos\omega t\right]$$

$$= Y\sqrt{k^2 + (c\omega)^2}\ \{\sin\omega t\cos\alpha + \cos\omega t\sin\alpha\}$$

$$= Y\sqrt{k^2 + (c\omega)^2}\ \sin(\omega t + \alpha)$$

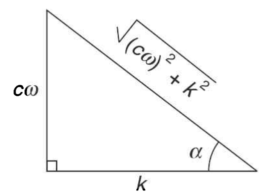


Fig. 4.9 *Absolute motion*

$$\therefore \qquad m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2 + (c\omega)^2}\ \sin(\omega t + \alpha) \qquad \text{... 4.14}$$

and phase angle is given by $\tan\alpha = \dfrac{c\omega}{k}$ or $\alpha = \tan^{-1}\left[2\xi\dfrac{\omega}{\omega_n}\right]$ ... 4.15

Equation 4.15 can be compared to a system excited by an external harmonic force, the steady-state amplitude $X = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$,

where $F_0 = Y\sqrt{k^2 + (c\omega)^2}$ from last equation

$$\therefore \qquad X = \frac{Y\sqrt{k^2 + c\omega^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \text{ or } \frac{X}{Y} = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \text{... 4.16}$$

Equation 4.16 can be expressed in nondimensional form by dividing numerator and denominator by 'k'.

$$\frac{X}{Y}=\frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}\;\therefore\; \text{T R}=\frac{F_{tr}}{F_0}=\frac{X}{Y}=\frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} \quad ...4.17$$

and $\tan\beta=\frac{c\omega}{k}$ or $\beta=\tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right]$ or it can be written as $\beta=\tan^{-1}[2\xi r]$

where $r=\frac{\omega}{\omega_n}$ frequency ratio and

phase lag is given by $(\alpha-\beta)=\tan^{-1}\left(\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)}\right)-\tan^{-1}\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]$ ... 4.18

Figure 4.10(a) shows the plots of **amplitude ratio** $\frac{X}{Y}$ against **frequency ratio** $\frac{\omega}{\omega_n}$ for various values of damping factor and Fig. 4.10(b) shows the plots of **phase angle** $(\alpha-\beta)$ against the **frequency ratio** $\frac{\omega}{\omega_n}$ for various values of damping factor. From the figure, it is seen that

(i) when $\frac{\omega}{\omega_n}<\sqrt{2}$, $\frac{X}{Y}>1$

(ii) when $\frac{\omega}{\omega_n}>\sqrt{2}$, $\frac{X}{Y}<1$

(iii) when $\frac{\omega}{\omega_n}=\sqrt{2}$, $\frac{X}{Y}=1$

(iv) when $\frac{\omega}{\omega_n}=1$, $\frac{X}{Y}=\infty$

2. Relative motion In Absolute motion, we assumed that the displacement of the mass 'm' relative to the support is 'z'.

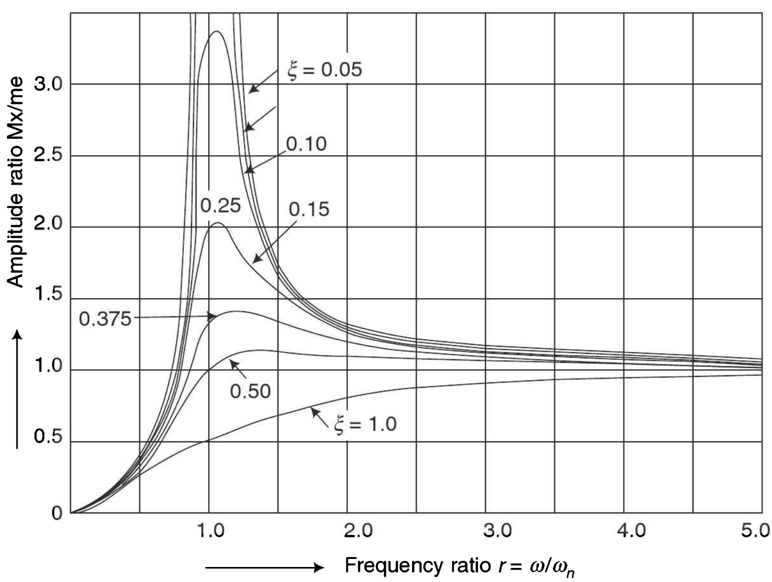


Fig. 4.10 (a) *Amplitude ratio verses frequency ratio*

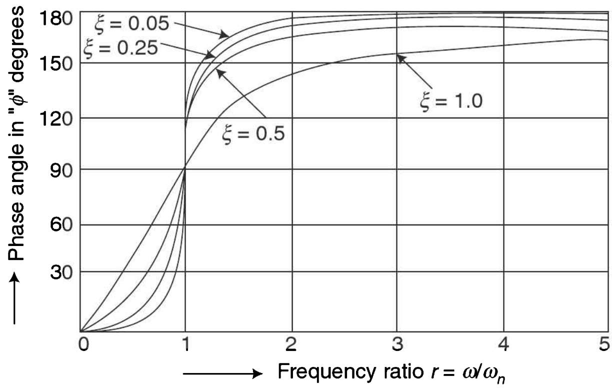


Fig. 4.10 (b) *Phase angle versus frequency ratio*

Then this can be written as

$$z = (x - y), \quad \dot{z} = (\dot{x} - \dot{y}), \quad \ddot{z} = (\ddot{x} - \ddot{y})$$

Substituting these valves in Eq. 4.14 and simplifying, we get

$$\therefore \qquad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad \text{...4.19}$$

We have $\quad y = Y \sin \omega t, \quad \dot{y} = Y\omega \cos \omega t \quad \therefore \ddot{y} = -Y\omega^2 \sin \omega t$

Hence Eq. 4.19 can be written as

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 y \sin \omega t \qquad \text{...4.20}$$

Eq. 4.20 is in the same form of rotating and reciprocating unbalance Eq. 4.10

Therefore, the steady-state relative amplitude is

$$Z = \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \text{ or } \frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega^n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \qquad \text{...4.21}$$

where $\frac{Z}{Y}$ is called relative displacement transmissibility

and phase angle is given by $\phi = \tan^{-1} \dfrac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$. ... 4.22

4.6 ENERGY DISSIPATED BY DAMPING

During forced vibration with viscous damping, energy is continuously absorbed by the damper so that power must be supplied to maintaining steady-state condition. So there is a need to evaluate the magnitude of power required and to see how it changes with the variables.

Work done by the force 'F' during an interval of time when the body moves through a displacement 'dx' is given by $dw = F\,dx$.

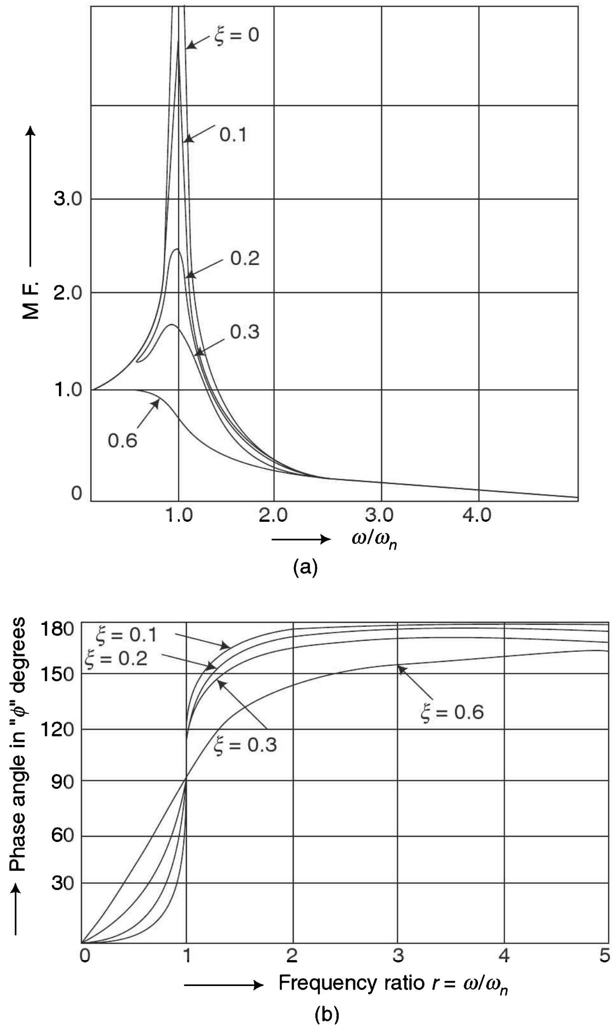


Fig. 4.11 *(a) Magnification factor versus frequency ratio (b) Phase angle versus frequency ratio*

$= F \cdot \dfrac{dx}{dt} \times dt$. Over a period of one cycle displacement, 'ωt' varies from 0 to 2π. Therefore, 't' varies from 0 to $\dfrac{2\pi}{\omega}$.

$\therefore$ work done per cycle is given by $w = \int_0^{2\pi/\omega} F \cdot \dfrac{dx}{dt} \cdot dt$ but $x = X \sin(\omega t - \phi)$

$$\therefore \qquad \frac{dx}{dt} = X\omega \cos(\omega t - \phi), \text{ and } F = F_0 \sin \omega t$$

$$w = \int_0^{2\pi/\omega} (F_0 \sin \omega t)(X\omega \cos(\omega t - \phi))\,dt \qquad \text{... 4.23,}$$

$$\omega = \pi F_0 \times \sin \phi$$

where X = Amplitude of vibratory motion

F_0 = Amplitude of vibrating force

ϕ = Phase angle by which the motion lags the force

The maximum work is absorbed when the face angle ϕ is 90° and when $\frac{\omega}{\omega_n} = 1$ and $\sin\phi = \sin 90° = 1$.

Therefore, work done per cycle or energy dissipated per cycle = $\pi(c\omega X)\,X$.

Energy dissipated per cycle = $\pi\, c\omega X^2$. ...4.24

4.7 FORCED VIBRATION WITH COULOMB DAMPING

As we know from Chapter 3, Sec. 3.3.1, Case (b) on different types of damping, Coulomb damping or dry friction damping is caused by friction between the surfaces that are dry or having insufficient lubrication. When a body slides on a dry surface, the force of resistance between the surfaces or the frictional force is proportional to the normal load. This damping is called Coulomb damping and is shown in Fig. 4.12.

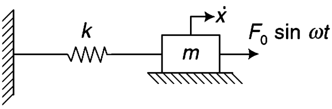


Fig. 4.12 *Coulomb damping*

When a single-degree-of-freedom system under Coulomb damping or dry friction damping is subjected to a harmonic force of '$F_0 \sin \omega t$' the differential equation of a motion is written as

$$m\ddot{x} + kx \pm \mu R_N = F_0 \sin \omega t \qquad ...4.25$$

In Eq. 4.25, the sign of friction force ($\pm\, \mu R_N$) is positive, when the body moves from left to right and it is vice versa as we know. In small values of Coulomb damping, the exact solution for small damping force is small so that the motion is continuous. On the other hand, in case of high value of Coulomb damping force, the motion does not remain continuous. Also, if dry friction force is small compared to harmonic force, an approximate solution is necessary. By determining an equivalent viscous damping 'c_{eq}' in case of forced vibration with Coulomb damping, the means of energy absorbed per cycle is same in both the cases.

Let 'X' be the amplitude of steady-state vibration and 'F' be the constant frictional force. Then the energy absorbed per cycle is $4FX$. For the similar amplitude of vibration, the energy absorbed per cycle for the case of equivalent viscous damping from Eq. 4.24 is

energy dissipated per cycle = $\pi\, c\omega X^2 = \pi\, c_{eq}\, \omega X^2$ since $c = c_{eq}$

Therefore, equating the two equations, $4FX = \pi\, c_{eq}\, \omega X^2, = c_{eq} = \dfrac{4F}{\pi\, \omega X}$ ...4.26

We have known that in case of viscous damping the steady-state amplitude is given by Eq. 4.5 in Sec. 4.2.

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}$$

Substituting for 'c' the value of equivalent viscous damping as obtained in Eq. 4.26 we get,

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{4F}{\pi X k}\right)^2}}$$

Putting $\omega^2_n = \frac{k}{m}$ and solving for 'X', we have

$$\frac{X}{F_0/k} = \frac{\sqrt{1 - \left(\frac{4F}{\pi F_0}\right)^2}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad \text{...4.27(a)}$$

The amplitude will have a real value if $\frac{4F}{\pi F_0} < 1$ or $\frac{F}{F_0} < \frac{\pi}{4}$ ...4.27(b)

In some cases the friction force 'F' is very small and therefore Eq. 4.27 is suitable. As friction force increases, this equation becomes approximate only till when $\left[\frac{F}{F_0} > \frac{\pi}{4}\right]$ and Eq. 4.27 ceases to hold good.

Now we can seen that at resonance i.e. $\omega = \omega_n$, the amplitude becomes infinity when friction damping is present in the system. This can be explained based on considering energy input or energy dissipated. The energy input per cycle is proportional to the amplitude of the system and the energy dissipated per cycle by Coulomb damping is also proportional to the amplitude $4FX$. Thus, if the friction damping force is small, the energy dissipated per cycle is always less than the input energy and hence, the amplitude increase without limit. Also we know that the energy dissipated in case of viscous damping is proportional to the square of the amplitude, i.e. $\pi c\omega X^2$ or even if the damping is very small, increase in amplitude makes the energy dissipated increase the amplitude with rapidly and a stage comes when the input and absorbed energies are equal.

EXAMPLE 4.14

A spring–mass system in horizontal position subject to dry friction damping has the mass of the system as 3.75, and spring stiffness as 7550 N/m. The coefficient of friction between the mass and horizontal plane is assumed as a 0.21, subjected to a sinusoidal force function of 19.7 N amplitute and 4.95 Hz frequency. Determine the amplitude of the mass and equivalent viscous damping.

Solution Spring stiffness k = 7550 N/m coefficient of friction $\mu = 0.21$,
$mg = 3.75 \times 9.81 = 36.88$ N, $F = \mu mg = 7.73$ N, $F_0 = 19.7$ N, $f = 4.95$ Hz, $\omega = 2\pi f = 2 \times \pi \times 4.95 = 31.1$,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7550}{3.75}} = 45 \text{ rad/s}, \quad \frac{\omega}{\omega_n} = \frac{31.1}{45} = 0.7$$

Substituting all these values in Eq. 4.27(a),

$$\frac{X}{F_0/k} = \frac{\sqrt{1 - \left(\frac{4F}{\pi F_0}\right)^2}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Amplitude of the mass $\dfrac{X}{19.7/7550} = \dfrac{\sqrt{1 - \left(\frac{4 \times 7.73}{\pi \times 19.7}\right)^2}}{1 - (0.7)^2}$

$\therefore$ amplitude $\quad X = 0.0042$ m

Equivalent viscous damping $c_{eq} = \dfrac{4F}{\pi \omega X} = \dfrac{4 \times 7.73}{\pi \times 31.1 \times 0.0042} \quad \therefore c_{eq} = 79$ N-s/m.

4.8 FORCED VIBRATION WITH COULOMB DAMPING AND VISCOUS DAMPING

Sometimes, there is a combination of damping consisting of viscous damping coefficient 'c_1' and Coulomb damping force 'F' in parallel combination. Then the equivalent damping coefficient in such situation, an addition of viscous damping coefficient and the equivalent damping coefficient is given by Eq. 4.26 in Sec. 4.7,

$$c_{eq} = c_1 + \frac{4F}{\pi \omega X} \qquad \text{...4.28}$$

We know that in case of viscous damping, the steady-state amplitude is given by Eq. 4.5 in Sec. 4.2.

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}} \qquad \because \quad \omega_n = \sqrt{\frac{k}{m}}.$$

Substituting the values of c_{eq} in place of 'c',

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left[\left(c_1 + \frac{4F}{\pi X \omega}\frac{\omega}{k}\right)\right]^2}} \qquad \text{...4.29}$$

Squaring Eq. 4.29 on both side and solving, then rewriting the equation as follows:

$$X = \left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{c_1\omega}{k}\right)^2\right\}X^2 + \left[\frac{8Fc_1\omega}{\pi k^2}\right]X + \left[\left(\frac{4F}{\pi k}\right)^2 - \left(\frac{F_0}{k}\right)^2\right] = 0 \qquad ...4.30$$

By solving the above quadratic equation for 'X', we get the amplitude of vibration of both the systems.

Case (i) In the above equation, if there is no viscous damping, i.e. $c_1 = 0$ then we get the amplitude

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1-\frac{m}{k}\omega^2\right)^2 + \left(\frac{4F}{\pi X k}\right)^2}} \qquad ...4.31$$

Case (ii) In the above equation if there is no coulomb damping, i.e. $F = 0$ then we get the amplitude

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}} \qquad ...4.32$$

$$\therefore \qquad \omega_n^2 = \frac{k}{m}$$

And the phase angle is obtained by substituting the values of c_{eq} in place of 'c'; we have

$$\phi = \tan^{-1}\left[\left(\frac{c_1 + \frac{4F}{\pi X\omega}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)\right]\frac{\omega}{k} \qquad ...4.33$$

EXAMPLE 4.15

A mass of 6 kg suspended by a spring of stiffness 1180 N/m is forced to vibrate by the harmonic force of 10 N. Assuming viscous damping coefficient of 85 N-s/m, determine the resonant frequency, amplitude of resonance, phase angle at resonance, a frequency corresponding to the peak amplitude, peak-amplitude and the phase angle-to-peak amplitude.

Solution $m = 6$ kg, $k = 1180$ N/m, $F_0 = 10$ N and $c = 85$ N-s/m.

(i) Resonant frequency $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1180}{6}} = 14.023$ rad/s

$$f_n = \frac{1}{2\pi}\omega_n = \frac{14.023}{2\pi} = 2.232 \text{ Hz.}$$

(ii) At resonance, we know $\frac{X_{res}}{X_{st}} = \frac{1}{2\xi}$ $\quad\therefore X_{res} = \frac{F_0}{k \times 2 \times \xi}$,

$$\xi = \frac{c}{c_c} = \frac{85}{2\sqrt{1180 \times 6}} = 0.505$$

$$\therefore \qquad X_{res} = \frac{10}{1180 \times 2 \times 0.505}, \quad X_{res} = 0.00839 \text{ m.}$$

(iii) Phase angle at resonance $r = \dfrac{\omega}{\omega_n} = 1$, $\phi = \tan^{-1}\left(\dfrac{2\xi r}{1 - r^2}\right) = \tan^{-1}\left(\dfrac{2 \times 0.505 \times 1}{1 - 1^2}\right)$

$= \tan^{-1}(\infty) = 90°$.

EXAMPLE 4.16

A 45 kg piston is supported by a spring of modulus k = 35 k N/m. A dashpot of damping coefficient c = 1250 N-s/m acts in parallel with the spring. A fluctuating pressure P = 4000 sin 30 t in Pa acts on the piston, whose top surface area is 50 × 10^{-3} m^2. Determine the steady-state maximum displacement and maximum force transmitted to the base.

Solution The natural frequency and damping ratio are,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{35 \times 10^3}{45}} = 27.9 \text{ rad/s}, \quad \xi = \frac{c}{2m\omega_n} = \frac{1250}{2 \times 45 \times 27.9} = 0.498$$

$$\omega = 30. \ \frac{\omega}{\omega_n} = \frac{30}{27.9}, \ F_0 = 4000 \text{ N}$$

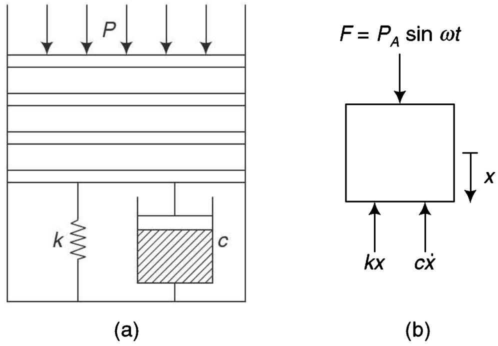


Fig. p-4.16 *Spring system*

The steady-state amplitude is $X = \dfrac{\dfrac{F_0}{k}}{\left\{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\dfrac{\omega}{\omega_n}\right]\right\}} = 0.00528$ m

(or 5.28 mm).

The phase angle is $\phi = \tan^{-1}\left[\dfrac{2\xi\dfrac{\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left\{\dfrac{2 \times 0.498 \times \dfrac{30}{27.9}}{1 - \left(\dfrac{30}{27.9}\right)^2}\right\}$

$= 1.716$ radians

The maximum value of force transmitted is $F_{tr\ max} = \sqrt{(kX)^2 + (c\omega X)^2}$

$$= X\sqrt{k^2 + c^2\omega^2} = 0.00528\sqrt{35000^2 + 1250^2 \times 30^2} = 271 \text{ N}$$

EXAMPLE 4.17

A weight of 55 N suspended by a spring stiffness of 1.1 k N/m is forced to vibrate by a harmonic force of 9 N. Assuming viscous damping coefficient c = 77 N S/m, determine: (i) the amplitude at resonance, (ii) frequency corresponding to peak amplitude, (iii) peak amplitude, and (iv) phase angle corresponding to peak amplitude.

Solution $W = 55$ N, $m = W/g = 55/9.81 = 5.61$ kg,

$k = 1100$ N/m, $F_0 = 9$ N, $c = 77$ N- s/m.

The frequency ratio $\omega/\omega_n = r$, at resonance $r = 1$; $\omega = \omega_n$

$$\therefore \quad = \sqrt{\frac{k}{m}} = \sqrt{\frac{1100}{5.61}} = 14 \text{ rad/s}$$

Also,
$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{77}{2(5.61)(14)} = 0.490$$

(i) The amplitude is given by the equation $X_{\text{res}} = \dfrac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

At resonance, $r = 1$.

$$\therefore \quad X_{res} = \frac{\frac{F_0}{k}}{\sqrt{(2\xi)^2}} = \frac{\frac{F_0}{k}}{2\xi} = \frac{\left(\frac{9}{1100}\right)}{2(0.490)} = 8.339 \times 10^{-3} \text{ m.}$$

(ii) Phase angle at resonance is given by the equation

$$\phi = \tan^{-1}\frac{2\xi r}{(1-r^2)} = \tan^{-1}\infty \quad \phi = 90° = 1.57 \text{ radians.}$$

(iii) Frequency corresponding to peak amplitude is given by the equation

$$r_{peak} = \frac{\omega_{peak}}{\omega_n} = \sqrt{1-2\xi^2} = \sqrt{1-2(0.490)^2}, \; \omega_{peak} = 10.09 \text{ rad/s.}$$

(iv) Peak amplitude is given by $X_{peak} = \dfrac{X_0}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$, $r_{peak} = \dfrac{10.09}{14} = 0.7207.$

$$X_{peak} = \frac{\frac{9}{1100}}{\sqrt{(1-0.7207^2)^2 + (2(0.49)(0.7207))^2}} = 9.57 \times 10^{-3} \text{ m.}$$

(v) Phase angle corresponding to peak amplitude is given by

$$\phi = \tan^{-1}\frac{2\xi r}{1-r_p^2}, \; \phi = \tan^{-1}\frac{2(0.49)(0.72)}{1-0.72^2} = 55.75° = 0.973 \text{ radians.}$$

4.8 FORCED VIBRATION WITH STRUCTURAL DAMPING OR HYSTERESIS DAMPING

As we learnt in Sec. 3.3.1, Chapter 3, case (c), different types of damping, structural damping occurs in all vibrating systems subjected to elastic restoring forces.

This type of damping is due to the internal friction of the molecules of elastic materials. Stress-strain graphs are different for loading and unloading, the energy dissipated within the material itself. When such a material is subjected to cyclic reversal of loading, a hysterisis loop appears on the stress-strain plot.

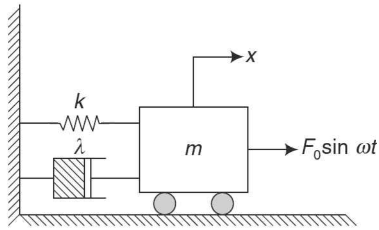


Fig. 4.13 *Spring-mass-damper system in horizontal position*

Let us consider a single-degree-of-freedom system with structural damping subject to a harmonic force of '$F_0 \sin \omega t$' as shown in Fig. 4.13. Then the differential equation of a motion of mass can be derived as $m\ddot{x} + \lambda \frac{k}{\omega}\dot{x} + kx = F_0 \sin \omega t$ ...4.34

where $\left(\lambda \frac{k}{\omega}\right)\dot{x}$ indicates the damping force. This damping force is a function of forcing frequency 'ω'.

In most of the structural elements "the energy dissipated per cycle is proportional to the square of the amplitude (X) and independent of frequency (β) over a wide range". That means energy dissipated per cycle = βX^2 ...4.35

In case of structural damping, 'c_{eq}' is the equivalent viscous damping, as we know in Sec. 4.6, Eq. 4.24.

Energy dissipated per cycle = $\pi c \omega X^2 = \pi c_{eq} \omega X^2$ since $c = c_{eq}$

$$\therefore \quad \beta X^2 = \pi c_{eq} \omega X^2 \text{ or } c_{eq} = \frac{\beta}{\omega \pi} \qquad ...4.36$$

In equivalent viscous damping, the energy dissipated by the damper in one complete cycle of motion can given by case (c), Sec. 3.3.1. Energy loss/cycle

$$E = \pi \lambda k X^2 \qquad ...4.37$$

From Eq. 4.35 and Eq. 4.37, $\beta X^2 = \pi \lambda k X^2 \quad \therefore \lambda = \frac{\beta}{\pi k}$ or $\beta = \pi k \lambda$ ... 4.38

Substituting the 'λ' value in the differential equation of a motion of the system under forced vibration subjected to structural damping, we get

$$m\ddot{x} + \left(\frac{\beta}{\pi\omega}\right)\dot{x} + kx = F_0 \sin \omega t \qquad ...4.39$$

where $\lambda = \left(\frac{\beta}{\pi\omega}\right)$ is known as the structural damping factor or loss factor.

The steady-state solution of Eq. 4.34 given as the steady-state amplitude for system when subject to structural damping or hysteresis damping, given by Eq. 4.5.

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}} \qquad ...(\text{Eq. } 4.5)$$

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{c_{eq}\omega}{k}\right)^2}} \quad ...4.40$$

Put $\omega_n^2 = \frac{k}{m}$

From Eq. 4.36, $c_{eq} = \frac{\beta}{\omega\pi}$

Substituting the value of $\beta = \pi k\lambda$, we get

$$c_{eq} = \frac{\pi k\lambda}{\pi\omega}, \; c_{eq}\omega = k\lambda \quad ...4.41$$

Substituting the value of $c_{eq}\omega = k\lambda$ in Eq. 4.40, we get

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \lambda^2}} \quad ...4.42(a)$$

$$\frac{X}{\frac{F_0}{k}} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \lambda^2}} \quad ...4.42(b)$$

As we know that the phase angle 'ϕ' is given by $\tan\phi = \frac{\left[\frac{c}{k}\right]\omega}{1 - \frac{m}{k}\omega^2}$

By substituting the values of 'c_{eq}' in place of 'c', we have $\phi = \tan^{-1}\left[\frac{\frac{c_{eq}\omega}{k}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$

But we know $c_{eq}\omega = k\lambda$ $\quad \therefore \; \phi = \tan^{-1}\left[\frac{\lambda}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$ ...4.43

If the harmonic excitation is assumed to be in the form of complex $Fe^{i\omega t}$,

$$m\ddot{x} + \lambda\frac{k}{\omega}\dot{x} + kx = F_0 \sin \omega t \quad ...4.44$$

$$m\ddot{x} + \lambda\frac{k}{\omega}\dot{x} + kx = Fe^{i\omega t} \quad ...4.45$$

So the response of 'x' in the above equation is also a harmonic function having the factor $e^{i\omega t}$.

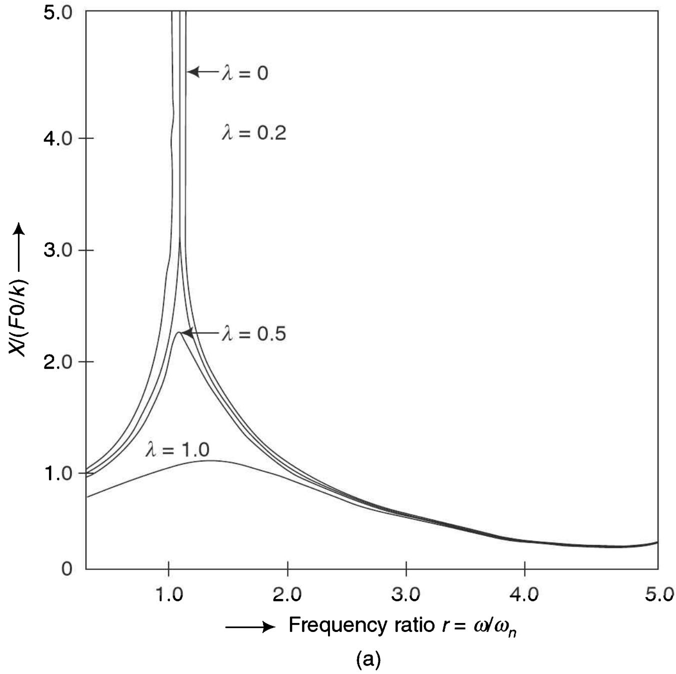
Therefore, $\dot{x}$ is given by $i\,\omega\,x$ and Eq. 4.45 will be as follows:

$$m\ddot{x} + k(1 + i\lambda)\,\dot{x} + kx = Fe^{i\omega t} \quad ...4.46$$

where $k(1 + i\lambda)$ is known as complex damping (complex stiffness). Then the steady-state solution of above Eq. 4.44 is as follows:

$$x = \frac{Fe^{i\omega t}}{k\left|1-\left(\frac{\omega}{\omega_n}\right)^2 + i\lambda\right|} \qquad \text{...4.47}$$

Equation 4.43 and Eq. 4.44 are plotted in Fig. 4.14(a) for various values of amplitude and 'λ' and phase angle versus frequency ratio respectively.



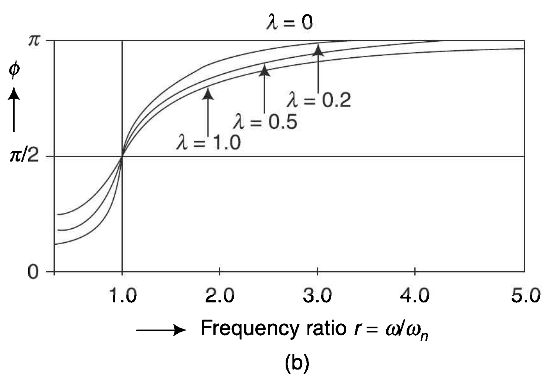


Fig. 4.14 *(a) Amplitude ratio versus frequency ratio (b) Phase angle versus frequency ratio*

EXAMPLE 4.18

A block of 35 kg mass is connected to a support through a spring of 1.4×10^6 N/m stiffness in parallel with a dashpot of damping coefficient 1.8×10^3 N-s/m. The support is given a harmonic displacement of 10 mm amplitude at a frequency of 35 Hz. What is the steady-state amplitude of the absolute displacement of the block?

Solution $m = 35$ kg, spring stiffness $k = 1.4 \times 10^6$ N/m,

damping coefficient $c = 1.8 \times 10^3$ N-s/m, $f = 35$ Hz, $Y = 10$ mm.

$$\omega_n = \sqrt{\frac{k}{m}} \text{ or } \omega_n = \sqrt{\frac{1.4 \times 10^6}{35}}$$

= 200 rad/s,

$$\omega = 2\pi f = 2 \times \pi \times 35 = 219.8 \text{ rad/s}$$

$$\xi = \frac{c}{c_c} = \frac{c}{2mw_n} = \frac{1.8 \times 10^3}{2 \times 35 \times 200} = 0.129, \qquad r = \frac{\omega}{\omega_n} = \frac{219.8}{200} = 1.10$$

$$X = Y\frac{\sqrt{(1 + 2\xi r)}}{\sqrt{(1 + r^2)^2 + (2\xi r)^2}}, \quad X = 10\frac{\sqrt{(1 + 2 \times 0.129 \times 1.1)}}{\sqrt{(1 + 1.1^2)^2 + (2 \times 0.129 \times 1.1)^2}},$$

$X = 29.4$ mm.

4.8 VIBRATION ISOLATION AND FORCE TRANSMISSIBILITY

When an unbalanced machine is mounted on the foundation, the vibration of the machine will be transmitted to the foundation. In order to minimise the transmission of forces to the foundation, machines are often mounted on springs and dampers as shown in the Fig. 4.15(a) The vibratory force transmitted to the foundation is by springs and dampers because these are the only connections. At any instant give a displacement 'x' to the mass 'm', and the FBD is as shown in Fig. 4.15(b).

Applying Newton's second law of motion to mass 'm', $\Sigma F = m\ddot{x}$.

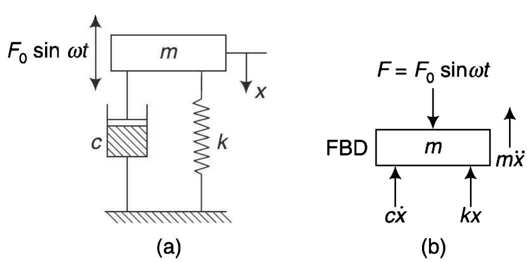


Fig. 4.15 *Vibration isolation and transmissibility*

$$\therefore \quad F - c\dot{x} - kx = m\ddot{x} \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad \text{...4.48}$$

Since $\quad F = F_0 \sin \omega t$

this is a second-order nonhomogeneous differential equation of motion. The solution is

$$x = x_c + x_p$$

where x_c = Complementary function (already discussed in **Sec. 4.2** case '*a*')

x_p = Particular integral (steady state response already discussed in **Sec. 4.2** case '*b*')

Considering the steady-state response or to find the particular integral 'x_p'

Let $x_p = x = X \sin(\omega t - \phi)$ $\quad [\because F$ is the external force which is a sinusoidal one]

$$\dot{x} = \omega X \cos(\omega t - \phi) = \omega X \sin\left(\frac{\pi}{2} + \omega t - \phi\right), \ \ddot{x} = -\omega^2 X \sin(\omega t - \phi)$$

Substituting these value in Eq. 4.13,

$$m[-\omega^2 X \sin(\omega t - \phi)] + c\left[\omega X \sin\left(\frac{\pi}{2} + \omega t - \phi\right)\right] + kX \sin(\omega t - \phi) = F_0 \sin \omega t$$

Rearranging the terms,

$$F_0 \sin \omega t - kX \sin(\omega t - \phi) - cX\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + m\omega^2 X \sin(\omega t - \phi) = 0 \qquad \text{...4.49}$$

These forces can be vectorially represented as shown in Fig. 4.16(a),

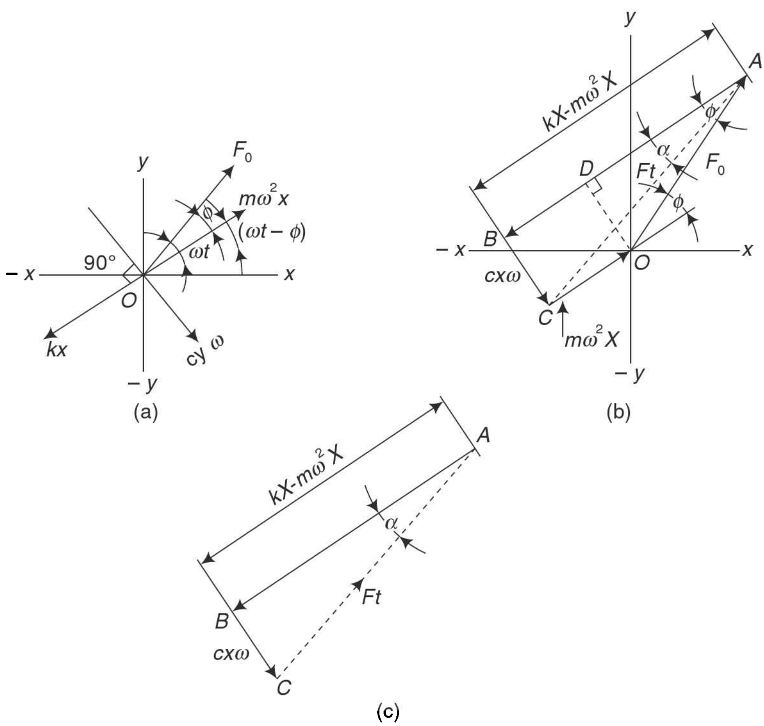


Fig. 4.16 *Force and vector polygon*

$$\overrightarrow{AC} = F_t = \sqrt{(kX)^2 + (cX\omega)^2} \quad \text{But } \overrightarrow{OA} = F_0, \overrightarrow{AB} = kX, \overrightarrow{BC} = C\omega X, \overrightarrow{CO} = mX\omega^2$$

The force transmitted to the foundation denoted by 'F_t' by joining 'AC' is the vectorial sum of the spring force and the damping force and it is shown in the Fig. 4.16(b) force polygon.

The *transmissibility ratio* or *transmissibility* is defined as the ratio of the force transmitted to the foundation 'F_t' through elastic supports to the force transmitted to the foundation through rigid supports 'F_0' (exiting force) **(See How to Draw the Vector and Force Polygon**.)

$$\therefore \text{ transmissibility TR} = \frac{F_1}{F_0}$$

From right-angled triangle OAD in Fig. 4.16(b),

$$OA^2 = AD^2 + DO^2 = (AB - BD)^2 + BC^2$$

$$\therefore \quad F_0{}^2 = (kX - mX\omega^2)^2 + (cX\omega)^2, F_0{}^2 = X^2\,[(k - m\omega^2)^2 + (c\omega)^2]$$

$$X^2 = \frac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}, \quad X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\therefore \quad X = \frac{F_0}{k\sqrt{\left[\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2\right]}}$$

But $\dfrac{m}{k} = \dfrac{1}{\omega_n^2}$ and $\dfrac{c}{k} = \dfrac{2\xi}{\omega_n}$, and let $\dfrac{\omega}{\omega_n} = r$.

$$\therefore \quad X = \frac{F_0}{k\sqrt{[(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...4.50}$$

$$\therefore \quad F_0 = kX\sqrt{[(1 - r^2)^2 + (2\xi r)^2} \qquad \text{...4.51}$$

From triangle ABC in Fig. 4.16(c),

$$F_t^2 = (kX)^2 + (cX\omega)^2, F_t^2 = k^2X^2\left[1 + \left(\frac{c}{k}\omega\right)^2\right], F_t = kX\sqrt{1 + (2\xi r)^2} \qquad \text{...4.52}$$

From Eq. 4.49 and Eq. 4.50,

$$\therefore \quad \text{TR} = \frac{F_t}{F_0} = \frac{kX\sqrt{1 + (2\xi r)^2}}{kX\sqrt{[(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...4.53}$$

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...4.54}$$

The phase angle $\phi = \tan^{-1}\left[\dfrac{2\xi r}{1 - r^2}\right]$

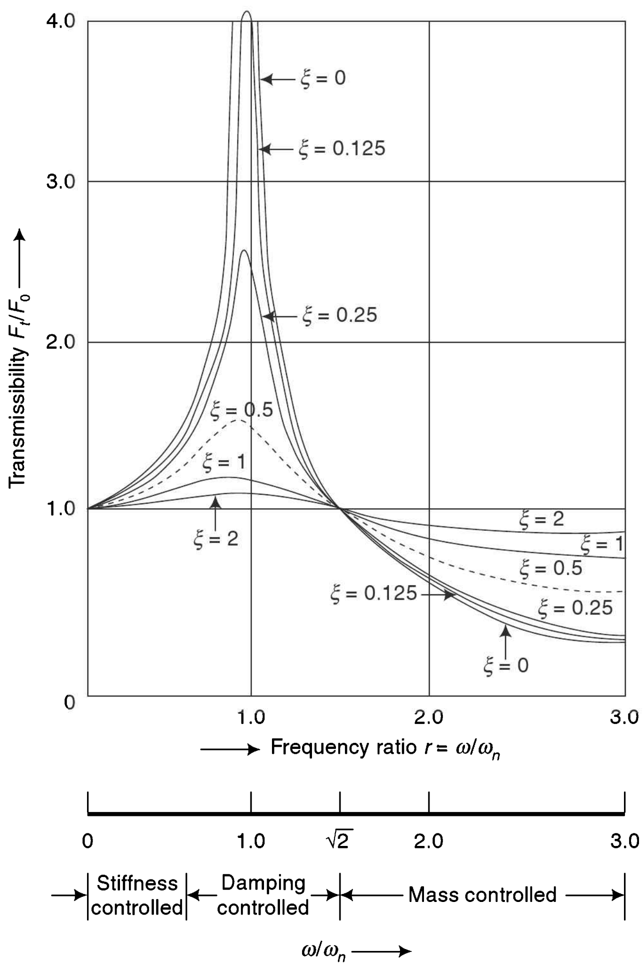
$$\therefore \qquad \phi = \tan^{-1}\left[\frac{2\xi r}{1-r^2}\right] \qquad \ldots 4.55$$

and $\qquad \tan\alpha = \dfrac{cX\omega}{kX},\ \alpha = \tan^{-1}(2\xi r).$

The angle of lag is given as a $(\phi - \alpha)$ $\quad \therefore\ \tan^{-1}\left[\dfrac{2\xi r}{1-r^2}\right] - \tan^{-1}(2\xi r)$ $\qquad$ ... 4.56

The equations 4.54 and 4.55 indicate the transmissibility and phase lag of transmitted force from the impressed force and can be plotted as shown in Fig. 4.17(a) and (b) for various values of damping factors.

1. Discussion on $\dfrac{F_t}{F_0}$ Versus $\dfrac{\omega}{\omega_n}$.



(a) Transmissibility versus frequency ration for various amount of damping factors

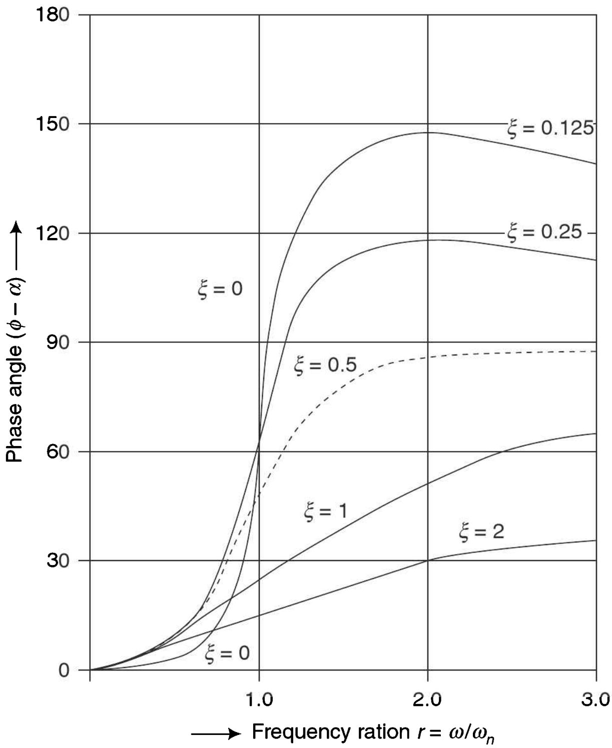


Fig. 4.17(b) *Phase angle frequency ratio for various amount of damping factors*

Case (i) When $r = 0$ in Eq. 4.54 $\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}}$.

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + 0}}{\sqrt{1 + 0}}, \quad \frac{F_t}{F_0} = 1,$$ which is independent of the damping ratio 'ξ'

Case (ii) When $r = 1$ (Resonance, i.e. $\omega = \omega_n$) in Eq. 4.54

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}}$$

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi)^2}}{\sqrt{1 + (2\xi)^2}} = \frac{\sqrt{1 + (2\xi)^2}}{(2\xi)^2}, \text{ at } \xi = 0 \ \frac{F_t}{F_0} = \infty$$

i.e. at resonance and at damping ratio = 0, $\frac{F_t}{F_0} = \infty$

Force transmitted to the foundation with elastic supports = ∞, F_t can be brought down to nominal value by introducing damping into the system.

Case (iii) When $r >> 1$ ($\omega >> \omega_n$) in Eq. 4.57

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

i.e. $r^2 >> 1 \quad \therefore \frac{1}{r^2} <<< 1$, i.e. $1/r^2 \approx 0$ and $1/r \approx 0$

$$\frac{F_t}{F_0} = \frac{\sqrt{r^2\left(\frac{1}{r^2} + (2r)^2\right)}}{\sqrt{r^2\left(\left(\frac{1}{r^2} - 1\right)^2 + (2r)^2\right)}}, \quad \frac{F_t}{F_0} = \frac{\sqrt{r^2\left(\frac{1}{r^2} + (2\xi)^2\right)}}{\sqrt{r^2\left(\left(\frac{1}{r^2} - 1\right)^2 + (2\xi)^2\right)}}$$

$$\frac{F_t}{F_0} = \frac{2\xi}{\sqrt{1 + (2\xi)^2}} \text{ for } \xi << 1, \xi^2 \approx 0 \text{ and } \xi \approx 0.$$

$\therefore$ for $\quad \xi << 1, \frac{F_t}{F_0} \approx 0$; for $\xi >> 1 \; \frac{F_t}{F_0} \approx 1$

2. How to draw vector diagram and force polygon

$$F_0 \sin \omega t - kX \sin (\omega t - \phi) - cX\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + m\omega^2 X \sin (\omega t - \phi) = 0 \quad \text{...(Eq. 4.49)}$$

(a) Vector diagram Draw an ordinate '$-xox$' and '$-yoy$' as shown in Fig. 4.17(a). From point 'o' draw an inclined line 'F_0' at an angle of sin ωt to a suitable scale, from reference line 'ox' indicating impressed force(F_0 sin ωt). Again from point 'o' draw an inclined line $m\omega^2 X$ at an angle of sin $(\omega t - \phi)$ with suitable scale from reference line ox, indicating inertia force $[m\omega^2 X \sin (\omega t - \phi)]$. On an inclined line, $o - m\omega^2 x$ sin $(\omega t - \phi)$, measure an angle 90° ($\pi/2$) in the clockwise direction. Draw a line equal to $(cX\omega)$ indicating damping force with suitable $[cX\omega \sin (\omega t - \phi + 90°)]$ scale as shown in Fig. 4.17(a). A spring force $(-kX)$ is always opposite to the displacement, i.e. $m\omega^2 x$ at an angle of $(\omega t - \phi)$ produce in the opposite direction with suitable scale indicating the spring force $[kX \sin (\omega t - \phi)]$ (see Eq. 4.49).

(b) Force polygon Draw a horizontal line and mark a point 'O' on the line. From point 'O', draw a parallel line 'OA' parallel to the 'OF_0' in Fig. 4.17(a) with suitable scale. From point 'A', draw a parallel line 'AB' parallel to '$O - kX$' with a suitable scale. From point 'B', draw a parallel line 'BC' parallel to '$O - CX\omega$' with suitable scale. From point 'C' draw a parallel line 'CO' parallel to '$O - m\omega^2 X$'. From point 'O', draw a parallel line 'OD' parallel to 'BC', angle 'ABC' = 90° as shown in Fig. 4.17(b). It is clear that the spring force is perpendicular to damping force and damping force is perpendicular to inertia force. Join 'AC' representing F_t as shown in Fig. 4.17(c).

From vector diagram, we can observe that the

(a) spring force is always opposite to the displacement
(b) damping force lags the displacement by 90°
(c) inertia force is in phase with the displacement

In Fig. 4.17(b), forced diagram from the right-angled triangle OAB

$$OA^2 = OB^2 + BA^2, \text{ where } OA = F_0,\ CD = OB = cX\omega,$$

$$BA = kX - mX\omega^2,\ F_0{}^2 = X^2[(k - m\omega^2)^2 + (c\omega)^2].$$

4.11 SPECIAL CASE, WHEN $r = \sqrt{2}$

Damping is useful in isolating forces only when the frequency ratio is less than $\sqrt{2}$.

From Eq. 4.19, $\dfrac{F_t}{F_0} = \dfrac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}}$ put $r = \sqrt{2}$

$$= \frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi\sqrt{2})^2}}{\sqrt{(1 - \sqrt{2^2})^2 + (2\xi\sqrt{2})^2}} = \frac{\sqrt{1 + 8\xi^2}}{\sqrt{1 + 8\xi^2}} = 1.$$

This is independent of 'ξ'.

For all the values 'ξ', all the curves will pass through $\dfrac{F_t}{F_0} = 1$when $r = \sqrt{2} = \dfrac{\omega}{\omega_n}$.

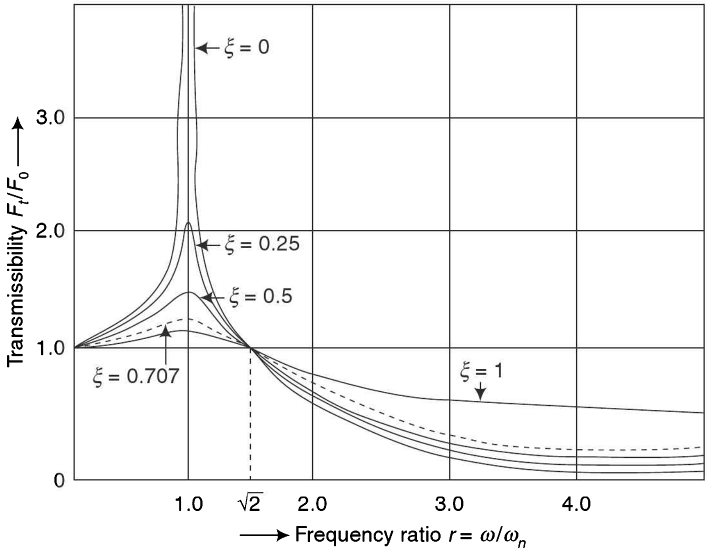


Fig. 4.18 *Frequency ratio versus transmissibility*

From the above plot, it is clear that the force transmitted to the foundation through the elastic supports is greater than the force transmitted through rigid supports for all values of $r < \sqrt{2}$.

In the region $r < \sqrt{2}$, $\dfrac{F_t}{F_0}$ value can be reduced only by increasing damping ratio 'ξ' i.e. by providing dampers.

But for all values of 'ξ' in the region $r > \sqrt{2}$, the force transmitted to the foundation through elastic supports is less than the force transmitted through rigid supports.

Thus vibration isolation is possible only in the range of $r > \sqrt{2}$.

These fact leads us to the conclusion that

(i) To provide vibration isolation when frequency ratio is less than $\sqrt{2}$, dampers are necessary

(ii) To provide vibration isolation when frequency ratio 'r' is greater than $\sqrt{2}$, dampers need not be provided (or should have a less value)

EXAMPLE 4.19

A machine weighing 750 N is mounted on springs of 1200 k N/m stiffness with an assumed damping factor of $\xi = 0.2$. A piston within the machine weighing 20 N has a reciprocating motion with a stroke of 0.075 m and a speed of 3000 rev/min. Assuming the motion of the piston to be a harmonic, determine (i) the amplitude of motion of the machine, (ii) Its phase angle with respect to the exciting force, (iii) the transmissibility and the force transmitted to the foundation, (iv) the phase angle of the transmitted force with respect to the exciting force, and (v) has vibration isolation achieved, if so how?

Solution Refer Fig. p-4.19. $W = 750$ N

$\therefore M = W/g = 76.45$ kg, $w = 20$ N

$\therefore m = w/g = 2.04$ kg.

$N = 3000$ rev/min, $\xi = 0.2$, Stroke $= 0.075$ m

$\therefore e = 0.075 / 2 = 0.0375$ m, $k = 1200\ k$ N/m $= 12 \times 10^5$ N/m.

(i) The amplitude of motion of the machine is given by

$$\frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}},\ X = \frac{mer^2}{M\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad \text{...(a),} \qquad \text{where } r = \frac{\omega}{\omega_n}.$$

The forcing frequency $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 3000}{60} = 100\pi$ rad/s

The natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{12 \times 10^5}{76.45}} = 125.28$ rad/s

The frequency ratio $r = \dfrac{\omega}{\omega_n} = \dfrac{100\pi}{125.28},\ r = 2.51$

Using all these values in Eq. (a),

$$X = \frac{2.04 \times 0.0375 \times (2.5)^2}{\sqrt{[1-(0.71)^2]^2 + (2 \times 0.2 \times 2.51)^2}} \times \frac{1}{76.45},\ X = 1.17 \times 10^{-3} \text{ m},\ X = 1.17 \text{ mm}.$$

(ii) The phase angle 'ϕ' w.r.t. the exciting force is given by

$$\phi = \tan^{-1}\left[\frac{2\xi r}{1-r^2}\right] = \tan^{-1}\left[\frac{2 \times 0.2 \times 2.51}{1-(2.571)^2}\right] = -10°43^1$$

or $\qquad \phi = 180° - 10°\,43',\ \phi = 169°16'$

(iii) The transmissibility ratio is given by

$$\text{TR} = \frac{F_t}{F_0} = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{\sqrt{1+(2\times 0.2\times 2.51)^2}}{\sqrt{(1-(2.51)^2)^2+(2\times 0.2\times 2.51)^2}},$$

TR = 0.263 or 26.3%

Force transmitted to the foundation $F_t = F_0 \times TR$, where $F_0 = me\omega^2$.

$$F_t = me\omega^2 \times TR, = \frac{20}{9.81} \times 0.0375 \times (100\pi)^2 \times 0.263,\ F_t = 1984.48 \text{ N}.$$

(iv) The phase angle of the transmitted force to the exciting force,

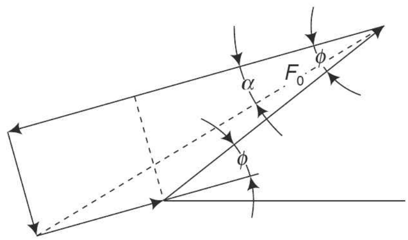


Fig. p-4.19 Phase angle

i.e. $\phi - \alpha$, where $= \tan^{-1}\dfrac{cx\omega}{kx}$ $\alpha = \tan^{-1}(2\xi r)$.

$$\alpha = \tan^{-1}[2\times 0.2\times 2.51],\ \alpha = 45°6',\ \phi - \alpha = 169°16' - 45°6',\ \phi - \alpha = 124°106'.$$

(v) Since the value of frequency ratio 'r' = 2.51 is greater than $\sqrt{2}$, vibration isolation is achieved and also only 26.3% of vibrating force 'F_0' is transmitted to the foundation.

EXAMPLE 4.20

Determine the power required to vibrate a spring–mass-damper system with an amplitude of 15 mm and at a frequency of 100 Hz. The system has a damping factor of 0.05 and a damped natural frequency of 22 Hz. The mass of the system is 0.5 kg.

Solution $X = 15 \text{ mm} \times 10^{-3}$ m, $f = 100$ Hz, $\xi = 0.05$, $f_d = 22$ Hz, $m = 0.5$ kg,

$\omega = 2\pi f$, $\omega = 2\pi \times 100 = 200\,\pi$ rad/s, $f_d = f_n\sqrt{1-\xi^2}$ or $f_n = \dfrac{f_d}{\sqrt{1-\xi^2}} = 22.055$ Hz.

$\omega_n = 2\pi f_n = 2\pi \times 22.055 = 138.576$ rad/s, $c = 2m\omega_n\xi = 2\times 0.5\times 138.576\times 0.05$ or

$c = 6.928$ N-s/m, $F_d = c\omega X = 6.928 \times 200\pi\,(15\times 10^{-3}) = 65.3026$ N, $T_d = c\omega X^2 = 0.9795$ N-m.

$$P = \frac{2\pi N}{60},\ N = \frac{60\omega}{2\pi},\ N = \frac{60\times 200\pi}{2\pi} = 6000 \text{ rev/min}.$$

The power is given by $P = \dfrac{2\pi N T_d}{60}$, $P = \dfrac{2\pi \times 6000 \times 0.9795}{60} = 615.5$ watts

EXAMPLE 4.21

A reciprocating pump of 200 kg is driven through a belt by an electric motor at 3000 rev/min. The pump is mounted on isolators with total stiffness of 5 M N/m and damping of 3.125 kN-s/m. Determine the vibrating amplitude of the pump at the running speed due to fundamental harmonic force of excitation, 1 kN. Also determine the maximum vibratory amplitude when the pump is switched 'ON' and the motor speed passes through resonant condition.

Solution $m = 200$ kg, $N = 3000$ rev/s, $k = 5$ MN/m $= 5 \times 10^6$ N/mm,

$c = 3.125$ kN-s/m $= 3.125 \times 1000$ N-s/mm. $F_0 = 1$ kN $= 1000$ N

Natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{5 \times 10^6}{200}} = 158.11$ rad/s

Damping factor $\xi = \dfrac{c}{c_c} = \dfrac{3.125 \times 1000}{2\sqrt{km}} = \dfrac{3.125 \times 1000}{2\sqrt{5 \times 10^6 \times 200}} = 0.049.$

The frequency ratio $r = \dfrac{\omega}{\omega_n} \cdot \omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 3000}{60} = 314$ rad/s,

$r = \dfrac{\omega}{\omega_n} = \dfrac{314}{158.11} = 1.987.$

Dynamic amplitude of the reciprocating pump is given by

$$X = \frac{\frac{F_0}{k}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{\frac{10000}{5 \times 10^6}}{\sqrt{(1 - 1.987^2)^2 + (2 \times 0.49 \times 1.987)^2}} = 6.769 \times 10^{-5} \text{ m or}$$

$X = 0.067$ mm.

At resonance $r = 1$, $X = X_r = \dfrac{\frac{F_0}{k}}{\sqrt{2\xi}} = \dfrac{\frac{10000}{5 \times 10^6}}{\sqrt{2 \times 0.069}}$, $X_r = 2.04 \times 10^{-3}$ m

or $X_r = 2.04$ mm.

4.11 MOTION ISOLATION (BASE EXCITATION)

As we know that earlier, vibrations are an undesirable phenomenon and hence they must be either eliminated or reduced. Vibration leads to excessive amplitude, stresses in parts of shafts, poles, chimneys, machineries, towers and even transmission line cables; complete elimination is impossible but their effect can be reduced. This can be done by proper choices of spring, proper balancing of mass and dashpot arrangements either to increase the natural frequency or to reduce the transmission of displacements and forces to the supports.

There are two basic requirements for vibration isolation.

(i) There should be no rigid contact between the vibrating unit such as shafts, machines, chimneys and the base. Rigid contact leads to higher transmission which is undesirable.

(ii) It is necessary to ensure that the isolation device remains together during operation, even if it fails. It is required to maintain the machinery in the safe position with respect to the support.

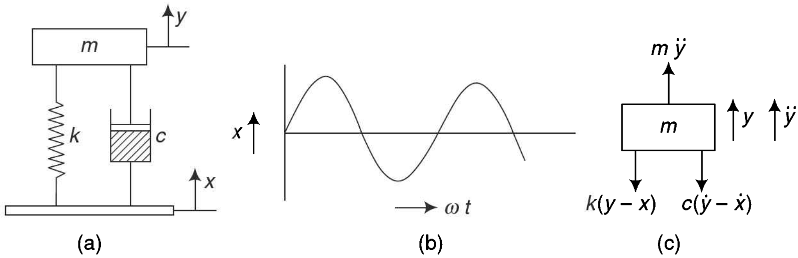


Fig. 4.19 *Vibration isolation*

Now at any instant, give a displacement to the system as shown in Fig. 4.19(a).

The equation of motion of the system with 'y' as the independent coordinate of mass 'm' and 'x' as the independent coordinate for the support, and applying Newton's second law of motion to mass 'm' is

$$k(y - x) + c(\dot{y} - \dot{x}) = -m\ddot{y}, \quad m\ddot{y} + c\dot{y} + ky - c\dot{x} - kx = 0.$$

$$m\ddot{y} + c\dot{y} + ky = c\dot{x} + kx \quad \text{...4.57}$$

Since we have assumed 'x' as sinusoidal, let $x = X \sin \omega t$

$$\dot{x} = X\omega \cos \omega t = X\omega \sin\left(\frac{\pi}{2} + \omega t\right).$$

Using these values in Eq. 4.57, we have

$$m\ddot{y} + c\dot{y} + ky = cX\omega \quad \sin\left(\frac{\pi}{2} + \omega t\right) + kX \sin \omega t \quad \text{... 4.58}$$

$\therefore$ right-hand side $= k\,X \sin \omega t + cX\omega \sin\left(\frac{\pi}{2} + \omega t\right)$

Representing right-hand side vectorially,

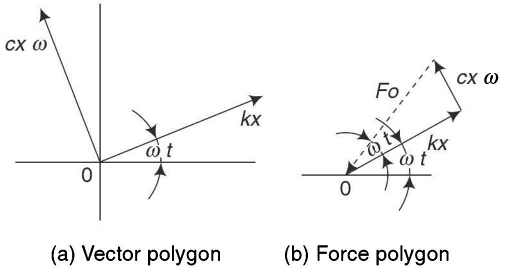


Fig. 4.20 *Force and vector polygon*

Let 'F_0' be the resultant vector in Fig. 4.20(b).

$\therefore \qquad F_0^2 = (kX)^2 + (cX\omega)^2$

$$F_0^2 = (kX)^2 + \left[1 + \frac{(cX\omega)^2}{(kX)^2}\right] = (kX)^2\left[1 + \left(\frac{c}{k}\,\omega\right)^2\right] \text{ where } \frac{c}{k} = \frac{2\xi}{\omega_n}.$$

$$\therefore \qquad F_0^2 = (kX)^2 \left[1 + \left(\frac{2\xi\omega}{\omega_n}\right)^2\right], \text{ let the frequency ratio } \frac{\omega}{\omega_n} = r.$$

$$\therefore \qquad F_0^2 = (kX)^2\,[1 + (2\xi r)^2], \quad F_0 = kX\sqrt{1 + (2\xi r)^2}$$

And $\tan \omega t = \dfrac{cX\omega}{kX} = \dfrac{c}{k}\,\omega, \quad \tan \omega t = 2\xi\,\dfrac{\omega}{\omega_n}, \quad \therefore\ \omega t = \tan^{-1}(2\xi r)$

$\therefore$ right-hand side will reduce to $= F_0 \sin \omega t$.

Therefore, Eq. 4.58 becomes

$$m\ddot{y} + c\dot{y} + ky = F_0 \sin \omega t \qquad \text{...4.59}$$

This is a second-order nonhomogeneous differential equation of motion, whose solution is given by

$$y = y_c + y_p$$

where y_c = Complementary function (transient response)

y_p = Particular integral (steady-state response already discussed in Article 4.2, case 'b')

Considering the steady-state response we find the particular integral 'y_p'

To determine 'y_p'

Let $\qquad y_p = y = y\sin(\omega t - \phi),\ \dot{y} = y\omega\cos(\omega t - \phi) = y\omega\sin\left(\dfrac{\pi}{2} + \omega t - \phi\right).$

$\ddot{y} = y\omega^2 \sin(\omega t - \phi)$ Using these values in Eq. 4.59,

$$-my\omega^2 \sin(\omega t - \phi) + c\dot{y}\,\omega\sin\left(\frac{\pi}{2} + \omega t - \phi\right) + ky\sin(\omega t - \phi) = F_0 \sin \omega t.$$

Rearranging the above terms,

$$my\omega^2 X\sin(\omega t - \phi + \pi) + cX\omega\sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX\sin(\omega t - \phi) - F_0\sin\omega t = 0 \qquad \text{...4.60}$$

These forces can be vectorially represented as follows:

In Fig. 4.21(b), from right-angled triangle *OAB*

$$OB^2 = AO^2 + AB^2,\ OB = F_0,\ OA = ky - my\,\omega^2,\ AB = cy\,\omega$$

$$\therefore \qquad F_0^2 = (ky - my\omega^2) + (cy\omega)^2 \quad \text{or } F_0^2 = (ky)^2\left[\left(1 - \frac{m}{k}\,\omega^2\right)^2 + \left(\frac{c}{k}\,\omega\right)^2\right] \qquad \text{... 4.61}$$

where $\dfrac{m}{k} = \dfrac{1}{\omega_n^2}$. $\dfrac{c}{k} = \dfrac{2\xi}{\omega_n}$ and $\dfrac{\omega}{\omega_n} = r$. Using these values in Eq. 4.61, we have

$$\therefore \qquad F_0^2 = (ky)^2\,[(1 - r^2)^2 + (2\xi r)^2]$$

$$F_0 = ky\sqrt{[1 - r^2]^2 + (2\xi r)^2},\ y = \frac{F_0 k}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

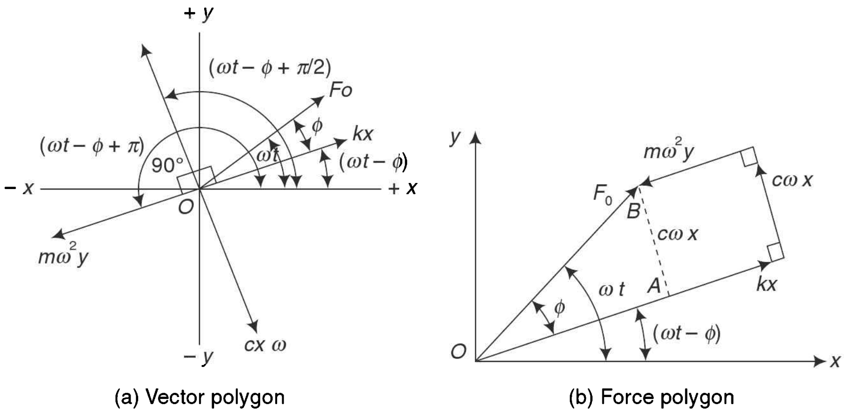


Fig. 4.21 *Force and vector polygon*

Case (i) At $r = 0, \dfrac{y}{X} = 1$

Using these value of 'F_0',

$$y = \frac{X\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad \text{or} \quad \frac{y}{X} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}.$$

Case (ii) At $r = 0$ (resonance), $\dfrac{y}{X} = \sqrt{1 + \dfrac{(2\xi)^2}{2\xi}}$ at $\xi = 0$ $\dfrac{y}{X} = \infty$.

Case (iii) For $r >> 1, \dfrac{y}{X} \approx 0.$

Special Case At $r = \sqrt{2}, \dfrac{y}{X} = 1$ (Independent of ξ).

For motion isolation, $y/X \to 0$

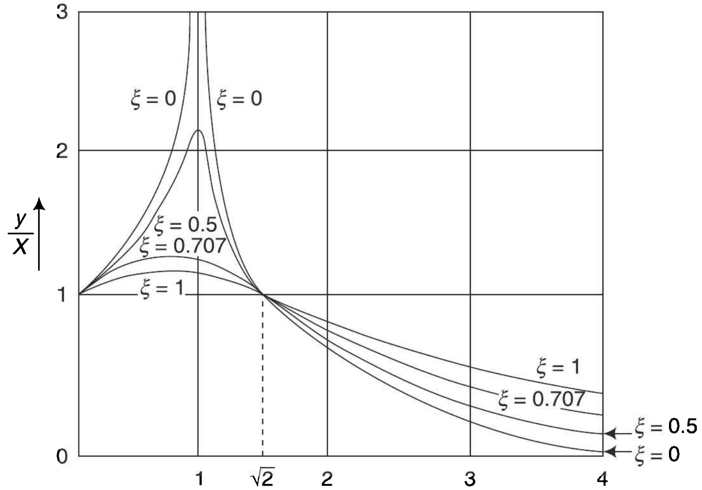


Fig. 4.22 *Frequency ratio*

From the above plot Fig. 4.22, it is clear that the displacement of the mass (y) is greater than the displacement of the base for all values of $r < \sqrt{2}$.

In this region $r < \sqrt{2}$, 'y/x' can be reduced by introducing damping into the system (i.e. by increasing 'ξ').

But for all values of $r < \sqrt{2}$, $\ddot{y}$ is always less than 'X'. Thus motion isolation is only possible in the region $r > \sqrt{2}$. From these facts, it can be concluded that

(i) To provide motion isolation when the frequency ratio is less than $\sqrt{2}$, dampers are necessary

(ii) To provide motion isolation when the frequency ratio is greater than $\sqrt{2}$, dampers need not be provided (or should have less value).

EXAMPLE 4.22

The body of car has a mass of 1500 kg and is mounted on 4 equal springs, which deflect through 225 mm under the weight of the body. The total damping coefficient of 4 shock absorbers is 4.6 N at a velocity of 1 cm/s. The car is placed with all four wheels on the platform which is moved up and down at resonance speed with amplitude of 25 mm. Find the amplitude of the car body on its spring assuming the centre gravity of the car to be at the centre of the wheel base.

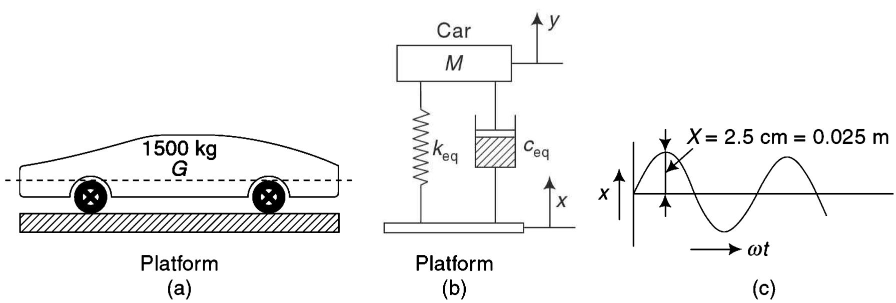


Fig. p-4.22 *Car-spring system*

Solution $w = Mg = 1500$ kg, $\Delta_{st} = 0.225$ m, $F_d = 4.6$ N, $v = 0.01$ m/s,

$\omega = \omega_n$ (resonance)

$\therefore$ for base excitation is given by $\dfrac{y}{X} = \dfrac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$, where $r = \dfrac{\omega}{\omega_n}$.

For equilibrium, $k_{eq}\,\Delta_{st} = Mg \quad \therefore \dfrac{k_{eq}}{M} = \dfrac{g}{\Delta_{st}}$

$$\therefore \omega_n = \sqrt{\frac{g}{\Delta_{st}}} = \sqrt{\frac{9.81}{0.225}}, = 6.60 \text{ rad/s}.$$

And $$k_{eq} = \frac{Mg}{\Delta_{st}} = \frac{1500}{0.225} = 6666.67 \text{ N/m}.$$

The each spring stiffness, $k = \frac{k_{eq}}{4} = \frac{6666.67}{4} = 1666.67$ N/m

For viscous damper, $F_d = c_{eq}v \;\therefore\; c_{eq} = \frac{F_d}{v} = \frac{4.6}{0.01} = 460$ N-s/m, $c_{eq} = 460$ N-s/m.

Damping ratio $\xi = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{2M\omega_n} = \frac{460}{2 \times 1500 \times 6.60} = 0.23$ at resonance $r = 1$ (given)

$$\therefore \; \frac{y}{X} = \frac{\sqrt{1 + (2\xi)^2}}{2\xi},\; y = \frac{0.025\sqrt{1 + (2 \times 0.23)^2}}{2 \times 0.23} = 5.98 \times 10^{-2}\text{m},\; y = 59.8 \text{ mm.}$$

EXAMPLE 4.23

A machine of 1000 kg mass is acted upon by an external force of 2450 N at a frequency of 1500 rev/min. To reduce the effect, vibration isolators of rubber having static deflection of 2 mm under the machine load and on estimated damping factor 0.2 are used. Determine

(i) force transmitted to the foundation, (ii) amplitude of vibration of machine, and (iii) the phase lag.

Solution $M = 1000$ kg, $F_0 = 2450$ N, $N = 1500$ rev/min.

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 1500}{60} = 157.07 \text{ rad/s,}$$

$$\delta = 2 \text{ mm} = 2 \times 10^{-3},\; \xi = 0.2,\; F_{tr} = ?,\; X = ?,\; \phi = ?$$

(i) The force transmitted to the foundation is given by $F_{tr} = \sqrt{(kX)^2 + (c\omega X)^2}$,

$$\therefore \qquad \omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{2 \times 10^{-3}}} = 70.03 \text{ rad/s} r = \frac{\omega}{\omega_n} = \frac{157.07}{70.03} = 2.242.$$

The transmissibility ratio is given by

$$\text{TR} = \frac{F_{tr}}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{\sqrt{1 + (2 \times 0.2 \times 2.242)^2}}{\sqrt{(1 - 2.242^2)^2 + (2 \times 0.2 \times 2.242)^2}} = 0.334$$

$$\text{TR} = \frac{F_{tr}}{F_0},\; F_{tr} = F_0 \times \text{TR},\; F_{tr} = 2450 \times 0.334 = 819.6 \text{ N}$$

(ii) The steady state amplitude is given by

$$\therefore \qquad X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\left(\frac{\omega}{\omega_n}\right)\right)^2}},\; \text{mg} = k\delta\; k = 4.96 \times 10^6 \text{ N /m.}$$

$$X = \frac{\frac{2450}{4.9 \times 10^6}}{\sqrt{\{1 - (2.24)^2\}^2 + (2 \times 0.2 \times 2.24)^2}},\; \omega_n = \sqrt{\frac{k}{m}},\; X = 1.22 \times 10^{-4} \text{ m.}$$

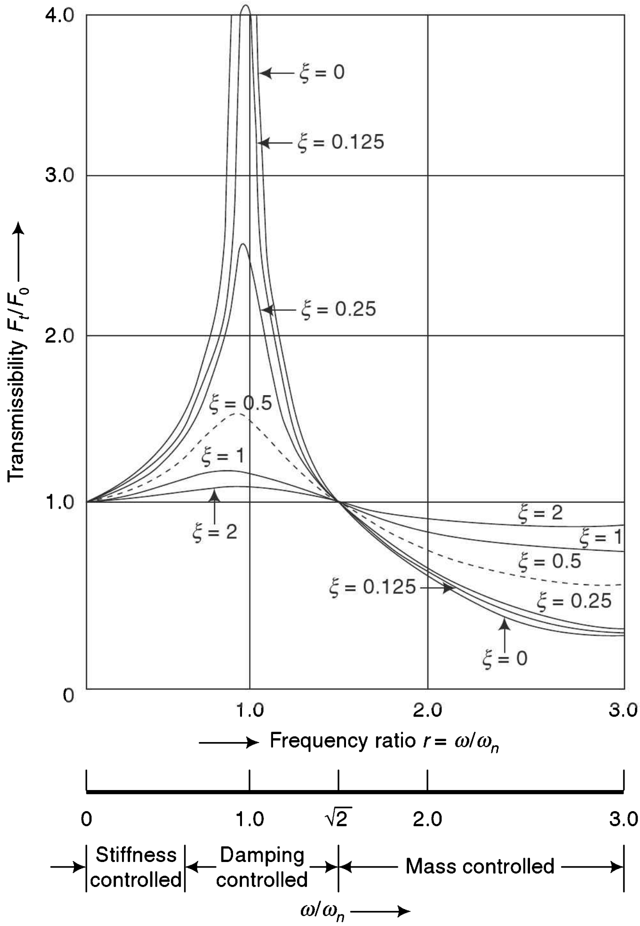
(iii) Phase angle is given by $\phi = \tan^{-1}\left\{\dfrac{2\xi r}{1 - r^2}\right\}$,

$$\phi = \tan^{-1}\left\{\frac{2 \times 0.2 \times 2.24}{1 - (2.24)^2}\right\} = -12.57 + 180 \quad \therefore \phi = 167.4°$$

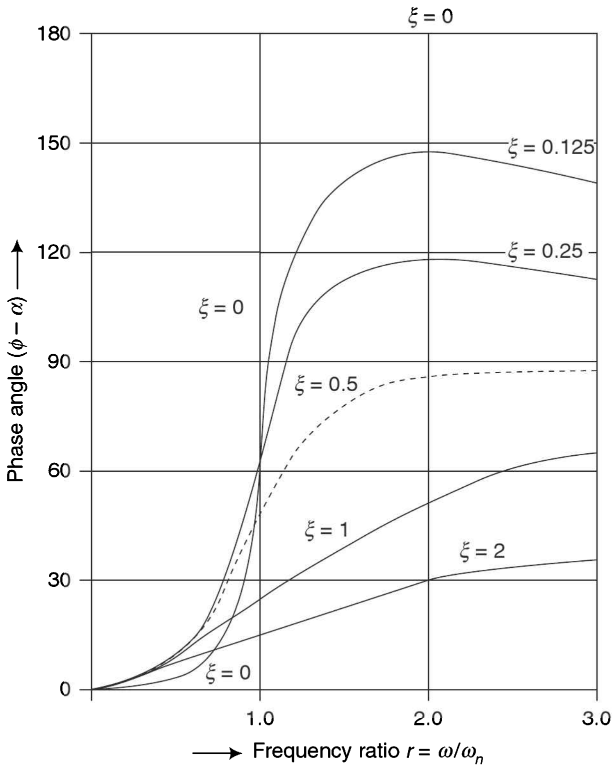
4.11 MOTION TRANSMISSIBILITY

We know that the forced vibration due to excitation of the support. In case of absolute motion the dimensionless form of equation is

$$\frac{X}{y} = \frac{\sqrt{1 + \left(\dfrac{2\xi\,\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left(\dfrac{2\xi\,\omega}{\omega_n}\right)^2}} \qquad \text{...4.62}$$



(a) Motion transmissibility versus frequency ratio for various amount of damping



(b)

Fig. 4.23 *Phase angle versus frequency ratio for various amount of damping*

'Motion transmissibility' and the phase angle is phase lag of the absolute motion of the body from the exciting motion. Ratio for various amount of damping is as shown in Fig. 4.23(a) and (b) and the phase lag is given by

$$(\alpha - \beta) = \tan^{-1}\left(\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)}\right) - \tan^{-1}\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right] \qquad ...4.63$$

will give the ratio of absolute amplitude of the mass to the base excitation amplitudes.

The above two equations are similar to the equations 4.17 and 4.18 in **Sec. 4.5** and is given by

$$\text{TR} = \frac{F_{tr}}{F_0} = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_2}\right)^2}} \qquad ...(\text{Eq. } 4.17)$$

Phase lag is given by $(\alpha - \beta) = \tan^{-1}\left(\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)}\right) - \tan^{-1}\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]$ ...(Eq. 4.18)

The respective plots show transmissibility versus frequency ratio for various amount of damping, also phase angle versus frequency ratio.

EXAMPLE 4.24

A machine of 500 kg mass is acted upon by an external force of 2000 N at a frequency of 1500 rev/min. To reduce the effects of vibration, an isolator of rubber having a static deflection of 2 mm under the machine load and an estimated damping factor $\xi = 0.2$ are used.

Determine (i) the force transmitted to the foundation, and (ii) the amplitude of vibration of machine.

Solution $m = 500$ kg, $F_0 = 2000$ N, $N = 1500$ rev/min, $\delta = 2$ mm, $\xi = 0.2$.

$$\omega = \frac{2\pi N}{60} = 2\pi \times \frac{1500}{60} = 157.1 \text{ rad/s}, \ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{981}{2}} = 70 \text{ rad/s.}$$

The frequency ratio $r = \dfrac{\omega}{\omega_n} = \dfrac{157.1}{70} = 2.3,$

The transmissibility ratio is given by

$$T_R = \frac{F_{TR}}{F_0} = 0.33, \ F_{TR} = T_R \times F_0 = 0.33 \times 2000 = 645.7 \text{ N.}$$

$$X = \frac{X_{st}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}},$$

$$X_{st} = \frac{F_o}{k}$$

$$X = \frac{\frac{F_0}{k}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{\frac{F_0}{k}}{\sqrt{(1 - 2.3^2)^2 + (2 \times 0.2 \times 2.3)^2}} = 02398 \times \frac{F_0}{k}.$$

(i) $\omega_n = \sqrt{\dfrac{k}{m}}, \ 70 = \sqrt{\dfrac{k}{500}}, \ k = 2450$ N/mm

(ii) $X_{st} = \dfrac{F_o}{k} = \dfrac{0.2398 \times F_0}{k}$, $k = 2450$ N/mm, the amplitude of vibration is $X = 0.2398$ $\left(\dfrac{2000}{2450}\right) = 0.196$ mm.

4.14 COMMERCIAL VIBRATION ISOLATION MATERIALS

There are many types of commercial isolators used practically to isolate vibrations. The main requirements of isolators are as follows:

(i) There should be no solid connection between the unit and the supporting structure through which sound may be conducted.

(ii) There should be provision to hold the isolator together when damping materials fails. The common materials used for vibration isolation are rubber, cork, felt, metal springs, etc.

1. Rubber Many commercial isolators are made up of rubber and are generally loaded in shear rather than in compression, to ensure greater flexibility. The stress in rubber is generally kept low. The properties vary widely with the load, the temperature, the shape of the piece and the impressed frequency. For higher temperature, the stress must be reduced to avoid excessive creep and deterioration. Rubber is generally not satisfactory for temperatures above 125 to 150°C.

Oil and gasoline attack rubber, and isolators made of it cannot be used in the pressure of these fluids. Rubber has low transmissibility and low modulus of elasticity. This makes it particularly good for light loads and high-frequency oscillation.

2. Cork Cork is generally used in compression or compression and shear. It is not perfectly elastic, being more flexible at high loads and its properties change with the frequency. In this context, it is similar to rubber. Generally, it is placed beneath a large concrete block to obtain good results.

3. Felt Felt is used in the form of small compression pads, which are placed under concrete or steel bases. It has a high damping factor and this is particularly satisfactory for low-frequency ratios, i.e. below $\sqrt{2}$.

4. Metal springs Helical springs have very little damping. There is more damping with leaf springs, because of friction between the leaves. But in either type, the damping is commonly considered to be negligible. They have a high sound transmissibility. This may be reduced by mounting the springs on pads of felt, cork or rubber. Metal springs are not affected by the presence of oil or water and they are quiet consistent in their properties. Metal springs are particularly satisfactory for frequency ratios, grater than $\sqrt{2}$, since damping factor 'ξ' is close to zero. For a frequency ratio less than $\sqrt{2}$, this is not suitable.

REVIEW QUESTIONS

(1) Derive the solution (both complementary 'x_c' and particular integral 'x_p') for a differential equation of an underdamped single-degree-freedom system with harmonic excitation.

(2) Derive an expression for the magnification factor and discuss its variation with frequency ratio.

(3) Derive an expression for the transmissibility and transmitted force for a spring-mass-damper system subjected to external excitation.

(4) For a spring-mass-damper system subjected to a harmonic force '$F_0 \sin \omega t$', sketch the plot of magnification factor v/s frequency ratio for different amounts of damping and describe the characteristics of the curves.

(5) The mass of spring-mass-damper system is excited by a harmonic excitation '$F_0 \sin \omega t$'. Derive an expression for magnification factor or amplitude ratio of forced vibration.

(6) Define the 'transmissibility'. Derive the expression for 'motion transmissibility'.

(7) What is transmissibility? At what frequency ratio is damping useful in isolating forces.

(8) What is magnification factor? Derive an expression for the same and discuss its variation with frequency ratio.

(9) Derive an expression for the transmissibility ratio and the phase angle for the transmitted force.

(10) Prove that damping is useful in isolating forces only when the frequency ratio is less than $\sqrt{2}$.

(11) Explain the term 'vibration Isolation' and 'transmissibility ratio'. Show that the vibration Isolation is possible only when the frequency ratio is more then $\sqrt{2}$.

(12) Show that for a spring-mass-damper system, the peak amplitude occurs at a frequency ratio (r) given by the expression, $r = \dfrac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2}$, when the system is excited by a harmonic force.

(13) Explain why a constant force on the vibrating mass has no effect on the steady-state vibration.

(14) How does the force transmitted to the base change as the speed of the machine increases? Explain.

(15) Explain the following with proper equations and graphs: (i) Magnification factor (ii) Transmissibility ratio (iii) Rotating and reciprocating unbalance (iv) Motion isolation (base excitation).

(16) What are the materials used in vibration isolation? Explain the vibration isolation materials.

(17) Write notes on commercial isolator materials.

PROBLEMS FOR PRACTICE

(1) A block of 35 kg mass is connected to a support through a spring of stiffness 1.4×10^6 N/m in parallel with a dashpot of 1.8×10^3 N-s /m damping coefficient. The support is given a harmonic displacement of 10 mm amplitude at a frequency of 35 Hz. What is the steady-state amplitude of the absolute displacement of the block?

Ans. $\omega_n = 200$ rad/s, $\xi = 0.129$, $X = 29.4$ mm.

(2) A machine of 110 kg mass is mounted on elastic foundation of spring stiffness $k = 2 \times 10^6$ N/m when operating at 150 rad/s. The machine is subjected to harmonic force of 1500 N. The steady-state amplitude of the machine is measured as 1.9 mm. What is the damping ratio of the foundation?

Ans. $\omega_n = 134.8$ rad/s, MF = 2.53, $r = 1.113$, $\xi = 0.142$.

(3) An electric motor of 40 kg mass is running at 50 rev/min. The motor is supported on a spring of 7 kN/m and a dashpot which offers a resistance of 500 N at 0.25 m/s. The unbalance of the rotor is equivalent to a mass of 1 kg located 4 cm from the axis of rotation. Knowing that the motor is constrained to move vertically, determine (i) damping factor (ii) amplitude of vibration, and (iii) phase angle.

Ans. $\xi = 1.89$, Amplitude of vibration, $X = 7.476 \times 10^{-4}$ m. Phase angle $\phi = -45.6°$.

(4) A machine of 500 kg mass is supported on spring of 10^6 N/m stiffness. If the machine has a rotating unbalance of 0.25 kg-m, determine (i) the force transmitted to the floor at 1200 rev/min, and (ii) the dynamic amplitude at this speed.

(5) When a free-vibration test is run on the system of Fig. p.p-4.5 as shown, the ratio of amplitudes on successive cycles is 2.5 to 1. Determine the response of the machine due to a rotating unbalance of 0.25 kg-m magnitude. When the machine operates at 2000

rev/min and the damping is assumed to be viscous, take $E = 200 \times 10^9$ N/m^2, I = 4.5 × 10^{-6} m^4.

Ans. $k = 5.27 \times 10^6$ N/m, $\omega_n = 250.3$ rad/s, $\delta = 0.916$, $\xi = 0.144$, $X = 7.02$ mm.

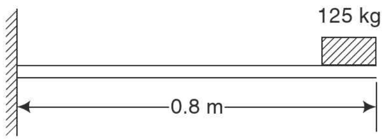


Fig. p.p-4.5 *Cantilever beam with mass on free end*

(6) A machine of 110 kg mass is mounted on elastic foundation of stiffness $k = 2 \times 10^6$ N/m. When operating at 150 rad/s, the machine is subjected to harmonic force of 150 N. The steady-state amplitude of the machine is measured as 1.9 mm. What is the damping ratio of the foundation?

Ans. $\omega_n = 138.8$ rad/s, MF = 2.53, $\xi = 0.142 < 1$ (underdamping).

(7) The damped natural frequency is 9.78 cps. During forced vibration of 9.57 cps, peak amplitude was observed. Determine damping ratio and natural frequency of the system.

Ans. $\xi = 0.1 < 1$ $\omega_n = 9.5$ cps.

(8) A vibrating 150 kg mass is supported on spring stiffness of 2M N/m and has a rotating unbalance of 600 N. If the damping factor is 0.3, determine (i) the amplitude of vibrations, and (ii) the transmissibility at an operating speed of 1500 rev/min.

Ans. $X = 2.54 \times 10^{-4}$ m, $T_R = 1.09$.

(9) A periodic torque of 0.57 N-m amplitude of 3.99 rad/s frequency is acting up a flywheel suspended to oscillate in rotary motion. The MI of the flywheel is 0.1 kg m^2, the spring stiffness is 1 Nm/rad, damping coefficient is 0.389 Ns/m. Determine (i) maximum angular displacement, (ii) maximum couple applied to the dashpot, and (iii) phase angle by which the angular displacement lags the torque.

Ans. (i) $\theta = .4$ rad (ii) Maximum torsional moment of dashpot = 0.6 N-m, $\phi = -69.5°$.

(10) A body of 75 kg mass is suspended by a spring which deflects by 2 cm under the self-weight of the body. It is having a damper with damping ratio of 0.25.It is distributed by a periodic force of amplitude 700 N and frequency which is 0.8 times the natural frequency of the system. Determine the amplitude, damping coefficient, damped natural frequency and the phase angle.

Ans. $c_c = 830.5$ N-s/m., $\omega_n = 17.72$, $\omega_d = 21.5$ rad/s. $\phi = 63.5°$ $x = 1.293 \times 10^{-8}$ m.

OBJECTIVE-TYPE QUESTIONS

1. In a forced damped vibration system
   (a) the spring force vector lags behind the displacement vector by 180°
   (b) the damping force vector lags behind the displacement vector by 90°
   (c) the inertia force vector is in phase with the displacement vector
   (d) all the above statements are true
2. In a forced damped vibration system, the excitation force lags behind the displacement vector by

(a) $\phi = \tan^{-1}\left(\dfrac{c\omega^2}{k - m\omega^2}\right)$

(b) $\phi = \tan^{-1}\left(\dfrac{m\omega^2}{k - c\omega^2}\right)$

(c) $\phi = \tan^{-1}\left(\dfrac{c\omega}{k - m\omega^2}\right)$

(d) $\phi = \tan^{-1}\left(\dfrac{c_t\omega^2}{k_t - J\omega^2}\right)$

3. In a forced damped vibration system the impressed force lags behind the displacement vector by

(a) $\phi = \tan^{-1}\left(\dfrac{r}{1 - r^2}\right)$

(b) $\phi = \tan^{-1}\left(\dfrac{\xi r}{1 - r^2}\right)$

(c) $\phi = \tan^{-1}\left(\dfrac{2\xi r}{1 - r^2}\right)$

(d) $\phi = \left(\dfrac{2\xi r}{1 - r^2}\right)$

4. For damping factor $\tau = 0$ and frequency ratio $\dfrac{\omega}{\omega_n} > \sqrt{2}$, the transmissibility is

(a) $\text{TR} = \left(\dfrac{r}{1 - r^2}\right)$

(b) $\text{TR} = \left(\dfrac{1}{1 - r^2}\right)$

(c) $\text{TR} = \left(\dfrac{r}{r^2 - 1}\right)$

(d) $\text{TR} = \left(\dfrac{r}{1 - r^2}\right)^2$

5. The transmissibility is same for all values of damping factors at frequency ratio ω/ω_n of

(a) 1
(b) 2
(c) $\sqrt{2}$
(d) none of the above

6. Damping is beneficial only when

(a) $\dfrac{\omega}{\omega_n} = 1$ (b) $\dfrac{\omega}{\omega_n} < 1$

(c) $\dfrac{\omega}{\omega_n} > \sqrt{2}$ (d) $\dfrac{\omega}{\omega_n} < \sqrt{2}$

7. For forced damped vibration system, the vibration isolation is possible only when

(a) $\dfrac{\omega}{\omega_n} = 1$ (b) $\dfrac{\omega}{\omega_n} < 1$

(c) $\dfrac{\omega}{\omega_n} > \sqrt{2}$ (d) $\dfrac{\omega}{\omega_n} < \sqrt{2}$

8. In vibration isolation system if $\dfrac{\omega}{\omega_n} > 1$ then the phase difference between the transmitted force and the disturbing force is

(a) 0° (b) 180°
(c) 90° (d) 145°

9. The ratio of the maximum displacement of the forced vibration to the deflection due to the static force is defined as

(a) damping factor
(b) Magnification factor
(c) Logarithmic decrement

10. In a vibration isolation system if $\dfrac{\omega}{\omega_n} > 1$, and less then $\sqrt{2}$ then for all values of the damping factor the transmissibility will be

(a) greater then unity
(b) equal to zero
(c) equal to unity
(d) less then unity

Answers

(1) d	(2) c	(3) c	(4) b	(5) c	(6) c
(7) c	(8) b	(9) b	(10) a		

VIBRATION MEASURING INSTRUMENTS (SEISMIC INSTRUMENTS)

5

5.1 INTRODUCTION

The most important characteristics of vibrating systems are its displacement, velocity and acceleration. The instruments used to measure these characteristic parameters are known as vibration measuring instruments. These instruments are usually the forced damped systems with small amount of damping or zero damping with support motion. They have mass and elastic springs. Basically, there are three types of vibration-measuring instruments, they are

(i) measuring the displacement, (ii) velocity and, (iii) acceleration.

Many measuring instruments consists of spring-mass–damper system as shown in Fig. 5.1.

A device measuring the displacement of the mass relative to the casing. The mass is constrained to move along a given axis. The displacement of the mass relative to the casing is generally measured electrically. Damping may be provided by a viscous fluid inside the casing.

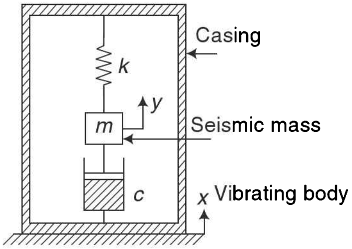


Fig. 5.1 *Spring-mass–damper system*

5.1 SEISMIC INSTRUMENTS

These are the vibration-measuring instruments. Seismic instruments are essentially vibratory systems consisting of the support of the base and mass with spring attached as shown in Fig. 5.1. The support or the base is attached to the body whose motion is to be measured. The relative motion between the maximum and the base, recorded by a rotating drum or some other devices inside the system, will indicate the motion of the body.

5.1 VIBROMETER

It is a device used for measuring the vibration characteristics of machine parts. It is designed for low natural frequency. It is used to measure any of the vibration characteristics and for large values of frequency ratio 'r'.

An accelerometer is used to measure acceleration because its natural frequency is high compared to that of vibration to be measured. Seismographs are used for recording earthquake vibration. In this system, a desired amount of damping is introduced. The relative motion between the seismic mass and the casing is utilised to measure or to find the quantity which we desired.

The simplified form of Fig. 5.1 is as shown in Fig. 5.2(a).

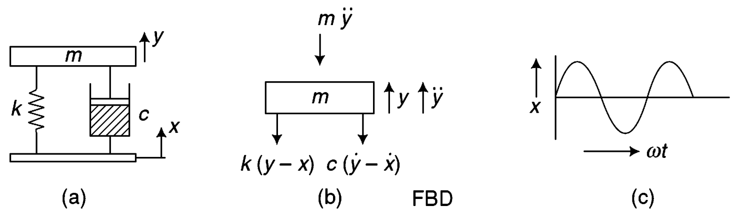


Fig. 5.2 *Seismic instruments*

Applying Newton's second law of motion to mass 'm' in Fig. 5.2(a), the FBD is as shown in Fig. 5.2 (b). $\Sigma F = m\ddot{x}$.

Let $y > x$

$$k(y - x) + c(\dot{y} - \dot{x}) = -m\ddot{y} \quad \text{...5.1}$$

Let $y - x, = z, \dot{y} - \dot{x} = \dot{z}, \ddot{y} - \ddot{x} = \ddot{z}, \quad \ddot{y} = \ddot{z} + \ddot{x}$, where z = Relative amplitudes

Substituting these values in Eq. 5.1,

$$kz + c\dot{z} = -m(\ddot{z} + \ddot{x})$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x} \quad \text{...5.2}$$

Since we have assumed 'x' as a simple harmonic motion as shown in Fig. 5.2(c),

let $\quad x = X \sin \omega t, \ \ddot{x} = -X\omega^2 \sin \omega t$

Substituting these values in Eq. 5.2,

$$m\ddot{z} + c\dot{z} + kz = mX\omega^2 \sin \omega t \quad \text{...5.3}$$

[$\therefore$ RHS of Eq. 5.3 is sinusoidal]

This is a second-order nonhomogeneous differential equation of motion, whose solution is given by

$$z = z_c + z_p$$

where z_c = Complementary function (transient response)

z_p = Particular integral (steady-state response already discussed in Article 4.2, case 'b'). Considering the steady-state response, we find the particular integral 'z_p'.

To find particular integral 'z_p'

Let $\quad z_p = Z \sin(\omega t - \phi)$

$\therefore z = Z \sin(\omega t - \phi), \dot{z} = Z\omega \cos(\omega t - \phi) = Z\omega \sin\left[\frac{\pi}{2} + \omega t - \phi\right], \ddot{z} = -Z\omega^2 \sin(\omega t - \phi)$

Substituting these values in Eq. 5.3,

$$-mz\omega^2 \sin(\omega t - \phi) + cz\omega \sin\left(\frac{\pi}{2} + \omega t - \phi\right) + kz \sin(\omega t - \phi) = mX\omega^2 \sin \omega t$$

Rearranging the terms,

$$mX\omega^2 \sin \omega t - kz \sin(\omega t - \phi) - cz\omega \sin\left(\frac{\pi}{2} + \omega t - \phi\right) + mz\omega^2 \sin(\omega t - \phi) = 0 \quad ...5.4$$

These forces can be vectorially represented as shown in Fig. 5.3.

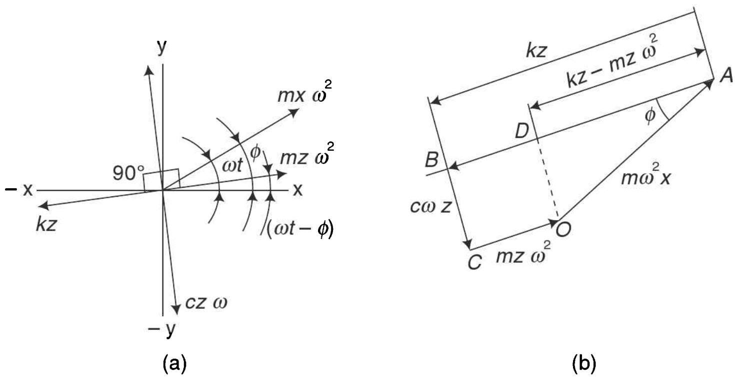


Fig. 5.3 *Force and vector polygon*

From triangle *OAD* in Fig. 5.3(b),

$$OA^2 = AD^2 + DO^2,\ OA = mX\omega^2,\ AD = (kz - \text{mz}\ \omega^2)^2,\ DO = (cz\ \omega^2)$$

$$\therefore \qquad (mX\omega^2)^2 = [kz - mz\omega^2]^2 + (cz\omega)^2 \quad ...5.5$$

Let $\quad F_o = mX\omega^2$, then Eq. 5.5, $F_o^2 = (kz)^2\left[\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2\right] \quad ...5.6$

where $\quad \frac{m}{k} = \frac{1}{\omega_n^2},\ \frac{c}{k} = \frac{2\xi}{\omega_n}$ and let $\frac{\omega}{\omega_n} = r$ (frequency ratio)

where $\quad \frac{m}{k} = \frac{1}{\omega_n^2}$

Substituting these values in Eq. 5.6,

$$F_0^2 = (kz)^2[(1 - r^2)^2 + (2\xi r)^2],\ F_o = kz\sqrt{(1 - r^2)^2 + (2\xi r)^2},\ Z = \frac{F_o/k}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

$$Z = \frac{\frac{m\omega^2.X}{k}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}},\ \frac{m}{k} = \frac{1}{\omega_n^2},\ \frac{Z}{X} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad ...5.7$$

And phase angle $\phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right]$

Frequency plot $= \frac{Z}{X}$ *versus* $\frac{\omega}{\omega_n}$.

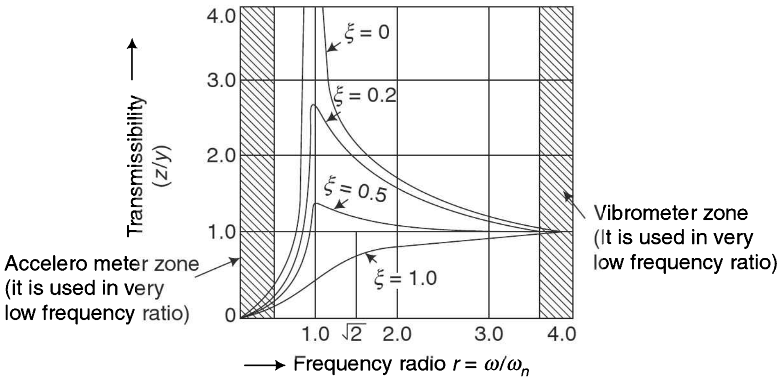


Fig. 5.4 *Frequency plot* $\frac{Z}{X}$ *versus* $\frac{\omega}{\omega_n}$ *(frequency ratio)*

From the frequency plot of Fig. 5.4, the following points can be noted.

1. The higher frequency ratio, the amplitude ratio $\frac{Z}{X}$, is almost equal to unity. Then relative amplitude 'Z' and the support amplitude 'X' are equal.
2. When $\frac{Z}{X} = 1$, $Z = X$ means that the mass of the instrument will have no displacement (absolute displacement).
3. For higher frequency ratio, the damping ratio will not have any effect.

5.3.1 Vibrometer or Seismometer

Vibrometer (instrument with low natural frequency) is an instrument used to measure the displacement of a vibrating body. The natural frequency ω_n of the vibrometer is very small as compared to that of the vibration (forcing frequency) to be measured.

From Eq. 5.7, $$\frac{Z}{X} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}.$$

Let $$\frac{\omega}{\omega_n} = r, \quad \omega_n << \omega, \quad r >>> 1, \quad r^2 >>> 1 \ \frac{1}{r^2} <<< 1 \quad \text{or} \quad \frac{1}{r^2} \approx 0$$

Substituting these values in Eq. 5.7,

$$\frac{z}{X} = \frac{r^2}{\sqrt{r^4(1/r^2 - 1)^2 + r^4(2\xi)^2/r^2}} \quad \text{or} \quad \frac{Z}{X} \approx 1.$$

Let $Z = X$ or $Z\alpha$ Amplitude, $Z\alpha$ displacement

The relative motion 'Z' of the mass will be a measure of the amplitude of the vibrating body.

The condition for vibrometer is that $\frac{\omega}{\omega_n} >>> 1$

This is possible only when 'ω_n' is small, i.e. if the spring stiffness is made very small, 'ω_n' will be small. Some amount of damping may be introduced to the vibrometer to minimise steady-state vibrations. The relative amplitude is equal to the amplitude of vibrating body for every value of damping ratio. This is as shown in Fig. 5.4.

The main disadvantages of a seismometer is its large size, because it is an instrument with low natural frequency. Thus the seismic mass remains stationary while the frame moves with the vibrating body. This instrument is mainly used to measure velocity and acceleration by incorporating differentiators.

5.3.2 Velocity–Measuring Instruments or Velocity Pick-ups

In Sec. 5.3, we know that the relative motion 'z' could be measured by means of a secondary strain-sensing transducer. However, in case of a velocity-sensing secondary transducer of the type of a magnet rigidly fixed to the seismic mass moving in a coil fixed to the frame, the resulting output voltage at the two ends of coil is proportional to the relative velocity. The output voltage is also proportional to that based on which the lines of magnetic force are cut, given by expression $V = \Phi L v$, where V = Volts, Φ = Magnetic flux density, L = Length of the conductor and v = Velocity of the conductor. The relative velocity is equal to the input velocity of the vibrating system at greater values of frequency ratio $\left(\frac{\omega}{\omega_n}\right)$. By these, the system or the instrument behaves like a velocity pick-up and everything else said in regard to the vibrometer holds good as a velocity pick-up.

5.3.3 Accelerometer (Instrument with High Natural Frequency)

An accelerometer is an instrument which is constructed and used like a vibrometer. Accelerometer measures the acceleration of the body with which it is in contact. The natural frequency of accelerometer is very high as compared to that of the vibration to be measured.

Let us consider Eq. 5.7, i.e. $\frac{Z}{X} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ...(5.7)

Assuming $\frac{\omega}{\omega_n} << 1$, we have, $\frac{Z}{X} = \left(\frac{\omega}{\omega_n}\right)^2 F$, or $Z = \left(\frac{\omega}{\omega_n}\right)^2 X.F$ ...5.8

where 'F' is a factor which remains constant for the useful range of the accelerometer.

Here, $$F = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad ...5.9$$

In the equation, $\omega^2 Y$ is the acceleration of the vibrating body. It is clearly seen that the acceleration is multiplied by the factor $\left(\frac{1}{\omega^2}\right)$. To keep the value of factor $f = 1$, for higher range of $\frac{\omega}{\omega_n}$, ξ should be high. Then amplitude 'Z' becomes proportional to the amplitude of acceleration to be measured, i.e. $Z\alpha(\omega^2 X)$ or $Z\alpha$ acceleration where ω_n is constant and $f = 1$.

From Eq. (5.9), the following plot can be drawn. Figure 5.5 shows the response of the accelerometer

It is seen that for $\xi = 0.7$, there is complete linearity of acceleration for $\dfrac{\omega}{\omega_n} < 0.25$.

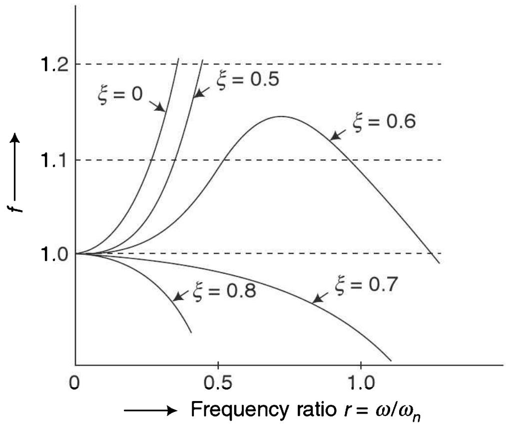


Fig. 5.5 *Response curve*

Since the natural frequency of the accelerometer is high, so it is very light in construction. Using integration circuits, one can get the display of velocity and displacement, Because of very light construction, the same instrument can be used to measure acceleration, velocity and displacement. An accelerometer is widely used as a vibration measuring device.

$$\frac{Z}{X} = \frac{r^2}{r^2\sqrt{\left(\frac{1}{r^2} - 1\right)^2 + 0}}, \text{ for a system '}\omega_n\text{' is constant}$$

$$\therefore \qquad \frac{Z}{X} \propto \omega^2, \; Z \propto \omega^2 X$$

For simple harmonic motion, $x = X \sin \omega t$, $\ddot{x} = -X\omega^2 \sin \omega t$ or $\ddot{x} \propto X\omega^2$,

$$\therefore \qquad Z \propto \ddot{x}, \; Z \propto \text{Acceleration}$$

Thus, the relative motion 'Z' of the mass will be a measure of acceleration of the vibrating body. The condition for an accelerometer is that $\dfrac{\omega}{\omega} << 1$.

This is possible only when 'ω_n' is high. If the spring stiffness is made very high, 'ω_n' will be high.

5.3.4 Principle and Difference between Vibrometer and Accelerometer

1. Vibrometer, or seismometer If $r > 1$ i.e. $\omega_n << \omega$, $\dfrac{Z}{X}$ approaches unity for all values of ξ. i.e. in that case $Z = X$

Or the relative motion of the seismic mass with respect to the frame is then equal to the displacement which is to be measured. (Place the base on the machine element vibration which you want to measure). Such an instrument is called a vibrometer. As it can be made low by smaller value of 'k', i.e. softer spring or higher value of 'm' or heavier seismic mass. One disadvantage of a seismometer is its large size because it is an instrument with low natural frequency. Thus, the seismic mass remains stationary while the frame moves with the vibrating body. These instruments can be used to measure velocity and acceleration by incorporating differentiators.

2. Accelerometer If $r << 1$, i.e. $\omega_n >> \omega$, the denominator becomes nearly unity for all values of ξ, and therefore, $\frac{Z}{X} = r^2 = \frac{\omega^2}{\omega_n^2}$ or $Z = \frac{\omega^2 X}{\omega_n^2} = \frac{\text{Accleration of base}}{\omega_n^2}$,

i.e. the relative motion of the seismic mass with respect to the base on multiplication by ω_n^2 gives the acceleration of the base. Such an instrument is called an accelerometer. It can be made very large either by having a harder spring or lighter seismic mass.

It is, therefore, evident that the only difference in a vibrometer and an accelerometer is in the natural frequency. In the former it is very small, whereas in the later it is very high. The principle and construction remain the same.

The denominator $\frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ approaches unity when $r << 1$. This factor is called the *accelerometer error* because its tending to unity depends on both 'r' and 'ξ' and if this is not very near unity then acceleration measured from an accelerometer will not be exact. Since the natural frequency of the accelerometer is high, so it is very light in construction. Using integration circuits, one can get the display of velocity and displacement. Becasue of very light construction and the same instrument can used to measure acceleration, velocity and displacment. Accelerometer is widely used as vibration measuring device.

EXAMPLE 5.1

A mass of 50 kg suspended from a spring produces a static deflection of 0.017 m and when in motion, it experiences a viscous damping force with a value of 250 N at a velocity of 0.3 m/s. Calculate the periodic time of damped vibration. If the mass is then subjected to a periodic disturbing force having a maximum value of 200 N and making 2cps, find the amplitude of the ultimate motion.

Solution $m = 50$ kg, static deflection $(\delta) = 0.017$ m,

$$F = 250 \text{ N}, \quad \dot{x} = 0.3 \text{ m/s},$$

At static deflection, $k\delta = mg$.

$$\frac{k}{m} = \frac{g}{\delta} \quad \therefore k = \frac{gm}{\delta} = 9.81 \times \frac{50}{0.017} = 28852.94 \text{ N/m}$$

$\therefore$ natural frequency $\omega_n = \sqrt{\frac{g}{\delta}}$, $\omega_n = \sqrt{\frac{9.81}{0.017}}$, $\omega_n = 24.02$ rad/s.

The damped natural frequency is $\omega_d = \omega_n \sqrt{1 - \xi^2}$.

To find damping coefficient 'ξ',

$$\xi' = \frac{c}{c_c} = \frac{c}{2\,m\omega_n} = \frac{F/\dot{x}}{2\,m\omega_n},\ \xi' = \frac{250/0.3}{2 \times 50 \times 24.02},\ \xi' = 0.35,$$

$$\therefore \qquad \omega_d = 24.02\sqrt{1-(0.35)^2},\ \omega_d = 22.5 \text{ rad/s.}$$

Periodic time $t_p = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{22.5}$ $\quad \therefore\ t_p = 0.28$ s.

Since the mass is subjected to forced vibration, the amplitude ratio is given by

$$\frac{X}{X_{\text{st}}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \text{ where } r = \frac{\omega}{\omega_n}$$

$$\therefore \qquad r = 2 \times \frac{2 \times 2\pi}{24.02} = 8.33 \times 10^{-2} \times 2\pi = 0.523,\ r = 0.523$$

and $\qquad X_{st} = \dfrac{F_0}{k} = \dfrac{200}{28852.94} = 6.93 \times 10^{-3}$

$\therefore$ amplitude of the ultimate motion is given by

$$X = \frac{6.93 \times 10^{-3}}{\sqrt{[1-(0.523)^2]^2 + (2 \times 0.35 \times 0.523)^2}} \text{ m } = 8.52 \times 10^{-3} \text{ m} = 8.52 \text{ mm.}$$

EXAMPLE 5.2

A refrigerator unit weighing 30 kgf is to be supported by three springs of stiffness '*k*' each. If the unit operates at 580 rev/min, what should be the value of spring constant '*k*' if only 10% of the shaking force of the unit is to be transmitted to the supporting structure?

Solution $\quad m = 30$ kg, $N = 580$ rev/min, 10% of $F_o = F_t$, $\dfrac{F_t}{F_0} = 0.1$

$\therefore$ transmissibility ratio is given by $\dfrac{F_t}{F_0} = \sqrt{\dfrac{1 + (2\xi r)^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}}$.

Since damping is not given, assume $\xi = 0$ $\quad \therefore \dfrac{F_t}{F_0} = \dfrac{1}{\pm(1-r^2)}$ where $r = \dfrac{\omega}{\omega_n}$,

$$\frac{1}{-(1-r^2)} = 0.1,\ -1 + r^2 = \frac{1}{0.1},\ r^2 = 10 + 1,\ r = \sqrt{11},\ r = 3.32,\ \frac{\omega}{\omega_n} = 3.32,$$

$$\omega_n = (3.32)^{-1} \times \omega, \text{ where } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 580}{60} = 60.74 \text{ rad/s,}$$

$$\therefore \qquad \omega_n = \frac{60.74}{3.32},\ \omega_n = 18.29 \text{ rad/s.}$$

But $\qquad \omega_n = \sqrt{\dfrac{k_{\text{eq}}}{m}} = 18.29 \quad \therefore \dfrac{k_{\text{eq}}}{m} = 334.69,$

$$k_{eq} = 334.69 \times 30,\ k_{eq} = 10040.57 \text{ N/m},$$

Since there are three springs, $k = \dfrac{k_{eq}}{3}$

$\therefore$ each spring stiffness $k = 3346.86$ N/m.

EXAMPLE 5.3

The springs of an automobile trailer are compressed 0.1 m under its own weight. Determine the critical speed when the trailer is traveling over a road with a profile approximated by a sine wave of 0.08 m amplitude and wavelength of 14 m. What will be the amplitude of vibration at 60 km/hour?

Solution $\delta = 0.1$ m, $y = 0.08$ m, $l = 14$ m, $v = 60$ km/h = 60000/3600 = 16.667 m/s

We have $mg = k\delta$ or $\dfrac{k}{m} = \dfrac{g}{\delta},\ \omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{g}{\delta}}\ \sqrt{\dfrac{9.81}{0.1}} = 9.905$ rad/s,

$$\omega_n = 2\pi f_n \text{ or } f_n = 0.634 \text{ cycles/s.}$$

Critical velocity $V_{cr} = \dfrac{1}{f_n} = \dfrac{14}{0.6343} = 22.07$ m/s, $V_{cr} = \dfrac{22.07 \times 3600}{1000} = 79.5$ km/h

$$f_n = \frac{1}{v} = \frac{14}{16.667} = 0.84 \text{ s},\ \omega = \frac{2\pi}{f} = \frac{2\pi}{0.84} = 7.5 \text{ rad/s}$$

$$\frac{X}{y} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \text{ where } r = \frac{\omega}{\omega_n} = \frac{9.5}{9.9} = 0.76$$

Damping is not given, so assume

$$\xi = 0,\ T_r = \frac{X}{0.08} = \frac{\sqrt{1 + (2 \times 0 \times r)^2}}{\sqrt{(1 - r^2)^2 + (2 \times 0 \times r)^2}} = 2.32$$

$\therefore$ $x = 0.186$ m.

EXAMPLE 5.4

A vehicle has a mass of 490 kg and the total spring constant of its suspension system is 58800 N/m. The profile of the road may be approximated to a sine wave of 40 mm amplitude and wavelength 4 m. Determine (i) critical speed of the vehicle, (ii) the amplitude of the steady state motion of the mass when the vehicle is driven at critical speed and $\xi = 0.5$, and (iii) the amplitude of steady state motion of mass when the vehicle is driven at 57 km/h and damping factor = 0.5.

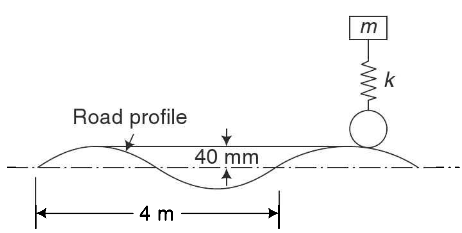


Fig. p-5.4 Vehicle-road system

Solution $m = 490$ kg, $k = 58800$ N/m, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{58800}{490}} = 10.95$ rad/s.

(i) Critical speed of the vehicle $\lambda = v\frac{1}{f}$, $\lambda = \frac{v}{\left(\frac{\omega}{2\pi}\right)}$,

$$\therefore \qquad \lambda = \frac{2\pi v}{\omega} \quad \therefore\ \omega = \frac{2\pi v}{\lambda}$$

At critical speed, $\omega = \omega_n = 10.95$ rad/s, $\frac{2\pi V}{\lambda} = 10.95$

$\therefore\ V = 6.973$ m/s, $V = 25.106$ km/h (Critical speed).

(ii) $\frac{\omega}{\omega_n} = 1$, $\xi = 0.5$, $Y = 40 \times 10^{-3}$ m, $\frac{X}{Y} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2) + (2\xi r)^2}}$

$$\therefore \qquad X = 40 \times 10^{-3}\frac{\sqrt{1+1}}{\sqrt{1}}$$

$$X = 56.57 \times 10^{-3} \text{ m.}$$

(iii) $\omega = \frac{2\pi v}{\lambda} = \frac{2\pi}{4}\left(\frac{57 \times 1000}{3600}\right) = 24.87$ rad/s, $\frac{\omega}{\omega_n} = 2.27$, $X = 40 \times 10^{-3}$.

$$= \frac{\sqrt{1 + [2(0.5)(2.27)]^2}}{\sqrt{(1 - 2.27^2)^2 + [2(0.5)\,(2.27)]^2}} = 20.96 \times 10^{-3} \text{ m.}$$

EXAMPLE 5.5

If a loaded automobile of 900 kg mass is running at 100 km/h over a rough road whose surface waves varies sinusoidally with 5 m/cycle and amplitude of 0.15 m, determine the amplitude ratio of the automobile when it is loaded and empty. It weighs 230 kg when it is empty, and the damping factor is 0.5 when it is loaded.

Solution $M_{\text{loaded}} = 900$ kg, $v = 100$ km/h, $X = 0.15$ m, $M_{\text{empty}} = 230$ kg,

$\xi_{\text{loaded}} = 0.5$.

Velocity $$v = \frac{100 \times 100}{3600} = 27.78 \text{ m/s}$$

To find the forcing frequency 'ω', considering the velocity of car, In 1s the distance moved is 27.78 m.

In 'τ' s, the distance moved is 5 m/cycle; see Fig. p-5.5.

$$\therefore \qquad \tau = \frac{5}{27.78} = 0.18 \text{ s}$$

But $$\tau = \left(\frac{\omega}{2\pi}\right)^{-1} = \frac{2\pi}{\omega} \quad \therefore\ \omega = \frac{2\pi}{0.18} = 34.91 \text{ rad/s.}$$

Motion isolation is given by $\frac{y}{X} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$.

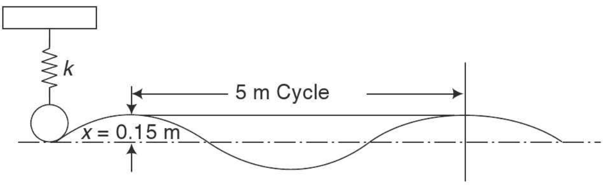


Fig. p-5.5 *Vehicle on rough road*

Let ω_n be the natural frequency of the automobile when empty,

ω_{2n} be the natural frequency of the automobile when loaded.

$$\therefore \qquad \omega_{1n} = \sqrt{\frac{k}{M_{\text{emty}}}},\ \omega_{2n} = \sqrt{\frac{k}{M_{\text{loaded}}}} \quad \therefore\ \frac{\omega_{1n}}{\omega_{2n}} = \sqrt{\frac{M_{\text{loaded}}}{M_{\text{empty}}}}$$

Let 'r_1' be the frequency ratio when empty, 'r_2' be the frequency ratio when loaded.

$$\therefore\ r_1 = \frac{\omega}{\omega_{1n}},\ r_2 = \frac{\omega}{\omega_{2n}},\ \frac{r_1}{r_2} = \frac{\omega_{2n}}{\omega_{1n}} = \sqrt{\frac{M_{\text{empty}}}{M_{\text{loaded}}}} = \sqrt{\frac{230}{900}} = \frac{15.17}{30.0},$$

$r_1 = 15.17$, $r_2 = 30$

Assuming $\xi = 0$ for empty automobile,

$$\frac{y_{\text{empty}}}{X} = \frac{1}{\sqrt{(1 - r_1^2)^2}} \qquad \text{...a}$$

$$\frac{y_{\text{loaded}}}{X} = \frac{\sqrt{1 + (2\xi r_2)^2}}{\sqrt{(1 - r_2^2)^2 + (2\xi r_2)^2}} \qquad \text{...b}$$

Dividing Eq. (a) by Eq. (b),

$$\frac{y_{\text{loaded}}}{y_{\text{empty}}} = \frac{\sqrt{1 + (2\xi)^2 r_2^2} \times \sqrt{(1 - r_1^2)^2}}{\sqrt{(1 - r_2^2)^2 + (2\xi)^2 (r_2^2)}} = \frac{\sqrt{1 + (2 \times 0.5)^2 (900)}\ \sqrt{(1 - 230)^2}}{\sqrt{(1 - 900)^2 + (2 \times 0.5)^2 (900)}}$$

$$= 0.505.$$

EXAMPLE 5.6

A vibrometer with a natural frequency of 2 Hz and with negligible damping is attached to a vibrating system which performs a harmonic motion. Assuming the difference between the maximum and minimum recorded values as 0.6 mm, determine the amplitude of motion of the vibrating system when its frequency is (i) 20 Hz, and (ii) 4 Hz.

Solution $f_n = 2$ Hz, $\xi = 0$ (negligible damping) $\therefore Z = 0.3$ mm, see Fig. p-5.6

(i) For vibrometer $\dfrac{Z}{X} = \dfrac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$, at $\xi = 0$. $\dfrac{Z}{X} = \dfrac{r^2}{\sqrt{(1 - r^2)^2}}$

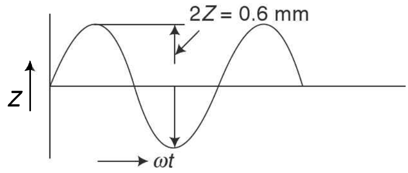


Fig. p-5.6 *Vibromater graph*

The frequency ratio $r = \dfrac{\omega}{\omega_n} = \dfrac{f}{f_n} = \dfrac{20}{2} = 10$, $r = 10$

Amplitude of the vibrating system, $X = \dfrac{Z\sqrt{(1-r^2)^2}}{r^2} = \dfrac{0.3\sqrt{(1-10^2)^2}}{10^2}$,

$X = 0.297 \times 10^{-3}$ m

(ii) Given $f = 4$ Hz; $\therefore\ r = \dfrac{4}{2} = 2$

$\therefore\quad X = \dfrac{0.3\sqrt{(1-2^2)^2}}{2^2} = 0.225 \text{ mm} = 0.225 \times 10^{-3}$ m.

EXAMPLE 5.7

A device used to measure torsional acceleration consists of a ring having a moment of inertia of 0.049 kg-m^2 connected to a shaft by a spiral spring having a scale of 0.98 N-s/rad and a viscous damper having a constant of 0.11 N-m-s/rad.When the shaft vibrates with a frequency of 15 cpm, the relative amplitude between the ring and the shaft is found to be 2° What is the maximum acceleration of the shaft?

Solution Moment of inertia of ring $J = 0.049$ kg-m^2,

Spring scale or stiffness $k_t = 0.98$ N-s/rad

Damping constant $c_t = 0.11$ N-m-s/rad, Frequency $f = 15$ cpm, Relative Amplitude $\theta_z = 2° = 0.0349$ radians. We have $\omega = \dfrac{2\pi f}{60}$, $\omega = 2\pi \times 15/60 = \pi/2$ rad/s

$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{0.98}{0.049}} = 4.47 \text{ rad/s},\ \xi = \frac{c_t}{c_{tc}} = \frac{c_t}{2\sqrt{k_t J}} = \frac{0.11}{2\sqrt{0.98 \times 0.049}} = 0.25$$

Relative amplitude $\theta_z = 2° = 0.0349$ radians.

The frequency ratio $r = \dfrac{\omega}{\omega_n} = \dfrac{\pi}{2 \times 4.47} = 0.352$

$$\frac{\theta_z}{\theta_y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1-\left(\frac{\omega}{\omega}\right)^2\right)^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}},\ \frac{0.0349}{\theta_y} = \frac{(0.352)^2}{\sqrt{(1-0.352)^2 + [2 \times 0.25 \times 0.352]^2}}$$

$\theta_y = 0.253$ radians.

Maximum acceleration of the shaft $= \omega^2\,\theta_y = (\pi/2)^2 \times 0.253 = 0.62$ rad/s^2

EXAMPLE 5.8

A commercial type vibration pick-up (vibrometer) has a natural frequency of 4.75 cps and a damping factor $\xi = 0.65$. What is the lowest frequency that (beyond which the amplitude) can be measured with 2% error?

Solution $f_n = 4.75$ cps, $\xi = 0.65$

For a seismic instrument, $\dfrac{Z}{X} = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

2% error in amplitude is $\dfrac{Z}{X} = 1.02$

$\therefore \quad Z = 1.02\,X$

$$\therefore \quad 1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}, \quad 1.02\left[\sqrt{(1-r^2)^2 + (2\xi r)^2}\right] = r^2,$$

Squaring on both sides and simplifying,

$$1.04(1-r^2)^2 + (2\xi r)^2 = r^4,\ 1.04\lfloor 1 + r^4 - 2r^2 \rfloor + (2 \times 0.65r)^2 = r^4$$

$$1.04 + 1.04\,r^4 - 2.08\,r^2 - r^4 + 1.69\,r^4 = 0, \quad 0.04\,r^4 - 0.39\,r^2 + 1.04 = 0$$

This is a quadratic equation in r^2, $\therefore\ r^2 = \dfrac{0.39 \pm \sqrt{(0.39)^2 - 4 \times 0.04 \times 1.04)}}{2 \times 0.04}$,

r^2 = imaginary. Therefore rearranging, $\dfrac{Z}{X} = 0.98$, $Z = 0.98\,X$

$$0.98 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}},\ 0.96 = \left[(1-r^2)^2 + (2 \times 0.65r)^2\right] = (r^2)^2$$

$0.96\left\{(1 + r^4 - 2r^2) + 1.69\,r^2\right\} = (r^2)^2$,

$0.96 + 0.96\,r^4 - 1.92\,r^2 + 1.62\,r^2 = r^4$ or $0.04\,r^4 + 0.30\,r^2 - 0.96 = 0$

This is a quadratic in equation in r^2.

$$\therefore \quad r^2 = \frac{-0.30 \pm \sqrt{(0.30)^2 + 4 \times 0.96 \times 0.04}}{2 \times 0.04}, \quad r^2 = \frac{0.30 \pm 0.49}{2 \times 0.04}$$

Considering +ve sign, $r^2 = \dfrac{-0.3 + 0.49}{2 \times 0.04} = 2.42$, $\quad r = 1.56$

But $\quad r = \dfrac{\omega}{\omega_n} = \dfrac{f}{f_n} \quad \therefore\ \dfrac{f}{f_n} = 1.56, \qquad f = 1.56 \times 4.75 = 7.39$ cps

Lowest frequency $f = 7.39$ cps.

EXAMPLE 5.9

A commercial type vibration pick-up has a natural frequency of 5.75 cps and a damping factor $\xi = 0.65$. What is the lowest frequency beyond which the amplitude can be measured with (i) 1% error, and (ii) 2% error?

Solution $f_n = 5.75$ cps, $\xi = 0.65$,

(i) $1\%\text{ error} = \dfrac{\text{Reading of instrument}}{\text{Actual reading}} = \dfrac{100+1}{100} = \dfrac{101}{100} = 1.01 = \dfrac{z}{y} = \dfrac{100-1}{100} = 0.99.$

$$\frac{z}{y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

Since 1% error is shown by the instrument then $\dfrac{z}{y} = 1.01$ or 0.99.

$$1.01 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}},\ 1.0201 = \frac{r^4}{(1-2r^2) + r^2 + (2 \times 0.65 \times r)^2}$$

$$1.0201 = \frac{r^4}{(1-2\,r^2) + r^4 + 1.69\,r^2},\ 1.0201 - 2.0402\,r^2 + 1.0201\,r^4 + 1.724\,r^2 = r^4$$

$$r^4 - 15.73\,r^2 + 50.75 = 0$$

This is in the form of quadratic equation of r^2.

$$\therefore \qquad r^2_{1.2} = \frac{15.73 \pm 6.667}{2},\ = 11.19 \text{ or } 4.532,\ r = 3.35 \text{ or } 2.12$$

$z/y = 1.01$ at frequency ratio $r = 3.35$ and at 2.12;
in between these two values, z/y will be > 1.01.

(i) Therefore, the lowest frequency beyond which the amplitude can measure within 1% error.

$$\frac{f}{f_n} = 3.35,\ \frac{f}{5.75} = 3.35. \quad \therefore f = 19.24\text{ Hz.}$$

(ii) With 2% error, $\dfrac{z}{y} = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$, 2% error $\dfrac{z}{y} = 1.02$ or 0.98.

$$1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2 \times 0.65 \times r)^2}}$$

Squaring on both sides,

$$1.0404 = \frac{r^4}{1 - 2\,r^2 + r^4 + 1.69\,r^2},\ 1.0404 - 2.0808\,r^2 + 1.0404\,r^4 + 1.76\,r^2 = r^4$$

$0.0404\,r^4 - 0323\,r^2 + 1.0404 = 0$, this is in the form of a quadratic equation.

$\therefore\ r^2 = 0.323 \pm \ldots$ since the roots are imaginary. Therefore, take $z/y = 0.98$

$$0.9804 = \frac{r^4}{1 - 2\,r^2 + r^4 + 1.69\,r^2},\ 0.9804 - 1.921r^2 + 0.9604\,r^4 + 1.5906\,r^2 = r^4$$

$r^4 - 8.338\,r^2 + 24.25 = 0$, this is in the form of a quadratic equation.

$r^2 = 8.34 \pm \ldots.$ since the roots are imaginary.

$r^2 = 2.44,\ r = 1.56,\ \dfrac{f}{f_n} = 1.56,\ \dfrac{f}{5.57} = 1.56. \quad \therefore f = 8.97$ Hz.

5.4 FREQUENCY-MEASURING INSTRUMENTS

Usually frequencies of vibration of structural components are measured by two types of instruments: (a) Fullartors tachometer, and (b) Frahm's tachometer. Most of these frequency-measuring instruments are of the mechanical type and the working of these instruments are based on the principal of resonance. At resonance, the amplitude of vibration is found to be maximum and then the forced frequency is equal to the natural frequency, i.e. $\omega = \omega_n$.

5.4.1 Fullartor's Tachometer

The construction of a Fullartor's tachometer is as shown in Fig. 5.6 and it consists of a single reed. The reed is made up of a thin metallic strip, acting as a cantilever with a small mass attached at one of its ends. The length of the cantilever strip can be adjusted by operating a screw mechanism. The reed is brought in contact with the vibrating body to find its frequencies of vibration and the length of the reed is varied till resonance occurs and the frequencies at resonance of the reed is read on the scale provided with it. The natural frequency (f_n) of the reed is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{ML^2}}\ \text{Hz},$$

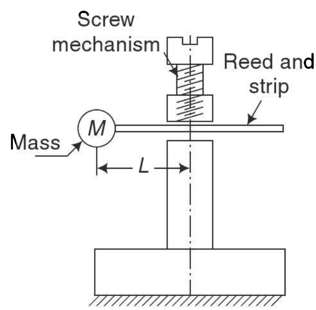


Fig. 5.6 *Fullartor's tachometer*

where E = Young's modulus of the material
L = length of the reed
I = Moment of inertia of the cross-section of the reed
M = Mass per unit length of the reed.

5.4.2 Frahm's Tachometer

Frahm's tachometer, also called multi-reed tachometer, is as shown in Fig. 5.7. It consists of a series of reeds carrying small masses at their free ends and each reed has a different natural frequency (f_n) of vibration marked in series. When it is brought in contact with the vibrating body, one of the reeds resonates and its resonating

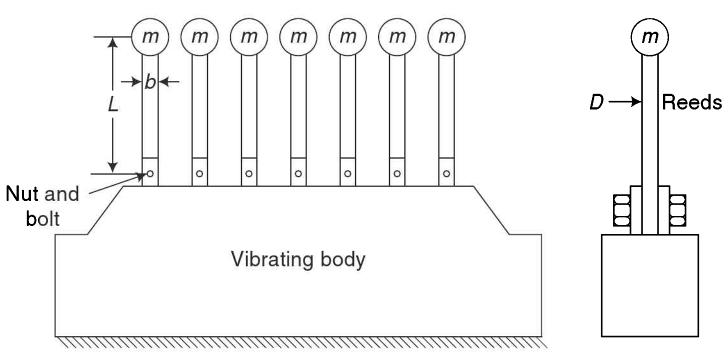


Fig. 5.7 *Frahm's tachometer*

frequency is the frequency of vibrating of the body. Hence, the frequency of the vibrating body can be found from the known frequency of the vibrating reed.

Let 'm' be the mass attached to each reed, 'E' is the modulus of elasticity of the reed material and 'L' is the length of each reed.

We know that the static deflection $k\delta = mg$, or $k = mg/\delta$.

where 'k' is the stiffness of the reed material. The static deflection of the cantilever beam when the mass acting at one end is given by $\delta = \dfrac{mgL^3}{3EI}$, $I = \dfrac{bD^3}{12}$,

b = Width of the reed and D = Depth of the reed.

$$\therefore \text{ natural frequency } f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{mg}{m\delta}},\ \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad \therefore f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}} \text{ Hz.}$$

5·5 CRITICAL SPEED, OR WHIRLING SPEED, OR WHIPPING SPEED OF THE SHAFT

Critical speed occurs when the speed of the rotation of the shaft is equal to the natural frequency of lateral vibration of the shaft. Whirling is defined as the rotation of the plane made by the bent shaft and the line of centre of bearings.

'Critical speed of a rotating shaft is the speed at which the shaft starts to vibrate violently in the transverse direction." It is very dangerous to continue to run the shaft at its critical speed as the amplitude of vibrations will build up to such a level that the system may break down. This phenomenon results from the following reasons: mass unbalance, gyroscopic forces, fluid friction in bearings, hysteresis damping in the shaft, etc.

1. *Critical speed of a light shaft having a single rotor—without damping*

Consider a light shaft carrying a rotor, rotating about the axis of rotation at a uniform speed of 'ω' rad/s, as shown in Fig. 5.8(a). Let 'G' be the centre of gravity of the disc, 'O' be the geometric centre of the disc, 'm' be the mass of the disc, 'e' be the eccentricity by which centre of gravity 'G' is shifted from the axis of rotation and 'k_t' be the stiffness of the shaft in the lateral direction, and 'y' be the deflection of the shaft due to centrifugal force as shown in Fig. 5.8(b).

Assumptions

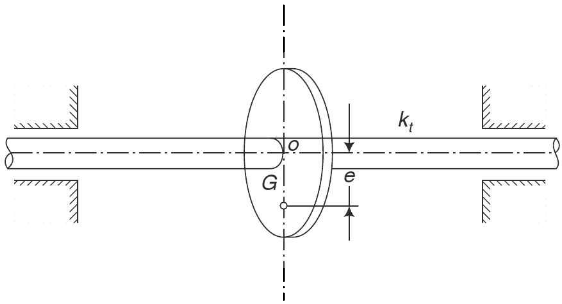
(a) Shaft is light and flexible. (b) Friction at shaft centre is very small.

(c) Gravity effect is negligible. (d) Damping due to air is neglected.

Under the above assumptions the rotor is being acted upon by only two forces, namely the

(i) Centrifugal force $m\,(y + e)\,\omega^2$ acting in outward direction through the centre of gravity 'G', and

(ii) Restoring force of the shaft $(k.y)$.



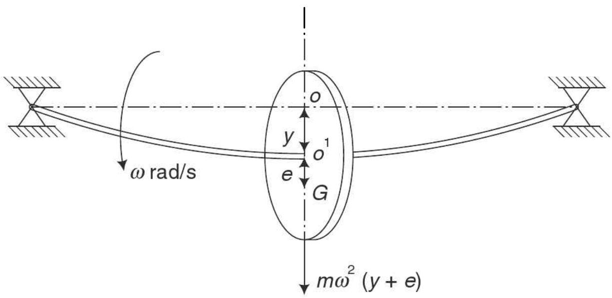


Fig. 5.8 *Critical speed of a light shaft having a single rotor—without damping*

For equilibrium, therefore, it is necessary that these two forces must be collinear, equal in magnitude and opposite in direction.

In Fig. 5.8(b), for equilibrium, Restoring force = Centrifugal force.

$ky = m(y + e)\,\omega^2$, $m\omega^2 y - ky + m\omega^2 e = 0$, $(m\omega^2 - \text{k})\text{y} + m\omega^2 e = 0$, $m\omega^2 e = y(k - m\omega^2)$ or

$$\frac{y}{e} = \frac{m\omega^2}{k - m\omega^2} \qquad \text{...5.10(a)}$$

Dividing the right-hand side numerator and denominator by 'k' we get,

$$\frac{y}{e} = \left[\frac{\dfrac{m\omega^2}{k}}{1 - \left(\dfrac{m\omega^2}{k}\right)}\right],$$

The circular frequency $\omega_n^2 = \dfrac{k}{m}$

$$\therefore \qquad \frac{y}{e} = \frac{\left(\dfrac{\omega}{\omega_n}\right)^2}{\left(1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right)} \quad \text{Let } r = \frac{\omega}{\omega_n} \quad \therefore \frac{y}{e} = \frac{r^2}{1 - r^2} \qquad \text{...5.10(b)}$$

Condition for critical speed is that $\omega = \omega_n$ (Resonance)

$$r = 1 \text{ in Eq. 5.10(b).} \quad \therefore \frac{y}{e} = \frac{1}{1 - 1}, \frac{y}{e} = \infty, y = \infty.$$

Case (i) If $\frac{\omega}{\omega_n} < 1$, the rotor is rotating with its heavy side out, Fig. 5.8(c).

Case (ii) If $\frac{\omega}{\omega_n} > 1$, the rotor is rotating with its heavy side in, Fig. 5.8(d).

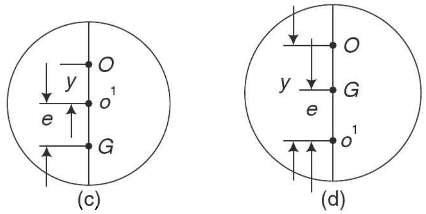


Fig. 5.8 *Phase relationships without damping*

2. *Critical speed of a light shaft having a single rotor—with damping*

When damping is present in the form of air resistance, the analysis becomes slightly more involved. Now three forces act on the rotor: they are

(a) Centrifugal force at 'G' due to mass of the rotor ($ma\ \omega^2$)

where a = Distance between centre of rotation(O) to the centre of gravity(G).

(b) Restoring force due to spring stiffness (ky)

(c) Damping force($cy\ \omega$) in a direction opposite to the velocity.

Due to these forces, the points O, O^1 and 'G' no longer lie in straight line. These forces are shown in Fig. 5.9(c).

Let 'G' be the centre of gravity of the disc, 'O^1' be the geometric centre of the rotor, 'm' be the mass of the rotor, 'e' be the eccentricity of 'G' from 'O^1', 'y' be the deflection of shaft due to $m\omega^2 a$, and 'k' be the stiffness of the shaft due to lateral vibrations.

$$OG = a,\ \angle GO\,O^1 = \alpha, \quad \angle GO^1B = \phi,\ \text{mw}^2\,a = \text{Centrifugal force}$$

ky = Spring force or restoring force, $cy\omega$ = Damping force due to damping coefficient 'c'. From the geometry of Fig. 5.9(c), we have

$$a \sin \alpha = e \sin \phi \text{ and } a \cos \alpha = y + e \cos \phi$$

Considering the forces acting on the rotor, $\Sigma V = 0$ and $\Sigma H = 0$

$$\therefore \qquad ky = m\omega^2 a \cos \alpha, \quad cy\ \omega = m\omega^2 a \sin \alpha$$

Using the values of $a \cos \alpha$ and $a \sin \alpha$

$$ky = m\omega^2 (y + e \cos \phi) \qquad \text{...5.11(a)}$$

$$cy\ \omega = m\omega^2\, e \sin \phi \qquad \text{...5.11(b)}$$

Equation 5.11(a) can be written as

$$ky = m\omega^2 y + m\omega^2\, e \cos \phi, \quad m\omega^2 e \cos \phi + ky - m\omega^2 y \qquad \text{...5.11(c)}$$

Squaring and adding equation 5.11(b) and equation 5.11(c), we have

$$(m\omega^2 e)^2\, [\cos^2 \phi + \sin^2 \phi] = (ky - m\omega^2 y)^2 + (cy\ \omega)^2$$

$$(m\omega^2 e)^2 = y^2[(k - m\omega^2)^2 + (c\omega)^2],\ m\omega^2 e = y\sqrt{(k - m\omega^2)^2 + (c\omega)^2},$$

$$\frac{y}{e} = \frac{m\omega^2}{\sqrt{k^2\left[1 - \frac{m}{k}\omega^2\right]^2 + \left(\frac{c}{k}\omega\right)^2}}$$

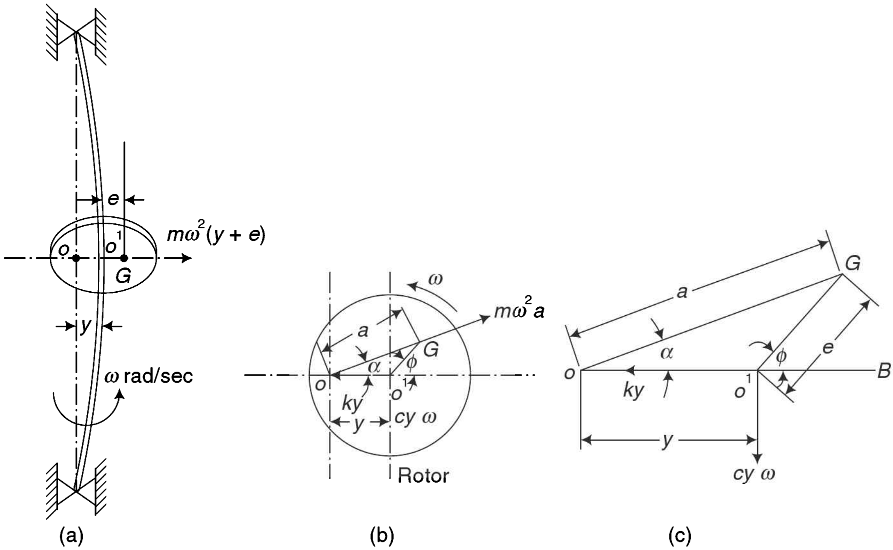


Fig.5.9 *Critical speed of a light shaft having a single rotor—with damping*

Dividing both numerator and denominator of the right-hand side by 'k', we get

$$\frac{y}{e} = \frac{\frac{m}{k}\omega^2}{\sqrt{\left[1 - \frac{m}{k}\omega^2\right]^2 + \left(\frac{c}{k}\omega\right)^2}}$$

We have $\omega_n^2 = \frac{k}{m}$ and $\frac{c}{k} = \frac{2\xi}{\omega_n}$ where 'ξ' is the damping ratio.

$$\therefore\ \frac{y}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}},\ \text{let}\ \frac{\omega}{\omega_n} = r,\ \frac{y}{e} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...5.11(d)}$$

and phase angle 'ϕ' is given by dividing Eq. 5.11(b) by Eq. 5.11(c).

$$\frac{\sin\phi}{\text{cas}\ \phi} = \frac{cy\,\omega}{ky - m\omega^2 y},\ \tan\phi = \frac{c\omega}{k - m\omega^2} = \frac{\frac{c}{k}\omega}{1 - \frac{m}{k}\omega^2}$$

$$\phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right] \qquad \text{...5.11(e)}$$

where $r = \dfrac{\omega}{\omega_n}$

Case (i) When $\omega << \omega_n$ $\phi \approx 0$ (heavy side out)

Case (ii) When $\omega < \omega_n$ $0 < \phi < 90°$ (heavy side out)

Case (iii) When $\omega = \omega_n$ $\phi = 90°$ (resonance system break)

Case (iv) When $\omega > \omega_n$ $90° < \phi < 180°$ (heavy side in)

Case (v) When $\omega >> \omega_n$ $\phi \approx 180°$ (heavy side in)

These cases are as shown in Fig. 5.9(d) and Fig. 5.9(e).

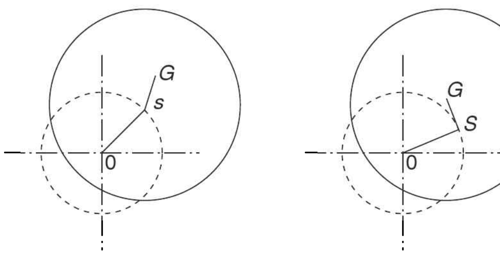


Fig. 5.9 *Phase relationships with damping*

EXAMPLE 5.10

A rotor of 10 kg mass is mounted midway on a 2 cm diameter, the horizontal shaft supported at the ends by two bearings. The bearing span is 80 cm, because of certain manufacturing defect, the centre of gravity of the disc is 0.01 mm away from the geometric centre of the rotor. If the system rotates at 3000 rev/min, determine the amplitude of the steady-state vibration and the dynamic force transmitted to the bearing.

Take $E = 2 \times 10^6$ kgf/cm^2.

Solution $m = 10$ kg, $d = 0.02$ m, $l = 0.8$ m, $e = 0.01 \times 10^{-3}$ m, $N = 3000$ rev/min.,

$$E = 2 \times 10^{6\,\text{kg}}\,\text{f/cm}^2 = 2 \times 9.81 \times 10^{10}\ \text{N/m}^2$$

Assuming the shaft is simply supported, the maximum deflection (δ) is given by

$$\delta = \frac{Wl^3}{48EI},\ \frac{W}{\delta} = \frac{48EI}{l^3} \quad \therefore\ \frac{W}{\delta} \text{ is spring stiffness, } k = \frac{48\,E\pi d^4}{64 l^3},$$

$$k = \frac{48 \times 2 \times 9.81 \times 10^{10} \times \pi \times [0.02]^4}{64 \times (0.8)^3},\ k = 144464.17\ \text{N/m}$$

Natural frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{144464.17}{10}}$ $\omega_n = 120.19$ rad/s

Forcing frequency $= \omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 3000}{60}$, $\omega_n = 314.16$ rad/s

Assuming damping is not present, i.e. $\xi = 0$. Then amplitude is given by equation

$\frac{y}{e} = \frac{r^2}{1 - r^2}$, where $r = \frac{\omega}{\omega_n}$ is a frequency ratio, $\therefore r = \frac{314.16}{120.19}$, $r = 2.61$.

$$\therefore \text{amplitude } y = \frac{er^2}{1 - r^2} = \frac{0.01x\, 10^{-3} x(2.61)^2)}{1 - (2.6)^2},\ y = -1.17 \times 10^{-5}\text{m}$$

Neglect the –ve sign, $y = 1.17 \times 10^{-5}$

Since the shaft is horizontal, dynamic load on the bearings is due to the deflection 'y' as well as the load due to mass of rotor.

Dynamic load on bearings = $ky = 144464.17 \times 1.17 \times 10^{-5} = 1.69$ N

$$\text{Total load on each bearing} = \frac{mg + ky}{2} = \frac{10 \times 9.81 + 1.69}{2} = 49.90 \text{ N}$$

$$\text{Dynamic load on each bearing} = \frac{1.69}{2} = 0.85 \text{ N.}$$

Note: (i) If the shaft is placed horizontally, the dynamic load transmitted $F_d = F_t$ + weight of rotor $F_d = F_t + mg$.

(ii) If the shaft is placed vertically then $F_d = F_t$, F_t = Force transmitted.

EXAMPLE 5.11

A rotor of weight 50 N is supported in bearings 400 mm apart. The diameter of the shaft is 20 mm. Due to manufacturing errors, the centre of gravity of disc is 0.1 mm away from the midpoint where the shaft is introduced. The shaft rotates 4500 rev/min. Determine the critical speed, maximum amplitude of steady state vibration and the bending stress induced, considering weight of the shaft. The shaft is horizontally supported and the support conditions may be assumed to be simply supported. Assume the density of the shaft as 8900 kg/m³. Take $E = 1.96 \times 10^{11}$ N/m².

Solution Mass of shaft/unit length $m_s = \rho \times v = 8900 \times \frac{\pi}{4}(20 \times 10^{-3}) \times 1$,

$m_s = 2.796$ weight/unit length of shaft, $W_s = m_s \times g$, $= 2.796 \times 9.81$, $W_s = 27.428$ N/m

$N = 4500$ rev/min $E = 1.96 \times 10^{11}$ N/m²

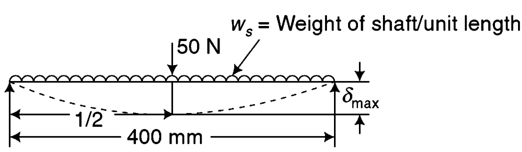


Fig. p-5.11 *Rotor system*

Let 'δ_1' be the static deflection of shaft under concentrated load $W = 50$ N at the mid span of the shaft, which is simply supported as shown in Fig. p-5.11.

$$\delta_1 = \frac{wl^3}{48\,EI},\quad \delta_1 = \frac{50 \times 0.4^3}{48 \times 1.96 \times 10^{11} \times \pi \frac{(0.02)^4}{64}} = 4.33 \times 10^{-5} \text{ m}$$

Let 'δ_2' be the deflection of the beam due to its weight 'W_s' For a simply supported beam under udl as shown in figure, the deflection

$$\delta_2 = \frac{5W_S l^4}{384EI} = \frac{5 \times 27.428 \times 0.4^4}{384 \times 1.96 \times 10^{11} \times \pi \dfrac{(0.02)^4}{64}} = 5.94 \times 10^{-6}\text{ m}$$

The total deflection of the shaft due to weight of rotor and weight of shaft,

$$\delta = \delta_1 + \delta_2 = 4.33 \times 10^{-5} + 5.94 \times 10^{-6} = 4.923 \times 10^{-5}\text{ m}$$

Natural frequency of the system, $\omega_n = \sqrt{\dfrac{g}{\delta}} = \sqrt{\dfrac{9.81}{4.923 \times 10^{-5}}} = 446.35$ rad/s.

Critical speed of system, $N_{cr} = \dfrac{\omega_n \times 60}{2\pi} = \dfrac{446.35 \times 60}{2\pi} = 4262.32$ rev/min.

$\therefore \quad \dfrac{y}{e} = \dfrac{r^2}{1 - r^2}$

Frequency ratio, $r = \dfrac{\omega}{\omega_n} = \dfrac{471.26}{446.35} = 1.0557$

For undamped system, $\dfrac{y}{e} = \dfrac{r^2}{1 - r^2}\ (r < 1) = \dfrac{r^2}{r^2 - 1}\ (r > 1)$

When $\dfrac{y}{0.1 \times 10^{-3}} = \dfrac{1.0557^2}{1.0557^2 - 1}$ $\quad\therefore$ steady-state amplitude, $y = 9.72 \times 10^{-4}$ m

Maximum bending stress (σ_{max}), $\sigma_{max} = M_{max} \times \dfrac{C}{I}$

$M_{max} = \dfrac{F_{max} \times l}{4}$ for simply supported beam.

$F_{max} = F_d +$ weight of rotor $= k_r + 50 = 1.238 \times 10^6 \times 9.72 \times 10^{-4} + 50$

$= 1.253 \times 10^3$ N

$$\therefore \qquad \sigma_{max} = \frac{\dfrac{1.253 \times 10^3 \times 0.4}{4}}{\dfrac{\pi(20 \times 10^{-3})}{64}} \times \frac{20 \times 10^3}{2} \quad \sigma_{max} = 1.27630 \times 10^9.\ \text{N/m}^2.$$

EXAMPLE 5.12

A rotor having a mass of 9.5 kg is mounted on a 12 mm horizontal steel shaft ($E = 1.96 \times 10^{11}$ N/m^2) midway between bearings that are 0.6 m apart. The CG of the disc is 6 mm from its geometric centre. If the damping factor is 0.1 and the shaft rotates at 690 rev/min, determine the maximum stress in the shaft and compare it with the dead load stress in the shaft. Also find the power required to drive the shaft at this speed.

Solution $\quad m = 9.5$ kg, $d = 12$ mm, $E = 1.96 \times 10^{11}$ N/m^2, $l = 0.6$ m, $e = 6$ mm, $\xi = 0.1$, $N = 690$ rev/min, $\sigma_{max} = ?$, $P = ?$

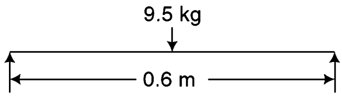


Fig. p.5.12 *Rotor system*

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 690}{60} = 72.26 \text{ rad/s}$$

The static deflection 'δ' for simply supported beam load at midway is given by $\delta = \frac{FL^3}{48EI}$

$$\frac{F}{\delta} = k = \frac{48EI}{L^3} = \frac{48 \times 1.96 \times 10^{11}}{(0.6)^3} \frac{\pi}{60} \times 12^4, = 44.33 \text{ N/mm}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{44.23}{9.5}} = 68.3 \text{ rad/s}, \; c_c = 2\sqrt{mk} = 2\sqrt{44.33 \times 9.5} = 1298 \text{ N-s/m}$$

$$r = \frac{\omega}{\omega_n} = \frac{72.26}{68.3} = 1.058$$

We have $\xi = \frac{c}{c_c}$ or $c = \xi c_c$ $\therefore$ $c = 0.1 \times 1298 = 129.8$ N-s/m

We know that $\frac{y}{e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$, $y = \frac{er^2}{\sqrt{[1-(1.058)^2]^2 + (2 \times 0.1 \times 1.058^2)}}$

or $y = 26.16$ mm.

$$F_d = ky\sqrt{1 + (2\xi r)^2} = 44.33 \times 26.16\sqrt{1 + (2 \times 0.1 \times 1.058^2)} = 1185.5 \text{ N.}$$

$$F_{max} = F_d + mg = 1185.3 + 9.5 \times 9.81 = 1278.5 \text{ N}, \quad F_{min} = mg = 9.5 \times 9.8 = 93.2 \text{ N}$$

$$\sigma = \frac{My}{I} = \left(\frac{FL}{4} \times \frac{\frac{d}{2}}{\frac{\pi}{64} \times d^4} = \frac{8FL}{\pi d^3} = 0.8841\, F \times 1278.5 \right)$$

$$\sigma_{max} = 1130 \text{ MPa}, \; \sigma_{min} = 0.8841 \times 93.2 = 82.4 \text{ MPa}$$

Power $P = T\omega$, $T = c\omega y = 129.8 \times 72.6 \times 26.16$,

$P = 129.8 \times 72.26 \times 26.16 \times 72.26$, $P = 463$ watts.

EXAMPLE 5.13

A 1.2 meter long vertical steel shaft of 22 mm diameter supported by two bearings at its ends carries a disc of 25 kg mass at its midspan. The eccentricity of the centre of gravity of the disc from the centre of the rotor is 0.20 mm. The modulus of elasticity for the shaft material is 200 GPa and the permissible stress is 80 N/m^2. Determine (i) the critical speed of the shaft, and (ii) the range of speeds over which it is unsafe to run the shaft. Neglect the mass of the shaft.

Solution Long bearings indicate that the shaft is with the fixed ends.

Bending moment and deflection at the midspan are $M = \frac{WL}{8}$ and $\delta = \frac{WL^3}{192\,EI}$

Moment of inertia of the section of the shaft is $I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times 22^4 = 11499 \text{ mm}^4$

Static deflection at the midspan of the shaft is $\delta = \frac{WL^3}{192\,EI} = \frac{25 \times 9.81 \times 1200^3}{192 \times 2 \times 10^5 \times 11499}$ $= 0.96$ mm.

Natural frequency of transverse vibration $\omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.96 \times 10^{-3}}} = 101.09$ rad/s

The critical speed $\omega_n = \frac{2\pi N_c}{60}$, $N_c = \frac{60\,\omega_n}{2\pi} = \frac{60 \times 101.9}{2\pi} = 973$ rev/min.

The bending moment equation is $\frac{M}{I} = \frac{\sigma_h}{y}$, $M = 83.63$ N-m

Therefore, maximum permissible bending moment is 83.63 N-m.

Bending moment at the midspan is $M = \frac{F_{max} \times 1.2}{8} = 83.63$, $F_{max} = 557.53$ N.

By linear proportion, deflection due to load F_{max} is $\frac{0.96 \times 10^{-3}}{25 \times 9.81} = \frac{\delta_1}{557.53}$, $\delta_1 = 2.182 \times 10^{-3}$ m

The relation for the dynamic deflection is $\pm \frac{2.182 \times 10^{-3}}{0.2 \times 10^{-3}} = \frac{r^2}{1 - r^2}$,

$r_1 = 0.96$ and $r_2 = 1.05$

$0.96 \le \frac{N}{N^c} \le 1.05$, $N_{max} = 1.05 \times 965.3 = 1013.6$ rev/min,

$N_{min} = 0.96 \times 965.3 = 926.7$ rev/min.

EXAMPLE 5.14

A disc of 4 kg mass is mounted midway between bearings which may be assumed to be simply supports. The bearing span is 0.5m. The shaft is 10 mm in diameter and is horizontal. The CG of the disk is displaced by 2 mm from the geometric centre. The damping ratio is 0.065. If the shaft rotates 250 rev/min, determine the critical speed assuming E = 196 GPa and the amplitude of vibration.

Solution $d = 10$ cm, $m = 4$ kg, $l = 0.5$ m, CG = 2 mm, $\xi = 0.065$, $N = 250$ rev/min.

Static deflection $\delta = \frac{mgl^3}{48El} = \frac{4 \times 9.81 \times 0.5^3}{48 \times 196 \times 10^9 \times \frac{\pi}{64} \times 0.01^4} = 1.0621 \times 10^{-3}$ m

Stiffness $k = \frac{mg}{\delta} = \frac{4 \times 9.81}{1.0621 \times 10^{-3}} = 36945.12$ N/m

Critical speed $N_c = \dfrac{0.4985}{\sqrt{\delta}} = \dfrac{0.4985}{\sqrt{1.0621 \times 10^{-3}}} = 15.3 \text{ Hz} = 917.8 \text{ rev/min}$

Amplitude $= \dfrac{e}{\left(\dfrac{N_c}{N}\right)^2 - 1} = \dfrac{2 \times 10^{-3}}{\left(\dfrac{917.8}{250}\right)^2 - 1} = 1.603 \times 10^{-4} \text{ m.}$

EXAMPLE 5.15

A vertical shaft 12.5 mm in diameter rotates in long bearings and a disc of 10 kg mass is attached to the midspan of the shaft. The span of the shaft between the bearings is 0.9 m. The mass centre of the disc is 0.25 mm away from the geometric axis. If the stress in the shaft is not to exceed 10.3×10^7 N/m^2, determine the range of speed within which it is unsafe to run the shaft. Determine the critical speed of the shaft. Neglect the mass of the shaft and the damping in the system. Assume $E = 2 \times 10^{11}$ N/m^2.

Given: $d = 12.5 \times 10^{-3}$ m, $m = 10$ kg, $e = 0.25 \times 10^{-3}$ m,

long bearings $E = 2 \times 10^{11}$ N/m^2

Note: For long bearings, the deflection, $\delta = \dfrac{wl^3}{192\,EI}$ and bending moment $M = \dfrac{wl}{8}$.

Considering the mass of the rotor, static deflection

$$\delta = \frac{wl^3}{192\,EI},\; k = \frac{w}{\delta} = \frac{192\,EI}{l^3} = \frac{192\,\pi d^4}{64 \times l^3}$$

$\therefore$ spring stiffness $k = \dfrac{3 \times 2 \times 10^{11} \times \pi \times (12.5 \times 10^{-3})^4}{(0.9)^3} = 63126.78$ N/m

$\therefore$ natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{63126.78}{10}} = 79.45$ rad/s.

At critical speed, $\omega = \omega_n \;\therefore\; \dfrac{2\pi N_c}{60} = 79.45,\; N_c = \dfrac{60 \times 79.45}{2\pi} = 75.69$ rev/min.

Let w_d be the dynamic load on bearings.

Since the stress in the shaft is given by $\sigma = 10.3 \times 10^7$ N/m^2,

from bending equation $\dfrac{M}{I} = \dfrac{\sigma}{y} = \dfrac{\left[w_d \dfrac{1}{8}\right]}{I} = \dfrac{\sigma}{y}$

$$w_d = \frac{\sigma \pi d^4 \times 8}{64 l \times \dfrac{d}{2}}, = \frac{\sigma \pi d^3}{4l} = \frac{10.3 \times 10^7 (12.5 \times 10^{-3})^3 \times \pi}{4 \times 0.9} = 175.56 \text{ N.}$$

But dynamic load transmitted to bearings, $w_d = ky$.

$\therefore$ deflection $y = \dfrac{w_d}{k} = \dfrac{175.56}{63126.78},\; y = 2.78 \times 10^{-3}$ m

We know that $\frac{y}{e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ at $\xi = 0$, $\frac{y}{e} = \frac{r^2}{\pm(1-r^2)}$ where $r = \frac{\omega}{\omega_n}$.

For finding out the range of speed, $\omega = \omega_n$ and $\omega_c = \omega_n$.

$$\frac{\pm y}{e} = \frac{1}{\left(\frac{\omega_c}{\omega}\right) - 1} = \frac{1}{\pm\left[\left(\frac{\omega_c}{\omega}\right) - 1\right]} = \frac{1}{\pm\left(\frac{1}{r^2} - 1\right)}$$

$$\frac{\pm 2.78 \times 10^{-3}}{0.25 \times 10^{-3}} = \frac{1}{\left(\frac{\omega_c}{\omega}\right) - 1} \quad \left(\frac{\omega_c}{\omega}\right)^2 - 1 = \pm 0.09 = \left(\frac{\omega_c}{\omega}\right)^2 = 1 \pm 0.09 \left(\frac{N_c}{N}\right)^2 = 1 \pm 0.09$$

Taking positive sign, $\left(\frac{N_c}{N}\right)^2 = 1 + 0.09$, $N_1 = \frac{N_c}{\sqrt{1.09}} = \frac{758.69}{1.044} = 726.72$ rev/min.

Taking negative sign, $\left(\frac{N_c}{N}\right)^2 = 1 - 0.09$, $N_2 = \frac{758.69}{\sqrt{0.91}} = 795.3$ rev/min.

Hence, the range of speed is 726.72 rev/min to 795.3 rev/min, within which it is dangerous to run.

3. Critical speed of a shaft having more than one disc or rotor

Consider a shaft carrying two rotors of masses 'm_1' and 'm_2' rotating about the axis of rotation at an angular velocity of 'ω' rad/s, as shown in Fig. 5.10(a).

Let 'y_1' and 'y_2' be the deflections of the two rotors from the centre line of the bearings.

Let 'F_{c1}' and 'F_{c2}' be the centrifugal forces on these two rotors as shown in Fig. 5.10(b).

Let 'e_1' and 'e_2' be the eccentricities of these two rotors displaced from the geometric centres. Then the centrifugal forces 'F_{c1}' and 'F_{c2}' are given by the equations

$$\left.\begin{aligned} F_{c1} &= m_1 \omega^2 (y_1 + e_1) \\ F_{c2} &= m_2 \omega^2 (y_2 + e_2) \end{aligned}\right\} \quad \text{...5.12(a)}$$

In above equations we have shown that e_1 and e_2 are along the same directions as that of y_1 and y_2 or otherwise e_1 and e_2 can be treated as a vectors with respect to y_1 and y_2.

By applying the influence coefficients method, write the deflections of the two rotors as follows:

$$\left.\begin{aligned} &\text{For rotor one,} & y_1 &= a_{11}F_{c1} + a_{12}F_{c2} \\ &\text{And for rotor two,} & y_2 &= a_{21}F_{c1} + a_{22}F_{c2} \end{aligned}\right\} \quad \text{...5.12(b)}$$

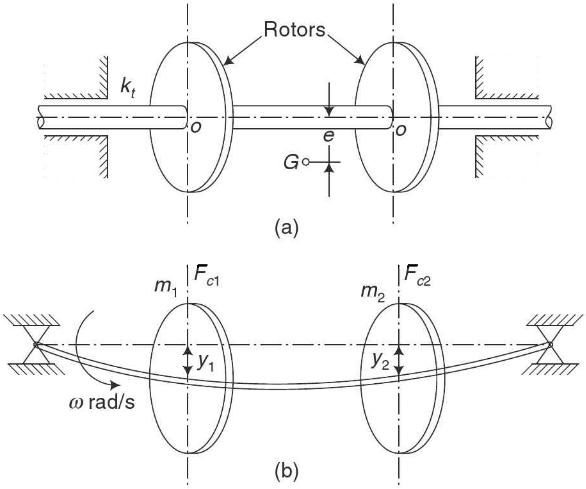


Fig. 5.10 *Critical speed of a shaft having more than one rotor*

Substituting the values of F_{c1} and F_{c2} in Eq. 5.12(b), we get

$$\left.\begin{aligned} y_1 &= a_{11}\, m_1 \omega^2 (y_1 + e_1) + a_{12}\, m_2\, \omega^2 (y_2 + e_2). \\ y_2 &= a_{21}\, m_1\, \omega^2 (y_1 + e_1) + a_{22}\, m_2\, \omega^2 (y_2 + e_2). \end{aligned}\right\} \qquad \text{...5.12(c)}$$

By using these equations we can determine the deflections of the two rotors 'y_1' and 'y_2' and is more important than these values of the critical speeds. That means it is clearly understood that at the critical speeds, the deflections of the two rotors y_1 and y_2 will be too much. Since we can ignore the values of eccentricities 'e_1' and 'e_2', compare with that of the deflections of the two rotors 'y_1' and 'y_2' respectively. Then the above equations 5.12(c) can be written as

$$\left.\begin{aligned} a_{11}\, m_1\, \omega^2 y_1 - y_1 + a_{12}\, m_2\, \omega^2 y_2 = 0 \text{ or } (a_{11}\, m_1\, \omega^2 - 1)\, y_1 + (a_{12}\, m_2\, \omega^2)\, y_2 = 0 \\ a_{21}\, m_1\, \omega^2 y_1 + a_{22}\, m_2\, \omega^2 y_2 - y_2 = 0 \text{ or } (a_{21}\, m_1 \omega_2)\, y_1 + (a_{22}\, m_2 \omega^2 - 1)\, y_2 = 0 \end{aligned}\right\}$$

...5.12(d)

The solution of the above equations

$$\begin{bmatrix} (a_{11}\, m_1\, \omega^2 - 1) & (a_{12}\, m_2\, \omega^2) \\ (a_{21}\, m_1\, \omega^2) & (a_{22}\, m_2\, \omega^2 - 1) \end{bmatrix} = 0 \qquad \text{... 5.12 (e)}$$

$$(a_{11}\, m_1\, \omega^2 - 1)\,(a_{22}\, m_2\, \omega^2 - 1) - (a_{21}\, m_1\, \omega^2)\,(a_{12}\, m_2\, \omega^2) = 0$$

$$a_{11}\, a_{22}\, m_1 m_2\, \omega^4 - a_{22}\, m_2\, \omega^2 - a_{11}\, m_1\, \omega^2 - a_{21}\, a_{12}\, m_1 m_2\, \omega^4 + 1 = 0$$

The above equation can be rearranged as

$$(a_{11}\, a_{22}\, m_1 m_2 - a_{21}\, a_{12} m_1 m_2)\,(\omega^2)^2 - (a_{22}\, m_2 + a_{11}\, m_1)\,\omega^2 + 1 = 0 \qquad \text{...5.12(f)}$$

This in the form of quadric equation of 'ω^2'.

$$\therefore \qquad \omega^2 = \left(\frac{y \pm \sqrt{y^2 - 4x}}{2x}\right) \qquad \text{...5.12(g)}$$

where $x = (a_{11}\, a_{22}\, m_1 m_2 - a_{21}\, a_{12} m_1 m_2)\, \omega^2$ and $y = -\,(a_{22}\, m_2 + a_{11}\, m_1) = 0$

$$\omega^2 = \frac{(a_{22}m_2 + a_{11}m_1) \pm \sqrt{(a_{22}m_2 + a_{11}m_1)^2 - 4(a_{11}a_{22}m_1m_2 - a_{21}a_{12}m_1m_2)\omega^2}}{2(a_{11}a_{22}m_1m_2 - a_{21}a_{12}m_1m_2)\omega^2} \quad \ldots 5.12(h)$$

In Eq. 5.12(h), the positive value gives the second critical speed and the negative value gives the first critical speed. These critical speeds can also be calculated by using Rayleigh's method by means of determining the natural frequency of lateral vibration of the shaft.

In case the shaft has more than two rotors, the solution becomes more complicated and in such cases we are more interested in lower critical speed. The solutions can easily obtained by means of Rayleigh's method considering the mass of shaft also.

5.6 SECONDARY CRITICAL SPEED

In Section 5.5, critical speed of a light shaft having a single rotor without damping – with damping and more than one disc or rotor are the main critical speeds of a shaft resulting from the centrifugal forces of the unbalanced masses. A good amount of vibration has been observed at half the critical speed. These effects will differ in two positions i.e. in horizontal and vertical shaft. These effects have been noticed in horizontal shaft only, whereas in case of vertical shafts these effects have been found totally absent. This is because gravity must be one of the important reasons of it. The importance and severity of the critical speed is called a '**secondary critical speed'** but this is less than that of the primary or main critical speed. This can be described as follows.

Let us consider a rotating shaft without vibration, as shown in Fig. 5.11. In this case, the geometric centre '*C*' of the rotor coincides with '*O*', the point where the bearing centre line intersects the plane of the rotor.

Let '*G*' be the centre of gravity of the rotor or disc at a distance of '*r*' from the geometric centre '*C*' or '*O*'. Let '*m*' be the mass of the shaft and '*I*' be the mass moment of inertia of the rotor about its geometric axis. Let 'ω' be the angular speed of the shaft as shown in Fig. 5.11. Now let us consider a shaft rotating in anticlockwise direction, the torque on the rotor due to the gravitational force '*mg*' accelerates the shaft when the point '*G*' lies on the left of '*C*' and retards the shaft when the point '*G*' lies on the right of '*C*', and magnitude of this torque is '$mgr \cos \omega t$'.

Due to variation of this torque, $mgr \cos \omega t$, the angular acceleration of the shaft is $\left(\frac{mg}{I}\right) r \cos \omega t$. Hence the tangential acceleration of point '*G*' is $\left(\frac{mg}{I}\right) r^2 \cos \omega t$. From this tangential acceleration, there must be an equivalent force $\left(\frac{m^2 g}{I}\right) r^2 \cos \omega t$. The component of this tangential force acting along the vertical direction will be $\cos \omega t$ as greater. Then vertical force $= \frac{m^2\, g}{I} r^2 \cos^2 \omega t = \frac{1}{2} \frac{m^2 g}{I} r^2 [1 + \cos 2\, \omega t]$ ...5.13

In Eq. 5.13, the first portion of the force is a constant, also taken up as a little bit of a additional deflection of the shaft. The latter variable part has a frequency 2ω. If the shaft is running on half of its critical speed, there is a variation in vertical force which occurs at the natural frequency where a larger amount of vibration occurs.

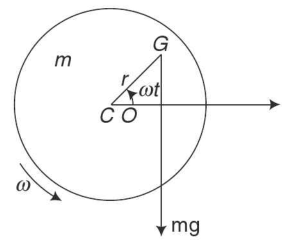


Fig 5.11 *Secondary critical speed*

5.7 CRITICAL SPEED OF A LIGHT CANTILEVER SHAFT WITH A LARGE HEAVY ROTOR AT ITS END

In a light shaft having two end supports and a central rotor or disc, the system has been shown to have only one critical speed. Even if the rotor is not in midposition, the mass of the rotor is assumed to be concentrated, and the system will have only one critical speed. On the other hand if the rotor has mass as well as moment of inertia and is not in mid-position then the system will have two critical speeds. The example given is of a light cantilever shaft having a rotor which has a mass as well as moment of inertia. Here, the critical speed is numerically equal to the natural frequency of lateral vibration and is determined by considering the light cantilever shaft as an example.

Let us consider a light cantilever beam with a large heavy rotor at its free end and the beam so as to be displaced from the equilibrium position as shown in Fig. 5.12. After given initial displacement (y) to the system, we consider two parameters of the system as mass of the rotor and moment of inertia of the beam.

Let m = Mass of the rotor, mr^2 = Moment of inertia of the rotor about an axis passing through the centre of gravity(CG) of the rotor and perpendicular to the plane of the paper.

y and $\ddot{y}$ = Displacement and acceleration of the centre of gravity(CG) of the rotor.

θ and $\ddot{\theta}$ = Angular displacement and angular acceleration of the centre of gravity of the rotor.

l = Length of the beam, E = Modulus of elasticity of the material of the beam.

I = Moment of inertia of the section of the beam about the neutral axis.

a_{11} = Deflection of the centre of gravity(CG) of the rotor per unity force acting on it in the lateral direction = $\dfrac{l^3}{3EI}$.

a_{12} = Deflection of the centre of gravity(CG) of the rotor per unit moment acting on it equal to the slope at the free end of the beam per unit force acting on it in a lateral direction = $\dfrac{l^2}{2EI}$.

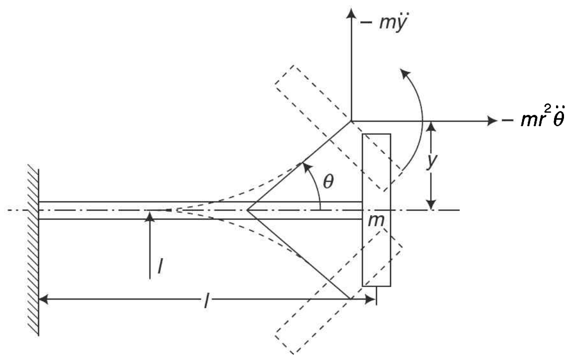


Fig. 5.12 *Critical speed of a light cantilever shaft with a large heavy rotor at its end*

a_{22} = Slope at the free end of the beam per unit moment acting on the centre of gravity of the rotor in the plane of the paper $= \dfrac{l}{EI}$.

Here, we consider two types of forces, inertia force and inertia torque, on the rotor after it is displaced from the equilibrium position as shown in Fig. 5.12 along with their direction as follows:

The Inertia force $= -m\ddot{y}$

The principle mode of vibration of the system is

$$y = y \sin \omega t \;\; \ddot{y} = -y\omega^2$$

∴ the Inertia force $= m\omega^2 y$

The inertia torque $= -mr^2\ddot{\theta}$

The principle mode of vibration of the system is

$$\theta = \theta \sin \omega t \text{ and } \ddot{\theta} = -\omega^2\theta$$

∴ the inertia torque $= m\, r^2\omega^2\, \theta$

where 'ω' is natural frequency of vibration of the system.

Then determining the deflection and rotation of the rotor or disc in the plane of paper is given $y = a_{11}\, m\omega^2 y + a_{12}\, m\, r^2\omega^2\theta$ (for deflection)

and $\theta = a_{21}\, m\omega^2 y + a_{22}\, mr^2\omega^2\theta$ (for rotation).

Substituting the values of a_{11} a_{12} a_{21} and a_{22} neglecting the 'y' and 'θ' in above two equations simply and let $G = \omega\sqrt{\dfrac{ml^3}{3EI}}$ and $C = \dfrac{3r^2}{l^2}$.

We get $CG^4 - 4(C+1)G^2 + 4 = 0$

This will give the two natural frequencies as $G_{1,2}^2 = \left[(C+1) \pm \sqrt{(C+1)^2 - C}\right]$

Figure 5.13 is a plot of the above equation and shows the vibration of the two natural frequencies of the system with the change in $C = \frac{3r^2}{l^2}$; it is noticed that 'r' is the radius of gyration of the rotor about an axis passing through its centre of gravity(CG) and perpendicular to the axis of the rotor. It may be seen that there will be two conditions, when $C = 0$, for the concentrated mass and $C = \infty$, the rotor having large radius of gyration.

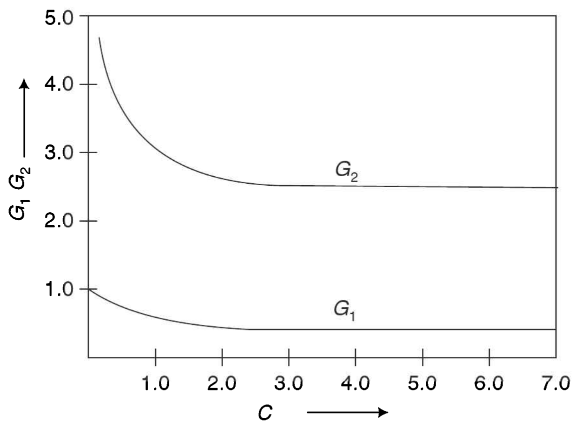


Fig. 5.13 *Vibration of the two natural frequencies with the change in* $C = \frac{3r^2}{l^2}$

REVIEW QUESTIONS

(1) What is the importance of vibration measurement? List out the vibration-measuring instrument.

(2) Derive an expression for the relative motion of the 'seismic' mass of a seismic instrument and indicate how it can be used to measure displacement and acceleration of a vibrating body.

(3) Explain the 'seismic' instrument and how it will be used to measure displacement and acceleration.

(4) Sketch the dimensionless amplitude versus frequency curves of a vibration–measuring instrument. Explain in what region it can be used as an accelerometer.

(5) Sketch the basic elements constituting a 'seismic instrument' and explain mathematically how it can be utilised to indicate the displacement and acceleration of a vibrating body.

(6) Explain with a neat the sketch working principle of vibrometers with their range of frequency of operation.

(7) Explain with a neat sketch the working principle of the accelerometer. Discuss the effects of amplitude distortion in such an instrument.

(8) Discuss the basic principle on which vibration-measuring instruments are designed. What are their practical limitations?

(9) What are the principle of the vibrometer and accelerometer? What is the difference between these two?

(10) How do you differentiate a displacement pick-up, velocity pick-up and acceleration pick-up?

(11) Write short notes on whirling of shaft.

(12) Derive an expression for the critical speed of a light shaft having a single disc without damping.

(13) Derive an expression for the critical speed of a light shaft having a single disc with damping.

(14) Prove that the_critical speed or whirling speed for a rotating shaft is same as the frequency of natural transverse vibration.

(15) What do you understand by 'critical speed' of shafts? Why does it occur?

PROBLEMS FOR PRACTICE

(1) A car has a vertical natural frequency of 13.9 cpm. This car is driven along a road whose deviation varies approximately as sinusoidal. The distance between peak to trough is 10 cm and distance between two peaks is 30 m as shown in Fig. p.p-5.1. Assuming the car has a single-degree-of freedom system having a damping ratio of 0.2 for shock absorber, what is the amplitude of vibration of the car at a speed of 50 km/h?

Ans. $y = 2.06$ cm.

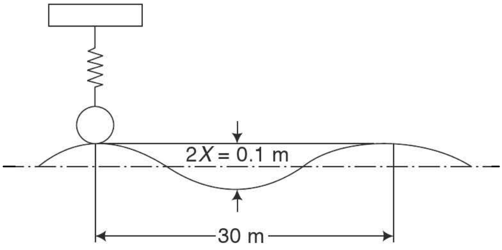


Fig. p.p-5.1 *Wavelength or distance between successive peaks or trough*

(2) A commercial type vibration pick-up (vibrometer) has a natural frequency of 6 cps and a damping factor $\xi = 0.6$. Calculate the relative displacement amplitude if the instrument is subject to a motion $x = 0.08 \sin 20t$.

Ans. Relative amplitude = 0.023 m.

(3) An undamped vibration pick-up having a natural frequency of 1 Hz is used to measure a harmonic vibration of 4 Hz. If the amplitude recorded is 0.52 mm, what is the correct amplitude?

Ans. Correct amplitude $X = 0.488$ mm.

(4) It is desired to measure the maximum acceleration of a machine part which vibrates violently with a frequency of 700 cpm. An accelerometer with negligible damping is

attached to it, the total travel of the indicator arm is 0.82 cm. If the accelerometer has a mass of 0.5 kg and spring of stiffness 18000 N/m, what is (i) the maximum amplitude of vibration, (ii) the maximum velocity of vibration, and (iii) the maximum acceleration?

Ans. (i) $X = 4.57 \times 10^{-2}$ m (ii) maximum velocity = 3.35 m/s (iii) maximum acceleration = 245.61 m/s^2.

(5) A vibrometrer is to measure amplitude at a lowest frequency of 10 cps with an accuracy of at least 1.5%. The seismic mass is to be about 1 kg. What is the spring stiffness of the spring? How much should be the damping in the system? Take $\xi = 0.3$.

Ans. 71.04 N/m.

(6) The distance between the bearings in a freely supported shaft of 50 mm diameter is 1.5 m. It carries a single concentrated load of 50 N at its centre. Determine its critical speed if the shaft material has a density of 10 g/cc, take $E = 2 \times 10^{11}$ N/m.

Ans. Nc = 2004 rev/min.

(7) A single rotor weighing 100 N is mounted midway between the bearings on a steel shaft of 10 mm diameter. The bearing span is 0.4 m. It is known that the centre of gravity of the rotor is 0.25 mm from its geometric centre. If the system rotates at 1000 rev/min and the damping ratio is estimated to be 0.05, determine (i) the amplitude of vibration, (ii) the dynamic load transmitted to bearings, and (iii) maximum stress induced in the shaft, when the shaft is supported vertically.

Neglect the weight of the shaft; assume the shaft to be simply supported and take $E = 1.96 \times 10^{11}$ N/m^2.

Ans. Amplitude of vibration, $y = 6.89 \times 10^{-4}$ m (ii) Dynamic load F_d = 50.1 N (iii) Maximum stress induced (σ_{max}) = 51.02×10^6 N/m^2.

(8) A rotor has an eccentricity of 12 mm. It is mounted on a shaft and bearing system whose stiffness is 3.2×10^5 N/m and has damping ratio of 0.07. What is the amplitude of whirling when the rotor operates at 1000 rev/min? Also find the critical speed of the shaft.

(9) A turbo supercharge rotor weighs 8.6 kgf and is keyed to the centre of a 2.12 cm diameter steel shaft mounted on two bearings 33.8 cm apart. Determine

(i) critical speed of shaft (ii) amplitude of vibration of rotor at a speed 2675 rev/min. if the eccentricity is 0.0012 cm, and (iii) vibratory force transmitted to the bearings at this speed.

Ans. (i) Critical speed of the shaft N_c = 5112.60 rev/min. (ii) Amplitude of vibration $y = 4.45 \times 10^{-6}$ m (iii)Vibratory force transmitted (ky) = 10.97 N.

(10) A rotor of a small high-speed steam turbine weighing 7.0 N is mounted at the midpoint of a steel shaft of 7.5 mm diameter, supported in self-aligning bearings over a span of 40 cm. Owing to slight manufacturing inaccuracy, the centre of gravity of the disc is 0.0125 cm from the centre of the disc. Determine (i) critical speed of the shaft, and (ii) the dynamic force transmitted to each bearing if the turbine rotates at 3500 rev/min. Take $E = 2.0 \times 10^7$ N/cm^2. Neglect the weight of the shaft.

Ans. (i) N_c = 2768.44 rev/min. (ii) Dynamic load on each bearing 10.35. Total load on each bearing = 13.64 N.

OBJECTIVE-TYPE QUESTIONS

(1) An accelerometer is used to measure acceleration
   (a) because of its natural frequency is high compared to that of vibration to be measured
   (b) because of its natural frequency is low compared to that of vibration to be measured
   (c) because of its natural frequency is peak value compared to that of vibration to be measured
   (d) all of the above

(2) Rotating shafts tend to vibrate violently in transverse directions at certain speed. This speed is called
   (a) whirling speed
   (b) whipping speed
   (c) critical speed
   (d) all of the above

(3) The critical speed of the shaft carrying a mass m at the centre of the span is given by
   (a) $\omega_n = \sqrt{\frac{k}{m}}$
   (b) $\omega_n = \sqrt{\frac{g}{\delta}}$
   (c) both (a) and (b)
   (d) $\omega_n = \sqrt{\frac{\delta}{g}}$

(4) The whirling speed of a rotating shaft carrying a mass m at the centre is
   (a) equal to natural frequency of transverse vibration of the system
   (b) more or less depending upon the stiffness of the shaft
   (c) more than the natural frequency of transverse vibration of the system
   (d) less than the natural frequency of transverse vibration of the system

(5) The following relationship holds for whirling of shafts:
   (a) $\frac{y}{e} = \frac{1}{(1 - r^2)}$ (b) $y = \frac{er^2}{(r^2 - 1)}$
   (c) $y = \frac{e}{(1 - r^2)}$ (d) $\frac{y}{e} = \frac{r^2}{1 - r^2}$

(6) For shaft speed less than the critical speed, the phase difference between displacement and centrifugal force is
   (a) 45° (b) 90°
   (c) 0° (d) 180°

(7) For shaft speed more than critical speed, the phase difference between displacement and centrifugal force is
   (a) 0° (b) 45°
   (c) 180° (d) 90°

(8) Natural frequency of transverse vibration of a shaft carrying load at the centre of the span is
   (a) $f_n = \frac{5.63}{\sqrt{\delta}}$ Hz
   (b) $f_n = \sqrt{\frac{4.97}{\delta}}$ Hz
   (c) $f_n = \sqrt{\frac{5.63}{\delta}}$ Hz
   (d) $f_n = \frac{4.4987}{\sqrt{\delta}}$ Hz.

(9) Accelerometer is an instrument which is used to measure
   (a) high natural frequency
   (b) low natural frequency
   (c) low and high natural frequency
   (d) none of these

(10) For a shaft having two discs, the critical speed is
   (a) one
   (b) two
   (c) more than one
   (d) none of these

Answers

(1) a	(2) d	(3) c	(4) a	(5) d	(6) c
(7) c	(8) d	(9) a	(10) b		

TWO-DEGREE-FREEDOM SYSTEMS

6

6.1 INTRODUCTION

Systems that require two independent coordinates to specify the system configuration at any instant are called **'two-degree-freedom systems'**. In such a system there are two masses which have two equations of motion, treated as coupled differential equations. Each mass will have its own natural frequency. Sometimes nonharmonic motion of the masses makes the system more complicated for solving problems.

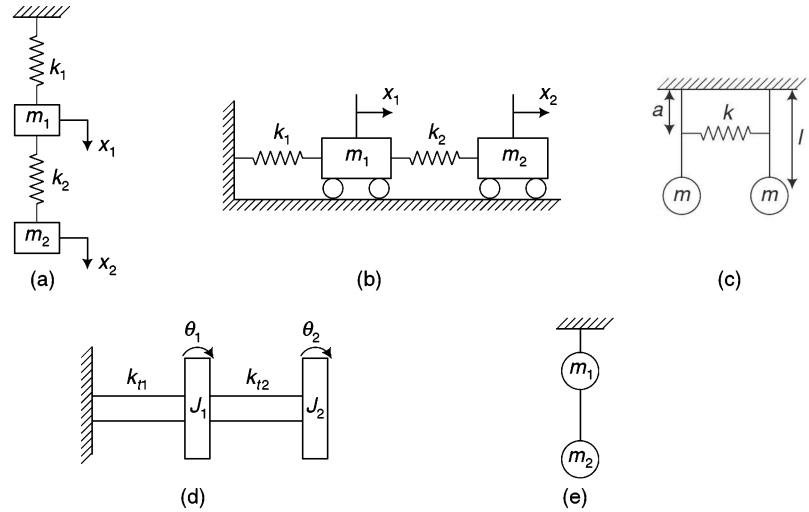


Fig. 6.1 *Two-degree-freedom systems*

Example In a spring-mass system, $k_1 - m_1$ and $k_2 - m_2$ are shown in Fig. 6.1(a) and Fig. 6.1(b). if the masses 'm_1' and 'm_2' are constrained to move vertically or horizontally (linear displacements), two independent coordinates 'x_1' and 'x_2' are necessary to specify their positions at any instant. 'k_1' and 'k_2' are stiffnesses of the spring. Therefore, the given system is a two-degree-freedom system. Other examples are double pendulum and two-rotor system as shown in Fig. 6.1(c), (d), (e).

In Fig. 6.1(c), two masses of a simple pendulum are coupled together by means of a spring 'k'. Similarly, a shaft of torsional stiffness 'k_t' [Fig. 6.1(d)] has two rotors which can have angular displacements 'θ_1' and 'θ_2' independent of each other. In Fig. 6.1(e), two masses 'm_1' and 'm_2' of a simple pendulum are constrained to move vertically. Thus, it is a two-degree-freedom system.

In general, the number of degrees of a freedom system can be stated as the number of mass or masses in a system and multiplied by number of possible types of motion of each mass or masses.

6.2 PRINCIPAL MODE OF VIBRATION, OR NORMAL MODE OF VIBRATION

There are two equations of motion for a two-degree-freedom system, one for each mass. As a result, there are two natural frequencies, for a two-degree-freedom system. The natural frequencies are found by solving the frequency equation of an undamped system or the characteristic equation of a damped system.

When the masses of a system are oscillating in such a manner that they reach maximum amplitudes simultaneously and pass their equilibrium points simultaneously or all the moving parts of the system are oscillating in the same frequency and phase, such a mode of vibration is called **principal mode of vibration,** or **normal mode of vibration.**

If at the principal mode of vibration, the amplitude of one of the masses is considered equal to unity, the mode of vibration is then called **'normal mode of vibration'**, i.e. the amplitude ratio, X_2/X_1 = Principal mode of vibration, if $X_1 = 1$(unity). Then the amplitude ratio, $X_2/1$ = Normal mode of vibration.

In case of two-degrees-freedom system, masses will vibrate in two different modes called '**principal modes**'. If masses 'm_1' and 'm_2' shown in Fig. 6.1(a) are vibrating in phase, such a mode of vibration is called **first principal mode**. When the masses 'm_1' and 'm_2' are vibrating in the opposite phase, such a mode of vibration is called **second principal mode** of vibration.

$\frac{X_2}{X_1}$ = Principal mode, and $\frac{X_2}{1} = \frac{1}{X_1}$ = Normal mode.

In first principal mode, $\frac{X_2}{X_1}$ or $\frac{X_1}{X_2}$ is positive, $X_1 \uparrow$ and $X_2 \uparrow$

In first principal mode, $\frac{X_2}{X_1}$ or $\frac{X_1}{X_2}$ is negative, $X_1 \downarrow$ and $X_2 \uparrow$

5.6 ORTHOGONALITY PRINCIPLE

The principal modes or normal modes of vibration for systems having two or more degrees of freedom are orthogonal. This is known as orthogonality principle.

This is an important property while finding the natural frequencies.

For a two-degree-freedom system, orthogonality principle can be written as

$$m_1 A_1 A_2 + m_2 B_1 B_2 = 0,$$

where A_1, B_1 and A_2, B_2 are the amplitudes of first and second modes of vibration.

EXAMPLE 6.1

Determine the natural frequencies for the following system as shown in Fig. p-6.1(a) and determine the ratio of amplitudes and locates the nodes for each mode of vibration and draw the mode shapes. Given, $m_1 = m$, $k_1 = 2k$, $m_2 = 2m$, $k_2 = k$.

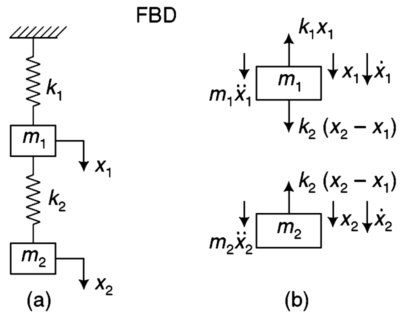


Fig. p-6.1 *Two-degree linear spring-mass system*

Solution Now at any instant, give displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.1(a). The FBD is as shown in Fig. p-6.1(b).

Applying Newton's second law of motion to mass 'm_1', assuming that $x_2 > x_1$

$$\Sigma F = m\,a$$

$$\therefore \quad k_2(x_2 - x_1) - k_1x_1 = m_1\ddot{x}_1$$

$$\therefore \quad m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0$$

$$\therefore \quad m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

But the given values of $m_1 = m,\ k_1 = 2k,\ k_2 = k.$

$$\therefore \quad m\ddot{x}_1 + (2k + k)x_1 - kx_2 = 0,\ m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \qquad ...6.1$$

This is the differential equation of motion of the mass 'm_1'.

Again apply Newton's second law of motion to the mass 'm_2'.

$$\Sigma F = m\,a$$

$$\therefore \quad -k_2(x_2 - x_1) = m_2\ddot{x}_2$$

$$\therefore \quad m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

But the given values of $k_1 = 2k,\ m_2 = 2m,\ k_2 = k$

$$\therefore \quad 2m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad ...\ 6.2$$

This is the differential equation of motion of the mass 'm_2'.

Assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = \omega A \cos \omega t \qquad \dot{x}_2 = \omega B \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 A \sin \omega t \quad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Using the values of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.1, we get

$$m(-A\omega^2 \sin \omega t) + 3kA \sin \omega t - kB \sin \omega t = 0$$

$$-m\omega^2 A \sin \omega t + 3\,kA \sin \omega t = kB \sin \omega t$$

$$A \sin \omega t\,(3k - m\omega^2) = kB \sin \omega t,\ \ A\,(3k - m\omega^2) = kB$$

The amplitude ratio $\therefore \dfrac{A}{B} = \dfrac{k}{3k - m\omega^2}$ .. 6.3

Again using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.2, we get

$$2m(-\omega^2 B \sin \omega t) + k(B \sin \omega t) - k(A \sin \omega t) = 0$$

$$-2m\omega^2 B \sin \omega t + kB \sin \ \omega t = kA \sin \omega t$$

$$B \sin \omega t\,(k - 2m\omega^2) = kA \sin \omega t,\ B(k - 2m\omega^2) = kA$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k - 2m\omega^2}{k}$ ...6.4

From equations 6.3 and 6.4,

$$\frac{k}{3k - m\omega^2} = \frac{k - m\omega^2}{k}$$

$$(3k - m\omega^2)\,(k - 2m\omega^2) = k^2$$

$$3k^2 - 2m\omega^2 \times 3k - m\omega^2 k + 2m^2\omega^4 = k^2,\ \ 2m^2\omega^4 - 7km\omega^2 + 2k^2 = 0$$

$$\omega^4 - \frac{7k}{2m}\,\omega^2 + \frac{k^2}{m^2} = 0$$

This is a quadratic equation in ω^2, where roots are given by

$$\omega^2 = \frac{+\dfrac{7k}{2m} \pm \sqrt{\left(\dfrac{7k}{2m}\right)^2 - \dfrac{4k^2}{m^2}}}{2},\ \omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{\dfrac{49k^2}{4m^2} - \dfrac{4k^2}{m^2}}{4}}$$

$$\omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{49k^2 - 16k^2}{16m^2}},\ \omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{33k^2}{16m^2}}$$

$$\omega^2 = \frac{7k}{4m} \pm \frac{5.74k}{4m},\ \ \omega_{1n}^2 = \frac{7k}{4m} - 5.74k/4m,\ \ \omega_{2n}^2 = \frac{7k}{4m} + \frac{5.74k}{4m}$$

$$\omega_{1n}^2 = 0.315\,\frac{k}{m}\ \omega_{2n}^2 = 3.185\,\frac{k}{m},\ \ \omega_{1n} = 0.56\sqrt{\frac{k}{m}}\ \text{rad/s},\ \ \omega_{2n} = 1.78\sqrt{\frac{k}{m}}\ \text{rad /s}$$

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

To draw the mode shapes

(i) First mode shape. From Eq. 6.3

$$\frac{A}{B} = \frac{k}{3k - m\omega^2}$$

At $\quad \omega^2 = \omega_{1n}^2 = 0.315\,\frac{k}{m}, \qquad \frac{A}{B} = \frac{k}{3k - m \times 0.315\,\frac{k}{m}}$

$\therefore \quad \frac{A}{B} = \frac{1}{2.69},$ i.e. at $A = 1$, $B = 2.6$ [Fig. p-6.1(c)]

(ii) Second mode shape. From Eq. 6.4, $\frac{A}{B} = \frac{k - 2m\omega^2}{k}$

At $\quad \omega^2 = \omega_{2n}^2 = 3.185\,\frac{k}{m}$

At $\quad A = 1$

$$\therefore \quad \frac{A}{B} = \frac{k - 2m \times 3.185\,\frac{k}{m}}{k} = \frac{k - 6.37k}{k}$$

$$\therefore \quad \frac{A}{B} = -5.37$$

$$\therefore \quad A = -5.37\,B \text{ [Fig. p-6.1(d)]}$$

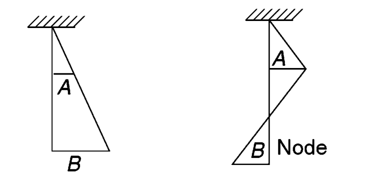


Fig. p-6.1 *Mode shape*

In the first mode of Fig. p-6.1(c) the full spring moves to the right side of the mean line as it is 'same phase'.

Whereas in the second mode of Fig. p-6.1(d), the second spring crosses the mean line as it is 'out of phase'. The crossed point is called **'node'** point, i.e. there is no displacement at that point.

Note: (i) Node is a point in a vibrating system which doesn't experience any displacements.

(ii) As number of modes (degree) increases, number of nodes also increases.

EXAMPLE 6.2

Find the natural frequency of the system as shown in Fig. p-6.2(a).

Solution Now at any instant give displacement 'x_1' to the mass '$2m$' and 'x_2' to the mass 'm' to Fig. p-6.2(a). The FBD is as shown in Fig. p-6.2(b).

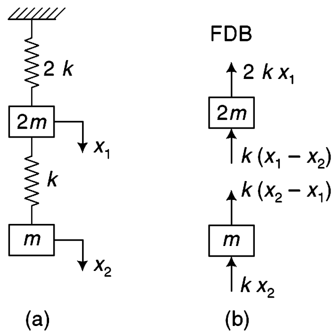


Fig. p-6.2 *Two-degree linear spring-mass system*

Applying Newton's second law of motion to mass '$2m$', assuming that $x_2 > x_1$

$$\Sigma F = m\,a,\; 2m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \qquad \text{...6.5(a)}$$

Newton's second law of motion to mass 'm', $\quad m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad$ ...6.5(b)

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = \omega A \cos \omega t \qquad \dot{x}_2 = \omega B \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 A \sin \omega t \qquad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Using the values of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.5 and again using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.5(a) and rearranging, we have $3k - 2m\,\omega^2\,A - kB = 0$, $-kA + (k - m\omega^2)$ $B = 0$.

The determents of the coefficient of 'A' and 'B' are

$$\begin{bmatrix} 3k - 2m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} = 0,\; (3k - 2m\omega^2)\,(k - m\omega^2) - k^2 = 0,$$

$$2m^2\omega^4 - 5\,km\,\omega^2 + 2\,k^2 = 0$$

$$m\omega\tau^2 = \frac{5k \pm \sqrt{(5k)^2 - 16k^2}}{4} = \frac{5k \pm 3k}{4} = \frac{k}{2} \text{ or } 2k$$

$$m\omega_{1n}^2 = \frac{k}{2} \text{ or } \omega_{1n}^2 = \frac{k}{2m},\; \omega_{1n} = \sqrt{\frac{k}{2m}} \text{ and mode shape } \left(\frac{B}{A}\right)_1 = \frac{3k - 2m\omega_{1n}^2}{k} = 2$$

$$m\omega_{2n}^2 = 2k,\; \omega_{2n} = \sqrt{\frac{2k}{m}}, \text{ and mode shape } \left(\frac{B}{A}\right)_1 = -1$$

EXAMPLE 6.3

Determine the natural frequency and normal modes for the system as shown in Fig. p-6.3(a). Draw the mode shape and locate the node.

Solution Let us at any instant give a vertical displacement 'x_1' to the first mass 'm' and 'x_2' be the second mass as shown in Fig. p-6.3(a). Then the FBD is as shown in Fig. p-6.3(b). Now apply Newton's second law of motion to 'm' (rectilinear motion).

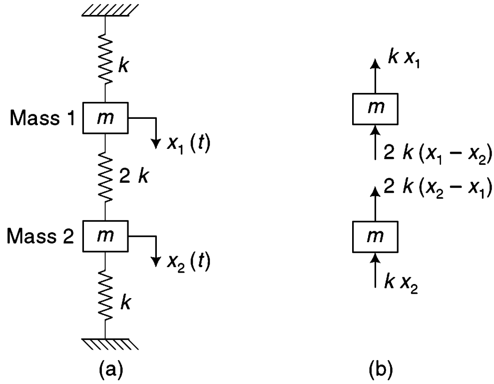


Fig. p-6.3 *Two-degree linear spring-mass system*

From Newton's second law of motion, $\Sigma F = ma$

$$m\ddot{x}_1 = -kx_1 - 2k(x_1 - x_2), \qquad m\ddot{x}_2 = -2k(x_2 - x_1) - kx_2$$

$$m\ddot{x}_1 + kx_1 + 2k(x_1 - x_2) = 0 \qquad \text{...6.6}$$

This is the differential equation of motion for the mass (1)

$$m\ddot{x}_2 + 2k(x_2 - x_1) + kx_2 = 0 \qquad \text{...6.7}$$

This is the differential equation of motion for the mass (2)

Assuming that the motion is periodic and is composed of harmonic motions of varies amplitudes and frequencies, let one of these components be,

$$x_1 = X_1 \sin \omega t \qquad x_2 = X_2 \sin \omega t$$

$$\dot{x}_1 = \omega X_1 \cos \omega t \qquad \dot{x}_2 = \omega X_2 \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin \omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$$

Using these values in Eq. 6.6 and Eq. 6.7, we have

$[-m\omega^2 x_1 + k_1 x_1 + 2k(x_1 - x_2)] \sin \omega t = 0,$

$[-m\omega^2 x_2 + 2k(x_2 - x_1)\ kx_2 +] \sin \omega t = 0$

$\sin \omega t \neq 0$

$$(3k - m\omega^2)\, x_1 - 2kx_2 = 0, \qquad (3k - m\omega^2)\, x_2 - 2kx_1 = 0 \qquad \text{...6.8}$$

The determinant of the coefficient $\begin{vmatrix} x_1 & x_2 \\ (3k - m\omega^2) & (-2k) \\ (-2k) & (3k - m\omega^2) \end{vmatrix} = 0,$

$$(3k - m\omega^2)(3k - m\omega^2) - (-2k)(-2k) = 0$$

$$9k^2 - 3km\omega^2 - 3\,km\omega^2 + m^2\omega^4 - 4k^2 = 0$$

$$9k^2 - 6km\omega^2 + m^2\omega^4 - 4k^2 = 0 \qquad m^2\omega^4 - 6km\omega^2 + 5k^2 = 0 \text{ divided by } m^2$$

$$\omega^4 - \left(\frac{6k}{m}\right)\omega^2 + 5\left(\frac{k}{m}\right)^2 = 0, \qquad \omega_{1,2}^2 = \frac{\frac{6k}{m} \pm \sqrt{\left(\frac{6k}{m}\right)^2 - 4 \times 5\left(\frac{k}{m}\right)^2}}{2}$$

$$\omega_{1,2}^2 = \frac{3k}{m} \pm \sqrt{\left(\frac{3k}{m}\right)^2 - 5\left(\frac{k}{m}\right)^2}, \quad \omega_{1,2}^2 = \frac{3k}{m} \pm \sqrt{9\left(\frac{k}{m}\right)^2 - 5\left(\frac{k}{m}\right)^2} = \frac{3k}{m} \pm 2\frac{k}{m}$$

$$\omega_1^2 = \frac{3k}{m} - 2\frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m} + 2\frac{k}{m}, \quad \omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{5k}{m}$$

$$\omega_1 = \sqrt{\frac{k}{m}} \text{ rad/s}, \quad \omega_2 = \sqrt{\frac{5k}{m}} \text{ rad/s}$$

From Eq. 6.8, $\left(\frac{x_1}{x_2}\right) = \frac{-2k}{(3k - m\omega_1^2)} = \frac{2k}{\left(3k - m \times \frac{k}{m}\right)} = \frac{2k}{2x} = +1$

$$\left(\frac{x_1}{x_2}\right)_2 = \frac{\left(3k - m\omega_2^2\right)}{2k} = \frac{\left(3k - m \times \frac{5k}{m}\right)}{2k} = \frac{-2k}{2x} = -1 \quad \frac{x_{11}}{x_{21}} = +1 \quad \frac{x_{12}}{x_{22}} = -1$$

To draw the mode shapes

Take $x_{11} = 1, \quad x_{21} = 1, \quad x_{12} = -1, \quad x_{22} = -1$

The first-mode and second-mode shapes are as shown in Fig. p-6.3(c) and Fig. p-6.3(d).

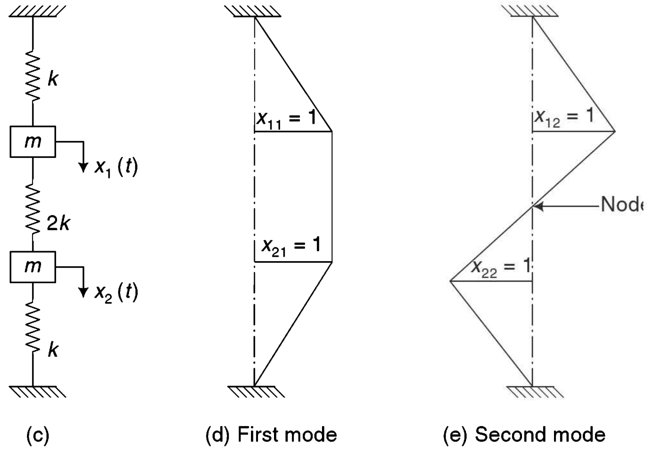


Fig. p-6.3 *Mode shape*

EXAMPLE 6.4

Determine the equation of motion and the natural frequencies of the two-degree-freedom system shown in Fig. p-6.4(a). Determine the displacements 'x_1' and 'x_2' in terms of natural frequencies.

Solution Now at any instant give linear displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.4(a). The FBD is as shown in Fig. p-6.4(b).

Now applying Newton's second law of motion to mass 'm_1', assuming that $x_2 > x_1$

$$\Sigma F = m\ddot{x} \quad \rightarrow \text{Take +ve}, \quad \leftarrow \text{Take −ve}$$

$$\therefore \quad -k_1x_1 + k_2(x_2 - x_1) = m_1\ddot{x}, \; m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$

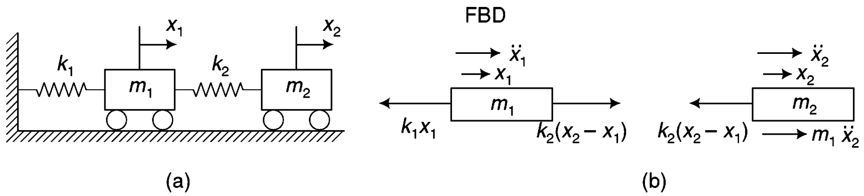


Fig. p-6.4 *Two-degree linear spring-mass system*

$$m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0,\ m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \qquad ...6.9$$

This is a differential equation of motion of mass 'm_1'.

Applying Newton's second law of motion to mass 'm_2'$\Sigma F = m\ddot{x}$,

$$\therefore \quad -k_2(x_2 - x_1) = m_2\ddot{x}_2,\ m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0,\ m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \qquad ...6.10$$

This is a differential equation of motion of mass 'm_2'.

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A\sin\omega t \qquad x_2 = B\sin\omega t$$

$$\ddot{x}_1 = -A\omega^2\sin\omega t \qquad \ddot{x}_2 = -B\omega^2\sin\omega t$$

Using the value of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.9,

$$m_1(-A\omega^2\sin\omega t) + (k_1 + k_2)A\sin\omega t - k_2B\sin\omega t = 0$$

$$-m_1\omega^2A + (k_1 + k_2)A - k_2B = 0,\ A[(k_1 + k_2) - m_1\omega^2] = k_2B$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k_2}{(k_1 + k_2) - m_1\omega^2}$ ...6.11

Using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.10,

$$m_2(-\omega^2B\sin\omega t) + k_2(B\sin\omega t) - k_2(A\sin\omega t) = 0$$

$$-m_2\omega^2B + k_2B - k_2A = 0, \quad B(k_2 - m_2\omega^2) = k_2A$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k_2 - m_2\omega^2}{k_2}$ ...6.12

From equations 6.11 and 6.12,

$$\frac{k_2}{(k_1 + k_2) - m_1\omega^2} = \frac{k_2 - m_2\omega^2}{k_2},\ (k_2 - m_2\omega^2)[(k_1 + k_2) - m_1\omega^2] = k_2^2$$

$$k_2(k_1 + k_2) - m_1k_2\omega^2 - m_2(k_1 + k_2)\omega^2 + m_1m_2\omega^4 - k_2^2 = 0$$

$$m_1m_2\omega^4 - m_1k_2\omega^2 - m_2k_1\omega^2 - m_2k_2\omega^2 + k_1k_2 + k_2^2 - k_2^2 = 0$$

$$\omega^4 - \frac{k_2}{m_2}\omega^2 - \frac{k_1}{m_1}\omega^2 - \frac{k_2}{m_1}\omega^2 + \frac{k_1k_2}{m_1m_2} = 0$$

$$\omega^4 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right]\omega^2 - \frac{k_1k_2}{m_1m_2} = 0$$

This is a quadratic equation in ω^2, whose roots are given by

$$\omega^2 = \frac{\left[\dfrac{k_1+k_2}{m_1}+\dfrac{k_2}{m_2}\right] \pm \sqrt{\left(\dfrac{k_1+k_2}{m_1}+\dfrac{k_2}{m_2}\right)^2 - 4\dfrac{k_1k_2}{m_1m_2}}}{2}$$

$$\therefore \quad \omega^2 = \left[\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right] \pm \sqrt{\left(\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}}$$

$$\therefore \quad \omega_{1n}^2 = \left[\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right] - \sqrt{\left(\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}},$$

$$\therefore \quad \omega_{2n}^2 = \left[\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right] + \sqrt{\left(\frac{k_1+k_2}{2m_1}+\frac{k_2}{2m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}},$$

Hence the general solutions x_1 and x_2 are given by,

$$x_1 = A_1 \sin \omega_{1n}t + A_2 \sin \omega_{2n}t, \quad x_2 = B_1 \sin \omega_{1n}t + B_2 \sin \omega_{2n}t$$

where A_1, A_2 and B_1, B_2 are constants and are evaluated by four initial conditions:

$$x_1(0), \quad \dot{x}_1(0), \quad x_2(0), \quad \dot{x}_2(0).$$

EXAMPLE 6.5

Determine the natural frequency and the amplitude ratios for the system as shown in Fig. p-6.5(a). If the mass 'm_1' is displaced 1 m from its static equilibrium position and released, determine the resulting displacements 'x_1' and 'x_2'. Given $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$.

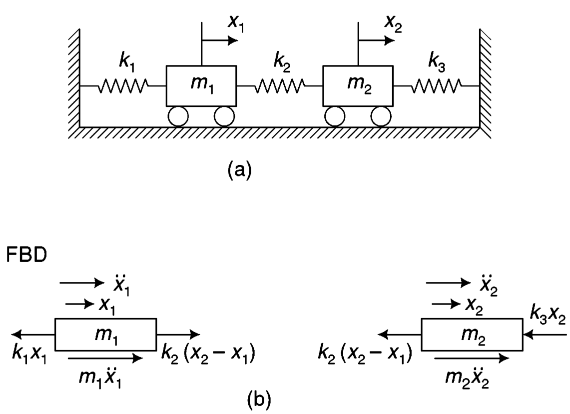


Fig. p-6.5 *Two-degree linear spring-mass system*

Solution Now at any instant give linear displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.5(a). The FBD is as shown in Fig. p-6.5(b).

Now applying Newton's second law of motion to mass 'm_1' assuming that $x_2 > x_1$,

$$k_2(x_2 - x_1) - k_1x_1 = m_1\ddot{x}_1,\ m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0$$

$$m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2x_2 = 0$$

Given $$m_1 = m,\ k_1 = k_2 = k$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \qquad \text{...6.13}$$

This is the differential equation of motion for the mass 'm_1'.

Again applying Newton's second law of motion to the mass 'm_2',

$$-k_2(x_2 - x_1) - k_3x_2 = m_2\ddot{x}_2,\quad m_2\ddot{x}_2 + k_3x_2 + k_2x_2 - k_2x_1 = 0$$

$$m_2\ddot{x}_2 + (k_3 + k_2)\,x_2 - k_2\,x_1 = 0$$

Given $$m_2 = m,\quad k_2 = k_3 = k$$

$$m\ddot{x}_2 + 2k\,x_2 - k\,x_1 = 0 \qquad \text{...6.14}$$

This is the differential equation of motion for the mass 'm_2'.

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A\cos\omega t,\quad x_2 = B\cos\omega t,\quad \ddot{x}_1 = -A\,\omega^2\cos\omega t,\quad \ddot{x}_2 = -B\,\omega^2\cos\omega t$$

Using the values of $x_1, x_2, \ddot{x}_1$ in Eq. 6.13,

$$m(-A\,\omega^2\cos\omega t) + 2kA\cos\omega t - kB\cos\omega t = 0$$

$$A(2k - m\omega^2) = Bk$$

The amplitude ratio $$\frac{A}{B} = \frac{k}{2k - m\omega^2} \qquad \text{...6.15}$$

Using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.14,

$$m(-B\cos\omega t)\,\omega^2 + 2kB\cos\omega t - kA\cos\omega t = 0,\ B(2k - m\omega^2) = kA$$

The amplitude ratio $$\frac{A}{B} = \frac{2k - m\omega^2}{k} \qquad \text{...6.16}$$

From equations 6.15 and 6.16,

$$\frac{k}{2k - m\omega^2} = \frac{2k - m\omega^2}{k},\qquad (2k - m\omega^2)^2 = k^2$$

$$4k^2 - 4mk\omega^2 + m^2\omega^4 = k^2,\ m^2\omega^4 - 4mk\omega^2 + 3k^2 = 0,\ \omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

This is the quadratic equation in ω^2, where $$\omega^2 = \frac{\frac{4k}{m} \pm \sqrt{\left(\frac{4k}{m}\right)^2 - \frac{4\times 3k^2}{m^2}}}{2}$$

$$\omega^2 = \frac{2k}{m} \pm \sqrt{\frac{4k^2}{m^2} - \frac{3k^2}{m^2}} = \frac{2k}{m} \pm \frac{k}{m}\sqrt{4 - 3}$$

$$\omega^2 = \frac{2k}{m} \pm \frac{k}{m}$$

$\therefore \qquad \omega_{1n}^2 = \frac{k}{m}$

$$\omega_{2n}^2 = \frac{3k}{m}$$

$\therefore \qquad \omega_{1n} = \sqrt{\frac{k}{m}}$ rad/s, $\omega_{2n} = 1.73\sqrt{\frac{k}{m}}$ rad/s

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

This resulting displacements 'x_1' and 'x_2' are given by

$$x_1 = A_1 \cos \omega_{1n}t + A_2 \cos \omega_{2n}t \qquad ...6.17$$

$$x_2 = B_1 \cos \omega_{1n}t + B_2 \cos \omega_{2n}t \qquad ...6.18$$

Given $\qquad x_1 = 1$ m at $t = 0$

$\therefore \qquad \dot{x}_1 = 0$ at $t = 0$

$$x_2 = 0 \text{ at } t = 0$$

$$\dot{x}_2 = 0 \text{ at } t = 0$$

Differentiating equations 6.17 and 6.18 with respect to time 't',

$$\dot{x}_1 - A_1\, \omega_{1n} \sin \omega_{1n}t + (-A_2\, \omega_{2n} \sin \omega_{2n}t) \qquad ...6.19$$

$$\dot{x}_2 = -B_1\, \omega_{1n} \sin \omega_{1n}t + (-B_2\, \omega_{2n} \sin \omega_{2n}t) \qquad ...6.20$$

From Eq. 6.15, $\frac{A}{B} = \frac{k}{2k - m\omega^2}$, $\quad$ at $\omega^2 = \omega_{1n}^2 = \frac{k}{m}$

$$\frac{A_1}{B_1} = \frac{k}{2k - m\frac{k}{m}}, \quad \frac{A_1}{B_1} = 1 \text{ or } A_1 = B_1$$

At $\qquad \omega^2 = \omega_{2n}^2 = \frac{3k}{m}, \quad \frac{A_2}{B_2} = \frac{k}{2k - 3k} = -1$

$\therefore \qquad A_2 = -B_2$

Using the initial conditions in equations 6.17, 6.18, 6.19 and 6.20,

$$A_1 + A_2 = 1, \quad B_1 + B_2 = 0, \quad A_1 - A_2 = 0, \quad 2A_1 = 1$$

$\therefore \qquad A_1 = \frac{1}{2}, \quad B_1 = \frac{1}{2}, \quad A_2 = +\frac{1}{2}, \quad B_2 = -\frac{1}{2}$

Thus, the motions of the masses are given by

$$X_1 = \frac{1}{2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{1}{2}\cos\left(1.73\sqrt{\frac{k}{m}}t\right), X_2 = \frac{1}{2}\cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{1}{2}\cos\left(1.73\sqrt{\frac{k}{m}}t\right).$$

EXAMPLE 6.6

Find the natural frequency of oscillation of the double pendulum as shown Fig. p-6.6(a) where $m_1 = m_2 = m$, and $l_1 = l_2 = l$. Draw the mode shapes and locate the nodes for each mode of vibration.

Solution Now at any instant give an angular displacement to the bobs in Fig. p-6.6(a) in the horizontal position.

Let 'θ_1' and 'θ_2' be the angular displacements of masses 'm_1' and 'm_2' respectively from vertical equilibrium positions 'x_1' and 'x_2'. Then the FBD is as shown in Fig. p-6.6(b).

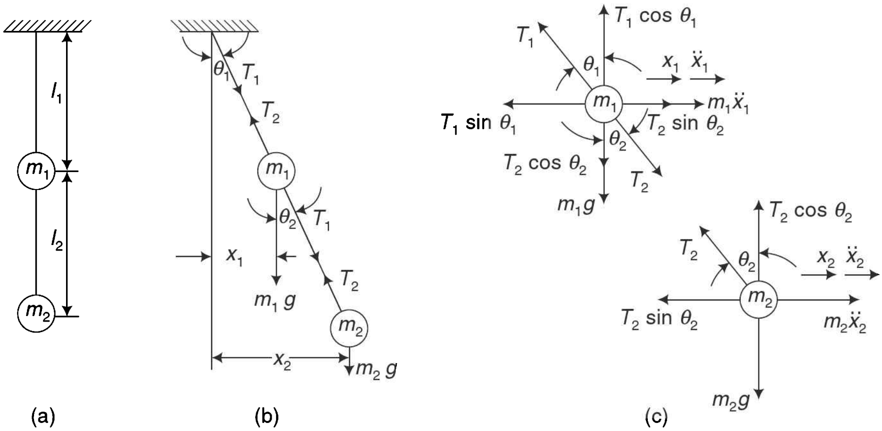


Fig. p-6.6 *Double pendulum*

Now applying Newton's second law of motion to the mass 'm_1' (considering only horizontal forces),

$$-T_1 \sin\theta_1 + T_2 \sin\theta_2 = m_1\ddot{x}_1$$

$$\therefore \quad m_1\ddot{x}_1 + T_1 \sin\theta_1 - T_2 \sin\theta_2 = 0$$

But considering masses 'm_2' and 'm_1',

$$\Sigma v = 0$$

$$\therefore \quad T_2 \cos\theta_2 = m_2 g, \quad T_1 \cos\theta_1 = T_2 \cos\theta_2 + m_1 g$$

For small angles of θ_1 and θ_2, $\cos\theta_1 \approx 1$ and $\cos\theta_2 \approx 1$

$$\therefore \quad T_2 = m_2 g, \quad T_1 = T_2 + m_1 g, \; T_1 = (m_1 + m_2)g$$

For the geometry of Fig. p-6.6(b),

$$\sin\theta_1 = \frac{x_1}{l_1} \text{ and } \sin\theta_2 = \frac{x_2 - x_1}{l_2}$$

$$m_1\ddot{x}_1 + (m_1 + m_2)g\frac{x_1}{l_1} - \frac{mg}{l}x_2 + \frac{mg}{l}x_1 = 0$$

$$m_1\ddot{x}_1 + \frac{3mg}{l_1}x_1 - \frac{mg}{l}x_2 = 0 \qquad \text{...6.21}$$

Applying Newton's second law of motion to the mass m_2 (considering only horizontal force),

$$-T_2 \sin \theta_2 = m_2 \ddot{x}_2$$

$$\therefore \quad m_2\ddot{x}_2 + m_2 g = \frac{x_2 - x_1}{l_2} = 0,\ m\ddot{x}_2 + \frac{mg}{l} x_2 - \frac{mg}{l} x_1 \qquad \text{...6.22}$$

Assuming that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$

$$\ddot{x}_1 = -A\omega^2 \sin \omega t \qquad \ddot{x}_2 = -B\omega^2 \sin \omega t$$

Using these values $x_1, x_2, \ddot{x}_1$ in Eq. 6.21,

$$m(-A\omega^2) + \frac{3mg}{l} A - \frac{mg}{l} B = 0, \quad A\left(\frac{3mg}{l} - m\omega^2\right) = \frac{mg}{l} B$$

$$\frac{A}{B} = \left(\frac{mg}{3mg - ml\omega^2}\right) \qquad \text{...6.23}$$

Using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.22,

$$m(-B\omega^2) + \frac{mg}{l} B - \frac{mg}{l} A = 0, \quad B\left(\frac{mg}{l} - m\omega^2\right) = \frac{mg}{l} A$$

$$\therefore \quad \frac{A}{B} = \frac{mg - ml\omega^2}{mg} \qquad \text{...6.24}$$

From equations 6.23 and 6.24,

$$\frac{mg}{3mg - ml\omega^2} = \frac{mg - ml\omega^2}{mg}$$

$$(3mg - ml\omega^2)(mg - ml\omega^2) = (mg)^2$$

$$3(mg)^2 - 3m^2gl\omega^2 - m^2gl\omega^2 + (ml\omega^2) = (mg)^2$$

$$m^2l^2\omega^4 - 4m^2gl\omega^2 + 2m^2g^2 = 0$$

$$l^2\omega^4 - 4gl\omega^2 + 2g^2 = 0,\ \omega^4 - \frac{4g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

This is a quadratic equation in ω^2.

$$\therefore \quad \omega^2 = \frac{\frac{4g}{l} \pm \sqrt{\left(\frac{4g}{l}\right)^2 - 4 \times 2\left(\frac{g}{l}\right)^2}}{2}$$

$$\therefore \quad \omega^2 = \frac{2g}{l} \pm \sqrt{\frac{16}{4}\left(\frac{g}{l}\right)^2 - \frac{4}{4} \times 2\left(\frac{g}{l}\right)^2}$$

$$\therefore \quad \omega^2 = \frac{2g}{l} \pm \frac{g}{l}\sqrt{4-2} \qquad \therefore \quad \omega^2 = \frac{2g}{l} \pm \sqrt{2}\,\frac{g}{l}$$

$$\therefore \quad \omega_{1n}^2 = \frac{g}{l}(2 - \sqrt{2}) \qquad \therefore \quad \omega_{2n}^2 = \frac{g}{l}(\sqrt{2} + 2)$$

$\therefore \quad \omega_{1n}^2 = 0.59 \dfrac{g}{l}$ $\qquad \therefore \; \omega_{2n}^2 = 3.41 \dfrac{g}{l}$

$\therefore \quad \omega_{1n}^2 = 0.77 \sqrt{\dfrac{g}{l}}$ rad/s $\qquad \therefore \; \omega_{2n}^2 = 1.85 \sqrt{\dfrac{g}{l}}$ rad/s

To draw the mode shapes

(i) First mode shape At $\omega^2 = \omega_{1n}^2 = 0.59 \dfrac{g}{l}$ in Eq. 6.23

$$\frac{A}{B} = \frac{mg}{3mg - ml\omega^2}$$

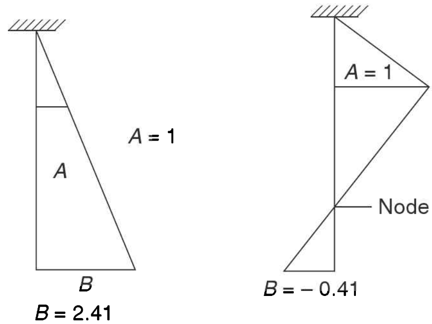
$$\therefore \quad \frac{A}{B} = \frac{g}{3g - ml\omega^2} = \frac{g}{3g - l \times 0.59 \dfrac{g}{l}} = \frac{g}{3g - 0.59\,g},$$

$$\frac{A}{B} = \frac{1}{2.41}, \quad \text{i.e. } A = 1, \; B = 2.41$$

(ii) Second mode shape At $\omega^2 = \omega_{2n}^2 = 3.41 \dfrac{g}{l}$, $\dfrac{A}{B} = \dfrac{g}{3g - 3.41g}$,

$$\frac{A}{B} = \frac{1}{-0.41}, \quad \text{i.e. } A = 1, B = -0.41$$

The first-mode and second-mode shapes are as shown in Fig. p-6.6(d) and Fig. p-6.6(e).



(d) Same phase (e) Out phase

Fig. p-6.6 *Mode shapes*

EXAMPLE 6.7

Determine the two natural frequencies and the corresponding mode shapes of the system shown in Fig. p-6.7(a).

Solution Let at any instant give an angular displacement 'θ_1' to the mass 'm' and 'θ_2' to the mass '$2m$' from the vertical position and 'x_1' and 'x_2' from the vertical equilibrium position to the mass 'm' and '$2m$' respectively in Fig. p-6.7(a). The FBD is as shown in Fig. p-6.7(b).

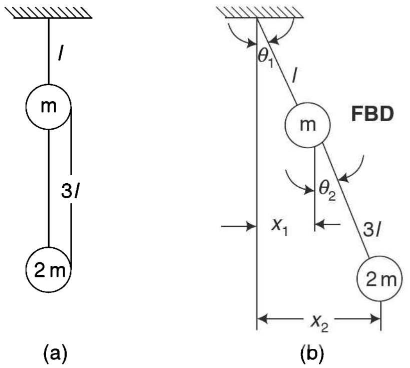


Fig. p-6.7 *Double pendulum*

Now applying Newton's second law of motion to the mass '*m*', assuming that $x_2 > x_1$,

$\sin\theta_1 = \frac{x_1}{l}$, $\sin\theta_2 = \frac{x_2 - x_1}{3l}$, $\theta_1 \approx \frac{x_1}{l}$, $\theta_2 \approx \frac{x_2 - x_1}{3l}$

If 'θ_1' and θ_2 are very small, $\sin\theta_1 \approx \theta_1$ and $\sin\theta_2 \approx \theta_2$

$$T_2 \cos\theta_2 = 2mg, \quad T_1 \cos\theta_1 = mg + T_2 \cos\theta_2$$

$$T_2 \approx 2mg\ T_1 \approx 3mg, \quad m\ddot{x}_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

$$m\ddot{x}_1 + \frac{11mg}{3l}x_1 - \frac{2mgx_2}{3l} = 0 \quad \text{...6.25}$$

$$x_2 = -T\sin\theta_2; \quad \ddot{x}_2 = \frac{9x_2}{3l} - \frac{9x_1}{3l} = 0 \quad \text{...6.26}$$

$$x_1 = X_1 \sin\omega t, x_2 = X_2 \sin\omega t$$

Equations 6.25 and 6.26 become $\left(\frac{119}{3l} - \omega^2\right)X_1 - \frac{29X_2}{3l} = 0$

Frequency equation $\omega^4 - \omega^2\left(\frac{48}{l}\right) + \frac{9^2}{l^2} = 0, -\frac{9}{3l}X_1 + \left(\frac{9}{3l} - \omega^2\right)X_2 = 0,$

$$\omega_1 = 518\sqrt{\frac{9}{l}}, \omega_2 = 1.932\sqrt{\frac{9}{l}}\ (\omega_{12})^2 = \frac{9}{l}(2 \pm \sqrt{3}) \quad \text{...6.27}$$

$$\left(\frac{X_1}{X_2}\right) = 0.196, -10.19 \quad \text{...6.28}$$

EXAMPLE 6.8

Find the natural frequencies of the system shown in Fig. p-6.8(a). Also determine the ratio of amplitudes and locate the nodes for each mode of vibration.

Draw the mode shapes also.

Solution Now at any instant give displacement 'y_1' to the mass '*m*' and 'y_2' to another mass '*m*' in vertical position to Fig. p-6.8(a). Then the FBD is as shown in Fig. p-6.8(b).

Now apply Newton's second law of motion to the mass '*m*' assuming that $y_2 > y_1$.

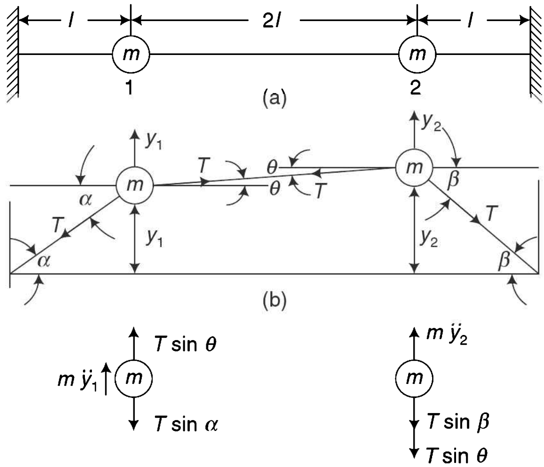


Fig. p-6.8 *System for Example 6.8*

Applying Newton's second law of motion to the mass (1),

$$\Sigma F = m\ddot{x},\ T\sin\alpha - T\sin\theta = -m\ddot{y}_1,$$

$$\therefore \qquad m\ddot{y}_1 + T\sin\alpha - T\sin\theta = 0$$

From the geometry of the FBD in Fig. p-6.8(b),

$$\sin\alpha = \frac{y_1}{l},\ \sin\beta = \frac{y_2}{l} \text{ and } \sin\theta = \frac{y_2 - y_1}{2l}$$

$$m\ddot{y}_1 + T\frac{y_1}{l} - T\left(\frac{y_2 - y_1}{2l}\right) = 0,\ m\ddot{y}_1 + T\frac{y_1}{l} - \frac{y_2}{2l} + T\frac{y_1}{2l} = 0,$$

$$m\ddot{y}_1 + \left(\frac{T}{l} + \frac{T}{2l}\right)y_1 - \frac{T_1}{2l}y_2 = 0$$

$$m\ddot{y}_1 + \frac{3}{2l}Ty_1 - \frac{T}{2l}y_2 = 0 \qquad \text{...6.29}$$

This is the differential equation of motion of the mass (1).

Applying Newton's second law of motion to the mass (2),

$$\Sigma F = m\ddot{x},\ T\sin\theta + T\sin\beta = -m\ddot{y}_2,\ m\ddot{y}_2 + T\sin\beta + T\sin\theta = 0$$

$$m\ddot{y}_2 + T\frac{y_2}{l} + T\left(\frac{y_2 - y_1}{2l}\right) = 0,\ m\ddot{y}_2 + \frac{3T\,y_2}{2l} - \frac{T}{2l}y_1 = 0 \qquad \text{...6.30}$$

This is the differential equation of motion of the mass (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$y_1 = A\sin\omega t \qquad y_2 = B\sin\omega t$$

$$\ddot{y}_1 = -A\omega^2\sin\omega t \qquad \ddot{y}_2 = -B\omega^2\sin\omega t$$

Using the values of y_1, y_2 and $\ddot{y}_1$ in Eq. 6.29,

$$m(-A\omega^2\sin\omega t) + \frac{3T}{2l}A\sin\omega t - \frac{T}{2l}B\sin\omega t = 0,\ A\left[\frac{3T}{2l} - m\omega^2\right] = B\frac{T}{2l}$$

The amplitude ratio $\frac{A}{B} = \frac{T}{3T - 2ml\omega^2}$ ...6.31

Using the values of $y_1, y_2, \ddot{y}_2$ in Eq. 6.30,

$$m(-B\omega^2 \sin \omega t) + \frac{3T}{2l} B \sin \omega t - \frac{T}{2l} A \sin \omega t = 0$$

$$B\left[\frac{3T}{2l} - m\omega^2\right] = A\,\frac{T}{2l}$$

The amplitude ratio $\frac{A}{B} = \frac{3T - 2ml\omega^2}{T}$ ...6.32

From equations 6.31 and 6.32, $\frac{T}{3T - 2ml\omega^2} = \frac{3T - 2ml\omega^2}{T}$

$\therefore\ (3T - 2ml\omega^2)^2 = T^2,\ 3T - 2ml\omega^2 = \pm T,\ 2ml\omega^2 = 3T \pm T,\ \omega^2 = \frac{3T \pm T}{2ml}$

$\therefore\ \omega_{1n}^2 = \frac{2T}{2ml} = \frac{T}{ml} \quad \therefore\ \omega_{1n} = \sqrt{\frac{T}{ml}}$ rad/s $\therefore\ \omega_{2n}^2 = \frac{4T}{2ml} = \frac{2T}{ml}$

$\therefore\ \omega_{2n}^2 = 1.41\sqrt{\frac{T}{ml}}$ rad/s

To draw the mode shapes

At $\quad \omega^2 = \omega_{1n}^2 = \frac{T}{ml}$ in Eq. 6.31 at $\omega^2 = \omega_{2n}^2 = \frac{2T}{ml}$ in Eq. 6.32,

$$\frac{A}{B} = \frac{T}{3T - 2ml\omega^2}\ \frac{T}{3T - 2ml \times \frac{2T}{ml}} = \frac{T}{3T - 2ml \times \frac{T}{ml}} = 1$$

i.e. $\quad A = 1, B = 1, \quad \frac{A}{B} = \frac{T}{3T - 4T} = -1, \quad$ i.e. $A = 1, \quad B = -1$

The first-mode and second-mode shapes are as shown in Fig. p-6.8(c) and Fig. p-6.8(d).

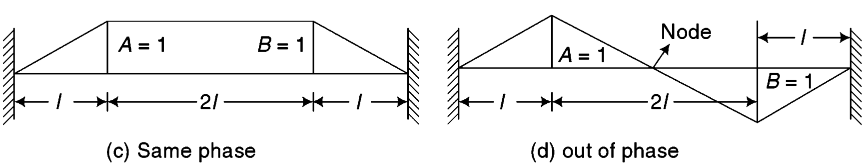


Fig. p-6.8 *Mode shape*

EXAMPLE 6.9

Determine the natural frequencies for the system shown in Fig. p-6.9(a).

Solution Let at any instant give an angular displacement 'θ_1' to the mass 'm' and 'θ_2' to the mass 'm' from the vertical position to Fig. p-6.9(a). The FBD is as shown in Fig. p-6.9(b).

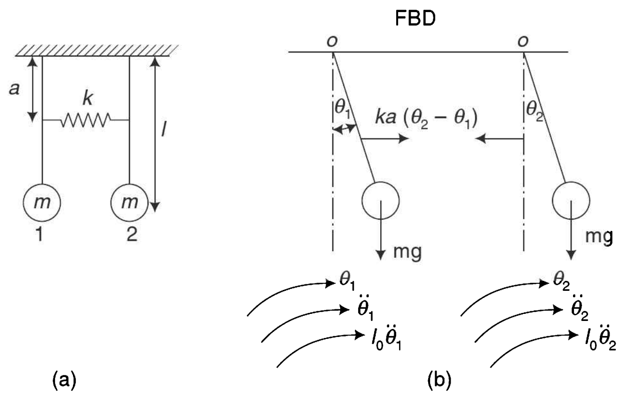


Fig. p-6.9 *Double coupled pendulum*

Now apply Newton's second law of motion to mass 'm' assuming that $\theta_2 > \theta_1$.

Applying Newton's second law of motion to the mass (1),

$$\Sigma M_o = I_0\ddot{\theta},\ ka\,(\theta_2 - \theta_1)\,a - mgl \sin\theta_1 = I_0\ddot{\theta}_1, \text{ if } \theta \text{ is very small, } \sin\theta \approx \theta$$

$$\therefore \quad I_0\ddot{\theta}_1 + mgl\,\theta_1 - ka^2(\theta_2 - \theta_1) = 0$$

But $\quad I_0 = ml^2,\ ml^2\ddot{\theta}_1 + mgl\,\theta_1 + ka^2\theta_1 - ka^2\theta_2 = 0$

$$ml^2\ddot{\theta}_1 + (ka^2 + mgl)\theta_1 - ka^2\theta_2 = 0 \qquad \text{...6.33}$$

This is the differential equation of motion for the mass (1).

Apply Newton's second law of motion to mass (2), $-ka\,(\theta_2 - \theta_1)\,a - mgl \sin\theta_2 = I_o\ddot{\theta}_2$, if θ is very small, $\sin\theta \approx \theta$

$$I_0\ddot{\theta}_2 + ka^2(\theta_2 - \theta_1) = mgl\theta_2$$

But $\quad I_0 = ml^2$

$$ml^2\ddot{\theta}_2 + ka^2\theta_2 + mgl\theta_2 - ka^2\theta_1 = 0,\ ml^2\ddot{\theta}_2 + (ka^2 + mgl)\,\theta_2 - ka^2\theta_1 = 0 \qquad \text{...6.34}$$

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A\sin\omega t \qquad \theta_2 = B\sin\omega t$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega t \qquad \ddot{\theta}_2 = -B\omega^2\sin\omega t$$

Using these value of θ_1, θ_2, and $\ddot{\theta}_1$ in Eq. 6.33, we have

$$ml^2(-A\omega^2\sin\omega t) + (ka^2 + mgl)\,A\sin\omega t - ka^2B\sin\omega t = 0$$

$$A[(ka^2 + mgl) - ml^2\omega^2] = ka^2B$$

The amplitude ratio $\dfrac{A}{B} = \dfrac{ka^2}{[(ka^2 + mgl) - ml^2\omega^2]}$ ...6.35

Using these values of θ_1, θ_2, and $\ddot{\theta}_1$ in Eq. 6.34, we have

$$ml^2(-B\omega^2\sin\omega t) + (ka^2 + mgl)\,B\sin\omega t - ka^2A\sin\omega t = 0$$

$$B[(ka^2 + mgl) - ml^2\omega^2] = ka^2A$$

The amplitude ratio $\therefore \dfrac{B}{A} = \dfrac{ka^2}{[(ka^2 + mgl) - ml^2\omega^2]}$ ...6.36

From equations 6.35 and 6.36, $[(ka^2 + mgl) - ml^2\omega^2]^2 = [ka^2]^2$

$$(ka^2 + mgl) - ml^2\omega^2 = \pm\, ka^2, \qquad ml^2\omega^2 = (ka^2 + mgl) \pm ka^2$$

$$\omega^2 = \frac{ka^2 + mgl \pm ka^2}{ml^2},\ \omega_{1n}^2 = \frac{ka^2 + mgl - ka^2}{ml^2},\ \omega_{2n}^2 = \frac{ka^2 + mgl + ka^2}{ml^2},\ \omega_{1n}^2 = \frac{g}{l}$$

$$\omega_{1n} = \sqrt{\frac{g}{l}}\ \text{rad/s}, \qquad \omega_{2n}^2 = \frac{2ka^2 + mgl}{ml^2}, \qquad \omega_{2n} = \sqrt{\frac{2ka^2 + mgl}{ml^2}}\ \text{rad/s.}$$

EXAMPLE 6.10

Two uniform slender rods weighing $w_1 = 131.4$ N and $w_2 = 65.7$ N are suspended at their upper ends and are connected by a spring of stiffness 876 N/m as shown in Fig. p-6.10. Compute the natural frequencies of the system.

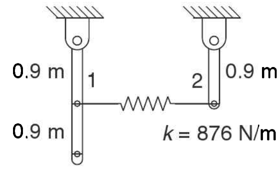


Fig. p-6.10 *Uniform rods*

Solution Assume that at any instant during the vibratory motion the bars make angles 'θ_1' and 'θ_2' where $\theta_1 > \theta_2$. The compressive force in the spring is 876 $k(\theta_2 - \theta_1)$. The mass moment of inertia are

$$I_1 = \frac{M_1L_1}{3} = \frac{(131.4) \times 1.8^2}{3 \times 9.81} = 14.47\ \text{kg-m}^2,\ I_2 = \frac{M_2L_2^2}{3} = \frac{(65.7) \times 0.9^2}{3 \times 9.81} = 1.81\ \text{kg-m}^2$$

Assume that both bars swing in the same direction and using the Newton's second law of motion for the moments about the hinges,

$$I_1\,\theta_1 = -w_1 \times 0.9 \times \theta_1 - k\,(\theta_1 - \theta_2) \times 0.9^2 \qquad ...6.37$$

$$I_2\,\theta_2 = -w_2 \times 0.9 \times \theta_2 - k\,(\theta_1 - \theta_2) \times 0.9^2 \qquad ...6.38$$

Assume the solution in the form of $\theta_1 = A \sin \omega t$ and $\theta_2 = B \sin \omega t$.

We have $\ddot{\theta}_1 = -A\omega^2 \sin \omega t$ and $\ddot{\theta}_2 = -B\omega^2 \sin \omega t$.

Substituting these values into equations 6.37 and 6.38, we get

$$(0.9\,W_1 + 0.81\,k - 1\,\omega^2)\,A - (0.81\,k)\,B = 0$$

$$-(0.81\,k)\,A + (0.45\,W_2 + 0.81\,k - I_2\,\omega^2)\,B = 0$$

Substituting the values of W_1, W_2, kI_1, I_2, we have

$$(827.82 - 14.47\,\omega^2)\,A - 709.5\,B = 0 \qquad ...6.39$$

$$-709.56\,A + (739.13 - 1.81\,\omega^2)\,B = 0 \qquad ...6.40$$

from which we get, $\omega^4 - 465.6\,\omega^2 + 4152.48 = 0$

After solving the above quadratic equation, we get frequencies

$$\omega_1 = 3.02 \text{ and } \omega_2 = 21.37 \text{ rad/s.}$$

EXAMPLE 6.11

Figure. p-6.11 shows two equal pendulums free to rotate about the *Y–Y* axis. A rubber hose of torsional stiffness k_t N-mm/rad couples together these pendulums. Find out the two natural frequencies and motion how the two principal modes may be started.

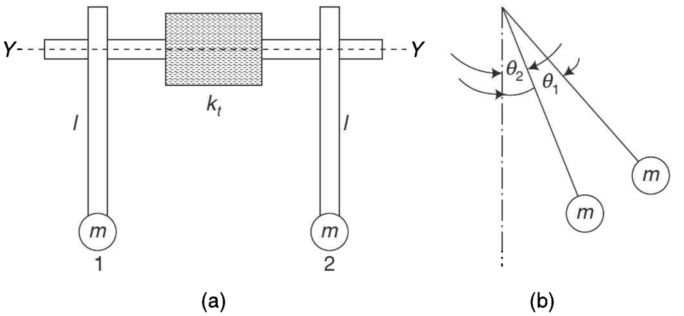


Fig. p-6.11 *Two equal pendulums*

Solution Let 'θ_1' and 'θ_2' be the angular displacements of the pendulum of mass (1) and mass (2). Applying Newton's second law of motion for mass (1) and mass (2),

$$ml^2 \ddot{\theta}_1 = -k_t(\theta_1 - \theta_2) - mgl\theta_1 \qquad ...6.41$$

is the differential equation for the mass (1).

$$ml^2 \ddot{\theta}_2 = k_t(\theta_1 - \theta_2) - mgl\theta_2 \qquad ...6.42$$

is the differential equation for the mass (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of θ_1, θ_2, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in equations 6.41 and 6.42,

$$ml^2 \ddot{\theta}_1 + k_t(\theta_1 - \theta_2) + mgl\theta_1 = 0, \qquad ml^2 \ddot{\theta}_2 - k_t(\theta_1 - \theta_2) + mgl\theta_2 = 0$$

$$(k_t + mgl - ml^2\omega^2) A_1 = k_t A_2, \qquad k_t A_1 = (k_t + mgl - ml^2\omega^2) A_2$$

The amplitude ratio, $\dfrac{A_1}{A_2} = \dfrac{k_t}{k_t + mgl - ml^2\omega^2}$, and also, $\dfrac{A_2}{A_1} = \dfrac{k_t + mgl - ml^2\omega^2}{k_t}$

From these two above equations the frequency is obtained as

$$k_t + mgl - ml^2\omega^2 = k_t, \quad \omega^2 = \frac{g}{l} + \frac{k_t}{ml^2}(1 \pm 1), \text{ or } \omega_{1,2} = \sqrt{\frac{g}{l} + \frac{k_t}{ml^2}(1 \pm 1)}$$

$$\therefore \qquad \omega_1 = \sqrt{\frac{g}{l}} \text{ rad/s}, \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k_t}{ml^2}} \text{ rad/s}$$

$$\frac{A_1}{A_2} = \frac{k_t}{k_t + mgl - ml^2 \frac{g}{l}} = 1, \text{ when } \omega_1 = \sqrt{\frac{g}{l}} \text{ rad/s}$$

with same direction and equal distance and leave to vibrate.

Also $\frac{A_1}{A_2} = \frac{k_t}{k_t + mgl - ml^2\left(\frac{g}{l} + \frac{2k_t}{ml^2}\right)} = -1$, when $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k_t}{ml^2}}$, giving equal and opposite angular displacements to the bobs.

EXAMPLE 6.12

Determine the natural frequency for the system shown in Fig. p-6.12(a) and draw the mode shapes and locate the node for each mode of vibration. Given $I_1 = I, I_2 = 2I, \quad kt_1 = kt_2, kt_2 = kt$.

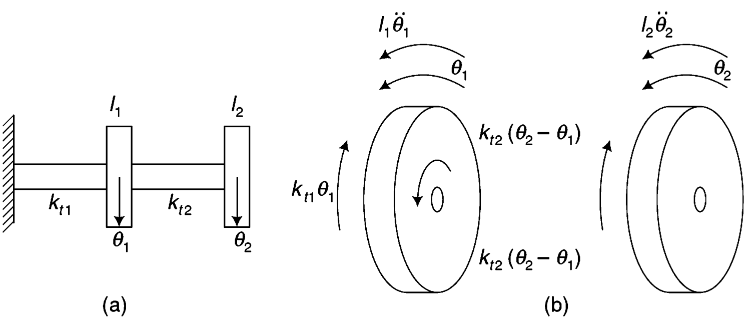


Fig. p-6.12 *System for Example 6.12*

Solution Let us compare at any instant 'θ_1' and 'θ_2' be the angular displacement of flywheels 'I_1' and 'I_2' respectively. Let 'k_t' is the torsional stiffness of the connecting shaft of flywheel 'I_1' and 'I_2', then the FBD as shown in Fig. p-6.12(b).

Applying Newton's second law of motion to disc (I_1), let $\theta_2 > \theta_1$

$$\Sigma M = I\ddot{\theta},\ kt_2(\theta_2 - \theta_1) - kt_1\theta_1 = I_1\ddot{\theta}_1,\ I_1\ddot{\theta}_1 + kt_1\theta_1 - kt_2(\theta_2 - \theta_1) = 0$$

$$I\ddot{\theta}_1 + 2kt\theta_1 - kt\theta_2 + kt\theta_1 = 0,\ I\ddot{\theta}_1 + 3kt\theta_1 - kt\theta_2 = 0 \qquad ...6.43$$

This is a differential equation of motion to the disc (I_1).

Applying Newton's second law of motion to the disc (I_2),

$$\Sigma M = I\ddot{\theta},\quad -kt_2(\theta_2 - \theta_1) = I_2\ddot{\theta}_2 \quad \therefore\ I_2\ddot{\theta}_2 + kt_2\theta_2 - kt_2\theta_1 = 0$$

$$2I\ddot{\theta}_2 + kt\theta_2 - kt\theta_1 = 0 \qquad ...6.44$$

This is a differential equation of motion to the disc (I_2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies.

Let one of these components be,

$$\theta_1 = A_1 \sin \omega t, \qquad \theta_2 = A_2 \sin \omega t$$

$$\ddot{\theta}_1 = -A_1\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -A_2\omega^2 \sin \omega t$$

Substituting these values in equations 6.43 and 6.44, we get

$$\therefore \qquad I(-A\omega^2) \sin \omega t + 3kt\, A \sin \omega t - kt\, B \sin \omega t = 0$$

Amplitude ratio $\dfrac{A}{B} = \dfrac{kt}{3kt - I\omega^2}$ ...6.45

$$2I(-B\omega^2 \sin \omega t) + kt\, B \sin \omega t - A\, kt \sin \omega t = 0$$

$$\therefore \qquad \frac{A}{B} = \frac{kt - 2I\omega^2}{kt} \qquad \text{...6.46}$$

From equations 6.45 and 6.46, $\dfrac{kt}{3kt - I\omega^2} = \dfrac{kt - 2I\omega^2}{kt}$.

$$2I^2\omega^4 - 7kt\,\omega^2 I + 2kt^2 = 0, \; \omega^4 - \frac{7kt}{2I} + \frac{kt^2}{I^2} = 0.$$

This is a quadratic equation in ω^2

$$\therefore \qquad \omega^2 = \frac{\dfrac{7kt}{2I} \pm \sqrt{\left(\dfrac{-7kt}{2I}\right)^2 - \dfrac{4kt^2}{I^2}}}{2}$$

$$\therefore \qquad \omega_{1n}^2 = 0.315\,\frac{kt}{I}, \quad \omega_{2n}^2 = 3.185\,\frac{kt}{I}$$

$$\omega_{1n} = 0.56\sqrt{\frac{kt}{I}} \text{ rad/s}, \quad \omega_{2n} = 1.78\sqrt{\frac{kt}{I}} \text{ rad/s}$$

To draw the mode shapes

At $\omega^2 = \omega_{1n}^2 = 0.315\,\dfrac{kt}{I}$ in Eq. 6.45,

$$\frac{A}{B} = \frac{kt}{3kt - I \times 0.315 \times \dfrac{kt}{I}} = \frac{1}{2.69}$$

$$\therefore \qquad A = 1,\; B = 2.69$$

at $\omega^2 = \omega_{2n}^2 = 3.185\,\dfrac{kt}{I}$, $\dfrac{A}{B} = \dfrac{kt}{3kt - I \times 3.185\,\dfrac{kt}{I}} = \dfrac{1}{-0.189}$ $\therefore\; A = 1,\; B = -0.189$

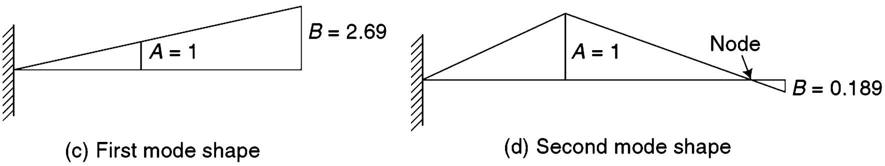


Fig. p-6.12 *Mode shapes*

EXAMPLE 6.13

Find the natural frequencies of the system shown in Fig. p-6.13(a). Also determine the ratio of amplitudes and the mode shapes. Given $I_1 = I_0$, $I_2 = 2I_0$, and $k_{t1} = k_{t2} = k_{t3} = kt$.

Solution Let 'θ_1' and 'θ_2' be the angular displacement of the disc (I_1) and the disc (I_2) respectively. Then the FBD is as shown in Fig. p-6.13(b). Then the equation of motion for the disc (I_1) may be written as

$$I_1\ddot{\theta}_1 = -kt_1\theta_1 + kt_2(\theta_2 - \theta_1), \quad I_1\ddot{\theta}_1 + kt\theta_1 + kt(\theta_1 - \theta_2) = 0$$

$$I_1\ddot{\theta}_1 + 2k_1\theta_1 - kt\theta_2 = 0 \qquad \text{...6.47}$$

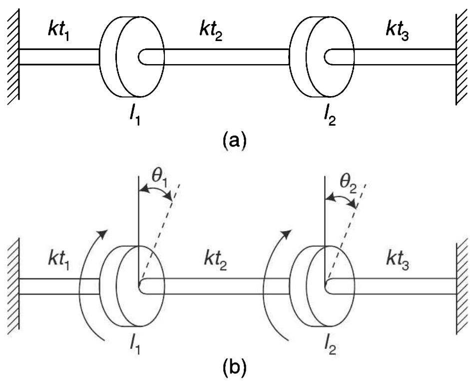


Fig. p-6.13 *System for Example 6.13*

Similarly, the equation of motion for the disc (I_2) may be written as

$$I_2\ddot{\theta}_2 = -kt_2(\theta_2 - \theta_1) - kt_3\theta_2, \quad I_2\ddot{\theta}_2 + kt(\theta_2 - \theta_1) + kt\theta_2 = 0$$

$$I_2\ddot{\theta}_2 + kt\theta_2 + kt\theta_2 - kt\theta_1 = 0, \quad I_2\ddot{\theta}_2 + 2kt\theta_2 - kt\theta_1 = 0$$

$$2I_2\ddot{\theta}_2 + 2kt\theta_2 - kt\theta_1 = 0 \qquad \text{...6.48}$$

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A_1 \sin \omega t, \qquad \theta_2 = A_2 \sin \omega t$$

$$\ddot{\theta}_2 = -A_1\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -A_2\omega^2 \sin \omega t$$

Using these values in equations 6.47 and 6.48, we have

$$-\omega^2 I_0 A_1 + 2ktA_1 - ktA_2 = 0, \; (-\omega^2 I_0 + 2kt)A_1 - ktA_2 = 0$$

$$(-2\omega^2 I_0 + 2kt)A_2 - ktA_1 = 0$$

The amplitude ratios are given by

$$\frac{A_1}{A_2} = \frac{kt}{-\omega^2 I_0 + 2kt} = \frac{(-2\omega^2 I_0 + 2kt)}{kt} \qquad \text{...6.49}$$

The frequency equation can be written as $2(-\omega^2 I_0 + 2kt)(-\omega^2 I_0 + kt) - kt^2 = 0$,

$$2(\omega^4 I_0^2 - \omega^2 I_0 kt + 2kt\omega^2 I_0 + 2kt^2) - kt^2 = 0,\ 2\omega^4 I_0^2 - 6\ ktI_0\omega^2 + 3kt^2 = 0$$

$$\omega^4 - \frac{3kt}{I_0}\omega^2 + \frac{3kt^2}{2I_0^2} = 0,\ \omega^2 = \frac{\frac{3kt}{I_0} \pm \sqrt{\left(\frac{3kt}{I_0}\right)^2 - 4\cdot\frac{3}{2}\frac{kt^2}{I_0^2}}}{2} = \frac{1.5kt}{I_0} + \frac{\sqrt{3}}{2}\frac{kt^2}{I_0}$$

So $\qquad \omega_1 = 1.5\sqrt{\frac{kt}{I_0}}\ \omega_2 = 0.80\sqrt{\frac{kt}{I_0}}$

The amplitude ratios are given by, $\left(\frac{A_1}{A_2}\right)_{\omega 1} = \frac{kt}{-\omega_1^2 I_0 + 2kt} = \frac{kt}{-\left(1.5\sqrt{\frac{kt}{I_0}}\right) I_0 + 2kt} = -4$

$$\left(\frac{A_1}{A_2}\right)_{\omega 2} = \frac{-\omega_1^2 I_0 + 2kt}{kt} = \frac{-2(0.8)^2\frac{kt}{I_0}I_0 + 2kt}{kt} = \frac{-2\times 0.64 + 2}{1} = +0.72$$

Mode shapes are as shown in Fig. p-6.13(b) and Fig. p-6.13(c).

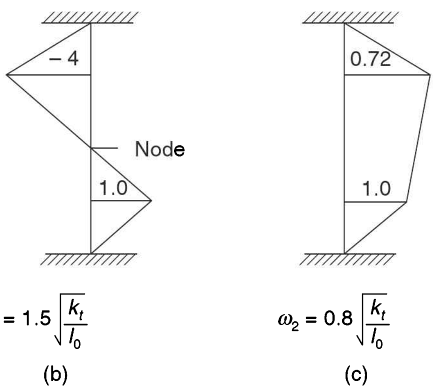


Fig. p-6.13 *Mode shapes*

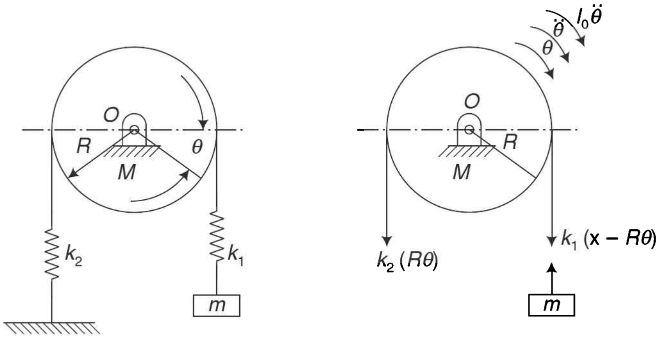
EXAMPLE 6.14

Derive the frequency equation for the pulley-mass system shown in Fig. p-6.14(a). The pulley has a mass of '*M*' and effective radius of '*R*'. Assume that the cord, which passes over the pulley, does not slip, if k_1 = 60 N/m, k_2 = 40 N/m, *m* = 2 kg and *M* = 10 kg. Determine the natural frequencies and mode shapes.

Solution Let us at any instant give a vertical displacement '*x*' to the mass '*m*' as shown in Fig. p-6.14(a). Since there is no slip between the card and cylinder of mass '*M*', so the vertical displacement '*x*' causes the cylinder to rotate by an angle '*θ*' as shown in FBD of Fig. p-6.14(b).

Now applying Newton's second law of motion to '*m*' (rectilinear motion),

$$\Sigma F = ma, \quad m\ddot{x} = -k\,(x - R\theta)$$



(a) (b)

Fig. p-6.14 *Pulley-mass system*

For mass M (rotational), $J\ddot{\theta} = \Sigma T$

But $$J = \frac{MR^2}{2}$$

$$\frac{MR^2}{2}\ddot{\theta} = -k_1 (R\theta - x) R - k_2 (R\theta) R \qquad ...6.50$$

$$J\ddot{\theta} + k_1R^2\theta - k_1Rx + k_2R^2\theta = 0 \qquad ...6.51$$

This is the differential equation of motion of mass and pulley

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x = \sin \omega t, \qquad \theta = B \sin \omega t$$

$$\ddot{x} = -A\omega^2 \sin \omega t, \quad \ddot{\theta} = -B\omega^2 \sin \omega t$$

Using these values in Eq. 6.50, we have

$$-JBw^2 \sin \omega t + k_1 R^2B \sin \omega t - k_1RA \sin \omega t + k_2R^2 B \sin \omega t = 0 \text{ , } \sin \omega t \neq 0$$

$$-JB\omega^2 + k_1R^2B - k_1RA + k_2 R^2B = 0$$

$$-k_1RA + [k_1R^2 + k_2R^2 - J\omega^2] B = 0, \quad k_1RA = \left[k_1R^2 + k_2R^2 - \frac{MR^2}{2}\cdot\omega^2\right] B$$

$$\frac{A}{B} = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega^2\right] R^2}{k_1R} = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega^2\right] R}{k_1}$$

The frequency equation is equating to two equations ratio,

$$\left(\frac{Mm}{2}\right) R^2\omega^4 - \left[\left(\frac{MR^2}{2}\right) k_1 + m (k_1 + k_2) R^2\right]\omega^2 + k_1k_2R^2 = 0$$

Dividing by $MmR^2/2$

$$\omega^4 - \left[\frac{k_1}{m} + \frac{2(k_1 + k_2)}{M}\right]\omega^2 + \frac{2k_1k_2}{Mm} = 0 \qquad ...6.52$$

This is in the form of a quadratic equation.

The solution is given by the roots of the equation as

$$\omega_{1,2}^2 = \frac{\left[\frac{k_1}{m} + \frac{2(k_1+k_2)}{M}\right] \pm \sqrt{\left[\frac{k_1}{m} + \frac{2(k_1+k_2)}{M}\right]^2 - \frac{4 \times 2k_1k_2}{Mm}}}{2 \times 1}$$

$$\omega_{1,2}^2 = \frac{\left[\frac{60}{2} + \frac{2(60+40)}{10}\right] \pm \sqrt{\left[\frac{60}{2} + \frac{2(60+40)}{10}\right]^2 - \frac{4 \times 2 \times 60 \times 40}{10 \times 2}}}{2}$$

$$\omega_{1,2}^2 = \frac{50 \pm 39.24}{2}$$

$$\omega_1^2 = \frac{50 + 39.24}{2} = 44.62$$

$$\therefore \quad \omega_1 = \sqrt{44.62} = 6.68 \text{ rad/s}$$

$$\omega_2^2 = \frac{50 - 39.24}{2} = 5.38, \ \omega_2 = \sqrt{5.38} = 2.319 \text{ rad/s}$$

The amplitude ratios are

$$\left(\frac{A}{B}\right)_1 = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega_1^2\right]R}{k_1} = \frac{\left[60 + 40 - \left(\frac{10}{2}\right) \times 44.62\right]1}{k60} = -2.052$$

$$\left(\frac{A}{B}\right)_2 = \frac{\left[k_1 + k_2 - \left(\frac{M}{2}\right)\omega_2^2\right]R}{k_1} = \frac{\left[60 + 40 - \left(\frac{10}{2}\right)5.38\right] \times 1}{60} = 1.22$$

The first-mode and second-mode shapes are as shown in Fig. p-6.14(c) and Fig. p-6.14(d).

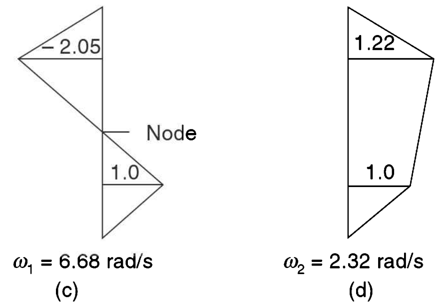


Fig. p-6.14 *Mode shapes*

EXAMPLE 6.15

Determine the frequency equation for the system shown in Fig. p-6.15(a) and determine the natural frequencies if $k_1 = k_2 = k_3 = k$, $m_1 = m_2 = m$ and $r_1 = r_2 = r$

Solution Let us at any instant give an angular displacement 'θ_1' to the mass 'm_1' and 'θ_2' of the mass 'm_2' as shown in Fig. p-6.15(a). Then the FBD is as shown in Fig. p-6.15(b). Let $\theta_2 > \theta_1$.

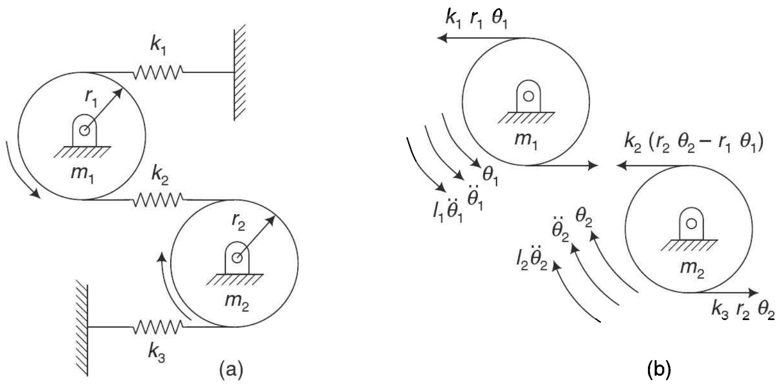


Fig. p-6.15 *Pulley system*

Applying Newton's second law of motion to the disc (1),

$$\Sigma M = I\ddot{\theta},\ k_2(r_2\theta_2 - r_1\theta_1)\, r_1 - k_1 r_1 \theta_1 \cdot r_1 = I_1\ddot{\theta}_1$$

$$\therefore \quad I_1\ddot{\theta}_1 + k_1 r_1^2\, \theta_1 - k_2 r_1\,(r_2\theta_2 - r_1\theta_1) = 0$$

$$\therefore \quad \frac{mr^2}{2}\,\ddot{\theta}_1 + k_1 r^2\theta_1 - k_2 r^2\theta_2 + k_2 r^2\theta_1 = 0$$

$$\therefore \quad \frac{m}{2}\,\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$$

$$\therefore \quad m\ddot{\theta}_1 + 4k\theta_1 - 2k\theta_2 = 0 \qquad \text{...6.53}$$

This is the differential equation of motion for the disc (1).

Applying Newton's second law of motion to the disc (2), $\Sigma M = I\ddot{\theta}$,

$$k_2(r_2\theta_2 - r_1\theta_1)\, r_2 + k_3 r_2\theta_2 r_2 = -I_2\ddot{\theta}_2$$

$$\therefore \quad I_2\ddot{\theta}_2 + k_2 r_2\,(r_2\theta_2 - r_1\theta_1) + k_3 r_2^2\theta_2 = 0$$

$$\therefore \quad \frac{mr^2}{2}\,\ddot{\theta}_2 + kr^2\theta_2 - kr^2\theta_1 + kr^2\theta_2 = 0$$

$$\frac{m}{2}\,\ddot{\theta}_2 + 2k\theta_2 - k\theta_1 = 0$$

$$\therefore \quad m\ddot{\theta}_2 + 4k\theta_2 - 2k\theta_1 = 0 \qquad \text{...6.54}$$

This is the differential equation of motion of the disc (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t, \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of 'θ_1', 'θ_2' and $\ddot{\theta}_1$ in Eq. 6.52,

$$-m\omega^2 A + 4kA - 2kB = 0, \quad A\,(4k - m\omega^2) = 2kB$$

$$\therefore \quad \frac{A}{B} = \frac{2k}{4k - m\omega^2} \qquad \text{...6.55}$$

Using the values of 'θ_1', 'θ_2' and $\ddot{\theta}_2$ in Eq. 6.54,

$$-m\omega^2 B + 4kB - 2kA = 0, \quad B(4k - m\omega^2) = 2kA$$

$$\therefore \quad \frac{A}{B} = \frac{4k - m\omega^2}{2k} \qquad \text{...6.56}$$

From equations 6.55 and 6.56,

$$\frac{2k}{4k - m\omega^2} = \frac{4k - m\omega^2}{2k}$$

$$\therefore \quad (4k - m\omega^2)^2 = (2k)^2, \quad 16k^2 - 8mk\omega^2 + m^2\omega^4 = 4k^2$$

$$\therefore \quad m^2\omega^4 - 8mk\omega^2 + 12k^2 = 0$$

$$\omega^4 - 8\frac{k}{m}\omega^2 + 12\frac{k^2}{m^2} = 0$$

This is the frequency equation and this is the quadratic equation in ω^2.

$$\therefore \quad \omega^2 = \frac{\frac{8k}{m} \pm \sqrt{\left(\frac{8k}{m}\right)^2 - 4\frac{12k^2}{m^2}}}{2}$$

$$\therefore \quad \omega^2 = \frac{4k}{m} \pm \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}$$

$$\therefore \quad \omega^2 = \frac{4k}{m} \pm \frac{2k}{m}$$

$$\therefore \quad \omega_{1n}^2 = \frac{2k}{m}, \ \omega_{2n}^2 = \frac{6k}{m}$$

$$\omega_{1n} = 1.41\sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{2n} = 2.45\sqrt{\frac{k}{m}} \text{ rad/s}$$

where ω_{1n} and ω_{2n} are first and second natural frequencies.

EXAMPLE 6.16

Determine the frequency equation for the system as shown in Fig. p-6.16(a).

Solution Let us at any instant give a displacement 'θ' to the mass 'M' and the attached mass 'm' as shown in Fig. p-6.16(a). Then the FBD as shown in Fig. p-6.16(b).

From the geometry of Fig. p-6.16(b), let $\theta > \phi$

$$X = r\phi, \quad y = x + l\sin\theta, \quad y = x + l\theta \text{ and } \ddot{y} = \ddot{x} + l\ddot{\theta}$$

For small angles of 'θ', let us assume the cylinder is oscillating about the point '0'. $\cos\theta \approx 1$, $\sin\theta \approx \theta$.

Considering mass 'm', $\Sigma V = 0$

$$T\cos\theta - mg = 0, \ T = mg, \text{ if '}\theta\text{' is very small } \cos\theta = 1$$

Applying Newton's second law of motion to cylinder, $\Sigma M_0 = I_0\ddot{\phi}$

$\therefore \quad 2kx.r - T\sin\theta.r = I_0\ddot{\phi}, \quad I_0\ddot{\phi} + 2kr\phi.r - mg\,\theta.r = 0$

where $\quad I_0 = I_G + Mr^2 = 1/2\,Mr^2 + Mr^2, \quad I_0 = 3/2\,Mr^2$

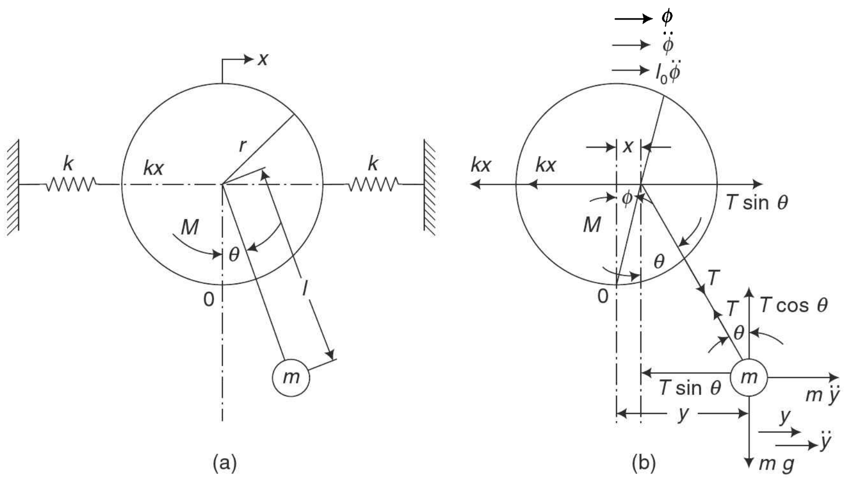


Fig. p-6.16 *System for Example 6.16*

$$\frac{3}{2}Mr^2\ddot{\phi} + 2kr^2\phi - mgr\theta = 0$$

$$\frac{3}{2}Mr\,\ddot{\phi} + 2kr\phi - mg\theta = 0 \qquad ...6.57$$

This is the differential equation of motion for the cylinder.

Applying Newton's second law of motion to the mass 'm',

$$\Sigma F = m\ddot{x}, -T\sin\theta = m\ddot{y}$$

$$\therefore \quad m\ddot{y} + T\theta = 0$$

$$m\ddot{x} + ml\ddot{\theta} + mg\theta = 0,\ l\ddot{\theta} + g\theta + \ddot{x} = 0,\ l\ddot{\theta} + g\theta + r\ddot{\phi} = 0 \qquad ...6.58$$

This is the second differential equation of motion for the bob.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\phi = A\sin\omega t, \qquad \theta = B\sin\omega t$$

$$\ddot{\phi} = -A\omega^2\sin\omega t, \qquad \ddot{\theta} = -B\omega^2\sin\omega t$$

Using the values of, $\ddot{\phi}$ and $\ddot{\theta}$ in Eq. 6.57,

$$\frac{3}{2}Mr(-A\omega^2) + 2krA - mgB = 0, \quad A\left[2kr - \frac{3}{2}Mr\omega^2\right] = mgB$$

$$\frac{A}{B} = \frac{2mg}{r[4k - 3M\omega^2]} \qquad ...6.59$$

Using the values of $\ddot{\theta}$, θ and $\ddot{\phi}$ in Eq. 6.58,

$$l(-B\omega^2) + gB - A\,\omega^2 r = 0, \quad B(g - \omega^2 l) = A\,\omega^2 r, \quad \frac{A}{B} = \frac{g - \omega^2 l}{r\omega^2} \qquad ...6.60$$

From equations 6.59 and 6.60, $\dfrac{2mg}{r[4k - 3M\omega^2]} = \dfrac{g - \omega^2 l}{r\omega^2}$

$$(g - \omega^2 l)(4k - 3M\omega^2) = 2m\,\omega^2 g,\ 4kg - 3Mg\,\omega^2 - 4k\,l\,\omega^2 - 2mg\,\omega^2 + 3M\,l\,\omega^4 = 0$$

$$3Ml\omega^4 - [3Mg + 4kl + 2mg]\,\omega^2 + 4kg = 0,$$

This is the frequency equation for the given system.

6.4 SEMIDEFINITE SYSTEM OR DEGENERATING SYSTEM

This is defined as a system where one natural frequency is equal to zero. This is also known as a degenerate system. Consider the system to represent two masses 'm_1' and 'm_2' and with a coupling spring 'k' as shown in Fig. 6.2(a).

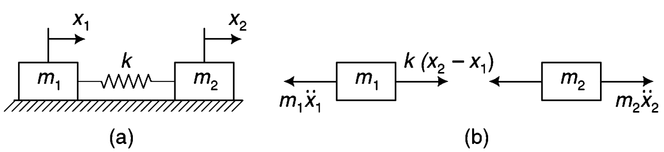


Fig. 6.2 *Semidefinite system*

Now at any instant, give displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to the Fig. 6.2(a). The FBD is as shown in Fig. 6.2(b).

Assuming that $x_2 > x_1$ or $x_1 > x_2$ also can be taken, but $x_2 > x_1$ is easy to writing down the differential equations.

Apply Newton's second law of motion to the mass 'm_1', i.e. $\Sigma F = ma$

$$m_1\ddot{x}_1 = -k(x_1 - x_2), \quad m_1\ddot{x}_1 + k\,(x_1 - x_2) = 0 \qquad ...6.61$$

Similarly apply Newton's second law of motion for the mass 'm_2',

$$m_2\ddot{x}_2 = -k(x_2 - x_1), \quad m_2\ddot{x}_2 + k\,(x_2 - x_1) = 0 \qquad ...6.62$$

Assume that motion is periodic and is composed of harmonic motions of varies amplitudes and frequencies. Let one of these components be,

$$x_1 = A\sin\omega t, \qquad x_2 = B\sin\omega t$$

$$\dot{x}_1 = A\omega\cos\omega t, \qquad \dot{x}_2 = B\omega\cos\omega t$$

$$\ddot{x}_1 = -A\omega^2\sin\omega t, \qquad \ddot{x}_2 = -B\omega^2\sin\omega t$$

Substituting these values in equations 6.61 and 6.62, we get

$$(k - m_1\omega^2)\,A - kB = 0 \qquad ...6.63$$

$$-kA + (k - m_2\omega^2)\,B = 0 \qquad ...6.64$$

The frequency equation is obtained by equating to zero the determinants of the coefficient 'A' and 'B' are

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0,\ (k - m_1\omega^2)\,(k - m_2\omega^2) - (-k)\,(-k) = 0$$

$$k^2 - km_2\omega^2 - km_1\omega^2 + m_1m_2\omega^4 - k^2 = 0,\ m_1m_2\omega^4 - km_2\omega^2 - km_1\omega^2 = 0$$

$\omega^2\,[m_1m_2\omega^2 - k\,(m_1 + m_2)] = 0 \quad \therefore\ \omega_1 = 0$, as one of their natural frequencies is equal to zero as the statement of the semidefinite system and $\omega_2 = \sqrt{\dfrac{k(m_1 + m_2)}{m_1m_2}}$ rad/s.

Mode shapes Dividing Eq. 6.64 by Eq. 6.63, we get

$$-kA + (k - m_2\omega^2)\,B/(k - m_1\omega^2)\,A - kB,\quad \frac{A}{B} = -\,(m_1/m_2).$$

EXAMPLE 6.17

Two identical cylinders are linked together as shown in Fig. p-6.17(a). Determine the natural frequencies of the system.

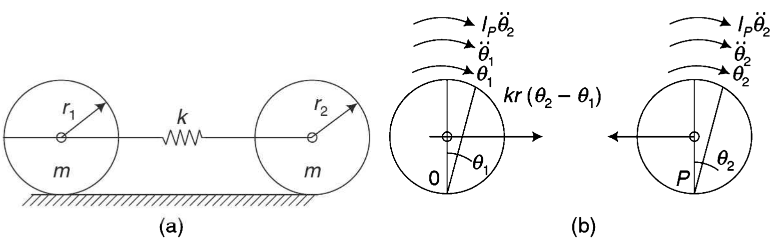


Fig. p-6.17 *Cylinder system*

Solution Let us at any instant give an angular displacement 'θ_1' to the first cylinder of mass 'm' and 'θ_2' to the second cylinder of mass 'm' as shown in Fig. p-6.17(a). Then FBD is as shown in Fig. p-6.17(b).

Applying Newton's second law of motion to the cylinder (1), let $\theta_2 > \theta_1$

$$\Sigma M_P = I_P\ddot{\theta},\ -kr\,(\theta_2 - \theta_1)\,r = -I_P\ddot{\theta},\quad -I_P\ddot{\theta} + kr^2\theta_2 - kr^2\theta_1 = 0$$

$$-\frac{3}{2}\,mr^2\ddot{\theta}_1 + kr^2\theta_2 - kr^2\theta_1 = 0,\quad -\frac{3}{2}\,m\ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0 \quad \text{...6.65}$$

This is the differential equation of motion for the cylinder (1).

Applying Newton's second law of motion to the cylinder (2),

$$\Sigma M_0 = I_0\ddot{\theta},\quad kr(\theta_2 - \theta_1)\,r = -I_0\ddot{\theta}_2,\quad I_0\ddot{\theta}_2 + kr^2\theta_2 - kr^2\theta_1 = 0$$

$$\frac{3}{2}\,m\ddot{\theta}_2 + k\theta_2 - k\theta_1 = 0 \quad \text{...6.66}$$

This is the differential equation of motion for the cylinder (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t \qquad \theta_2 = B \sin \omega t$$
$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of θ_1, θ_2, $\ddot{\theta}_1$ in Eq. 6.65,

$$\frac{3}{2} m (-A\omega^2) + kA - kB = 0, \quad A\left(k - \frac{3}{2} m\omega^2\right) = kB, \quad \frac{A}{B} = \frac{2k}{2k - 3m\omega^2} \qquad \text{...6.67}$$

Using the values of θ_1, θ_2, $\ddot{\theta}_2$ in Eq. 6.66,

$$\frac{3}{2} m(-B\omega^2) + kB - kA = 0, \quad B\left(k - \frac{3}{2} m\omega^2\right) = kA, \quad \frac{A}{B} = \frac{2k - 3m\omega^2}{2k} \qquad \text{...6.68}$$

From equations 6.67 and 6.68,

$$\frac{2k}{2k - 3m\omega^2} = \frac{2k - 3m\omega^2}{2k}, \ (2k - 3m\omega^2)^2 = (2k)^2, \ 2k - 3m\omega^2 = \pm\, 2k, \ 3m\omega^2 = 2k \pm 2k$$

$$\omega_{1n}^2 = 0, \quad \omega_{1n} = 0 \ \omega_{2n}^2 = \frac{4k}{3m}, \quad \omega_{2n} = \sqrt{\frac{4k}{3m}} \text{ rad/s}$$

Since one of the natural frequencies is zero, the system is a semidefinite system.

EXAMPLE 6.18

Two flywheels of moment of inertia 'I_1' and 'I_2' are keyed to the ends of a steel shaft. Derive an expression for the frequency of free torsional vibrations of the system shown in Fig. p-6.18(a) and describe the modes.

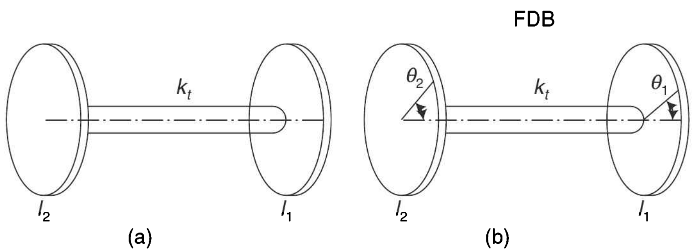


Fig. p-6.18 *Flywheel system*

Solution Let at any instant 'θ_1' and 'θ_2' be the angular displacement of flywheels 'I_1' and 'I_2' respectively. Let 'k_t' is the torsional stiffness of the connecting shaft of flywheel 'I_1' and 'I_2'. Then the FBD is as shown in Fig. p-6.18(b); assume $\theta_1 > \theta_2$

Then twist of the shaft = $\theta_1 - \theta_2$.

Then equation of motion is,

$$I_1\ddot{\theta}_1 = -k_t(\theta_1 - \theta_2) \qquad \text{...6.69}$$
$$I_2\ddot{\theta}_2 = k_t(\theta_1 - \theta_2) \qquad \text{...6.70}$$

Assuming that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t, \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Substituting these values of θ_1, θ_2, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in equations 6.69 and 6.70,

$$(k_t - \omega^2 I_1)\, A_1 = k_t A_2, \; (k_t - \omega^2 I_2)\, A_2 = k_t A_1$$

From these two equations, we obtained the amplitude ratios as

$$\frac{A_1}{A_2} = \frac{k_t}{k_t - \omega^2 I_1} = \frac{k_t - \omega^2 I_2}{k_t} \qquad \text{...6.71}$$

which gives the frequency equation as

$$(k_t - \omega^2 I_1)\,(k_t - \omega^2 I_2) = k_t^2, \quad \omega^2\,[I_1 I_2 \omega^2 - k_t\,(I_1 + I_2)] = 0$$

$$\omega_1 = 0 \text{ rad/s and } \omega_2 = \sqrt{\frac{k_t\,(I_1 + I_2)}{I_1 I_2}} \text{ rad/s.}$$

Since one of the natural frequencies is zero, the system is a semidefinite system.

Substituting these values of the natural frequencies in the amplitude ratio A_1/A_2 in the above equation 6.71, we obtained the two conditions for the principal modes as shown in Fig. p-6.18(c).

$$\frac{A_1}{A_2} = \frac{k_t - \omega^2 I_2}{k_t} = 1 \text{ when } \omega_1 = 0 \text{ and } \frac{A_1}{A_2} = \frac{-I_2}{I_1}, \text{ when } \omega_2 = \sqrt{\frac{k_t\,(I_1 + I_2)}{I_1 I_2}} \text{ rad/s.}$$

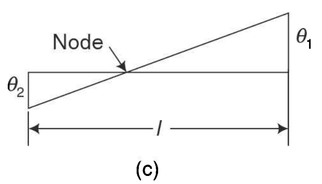


Fig. p-6.18 *Principal modes*

EXAMPLE 6.19

An electric train made of two cars weighing 30 kN each has got a spring coupling of 3000N/mm stiffness as shown in Fig. p-6.19. Determine the natural frequency of vibration of the system.

Solution For small displacements x_1 and x_2 of the two cars, applying Newton's second law of motion, we get

$$m\ddot{x}_1 = k\,(x_2 - x_1) \qquad \text{...6.72}$$

$$m\ddot{x}_2 = -k\,(x_2 - x_1) \qquad \text{...6.73}$$

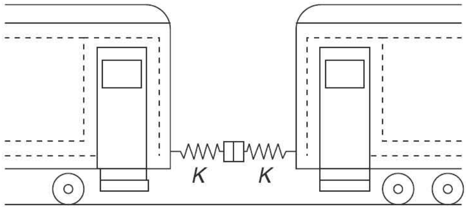


Fig. p-6.19 *Electric train*

This is the differential equation for the motion

where k = Stiffness of coupling spring, m = Mass of each car

For principal mode of vibration, let $x_1 = X_1 \sin \omega t, \quad x_2 = X_2 \sin \omega t$

Putting these values of x's and their derivatives above equations 6.72 and 6.73, we get,

$$X_1(k - m\omega^2) = kX_2, \quad kX_1 = X_2(k - m\omega^2)$$

The amplitude ratios are $\dfrac{X_1}{X_2} = \dfrac{k}{k - m\omega^2} = \dfrac{k - m\omega^2}{k}$

So, the frequency equation is,

$$k^2 = (k - m\omega^2)^2, \text{ or } \omega^2 = \frac{2k}{m}, \omega_n^2 = \frac{2 \times 3000}{30000} = 0.2$$

$$\therefore \qquad \omega_n = 0.447 \text{ rad/s}$$

$$\therefore \qquad f_n = \frac{\omega_n}{2\pi} = \frac{0.447}{2\pi} = 0.07 \text{ Hz}$$

6.4 COMBINED RECTILINEAR AND ANGULAR MODES

In this chapter, almost all cases have been discussed, where two coordinates have been either both linear systems or both angular systems. Now in this section we will discuss a system having combined rectilinear and as well as angular modes.

Let us consider a body having a mass 'm' and momentum of inertia 'J' (where $J = mr^2$, r = Radius of gyration about the centre of gravity of the body CG) supported by springs of stiffness 'k_1' and 'k_2' and capable of oscillating in the directions 'x' (linear) and 'θ' (angular); and 'l_1' and 'l_2' are the distances between the centre of gravity of the body and the springs 'k_1' and 'k_2' respectively as shown in Fig. 6.3(a).

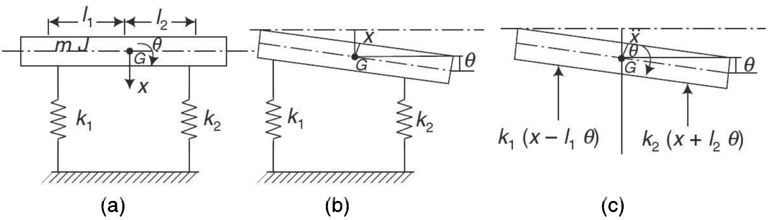


Fig. 6.3 *Combined rectilinear and angular modes system*

Suppose at any instant, the body be displaced through a linear distance 'x' and angular distance 'θ' as shown in Fig. 6.3(b).

Now assuming that at this instant, taking 'θ' to be small, the springs 'k_1' and 'k_2' will be compressed through an amount of $(x - l_1\theta)$ and $(x + l_2\theta)$ respectively from the equilibrium as shown. Then the FBD of entire system is as shown in Fig. 6.3(c).

Now apply Newton's second law of motion and write down the differential equations of motion for the system by considering the 'x' and 'θ' direction by taking the forces and the moments in the respective directions acting on the system.

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta), \quad J\ddot{\theta} = k_1 l_1(x - l_1\theta) - k_2 l_2(x + l_2\theta) \text{ or}$$

$$\left.\begin{aligned} m\ddot{x} + x(k_1 + k_2) &= (k_1 l_1 - k_2 l_2)\theta \\ J\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\,\theta &= (k_1 l_1 - k_2 l_2)\,x \end{aligned}\right\} \quad \text{...6.74}$$

$$\text{Put} \begin{bmatrix} P = \dfrac{k_1 + k_2}{m} \\ Q = \dfrac{k_1 l_1 - k_2 l_2}{m} \\ R = \dfrac{k_1 l_1^2 + k_2 l_2^2}{J} \end{bmatrix} \quad \text{...6.75}$$

Also we known that $J = mr^2$

Substitute all these values in equations 6.74 and it will

$$\text{reduce to} \begin{bmatrix} \ddot{x} + Px = Q\theta \\ \ddot{\theta} + R\theta = \left(\dfrac{Q}{r^2}\right)x \end{bmatrix} \quad \text{...6.76}$$

Almost in all earlier cases, we ended up with two differential equations, one for each mass which are coupled with respect to the two coordinates. In the above cases 'Q' is termed as the coupling coefficient since if $Q = 0$, the two equations are independent or uncoupled of each other and therefore give the two motions, one is rectilinear and the other one is angular. These can exist independently of each other with their respective natural frequencies $\sqrt{P}$ and $\sqrt{R}$ and in case of uncoupled system when $Q = 0$, it means $k_1 l_1 = k_2 l_2$ the natural frequencies in the rectilinear and angular modes respectively, are given as below:

$$\begin{bmatrix} \omega_{n1} = \sqrt{P} = \dfrac{\sqrt{k_1 + k_2}}{m} \\ \omega_{n2} = \sqrt{R} = \sqrt{\dfrac{k_1 l_1^2 + k_2 l_2^2}{J}} \end{bmatrix} \quad \text{...6.77}$$

Now consider the coupled equation 6.76 and let us assume the principal mode of vibration.

Let $x = X \sin \omega t, \quad \ddot{x} = -X\omega^2 \sin \omega t$ and

$$\theta = \beta \sin \omega t, \quad \ddot{\theta} = -\beta \omega^2 \sin \omega t \qquad \text{...6.78}a$$

$$\begin{bmatrix} \ddot{x} = -X\omega^2 \sin \omega t \\ \ddot{\theta} = -\beta\omega^2 \sin \omega t \end{bmatrix} \qquad \text{...6.78}b$$

Substutiting these values in equations 6.76 and simplifying, we get

$$\begin{bmatrix} [-\omega^2 + P]\, X = Q\beta \\ [-\omega^2 + R] = \left(\dfrac{Q}{r^2}\right) X \end{bmatrix} \qquad \text{...6.79}$$

By these equations we get the amplitude ratios:

$$\frac{X}{\beta} = \frac{Q}{P - \omega^2} \qquad \text{...6.80}$$

$$\frac{X}{\beta} = \frac{R - \omega^2}{\dfrac{Q}{r^2}} \qquad \text{...6.81}$$

Therefore, $\dfrac{Q}{P - \omega^2} = \dfrac{R - \omega^2}{\dfrac{Q}{r^2}}$

By simplifying these we get the frequency equation as

$$\omega^4 - (P + R)\, \omega^2 + \left\{ PR - \frac{Q^2}{r^2} \right\} = 0 \qquad \text{...6.82}$$

This is in the form of a quadric equation of ω^2, and the roots of the above equation gives the following two natural frequencies of the system.

$$\omega^2_{n1} = \frac{1}{2}(P + R) - \sqrt{\frac{1}{4}(R - P)^2 + \frac{Q^2}{2}}$$

$$\omega^2_{n2} = \frac{1}{2}(P + R) + \sqrt{\frac{1}{4}(R - P)^2 + \frac{Q^2}{r^2}} \qquad \text{...6.83}$$

These two natural frequencies reduce to that of equations 6.77 when $Q = 0$ for the uncoupled case and the mode shape can be got in the usual manner. Also the expression will not be much meaningful in this particular case due to complexity.

6.6 GEARED SYSTEM

Consider a geared system as shown in Fig. 6.4(a). This system may be replaced by an equivalent system shown in Fig. 6.4(b) by assuming the following.

1. The gear teeth are rigid and are always in contact,
2. There is no backlash in the gearing, and
3. The inertia of the shafts and gears is negligible.

The following two conditions must be satisfied by an equivalent system.

1. The kinetic energy of the equivalent system must be equal to the kinetic energy of the original system.
2. The potential energy of the equivalent system must be equal to the PE of the original system.

∴ KE of section (1) + KE of section (2) = KE of section (1) + KE of section (3)

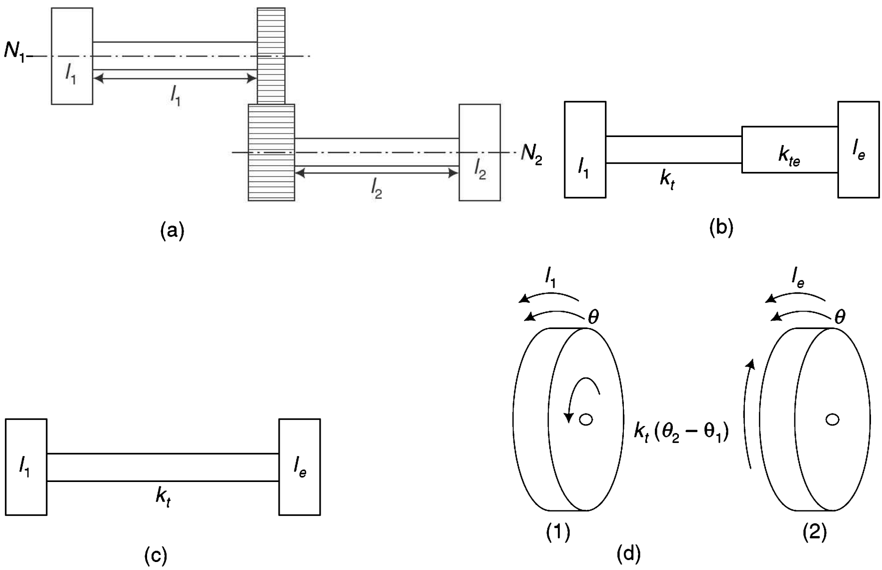


Fig. 6.4 *Geared system*

Note: Let N_1 and N_2 be the speeds of the pinion and gear respectively.

$$\therefore \text{ angular velocity } \omega_1 = \frac{2\pi N_1}{60}, \quad \omega_2 = \frac{2\pi N_2}{60} \text{ rad/s}$$

$$\text{Gear ratio} = n = \frac{\text{Speed of pinion}}{\text{Speed of gear}}$$

$$\therefore \qquad n = \frac{N_1}{N_2} = \frac{\omega_1}{\omega_2}$$

∴ KE of section (2) = KE of section (3)

$$\frac{1}{2} I_2 \dot{\theta}_2^2 = \frac{1}{2} I_e \dot{\theta}_e^2$$

where $\dot{\theta}_2$ and $\dot{\theta}_e$ are the angular velocity of the sections (2) and (3) respectively.

But the rotor I_e is rotating at the same speed as I_1.

$$\dot{\theta}_e = \dot{\theta}_1 \quad \therefore \frac{1}{2} I_2 \dot{\theta}_2^2 = \frac{1}{2} I_e \dot{\theta}_1^2, \quad I_2\omega_2^2 = I_e\omega_1^2$$

$$I_e = \frac{I_2}{\left(\frac{\omega_1}{\omega_2}\right)^2}, I_e = \frac{I_2}{n^2} \text{ inertia of equivalent system}$$

Since potential energy of both the systems are same,

$$\frac{1}{2} k_{t2}\theta_2^2 = \frac{1}{2} k_{te}\theta_e^2, k_{t2}\theta_2^2 = k_{te}\theta_1^2,$$

$$k_{te} = \frac{k_{t2}}{\left(\frac{\theta_1}{\theta_2}\right)^2}, k_{te} = \frac{k_{t2}}{n^2} \text{ equivalent torsional spring stiffness of the section (3).}$$

EXAMPLE 6.20

An electric motor rotating at 1500 rev/min. drives a centrifugal pump at 500 rev/min, through a single-stage reduction gearing is as shown Fig. 6.4(a). The moments of inertia of the pump impeller and the electric motor are 1400 kg-m^2 and 400 kg-m^2 respectively. The pump shaft and the motor shaft are 45 cm and 18 cm long respectively and their respective diameters are 9 cm and 4.5 cm. Determine the natural frequencies of oscillation. Neglect inertia of gears and $G = 0.8 \times 10^6$ kg/cm^2.

Solution N_1 = 1500 rpm, N_2 = 500 rpm, I_1 = 400 kg-m^2, I_2 = 1400 kg-m^2, l_1 = 18 cm, l_2 = 45 cm, d_1 = 4.5 cm, d_2 = 9 cm, $G = 0.8 \times 10^6$ kg/cm^2.

Gear ratio $$n = \frac{N_1}{N_2}, \quad n = \frac{1500}{500} = 3$$

By torsional equation, $$\frac{T}{I_P} = \frac{G\theta}{l}, \quad \frac{T}{\theta} = k_t = \frac{GI_P}{l} = \frac{G\pi d^4}{32l}$$

$$\therefore \quad k_{t1} = \frac{0.84 \times 10^{11} \times \pi \times (0.045)^4}{32 \times 0.45}, k_{t1} = 187869.70 \text{ N-m/rad}$$

$$\therefore \quad k_{t2} = \frac{0.84 \times 10^{11} \times \pi \times (0.09)^4}{32 \times 0.45}, k_{t2} = 1202366.05 \text{ N-m/rad}$$

The given system can be reduced as shown in Fig. 6.4(b),

where $$I_e \frac{I_2}{n^2} = \frac{1400}{3^2}, I_e = 155.56 \text{ kg-m}^2$$

$$k_{te} = \frac{k_{t2}}{n^2} = \frac{1202366.05}{3^2}, k_{te} = 133596.23 \text{ N-m/rad}$$

Again the system reduces to Fig. 6.4(c). Then the FBD is as shown in Fig. 6.4(d),

where $$\frac{1}{k_t} = \frac{1}{k_{t1}} + \frac{1}{k_{te}} \qquad \therefore \frac{1}{k_t} = \frac{1}{187869.70} + \frac{1}{133596.23}$$

$\therefore \quad k_t = 78075.72$ N-m/rad

Applying Newton's second law of motion to the disc (1), $\theta_2 > \theta_1$

$$\Sigma M = I\ddot{\theta} \qquad \therefore\ k_t(\theta_2 - \theta_1) = I_1\ddot{\theta}_1$$

$$I_1\ddot{\theta}_1 + k_t\theta_1 - k_t\theta_2 = 0 \qquad ...6.84$$

This is the differential equation of motion for the disc (1).

Applying Newton's second law of motion to the disc (2),

$$\Sigma M = I\ddot{\theta},\ -k_t(\theta_2 - \theta_1) = I_e\ddot{\theta}_2, \qquad I_e\ddot{\theta}_2 + k_t\theta_2 - k_t\theta_1 = 0 \qquad ...6.85$$

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A\sin\omega t, \qquad \theta_2 = B\sin\omega t$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega t, \qquad \ddot{\theta}_2 = -B\omega^2\sin\omega t$$

Using these values in Eq. 6.84,

$$I_1(-A\omega^2) + k_t A = k_t B, \qquad \frac{A}{B} = \frac{k_t}{k_t - I_1\omega^2} \qquad ...\ 6.86$$

Using the values of θ_1 $\theta_2, \ddot{\theta}_2$ in Eq. 6.85,

$$I_e(-B\omega^2) + k_t B = k_t A, \quad \frac{A}{B} = \frac{-I_e\omega^2 + k_t}{k_t} = \frac{k_t - I_e\omega^2}{k_t} \qquad ...6.87$$

From equations 6.86 and 6.87, $\dfrac{k_t}{k_t - I_1\omega^2} = \dfrac{k_t + -I_e\omega^2}{k_t}$, $(k_t - I_1\omega^2)(k_t - I_e\omega^2) = k_t^2$

$$I_1 I_e\omega^4 - [k_t I_1 + k_t I_e]\,\omega^2 + k_t^2 = k_t^2, \quad \omega^2[I_1 I_e\omega^2 - (k_t I_1 + k_t I_e)] = 0$$

$$\omega_{1n}^2 = 0, \quad \omega_{2n}^2 = \frac{k_t I_1 + k_t I_e}{I_1 I_e},$$

$$\omega_{2n}^2 = \frac{78075.72\,[400 + 155.56]}{400 \times 155.56},$$

$$\omega_{2n}^2 = 697.10 \quad \therefore\ \omega_{1n} = 0,\ \omega_{2n} = 26.40 \text{ rad/s}$$

Since one of the natural frequencies of the system is equal to zero, the system is semidefinite.

6.7 VIBRATION ABSORBER OR UNDAMPED DYNAMIC VIBRATION ABSORBER OR FRAHM VIBRATION ABSORBER

When a single-degree-freedom system $k_1 - m_1$ as shown in Fig. 6.4(a) subjected to a harmonic force $F = F_0\sin\omega t$, it will undergo resonance, when the forcing frequency equals to the natural frequency of the system.

It is possible to make the amplitude of vibration of mass 'm_1' to become zero, by adding a sub-system '$k_2 - m_2$' as shown in Fig. 6.4(b) thereby converting the original single-degree-freedom system into a two-degree-freedom system. The system which is attached to the main system is termed the **dynamic vibration absorber,** also known as Frahm vibration absorber after the name of its inventor and it has to

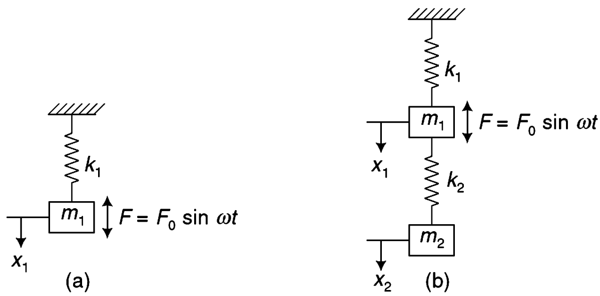


Fig. 6.5 *Dynamic vibration absorber*

be properly designed so that the amplitude of vibration of the main system should be equal to zero.

The disadvantage of this system is the occurrence of two resonant conditions. By keeping away of these two resonant conditions, one can use this principle for vibration reduction. Also this type of absorber is extremely effective at one speed only and thus it is suitable only for constant speed machines.

Let us consider a two-degree-freedom system as shown in Fig. 6.5(b). The spring-mass system $k_1 - m_1$ is considered as a main system subjected to a harmonic force $F = F_0 \sin \omega t$. The spring-mass system $k_2 - m_2$ is considered as an absorber system.

Write down the differential equation of motion for the system by applying Newton's second law of motion to the mass 'm_1' and FBD is as shown in Fig. 6.5(c).

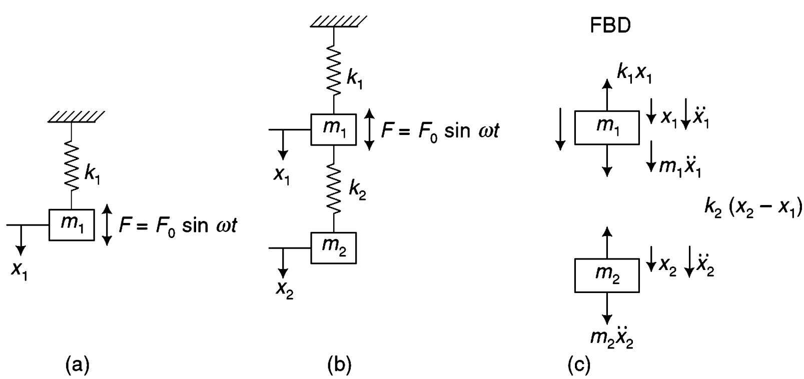


Fig. 6.6 *Spring-mass system*

$$\Sigma F = m\ddot{x}, \quad F + k_2 (x_2 - x_1) - k_1 x_1 = m\ddot{x}_1$$

$$m\ddot{x}_1 + k_1 x_1 - k_2 x_2 + k_2 x_1 - F = 0, \quad m\ddot{x}_1 + (k_1 + k_2)\, x_1 - k_2 x_2 = F \quad \text{...6.88}$$

This is the differential equation of motion of the mass 'm_1'.

Applying Newton's second law of motion to the mass 'm_2'.

$$\Sigma F = m\ddot{x}, \quad k_2 (x_2 - x_1) = m_2\, \ddot{x}_2, \quad m_2\, \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad \text{...6.89}$$

This is the differential equation of motion of the mass 'm_2'.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be

$$x_1 = X_1 \sin \omega t, \qquad x_2 = X_2 \sin \omega t$$
$$\ddot{x}_1 = -X_1 \omega^2 \sin \omega t, \quad \ddot{x}_2 = -X_2 \omega^2 \sin \omega t$$

Using these values in equations 6.88 and 6.89, we have

$$m_1(-X_1\omega^2 \sin \omega t) + (k_1 + k_2)(X_1 \sin \omega t) - k_2 X_2 \sin \omega t = F$$

But $F = F_0 \sin \omega t$

$\therefore$ $X_1[(k_1 + k_2) - m_1\omega^2] \sin \omega t - X_2 k_2 \sin \omega t = F_0 \sin \omega t$

$$X_1[(k_1 + k_2) - m_1\omega^2] - X_2 k_2 = F_0$$
$$X_1[(k_1 + k_2) - m_1\omega^2] = F_0 + X_2 k_2 \qquad ...6.90$$

Equation 6.89 becomes $m_2(-X_2\,\omega^2 \sin \omega t) + k_2(X_2 \sin \omega t) - k_2(X_1 \sin \omega t) = 0$

$$X_2[(k_2 - m_2\,\omega^2] = X_1 k_2, \quad X_2 = X_1\left[\frac{k_2}{k_2 - m_2\omega^2}\right] \qquad ...6.91$$

Using the values of Eq. 6.91 in 6.90, we have

$$X_1(k_1 + k_2) - m_1\omega^2 = F_0 + k_2\left[\frac{X_1 k_2}{k_2 - m_2\omega^2}\right]$$

$$X_1\left[(k_1 + k_2 - m_1\omega^2) - \frac{k_2^2}{k_2 - m_2\omega^2}\right] = F_0, \; X_1\left[\frac{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}{k_2 - m_2\omega^2}\right] = F_0$$

$$X_1 = \left[\frac{F_0(k_2 - m_2\omega^2)}{[(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2]}\right] \qquad ...6.92$$

The condition for dynamic vibration absorber is that the amplitude of mass 'm_1' = 0,

i.e. $X_1 = 0$

$\therefore$ $F_0(k_2 - m_2\,\omega^2) = 0$

$kF_0 \neq 0 \qquad \therefore k_2 - m_2\,\omega^2 = 0$

Or $\omega^2 = \dfrac{k_2}{m_2}$, i.e. $\omega = \sqrt{\dfrac{k_2}{m_2}}$ rad/s

But $\omega_1 = \sqrt{\dfrac{k_1}{m_1}}$ and $\omega_2 = \sqrt{\dfrac{k_2}{m_2}} \quad \therefore \omega = \omega_2$

i.e. when the forcing frequency is equal to the natural frequency of the sub-system then the dynamic vibration absorption is attained as shown in Fig. 6.6(d).

The static deflection $x_{st} = \dfrac{F_0}{k_1}$.

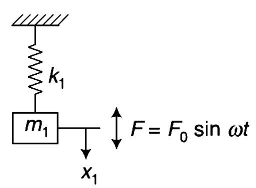


Fig. 6.6(d) *Dynamic vibration absorption*

From Eq. 6.92, $X_1 = \dfrac{F_0 k_2\left[1-\dfrac{m_2}{k_2}\omega^2\right]}{\left[k_1k_2 - m_2k_1\omega^2 + k_2^2 - m_2k_2\omega^2 - m_1k_2\omega^2 + m_1m_2\omega^4 - k_2^2\right]}$

$$X_1 = \frac{k_2\left[1-\dfrac{m_2}{k_2}\omega^2\right]F_0}{m_1m_2\omega^4 - [m_1k_2 + m_2k_1 + m_2k_2]\,\omega^2 + k_1k_2]} \qquad \text{...6.93}$$

$$X_1 = \frac{k_2\left[1-\dfrac{m_2}{k_2}\omega^2\right]F_0}{k_1k_2 - \left[\dfrac{m_1}{k_1}\cdot\dfrac{m_2}{k_2}\,\omega^4 - \left[\dfrac{m_1}{k_1}+\dfrac{m_2}{k_2}+\dfrac{m_2}{k_1}\right]\omega^2 + 1\right]}$$

$$\frac{X_1}{X_{st}} = \frac{\left[1-\dfrac{m_2}{k_2}\omega^2\right]}{\dfrac{m_1}{k_1}\omega^2\cdot\dfrac{m_2}{k_2}\omega^2 - \left[\dfrac{m_1}{k_1}\omega^2 + \dfrac{m_2}{k_2}\omega^2 + \dfrac{m_2}{k_1}\omega^2\right] + 1}$$

Let $\qquad \dfrac{m_2}{m_1} = \mu$

$\therefore \qquad \dfrac{m_2}{k_1} = \dfrac{m_2}{m_1}\cdot\dfrac{m_1}{k_1} = \mu\,\dfrac{m_1}{k_1} \quad \therefore\ \dfrac{m_2}{k_1} = \dfrac{\mu}{\omega_1^2}$

$$\frac{X_1}{X_{st}} = \frac{\left[1-\left(\dfrac{\omega}{\omega_2}\right)^2\right]}{\left(\dfrac{\omega}{\omega_1}\right)^2\left(\dfrac{\omega}{\omega_2}\right)^2 - \left[\left(\dfrac{\omega}{\omega_1}\right)^2 + \left(\dfrac{\omega}{\omega_2}\right)^2 + \mu\left(\dfrac{\omega}{\omega_1}\right)^2\right] + 1} \qquad \text{...6.94}$$

For dynamic vibration, $X_1 = 0$, $\dfrac{\omega}{\omega_2} = 1$, $\omega = \omega_2$.

The natural frequencies of the combined system is obtained by comparing Eq. 6.90 and Eq. 6.91 or by equating the denominator of either equations 6.93 or 6.94 to zero.

$\therefore \qquad m_1m_2\,\omega^4 - [m_1k_2 + m_2k_1 + m_2k_2]\,\omega^2 + k_1k_2 = 0$

$$\omega^4 - \left[\frac{k_2}{m_2}+\frac{k_1}{m_1}+\frac{k_2}{m_1}\right]\omega^2 + \frac{k_1k_2}{m_1m_2} = 0$$

This is a quadratic equation in ω^2.

Assuming a restricted case $\omega_1 = \omega_2$ or $\omega_1/\omega_2 = 1$

$$\omega^4 - \left[\omega_2^2 + \omega_1^2 + \mu\omega_2^2\right]\omega^2 + \omega_1^2\,\omega_2^2 = 0$$

$$\omega^4 - \left[2\omega_2^2 + \mu\omega_2^2\right]\omega^2 + \omega_2^4 = 0, \quad \left(\frac{\omega}{\omega_2}\right)^4 - [2+\mu]\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

Let $\qquad R = \left(\dfrac{\omega}{\omega_2}\right)^2$

$$\therefore \qquad R^2 - (2 + \mu)\, R + 1 = 0$$

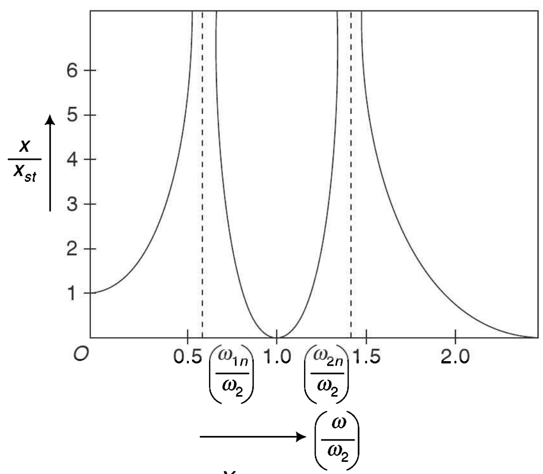
This is a quadratic equation in R.

$$\therefore \qquad R = \frac{2 + \mu \pm \sqrt{(2 + \mu)^2 - 4}}{2}$$

$$R = \frac{2 + \mu}{2} \pm \frac{\sqrt{4 + \mu^2 + \mu - 4}}{2} = \left[1 + \left(\frac{\mu}{2}\right)\right] \pm \sqrt{\mu + \left(\frac{\mu}{2}\right)^2}$$

$$\therefore \qquad \left(\frac{\omega}{\omega_2}\right)^2 = 1 + \left(\frac{\mu}{2}\right) \pm \sqrt{\mu + \left(\frac{\mu}{2}\right)^2}$$

$$\left(\frac{\omega_{1n}}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\mu + \left(\frac{\mu}{2}\right)^2} \quad \left(\frac{\omega_{2n}}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) + \sqrt{\mu + \left(\frac{\mu}{2}\right)^2}$$



Plot of $\dfrac{X_1}{X_{st}}$ versus $\dfrac{\omega}{\omega_2}$

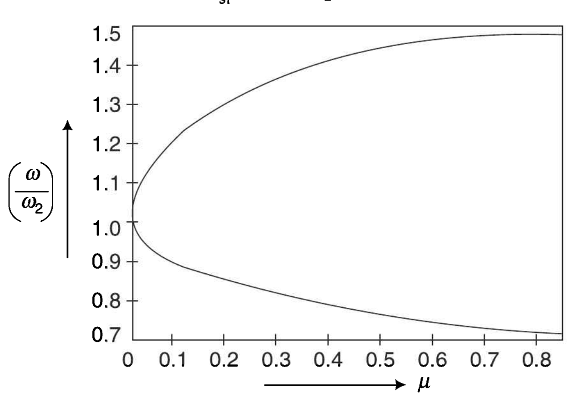


Fig. 6.6(e) *Plots for Frahm vibration absorber*

These are the two natural frequencies of the combined system for a special case $\omega_1 = \omega_2$.

EXAMPLE 6.21

A spring-mass ($k_1 - M$) system is being acted upon by a harmonic force $F = F_0 \sin \omega t$ (force acting on the mass) as shown in Fig. p-6.21. Another ($k_2 - m$) system is attached to the mass 'M'. Analyse the system to show that the second system may act as a vibration absorber if properly designed. Mention how to design it.

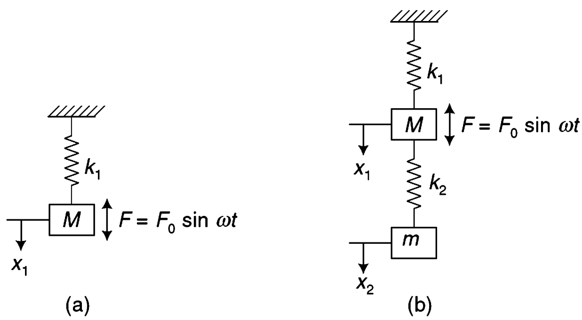


Fig. p-6.21 *Spring-mass system*

Solution As per the statement of the problem, the system is as shown in Fig. p-6.21.

The system is of two degrees of freedom with a forcing function acting on mass 'M'.

Applying Newton's second law of motion to mass 'M' in Fig. p-6.21, the equations of motion are

$$M\ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = F_0 \sin \omega t \quad \text{...6.95}$$

$$m\ddot{x}_2 + k_2 (x_2 - x_1) = 0 \quad \text{...6.96}$$

This is the differential equation of motion of masses 'M' and 'm'.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = A \sin \omega t, \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = \omega A \cos \omega t, \qquad \dot{x}_2 = \omega B \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 A \sin \omega t, \quad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Substituting these values in equations 6.95 and 6.96,

$$M(-\omega^2 A \sin \omega t) + k_1 A \sin \omega t + k_2 (A \sin \omega t - B \sin \omega t) = F_0 \sin \omega t$$

$$m(-\omega^2 B \sin \omega t) + k_2 (B \sin \omega t - A \sin \omega t) = 0$$

$$(k_1 + k_2 - M\omega^2) A - k_2 B = F_0, \quad -k_2 A + (k_2 - m\omega^2) B = 0$$

Solving the above equation, $A = \dfrac{F_0(k_2 - m\omega^2)}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2}$

In order to cut down the amplitude of vibration of mass 'M', i.e. $A = 0$, $(k_2 - m\omega^2)$ must be equal to zero. Hence, $k_2 = m\omega^2$ or $\omega^2 = \sqrt{\dfrac{k_2}{m}}$

$$\therefore \qquad \omega = \sqrt{\frac{k_2}{m}} \text{ rad/s}$$

The absorber must be, therefore, so designed that its natural frequency is equal to the impressed frequency. When this happens, the amplitude of vibration of mass 'M' is practically zero.

In general, an absorber is used only when the natural frequency of the original system is close to the forcing frequency. Hence, $\dfrac{k_1}{M} = \dfrac{k_2}{m}$ is approximately true for the entire system.

EXAMPLE 6.22

A two-degree-freedom system is as shown in Fig. p-6.22(a).

Determine the amplitude of masses 'M' and 'm'. What modifications are necessary if the sub-system is to act as a dynamic vibration absorber under the following condition?

(i) Spring stiffness kept constant.

(ii) The amplitude of the absorber mass is limited to 0.01 cm.

Solution $M = 10$ kg, $m = 0.5$ kg, $k_1 = 4000$ N/m, $k_2 = 500$ N/m, $F = 50 \cos 21t$

Let us at any instant give a vertical displacement 'x' to the mass 'm_1' as shown in Fig. p-6.22(a). The FBD is as shown in Fig. p-6.22(b).

Applying Newton's second law of motion to mass 'M' $\Sigma F = M\ddot{x}$

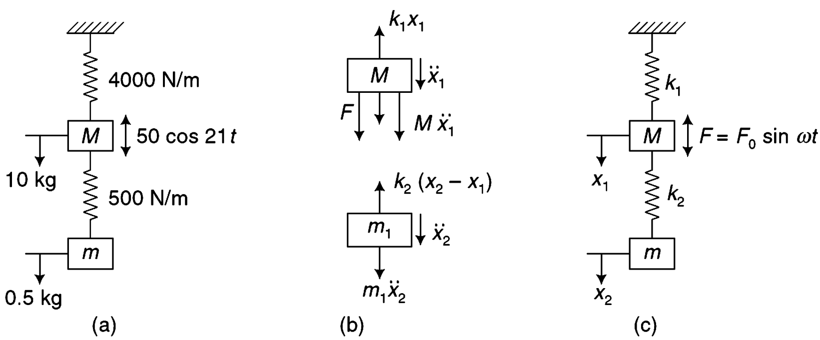


Fig. p-6.22 *Two-degree-freedom system*

$$k_2(x_2 - x_1) + F - k_1x_1 = M\ddot{x}, \quad M\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F \qquad \text{...6.97}$$

Applying Newton's second law of motion to the mass 'm',

$$\Sigma F = m\ddot{x} \; -k_2 (x_2 - x_1) = m\ddot{x}_2, m\ddot{x}_2 + k_2 x_2 - k_2 x_1 \qquad \text{...6.98}$$

This is the differential equation of motion of masses 'M' and 'm'.

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be

$$x_1 = X_1 \cos \omega t, \qquad x_2 = X_2 \cos \omega t$$

$$\ddot{x}_1 = -X_1 \omega^2 \cos \omega t, \qquad \ddot{x}_2 = -X_2 \omega^2 \cos \omega t$$

Using these values in equations 6.97 and 6.98,

$$M(-X_1 \omega^2 \cos \omega t) + (k_1 + k_2) X_1 \cos \omega t - k_2 X_2 \cos \omega t = F_0 \cos \omega t$$

$$X_1 [k_1 + k_2 - M\omega^2] - k_2 x_2 = F_0 \qquad \text{...6.99}$$

$$m(-X_2 \omega^2 \cos \omega t) + k_2 X_2 \cos \omega t - k_2 X_1 \cos \omega t = 0$$

$$X_2 [k_2 - m\omega^2] - k_2 X_1 = 0, X_2 = \left[\frac{k_2}{k_2 - m\omega^2}\right] X_1 \qquad \text{...6.100}$$

Using Eq. 6.100 in Eq. 6.99, $X_1 = \left[(k_1 + k_2 - M\omega^2) - \frac{k_2^2}{(k_2 - m\omega^2)}\right] = F_0$

$$X_1 = \left[\frac{F_0 (k_2 - m\omega^2)}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2}\right] \qquad \text{... 6.101}$$

Using Eq. 6.101 in Eq. 6.100,

$$X_2 = \left[\frac{F_0 k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2}\right] \qquad \text{...6.102}$$

∴ amplitude of the mass 'M' is

$$X_1 = \left[\frac{50\,(500 - 0.5 \times 21^2)}{(4000 + 500 - 10 \times 21^2)(500 - 0.5(21)^2) - (500)^2}\right]$$

$$X_1 = -\,6.215 \times 10^{-2}\ \text{m}$$

∴ amplitude of the mass 'm' is

$$X_2 = \left[\frac{50 \times 500}{(4000 + 500 - 10 \times 21^2)(500 - 0.5(21)^2) - k_2^2}\right]$$

$$X_2 = -1.112 \times 10^{-1}\ \text{m}$$

(i) For the sub-system to act as dynamic vibration absorber, $x_1 = 0$

From Eq. 6.101, $F_0 (k_2 - m\,\omega^2) = 0, \omega^2 = \frac{k_2}{m}$

Keeping k_2 constant, $m = \frac{k_2}{\omega^2}$

$\therefore$ mass 'm' $\frac{500}{(21)^2}$, $m = 1.13$ kg

(ii) Given $X_2 = 0.001$ m

For dynamic vibration absorber, $X_1 = 0$

i.e. $(k_2 - m\omega^2) = 0$

From Eq. 6.102, $X_2 = \frac{-F_0}{k_2}$ $\therefore k_2 = \frac{-F_0}{X_2}$

$$k_2 = \frac{-50}{0.0001} = 5 \times 10^5 \text{ N/m. Negative sign may be neglected.}$$

Note: The vibration absorber must be so designed that its natural frequency is equal to the forcing frequency, i.e. $\omega = \omega_2$, $\omega = \sqrt{\frac{k_2}{m_2}}$ rad/s

When this happens, the amplitude of vibration of the mass 'm_1' of the original system is equal to zero. In general, a dynamic vibration absorber is used only when the natural frequency of the original system is close to the forcing frequency (resonance).

Hence, $\omega \approx \omega_1 \approx \omega_2$, or $\frac{k_1}{m_1} \approx \frac{k_2}{m_2}$ is true for the entire system.

6.8 CENTRIFUGAL PENDULUM ABSORBER

The undamped dynamic vibration absorber discussed earlier is fully effective at only one frequency of design. They lose their effect if the speed changes. In cases where either the speed changes or the speed fluctuates, the centrifugal vibration absorbers are very effective.

In case of a torsional system having torsional oscillations superimposed upon its rotation, it is possible to use an undamped dynamic vibration absorber that will be effective at all rotating speeds. A centrifugal pendulum absorber is shown in Fig. 6.7.

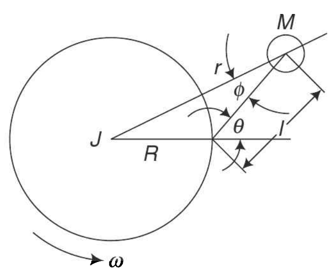


Fig. 6.7 *Centrifugal pendulum absorber*

The centrifugal pendulum absorber consists of a small pendulum of mass 'M' and length 'l' attached to the main rotor (J) of radius 'R' rotating at an angular velocity of 'ω' rad/s as shown in Fig. 6.7. Assume that the mass of the pendulum bob is 'M' and the string has negligible mass. Resolving the centrifugal force of the pendulum mass as $Mr\omega^2$ into two components, $Mr\omega^2 \sin\phi$ in the tangential direction and $Mr\omega^2 \cos\phi$ in the radial direction, the moments leads to

$$Ml^2\ddot{\theta} + Mr\omega^2 (\sin\phi)\, l = 0 \text{ or } \ddot{\theta} + \frac{r}{l}\omega^2 \sin\phi = 0$$

But $$\frac{R}{\sin\phi} = \frac{r}{\sin(180 - \theta)} = \frac{r}{\sin\theta} \text{ or}$$

$$R\sin\theta = r\sin\phi,\ \ddot{\theta} + \left(\frac{R}{l}\omega^2\right)\theta = 0,\ \omega_n = \omega\sqrt{\frac{R}{l}}$$

Speed $$= \frac{1}{f_n} = \frac{2\pi}{\omega_n} = \frac{2\pi}{\omega\sqrt{\dfrac{R}{l}}} \text{ rps.}$$

6.9 UNTUNED DRY FRICTION DAMPER (LANCHESTER DAMPER)

This type of damper is very advantageous to use for torsional vibrations near resonance conditions. This dry friction damper is very useful in reducing the amplitudes of torsional vibrations of the systems at critical speed or at resonance conditions.

The main parts of this damper are two flywheels and hub, shaft with key, spring loaded bolts and friction material as shown in Fig. 6.8.

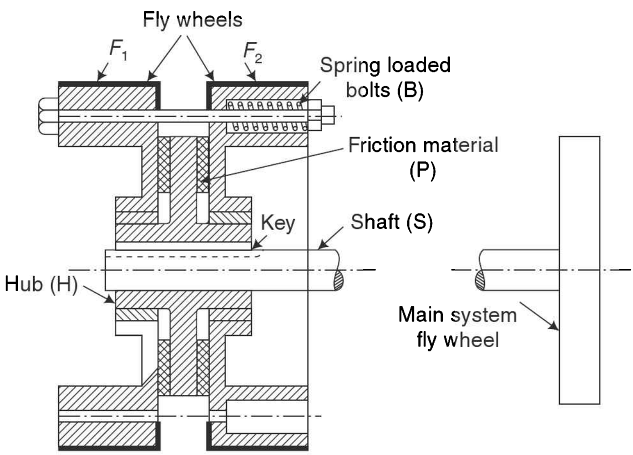


Fig. 6.8 *Dry friction torsional vibration damper*

The construction of the friction damper is simple and it consists of two flywheels (F_1 and F_2) mounted freely over a hub (H). The hub is rigidly fixed to the shaft (S) undergoing vibrations and then the flywheels are driven by the friction plates (P) fixed to the extension of the hub. The pressure between the friction plates and flywheels being adjustable by means of the spring-loaded bolts (B) which hold both the flywheels together.

The working principle of dry friction damper is based on two conditions:

1. If the frictional torque is more then the pressure between the friction plates and the flywheels is very large. In such a situation, the flywheels become rigid with the shaft and possess the same oscillations as that of the shaft. In this condition, there is no energy dissipated during vibrations due to the zero

relative rubbing. Then the amount of energy dissipated is proportional to the frictional torque times the relative velocity.

2. If the pressure between the friction plates and the flywheels becomes zero, the relative velocity is maximum but the frictional torque is zero. Here also, there is no further energy dissipation. But, for an intermediate value of the pressure, there is relative rubbing as well as frictional torque because of the inertia effect of the flywheels, and therefore some amount of energy is dissipated. Hence, there is reduction in amplitude of torsional oscillations. Hence, it is understood that greater the amount of energy dissipated, larger will be the amplitude reduction of the main system and this energy dissipated versus frictional torque curve is as shown in Fig. 6.9.

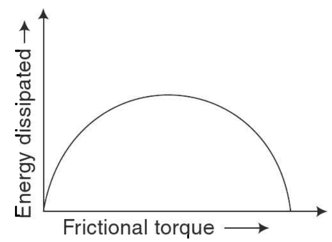


Fig 6.9 *Energy dissipated versus frictional torque curve for dry friction damper*

Though the system looks to be simple, its mathematical analysis is complicated. This is because of the flywheels slipping fully or partially or not at all depending upon the pressure of the spring bolts.

Now let us consider the shaft to be oscillating about its normal speed as shown by the time-speed curve of Fig. 6.10.

Let 'T' be the constant torque, 'J_1' and 'J_2' be the mass moment of inertia of the flywheels and 'ω_R' is the relative velocity at any instant.

If the flywheels are continuously slipping over the shaft then they will be acted upon by a constant torque. This will give a constant angular acceleration to the flywheels (T/J).

This is shown in Fig. 6.10, the constant angular acceleration of the flywheels gives a constant slope of the velocity curve. This can be indicated by BDF in Fig. 6.10.

The linear velocity of the flywheels will continue as long as to increase the shaft speed and make it greater than that of the flywheels in *BDEFG*. When the shaft speed is lower than that of flywheels, the latter's speed will continue to decrease.

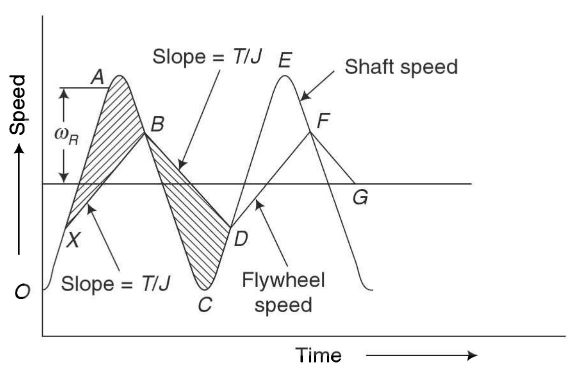


Fig. 6.10 *Relative slipping in a Lanchester damper*

Then work done per cycle or energy dissipation per cycle is given by the expression

$$E = \int_{cyc} T \times \omega_R dt \text{ or } E = T \times \int_{cyc} \omega_R dt,\ E = T \times \text{Area shaded } (XABCD).$$

On the other hand for decreasing pressure between the flywheels and friction plates, the curve of flywheel speed becomes flatter and at the end this will coincide with the mean line at the zero pressure as indicated by the line *DFG* shown in Fig. 6.12.

Next in case of increasing pressure, the same curve becomes steeper and at the end it will coincide with the shaft speed curve.

6.9 UNTUNED VISCOUS DAMPER, OR HOUDAILLE DAMPER

The principle of a viscous damper, also called Houdaille damper, is almost the same as that of the Lanchester damper, except for small changes: instead of using frication plates for dry friction damping, a viscous damper is used in this system. These types of dampers are helpful for damping out torsional oscillations. The viscous damper is added to a dynamic system to alter its vibrational response.

The construction of the viscous damper is simple and essentially consists of a rotating rotor or disc enclosed in a close-fitting case and which is keyed in the shaft as shown in Fig. 6.11.

The working principle of a viscous damper is that if the shaft rotates, normally the disc also rotates at the shaft's speed owing to viscous drag of the oil between the case and disc or rotor.

If the shaft vibrates torsionallly, the viscous action of the oil takes place between the disc and casing which produces the damping of the system.

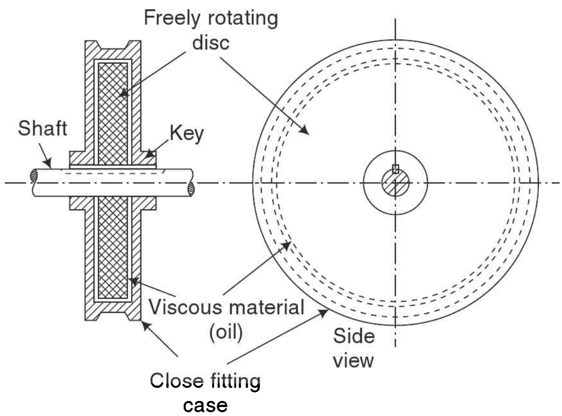


Fig. 6.11 *Untuned viscous damper or Houdaille damper*

Now there will be two cases arises in this system:

1. If there is zero damping in the damper, this is ineffective and the system corresponds to a single-degree-freedom system.

2. If the damping is too large, the damper mass becomes virtually integral with the main mass. Even then, it remains a single-degree-freedom system, whose effective mass becomes larger. When the main system rotates at such a speed then torsional vibrations are produced, and there will be energy dissipated due to the viscous drag of the viscous material filled between the casing and the disc. In this condition, the energy of the main system is reduced; hence there is a reduction in amplitude of torsional vibrations of the main system.

Now by the above working principle of a viscous damper, the optimum damping is produced in the system and we get the maximum response of the damper over the entire frequency range, and it will not go beyond a certain permissible limit. Usually, a Houdaille damper is used for machines of fluctuating or variable speed, and also maximum response being controlled by the ratio of inertia of damper to the inertia of main system is equal to J_2/J_1, where 'J_1' and 'J_2' are the mass moments of inertia of the flywheels.

The magnification factor versus frequency response curve for an untuned viscous damper is as shown in Fig. 6.12.

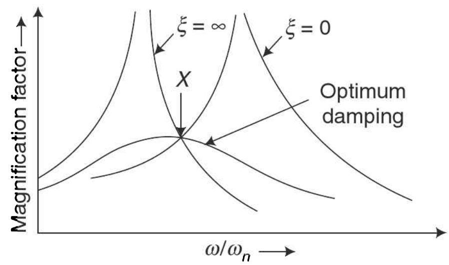


Fig. 6.12 *Magnification factor versus frequency response curve for untuned viscous damper*

From Fig. 6.12, we can see that frequency response curve is of same nature for the above two cases except that the peak now shifts towards the left by an equal amount based upon the ratio of the damper inertia to the main system inertia. It can be shown that the point of intersection 'X' of the above two curves, corresponding to $\xi = 0$ and $\xi = \infty$, is the point through which the frequency response curve of different damping values pass through it. A system having optimum damping has its frequency response curve 'X' as the maximum point. The idea is to add optimum damping in the system in order that the maximum frequency response curve over the entire frequency range does not go beyond the certain limit only.

6.11 TORSIONAL VIBRATION ABSORBER

In case of undamped dynamic vibration absorber of rectilinear system, a torsional vibration absorber can be used to minimise or completely eliminate torsional oscillation of a system.

Let us consider a single rotor system of torsional stiffness 'k_{t1}' and mass moment of inertia of rotor 'J_1' subjected to a periodic torque '$T \sin \omega t$' as shown in Fig. 6.13(a).

It is possible to make the amplitude of torsional vibration of rotor 'J_1' to become zero, by adding a torsional vibration absorber of sub-system '$k_{t2} - J_2$' as shown in Fig. 6.13(b) thereby converting the original single-degree-freedom system into a two-degree-freedom system. This analysis holds good for a dynamic vibration absorber as we already studied in Section 6.7.

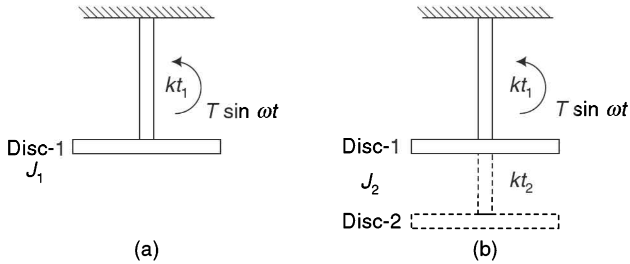


Fig. 6.13 *Torsional vibration absorber*

Consider Fig. 6.13(a). In this system we see that a harmonic torque $T \sin \omega t$ is impressed on the rotor, and the natural frequency of the main system is given by the relation

$$\omega_n = \sqrt{\frac{k_{t1}}{J_1}} \text{ rad/s} \qquad \text{...6.103}$$

Then the natural frequency of the main system coincides with that of the impressed torque frequency, and resonance occurs. However, the system requires some modification in which one modification is to change the stiffness of the shaft 'k_{t1}' or otherwise the inertia of the rotor 'J_1' to change the natural frequency of the system, because these are the two parameters in the whole system. But the above two modifications are somehow difficult to implement due to operating problem restrictions such as limited space, etc. Then an absorber rotor 'J_2' and the stiffness of the shaft 'k_{t2}' are to be added to the original system rotor 'J_1' to absorb the impressed torque ($T \sin \omega t$) such that the rotor 'J_1' does not vibrate. Here, the absorber should be tuned to the impressed frequency of 'ω' so that $\omega_{n2} = \omega = \sqrt{\frac{k_{t2}}{J_2}}$. ...6.104

This absorber should be strong enough to carry the amount of applied impressed torque to the absorber rotor or disc.

It can be seen that any value of 'k_{t2}' and 'J_2' will meet the requirement as long as their ratio is equal to the square of the frequency (ω^2). Suppose a shaft is to be added to a system. It may require a very long axial length which may not possible. Several ways are used to replace it out of which the most important torsional absorbers are ring torsional absorber, four-spring torsional vibration absorber, etc.

1. Ring torsional vibration absorber The construction of a ring torsional absorber is very simple and it consists of a ring (R) attached to the rotors or discs (J_1, J_2) of the original system. A mass is connected to this ring by means of a number of springs (k) as shown. If no vibration is present, the entire unit revolves at a constant speed. When torsional vibration occurs in the system, the mass tends to continue to revolve at constant speed and hence the springs are deflected and it acts as an torsional absorber. When the springs are deflected due the vibration caused by the main system, there is a change in their potential energy and thus they can absorb the energy of vibration of the main system. Hence, the vibrations of the main system

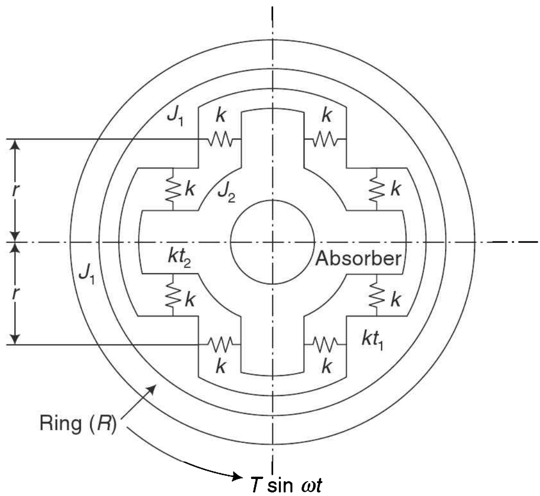


Fig. 6.14 *Ring torsional vibration absorber*

are eliminated or completely reduced. The coil springs are replaced suitably for the length of shafting.

2. Four-spring torsional vibration absorber The four-spring torsional vibration absorber is as shown in Fig. 6.14. It is observed that the stiffness of the torsional vibration 'k_{t2}' is given by

$$`k_{t2}\text{'} = 4 \times k \times r^2 \qquad \text{...6.105}$$

or
$$k = \frac{k_{t2}}{4r^2} \qquad \text{...6.106}$$

We know that earlier when springs are in parallel, the equivalent spring stiffness is the summation of their individual stiffness.

Therefore, $k_{eq} = k + k + k + k = 4k$ and the spring force is given by $F = k_{eq} \times x$, or $F = 4kx$

After small displacement 'x' given to the system, $x = r\theta$

where x = Linear displacement of equivalent system

θ = Angular displacement of equivalent system or the absorber system

Then
$$F = 4kx, F = 4k \times r\theta \qquad \text{...6.107}$$

Therefore, torque exerted on the absorber

$$T_2 = F \times r = 4kr^2\theta \qquad \text{...6.108}$$

Therefore, torque exerted per unit twist on the absorber

$$k_{t2} = \frac{T_2}{\theta} = \frac{4kr^2\theta}{\theta}, \quad k_{t2} = 4kr^2 \qquad \text{...6.109}$$

The amplitude of the torsional vibration absorber at the existing frequency 'ω' is given by the equation $F = -A_2\, k_2$ based on a same analysis of a vibration absorber after translational quantities into torsional quantities is $T = -X_2\, k_{t2}$

where T = Maximum value of applied torque

X_2 = Amplitude of vibration of torsional absorber

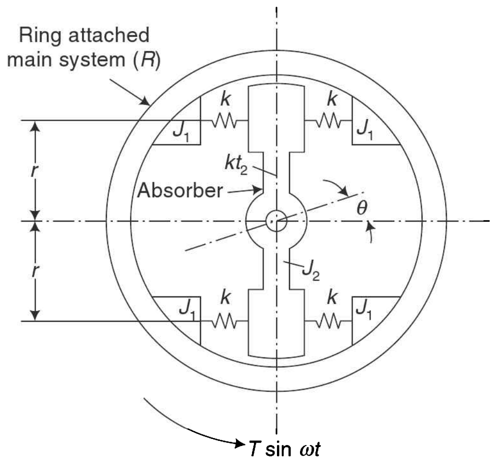


Fig. 6.15 *Four-spring torsional vibration absorber*

6.12 GENERALISED COORDINATES

Sometimes it is possible to specify the configuration of a system by more than one set of independent coordinates or parameters such as length, angle or some other physical parameters and any set of such coordinates may be called 'generalised coordinates'.

Consider a body 'AB' shown in Fig. 6.16 having a mass 'm' is supported by two springs 'k_1' and 'k_2' at its ends. The system is a two-degree-freedom system, since two independent coordinates will be necessary to describe its configuration.

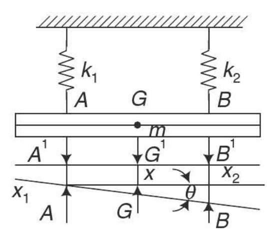


Fig. 6.16 *Generalised coordinates*

There may be any one of the following sets:

1. Deflection 'x_1' and 'x_2' of the two ends of the body AB,
2. Deflection 'x' of its centre of gravity and rotation 'θ'
3. Deflection of its left end 'x_2' and rotation 'θ'

Therefore, any set of coordinates, namely (x_1, x_2), (x_1, θ), (x_2, θ) etc., represent generalised coordinates.

6.12 COORDINATE COUPLING

This is a concept of coupling action where a vibration in one part of the system induces vibration in another part of the same system due to the force transmitted through the coupling spring or dashpot. In other words, the displacement of one type to the system results in a different type of displacement as shown in Fig. 6.17. The generalised coordinates are used to analyse the problems of vibration. To analyse the vibration problems, first obtain the governing differential equation by using different methods, like energy method, equilibrium method, etc. One differential equation represents the forces such as inertia, spring, damping forces acting on the mass. The equation having the mass 'm' must contain generalised coordinates like displacement 'x' and may or may not contain coordinates other than 'x'. If coordinates other than 'x' appear in the equation for 'm' then the equation is said to be **'coupling'** and the respective terms are known as **'coupling terms'.**

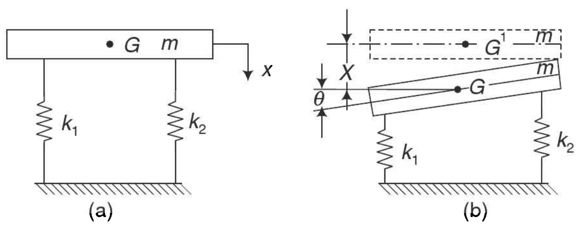


Fig. 6.17 *Coordinate coupling*

Examples There are two types of coordinate coupling

(i) Static coupling due to static displacements [Fig. 6.17(a)]

(ii) Dynamic coupling due to inertia forces [Fig. 6.17(b)]

1. Static coupling It contains coupling terms as functions of coordinates like springs force coupling. The above type of coordinate couplings can be represented by taking a few examples:

(a) $m_1\ddot{x}_1 + x_1(k_1 + k_2) - k_2x_2 = 0, \quad m_1\ddot{x}_1 + k_1x_1 + k_2x_1 - k_2x_2 = 0$

$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0, \quad m_2\ddot{x}_2 + k_2x_2 + k_3x_2 - k_2x_1 = 0$

In above equations, k_2x_2 and k_2x_1 are the static coupling

(b) $I_1\ddot{\theta}_1 + kt_1\theta_1 - kt_2(\theta_2 - \theta_1) = 0, \quad I_1\ddot{\theta}_1 + kt_1\theta_1 - k_{t2}\theta_2 + k_{t2}\theta_1$

$I_2\ddot{\theta}_2 + kt_2\theta_2 - kt_2\theta_1 = 0$

In above equations $k_{t2}\theta_2$ and $k_{t1}\theta_1$ are the static couplings:

(c) $mL^2\ddot{\theta}_1 + (kL^2 + mgL)\theta_1 - kL^2\theta_2 = 0$ ...6.110

(d) $4mL^2\ddot{\theta}_2 + (kL^2 + 2\,mgL)\theta_2 - kL^2\theta_1 = 0$ ...6.111

In equations 6.110 and 6.111, $kL^2\theta_2$ and $kL^2\theta_1$ are the static couplings.

2. *Dynamic coupling* It contains coupling terms as a function of time derivatives of coordinates, like inertia coupling, damping. The above types of coordinate couplings can be represented by taking a few examples as follows:

$$(M+m)\ddot{x} + kx + ml\ddot{\theta} = 0 \qquad ...6.112$$

$$l_2\ddot{\theta} + gl\theta + l\ddot{x} = 0 \qquad ...6.113$$

In equations 1.112 and 6.113 $ml\ddot{\theta}$ and $l\ddot{x}$ are the dynamic coupling.

6.12 PRINCIPAL COORDINATES

An 'n'-degree freedom system requires 'n' independent coordinates and there will be 'n' number of differential equations of motion. It is always possible to find a particular set of coordinates such that each equation of motion contains only one unknown quantity. Then the equations of motion can be solved independently and the unknown quantity can be found out. Such a particular set of coordinates is called **'principal coordinates'.**

EXAMPLE 6.23

Determine the pitch and bounce frequencies and the location of oscillation centers of an automobile with the following data: m = 1000 kg, r_g = 0.9 m, distance between the front axle and centre of gravity = 1 m, distance between the rear axle and centre of gravity = 1.5 m. Front spring stiffness, k_1 = 8 kN/m, rear spring stiffness k_2 = 22 kN/m.

Solution The equation of motion can be written as

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta),\ m\ddot{x} + (k_1+k_2)x + (k_2l_2 - k_1l_1)\theta = 0$$

$$\ddot{x} + \left(\frac{k_1+k_2}{m}\right)x + \left(\frac{k_2l_2 - k_1l_1}{m}\right)\theta = 0 \qquad ...6.114$$

$$J\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2,\ mr_g^2\ddot{\theta} + (k_2l_2 - k_1l_1)x + (k_1l_1^2 + k_2l_2^2)\theta = 0$$

$$\ddot{\theta} + \left(\frac{k_2l_2 - k_1l_1}{mr_g^2}\right)x + \left(\frac{k_1l_1^2 - k_2l_2^2}{mr_g^2}\right)\theta = 0 \qquad ...6.115$$

Substitute the values of m, k, r and l in equations 6.114 and 6.115.

$$\ddot{x} + 30x + 25\theta = 0, \quad \ddot{\theta} + 30.86x + 70.98\theta = 0$$

Assuming the solution as $x = X\sin\omega t$ $\ddot{x} = -\omega^2 X\sin\omega t$

$$\theta = \beta\sin\omega t, \quad \ddot{\theta} = -\omega^2\beta\sin\omega t, -\omega^2X + 30X + 25\beta = 0,$$

$$-\omega^2\beta + 30.86X + 70.98\beta = 0$$

$$\frac{X}{\beta} = \frac{25}{\omega^2 - 30} \qquad ...6.116$$

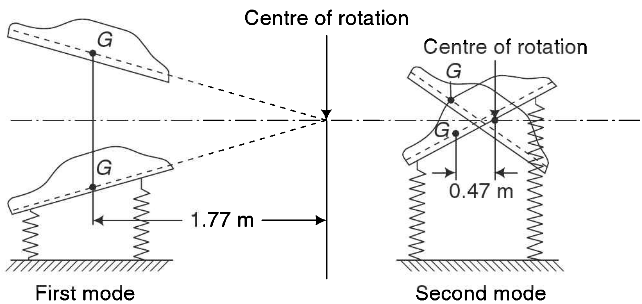


Fig. p-6.23 *Oscillation centers of automobile*

$$\frac{X}{\beta} = \frac{\omega^2 - 70.98}{30.86} \qquad ...6.117$$

Equating equations 6.116 and 6.117, we get $\omega^4 - (30 + 70.98)\omega^2 + 1357.9 = 0$

$$\omega^4 - 100.98\ \omega^2 + 1357.9 = 0$$

$$\omega_{1,2}^2 = \frac{100.98 \pm \sqrt{100.98^2 - 4 \times 1357.9}}{2} \text{ or } \omega_1 = 3.99\text{rad/s}$$

and $\omega_2^2 = 85.0$ or $\omega_2 = 9.24$ rad/s

$$\left(\frac{X}{b}\right)_1 = \frac{25}{\omega_1^2 - 30} = \frac{25}{3.99^2 - 30} = -\frac{1.77}{1} \text{ m/rad}$$

$$\left(\frac{X}{\beta}\right)_2 = \frac{\omega_2^2 - 70.98}{30.86} = \frac{9.24^2 - 70.98}{30.86} = \frac{0.466}{1} \text{ m/rad.}$$

EXAMPEL 6.24

A schematic diagram representation of an automobile is shown in Fig. p-6.24(a) if the automobile weighs 4000 N and has a radius of gyration about the centre of gravity of 4.5 m. The combined spring stiffness of front springs k_1 and k_2 are 3000 N/m and 3250 N/m respectively. Determine the natural frequency of the system.

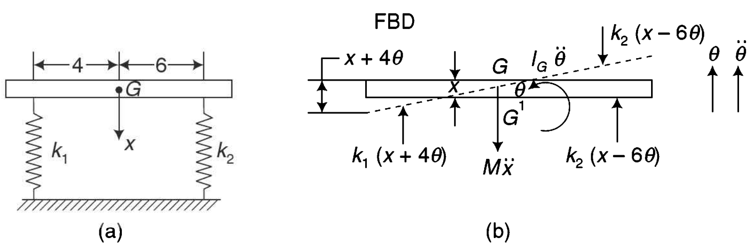


Fig. p-6.24 *Automobile system*

Solution Here, $I_G = MK^2$, K = Radius of gyration

$$I_G = \frac{4000}{9.81} \times (4.5)^2, \quad I_G = 8256.88 \text{ kg} - m^2, \quad M = 407.75 \text{ kg}$$

Considering the linear moment of the mass, apply Newton's second law of motion. The FBD is as shown in Fig. p-6.24(b).

$$-k_1 (x + 4\theta) - k_2 (x - 6\theta) = M\ddot{x}$$

$$\therefore \quad M\ddot{x} + k_1 x + k_2 x + 4k_1\theta - 6k_2\theta = 0$$

$$M\ddot{x} + (k_1 + k_2)\, x + (4k_1 - 6k_2)\theta = 0$$

$$407.75\ddot{x} + (3000 + 3250)\, x + (4 \times 3000 - 6 \times 3250)\, \theta = 0$$

$$407.75\ddot{x} + 6250x + (-7500)\theta = 0, \quad \ddot{x} + 15.33x - 18.39\theta = 0 \qquad \text{...6.118}$$

Considering the rotation of the mass about its centre of gravity,

$$\Sigma M_G = I_G\ddot{\theta}, \quad -k_1 (x + 4\theta)\, 4 + k_2 (x - 6\theta)\, 6 = I_G\ddot{\theta}$$

$$I_G\, \ddot{\theta} + 4k_1 (x + 4\theta) - 6k_2 (x - 6\theta) = 0, \; I_G\, \ddot{\theta} + (4k_1 - 6k_2)x - (16k_1 + 36k_2)\theta = 0$$

$$8256.88\, \ddot{\theta} + (4 \times 3000 - 6 \times 3250)\, x + (16 \times 3000 + 36 \times 3250)\theta = 0$$

$$\ddot{\theta} + 19.98\theta + 0.91x = 0 \qquad \text{...6.119}$$

Put $\quad x = \sin \omega t \qquad \theta = B \sin \omega t$

$$\ddot{x} = -A\omega^2 \sin \omega t, \; \ddot{\theta} = -B\omega^2 \sin \omega t$$

Using these value of x, θ and $\ddot{\theta}$ in Eq. 6.118,

$$-A\omega^2 + 15.33A - 18.39B = 0, \quad A\,[15.33 - \omega^2] = 18.39B$$

$$\frac{A}{B} = \frac{18.39}{15.33 - \omega^2} \qquad \text{...6.120}$$

Using the value of x, θ and $\ddot{\theta}$ in Eq. 6.119,

$$-B\omega^2 + 19.98\,B - 0.91A = 0, \quad B\,[19.98 - \omega^2] = 0.91A$$

$$\frac{A}{B} = \frac{19.98 - \omega^2}{0.91} \qquad \text{...6.121}$$

From equations 6.120 and 6.121,

$$\frac{18.39}{15.33 - \omega^2} = \frac{19.91 - \omega^2}{0.91}, \; (15.33 - \omega^2)\,(19.98 - \omega^2) = 18.39 \times 0.91$$

$$15.33 \times 19.98 - (15.33 + 19.98)\, \omega^2 + \omega^4 = 18.39 \times 0.91, \; \omega^4 - 35.31\, \omega^2 + 289.56 = 0$$

This is the frequency equation which is quadratic in ω^2.

$$\therefore \quad \omega^2 = \frac{35.31 \pm \sqrt{(35.31)^2 - 4 \times 289.56}}{2},$$

$$\omega_{1n}^2 = 12.95\; \omega_{1n} = 3.6 \text{ rad/s}, \quad \omega_{2n}^2 = 22.36, \quad \omega_{2n} = 4.73 \text{ rad/s}$$

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

To draw the principal mode shapes,

$$\frac{A}{B} = \frac{18.39}{15.33 - \omega^2} \qquad \frac{A}{B} = \frac{18.39}{15.33 - \omega^2}$$

At $\omega^2 = \omega_{1n}^2 = 12.95$ At $\omega^2 = \omega_{2n}^2 = 22.36$

$$\frac{A_1}{B_1} = \frac{18.39}{15.33 - 12.95} \qquad \frac{A_2}{B_2} = \frac{18.39}{15.33 - 22.36}$$

$$\frac{A_1}{B_1} = \frac{18.39}{2.38} \qquad \frac{A_2}{B_2} = \frac{18.39}{-7.03}$$

i.e. $A_1 = 18.39, B_1 = 2.38$, i.e. $A_2 = 18.39, B_2 = -7.03$.

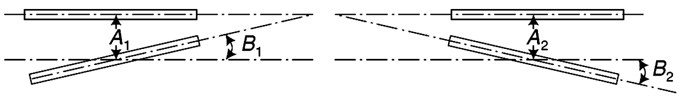


Fig. p-6.24(c) *Principal modes*

6.12 LAGRANGE'S METHOD, OR ENERGY METHOD

Some problems of two or more degrees of freedom systems are not very convenient to be formulated using force equilibrium method as the mass will undergo linear and rotary motions and their combinations. For this purpose for all systems, Lagrange's method is very suitable for the presence of force in function and damping forces present in a system. For use of this energy method or Lagrange's method, the equation will readily yield the directly as many equations of motion as the number of degrees of freedom of the system when the basic energy equations of the system are known.

The general form of this equation in terms of generalised coordinates is given by the expression.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial}{\partial q_i}(L) + \frac{\partial F}{\partial q_i} = Q_i \text{ or } \frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

Let T = KE = Kinetic energy of the entire system $= \frac{1}{2}\sum_i m_i x_i^2 \quad i = 1, 2, 3, 4, \dots n$

V = PE = Potential energy of the entire system $= \frac{1}{2}\sum_i k_i x_i$.

F = Frictional energy dissipated $= \frac{1}{2}\sum_i c_i x_i^2$.

By defining Lagrange's equation $L = T - v$, q_i = Generalised coordinates.

Q_i = Generalised external forces acting on the masses 'm_i' in the direction of 'x_i'

As 'T' is a function of '$\dot{x}_i$', and 'V' is a function of 'x_i' the above equation can be written as $\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial q_i}\right) + \frac{\partial}{\partial q_i}(V) + \frac{\partial F}{\partial q_i} = Q_i$

If damping force 'F' and external force 'Q'$_i$ are absent, the equation can be further simplified as $\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial q_i}\right) + \frac{\partial V}{\partial q_i} = 0$. This is very powerful principle and can be applied to any system.

EXAMPLE 6.25

Determine the equation of motion of the double pendulum as shown in Fig. p-6.25(a) for small oscillation by using Lagrange's method.

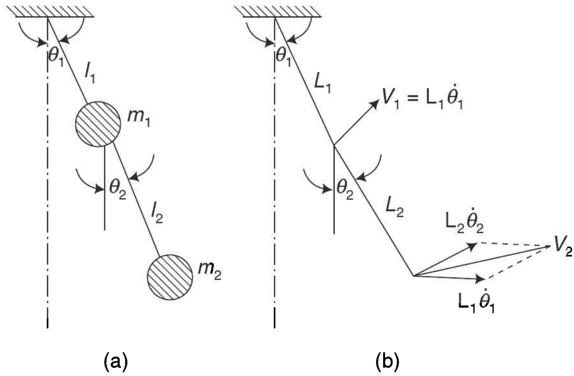


Fig. p-6.25 *Double pendulum*

Solution The KE of the system is KE $= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$$v_1^2 = (L_1\dot{\theta}_1)^2$$

where $$v_2^2 = (L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 + 2L_1L_2\,\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$$

Which are velocities of the masses m_1 and m_2 respectively.

$$\text{PE} = m_1 q L_1 (1 - \cos\theta_1) + m_2 g [L_1 (1 - \cos\theta_1) + L_2 (1 - \cos\theta_2)$$

Lagrange's equation is

$$\frac{d}{dt}\frac{\partial(\text{KE})}{\partial q_i} + \frac{\partial(\text{PE})}{\partial q_i} + \frac{\partial(\text{KE})}{\partial q_i} = 0$$

$$\frac{d}{dt}\frac{\partial(\text{KE})}{\partial q_i} = \frac{d}{dt}(m_1L_1^2\,\dot{\theta}_1 + m_2\,[L_1^2\,\dot{\theta}_1 + L_1L_2\,\theta_2\cos(\theta_2 - \theta_1)])$$

$$= m_1L_1^2\,\dot{\theta}_1 + m_2\left[L_1^2\,\dot{\theta}_1 + L_1L_2\,\dot{\theta}_2\cos(\theta_2 - \theta_1) + L_1L_2\,\dot{\theta}_2\frac{d}{dt}(m_2\cos(\theta_2 - \theta_1))\right]$$

$$= mL_1^2\,\ddot{\theta}_1 + m_2L_1^2\,\ddot{\theta}_1 + m_2L_1L_2\,\dot{\theta}_2$$

where $\sin\theta = \theta$, $\cos(\theta_2 - \theta_1) = 1$ and $\frac{d}{dt}[\cos(\theta_2 - \theta_1)] = 0$ since θ is small.

Also $$\frac{\partial(\text{KE})}{\partial\theta_i} = 0. \quad \frac{\partial(\text{PE})}{\partial\theta_i} = m_1gL_1\sin\theta_1 - m_2gL_1\sin\theta_1$$

Then the first equation of motion is given by

$$(m_1 + m_2)\,L_1\,\ddot{\theta}_1 + m_1L_2\,\ddot{\theta}_2 + (m_1 + m_2)\,g\theta_1 = 0$$

Similarly, $$\frac{d}{dt}\frac{\partial(\text{KE})}{\partial\theta_2} = \frac{d}{dt}[m_2L_2^2\,\ddot{\theta}_2 + m_2L_1L_2\,\dot{\theta}_1\cos(\theta_2 - \theta_1)]$$

$$= m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1$$

$$\frac{\partial(\text{KE})}{\partial \theta_2} = 0$$

$$\frac{\partial(\text{PE})}{\partial \theta_2} = m_2 g L_2 \sin \theta_2$$

And so the second equation of motion becomes

$$L_2 \ddot{\theta}_2 + g\theta_2 + L_1 \ddot{\theta}_1.$$

REVIEW QUESTIONS

(1) Explain:
  (i) Coordinate coupling
  (ii) Semidefinite system or degenerating system
  (iii) Principal mode of vibration or normal mode of vibration
  (iv) Generalised coordinates
  (v) Principle coordinates
  (vi) Orthogonality principle as applied to two-degree-freedom systems.

(2) Explain with a neat sketch, the basic working principle of a dynamic vibration absorber (Frahm vibration absorber).

(3) What is the main disadvantage of a dynamic vibration absorber? Show that for such an absorber, its natural frequency should be equal to the applied frequency.

(4) How can we make a system vibrate in one of its natural modes?

(5) What are meant by static and dynamic couplings?

(6) Define mass coupling, velocity coupling and elasticity or static coupling.

PROBLEMS FOR PRACTICE

(1) Find the natural frequency and mode shape of the system as shown in Fig. p.p-6.1, if m = 2 kg, k = 400 N/m.

Ans. ω_{n1} = 10 rad/s, ω_{n2} = 20 rad/s.

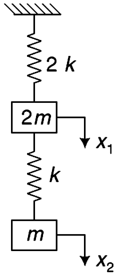


Fig. p.p-6.1

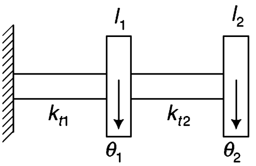


Fig. p.p-6.2

(2) Determine the natural frequency for the system shown in Fig. p.p-6.2 and draw the mode shapes and locate the node for each mode of vibration. Given I_1 = 200 kgm^2, I_2 100 kgm^2, kt_1 = 2000 Nm/rad, kt_2 = 500 Nm/rad.

Ans. ω_{n1} = 3.6 rad/s, ω_{n2} = 13.9 rad/s.

(3) An electric train made of two cars, each of mass 2000 kg, has got a spring coupling of 40 × 10^6 N/m stiffness as shown in Fig. p.p-6.3. Determine the amplitude ratios and natural frequency of the system.

Ans. $\omega_{n1} = 0$, ω_{n2} = 200 rad/s.

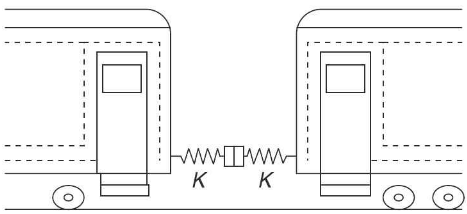


Fig. p.p-6.3

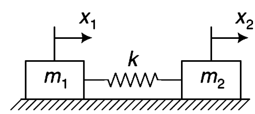


Fig. p.p-6.4

(4) Determine the amplitude ratios and natural frequencies of the system shown in Fig. p.p-6.4. If w_1 = 45 kg, w_2 = 65 kg and k = 16 kg/cm.

Ans. f = 3.5 cps.

(5) Derive the frequency equation for the following system and determine the natural frequencies. Assume the chord passing over the cylinder does not slip as shown in Fig. p.p-6.5.

Ans. $$\omega_{1n}^2 = \frac{k_1 + k_2}{m_2} + \frac{k_1}{2m_1} - \sqrt{\left[\frac{k_1 + k_2}{m_2} + \frac{k_1}{2m_1}\right]^2 - \frac{2k_1k_2}{m_1m_2}} = \frac{k_1 + k_2}{m_2} + \frac{k_1}{2m_1}$$

$$\omega_{2n}^2 + \sqrt{\left[\frac{k_1 + k_2}{m_2} + \frac{k_1}{2m_1}\right]^2 - \frac{2k_1k_2}{m_1m_2}}.$$

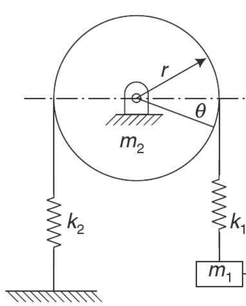


Fig. p.p-6.5

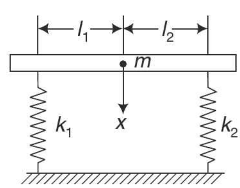


Fig. p.p-6.6

(6) An automobile of 2000 kg mass has a wheel base of 3.0 m. Its centre of gravity is located 1.4 m behind the front wheel axis and has a radius of gravitation about its c.g. as 1.1 m. The front springs have a combined stiffness of 5.88 × 10^6 N/m and the rear springs 6.37 × 10^6 N/m as shown in Fig. p.p-6.6. Find the two natural frequencies of vibrations.

Ans. $\omega_1 = 107.9$ rad/s, $\omega_2 = 77.34$ rad/s

(7) Determine the natural frequencies of the system shown in Fig. p.p-6.7. Neglect the friction at rollers.

Ans. $\omega_1 = \sqrt{\frac{2k}{m}}$ rad/s, $\omega_2 = \sqrt{\frac{4k}{m}}$ rad/s.

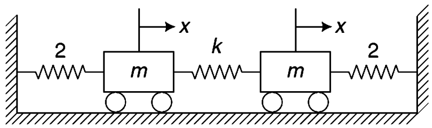


Fig. p.p-6.7

(8) Find the natural frequencies of the system shown in Fig. p.p-6.8. Also determine the ratio of amplitudes and the mode shapes. Given $I_1 = I_2 = I$.

Ans. $\omega_1 = \sqrt{\frac{2k_t}{I}}$ rad/s, $\omega_2 = \sqrt{\frac{5k_t}{I}}$ rad/s.

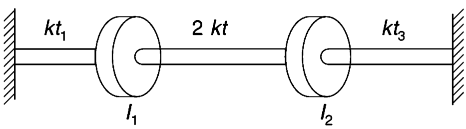


Fig. p.p-6.8

(9) Determine the natural frequencies of the double pendulum as shown in Fig. p.p-6.9, if $l_1 = l_2 = 0.5$ m and $m_1 = m_2 = 2$ kg.

Ans. $\omega_1 = 3.4$ rad/s, $\omega_2 = 8.8$ rad/s.

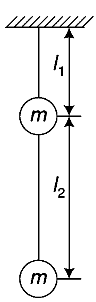


Fig. p.p-6.9

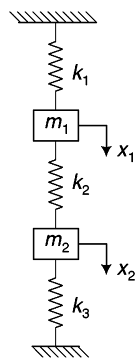


Fig. p.p-6.10

(10) Determine the natural frequencies of the system as shown in Fig. p.p-6.10 $m_1 = m_2 = m = 10$ kg, $k_1 = k_3 = 9000$ N/m and $k_2 = 3000$ N/m.

Ans. $\omega_1 = 30$ rad/s, $\omega_2 = 38.74$ rad/s.

(11) Reduce the following gear system as shown in Fig. p.p.6.11 considering the inertia of geared system as 'I',

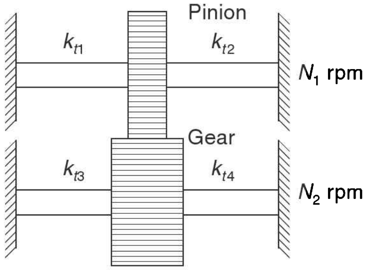
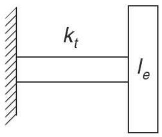


Fig. p.p-6.11



Ans. Fig. p.p-6.11

(12) A block of mass '*m*' resting on a frictionless horizontal plane is connected through a spring of constant '*k*' to a homogenous uniform rod of mass '*M*' and length '*L*' as shown in Fig. p.p-6.12. Determine the frequency equation.

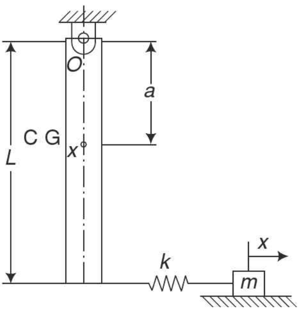


Fig. p.p-6.12

Ans. $I_0 m\omega^4 - [I_0 k\omega^2 + Mmga + mkL^2]\,\omega^2 + mkga = 0.$

OBJECTIVE-TYPE QUESTIONS

(1) All the moving parts of the system oscillating in the same frequency and phase are known as
  (a) principle coordinates
  (b) first principal mode of vibration
  (c) generalised coordinates
  (d) principal mode of vibration

(2) Static coupling occurs due to
  (a) static displacements and dynamic inertia forces
  (b) static displacements
  (c) dynamic inertia forces
  (d) all the above statements are true

(3) Dynamic vibration absorber means
  (a) it is possible to make the amplitude of vibration of first mass to become zero
  (b) it is possible to make the amplitude of vibration of second mass to become zero
  (c) it is possible to make the amplitude of vibration of first mass to become maximum
  (d) it is possible to make the amplitude of vibration of mass become zero

(4) In case of a two-degree-freedom system, masses will vibrate in two different modes called as
   (a) principal-mode vibration
   (b) normal-mode vibration
   (c) first-mode vibration
   (d) none of the above

(5) Centrifugal vibration absorbers are very effective when
   (a) at only one frequency of design
   (b) either the speed changes or the speed fluctuates
   (c) only the speed fluctuates
   (d) all of the above cases

(6) In case of a semidefinite system, natural frequency becomes
   (a) one of their natural frequencies is equal to zero
   (b) their natural frequency becomes maximum
   (c) both natural frequencies become zero
   (d) both of their natural frequencies are equal

(7) The principal modes or normal modes of vibration for systems
   (a) having two or more degrees of freedom are orthogonal
   (b) are an important property while finding the mode shape
   (c) are an important property while finding the mode shapes and nodes
   (d) all of the above cases

(8) In a two-rotor system as shown in Fig. p.6.8.1, if $I_1 > I_2$, a node of vibration lies in

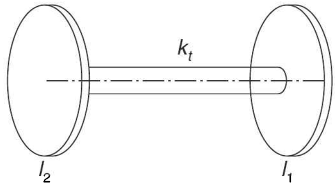


Fig. p.6.8.1

   (a) between I_1 and I_2 but near to I_1
   (b) between I_1 and I_2 but near to I_2
   (c) exactly center between the rotor I_1 and I_2
   (d) near I_2 but outside

(9) In a two-rotor system as shown in Fig. p.6.8.1, the frequency equation is given by

   (a) $\omega_1 = \sqrt{\dfrac{I_1 I_2}{k_t\,(I_1 + I_2)}}$ rad/s

   (b) $\omega_2 = \sqrt{\dfrac{k_t\,(I_1 + I_2)}{I_1 I_2}}$ rad/s

   (b) $\omega_1 = \sqrt{\dfrac{k_t(I_1 + I_2)}{I_1 I_2}}$ rad/s

   (d) $\omega_2 = \sqrt{\dfrac{I_1 I_2}{k_t\,(I_1 + I_2)}}$ rad/s

(10) Lagrange's method is very suitable for
   (a) the presence of damping force present in a system
   (b) the presence of force present in a system
   (c) the presence of force in function and damping forces present in a system
   (d) all of the above cases

Answers

(1) d	(2) b	(3) d	(4) a	(5) b	(6) a
(7) a	(8) d	(9) b	(10) c		

MULTI-DEGREE-FREEDOM SYSTEMS: EXACT ANALYSIS

7

7.1 INTRODUCTION

A multi-degree-freedom system means a system having more than one degree of freedom system. Accordingly, two-degree-freedom systems are also multi-degree-freedom systems. Still they have been discussed separately, due to the methods of analysis being different. A system will have as many equations of motion as the number of degrees of freedom and also as many natural frequencies. In principle, the vibration analysis of two-degree-freedom systems is not much different to that of multi-degree-freedom systems except that the latter requires much more mathematical analysis. As the number of degrees of freedom increases, it becomes very tedious in solving the equations of motion and to determine the natural frequencies and mode shapes. The natural frequencies and mode shapes can be determined easily and quickly with the help of computers. Figure 7.1 shows the example a of multi-degree-freedom system.

The following few methods are employed to determine the natural frequencies, mode shapes, etc., in multi-degrees exact analysis. Also a few methods employed to determine the natural frequencies; mode shapes, etc., in multi-degree numerical methods are given in Chapter 8.

(a) By Newton's method (b) Influence coefficient (c) Maxwell's reciprocal theorem (d) Matrix iteration method (e) Matrices.

7.1 BY NEWTON'S METHOD

When '*n*' independent coordinates are required to specify the positions of the masses of a system, it is an *n*-degrees-of-freedom system or a multi-degree-freedom system.

The vibration analysis of a multi-degree-freedom system by first principle is more complex as the number of equations of motion increases. Therefore, iterative numerical procedures are employed to eliminate the tedious mathematical work.

Example A typical multi-degree-freedom system having '*n*' degrees of freedom as shown in Fig. 7.2(a) shows a rectilinear system and Fig. 7.2(b) shows a torsional system.

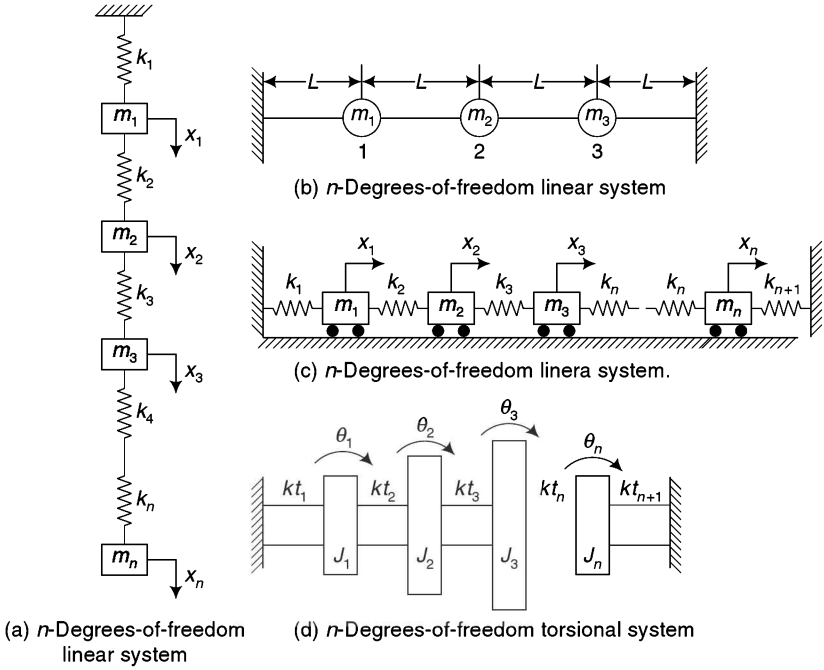


Fig. 7.1 *Undamped multi-degree-freedom systems*

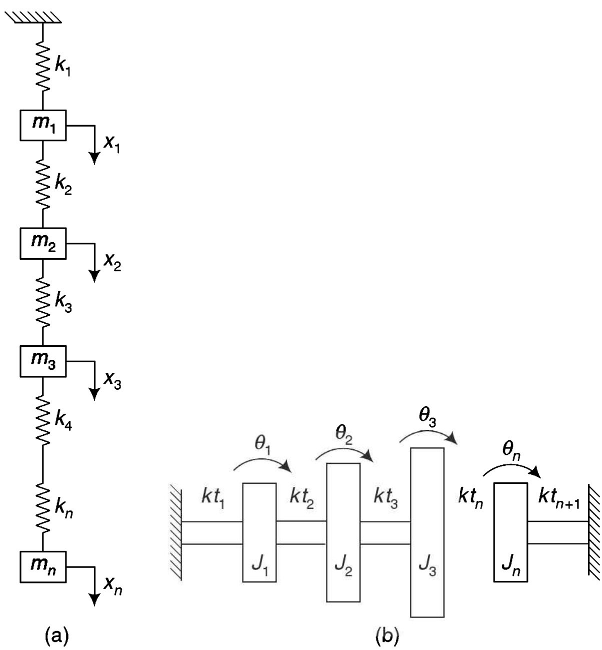


Fig. 7.2 *Undamped multi-degree-freedom system*

The equations of motion can be written by using Newton's second law of motion as follows, i.e. $\Sigma F = ma$.

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) &= 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - x_3) &= 0 \\ m_3\ddot{x}_3 + k_3(x_3 - x_2) + k_4(x_3 - x_4) &= 0 \end{aligned}\right\} \quad \text{...7.1(a)}$$

$$\vdots$$

$$m_n\ddot{x}_n + k_n(x_n - x_{n-1}) + k_{n+1}(x_n) = 0$$

Similarly, for torsional system equations of motion can be written by replacing 'm' by 'j', 'x' by 'θ' and 'k' by 'k_t'. Equation 7.1(a) is represented in the matrix form as follows:

$$[M]\{\ddot{x}\} + [k]\{x\} = 0 \qquad \text{...7.1(b)}$$

where $[M]$ is a square matrix of size '$n \times n$', known as a mass matrix. A mass matrix contains only diagonal elements given as follows:

$$[M] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & \cdot & \cdot & 0 \\ 0 & m_2 & 0 & \cdots & \cdot & \cdot & 0 \\ 0 & 0 & m_3 & \cdots & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 \cdots & \cdots & \cdots & \cdots & \cdots & m_n \end{bmatrix}$$

$[k]$ is a square-banded matrix of size '$n \times n$' and is given as follows:

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & 0 & \cdots \\ -k_2 & k_2 + k_3 & -k_3 & 0 & 0 & \cdots \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 & \cdots \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -k_n & k_n + k_{n+1} \end{bmatrix}$$

$\{\ddot{x}\}$ is a column matrix, $\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_n \end{Bmatrix}$ $\{x\}$ is a column matrix $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_n \end{Bmatrix}$

For free vibrations, the solution of Eq. 7.1(b) can be written as

$$\{x\} = \{X\} \sin \omega t,\ \{\dot{x}\} = \omega^2 \{X\} \cos \omega t\ \{\ddot{x}\} = -\omega^2 \{X\} \sin \omega t.$$

Substitute these values in Eq. 7.1(b), $[[k] - \omega^2 [M]]\{X\} = 0$ ...7.1(c).

Equation 7.1(c) is known as **eigenvalue problem** and it can be solved with the help of a computer for a large number of masses. ω^2 is called the eigenvalue or characteristic value of the equation. There will be 'n' such values for an 'n' degree-freedom system.

For each eigenvalue or natural frequency, there exists a corresponding eigenvector $\{x\}$, also called *characteristic vector*. The eigenvalue will be represented by the natural frequency while the eigenvector will represent the mode shape for a given frequency of the system.

7.3 INFLUENCE COEFFICIENT

It has been seen that in Section 7.2 that the differential equations of motion of a system can be written in matrix form and this matrix will include the mass matrix $[M]$ and **stiffness matrix** $[K]$. Suppose if a damper is present in a system then there will be a damping matrix denoted by$[C]$, also added in the matrix equation. We can observe that earlier the differential equations can also be written in the form of **flexibility matrix** denoted by $[A]$ instead of stiffness matrix $[K]$. This flexibility matrix is inverse of stiffness matrix. This can be written as follows.

$[A] = \{K\}^{-1}$This is also equal to $[K] = [A]^{-1}$ ...7.2

Equation 7.2 is same as that of the following relationship for a single-degree-freedom system, i.e. Stiffness $= \dfrac{1}{\text{Flexibility}}$

The terms or elements k_{ij}, a_{ij} and c_{ij} of stiffness, flexibility and damping matrices respectively are called '**influence coefficients**' and are stated as follows.

An influence coefficient, denoted by 'a_{ij}', is defined as the static deflection of the system at the position 'i' due to a unit force (unit load) at the position 'j'.

By Maxwell's reciprocal theorem, $a_{ij} = a_{ji}$

where a_{ij} = Deflection at 'i' due to unit load at 'j'.

a_{ji} =Deflection at 'j' due to unit load at 'i'.

The influence coefficients are very useful for writing differential equations of motion of multi-degree-freedom systems directly in matrix form, but this can be done by inspection for quite a few cases only. More laborious work is saved as in case of a higher-degree-freedom system, and also we know that the use of equations in matrix format facilitates the application of computer methods for their solution.

7.4 FLEXIBILITY COEFFICIENTS AND FLEXIBILITY MATRIX

Let us consider two points, point 'i' and point 'j' in any system; then a_{ij} is defined as the flexibility influence coefficient or the deflection at the point 'i' due to a unit load applied at the point 'j' of the system. In the same manner, 'a_{ji}' will be the deflection at the point 'j' due to the unit load applied at the point 'i' of the system. Based on the same principle, 'a_{jj}' or 'a_{ii}' will be the deflection at the point 'j' or 'i' due to a unit load applied at the same point of the system, wherein these a's are termed the '**flexibility influence coefficients**'. Whereas 'a_{ii}', 'a_{jj}', etc., are known as the '**direct influence coefficients**' and 'a_{ij}', 'a_{ji}', etc., are known as '**cross-influence coefficients**.'

The following matrix indicates a **flexibilwity coefficient** representing the **flexibility matrix.**

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

7.4.1 Maxwell's Reciprocal Theorem

Now we will discuss Maxwell's reciprocal theorem as follows.

Maxwell's reciprocal theorem states that **'the deflection at any point in the system due to a unit load acting at any other point of the same system is equal to the deflection at the second point due to the unit load acting at the first point.'**

Proof Consider a simply supported beam as shown in Fig. 7.3(a) above. Let 'w_1' and 'w_2' be the loads acting at points '1' and '2' respectively.

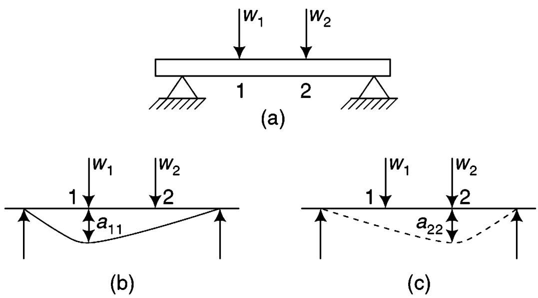


Fig. 7.3 *Simply supported beam with point load*

1. First cycle of application of load (w_1 first and then w_2)

Fig. 7.3(b) For unit load, deflection at the point '1' = a_{11}

For a load w_1, deflection at the point '1' = $w_1\, a_{11},\ PE = \frac{1}{2}(w_1 a_{11})$

$$P_1 = \frac{1}{2} w_1^2\, a_{11}$$

When 'w_2' is applied after 'w_1' is on, potential energy at the point '2' is

$$P_2 = \frac{1}{2} w_2^2\, a_{22} + w_1 (w_2 a_{12})$$

$$\therefore \text{ total energy } PE = \frac{1}{2} w_1^2\, a_{11} + \frac{1}{2} w_2^2\, a_{22} + w_1 w_2 a_{12}) \qquad \text{...7.3(a)}$$

2. Second cycle of application of loads (w_2 first and then w_1)

Fig. 7.3(c) Potential energy at position due to the load 'w_2'at the point '2', $P_2 = \frac{1}{2} w_2^2\, a_{22}$

Potential energy at the point '1' when 'w_1' is applied after 'w_2' is on,

$$P_1 = \frac{1}{2} w_1^2 a_{11} + w_2 (w_1 a_{21})$$

$$\therefore \text{ total energy } PE = \frac{1}{2} w_2^2 a_{22} + \frac{1}{2} w_1^2 a_{11} + w_2 w_1 a_{21}) \qquad ...7.3(b)$$

Equating equations 7.3(a) and 7.3(b), at the end of cycles the state of the beam will be same.

$$\therefore \qquad \frac{1}{2} w_1^2 a_{11} + \frac{1}{2} w_2^2 a_{22} + w_1 w_2 a_{12} = \frac{1}{2} w_2^2 a_{22} + \frac{1}{2} w_1^2 a_{11} + w_2 w_1 a_{21}$$

$\therefore a_{12} = a_{21}$ or in general $a_{ij} = a_{ji}$. Hence, the equation is proved.

7.4.2 Stiffness Coefficient and Stiffness Matrix

The stiffness coefficient is denoted by 'k_{ij}' of a system is defined as **"the location of coordinate q_i when a unity displacement is linear or angular given to the coordinate q_j with all other coordinates being held fixed, when displacement of all other coordinates is zero."**

A matrix of stiffness coefficient as shown in the matrix table referring to the stiffness matrix.

$$[K] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1j} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2j} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{i1} & ki_2 & \cdots & k_{ij} & \cdots & k_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ kn_1 & k_{n2} & \cdots & k_{nj} & \cdots & k_{nn} \end{bmatrix} \qquad ...7.4$$

This can be clearly explained by taking one simple example as follows.

Let us consider a two-degree-freedom linear system as shown in Fig. 7.4(a). In the figure, the direction of the two coordinates 'q_1' and 'q_2' is shown towards the right side treating them as positive. Now give a unit displacement to the mass 'm_1' treating all other masses to their zero position (in this case the other mass is 'm_2' only). Let 'F_1' and 'F_2' be the respective forces to hold the masses in positions, as shown in Fig. 7.4(b). Now as per the statement of stiffness coefficient, we have

$F_1 = k_{11}$ and $F_2 = k_{21}$. After unit displacement given to mass 'm_1' and mass 'm_2' the FBD is as shown in Fig. 7.4(c). It will give

$$\left.\begin{aligned} k_{11} &= k_1 + k_2 \\ k_{21} &= -\, k_2 \end{aligned}\right\} \qquad ...7.5$$

In a similar way in Fig. 7.4(d), apply unit displacement to the mass 'm_2' with the other mass held in zero position. Now here also 'F_1' and 'F_2' are the respective forces to hold the masses in their displaced positions.

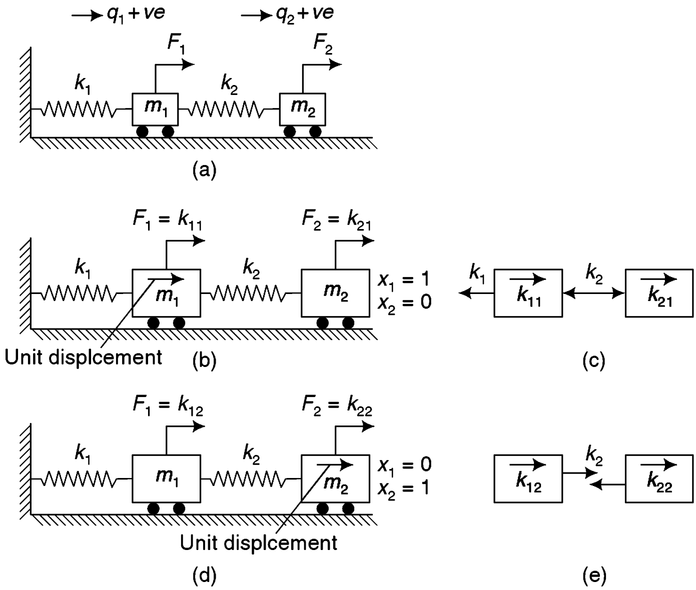


Fig. 7.4 *Stiffness coefficient of two-degree-freedom linear system*

Now by the statement of stiffness coefficient, we have

$$F_1 = k_{12} \text{ and } F_2 = k_{22}$$

After unit displacement, the FBD is as shown in Fig. 7.4(e). It will give

$$\left.\begin{aligned} k_{12} &= -k_2 \\ k_{22} &= k_2 \end{aligned}\right\} \qquad ...7.6$$

The stiffness matrix for the system, as per the statement of stiffness coefficient and from Eq. 7.4, may be written as

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad ...7.7$$

7.4.3 Stiffness Coefficients—Combined Rectilinear and Angular System

Let us take another example of both combined linear and angular system as shown in Fig. 7.5(a).

Let the two coordinates for the two-degree-freedom system be

$$q_1 = x \text{ (rectilinear coordinate) and } \quad q_2 = \theta \text{(angular coordinate)}$$

Now let us give a unit rectilinear displacement (x) to the first coordinate 'q_1' to the mass 'm' holding the other coordinate, i.e. the pulley, to its zero position as shown in Fig. 7.5(b). Let 'F' and 'M' be the respective force and moment to hold the system in the above position and respective FBD is as shown in Fig. 7.5(c).

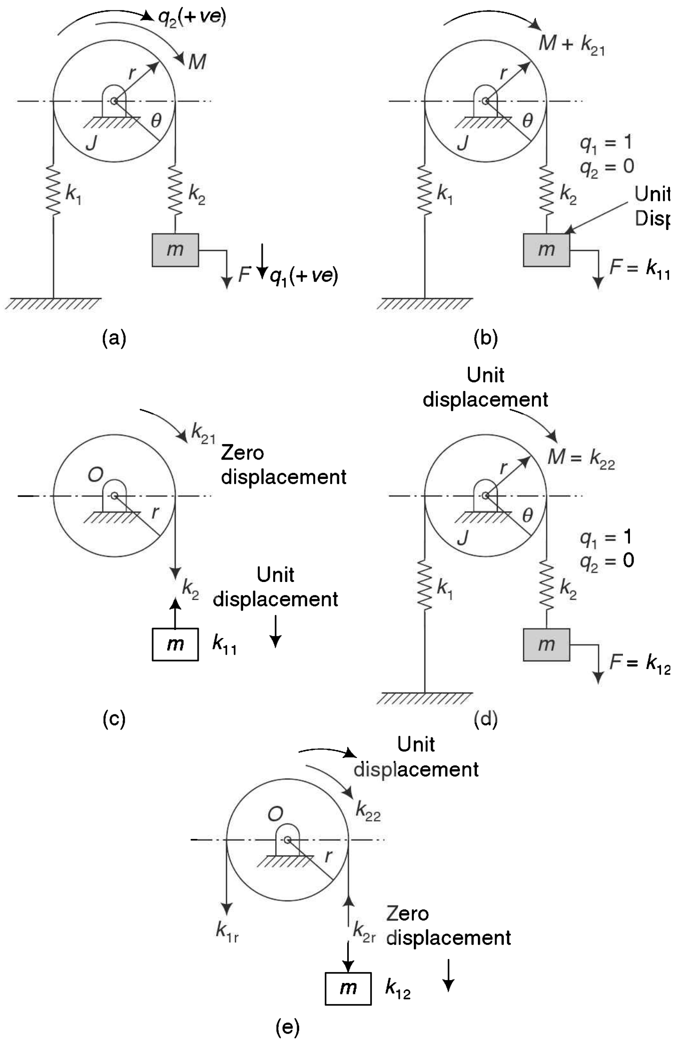


Fig. 7.5 *Stiffness coefficients—combined rectilinear and angular system*

Then as per the statement of stiffness coefficients,

$$F = k_{11} \text{ and } M = k_{21}$$

Since from this case the FBD for the mass and the pulley will give

$$\left.\begin{aligned} k_{11} &= k_2 \\ k_{21} &= -k_2 r \end{aligned}\right\} \qquad \ldots 7.8$$

In a similar way, give a unit angular displacement to the second coordinate 'q_2', i.e. rotation of the pulley 'J' by one radian (unit), holding the other coordinate i.e. the mass M, to its zero displacement position, shown in Fig. 7.5(d). In this system, 'F' and 'M' are the respective force and moments to hold the system in this position.

Then by the definition of stiffness coefficients, we get

$$F = k_{12} \text{ and } M = k_{22}$$

Figure 7.5(e) indicates the FBD for the mass and pulley in this angular displacement giving

$$\left.\begin{aligned} k_{12} &= -k_2 r\} \\ k_{22} &= (k_1 + k_2)\, r^2 \end{aligned}\right\} \qquad \ldots 7.9$$

The stiffness matrix for the system and its definition of stiffness coefficients from Eqs. 7.4 can be written as

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} k_2 & -k_{2r} \\ -k_{2r} & (k_{21} + k_2)\, r^2 \end{bmatrix} \qquad \ldots 7.10$$

The stiffness matrix for the above system can be obtained independently from stiffness coefficients as in equations 7.7 and 7.10, and then the differential equations of motion for the system in the matrix form can be directly written as per the Eqs. 7.1(b), in Section 7.2, i.e. $[M]\,\{\ddot{x}\} + [k]\,\{x\} = 0$ ...(7.2)

Usually, mass matrix is a diagonal matrix.

7·3 TYPES OF COUPLINGS

7.5.1 Generalised Coordinates and Coordinate Coupling

As we know that an n-degree of freedom system requires n-independent coordinates to specify the system completely at any instant. Almost in all cases, these coordinates need to be considered at equilibrium position only. However, we can have any other set of n-independent coordinates to specify the configuration of the system. Hence, any of these sets is known as '**generalised coordinates**'.

As we know that in a two-degree-freedom system explained in Section 6.5, we take the two coordinates as in Fig. 7.6(a) and Fig. 7.6(b) showing the displacement of the two coordinates at any instant and Fig. 7.6(c) with the external forces acting on system.

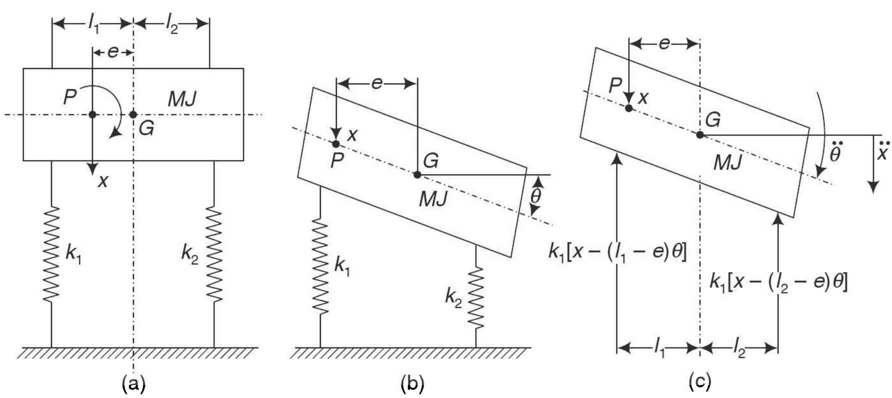


Fig. 7.6 *Generalised coordinates and coordinate coupling*

The displacement of the centre of gravity (CG) of the system is $(x + e\theta)$; then the differential equation of rectilinear motion is

$$M(\ddot{x} + e\ddot{\theta}) = -k_1 [x - (l_1 - e)\theta] - k_2 [x + (l_2 + e)\theta] \qquad \text{...7.11}$$

and the equation for the angular motion is

$$J\ddot{\theta} = k_1 [x - (l_1 - e)\theta] l_1 - k_2 [x + (l_2 + e)\theta] l_2 \qquad \text{...7.12}$$

J is the mass moment of inertia of the system about its centre of gravity (CG). In order to write Eq. 7.12 in terms of 'J_p', the mass moment of inertia about the point 'P', add $Me^2\ddot{\theta}$ to both sides of the equation or multiply Eq. 7.11 by an eccentricity 'e' and add this to Eq. 7.12. We will add $Me^2\ddot{\theta}$ to both sides of Eq. 7.11. We have,

$$J\ddot{\theta} + Me\ddot{x} + Me^2\ddot{\theta} = k_1 [x - (l_1 - e)\theta] l_1 - k_2 [x + (l_2 + e)\theta] l_2 - k_1 [x - (l_1 - e)\theta] e - k_2 [x + (l_2 + e)\theta] e = 0 \qquad \text{...7.13}$$

Now the equations 7.11 and 7.13 can be simplified as the following forms and after substituting $(J + Me^2) = J_p$ in Eq. 7.13, we have

$$\left.\begin{array}{r} M\ddot{x} + Me\ddot{\theta} + (k_1 + k_2)x + [k_2(l_2 + e) - k_1(l_1 - e)\theta] = 0 \\ J_p\ddot{\theta} + Me\ddot{x} + [k_1(l_1 - e)^2 + k_2(l_2 + e)^2]\theta + [k_2(l_2 + e) - k_1(l_1 - e)]x = 0 \end{array}\right] \qquad \text{...7.14}$$

The equations 7.14 are the two general differential equations for linear motion and also an angular motion respectively. This can be analysed in three different ways of generalised coordinates called **only static coupling, no dynamic coupling; only dynamic coupling, no static coupling** and **static and dynamic coupling.** These three different sets of generalised coordinates are explained separately as follows.

7.5.2 Only Static Coupling, No Dynamic Coupling

As we already stated the meaning of static and dynamic coupling, in Section 6.13, now in this section we will discuss about static and dynamic coupling in more detail. Let us consider system coordinates 'x' and 'θ' having static coupling in the above two equations 7.14, containing the terms 'x' and 'θ'. These equations do not having dynamic coupling if the terms $\ddot{x}$ and $\ddot{\theta}$ occur only in respect of the above two equations 7.14. In case of static coupling, first let us take the point 'P' lying on the centre of gravity (CG) of the system, meaning $e = 0$ (no eccentricity). Then the equations 7.14 will reduce as follows:

$$\begin{bmatrix} M\ddot{x} + x(k_1 + k_2) + (k_2 l_2 - k_1 l_1)\theta = 0 \\ J_p\ddot{\theta} + (k_1 1^2{}_1 + k_2 l^2{}_2)\theta + (k_2 l_2 - k_1 l_1)x = 0 \end{bmatrix} \qquad \text{...7.15}$$

This combination of generalised coordinates is as shown in Fig. 7.7(a) and this combination is similar to Section 6.5, and equation 6.74. Hence, the point 'A' coincides with the CG. Then we consider 'J_A' = J. Then equations 7.15 are similar to the equations 6.74, already discussed. The equations 7.15, both of which have, 'x' and 'θ', are called the statically or elastically coupled equations and are called a static or elastic coupling between the coordinates. On the other hand, since $\ddot{x}$ and $\ddot{\theta}$ terms occur in the same equations only; therefore there is no dynamic coupling. Hence the coordinates have static coupling only.

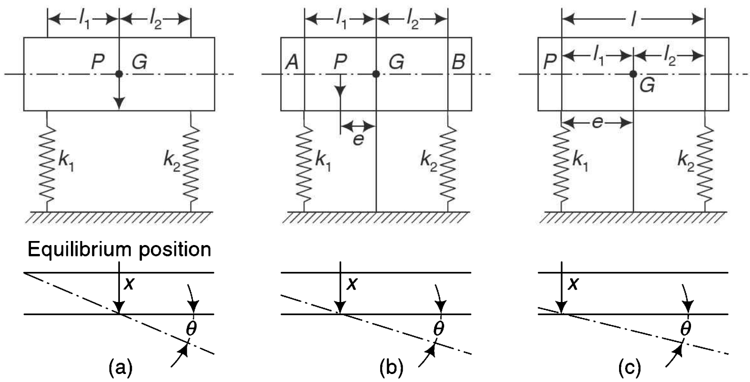


Fig. 7.7 *Coordinate coupling (a) Only static coupling (b) Only dynamic coupling (c) Static and dynamic coupling*

The physical concept of the static coupling is explained by considering Fig. 7.7(a).

In this system, the static coupling is that if one coordinate is given a displacement, the other coordinate also undergoes a small amount of displacement. This can be observed clearly in Fig. 7.7(a). Here in this position, if a displacement 'x' is given to the system at the point 'G', i.e. centre of gravity, the system does not go down horizontally but is tilted which means there is displacement of coordinate 'θ' also. On the other hand, if we give an angular displacement 'θ' to the system, the point 'G' does not stay in its position but undergoes displacement in 'x' direction also. Finally we can see that in equations 7.15, there is no static coupling if the first of the equations 7.15 contains no 'θ' term and the second equation does not have 'x' term, which means $k_1 l_1 = k_2 l_2$.

7.5.3 Only Dynamic Coupling, No Static Coupling

A system of generalised coordinates will have inertia or dynamic coupling if both the equations 7.14 contain the terms $\ddot{x}$ and $\ddot{\theta}$. It will be seen that there is no static coupling if the terms 'x' and 'θ' occur only in the respective equations. Then the coefficient of 'θ' in the first of generalised coordinates equation 7.14 and the coefficient of 'x' in the second equation are similar.

Therefore, if these coefficients are equated to zero, we have the equation

$$k_2 (l_2 + e) - k_1 (l_1 - e) = 0$$

Then these coordinates will have only dynamic coupling and there is no static coupling, and the equation reduces to

$$\left.\begin{aligned} M\ddot{x} + Me\ddot{\theta} + (k_1 + k_2)\, x = 0 \\ J_p\ddot{\theta} + Me\ddot{x} + [k_1 (l_1 - e)^2 + k_2 (l_2 + e)^2]\, \theta = 0 \end{aligned}\right\} \quad \text{...7.16}$$

It is seen that in equations 7.16, there is only dynamic coupling and no static coupling.

And it can be written as $k_1 (l - e) = k_2 (l_2 + e)$, and $e \neq 0$.

This can be seen in Fig. 7.6(b) and from the above equation $k_1AP = k_2BP$.

The point 'P' is selected to satisfy the above expression, and the coordinates as shown in Fig. 7.6(b) having only dynamic coupling and no static coupling.

Similar to the static coupling, in dynamic coupling also the physical concept of dynamic coupling is explained by considering Fig. 7.7(b) as follows. If acceleration is given to any one coordinate, the other coordinate also gets an acceleration. If an acceleration '$\ddot{x}$' is given at the point 'P' referring Fig. 7.7(b), there is an inertia force '$M\ddot{x}$' at the centre of gravity 'G' in opposition direction to '$\ddot{x}$' giving a torque on the system which gives finally an angular acceleration '$\ddot{\theta}$' to the system. On the other hand, this can also be seen in the same way that an angular acceleration '$\ddot{\theta}$' to the system causes a translational acceleration at the point 'P'. Hence, there will be no static coupling in this case; this can also be seen clearly.

7.5.4 Static and Dynamic Coupling

In the previous two cases based on the point 'P' in some position, we consider whether it is dynamic coupling or static coupling. In this case, let us take the point 'P' exactly lying above the spring 'k_1' giving $l_l = e$ as indicated in Fig. 7.7(c). Then substituting for 'e' in Eq. 7.14, we have

$$\left.\begin{aligned} M\ddot{x} + Ml_1\ddot{\theta} + (k_1 + k_2)\,x + k_2l\theta = 0 \\ J_D\ddot{\theta} + Ml_1\ddot{x} + k_2l^2\theta + k_2lx = 0 \end{aligned}\right\} \qquad ...7.17$$

where $l = l_1 + l_2$

Therefore, for the coordinates selected as in Fig. 7.7(c) the corresponding differential equations are obtained in equations 7.17. In equations 7.17, we have both the $\ddot{x}$ and $\ddot{\theta}$ terms, hence there is dynamic coupling between the coordinates. Also both the equations have both the terms 'x' and 'θ' and there is static coupling between the coordinates.

Hence, the system of coordinates chosen has both dynamic and static coupling. This is the general procedure for both dynamic and static coupling.

Let us now take an example of undamped two degree-freedom-system having q_1, q_2 as a generalised coordinates, and the equations of undamped free vibration can be written as

$$\left.\begin{aligned} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + b_{11}\ddot{q}_1 + b_{12}\ddot{q}_2 = 0 \\ a_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + b_{21}\ddot{q}_1 + b_{22}\ddot{q}_2 = 0 \end{aligned}\right\} \qquad ...7.18$$

In the above equations, a_{12} and a_{21} are the dynamic coupling coefficients and b_{12} and b_{21} are the static coupling coefficients. We compare these equations with the equations of 7.14 to determine the coupling coefficient between the 'x' and 'θ' of two coordinates using the principal mode of vibration in equations 7.18 by assuming the solution

$$q_i = Q_1 \sin(\omega t - \phi) \text{ and } q_2 = Q_2 \sin(\omega t - \phi)$$

Substituting these values in equations 7.18 and simplifying by using the following usual procedure,

$$\left.\begin{aligned}
&a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{1n}\ddot{q}_n + b_{11}\ddot{q}_1 + b_{12}\ddot{q}_2 + \ldots\ b_{1n}\ddot{q}_\text{n} = 0\\
&a_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + a_{2n}\ddot{q}_n + b_{21}\ddot{q}_1 + b_{22}\ddot{q}_2 + \ldots\ b_{2n}\ddot{q}_n = 0\\
&-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\\
&a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{nn}\ddot{q}_n + b_{n1}\ddot{q}_1 + b_{n2}\ddot{q}_2 + \ldots\ b_{nn}\ddot{q}_n = 0
\end{aligned}\right\}\quad \ldots.7.19$$

The generalised coordinates may be lengths or angles or a combination of both.

7.4 NATURAL FREQUENCIES AND MODE SHAPE (EIGENVALUES AND EIGENVECTORS)

1. For discussing eigenvalues and eigenvectors, we will take an example of multi-degree-of-freedom system as shown in Fig. 7.8(a) for generalising the method for finding its natural frequencies and mode shapes. The differential equations of motion for multi-degree-of-freedom system as obtained in equations 7.20 will be rearranged as follows:

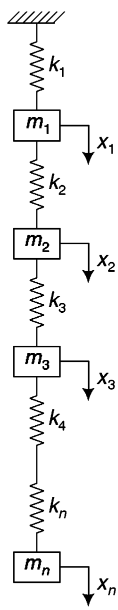


Fig. 7.8 *Natural frequencies and mode shape in multi-degree-freedom systems*

$$\left.\begin{aligned}
&[m_1\ddot{x}_1 + (k_1 + k_2)\,x] - k_2x_2 = 0\\
&k_2x_1 + [m_2\ddot{x}_2 + (k_2 + k_3)\,x_2] - k_3x_3 = 0\\
&k_3x_2 + [m_2\ddot{x}_3 + (k_3 + k_4)\,x_3] - k_4x_4 = 0\\
&\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\\
&-k_nx_{n-1} + (m_n\ddot{x}_2 + k_nx_n) = 0
\end{aligned}\right\}\quad \ldots 7.20$$

Assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\left.\begin{array}{ll} x_1 = X_1 \sin \omega t & x_2 = X_2 \sin \omega t \\ \dot{x}_1 = \omega X_1 \cos \omega t & \dot{x}_2 = \omega X_2 \cos \omega t \\ \ddot{x}_1 = -\omega^2 X_1 \sin \omega t & \ddot{x}_2 = -\omega^2 X_2 \sin \omega t \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_n = X_n \sin \omega t & x_2 = X_2 \sin \omega t \end{array}\right\} \qquad \text{...7.21}$$

Using the values of $x_1, x_2\, x_3 \ldots\ldots x_n$ in Eq. 7.21, and cancelling out the some common term sin ωt we get

$$\left.\begin{array}{l} [(k_1 + k_2) - m_1\omega^2]\, X_1 - k_2X_2 = 0 \\ -k_2X_1 + [(k_2 + k_3) - m_2\omega^2]\, X_2 - k_3X_3 = 0 \\ -k_3X_2 + [(k_3 + k_4) - m_3\omega^2]\, X_3 - k_4X_4 = 0 \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ -k_nX_{n-1} + (k_n + -m_3\omega^2)\, X_n = 0 \end{array}\right\} \qquad \text{...7.22}$$

In equations 7.21, the solution other than $X_1 = X_2 = X_3 = \ldots X_n = 0$ is possible only when the determinant composed of the coefficients of X's vanishes or

$$\begin{bmatrix} [(k_1 + k_2) - m_1\omega^2] & -k_2 & \cdots & 0 & 0 \\ -k_2 & [(k_1 + k_2) - m_1\omega^2] & \cdots & 0 & 0 \\ 0 & -k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -k_n & (k_n - m_n\omega^2) \end{bmatrix} = 0 \qquad \text{...7.23}$$

It is the frequency equation of 'n^{th}' degree in 'ω^2' and this equation will give n values of 'ω^2' corresponding to n natural frequencies also. The mode shapes can be obtained by using Eq. 7.22, one at a time, and the various values of 'ω' as obtained by Eq. 7.23.

2. We also know that in the matrix form, the differential equation of motion for a system or any other system, can be written in the matrix form as follows:

$$[M]\{\ddot{x}\} + [k]\,\{x\} = 0 \qquad \text{...7.24}$$

Then pre-multiplying Eq. 7.24 by $[M]^{-1}$, we have

$$[I]\{\ddot{x}\} + [D]\,\{x\} = 0 \qquad \text{...7.25}$$

Here, $[M]^{-1}[M] = [I]$, a unit matrix ...7.26

And $[M]^{-1}[K] = [D]$, a dynamic matrix ...7.27

For free vibration assuming that the harmonic motion of frequency 'ω'

$$\{x\} = \{X\} \sin \omega t \qquad \text{...7.28}$$

we have $\{\ddot{x}\} = -\omega^2\{x\}$

Let $\lambda = \omega^2$ is the eigenvector.

Then $\{\ddot{x}\} = -\lambda\{x\} = -\lambda\{X\} \sin \omega t$ ...7.29

$\{X\}$ is the column and will giving the respective amplitude of masses.

In Eq. 7.25, the reference of equations 7.28 and 7.29 will give

$$-\lambda[I]\{X\} + [D]\{X\} = 0 \text{ or}$$

$$[[D] - \lambda[I]]\{X\} = 0 \qquad ...7.30$$

The determinant formed from Eq. 7.30 is

$$[[D] - \lambda[I]\,] = 0 \qquad ...7.31$$

The equation 7.31 is the frequency equation where $\lambda = \omega^2$ is the eigenvector n and it will give n values of 'λ_i' = ω_i^2 for n degree freedom system. By substituting 'λ_i' in the matrix equation 7.30, we get the mode shape $\{X\}_i$, treated as an eigenvector, for the i^{th} mode of vibration. Thus, we can understand an n-degree-freedom system having n number of eigenvalues and the corresponding n number of eigenvectors.

This is another method of obtaining eigenvalues and eigenvectors.

3. The eigenvalues and eigenvectors can also be obtained from the adjoint matrix method and the definition of inversion matrix $[B]$ as

$[B]^{-1} = \dfrac{1}{B}\, adj\,[B]$ or for the case under consideration

$$[[D] - \lambda\,[I]]^{-1} = \frac{1}{[[D] - \lambda\,[I]]}\, adj\; [[D] - \lambda[I]] \qquad ...7.32$$

Pre-multiplying both sides of the equation by the terms

$[[D] - \lambda\,[I]1\;[[D] - \lambda\,[I]]$, we have

$$[[D] - \lambda\,[I]\;1\;[I] = [[D] - \lambda\,[I]]\; adj[[D] - \lambda\,[I]] \qquad ...7.33$$

For $\lambda = \lambda_i$, an eigenvalue, the determinants on the LHS of the equation is zero. Then it will be changed as follows:

$$[[D - \lambda_i\,[I]]\; adj\,[[D] - \lambda_i[I]] = [0] \qquad ...7.34$$

This equation is applicable for all values of 'λ_i' and this will give the frequency equation of the system.

For n values of 'λ_i' Eq. 7.34, represents n number of equations, and also equations 7.30, for the i^{th} mode can be written as

$$[[D] - \lambda_i[I]]\{X\}_i = (0) \qquad ...7.35$$

By comparing equations 7.34 and 7.35, it can be easily understood that the adjoint matrix.

$$adj\,[[D] - \lambda_i [I]] \qquad ...7.36$$

The equation must consist of columns, each of the columns having an eigenvector $\{X\}_i$.

4. The advantage of solving the differential equations by converting the system in matrix form as in cases 2 and 3 in Section 7.6 is that a computer program can be developed which could simplify Eq. 7.30 easily and get the directly eigenvalues and eigenvectors for the entire system. Also these matrix methods help to simplify the higher-degree-of-freedom problems with more ease. Standard sub-routines are available in the software of most modern computers to get eigenvalues and eigenvectors of Eq. 7.30.

7.4 ORTHOGONAL PROPERTIES OF THE NORMAL MODES

The normal modes, or the eigenvectors of the system, can be shown to be orthogonal with respect to mass or the stiffness matrices.

In general, the differential equations of motion for any vibrating system can be written in matrix form as follows (Equation 7.24 as we know):

$$[M]\,\{\ddot{x}\} + [k]\,\{x\} = (0) \qquad ...7.37$$

For free vibration, assuming that the harmonic motion of frequency 'ω'

$$\{x\} = \{X\} \sin \omega t$$

$$\{\ddot{x}\} = -\omega^2\{x\}$$

Let $\lambda = \omega^2$ be the eigenvector.

Then $$\{\ddot{x}\} = -\lambda\{x\} = -\lambda\{X\} \sin \omega t \qquad ...7.38$$

where 'λ' is the eigenvalue (= ω^2), $\{X\}$ is the column it will give the respective amplitudes of various masses. Substituting the value of Eq. 7.38 in Eq. 7.37 and simplifying, we get

$$[K]\,\{X\} = \lambda[M]\{X\} \qquad ...7.39$$

For 'r^{th}' mode, Eq. 7.39 becomes as follows:

$$[K]\,\{X\}_r = \lambda_r[M]\{X\}_r \qquad ...7.40$$

Pre-multiplying both sides of Eq. 7.40 by the transpose of 's^{th}' terms mode, we get

$$\{X\}s^1[K]\,\{X\}_r = \lambda_r\,\{X\}s^1\,[M]\{X\}_r \qquad ...7.41$$

Similarly, starting with the equation for 's^{th}' mode and pre-multiplying by the transpose of 'r^{th}' mode, we get

$$\{X\}r^1[K]\,\{X\}_s = \lambda_s\,\{X\}r^1\,[M]\{X\}_s \qquad ...7.42$$

Here, [K] and [M] are the symmetric matrices, having the following relationships applicable.

$$\left.\begin{aligned} &\{X\}s^1[K]\,\{X\}_r = \{X\}r^1\,[K]\{X\}_s \\ \text{and} \quad &\{X\}s^1[M]\,\{X\}_r = \{X\}r^1\,[M]\{X\}_s \end{aligned}\right\} \qquad ...7.43$$

Now subtracting Eq. 7.42 from Eq. 7.41 and using the Eq. 7.43, we get

$$0 = (\lambda_r - \lambda_s)\,\{X\}_r^1\,[M\,]\{X\}_s \qquad ...7.44$$

Suppose if 'λ_r' and 'λ_s' are varying, we must have

$$\{X\}r^1[M]\,\{X\}_s = 0, \quad r \neq s \qquad ...7.45$$

But also as a consequence, we can show that

$$\{X\}r^1[K]\,\{X\}_s = 0, \quad r \neq s \qquad ...7.46$$

The equations 7.45 and 7.46 define the orthogonal properties of the normal modes or eigenvector.

Suppose if $r = s$, equations 7.44 become an identity matrix.

Then

$$\left.\begin{aligned} &\{X\}r^1[M]\,\{X\}_r = M_r \text{ (say)} \\ \text{and} \quad &\{X\}r^1[K]\,\{X\}_r = K_r \text{ (say)} \end{aligned}\right\} \qquad ...7.47$$

where M_r and K_r are denoted as a generalised mass and generalised stiffness respectively. M_r and K_r are actually unit matrices [1 × 1] and the equations 7.47 are usually used to finding out the single elements of these matrices.

In case if [M] is a diagonal matrix, which is usually the case, the orthogonality equations are also written as $\sum_{i=1}^{n} m_i\,(X_i)_r\,(X_i)_s = 0, \quad r \neq s$ ...7.48

and for the first of Eq. 7.47 as

$$\sum_{i=1}^{n} m_i\,[(X_i)_r]^2 = M_r \qquad ...7.49$$

This orthogonality principle is of great value and is very useful in the study of multi-degree-freedom systems. Orthogonality of the normal modes means it defines perpendicularity of modes. The obtained equations like 7.45, 7.46 and 7.48, are based on the mathematical conditions of orthogonality of the normal modes. But also in certain degrees like 2-degree-of-freedom systems, 3-degree-freedom systems, the normal modes can be physically seen to be perpendicular to each other.

7.7.1 Orthogonality Principle

The principle modes or normal modes of vibration for systems having two or more degrees of freedom are orthogonal. This is known as Orthogonality principle.

This is an important property while finding the natural frequencies.

For a two-degree-freedom system, orthogonality principle can be written as

$$m_1A_1A_2 + m_2B_1B_2 = 0$$

where in the above equation A_1, B_1 and A_2, B_2 are the amplitudes of first and second modes of vibration. For a 3-degree or multi-degree-freedom system, the orthogonality principle can be written as

$$m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$$

$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

where in the above equations A_1, B_1, C_1, A_2, B_2, C_2 and A_3, B_3, C_3 are the amplitudes of first, second and third modes of vibration respectively.

7.8 MATRIX ITERATION METHOD

This is an iterative procedure to determine the principal modes of the system and its natural frequencies. Displacements of the masses are estimated from which the matrix equations of the system are written. The influence coefficients of the systems are substituted into the matrix equation, which is then expanded. Normalisation of the displacement and expansion of matrix is repeated.

The process is continued until the first mode repeats itself to any desired degree of accuracy. For next higher modes and the natural frequencies, the orthogonality principle is used to obtain a new matrix equation that is free from any lower modes. Then the procedure is repeated. Matrix methods in the analysis of problems in structure vibrations, fluid dynamics and design are becoming more popular with the advent of high-speed and large-memory digital computers. Here multiplication, inversion and iteration of large-size matrices can be done very easily. The method needs the following for undamped, free vibration orthogonality conditions of modes, influence coefficients for deflections, sweeping matrix to eliminate a certain mode, iteration for eigenvalues and eigenvectors, etc.

7.9 MODEL ANALYSIS

7.9.1 Undamped Free Vibration

In Section 7.5.1, for generalised coordinates and coordinate coupling, it can be seen that static and dynamic coupling is based on the choice of coordinates. Also we can see that there exists a set of coordinates, called the principal coordinates, which express the equations of motion in an uncoupled way. In such coordinates, each has equation to be solved individually to each others. As we already know in Section 7.5.1, in a two-degree-freedom system it is possible to uncouple the equations of motion of n-degree-freedom system provided the eigenvectors of the system. A model matrix $[U]$ is referred to a square matrix. Here, each column represents an eigenvector. Therefore, in case of an n-degree-freedom system,

$$[U] = \begin{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \\ \vdots \\ X_s \\ \vdots \\ \lambda_1 \end{Bmatrix}_1 & \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \\ \vdots \\ X_s \\ \vdots \\ X_n \end{Bmatrix}_2 & \begin{Bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix} & \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \\ \vdots \\ X_s \\ \vdots \\ X_n \end{Bmatrix}_r & \begin{Bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix} & \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \\ \vdots \\ X_s \\ \vdots \\ X_n \end{Bmatrix}_s & \begin{Bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix} & \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \\ \vdots \\ X_s \\ \vdots \\ X_n \end{Bmatrix}_n \end{bmatrix} \qquad ...7.50$$

The transpose of the matrix equations 7.50 can be written as

$$[U]^1 = \begin{bmatrix} [X_1 & X_2 & \ldots & X_r & \ldots & X_s & \ldots & X_n]_1 \\ [X_1 & X_2 & \ldots & X_r & \ldots & X_s & \ldots & X_n]_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ [X_1 & X_2 & \ldots & X_r & \ldots & X_s & \ldots & X_n]_r \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ [X_1 & X_2 & \ldots & X_r & \ldots & X_s & \ldots & X_n]_s \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ [X_1 & X_2 & \ldots & X_r & \ldots & X_s & \ldots & X_n]_n \end{bmatrix} \qquad ...\ 7.51$$

For undamped n-degree freedom system, the differential equation of motion is written as follows and also we already know,

$$[M]\{\ddot{x}\} + [K\}\ \{x\} = 0 \qquad ...7.52$$

In case Eq. 7.52 is to decouple the equations, let us use the linear transformation,

i.e. $$\{x\} = [U]\ \{y\} \qquad ...7.53$$

Here, $\{y\}$ is the principal coordinate and it can be determined by pre-multiplying Eq. 7.53, by $[U]^{-1}$, then we get

$$[U]^{-1}\{x\} = [U]^{-1}\ [U]\ (y),\text{ or } (y) = [U]^{-1}(x) \qquad ...7.54$$

Substituting the values of Eq. 7. In Eq. 7.53, we get

$$[M]\ [U]\ \{\ddot{y}\} + [K]\ [U]\ \{y\} = 0 \qquad ...7.55$$

Pre-multiplying Eq. 7.55 by $[U]^1$ we get

$$[U]^1\ [M]\ [U]\ \{\ddot{y}\} + [U]\ [K]\ [U]\ \{y\} = 0 \qquad ...7.56$$

The terms $[U]^1[M]\ [U]$ and $[U]^1[K]\ [U]$ in Eq. 7.56 are each a diagonal matrix. Therefore, the off-diagonal terms which include r^{th} row, where r^{th} is the eigenvector and the s^{th} column where s^{th} is the eigenvector, express the orthogonality relationships treated as a zero;

i.e.
$$\left.\begin{aligned} \{X\}_r^1\,[M]\{X\}_s = 0 \qquad r \neq s \\ \{X\}_r^1 = [K]\{X\}_s = 0 \qquad r \neq s \end{aligned}\right\} \qquad ...7.57$$

The equations 7.57 are similar to the equations of 7.45 and 7.46, in Section 7.7.

The diagonal terms in $[U]$ $[M]$ $[U]$ and $[U]^1$ $[K]$ $[U]$ include r^{th} row, where r^{th} is the eigenvector and the r^{th} column where r^{th} is the eigenvector, would give the generalised mass and the generalised stiffness terms

i.e.
$$\left.\begin{aligned} \{X\}_r^1\,[M]\{X\}_r = M_r \\ \{X\}_r^1\,[K]\{X\}_r = K_r \end{aligned}\right\} \qquad ...7.58$$

The equations 7.58 are similar to the equations of 7.47.

Thus Eq. 7.56 and therefore Eq. of 7.55 combinations are given as follows:

$$\begin{bmatrix} M_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & M_n \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \\ \vdots \\ y_n \end{Bmatrix} + \begin{bmatrix} K_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & K_r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & K_n \end{bmatrix}$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \\ \cdots \\ y_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \end{Bmatrix}$$

Or
$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} \{\ddot{y}\} + \begin{bmatrix} \ddots & & \\ & K_r & \\ & & \ddots \end{bmatrix} \{y\} = \{0\} \qquad ...7.59$$

Further, it can be easily written as follows

$$K_r = \lambda_r.\, M_r \qquad ...7.60$$

By rewriting Eq. 7.40, we can prove the relationship of Eq. 7.60.

$$[K]\{X\}_r = \lambda_r\,[M]\,\{X\}_r \qquad ...7.61$$

Pre-multiply Eq. 7.61 by transpose of r^{th} mode to get

$$\{X\}_r^1\,[K]\,\{X\}r = \lambda_r\{rX\}_r^1\,[M]\,\{X\}_r \qquad ...7.62$$

Equation 7.62 from Eq. 7.47 gives $K_r = \lambda_r\, M$;

this equation is similar to Eq. 7.60 that we derived.

Substituting the values of Eq. 7.60 in Eq. 7.59, we have

$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} \{\ddot{y}\} + \begin{bmatrix} \ddots & & \\ & \lambda_r M_r & \\ & & \ddots \end{bmatrix} \{y\} = \{0\} \qquad ...7.63$$

or

$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} \{\ddot{y}\} + \begin{bmatrix} \ddots & & \\ & \omega_r^2 M_r & \\ & & \ddots \end{bmatrix} \{y\} = \{0\} \qquad ...7.64$$

Here, $\lambda_r = \omega_r^2$ equals the eigenvalue for the r^{th} mode.

Thus, the equations 7.64 as detailed below are the n-uncoupled differential equations of motion for n-degree-freedom system in terms of the principal coordinates 'y'

$$\ddot{y}_r + \omega_r y_r = 0 \text{ (where } r = 1, 2, 3, \ldots n) \qquad ...7.65$$

The solution of Eq. 7.65 is,

$$y = A_r \cos \omega_r t + B_r \sin \omega_r t \text{ (where } r = 1, 2,3, \ldots n) \qquad ...7.66$$

Eq. 7.66 can also be written as follows:

$$\{y\} = \{A \cos \omega t + B \sin \omega t\} \qquad ...7.67$$

From equations 7.53 and 7.66, we get

$$\begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix} = [U] \begin{Bmatrix} A_1 \cos \omega_{1t} + B_1 \cos \omega_{1t} \\ A_2 \cos \omega_{2t} + B_2 \cos \omega_{2t} \\ \cdots \cdots \cdots \cdots \cdots \cdots \\ A_n \cos \omega_{nt} + B_n \cos \omega_{nt} \end{Bmatrix} \qquad ...7.68$$

The matrix equations 7.68 would give the vibratory response of undamped free vibrations. A_r and B_r (where $r = 1, 2, 3 \ldots n$) can be obtained from the initial boundary conditions.

Equation 7.53 can also be seen to be expanded and put in the form of matrix as follows and it will be more suitable also.

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{Bmatrix}_1 y_1 + \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{Bmatrix}_2 y_2 + \ldots \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{Bmatrix}_r y_r + \ldots \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{Bmatrix}_n y_n \qquad ...7.69$$

where $y_1, y_2, y_3, \ldots\ldots\ldots y_n$ as in Eq. 7.66.

Equation 7.69 can be written in short form as follows.

$$\{x\} = \{X\}_1 y_1 + \{X\}_2 y_2 + \{X\}_3 y_3 + \ldots \{X\}_r y_r \ldots \{X\}_n y_n \qquad ...7.70$$

This is a detailed discussion in undamped free vibration and model analysis.

7.9.2 Damped Free Vibration

In Section 7.9.1, Eq. 7.52, for an n-degree-freedom undamped system will be given by $[M]\{\ddot{x}\} + [K]\{x\} = 0$.

Let the differential equation of motion, for an n-degree-freedom damped system be given by the equation

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \qquad \text{...7.71}$$

Here, $[C]$ is the damping matrix and is given as follows.

$$[C] = \begin{bmatrix} c_{11} & c_{12} & \cdots & \cdots & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \cdots & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & \cdots & \cdots & c_{nn} \end{bmatrix} \qquad \text{...7.72}$$

Apart from the static and/or dynamic coupling amongst the generalised coordinates $x_1, x_2, x_3, \ldots.. x_n$, there now exists a damping coupling also. Differential equations can get uncoupled with regard to damping, if the damping matrix has only diagonal terms.

That means if $c_d = 0$ and $i \neq j$, then they decouple Eq. 7.71. Now let us assume the linear transformation as

$$\{x\} = [U]\{y\} \qquad \text{...7.73}$$

As earlier in case of undamped case, $\{y\}$ is the column of principal coordinates.

Now substituting the values of Eq. 7.73 in Eq. 7.71, and then pre-multiplying by $[U]$, we get the following equation.

$$[U]^1 [M][U]\{\ddot{y}\} + [U]^1 [C][U]\{\dot{y}\} + [U]^1 [K][U]\{y\} = 0 \qquad \text{...7.74}$$

As we know that in equations 7.56, 7.59 and 7.64, the first and the last terms of Eq. 7.74 reduces to the diagonal matrices.

$$\left.\begin{aligned} [U]'[M][U] &= \begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} \\ [U]'[M][U] &= \begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \omega r^2 M_r & \\ & & \ddots \end{bmatrix} \end{aligned}\right\} \qquad \text{...7.75}$$

In the above diagonal matrices, usually the terms $[U]^1[C][U]$ do not reduce, unless when we use the concept of proportional damping, i.e. $[C]$ being proportional to $[M]$ or to the $[K]$ or to a linear combination of both.

Let $$[C] = \alpha[M] + \beta[K] \quad ...7.76$$

where α and β are constant.

Then $[U]^1 [C [U] = \alpha [U] [M] [U] + \beta [U]^1 [K][U]$

$$= \begin{bmatrix} \ddots & & \\ & \alpha M_r & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \beta \alpha r^2 M_r & \\ & & \ddots \end{bmatrix} \quad \text{... from Eq. 7.75}$$

or $$[U]'[C][U] = \begin{bmatrix} \ddots & & \\ & (\alpha + \beta + \omega_r^2) M_r & \\ & & \ddots \end{bmatrix} \quad ...7.77$$

By expressing $(\alpha + \beta\,\omega_r^2)$ treating as a modal damping 'ξ_r' by using the relation

$$(\alpha + \beta\,\omega_r^2) = 2\,\xi_r a_r \quad ...7.78$$

and also Eq. 7.77 will be written as

$$[U]'[C][U] = \begin{bmatrix} \ddots & & \\ & 2\,r\omega_r M_r & \\ & & \ddots \end{bmatrix} \quad ...7.79$$

Finally the equations 7.74, 7.75 and 7.79 will be given as follows:

$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} \{\ddot{y}\} + \begin{bmatrix} \ddots & & \\ & 2\,\xi_r \omega_r M_r & \\ & & \ddots \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} \ddots & & \\ & \omega_r^2 M_r & \\ & & \ddots \end{bmatrix} = \{y\}\ \{0\} \quad ...7.80$$

The equations 7.80 are n-uncoupled differential equations in the form of principal coordinates given below:

$$\ddot{y}_r + (2\xi\omega_r)\,\dot{y}_r + \omega_r^2\,y_r = 0,\ (r = 1,2,3.4\ n) \quad ...7.81$$

ω_r is the undamped natural frequency in r^{th} mode and ζ_r is the model damping ratio in 'r^{th}' mode.

The solution of Eq. 7.81 is same as the solution of damped free vibration for a single degree-of-freedom system as already discussed and obtained in case of under damped system as follows:

$$y_r = e^{-\xi_r \omega_r t}\left[A_r \cos\sqrt{1-\xi_r^2}\ \omega_r t + B_r \sin\sqrt{1-\xi_r^2}\ \omega_r t\right] (r = 1,2,3,\ldots n) \quad ...7.82$$

Therefore, the complete solution is given by Eq. 7.83

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = [U] \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \quad ...7.83$$

Here, $y_1, y_2. y_3. \ldots y_n$ are as in Eq., 7.82, A_r, B_r $(r = 1,2,3,\ldots n)$ can be obtained from initial conditions.

Finally the solution can also be written as in Eq. 7.69 as below:

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_1 y_1 + \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_2 y_2 + \ldots \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_n y_n \qquad \ldots 7.84$$

7.9.3 Forced Vibrations

As we know that the differential equation of motion of a damped forced vibration for a multi-degree or n-degree freedom system can be written as

$$[M]\,\{\ddot{x}\} + [C]\,\{\dot{x}\} + [K]\,\{X\} = \{F\} \qquad \ldots 7.85$$

where

$$(F) = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{Bmatrix} \{F_1(t)\} \qquad \ldots 7.86$$

is the column of forces/torques pertaining to the coordinates $x_1, x_2, x_3, \ldots x_n$.

Then decouple Eq. 7.85, take the linear transformation as before and then,

$$\{x\} = [U]\,\{y\} \qquad \ldots 7.87$$

Now substituting Eq. 7.87 in Eq. 7.85 and then pre-multiplying by $[U]^1$, we have

$$[U]^1\,[M]\,[U]\{\ddot{y}\} + [U]^1[C]\,[U]\,\{\dot{y}\} + [U]^1\,[K]\,[U]\{Y\} = [U]^1\,\{F\} \qquad \ldots 7.88$$

Using the principals of proportional damping as in Eq. 7.76 then the Eq. 7.88 becomes

$$\begin{vmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{vmatrix} \{\ddot{y}\} + \begin{vmatrix} \ddots & & \\ & 2\,\xi_r \omega_r M_r & \\ & & \ddots \end{vmatrix} \{\dot{y}\} + \begin{vmatrix} \ddots & & \\ & \omega_r^2 M_r & \\ & & \ddots \end{vmatrix} \{y\} = [U]\,\{F\} \qquad \ldots 7.89$$

Equation. 7.89 is similar to Eq. 7.80. Therefore, the right-hand side of Eq. 7.89 can be written as follows with the reference of equations 7.51 and 7.86.

$$[U]^1\{F\} = \begin{vmatrix} X_{11} & X_{21} & \cdots & X_{r1} & \cdots & X_{n1} \\ X_{12} & X_{22} & \cdots & X_{r2} & \cdots & X_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1r} & X_{2r} & \cdots & X_{rr} & \cdots & X_{nr} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1n} & X_{2n} & \cdots & X_{rn} & \cdots & X_{nn} \end{vmatrix} \begin{Bmatrix} F_{1r}(t) \\ F_2(t) \\ \vdots \\ F_r(t) \\ \vdots \\ F_n(t) \end{Bmatrix} = \begin{Bmatrix} G_1 \\ G_2 \\ \cdots \\ G_r \\ \cdots \\ G_n \end{Bmatrix} \text{(say)} \qquad \ldots 7.90$$

Where $G_r = X_{1r}\,F_1(t) + X_{2r}\,F_2(t) + X_{3r}\,F_3(t) + \ldots X_{rr}\,F_r(t) + \ldots X_{nr}\,F_n(t)$ or

$$G_r = \sum_{i=1}^{n} X_{ir} F_i(t) \qquad \ldots 7.91$$

Note: In matrix Eq. 7.90, each row represents a mode shape with the second subscript giving the mode number.

In Eq. 7.89 and the combination of Eq. 7.90 becomes,

$$\left.\ddot{y}_r + 2\xi_r\omega_r\dot{y}_r + \omega_r^2 y = \frac{\sum_{i=1}^{n} X_{ir}F_i(t)}{M_r = E_r(\text{say})}\right\} \ldots(r = 1, 2, 3,\ldots n) \qquad \ldots 7.92$$

The above Eq. 7.92 gives the n-uncoupled equations of vibrations, where 'E_r' is referred to as excitation function for the r^{th} mode.

By the help of Eq. 7.92, the solution 'y_r' can be easily obtained for each equation is an uncoupled equation for the particular mode and also by using Eq. 7.87 the final solution $\{x\}$ can be got easily.

7.10 TORSIONAL VIBRATION OF MULTI-ROTOR SYSTEMS

Torsional vibration of multi-rotor systems are most commonly encountered in almost all machinery. The common examples are the internal combustion engines.

Let us consider 'n' rotor systems and these 'n' rotor's are connected by $(n - 1)$ shafts. Let $k_{t1}, k_{t2}, k_{t3}\ldots k_{tn}$ be the torsional stiffness of shafts respectively. $J_1, J_2, J_3 \ldots J_n$, be the moment of inertia of the rotors $1, 2, 3, \ldots n$, respectively. At any instant we assume that $\theta_1, \theta_2, \theta_3,\ldots \theta_n$ are the angular displacements given to the various rotors from their equilibrium position respectively, when the system is having torsional vibrations. The twist of the respective shafts at the instants are given as $(\theta_1 - \theta_2)$, $(\theta_2 - \theta_3)$, $(\theta_3 - \theta_4),\ldots(\theta_{n-1} - \theta_n)$ as shown in Fig. 7.9, assuming that $\theta_1 > \theta_2 > \theta_3 > \theta_4$.

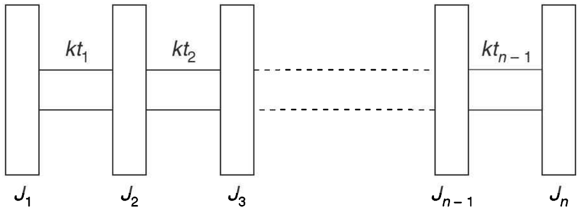


Fig. 7.9 *Torsional vibration of multi-rotor systems*

After displacement given to the various rotors, apply Newton's second law of motion and write down the differential equations of motion for each rotor as follows:

Now applying Newton's second law of motion $J\ddot{\theta} = -\Sigma T$, the equations of motion are

$$\left.\begin{aligned}
J_1\ddot{\theta}_1 &= -k_{t1}(\theta_1 - \theta_2)\\
J_2\ddot{\theta}_2 &= k_{t1}(\theta_1 - \theta_2) - k_{t2}(\theta_2 - \theta_3)\\
J_3\ddot{\theta}_3 &= k_{t2}(\theta_2 - \theta_3) - k_{t3}(\theta_3 - \theta_4)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
J_{n-1}\ddot{\theta}_{n-1} &= k_{tn-2}(\theta_{n-2} - \theta_{n-1}) - k_{t1}(\theta_{n-1} - \theta_n)\\
J_n\ddot{\theta}_n &= k_{tn-1}(\theta_{n-1} - \theta_n)
\end{aligned}\right\} \qquad \ldots 7.93$$

From the above equations 7.50, when we are going to solve there will be two cases: free vibrations and forced vibrations.

Case (i) Free vibrations

Adding of all the above equations 7.93, we get

$$\sum_{i=1}^{n} J_i\ddot{\theta}_i = 0 \text{ (for free vibrations)} \qquad \ldots 7.94$$

In Eq. 7.94, we can see that the sum of the inertia torques on all the rotors must be equal to zero since there is no external exciting torque on the system.

Solutions of equations 7.93

Assuming that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies, let one of these components be

$$\left.\begin{array}{l} \theta_1 = A_1 \sin \omega t,\ \ddot{\theta}_1 = -\omega^2 A_1 \sin \omega t \\ \theta_2 = A_2 \sin \omega t,\ \ddot{\theta}_i = -\omega^2 A_2 \sin \omega t \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \theta_n = A_n \sin \omega t,\ \ddot{\theta}_n = -\omega^2 A_n \sin \omega t \end{array}\right\} \qquad \ldots 7.95$$

Substituting the values of equations 7.95 in equations 7.93 and simplifying, we get

$$\left.\begin{array}{l} J_1\omega^2 A_1 - k_{t1}(A_1 - A_2) = 0 \\ J_2\omega^2 A_2 + k_{t1}(A_1 - A_2) - k_{t_2}(A_2 - A_3) = 0 \\ J_3\omega^2 A_3 + k_{t_2}(A_2 - A_3) - k_{t3}(A_3 - A_4) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ J_n\omega^2 A_n + k_{tn-1}(A_{n-1} - A_n) = 0 \end{array}\right\} \qquad \ldots 7.96$$

On rearranging the equations 7.96, we get

$$\left.\begin{array}{l} (J_1\omega^2 - k_{t1}) A_1 + k_{t1}A_2 = 0 \\ k_{t1}A_1 + (J_2\omega^2 - k_{t1} - k_{t2}) A_2 + k_{t2}A_3 = 0 \\ k_{t2}A_2 + (J_3\omega^2 - k_{t2} - k_{t2}) A_3 + k_{t2}A_4 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ k_{tn-1}A_{n-1} + (J_n\omega^2 - k_{tn-1}) \end{array}\right\} \qquad \ldots 7.97$$

This is a homogeneous set of equations *in* $A_1, A_2, A_3, A_4, \ldots A_n$, and this can have a solution only if the determinant formed with their coefficient vanishes.

Then eliminating $A_1, A_2, A_3, A_4, \ldots A_n$, from the above '$n$' homogeneous set of equations in $A_1, A_2, A_3, A_4, \ldots A_n$, and then the resulting 'n^{th}' degree equation in 'ω^2' would then give 'n' natural frequencies of the system. This is the frequency equation. Now again taking equations 7.93 and rearranging, we get equations as follows:

$$\left.\begin{array}{l} J_1\ddot{\theta}_1 + (k_{t1})\,\theta_1 - (k_{t2})\,\theta_2 = 0 \\ J_2\ddot{\theta}_2 - (k_{t1})\,\theta_1 + (k_{t1} + k_{t2})\,\theta_2 - (k_{t2})\,\theta_3 = 0 \\ J_n\ddot{\theta}_n - (k_{tn-1})\,\theta_{n-1} + (k_{tn-1})\,\theta_n = 0 \end{array}\right\} \qquad \ldots 7.98$$

From the equations 7.98, the determinant form of the problem can be derived.

Then the matrix form can be written as follows.

$$[M] = \begin{vmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & J_n \end{vmatrix} \text{ and } [K] = \begin{vmatrix} k_{t1} & -k_{t1} & 0 & \dots & 0 & 0 \\ -k_{t1} & k_{t1}+k_{t2} & -k_{t2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & k_{tn-1} & k_{tn-1} \end{vmatrix}$$

From Eq. 7.96, we get the dynamic matrix $[[D] - \lambda\,[I]^1]$, Where $[D] = [M]^{-1}[K]$. After substituting the matrix values, we get the frequency equation directly.

Case (ii) Forced vibrations If there is external excitation torque ($T \sin \omega t$) acting on the system at different positions along the system, then in such conditions we must have,

$$\sum_{i=1}^{n} J_i \theta_i = T_{ext} \text{ (for forced vibrations)} \qquad \dots 7.99$$

Here 'T_{ext}' is the sum of all external excitation torques acting on the system.

EXAMPLE 7.1

Determine the natural frequencies of the three-degree-freedom spring-mass (linear) system by using Newton's method as shown in Fig. p-7.1.

Solution Now at any instant give vertical displacement 'x_1'to the mass '$4m$', 'x_2' to the mass '$2m$'and 'x_3' to the mass 'm' as shown in Fig. p-7.1(a). The FBD is as shown in Fig. p-7.1(b) assuming that $x_1 > x_2 > x_3$.

Then the two lower springs are in compression and the top spring is in tension for the direction of x_1 as shown in Fig. p-7.1(b). Then the various spring forces acting are as shown in FBD of Fig. p-7.1(b).

Now applying Newton's second law of motion, $\Sigma F = m\ddot{x}$, the equations of motion are

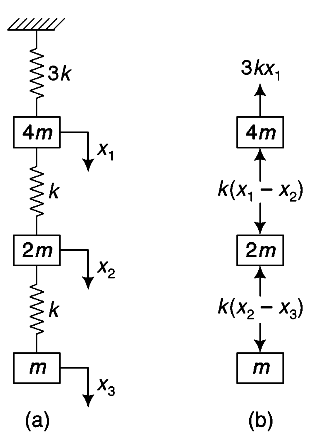


Fig. p-7.1 *Multi-degree linear spring-mass system*

$$4m\ddot{x}_1 = -3kx_1 - k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 = k(x_1 - x_2) - k(x_2 - x_3) = 0,\ m\ddot{x}_3 = k(x_2 - x_3) = 0$$

$$4m\ddot{x}_1 + 3kx_1 + k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Rearranging the above equations,

$$\left.\begin{aligned} 4m\ddot{x}_1 + 4kx_1 - kx_2 &= 0 \\ 2m\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 &= 0 \\ m\ddot{x}_3 + kx_3 - kx_2 &= 0 \end{aligned}\right\} \quad \text{...7.100}$$

This is the differential equation of motion of the masses 'm_1', 'm_2'and 'm_3'.

For solution of equations 7.100, we assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = X_1 \sin \omega t,\ \ddot{x}_1 = -\omega^2 X_1 \sin \omega t$$

$$x_2 = X_2 \sin \omega t,\ \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$$

$$x_3 = X_3 \sin \omega t,\ \ddot{x}_3 = -\omega^2 X_3 \sin \omega t$$

Substituting these values in equations 7.100

$\because$ $\sin \omega t \neq 0$

$$\left.\begin{aligned} (4k - 4m\omega^2) X_1 - kX_2 &= 0 \\ (2k - 2m\omega^2) X_2 - kX_1 - kX_3 &= 0 \\ (k - m\omega^2) X_3 - kX_2 &= 0 \end{aligned}\right\} \quad \text{...7.101}$$

To find the natural frequency equation, the determinant of the coefficient of x_1, x_2, and x_3 must be equated to zero.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 4(k - m\omega^2) & -k & 0 \\ -k & 2(k - m\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0$$

Expand the determinant to get the frequency equations:

$$4(k - m\omega^2)\lfloor 2(k - m\omega^2)(k - m\omega^2) - k^2 \rfloor + k\lfloor -k(k - m\omega^2) - 0 \rfloor + 0 = 0$$

$$(k - m\omega^2)\{\lfloor 8(k - m\omega^2)^2 - 4k^2 \rfloor - k^2\} = 0$$

$$(k - m\omega^2)\lfloor 8k^2 + 8m^2\omega^4 - 16\,km\omega^2 - 5k^2 \rfloor = 0$$

$$(k - m\omega^2)\lfloor 8m^2\omega^4 - 16km\omega^2 - 3k^2 \rfloor = 0$$

$$(k - m\omega^2) = 0,\ 8m^2\omega^4 - 16km\omega^2 - 3k^2 = 0$$

$$k = m\omega^2 \quad \omega^2 = \frac{k}{m} \quad \omega = \sqrt{\frac{k}{m}}\ \text{rad/s}$$

$$\omega_{a,b}^2 = \frac{16km \pm \sqrt{(16km)^2 - 4(8m^2)(3k^2)}}{2 \times 8m^2} = \frac{16km \pm \sqrt{256k^2m^2 - 96k^2m^2}}{16m^2}$$

$$\omega_{a,b}^2 = \frac{16km \pm \sqrt{256k^2m^2 - 96k^2m^2}}{16m^2} = \frac{16km \pm 12.65km}{16m^2}, = \frac{k}{m} \pm \frac{12.65}{16}\frac{k}{m}$$

$$\omega_a^2 = 0.2094\,\frac{k}{m} \qquad \omega_b^2 = 1.79\,\frac{k}{m}$$

$$\omega_a = 0.4576\sqrt{\frac{k}{m}}\text{ rad/s} \qquad \omega_b = 1.338\sqrt{\frac{k}{m}}\text{ rad/s}$$

Hence, the natural frequencies are

$$\omega_{n1} = 0.46\sqrt{\frac{k}{m}}\text{ rad/s},\ \omega_{n2} = \sqrt{\frac{k}{m}}\text{ rad/s},\ \omega_{n3} = 1.34\sqrt{\frac{k}{m}}\text{ rad/s.}$$

EXAMPLE 7.2

Determine the natural frequencies of the three-degree-freedom spring-mass (linear) system by using Newton's method as shown in Fig. p-7.2(a).

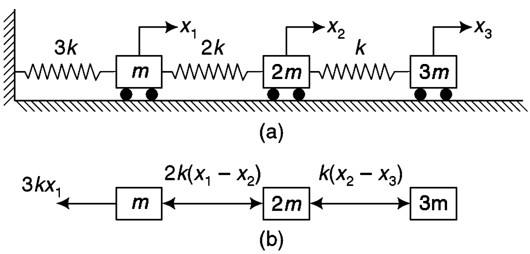


Fig. p-7.2 *Multi-degree linear spring-mass system*

Solution Now at any instant give linear displacement 'x_1' to the mass 'm', 'x_2' to the mass '$2m$'and 'x_3' to the mass '$3m$' as shown in Fig. p-7.2(a). The FBD is as shown in Fig. p-7.2(b) assuming that $x_1 > x_2 > x_3$.

Then the other two springs ($2k$ and k) are in compression and the top spring is in tension for the direction of x_1 as shown in Fig. p-7.2(b).Then the various spring forces acting are as shown in FBD of Fig. p-7.2(b).

Now applying Newton's second law of motion, $\Sigma F = m\ddot{x}$, the equations of motion are

$$m\ddot{x}_1 + 3kx_1 + 2k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 + 2k(x_2 - x_1) + k(x_2 - x_3) = 0 \; 3m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Rearranging the above equations,

$$\left.\begin{aligned} m\ddot{x}_1 + 5kx_1 - 2kx_2 &= 0 \\ 2m\ddot{x}_2 + 3kx_2 - 2kx_1 - kx_3 &= 0 \\ 3m\ddot{x}_3 + kx_3 - kx_2 &= 0 \end{aligned}\right\} \qquad \text{...7.102}$$

This is the differential equation of motion of the masses 'm_1', 'm_2'and 'm_3'.

For solutions of equations 7.102, we assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies.

Let one of these components be,

$$x_1 = X_1 \sin \omega t, \quad \ddot{x}_1 = -\omega^2 X_1 \sin \omega t$$
$$x_2 = X_2 \sin \omega t, \quad \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$$
$$x_3 = X_3 \sin \omega t, \quad \ddot{x}_3 = -\omega^2 X_3 \sin \omega t$$

Substituting these values in equations 7.102

$$-m\omega^2 X_1 + 5kX_1 - 2kX_2 = 0 \qquad \because \sin \omega t \neq 0$$
$$-2m\omega^2 X_2 + 3kX_2 - 2kX_1 - kX_3 = 0$$
$$-3m\omega^2 X_3 + kX_3 - kX_2 = 0$$

$$\left.\begin{aligned} (5k - m\omega^2) X_1 - 2kX_2 &= 0 \\ (3k - 2m\omega^2) X_2 - 2kX_1 - kX_3 &= 0 \\ (k - 3m\omega^2) X_3 - kX_2 &= 0 \end{aligned}\right\} \quad \text{...7.103}$$

To find the natural frequency equation, the determinant of the coefficients of x_1, x_2 and x_3 must be equated to zero.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ (5k - m\omega^2) & -2k & 0 \\ -2k & (3k - 2m\omega^2) & -k \\ 0 & -k & (k - 3m\omega^2) \end{vmatrix} = 0$$

Expand the determinant to get the frequency equation

$$(5k - m\omega^2)\lfloor(3k - 2\,m\omega^2)\,(k - 3m\omega^2) - k^2\rfloor + 2k\lfloor -2k\,(k - 3m\omega^2)\rfloor = 0$$
$$(5k - m\omega^2)\,[2k^2 - 9\,mk\omega^2 - 2mk\omega^2 + 6m^2\omega^4] + 2k[-2k^2 + 6mk\omega^2] = 0$$
$$\omega^6 - 6.83\,\frac{k}{m}\,\omega^4 + 7.5\,\frac{k^2}{m^2}\,\omega^2 - \frac{k^3}{m^3} = 0$$

By solving the above equation, the natural frequencies are

$$\omega_{n1} = 0.396\sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{n2} = 1.084\sqrt{\frac{k}{m}} \text{ rad/s}, \ \omega_{n3} = 2.35\sqrt{\frac{k}{m}} \text{ rad/s.}$$

EXAMPLE 7.3

Determine the natural frequencies of the three-rotor (semi-definite) system by using Newton's method as shown in Fig. p-7.3.

Solution Now at any instant give angular displacement 'θ_1'to the disc'J_1', 'θ_2' to the disc 'J_2'and 'θ_3' to the disc 'J_3' as shown in Fig. p-7.3(a). The FBD is as shown in Fig. p-7.3(b) assuming that $\theta_1 > \theta_2 > \theta_3$.

Now applying Newton's second law of motion, $J\ddot{\theta} = -\Sigma T$, the equations of motion are

$$J_1\,\ddot{\theta}_1 = -k_t(\theta_1 - \theta_2)$$
$$J_2\ddot{\theta}_2 = k_t\,(\theta_1 - \theta_2) - k_t\,(\theta_2 - \theta_3)$$
$$J_3\ddot{\theta}_3 = -k_t\,(\theta_2 - \theta_3)$$

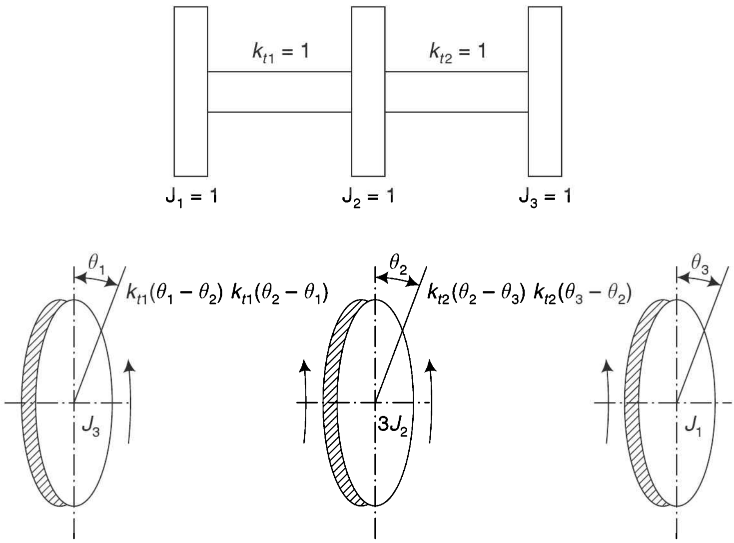


Fig. p-7.3 *Multi-degree torsional system*

$$\left.\begin{aligned} J_1\ddot{\theta}_1 + k_{t1}(\theta_1 - \theta_2) &= 0 \\ J_2\ddot{\theta}_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) &= 0 \\ J_3\ddot{\theta}_3 + k_{t2}(\theta_3 - \theta_2) &= 0 \end{aligned}\right\} \quad \text{...7.104}$$

This is the differential equation of motion of the discs 'J_1', 'J_2'and 'J_3'.

For solutions of equations 7.104, we assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be

$$\theta_1 = a \sin \omega t, \quad \ddot{\theta}_1 = -\omega^2 a \sin \omega t$$
$$\theta_2 = b \sin \omega t, \quad \ddot{\theta}_2 = -\omega^2 b \sin \omega t$$
$$\theta_3 = c \sin \omega t, \quad \ddot{\theta}_3 = -\omega^2 c \sin \omega t$$

Substituting these values in equations 7.104,

$$(k_{t1} - J_1\omega^2)\, a - k_{t1}b = 0$$
$$(k_{t1} + k_{t2} - 3J_2\omega^2)\, b - 2k_{t1}a - k_{t2}c = 0$$
$$(k_{t2} - J_3\omega^2)\, c - k_{t2}b = 0$$

To find the natural frequency equation, the determinant of the coefficient of a, b and c must be equated to zero.

$$\begin{vmatrix} \theta_1 & \theta_2 & \theta_3 \\ (k_{t1} - J_1\omega^2) & -k_{t1} & 0 \\ -k_{t1} & (k_{t1} + k_{t2} - J_2\omega^2) & -k_{t2} \\ 0 & -k_{t2} & (k_{t2} - J_3\omega^2) \end{vmatrix} = 0$$

$(k_{t1} - J_1\omega^2)\,[(k_{t1} + k_{t2} - J_2\omega^2)\,(k_{t2} - J_3\omega^2) - k_{t2}^{\,2}] + k_{t1}[-\,k_{t1}\,(k_{t2} - J_3\omega^2) - 0] + 0 = 0$

By simplying the above equation, we get

$$\omega^6 - \left[\frac{k_{t1}}{J_1} + \frac{k_{t1}}{J_2} + \frac{k_{t2}}{J_2} + \frac{k_{t2}}{J_3}\right]\omega^4 + \left[\frac{k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1\,J_2\,J_3}\right]\omega^2 = 0$$

$$\omega^2\,[\omega^4 - \left\{\frac{k_{t1}}{J_1} + \frac{k_{t1} + k_{t2}}{J_2} + \frac{k_{t2}}{J_3}\right\}\omega^2 + \left\{\frac{k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1J_2J_3}\right\} = 0$$

$\omega_1^2 = 0$ ($\because$ semidefinite system).

$$\omega_{2,3}^2 = +\left\{\frac{k_{t1}}{2J_1} + \frac{k_{t1} + k_{t2}}{2J_2} + \frac{k_{t2}}{2J_3}\right\} \pm \sqrt{\left\{\frac{k_{t1}}{2J_1} + \frac{k_{t1} + k_{t2}}{2J_2} + \frac{k_{t2}}{2J_3}\right\}^2 - \frac{k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1J_2J_3}}$$

By solving the above equation, the natural frequencies are

$$\omega^2_{n1} = 0 \text{ rad/s},\ \omega_{n2} = \sqrt{\frac{k_{t2}}{J_2}} \text{ rad/s},\ \omega_{n3} = 1.74\sqrt{\frac{k_{t2}}{J_3}} \text{ rad/s.}$$

EXAMPLE 7.4

Find the influence coefficients of the spring-mass system as shown in Fig. p-7.4.

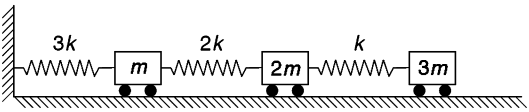


Fig. p-7.4 *Spring-mass system*

Solution Apply unit load at position '1'of Fig. p-7.4.

The influence coefficient, $a_{11} = \dfrac{1}{3k}$ (deflection at 1 due to unit load at 1)

$a_{21} = \dfrac{1}{3k}$ (deflection at 2 due to unit load at 1), $a_{31} = \dfrac{1}{3k}$ (deflection at 3 due to unit load at 1)

By Maxwell's reciprocal theorem, $a_{ij} = a_{ji}$,

$\therefore \qquad a_{21} = a_{12} = \dfrac{1}{3k}$

$\therefore \qquad a_{31} = a_{13} = \dfrac{1}{3k}$

Apply unit load at position '2' of Fig. p-7.4.

Neglecting the mass at 1, springs '3*k*' and '*k*' are in series.

$\therefore \qquad a_{22} = \dfrac{1}{k_{eq}}$

where $$\frac{1}{k_{eq}} = \frac{1}{3k} + \frac{1}{2k} \text{ or } a_{22} = \frac{5}{6k} \text{ (deflection at 2 due to load at 2)}$$

$$a_{32} = \frac{5}{6k} \text{ (deflection at 3 due to load at 2),}$$

By Maxwell's reciprocal theorem, $a_{32} = a_{23} = \frac{5}{6k}$

Apply unit load at position '3' of Fig. p-7.4.

Neglecting the masses at points '1' and '2', springs '$3k$', '$2k$' and 'k' are in series

$$\therefore \quad \frac{1}{k_{eq}} = \frac{1}{3k} + \frac{1}{2k} + \frac{1}{k} = \frac{11}{6k},$$

$$\therefore \quad a_{33} = \frac{1}{k_{eq}}$$

$$\therefore \quad a_{33} = \frac{11}{6k} \text{ (deflection at 3 due to unit load at 3).}$$

EXAMPLE 7.5

Find the influence coefficient of the system as shown in Fig. p-7.5(a) and thus find the values of natural frequencies.

Solution Apply unit load at the position 1.

Considering mass $m_1 = m$ in Fig. p-7.5(b), $T \sin \theta = 1$, $T \cos \theta = 3mg$,

$$\therefore \quad \tan \theta = \frac{1}{3mg}$$

For small angles of 'θ' $\tan \theta \approx \sin \theta = \frac{1}{3mg}$

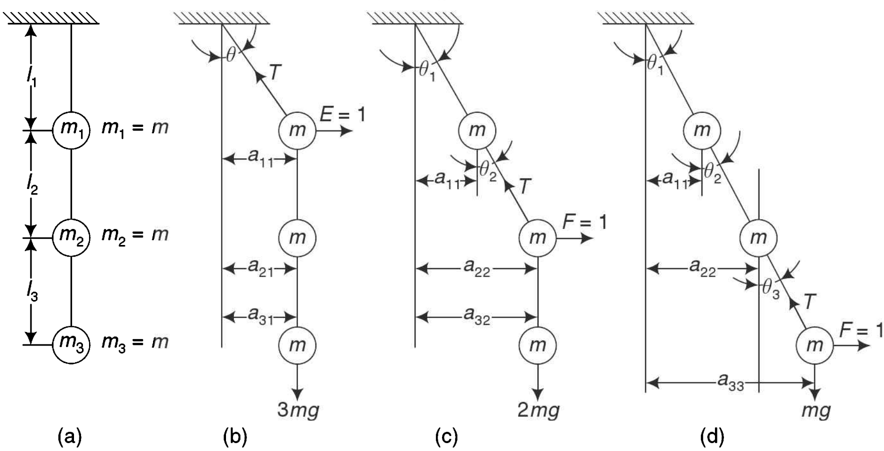


Fig. p-7.5 *System of multiple masses*

From the geometry of Fig. p-7.5(b), $\sin \theta = \frac{a_{11}}{l}$

$$\therefore \quad a_{11} = l \sin \theta$$

$$a_{11} = \frac{1}{3mg} = a_{21} = a_{31} = a_{12} = a_{13}$$

Apply unit load at the position 2.

Considering mass $m_2 = m$, $\Sigma V = 0$ and $\Sigma H = 0$

$$\therefore \qquad T \sin \theta_1 = 1, T \cos \theta_1 = 2\,mg$$

$$\therefore \qquad \tan \theta_1 = \frac{1}{2mg}$$

For small angles of θ_1 $\tan \theta_1 \approx \sin \theta_1 = \dfrac{1}{2mg}$

From the geometry of Fig. p-7.5(c), $a_{22} = a_{11} + l \sin \theta_1$

$$a_{22} = \frac{1}{3mg} + \frac{1}{2mg}, a_{22} = \frac{5l}{6mg} = a_{32} = a_{23}$$

Apply unit load at the position 3.

Considering mass $m_3 = m$, $\Sigma V = 0$ and $\Sigma H = 0$

$$\therefore \qquad T \sin \theta_2 = 1, T \cos \theta_2 = mg \quad \therefore \tan \theta_2 = \frac{1}{mg}$$

For small angles of θ_2, $\tan \theta_2 \approx \sin \theta_2 = \dfrac{1}{mg}$

From the geometry of Fig. p-7.5(d),

$$a_{33} = a_{22} + l \sin \theta^2, a_{33} = \frac{5l}{6mg} + \frac{1}{mg}, = \frac{11l}{6mg}$$

The equation of motion using influence coefficient in matrix form is as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} m_1 a_{11} & m_2 a_{12} & m_3 a_{13} \\ m_1 a_{21} & m_2 a_{22} & m_3 a_{23} \\ m_1 a_{31} & m_2 a_{32} & m_3 a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} \frac{ml}{3mg} & \frac{ml}{3mg} & \frac{ml}{3mg} \\ \frac{ml}{3mg} & \frac{5ml}{6mg} & \frac{5ml}{6mg} \\ \frac{ml}{3mg} & \frac{5ml}{6mg} & \frac{11ml}{6mg} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{ml\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To find the first principal mode and first natural frequency,

let $\qquad x_1 = x_2 = x_3 = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \frac{l\omega^2}{g} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$B_2 = 0A_2 + B_2 + 0C_2$$

$$C_2 = 0A_2 + 0B_2 + C_2$$

This can be written in matrix form as follows:

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & -2.29 & -3.92 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} \text{(Sweeping matrix)}$$

To obtain the second natural frequency, the sweeping matrix is combined with the matrix of the first principal mode.

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 0 & -2.29 & -3.92 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \frac{l\omega^2}{6g}$$

For first iteration, let $x_1 = 1, x_2 = 1, x_3 = 1$.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -8.42 \\ -2.42 \\ 3.58 \end{bmatrix} = \frac{8.42\, l\omega^2}{6g} \begin{bmatrix} -1 \\ -0.29 \\ 0.43 \end{bmatrix}$$

For second iteration, let $x_1 = -1,\ x_2 = 0.29, x_3 = 0.43$

$$\begin{bmatrix} -1 \\ -0.29 \\ 0.43 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -0.29 \\ 0.43 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} -1.76 \\ -1.34 \\ 1.24 \end{bmatrix} = \frac{1.76\, l\omega^2}{6g} \begin{bmatrix} -1 \\ -0.76 \\ 0.70 \end{bmatrix}$$

For third iteration, $x_1 = -1, x_2 = -0.76, x_3 = 0.70$

$$\begin{bmatrix} -1 \\ -0.76 \\ 0.70 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} -1 \\ -0.76 \\ 0.70 \end{bmatrix}$$

For second iteration, $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 12 \\ 27 \\ 45 \end{bmatrix} = \frac{12 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.25 \\ 3.75 \end{bmatrix}$$

For third iteration, $x_1 = 1,\ x_2 = 2.25,\ x_3 = 3.75$

$$\begin{bmatrix} 1 \\ 2.25 \\ 3.75 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2.25 \\ 3.75 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 14 \\ 32 \\ 54.70 \end{bmatrix} = \frac{14 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.29 \\ 3.89 \end{bmatrix}$$

For fourth iteration, $x_1 = 1,\ x_2 = 2.29,\ x_3 = 3.89$

$$\begin{bmatrix} 1 \\ 2.29 \\ 3.89 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2.29 \\ 3.89 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 14.36 \\ 32.90 \\ 56.24 \end{bmatrix} = \frac{14.36 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix}$$

For fifth iteration, $x_1 = 1,\ x_2 = 2.29,\ x_3 = 3.92$

$$\begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 14.42 \\ 33.04 \\ 56.57 \end{bmatrix} = \frac{14.42 l\omega^2}{6g} \begin{bmatrix} 1 \\ 2.29 \\ 3.92 \end{bmatrix}$$

Since the assumed values is approximately equal to the obtained values, the first principal modes will be, $A_1 = 1$, $B_1 = 2.29$, $C_1 = 3.92$.

The first natural frequency is

$$\frac{14.42 l\omega^2}{6g} = 1, \text{ or } \omega_{1n}^2 = \frac{6}{14.92} \cdot \frac{g}{l} \text{ or } \quad \therefore \ \omega_{1n} = 0.65\sqrt{\frac{g}{l}} \text{ rad/s.}$$

To obtain the second principal modes, the orthogonality principle is used.

$\therefore \ m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$, or $m\,(1)\,A_2 + m\,(2.29)\,B_2 + m\,(3.92)\,C_2 = 0$

$$A_2 = -2.29,\ B_2 = -3.92\ C_2,\ A_2 = 0A_2 - 2.29B_2 - 3.92C_2$$

$$= \frac{l\omega^2}{6g} = \begin{bmatrix} -2.13 \\ -2.31 \\ 1.89 \end{bmatrix} = \frac{2.13 l\omega^2}{6g} = \begin{bmatrix} -1 \\ -1.08 \\ 0.89 \end{bmatrix}$$

For fourth iteration, $x_1 = -1$, $x_2 = -1.08$, $x_3 = 0.89$

$$\begin{bmatrix} -1 \\ -1.08 \\ 0.89 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix}\begin{bmatrix} -1 \\ -1.08 \\ 0.89 \end{bmatrix} = \frac{l\omega^2}{6g}\begin{bmatrix} -2.41 \\ -2.98 \\ 2.36 \end{bmatrix} = \frac{2.41 l\omega^2}{6g}\begin{bmatrix} -1 \\ -1.24 \\ 0.98 \end{bmatrix}$$

For fifth iteration, $x_1 = -1$, $x_2 = -1.24$, $x_3 = 0.98$

$$\begin{bmatrix} -1 \\ -1.24 \\ 0.98 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix}\begin{bmatrix} -1 \\ -1.24 \\ 0.98 \end{bmatrix} = \frac{l\omega^2}{6g}\begin{bmatrix} -2.52 \\ -3.30 \\ 2.58 \end{bmatrix} = \frac{2.52 l\omega^2}{6g}\begin{bmatrix} -1 \\ -1.31 \\ 1.02 \end{bmatrix}$$

For sixth iteration, $x_1 = -1$, $x_2 = -1.31$, $x_3 = 1.02$

$$\begin{bmatrix} -1 \\ -1.31 \\ 1.02 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix}\begin{bmatrix} -1 \\ -1.31 \\ 1.02 \end{bmatrix} = \frac{l\omega^2}{6g}\begin{bmatrix} -2.58 \\ -3.45 \\ 2.67 \end{bmatrix} = \frac{2.58 l\omega^2}{6g}\begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix}$$

For seventh iteration, $x_1 = -1$, $x_2 = -1.34$, $x_3 = 1.04$

$$\begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix} = \frac{l\omega^2}{6g} = \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix}\begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix} = \frac{l\omega^2}{6g}\begin{bmatrix} -2.62 \\ -3.52 \\ 2.72 \end{bmatrix} = \frac{2.62 l\omega^2}{6g}\begin{bmatrix} -1 \\ -1.34 \\ 1.04 \end{bmatrix}$$

Since the assumed values and the obtained values are approximately equal, the second principal modes are given by $A_2 = -1$, $B_2 = -1.34$, $C_2 = 1.04$

$\therefore$ second natural frequency is given by

$$\frac{2.62 l\omega^2}{6g} = 1 \text{ or } \omega_{2n}^2 = \frac{6g}{2.62 l} \quad \therefore \ \omega_{2n} = 1.51\sqrt{\frac{g}{l}} \text{ rad/s}$$

To obtain the third natural frequency and third principal modes, the orthogonality principle should be used.

$$\therefore \qquad m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

Using the values,

$$m(-1)A_3 + m(1.34)B_3 + m(1.04)C_3 = 0,$$

$$-A_3 - 1.34\,B_3 + 1.04\,C_3 = 0 \qquad ...7.105$$

$$m(1)A_3 + m(2.29)B_3 + m(3.92)C_3 = 0$$

$$-A_3 + 2.29\,B_3 + 3.92\,C_3 = 0 \qquad ...7.106$$

Adding equations 7.105 and 7.106, we get

$$0.95B_3 + 4.96C_3 = 0$$

$$B_3 = -5.22C_3 \qquad ...7.107$$

Substituting the value of B_3 in Eq. 7.105, we get

$$-A_3 - 1.34(-5.22)C_3 + 1.04C_3 = 0$$

$$A_3 = 8.03\,C_3 \qquad ...7.108$$

Writing the equation in terms of C_3 from equations 7.107 and 7.108,

$$A_3 = 0A_3 + 0B_3 + 8.03\,C_3$$

$$B_3 = 0A_3 + 0B_3 - 5.22\,C_3$$

$$C_3 = 0A_3 + 0B_3 + C_3$$

The sweeping matrix will become

$$\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8.03 \\ 0 & 0 & -5.22 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}$$

When this sweeping matrix is combined with the matrix equation of the second mode, we get the matrix of third mode.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & -2.58 & -5.84 \\ 0 & 0.42 & -2.84 \\ 0 & 0.42 & 3.16 \end{bmatrix} \begin{bmatrix} 0 & 0 & 8.03 \\ 0 & 0 & -5.22 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & 0 & 7.63 \\ 0 & 0 & -5.03 \\ 0 & 0 & 0.97 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For first iteration, $x_1 = 1, x_2 = 1, x_3 = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & 0 & 7.63 \\ 0 & 0 & -5.03 \\ 0 & 0 & 0.97 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 7.63 \\ -5.03 \\ 0.97 \end{bmatrix} = \frac{7.63 l\omega^2}{6g} \begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix}$$

For second iteration, $x_1 = 1, x_{2,} = -0.66, x_{3,} = 0.13$

$$\begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0 & 0 & 7.63 \\ 0 & 0 & -5.03 \\ 0 & 0 & 0.97 \end{bmatrix} \begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix} = \frac{l\omega^2}{6g} \begin{bmatrix} 0.99 \\ -0.65 \\ 0.13 \end{bmatrix} = \frac{0.99 l\omega^2}{6g} \begin{bmatrix} 1 \\ -0.66 \\ 0.13 \end{bmatrix}$$

Since the assumed value is approximately equal to the obtained value, the third principal modes will be, $A_3 = 1$, $B_3 = -0.66$, $C_3 = 0.13$

The third natural frequency will be $0.99l\omega^2/6g = 1$, $\omega^2_{3n} = 6g/0.99l$,

$$\therefore \qquad \omega_{3n} = 2.46\sqrt{\frac{g}{l}} \text{ rad/s.}$$

EXAMPLE 7.6

Determine the influence coefficient of the triple pendulum of lengths 'l_1','l_2' and 'l_3' and masses 'm_1', 'm_2' and 'm_3' as shown in Fig. p-7.6(a).

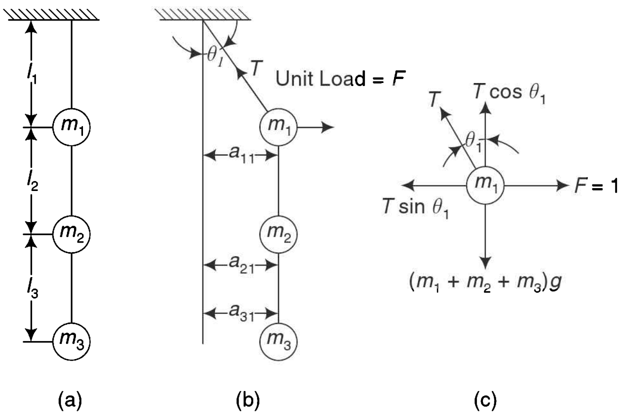


Fig. p-7.6 *Triple pendulum*

Solution Apply unit load to mass 'm_1'.

Considering mass 'm_1',

$$\Sigma V = 0 \text{ and } \Sigma H = 0$$

$$\therefore \qquad T \sin \theta_1 = 1 \qquad ...7.109$$

$$\therefore \qquad T \cos \theta_1 = (m_1 + m_2 + m_3)g \qquad ...7.110$$

Divide Eq. 7.109 by Eq. 7.110, $\tan \theta_1 = \dfrac{1}{(m_1 + m_2 + m_3)g}$

For small angles of θ_1, $\tan \theta_1 \cong \sin \theta_1$

From the geometry of Fig. p-7.6(b), $\sin \theta_1 = \dfrac{a_{11}}{l_1}$, $a_{11} = l_1 \sin \theta_1$

The influence coefficient, $a_{11} = \dfrac{l_1}{(m_1 + m_2 + m_3)g}$

From the geometry of Fig. p-7.6(c), $a_{21} = \dfrac{l_1}{(m_1 + m_2 + m_3)g}$, $a_{31} = \dfrac{l_1}{(m_1 + m_2 + m_3)g}$

By Maxwell's reciprocal theorem, $a_{21} = a_{12}$ and $a_{31} = a_{13}$

Applying unit load to mass 'm_2', neglecting mass 'm_1',

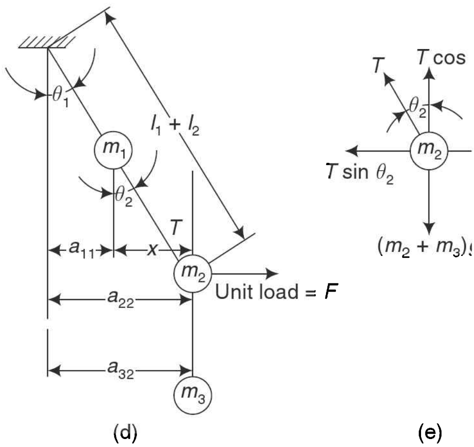


Fig. p-7.6 *Contd.*

Considering mass 'm_2', Fig. p-7.6(d), ΣV and $\Sigma H = 0$

$\therefore \quad T \sin \theta_2 = 1$

$\therefore \quad T \cos \theta_2 = (m_2 + m_3)g$

$\therefore \quad \tan \theta_2 = \dfrac{1}{(m_2 + m_3)g}$

For small angles of θ_2, $\tan \theta_2 \cong \sin \theta_2$

From the geometry of Fig. p-7.6(e), $\sin \theta_2 = \dfrac{x}{l_2}$, $x = l_2 \sin \theta$, $x = \dfrac{l_2}{(m_2 + m_3)g}$

But influence coefficient,

$$a_{22} = a_{11} + x,\; a_{22} = \frac{l_1}{(m_1 + m_2 + m_3)g} + \frac{l_2}{(m_2 + m_3)g}$$

From the geometry of the figure, $a_{32} = a_{22} = \dfrac{l_1}{(m_1 + m_2 + m_3)g} + \dfrac{l_2}{(m_2 + m_3)g}$

Applying unit load at the position '3', neglecting masses 'm_1' and 'm_2'.

Consider mass m_3, Fig. p-7.6(*f*) $\Sigma V = 0$ and $\Sigma H = 0$

$\therefore \quad T \sin \theta_3 = 1$

$\therefore \quad T \cos \theta_3 = m_3 g$

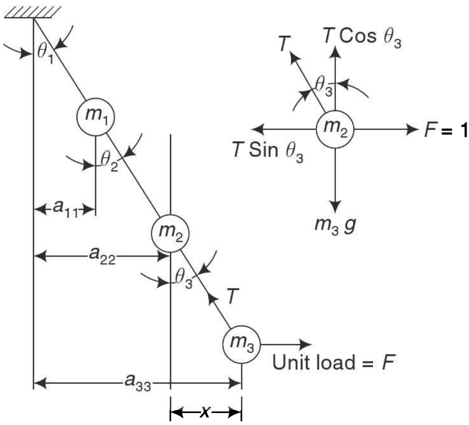
$\therefore \quad \tan \theta_3 = \dfrac{1}{m_3 g}$

For small angle of θ_3, $\tan \theta_3 \cong \sin \theta_3$

From the geometry of Fig. p-7.6(g)

$$\sin \theta_3 = \frac{x}{l_3},\quad x = l_3 \sin \theta_3,\; x = \frac{l_3}{m_3 g}$$

But $\quad a_{33} = a_{22} + x$



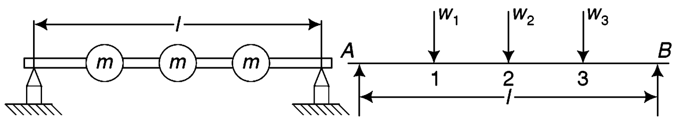
(f) (g)

Fig. p-7.6 *Contd.*

$$a_{33} = \frac{l_1}{(m_1 + m_2 + m_3)g} + \frac{l_2}{(m_2 + m_3)g} + \frac{l_3}{m_3 g}.$$

EXAMPLE 7.7

A simply supported beam of length '*l*' has three equal masses attached to it at equal distances as shown in Fig. p-7.7(a). Determine the influence coefficient.



(a) (b)

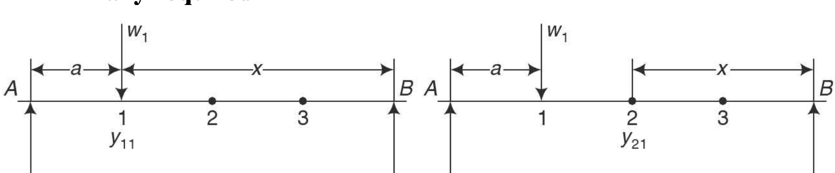
Fig. p-7.7 *Simply supported beam*

Note: Deflection at any point '*x*' is given by a simply supported beam.

$$y_x = \frac{wax(l^2 - a^2 - x^2)}{6EIl} \text{ for } x \le (1 - a)$$

where w **= Load applied at a distance '*a*' from the end *A* or *B***

x **= Distance to the point from end *B* or *A*, where the deflection is actually required**



(c) (d)

Fig. p-7.7 *Contd.*

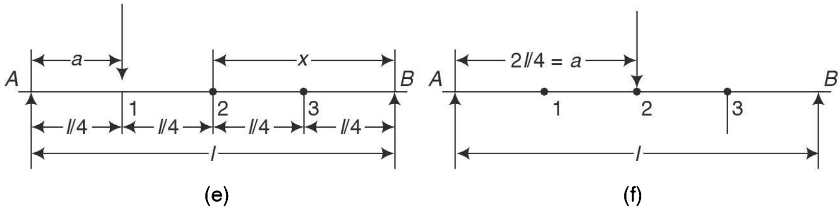


Fig. p-7.7 *Contd.*

***E* = Young's modulus of the beam material and *I* = Moment of inertia of the beam**

Applying unit load at the point 1, Fig. p-7.7(c).

Solution Influence coefficient, $a = \frac{l}{4}, x = \frac{3l}{4} \quad x \leq 1 - a$

i.e. $\frac{3l}{4} \leq l - \frac{l}{4}$ and $a_{11} = \dfrac{\frac{l}{4} \times \frac{3l}{4} (l_2 - \left(\frac{l}{l_4}\right)^2 - \left(\frac{3l}{4}\right)^2}{6EIl}$

$\therefore$ the condition is satisfied, $a_{11} = \dfrac{\frac{3l}{16}\left(l_2 - \frac{l^2}{16} - \frac{9l^2}{16}\right)}{6EI} = \dfrac{3l^3 \times 6}{16 \times 16 \times 6EI} = \dfrac{3l^3}{256EI}$

Deflection at the position '1' due to unit load at position '1', $a = \frac{l}{4}, x = \frac{2l}{4}$

Condition $x \leq l - a$, i.e. $\frac{2l}{4} \leq \left(l - \frac{l}{4}\right), \frac{2l}{4} \leq \frac{3l}{4}$ (true)

$$\therefore \quad a_{21} = \dfrac{l \times \frac{l}{4} \times \frac{2l}{4}\left(l_2 - \left(\frac{l}{4}\right)^2 - \left(\frac{2l}{4}\right)^2\right)}{6EI} = \dfrac{\frac{2l}{16}\left(l^2 - \frac{l^2}{16} - \frac{4l^2}{16}\right)}{6EI}$$

$a_{21} = \dfrac{2 \times 11l^3}{256EI} \; \dfrac{l}{6} = \dfrac{3.67l^3}{256EI}$ (deflection at 2 due to unit load at 1)

$a = \frac{l}{4}, x = \frac{1}{4}$

Condition $x \leq l - a$, i.e. $\frac{l}{4} \leq \left(l - \frac{l}{4}\right) \Rightarrow \frac{l}{4} \leq \frac{3l}{4}$ (True)

$$\therefore \quad a_{31} = \dfrac{\frac{l}{4} \times \frac{l}{4}\left(l^2 - \left(\frac{l}{4}\right)^2 - \left(\frac{l}{4}\right)^2\right)}{6EI} = \frac{14}{6} \times \frac{l^3}{256EI}$$

$a_{31} = \dfrac{2.33l^3}{256EI}$ (deflection at 3 due to unit load at 1)

By Maxwell's reciprocal theorem, $a_{21} = a_{12} = \dfrac{3.67l^3}{256EI} \quad a_{31} = a_{13} = \dfrac{2.33l^3}{256EI}$

Applying unit load at the point '2', influence coefficient a_{22}, $a = \frac{2l}{4}, x = \frac{2l}{4}$, condition $x \le (l - a)$,

i.e. $$\frac{2l}{4} \le \left(l - \frac{2l}{4}\right) \Rightarrow \frac{2l}{4} \le \frac{2l}{4} \text{ (True)}$$

$$\therefore\ a_{22} = \frac{\frac{2l}{4}\frac{2l}{4}\left[l^2 - \left(\frac{2l}{4}\right)^2 - \left(\frac{2l}{4}\right)^2\right]}{6EIl,} = \frac{4l(16l^2 - 4l^2 - 4l^2)}{6x256EI} = \frac{5.33l^3}{256EI}$$

(deflection at 2 due to unit load at 2)

$$a = \frac{2l}{4}, x = \frac{l}{4} \text{ condition } x \le (l - a), \text{ i.e. } \frac{l}{4} \le \frac{2l}{4} \text{ (True)}$$

$$\therefore \quad a_{32} = \frac{\frac{2l}{4} \times \frac{l}{4}\left[l^2 - \left(\frac{2l}{4}\right)^2 - \left(\frac{l}{4}\right)^2\right]}{6EIl} = \frac{2l[16l^2 - 4l^2 - l^2]}{6 \times 256EI} = \frac{3.67l^3}{256EI}$$

(deflection at 3 due to load at 2)

Applying unit load at point 3, $a = \frac{1}{4}, x = \frac{3l}{4}$

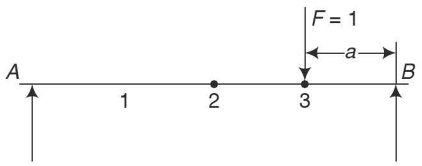


Fig. p-7.7(g) *Contd.*

Condition $x \le (l - a)$, $\therefore \frac{3l}{4} \le \left(l - \frac{l}{4}\right)$

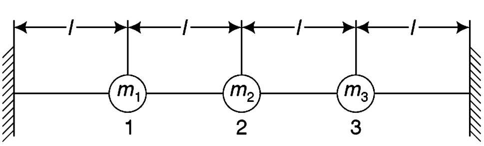
$$\therefore \quad \frac{3l}{4} \le \frac{3l}{4} \text{ (True)}$$

$$\therefore\ a_{33} = \frac{\frac{l}{4} \times \frac{3l}{4}\left(l^2 - \left(\frac{l}{4}\right)^2 - \left(\frac{3l}{4}\right)^2\right)}{6EIl} = \frac{3l^3}{256EI} \text{ (deflection at 3 due to unit load at 3)}$$

By Maxwell's reciprocal theorem, $a_{32} = a_{23} = \frac{3.67l^3}{256EI}$.

EXAMPLE 7.8

Determine the influence coefficient of a dynamic system consisting of three equal masses attached to a taut string as shown in Fig. p-7.8(a).



(a)

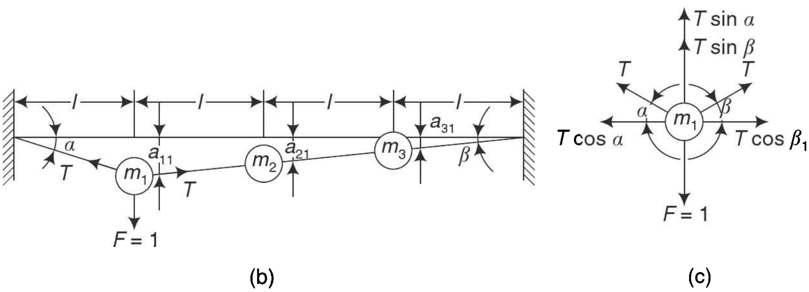


Fig. p-7.8 *Dynamic system*

Solution Applying unit load at the point 1, let 'T' be the tension in the string.

For small angles of 'α'and 'β', $\tan \alpha \approx \sin \alpha$, $\tan \beta \approx \sin \beta$

Considering the mass 'm_1' in Fig. 7.8(b) $\Sigma H = 0$

$$\therefore \quad T \cos \alpha = T \cos \beta$$

Considering vertical movement of the mass $m_{1,}$

$$\Sigma V = 0$$

$$\therefore \quad T \sin \alpha + T \sin \beta = 1$$

But from the geometry of Fig. 7.8(c), $\sin \alpha = \dfrac{a_{11}}{l}$, $\sin \beta = \dfrac{a_{11}}{3l}$

$T\left(\dfrac{a_{11}}{l} + \dfrac{a_{11}}{3l}\right) = 1$, $a_{11} = \dfrac{3l}{4T}$ (deflection at 1 due to unit load at the point '1')

Comparing similar triangles,

$$a_{21} = \frac{2}{3} a_{11} = \frac{2}{3} \times \frac{3}{4T} \quad \therefore a_{21} = \frac{l}{2T} \text{ (deflection at 2 due to unit load at 1)}$$

Comparing similar triangles,

$\dfrac{a_{11}}{3l} = \dfrac{a_{31}}{l}$, $a_{31} = \dfrac{1}{3} a_{11}$, $a_{31} = \dfrac{l}{4T}$ (deflection at 3 due to unit load at 1)

By Maxwell's reciprocal theorem, $a_{12} = a_{21} = \dfrac{l}{2T}$ (deflection at 1 due to unit load at 2)

$a_{31} = a_{13} = \dfrac{l}{4T}$ (deflection at 1 due to unit load at 3) applying unit load at the point 2.

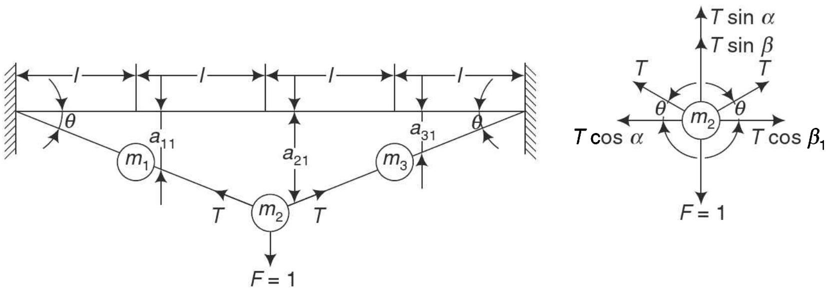
Considering the vertical movement of mass 'm_2' in Fig. p-7.8(d)

$$\Sigma V = 0, T \sin \theta + T \sin \theta = 1$$

From the geometry of Fig. p-7.8(e),

$$\sin \theta = \frac{a_{22}}{2l}$$

$$\therefore \quad T\left(\frac{l}{2l}\right) + \left(\frac{l}{2l}\right) a_{22} = 1$$



(d) (e)

Fig. p-7.8 *Contd.*

$$\therefore \qquad a_{22} = \frac{2l}{2T} = \frac{l}{T}$$

Comparing the similar triangles,

$$\frac{a_{22}}{2l} = \frac{a_{32}}{l}, \; a_{32} = \frac{a_{22}}{2}, \; a_{32} = \frac{l}{2T} \text{ (deflection at 3 due to unit load at 2)}$$

By Maxwell's reciprocal theorem,

$$a_{23} = a_{32} = \frac{l}{2T} \text{ (deflection at 2 due to unit load at 2)}$$

Applying unit load point '3' by symmetry, $a_{33} = a_{11} = \dfrac{3l}{4T}$

Note: For the system shown in Fig. p-7.8.(f)

$m_1\ddot{x}_1$ is the inertia force of the mass m_1

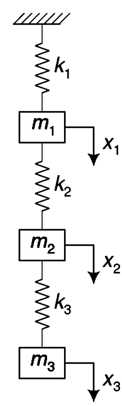
$m_2\ddot{x}_2$ is the inertia force of the mass m_2

$m_3\ddot{x}_3$ is the inertia force of the mass m_3

Let $\qquad x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t, \quad x_3 = C \sin \omega t$

$$\ddot{x}_1 = -\omega^2 x_1, \; \ddot{x}_2 = -B\omega^2 \sin \omega t, \; \ddot{x}_3 = -\omega^2 x_2$$

$\therefore$ the inertia forces will be $-m_1\omega^2 x_1$, $-m_2\omega^2 x_2$ and $-m_3\omega^2 x_3$



(f)

Fig. p-7.8 *Contd.*

For unit load, influence coefficient = a_{ij}

For inertia force, influence coefficient = (inertia force) a_{ij}

For a three-degree-freedom system shown in Fig. p-7.8.(f),

there are nine influence coefficients:

a_{11}, a_{12}, a_{13} for the mass m_1, a_{21}, a_{22}, a_{23} for the mass m_2, a_{31}, a_{32}, a_{33} for the mass m_3.

$\therefore$ total deflection of masses 'm_1', 'm_2' and 'm_3' is given by

$$\left.\begin{aligned} a_1 &= a_{11} + a_{12} + a_{13} \\ a_2 &= a_{21} + a_{22} + a_{23} \\ a_3 &= a_{31} + a_{32} + a_{33} \end{aligned}\right\} \text{For unit force}$$

Considering inertia forces, displacements are given by

$$-x_1 = a_{11}m_1\ddot{x}_1 + a_{12}m_2\ddot{x}_2 + a_{13}m_3\ddot{x}_3 \quad \therefore\ x_1 = a_{11}m_1\omega^2 x_1 + a_{12}m_2\omega^2 x_2 + a_{13}m_3\omega^2 x_3$$

$$\therefore\ x_2 = a_{21}m_1\omega^2 x_1 + a_{22}m_2\omega^2 x_2 + a_{23}m_3\omega^2 x_3$$

$$\therefore\ x_3 = a_{31}m_1\omega^2 x_1 + a_{32}m_2\omega^2 x_2 + a_{33}m_3\omega^2 x_3$$

This can be written in matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} a_{11}m_1\omega^2 x_1 & a_{12}m_2\omega^2 x_2 & a_{13}m_3\omega^2 x_3 \\ a_{21}m_1\omega^2 x_1 & a_{22}m_2\omega^2 x_2 & a_{23}m_3\omega^2 x_3 \\ a_{31}m_1\omega^2 x_1 & a_{32}m_2\omega^2 x_2 & a_{33}m_3\omega^2 x_3 \end{bmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \omega^2 \begin{bmatrix} a_{11}m_1 x_1 & a_{12}m_2 x_2 & a_{13}m_3 x_3 \\ a_{21}m_1 x_1 & a_{22}m_2 x_2 & a_{23}m_3 x_3 \\ a_{31}m_1 x_1 & a_{32}m_2 x_2 & a_{33}m_3 x_3 \end{bmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

EXAMPLE 7.9

Calculate the natural frequencies of the system as shown in Fig. p-7.9 by using matrix method.

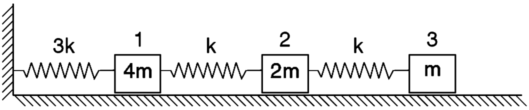


Fig. p-7.9 *Spring-mass system*

To calculate influence coefficient, applying unit load to the position '1'

$$a_{11} = \frac{1}{3k} = a_{21} = a_{31} = a_{12} = a_{13}$$

Applying unit load at the position '2', neglect mass at '1',

$$a_{22} = \frac{1}{3k} + \frac{1}{k} = \frac{4}{3k} = a_{32} = a_{23}$$

Applying unit load at the position 3, neglecting masses at '2' and '3',

$$a_{33} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k} = \frac{7}{3k} \text{ given } m_1 = 4m,\ m_2 = 2m,\ m_3 = m$$

The equation of motion for a three-degree-freedom system in matrix form is written as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} m_1 a_{11} & m_2 a_{12} & m_3 a_{13} \\ m_1 a_{21} & m_2 a_{22} & m_3 a_{23} \\ m_1 a_{31} & m_2 a_{32} & m_3 a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \begin{bmatrix} \frac{4m}{3k} & \frac{2m}{3k} & \frac{m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{4m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{7m}{3k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assuming $x_1 = 1, x_2 = 1, x_3 = 1$ for the first iteration,

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4+2+1\\4+8+4\\4+8+7\end{bmatrix}\begin{bmatrix}7\\16\\19\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}7\\16\\19\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{7m\omega^2}{3k}\begin{bmatrix}1\\2.29\\2.71\end{bmatrix}$$

For second iteration, $x_1 = 1, x_2 = 2.29, x_3 = 2.71$

$$\begin{bmatrix}1\\2.29\\2.71\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\2.29\\2.71\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}11.29\\33.16\\41.29\end{bmatrix} = \frac{11.29m\omega^2}{3k}\begin{bmatrix}1\\2.29\\3.66\end{bmatrix}$$

For third iteration, $x_1 = 1, x_2 = 2.94, x_3 = 3.66$

$$\begin{bmatrix}1\\2.29\\3.66\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix} = \frac{11.29m\omega^2}{3k}\begin{bmatrix}1\\2.29\\3.66\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}13.54\\42.16\\53.14\end{bmatrix}$$

$$\begin{bmatrix}1\\2.29\\3.66\end{bmatrix} = \frac{13.54m\omega^2}{3k}\begin{bmatrix}1\\3.11\\3.92\end{bmatrix}$$

For fourth iteration, $x_1 = 1, x_2 = 3.11, x_3 = 3.92$

$$\begin{bmatrix}1\\3.11\\3.92\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\3.11\\3.92\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}14.14\\44.56\\56.32\end{bmatrix} = \frac{14.14m\omega^2}{3k}\begin{bmatrix}1\\3.15\\3.98\end{bmatrix}$$

For fifth iteration, $x_1 = 1, x_2 = 3.15, x_3 = 3.98$

$$\begin{bmatrix}1\\3.15\\3.98\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\3.15\\3.98\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}14.28\\45.12\\57.06\end{bmatrix} = \frac{14.28\,m\omega^2}{3k}\begin{bmatrix}1\\3.16\\4\end{bmatrix}$$

For sixth iteration, $x_1 = 1, x_2 = 3.16, x_3 = 4$

$$\begin{bmatrix}1\\3.16\\4\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}4 & 2 & 1\\4 & 8 & 4\\4 & 8 & 7\end{bmatrix}\begin{bmatrix}1\\3.16\\4\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}14.32\\45.28\\57.28\end{bmatrix} = \frac{14.32\,m\omega^2}{3k}\begin{bmatrix}1\\3.16\\4\end{bmatrix}$$

Since the assumed value is very close to the obtained value

$$\frac{14.32\,m\omega^2}{3k} = 1$$

$$\therefore\ \omega_{1n}^2 = 0.21\,\frac{k}{m}\ \therefore\ \omega_{1n} = 0.46\sqrt{\frac{k}{m}}\ \text{rad/s (first natural frequency)}$$

The first principal modes are given by $A_1 = 1, B_1 = 3.16$ and $C_1 = 4.0$

To obtain the second principal mode, the orthogonality principle is used,

i.e. $m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$

$\therefore$ $4m(1)A_2 + 2m(3.16) + B_2 + m(4)C_2 = 0$

or $4A_2 + 6.32B_2 + 4C_2 = 0, A_2 = -1.58B_2 - C_2$ or $A_2 = 0A_2 - 1.58B_2 - C_2$

$B_2 = B_2,\quad B_2 = 0A_2 + B_2 + 0C_2,\quad C_2 = C_2,\quad C_2 = 0A_2 + 0B_2 + C_2$

These can be written in matrix form as

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}$$

This matrix, if combined with the matrix of first mode, is called sweeping matrix.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Starting the first iteration, let the second principal modes be

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1$$

$$\therefore \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -7.32 \\ 1.68 \\ 1.68 \end{bmatrix} = \frac{7.32m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.23 \\ 0.64 \end{bmatrix}$$

For second iteration, $x_1 = -1, \quad x_2 = 0.23, \quad x_3 = 0.64$

$$\begin{bmatrix} -1 \\ 0.23 \\ 0.64 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.23 \\ 0.64 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.91 \\ 0.39 \\ 2.31 \end{bmatrix} = \frac{2.91m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.13 \\ 0.79 \end{bmatrix}$$

For third iteration, $x_1 = -1, \quad x_2 = 0.13, \quad x_3 = 0.79$

$$\begin{bmatrix} -1 \\ 0.13 \\ 0.79 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.13 \\ 0.79 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.93 \\ 0.22 \\ 2.59 \end{bmatrix} = \frac{2.93\, m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.08 \\ 0.88 \end{bmatrix}$$

For fourth iteration, $x_1 = -1, \quad x_2 = 0.08, \quad x_3 = 0.88$

$$\begin{bmatrix} -1 \\ 0.08 \\ 0.88 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.08 \\ 0.88 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.93 \\ 0.13 \\ 2.77 \end{bmatrix} = \frac{2.99\, m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.04 \\ 0.93 \end{bmatrix}$$

For fifth iteration, $x_1 = -1, \quad x_2 = 0.04, \quad x_3 = 0.93$

$$\begin{bmatrix} -1 \\ 0.04 \\ 0.93 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.04 \\ 0.93 \end{bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} -2.96 \\ 0.07 \\ 2.86 \end{bmatrix} = \frac{2.96m\omega^2}{3k} \begin{bmatrix} -1 \\ 0.02 \\ 0.97 \end{bmatrix}$$

For sixth iteration, $x_1 = -1, \quad x_2 = 0.02, \quad x_3 = 0.97$

$$\begin{bmatrix} -1 \\ 0.02 \\ 0.97 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0.02 \\ 0.97 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -3 \\ 0.03 \\ 2.94 \end{bmatrix} = \frac{3m\omega^2}{3k} - \begin{bmatrix} 1 \\ 0.01 \\ 0.98 \end{bmatrix}$$

For seventh iteration, $x_1 = -1, \quad x_2 = 0.01, \quad x_3 = 0.98$

$$\begin{bmatrix} -1 \\ 0.01 \\ 0.98 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0.01 \\ 0.98 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -2.98 \\ 0.02 \\ 2.96 \end{bmatrix} = \frac{2.98m\omega^2}{3k}\begin{bmatrix} -1 \\ 0.01 \\ 0.99 \end{bmatrix}$$

Since the values of $x_{1=-1}, \quad x_{2=}0, \quad x_3 = 1$

Let for eighth iteration, $x_1 = -1, \quad x_2 = 0, \quad x_3 = 1$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = \frac{3m\omega^2}{3k}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Since the obtained modes is equal to the assumed modes,

$$= \frac{3m\omega^2}{3k} = 1, \quad \omega_{2n}^2 = \frac{k}{m}, \quad \omega_{2n} = \sqrt{\frac{k}{m}} \text{ rad/s. (second natural frequency)}$$

To obtain third principal modes, the orthogonality principle is

$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

But $\quad A_1 = 1, \ B_1 = 3.16, \quad C_1 = 4, \quad A_2 = -1, \ B_2 = 0, \quad C_2 = 1$

$\therefore \quad 4m(-1)A_3 + 2m(0)B_3 + m(1)C_3 = 0$

$$-4A_3 + 0\ B_3 + C_3 = 0 \qquad \text{...7.111}$$

$$4m(1)A_3 + 2m(3.16)B_3 + m(4)C_3 = 0$$

$$4A_3 + 6.32B_3 + 4C_3 = 0 \qquad \text{...7.112}$$

Solving equations 7.111 and 7.112 and adding, we get

$$6.32\ B_3 + 5C_3, \quad B_3 = \frac{-5}{6.32}C_3, \quad B_3 = -0.79\ C_3$$

Or $\quad A_3 = 0A_3 + 0B_3 + 0.25C_3, B_3 = 0A_3 + 0B_3 - 0.79C_3, C_3 = 0A_3 + 0B_3 + 1C_3$

Writing in matrix form,

$$\left.\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}\right\} \text{(sweeping matrix)}$$

This matrix is combined with the matrix of the second mode.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & 0 & 0.41 \\ 0 & 0 & -1.33 \\ 0 & 0 & 1.67 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For first iteration, $x_1 = 1.0, \quad x_2 = 1.0, \quad x_3 = 1.0$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0 & 0 & 0.41\\0 & 0 & -1.33\\0 & 0 & 1.67\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0.41\\-1.33\\1.67\end{bmatrix} = \frac{0.41m\omega^2}{3k}\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix}$$

For second iteration, $x_1 = 1.0, \quad x_2 = -3.24, \quad x_3 = 4.07$

$$\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}0 & 0 & 0.41\\0 & 0 & -1.33\\0 & 0 & 1.67\end{bmatrix}\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix}1.67\\-5.41\\6.80\end{bmatrix} = \frac{0.41m\omega^2}{3k}\begin{bmatrix}1\\-3.24\\4.07\end{bmatrix}$$

Since the assumed amplitudes is equal to the obtained values,

$$\therefore \qquad \frac{1.67m\omega^2}{3k} = 1, \quad \omega^2 = \frac{3}{1.67}\frac{k}{m}, \quad \omega_{3n}^2 = 1.80\frac{k}{m}$$

$$\therefore \qquad \omega_{3n} = 1.34\sqrt{\frac{k}{m}} \text{ rad/s (third natural frequency)}$$

Third principal mode will be $A_3 = 1, B_3 = -3.24, C_3 = 4.07$.

EXAMPLE 7.10

Determine the natural frequencies and principal modes of vibration for the 3-degree-freedom system as shown in Fig. p-7.10 by using matrix iteration method.

Solution Determine the influence coefficient system shown in Fig. p-7.10.

We know that earlier.

$$[\alpha_{ij}] = \begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix} = \begin{bmatrix}\frac{1}{3k} & \frac{1}{3k} & \frac{1}{3k}\\ \frac{1}{3k} & \frac{4}{3k} & \frac{4}{3k}\\ \frac{1}{3k} & \frac{4}{3k} & \frac{7}{3k}\end{bmatrix}$$

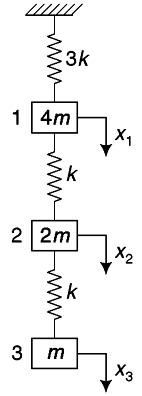


Fig. p-7.10 *Three-degree freedom system*

In the next step, write down the equation of motion using influence coefficients.

$$\alpha_{11}m_1\ddot{x}_1 + \alpha_{12}m_2\ddot{x}_2 + \alpha_{13}m_3\ddot{x}_3 + x_1 = 0$$

$$\alpha_{21}m_1\ddot{x}_1 + \alpha_{22}m_2\ddot{x}_2 + \alpha_{23}m_3\ddot{x}_3 + x_2 = 0$$

$$\alpha_{31}m_1\ddot{x}_1 + \alpha_{32}m_2\ddot{x}_2 + \alpha_{33}m_3\ddot{x}_3 + x_3 = 0$$

Substitute $\quad \ddot{x}_1 = -\omega^2 x_1, \quad \ddot{x}_2 = -\omega^2 x_2$ and $\ddot{x}_3 = -\omega^2 x_3$

$$x_1 = \alpha_{11}m_1x_1 + \alpha_{12}m_2x_2 + \alpha_{13}m_3x_3$$

$$x_2 = \alpha_{21}m_1x_1 + \alpha_{22}m_2x_2 + \alpha_{23}m_3x_3$$

$$x_3 = \alpha_{31}m_1x_1 + \alpha_{32}m_2x_2 + \alpha_{33}m_3x_3$$

The above equation can be written in the matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} \alpha_{11}m_1 & \alpha_{12}m_2 & \alpha_{13}m_3 \\ \alpha_{21}m_1 & \alpha_{22}m_2 & \alpha_{23}m_3 \\ \alpha_{31}m_1 & \alpha_{32}m_2 & \alpha_{33}m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Substitute the value of influence coefficients in the above equation.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} \frac{4m}{3k} & \frac{2m}{3k} & \frac{m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{4m}{3k} \\ \frac{4m}{3k} & \frac{8m}{3k} & \frac{7m}{3k} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

To start the iteration process, assume the configuration in the first mode as

$x_1 = 1, x_2 = 2, x_3 = 3$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 11 \\ 32 \\ 41 \end{Bmatrix} = (11)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 2.91 \\ 3.72 \end{Bmatrix}$$

Second iteration

$$\begin{Bmatrix} 1 \\ 2.91 \\ 3.72 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.91 \\ 3.72 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 13.5 \\ 42.16 \\ 53.32 \end{Bmatrix} = (13.5)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 3.12 \\ 3.95 \end{Bmatrix}$$

Third iteration

$$\begin{Bmatrix} 1 \\ 3.12 \\ 3.95 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.12 \\ 3.95 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 14.19 \\ 44.76 \\ 56.81 \end{Bmatrix} = (14.19)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix}$$

Fourth iteration

$$\begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.15 \\ 3.99 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{Bmatrix} 14.3 \\ 45.16 \\ 57.13 \end{Bmatrix} = (14.3)\frac{m\omega^2}{k} \begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix}$$

The ratio obtained is very close to the initial value.

$$\therefore \quad \begin{bmatrix} 1 \\ 3.15 \\ 3.99 \end{bmatrix} = 14.3\frac{m\omega^2}{3k} \begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix} \text{ or } \frac{14.3\, m\omega^2}{3k} = 1,\ \omega^2 = \omega_1^2 = \frac{3}{14.3}\frac{k}{m}$$

$$\therefore \quad \omega_1 = 0.458\sqrt{\frac{k}{m}} \text{ rad/s}$$

The first principal mode is given by $\begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix}$

The second natural frequency and principal mode is found by using orthogonality principle.

$$m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$$

The value is converging to $\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$ and hence $\dfrac{3m\omega^2}{3k} = 1$

$$\therefore \qquad \omega_2^2 = \frac{k}{m},\ \omega_2 = \sqrt{\frac{k}{m}} \text{ rad/s}$$

and the second principle mode is $\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$

To get the third mode, use orthogonality principle.

$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$$

Substitute $A_1 = 1,\ A_2 = -1,\ B_1 = 3.158,\ B_2 = 0,\ C_1 = 4.0,\ C_2 = 1$

$$4m\,(-1)A_3 + 2m(0)B_3 + m(1)C_3 = 0,\ -4A_3 + C_3 = 0 \quad \therefore A_3 = \frac{+C_3}{4}$$

$4m(1)\,A_3 + 2m(1)\,B_3 + m(4)\,C_3 = 0,\ \ 4m\left(\dfrac{+C_3}{4}\right) + 2\,(3.158)\,B_3 + 4C_3 = 0,\ 6.316\ B_3 = -5C_3$

Then

$$\begin{matrix} A_3 \\ B_3 \\ C_3 \end{matrix} = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \end{Bmatrix}$$

When this is contained with the matrix equation for the second mode, it will yield the third mode.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$4m(1)A_2 + 2m(3.158)B_2 + m(3.99)C_2 = 0$$

$$\therefore \qquad 4A_2 + 6.316\,B_2 + 3.99\,C_2 = 0$$

$$A_2 = -1.58\,B_2 - C_2,\ B_2 = B_2 \text{ and } C_2 = C_2$$

The same can be written in matrix form as

$$\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix} = \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix}$$

When this is combined with the matrix equation for first mode, it will converge to second mode.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

For first iteration, assume $x_1 = 1, x_2 = 1, x_3 = 1$.

$$\therefore \quad \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} -7.32 \\ 1.68 \\ 4.68 \end{bmatrix} = \frac{7.32m\omega^2}{k}\begin{Bmatrix} -1 \\ 0.23 \\ 0.64 \end{Bmatrix}$$

For second iteration

$$\begin{Bmatrix} -1 \\ 0.23 \\ 0.64 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ 0.23 \\ 0.64 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} -2.91 \\ 0.3864 \\ 2.31 \end{Bmatrix} = \frac{2.91m\omega^2}{3k}\begin{Bmatrix} -1 \\ 0.13 \\ 0.8 \end{Bmatrix}$$

For third iteration

$$\begin{Bmatrix} -1 \\ 0.13 \\ 0.8 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ 0.13 \\ 0.64 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} -2.48 \\ 0.218 \\ 2.14 \end{Bmatrix} = \frac{2.48m\omega^2}{3k}\begin{Bmatrix} -1 \\ 0.08 \\ 0.86 \end{Bmatrix}$$

For fourth iteration

$$\begin{Bmatrix} -1 \\ 0.08 \\ 0.86 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ 0.08 \\ 0.86 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} -2.92 \\ 0.1344 \\ 2.72 \end{Bmatrix} = \frac{2.92m\omega^2}{3k}\begin{Bmatrix} -1 \\ 0.04 \\ 0.93 \end{Bmatrix}$$

Assume the configuration in third mode as, $x_1 = 1, x_2 = 2, x_3 = 3$

Then first iteration

$$\begin{Bmatrix} -1 \\ 3.14 \\ 4.00 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix}\begin{Bmatrix} 1 \\ 3.14 \\ 4.00 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{Bmatrix} 1.68 \\ -5.28 \\ 6.72 \end{Bmatrix} = \frac{1.68m\omega^2}{3k}\begin{Bmatrix} 1.0 \\ -3.143 \\ 4.0 \end{Bmatrix}$$

Hence, the third principal mode $\begin{bmatrix} 1.0 \\ -3.143 \\ 4.0 \end{bmatrix}$

The third natural frequency is given by $1.68\,\dfrac{m\omega^2}{3k} = 1,\ \omega^2 = \omega_3^2 = \dfrac{3}{1.68}\dfrac{k}{m} = 1.785\,\dfrac{k}{m}$

$$\therefore \qquad \omega_3 = 1.336\sqrt{\frac{k}{m}} \text{ rad/s}$$

Hence the natural frequencies are

$$\omega_1 = 0.458\sqrt{\frac{k}{m}} \text{ rad/s},\ \omega_2 = \sqrt{\frac{k}{m}} \text{ rad/s},\ \omega_3 = 1.336\sqrt{\frac{k}{m}} \text{ rad/s}$$

The principal modes are $\begin{Bmatrix} 1 \\ 3.158 \\ 3.99 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$ and $\begin{Bmatrix} 1.0 \\ -3.143 \\ 4.0 \end{Bmatrix}$

REVIEW QUESTIONS

(1) What are the different methods by which a vibrating system having several degrees of freedom can be analysed?

(2) Name the methods available for the frequency analysis of the systems with several degrees of freedom.

(3) State and prove Maxwell's reciprocal theorem.

(4) Explain orthogonality principle in case of a multi-degree-freedom system.

(5) Distinguish between flexibility influence coefficient and stiffness coefficient.

(6) Define stiffness influence coefficient as applicable to multi-degree-freedom vibrations.

(7) Explain briefly: (i) Newton's method (ii) Method of Influence coefficient (iii) Maxwell's reciprocal theorem with example.

(8) Write short notes on (i) Matrix iteration method, and (ii) Orthogonality principle as applied to multi-degree-freedom system.

(9) What is mode shape? How is it computed? Explain.

(10) State and prove the orthogonality principle in case of a multi-degree-freedom vibrating system.

PROBLEMS FOR PRACTICE

(1) Determine the natural frequency of the system by using influence coefficients of the system as shown in Fig. p.p-7.1.

Ans. $\omega_{n1} = 0.45\sqrt{\frac{k}{m}}$ rad/s, $\omega_{n2} = \sqrt{\frac{k}{m}}$ rad/s, $\omega_{n3} = 1.34\sqrt{\frac{k}{m}}$ rad/s.

(2) Determine the influence coefficient of the triple pendulum of lengths 'l_1', 'l_2' and 'l_3' and masses 'm_1', 'm_2' and 'm_3' as shown in Fig. p.p-7.2.

Ans. Influence coefficients are expressed in the matrix

$$[\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{3k} & \frac{1}{3k} & \frac{1}{3k} \\ \frac{1}{3k} & \frac{4}{3k} & \frac{4}{3k} \\ \frac{1}{3k} & \frac{4}{3k} & \frac{7}{3k} \end{bmatrix}.$$

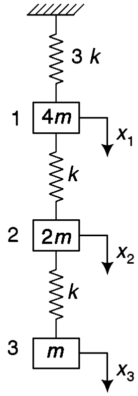


Fig. p.p-7.1

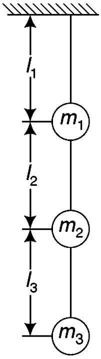


Fig. p.p-7.2

(3) Determine the natural frequency and mode shape of the system by using Newton's method as shown in Fig. p.p-7.3 as m_1 = 4 kg, m_2 = 2 kg, m_3 = 1 kg, k_1 = 300 N/m, k_2 = 100 N/m, k_3 = 100 N/m.

Ans. $\omega_{n1} = 4.6$ rad/s, $\omega_{n2} = 10$ rad/s, $\omega_{n3} = 13.4$ rad/s.

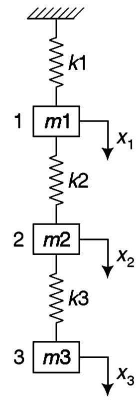


Fig. p.p-7.3

(4) Use matrix iteration method to determine the natural frequency of the system as shown in Fig. 7.4.

Ans. $\omega_{n1} = 0.8\sqrt{\dfrac{T}{ml}}$ rad/s, $\omega_{n2} = 1.142\sqrt{\dfrac{T}{ml}}$ rad/s,

$\omega_{n3} = 1.85\sqrt{\dfrac{T}{ml}}$ rad/s.

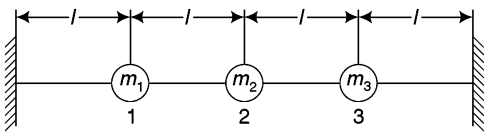


Fig. p.p-7.4

(5) Use matrix iteration method to determine the natural frequency of the system as shown in Fig. p.p-7.5.

Ans. $\omega_{n1} = 0.54\sqrt{\dfrac{k}{m}}$ rad/s, $\omega_{n2} = 1.12\sqrt{\dfrac{k}{m}}$ rad/s, $\omega_{n3} = 1.45\sqrt{\dfrac{k}{m}}$ rad/s.

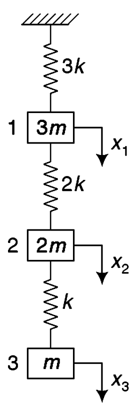


Fig. p.p-7.5

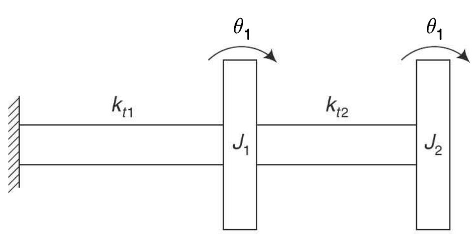


Fig. p.p-7.6

(6) Determine the natural frequencies of the two-degree torsional system by using matrix method as shown in Fig. p.p-7.6.

Ans. $\omega_{n1} = 0.54\sqrt{\dfrac{k_t}{J}}$ rad/s, $\omega_{n2} = 1.3\sqrt{\dfrac{k_t}{J}}$ rad/s.

(7) Determine the natural frequencies of the three-degree torsional system by using Newton's method as shown in Fig. p.p-7.7.

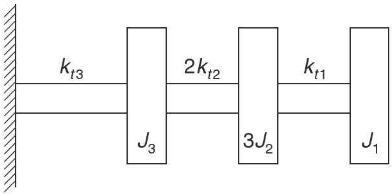


Fig. p.p-7.7

Ans. $\omega_{n1} = 0.38\sqrt{\frac{k_t}{J}}$ rad/s, $\omega_{n2} = 1.142\sqrt{\frac{k_t}{J}}$ rad/s, $\omega_{n3} = 1.884\sqrt{\frac{k_t}{J}}$ rad/s.

(8) Determine all the influence coefficients of a 3-degree freedom system as shown in Fig. p.p-7.8.and hence find the first natural frequency of the system by the method of matrix iteration.

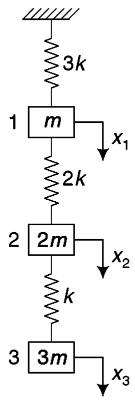


Fig. p.p-7.8

OBJECTIVE-TYPE QUESTIONS

(1) A shaft carrying three rotors will have
  (a) no node (b) three nodes
  (c) two nodes (d) one node

(2) The natural frequencies and mode shapes can be determined easily and quickly in case of multi-degree-freedom systems
  (a) by solving the number of equations easily
  (b) easily and quickly with the help of computers
  (c) by applying Newton's second law of motion only
  (d) none of the above cases

(3) Influence coefficient can be determine by using
  (a) Maxwell's reciprocal theorem
  (b) matrix iteration method
  (c) orthogonality principle
  (d) Rayleigh–Ritz method

(4) Matrix iteration methods are used to determine
  (a) analysis of problems in natural frequencies only
  (b) analysis of problems in structures, vibrations, fluid dynamics and design
  (c) large number of mathematical equations
  (d) all of the above cases

(5) Matrix iteration method is used in
  (a) determining amplitudes of second and third modes of vibration
  (b) determining the principle modes or normal modes of vibration
  (c) an iterative procedure to determine the principal modes of the system and its natural frequencies
  (d) an important property while finding the fundamental natural frequencies

(6) In a four-degree-freedom system, the eigenvalue will be
  (a) two eigenvalues
  (b) three eigenvalues
  (c) four eigenvalues
  (d) zero eigenvalue

(7) In eigenvalue problems
  (a) the eigenvector will represent the mode shape
  (b) the eigenvector will represent the natural frequency
  (c) the eigenvector will represent the mode shape as well as natural frequency
  (d) all of the above cases

(8) For a 3-degree freedom system, orthogonality principle can be written as
  (a) $m_1A_1A_2 + m_2B_1B_2 = 0$
  (b) $m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$
  (c) $m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$
  $m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$
  $m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0$
  (d) None of the above cases

Answers

(1) c	(2) b	(3) a	(4) b	(5) c	(6) c
(7) a	(8) c				

MULTI-DEGREE-FREEDOM SYSTEMS—NUMERICAL METHOD

8

In Chapter 7 analysis, we discussed the exact method for determining the natural frequencies, mode shapes, etc, in a multi-degree-freedom system. As the number of degrees of freedom increases, it becomes very tedious to solve the equations of motion and to determine the natural frequencies and mode shapes. The natural frequencies and mode shapes can be determined easily and quickly by using the following numerical methods in a multi-degree-freedom system:

Dunkerley's method, Rayleigh's method (energy method), Holzer's method, Stodola's method, Rayleigh–Ritz method and method of matrix iteration.

8.1 DUNKERLEY'S METHOD

This is an approximate equation and can be derived from the algebraic rules, i.e. if the coefficient of the highest term of the n^{th} degree freedom equation is unity, the coefficient of the second highest term equals the sum of the roots of the equations. Dunkerley's equation is capable of giving only the fundamental natural frequency.

Consider a three-degree-freedom system as shown in Fig. 8.1. The equation of motion in terms of influence coefficient may be written as

$$x_1 = a_{11} m_1 \omega^2 x_1 + a_{12} m_2 \omega^2 x_2 + a_{13} m_3 \omega^2 x_3$$
$$x_2 = a_{21} m_1 \omega^2 x_1 + a_{22} m_2 \omega^2 x_2 + a_{23} m_3 \omega^2 x_3$$
$$x_3 = a_{31} m_1 \omega^2 x_1 + a_{32} m_2 \omega^2 x_2 + a_{33} m_3 \omega^2 x_3$$

Rearranging,

$$x_1 [a_{11} m_1 \omega^2 - 1] + a_{12} m_2 \omega^2 x_2 + a_{13} m_3 \omega^2 x_3 = 0$$
$$a_{21} m_1 \omega^2 x_1 + x_2 [a_{22} m_2 \omega^2 - 1] + a_{23} m_3 \omega^2 x_3 = 0$$
$$a_{31} m_1 \omega^2 x_1 + a_{32} m_2 \omega^2 x_2 + x_3 [a_{33} m_3 \omega^2 - 1] = 0$$

Dividing the above equations by ω^2,

$$x_1 [a_{11} m_1 - 1/\omega^2] + a_{12} m_2 x_2 + a_{13} m_3 x_3 = 0$$
$$a_{21} m_1 x_1 + x_2 [a_{22} m_2 - 1/\omega^2] + a_{23} m_3 x_3 = 0$$
$$a_{31} m_1 x_1 + a_{32} m_2 x_2 + x_3 [a_{33} m_3 - 1/\omega^2] = 0$$

Writing the above equations in matrix form,

$$\begin{bmatrix} \left(a_{11}m_1 - \frac{1}{\omega^2}\right) & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & \left(a_{22}m_2 - \frac{1}{\omega^2}\right) & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & \left(a_{33}m_3 - \frac{1}{\omega^2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq 0$$

$$\therefore \quad \begin{vmatrix} \left(a_{11}m_1 - \frac{1}{\omega^2}\right) & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & \left(a_{22}m_2 - \frac{1}{\omega^2}\right) & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & \left(a_{33}m_3 - \frac{1}{\omega^2}\right) \end{vmatrix} = 0$$

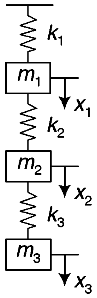


Fig. 8.1 *Dunkerley's method*

Expanding the determinant and rearranging,

$$\left(\frac{1}{\omega_2^2}\right)^3 - [a_{11}m_1 + a_{22}m_2 + a_{33}m_3]\left(\frac{1}{\omega^2}\right)^2 - [a_{12}m_2m_{21}m_1 + a_{13}m_3a_{31}m_1 + \ldots] = 0$$

By the algebraic rule mentioned above,

$$\left(\frac{1}{\omega^2}\right)^3 = (a_{11}m_1 + a_{22}m_2 + a_{33}m_3)\left(\frac{1}{\omega^2}\right)^2$$

or $$\frac{1}{\omega_{1n}^2} + \frac{1}{\omega_{2n}^2} + \frac{1}{\omega_{3n}^2} = a_{11}m_1 + a_{22}m_2 + a_{33}m_3$$

For $$\omega_{2n}, >> \omega_{1n}, \text{ i.e. } \frac{1}{\omega_{1n}} >> \frac{1}{\omega_{2n}} \quad \text{i.e. } \frac{1}{\omega_{2n}^2} \approx 0$$

For $$\omega_{3n}, >> \omega_{1n}, \quad \text{i.e. } \frac{1}{\omega_{1n}} >> \frac{1}{\omega_{3n}} \quad \text{i.e. } \frac{1}{\omega_{3n}^2} \approx 0$$

$$\frac{1}{\omega_{1n}^2} = a_{11}m_1 + a_{22}m_2 + a_{33}m_3 \text{ (approximately)}$$

Considering k_1m_1, k_2m_2 and k_3m_3 individually,

$$a_{11} = \frac{1}{k_1}\ a_{22} = \frac{1}{k_2}\ a_{33} = \frac{1}{k_3} \quad \therefore \frac{1}{\omega_{1n}^2} = \frac{m_1}{k_1} + \frac{m_2}{k_2} + \frac{m_3}{k_3} \text{ or } \frac{1}{\omega_{1n}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2}$$

where $$\omega_1^2 = \frac{k_1}{k_2},\ \omega_2^2 = \frac{k_2}{m_2},\ \omega_3^2 = \frac{k_3}{m_3}, \cdots$$

For an 'n' degree freedom system, it can be written as

$$\frac{1}{\omega_{1n}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \ldots \frac{1}{\omega_n^2}$$

This is the Dunkerley's equation.

8.2 RAYLEIGH'S METHOD (ENERGY METHOD)

Rayleigh's method has been used in analysing single-degree-of-freedom systems where the distributed mass was lumped up at places of known stiffness. This is a numerical method and is also used to determine the fundamental natural frequency of the multi-degree-freedom system. The very first trial gives fundamental natural frequency with sufficient accuracy. It is based upon equating the maximum kinetic energy of the vibrating system with the maximum potential energy of the system as in case of single degree of freedom systems. In case of a multi-degree-freedom system, there are many masses and thus many components of kinetic potential energies, but all the masses will have simple harmonic motions passing through their mean position at the same instant, for any principal mode of vibration.

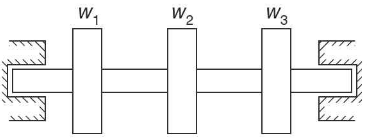


Fig. 8.2 *Rayleigh's method*

Let 'W_1', 'W_2' and 'W_3' be the weights acting on the beam as shown in Fig. 8.2 above, 'y_1,' 'y'_2 and 'y'_3 be the maximum deflections under the weights 'W_1', 'W_2' and 'W_3' respectively.

$$\therefore \text{ (Kinetic energy)} = \frac{1}{2}\frac{W_1}{g}\dot{y}_1^2 + \frac{1}{2}\frac{W_2}{g}\dot{y}_2^2 + \frac{1}{2}\frac{W_3}{g}\dot{y}_3^2$$

$$\therefore \quad \text{(KE)} = \frac{1}{2g}[W_1\dot{y}_1^2 + W_2\dot{y}_2^2 + W_3\dot{y}_3^2] \quad \text{...8.1}$$

$$\therefore \text{ (Potential Energy)} = \frac{1}{2}\left[\frac{W_1}{g}g\cdot y_1 + \frac{W_2}{g}g\cdot y_2 + \frac{W_3}{g}g\cdot y_3\right]$$

$$\therefore \text{ (Potential Energy)} = \frac{1}{2}[w_1\cdot y_1 + W_2\cdot y_2 + W_3\cdot y_3] \quad \text{...8.2}$$

By Rayleigh's principle,

$$\text{(KE)}_{\max} = \text{(PE)}_{\max}$$

Let $\quad y_i = Y_i \sin \omega t, \dot{y}_i = Y_i\omega \cos \omega t.$

For $(\text{KE})_{\max}$ $(\dot{y}_i)_{\max}$ and for $(\text{PE})_{\max}$, $(y_i)_{\max}$

$$(\dot{y}_i)_{\max} = Y_i\omega \quad \therefore (y_i)_{\max} = Y_i$$

Using these values in the above two equations,

$$\text{(KE)}_{\max} = \frac{1}{2g}[W_1(Y_1\omega)^2 + W_2(Y_2\omega)^2 + W_3(y_3\omega)^2]$$

$$\text{(KE)}_{\max} = \frac{\omega^2}{2g}\left[W_1Y_1^2 + W_2Y_2^2 + W_3Y_3^2\right] = \frac{\omega^2}{2g}\sum_{i=1}^{n} W_iY_i^2$$

$$\therefore \text{ for '}n\text{' number of weights on the beam } \text{(KE)}_{\max} = \frac{\omega^2}{2g}\sum_{i=1}^{n} W_1Y_i^2 \quad \text{...8.3}$$

$$\text{(PE)}_{\max} = \frac{1}{2}[W_1Y_1 + W_2Y_2 + W_3Y_3]$$

$\frac{1}{2}\sum_{i=1}^{n} W_i Y_i$, for '$n$' number of weights on the beam, $(PE)_{max} = \frac{1}{2}\sum_{i=1}^{n} W_i Y_i$ ...8.4

From Eqs. 8.3 and 8.4,

$$\frac{\omega^2}{2g}\sum_{i=1}^{n} W_i Y_i^2 = \frac{1}{2}\sum_{i=1}^{n} W_i Y_i, \quad \omega^2 = \frac{g\sum_{i=1}^{n} W_i Y_i}{\sum_{i=1}^{n} W_i Y_i^2}$$

or $$\omega_n = \sqrt{\frac{g\sum_{i=1}^{n} W_i Y_i}{\sum_{i=1}^{n} W_i Y_i^2}} \text{ rad/s} \quad ...8.5$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g\sum_{i=1}^{n} W_i Y_i}{\sum_{i=1}^{n} W_i Y_i^2}} \text{ Hz}$$

In Eq. 8.5, 'W' is the weight of the mass and 'y' is the deflection of the shaft under the weight as shown in Fig. 8.3(a). These deflections can be found out by the formula of mechanics of materials. In a majority of the cases the shaft is considered to be a simply supported one.

$\therefore$ net deflections are given by $y_1 = y_{11} + y_{12} + y_{13}$, $y_2 = y_{21} + y_{22} + y_{23}$, $y_3 = y_{31} + y_{32} + y_{33}$ in Fig. 8.3(b),(c) and (d).

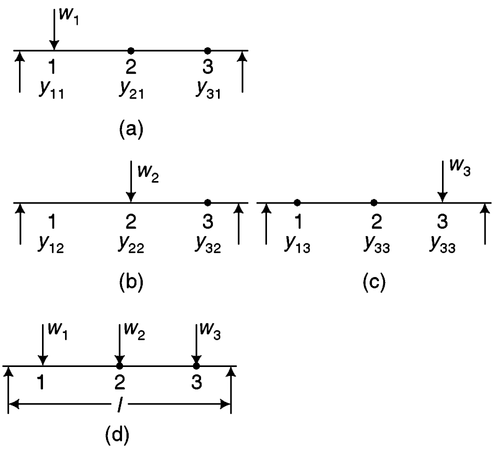


Fig. 8.3 *Net deflections*

These values can be found out by the formula. $y_x = \omega ax \frac{(l^2 - a^2 - x^2)}{6EI}$ for $x \leq (l - a)$.

On the other hand, a more accurate value of the natural frequency could be obtained by considering the deflection curve due to inertia loading, this can be obtained from the frequency calculated in equation 8.5. This method converges very fast and a very accurate value of the natural frequency can be obtained. Also for most practical purposes the natural frequency as obtained from Eq. 8.5 serves the purpose.

Equation 8.5 is also generalised to include the distributed masses as in case of beams with some assumption of a reasonable deflection curve.

Let us consider a uniform beam. Let '*l*' be the length of the beam, '*m*' be the mass per unit length. The maximum Potential Energy (PE) of the beam in bending is given by

$$\text{PE} = \frac{1}{2}\int_0^1 M d\theta \qquad ...8.6$$

where '*M*' is the bending moment and '$d\theta$' is the change in slope over a distance '*dx*'.

We know that from bending moment, the bending equation $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$ or it can be rewrite and apply the beam theory as $\frac{M}{EI} = \frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$.

Substituting from the above relation in Eq. 8.6, we get

$$\text{PE} = \frac{1}{2}\text{EI}\int_0^1 \left(\frac{dy^2}{dx^2}\right)^2 dx \qquad ...8.7$$

The maximum kinetic energy (KE) of the beam due to the mass of the beam is given by

$$\text{KE} = \frac{1}{2}\int_0^1 m(\omega_n y)^2\, dx \qquad ...8.8$$

where 'ω_n' is the natural frequency of the system corresponding to the assumed deflection curve '*y*'. Now equating the maximum potential energy (PE) and the maximum kinetic energy (KE) of the equations 8.7 and 8.8, we get,

$$\omega_n^2 = \frac{\text{EI}}{m} \frac{\int_0^1 \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^1 y^2 dx}. \qquad ...8.9$$

8.2 HOLZER'S METHOD

Holzer's method is a trial-and-error or tabular method used for the determination of natural frequency for free or forced vibration, with or without damping free-free, fixed-fixed, fixed-free and branched system, including linear and angular displacement. It is based on successive assumption of the natural frequency of the system, each followed by the calculation of the configuration governed by that assumed frequency. It can be used to compute all the natural frequencies of a system. Holzer's method is particularly useful for calculating the frequencies of torsional vibrations in shaft.

There are three different cases for analyzing the given problem.

- **Case 1** When both ends of the system are free (free-free system or semi-definite system)

- **Case 2** When one end is fixed and other end is free (fixed-free system)
- **Case 3** When both ends are fixed (fixed-fixed system). All three cases are discussed as follows by taking an example.

Case (i) When both ends are free [free-free system or semidefinite system]

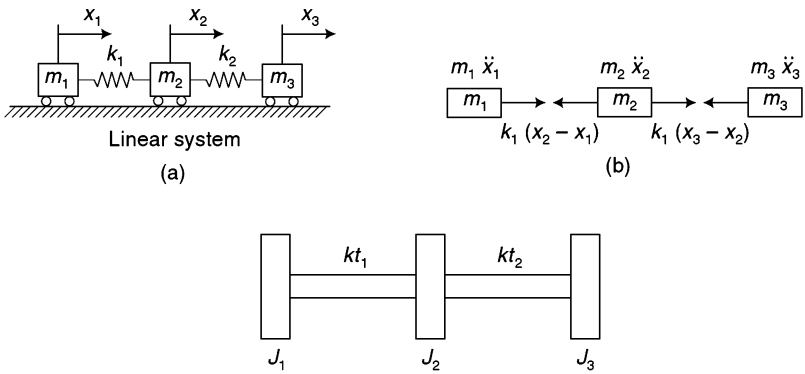


Fig. 8.4 *Holzer's method: Free-Free system*

Let us considering the system to represent masses 'm'$_1$ 'm'$_2$ and 'm_3' with a coupling spring 'k_1' and 'k_2'as shown in Fig. 8.4(a).

Now at any instant give displacement 'x_1' to the mass 'm_1', 'x_2' to the mass 'm_2'and 'x_3'to the mass 'm_3' as shown in Fig. 8.4(a). The FBD is as shown in Fig. 8.4(b).

Assuming that $x_2 > x_1$ or $x_1 > x_2$ also can be taken, but $x_2 > x_1$ is easy to writing down the differential equations.

Apply Newton's second law of motion to the mass 'm_1', i.e. $\Sigma F = ma$.

Let us assume that $x_3 > x_2 > x_1$ and the FBD is as shown in Fig. 8.4(b)

to writing down the differential equations of motion:

$$\therefore \quad k_1(x_2 - x_1) = m_1\ddot{x}_1,\ m_1\ddot{x}_1 - k_1(x_2 - x_1) = 0 \quad ...8.10$$

Similarly, applying Newton's second law of motion to mass 'm_2' , $\Sigma F = m_2\ddot{x}_2 - k_1 (x_2 - x_1) + k_2(x_3 - x_2) = m_2\ddot{x}_2,\quad m_2\ddot{x}_2 - k_2(x_3 - x_2) + k_1(x_2 - x_1) = 0$ ...8.11

Similarly, applying Newton's second law of motion to mass 'm_3'

$$\Sigma F = m_3\ddot{x}_3 - k_2(x_3 - x_2) = m_3\ddot{x}_3,\ m_3\ddot{x}_3 + k_2(x_3 - x_2) = 0 \quad ...8.12$$

Assume that motion is periodic and is composed of harmonic motions of varies amplitudes and frequencies. Let one of these components be,

$$x_1 = X_1 \sin \omega t,\ x_2 = X_2 \sin \omega t,\ x_3 = X_3 \sin \omega t$$

$$\ddot{x}_1 = -X_1\omega^2 \sin \omega t,\ \ \ddot{x}_2 = -X_2\omega^2 \sin \omega t,\ \ \ddot{x}_3 = -X_3\omega^2 \sin \omega t$$

Using these values in equations 8.10, 8.11 and 8.12,

$$-m_1X_1\omega^2 - k_1(X_2 - X_1) = 0,\ -m_1X_1\omega^2 - k_1X_2 + k_1X_2 = 0$$

$$k_1X_2 + X_1[m_1\omega^2 - k_1] = 0,\ X_2 = \frac{-X_1[m_1\omega^2 - k_1]}{k_1} = X_1 - \frac{m_1X_1\omega^2}{k_1} \quad ...8.13$$

Using Eq. 8.10 in Eq. 8.11,

$$m_2\ddot{x}_2 - k_2(x_3 - x_2) + m_1\ddot{x}_1 = 0,\ m_1\ddot{x}_1 + m_2\ddot{x}_2 + k_2x_3 + k_2x_2 = 0$$

Using the values of $\ddot{x}_1$, $\ddot{x}_2$ and x_3 x_2,

$$-m_1X_1\omega^2 - m_2X_2\omega^2 - k_2X_3 + k_2X_2 = 0,\ k_2X_3 = k_2X_2 - [m_1X_1\omega^2 + m_2X_2\omega^2]$$

$$X_3 = X_2 - \frac{[m_1X_1\omega^2 + m_2X_2\omega^2]}{k_2} \quad ...8.14$$

From equations 8.13 and 8.14, it can be generalised that

$$X_n = X_{n-1} - \sum_{p=1}^{n-1} \frac{m_pX_p\omega^2}{k_{n-1}} \text{ where } n = 2, 3, 4, 5, \ldots\ldots$$

where 'X', 'ω', 'm' and 'k' are the displacement, natural frequency, mass and spring constant of the given system respectively. For an assumed value of 'ω' being the process by assuming unit amplitude of vibration for the first mass. The amplitude and inertia force for all the remaining masses are then calculated. For the last mass of the system, its amplitude of vibration should be zero for fixed ends. For the last mass of the system, the total inertia force is zero for free ends.

The remaining values (amplitude or inertia forces) for each of the assumed frequencies are then plotted against the assumed value of natural frequency to give the true frequencies of the system.

Similarly for a torsional system, $\phi_n = \phi_{n-1} - \sum_{p=1}^{n-1} \frac{I_p\theta_p\omega^2}{k_{tn-1}}$ where $n = 2, 3, 4, 5, \ldots..$

Using Eq. 8.10 in Eq. 8.11, $m_2\ddot{x}_2 - k_2(x_3 - x_2) + m_1\ddot{x}_1 = 0$

or $$+ k_2(x_3 - x_2) = + [m_1\ddot{x}_1 + m_2\ddot{x}_2]$$

Using these values in Eq. 8.12,

$$m_1\ddot{x}_1 + m_3\ddot{x}_3 + m_2\ddot{x}_2 = 0,\ m_1X_1\omega^2 + m_2X_2\omega^2 + m_3X_3\omega^2 = 0$$

or in general, $\sum_{p=1}^{n} m_pX_p\omega^2 = 0$ where 'n' is the number of degrees of freedom of the given system.

Similarly, for a torsional system $\sum_{p=1}^{n} I_p\theta_p\omega^2 = 0$.

This condition is to be satisfied during iteration.

The graph for a free-free system is as shown in Fig. 8.4(c).

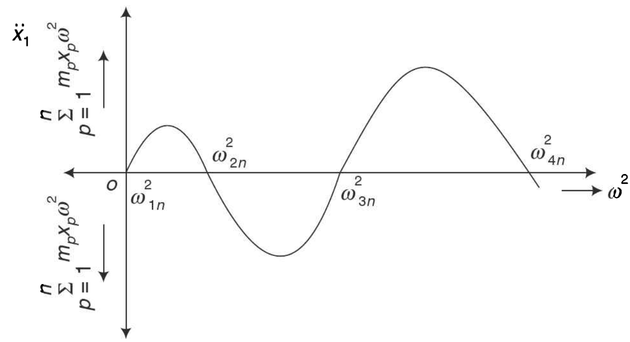


Fig. 8.4(c) *(Contd.) Graph of frequencies (ω) versus displacement (x_i or θ_i)*

Note: In graph for free-free system as shown in Fig. 8.4(c) it is seen that one of the natural frequencies becomes zero ($\omega_1 = 0$), this indicates that the system should be a semidefinite system.

Case (ii) When one end is fixed and the other end is free (fixed-free system) as shown in Fig. 8.5(a) and (b)

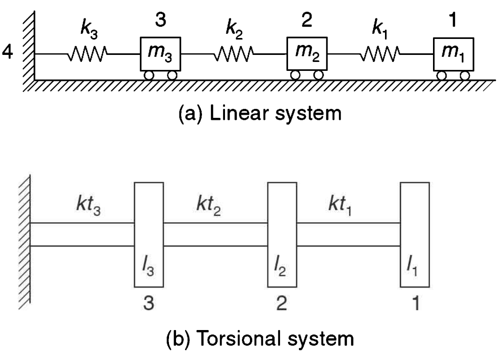


Fig. 8.5 *Fixed-free system*

The condition to be satisfied to get the natural frequencies is that

For a linear system $X_{n+1} = 0$

For a torsional system $\theta_{n+1} = 0$ (where n = number of degrees of freedom).

At the fixed point the amplitude may be found out by using the formula for linear system, $X_n = X_{n-1} - \sum_{p=1}^{n-1} \frac{m_p X_p \omega^2}{k_{(n-1)}}$ and for torsional system, $\theta_n = \theta_{n-1} - \sum_{p=1}^{n-1} \frac{I_p \theta_p \omega^2}{kt_{(n-1)}}$

The graph for fixed-free system is as shown in Fig. p-8.5(c).

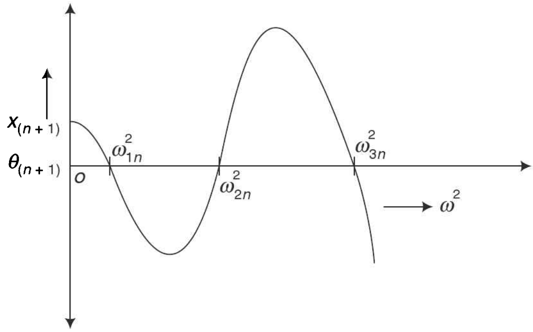
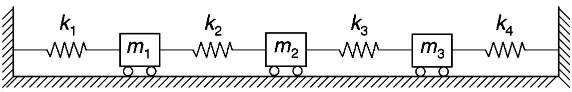


Fig. 8.5(c) *(Contd.) Graph of frequencies (ω) versus displacement (x_i or θ_i)*



(a)

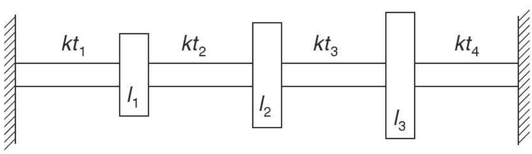


Fig. 8.6 *Fixed-fixed system*

Case (iii) When both ends are fixed (fixed-fixed system) as shown in Fig. 8.6 (a) and (b)

The condition to be satisfied to get the natural frequencies is that

$X_{n+1} = 0$ for a linear system

$\theta_{n+1} = 0$ for torsional system

where 'n' is the number of degrees of freedom.

The amplitude is obtained by the formula

$$X_n = X_{n-1} + \left\{ \frac{k_1 X_1 - \sum_{P=1}^{n-1} m_p X_p \omega^2}{k_n} \right\} \text{ for a linear system}$$

$$\theta_n = \theta_{n-1} + \left\{ \frac{k_{t1} \theta_1 - \sum_{P=1}^{n-1} I_p \theta_p \omega^2}{k_{tn}} \right\} \text{ for torsional system}$$

The graph for fixed-fixed system is as shown in Fig. 8.6(c).

Note: The tabular column for all three cases remains common.

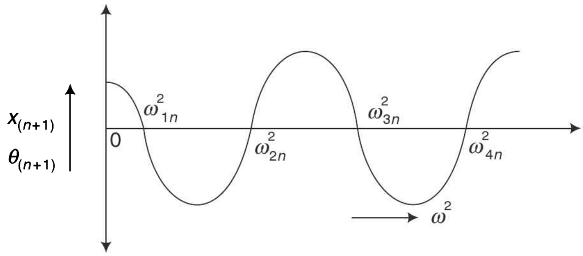


Fig. 8.6(c) *(Contd.) Graph of frequencies (ω) versus displacement (x_i or θ_i)*

8.4 STODOLA'S METHOD

Stodola's method is an iterative process used for the calculation of the principal modes and fundamental natural frequency of free undamped vibrating systems. It is a physical approach and there is no need to derive the differential equation of motion which starts of with a set of assumed deflections.

The procedure here is to assume any configuration for the principal mode. Find out the corresponding inertia forces and spring forces and from the latter, the spring deflections. The next tabulation is done with the resulting deflections. The procedure is continued until these two coincide giving the required principal-mode configuration which yields the natural frequency.

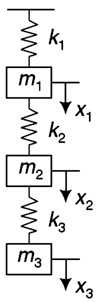


Fig. 8.7 *Stodola's method*

Unlike Holzer's method, Stodola's method yields only the fundamental mode of vibration and hence the first natural frequency. This method is usually applicable for fixed-free system.

Table 8.1 *General table for linear system*

Sl. No.	Details	k_1	m_1	k_2	m_2	k_3	m_3
1	Assumed deflection 'x'		x_1		x_2		x_3
2	Inertia force '$m\omega^2 x$'		$m_1x_1\omega^2$		$m_2x_2\omega^2$		$m_3x_3\omega^2$
3	Spring force 's'	$s_1 = m_1x_1\omega^2 + m_2x_2\omega^2 + m_3x_3\omega^2$		$s_2 = m_2x_2\omega^2 + m_3x_3\omega^2$		$s_3 = m_3x_3\omega^2$	
4	Spring deflection δ	$\delta 1 = \frac{S_1}{k_1}$		$\delta_2 = \frac{S_2}{k_2}$		$\delta_3 = \frac{S_3}{k_3}$	
5	Calculated deflection		δ_1		$\delta_1 + \delta_2$		$\delta_1 + \delta_2 + \delta_3$
6	If $\delta_1 = x_1$		$\frac{\delta_1}{\delta_1}$		$\frac{\delta_1 + \delta_2}{\delta_1}$		$\frac{\delta_1 + \delta_2 + \delta_3}{\delta_1}$

8.4 RAYLEIGH–RITZ METHOD

The Rayleigh –Ritz method is a modification of Rayleigh's energy method, by taking more than one admissible function simultaneously and minimising the total potential energy. It is based on the premise that a closer approximation to the exact natural mode can be obtained by superposing a number of assumed functions than by using a single assumed functions, as in case of Rayleigh's energy method. If the assumed functions are suitably chosen this method provides not only the approximate value of the fundamental frequency but also the approximate values of the higher natural frequencies and mode shapes. In this method the number of frequencies calculated is equal to the number of arbitrary functions used.

We know that, for lateral vibrations of beam expression derived from Eq. 8.9,

$$\omega_n^2 = \frac{EI}{m} \frac{\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l y^2 dx} \qquad \ldots 8.15$$

In the above expression 'EI' and 'm' (mass per unit length) were treated as a constant also taken out of the integral signs.

Then the equation becomes PE $= \frac{1}{2} EI \int \left(\frac{d^2y}{dx^2}\right)^2 dx$

or otherwise if 'EI' and 'm' are not treated as a constant then Eq. 8.15 becomes

$$\omega_n^2 = \frac{\int_0^l EI \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l my^2 dx} \qquad \ldots 8.16$$

Here, there will be two cases:

Case (i) Suppose in case the fundamental deflection curve 'y_1' is known then substituting these values in the expression 8.16, we get the value of 'ω_{n1}^2' where 'ω_{n1}' is the fundamental natural frequency of the system.

If a deflection curve 'y_1' is known and then substituting these values in the above expression 8.16, the resulting value will be that of 'ω_{n1}^2' corresponding to 'i^{th}' mode of vibration.

Case (ii) In this case, once again these will be subdivided into two cases, dependent variable 'ω_n' as a function of independent variable of deflection curve 'y'. Suppose in case if deflection curve 'y_1' is not known exactly and an approximate value that satisfies the boundary conditions is taken and subjected in the expression 8.16, the resulting value of 'ω_n' will always be higher than 'ω_{n1}' the true fundamental natural frequency. The expression 8.16 can therefore be considered to define 'ω_n' (dependent variable) as a function of deflection curve 'y' (independent variable). Thus, $\omega_n = \omega_n$ (y) is the relation which satisfies the required boundary conditions. The minimum of this function is equal to the fundamental natural frequency 'ω_{n1}' and the remaining corresponding value at which this function is attained is the deflection curve corresponding to the fundamental mode of vibration. This is some explanation about the Rayleigh–Ritz method regarding fundamental natural frequency, fundamental mode and the deflection curve corresponding to the fundamental mode of vibration of beams.

Now we come to the producer and calculation part of the lateral vibration of beam as follows. Let us consider $f_1(x), f_2(x), f_3(x)\ldots f_i(x)$ be the series of given functions satisfying the boundary conditions. The values $\omega_n(f_1)$, $\omega_n(f_2)$ $\omega_n(f_3)\ldots\omega_n(f_i)$ obtained from Eq. 8.16 are all greater than ω_{n1}.

A linear combination of the function f_i that is

$$A_1 f_1(x) + A_2 f_2(x) + A_3 f_3(x) \cdots + A_i f_i(x) \qquad ...8.17$$

where $f_1(x) + f_2(x) + f_3(x) + \cdots + f_i(x)$ are admissible functions.

[In case of transverse vibration of beams, if 'i' functions are chosen for approximating the deflection then calculate KE (T), PE(v) and apply the minimisation principle. This leads to lengthy calculations but leads to values closer to the exact values.]

Also, this satisfies the boundary conditions and also respective values of natural frequency 'ω_n' that is obtained by Eq. 8.16. Then

$$\omega_n [A_1 f_1(x) + A_2 f_2(x) + A_3 f_3(x) + \cdots + A_i f_i] \qquad ...8.18$$

This value will also be greater than 'ω_{n1}' for all other possible combinations of the coefficients of A's. Proper combinations of A's will give the lowest value of the expression 8.18 This will be still higher than 'ω_{n1}'.

To find the proper combination of the coefficients A_1, A_2, A_3,....A_i substitute the expression 8.17 in terms of A's for y, into Eq. 8.16 and after performing the integrating, we get

$$\omega_n^2 = F(A_1, A_2, A_3, \ldots A_i) \qquad ...8.19$$

Here the right-hand side is a function of A_1, A_2, A_3,......A_i.

For minimum value of ω_n^2, we must have

$$\frac{\partial \omega_n^2}{\partial A_1} = 0 \quad \frac{\partial \omega_n^2}{\partial A_2} = 0 \quad \frac{\partial \omega_n^2}{\partial A_3} = 0 \cdots \frac{\partial \omega_n^2}{\partial A_i} = 0 \qquad ...8.20$$

Since the right-hand side of the expression 8.16 is a function of A_1, A_2, A_3,...A_i it is a partial derivatives with respect to A_1, A_2, A_3,...A_i all will be zero when the value of the derivative is ω_n^2.

When the setting of its derivatives equal to zero, we have

$$\left[\int_0 my^2 dx\right] \frac{\partial}{\partial A_1}\left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] - \left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] \frac{\partial}{\partial A_1}\left[\int_0^l my^2 dx\right] = 0$$

$$(i = 1, 2, 3, \ldots i) \qquad ...8.21$$

But from the expression 8.16,

$$\left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] = \omega_n^2\left[\int_0^l my^2dx\right]$$

Therefore, Eq. 8.21 simplifies to

$$\frac{\partial}{\partial A_i}\left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] - \omega_n^2\frac{\partial}{\partial A_i}\left[\int_0^l my^2dx\right] = 0 \quad (i = 1, 2, 3,\ldots i) \qquad \ldots 8.22$$

Equations 8.22 are a set of 'i' equations which are linear homogeneous in 'i' unknown $A_1, A_2, A_3,\ldots A_i$. These equations will have a non trivial solution if the determinant of its coefficients is equal to zero.

That leads to an i^{th} degree equation in terms of natural frequency of ω_n^2, out of which the lowest root of which is a good approximation to ω_{n1}^2 On the other hand, the next larger root is a somewhat poor approximation to ω_{n2}^2. In most of the practical cases, only two terms in the expression 8.17 give a very better approximation for the fundamental natural frequency. Although with increasing number of terms, the lower root converges to ω_{n1}^2. and the further next higher root converges to ω_{n2}^2 although not so fast.

The functions selected in expression 8.17 should be simple enough to facilitate the integration process, and they must also satisfy the boundary conditions.

8.6 METHOD OF MATRIX ITERATION

The matrix iteration method is an iterative procedure used, instead of solving the characteristic equation. The matrix iteration method is one of the most commonly used methods amongst iterative methods for determining the eigenvalues (natural frequency) and eigenvectors (mode shape). Also matrix iteration method in the analysis of problems in structures, vibrations, fluid dynamics and design are becoming more and more popular with the advent of high speed and large memory digital computers. In this method of multiplication, inversion and iteration of large size matrices can be done so easily.

With the use of **flexibility matrix** in case of a differential equation, matrix iteration method is used when only the lowest eigenvalue and eigenvectors of a multi-degree-freedom system are desired.

The advantages of this method are (i) the iterative process here results in the principal mode of vibration of the system and corresponding natural frequency simultaneously, and (ii) rather than by separating operation as in the case of polynomial method explained earlier.

The equation of motion in terms of flexibility matrix can be written as we know in earlier the equation,

i.e. $$[A]\,[M] + \{\ddot{x}\} = \{0\} \qquad \ldots 8.23$$

Using principal mode of vibration differentiating twice $\{x\} = \{X\} \sin \omega t$, Eq. 8.23 becomes,

$$\{X\} = \omega^2 [A] [M]\{x\} \qquad ...8.24$$

Equation 8.24 can be written as $\{X\} = \omega^2[B] \{X\}$ ...8.25

where $[B] = [A] [M]$

Equation 8.25 is in the form of a matrix as follows:

$$\begin{Bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{Bmatrix} = \omega^2 \begin{Bmatrix} b_{11} & b_{12} & & & b_{in} \\ b_{22} & b_{22} & \cdots & \vdots & b_{2n} \\ \cdots & \cdots & & \vdots & \cdots \\ b_{n1} & b_{n_2} & & \cdots & b_{nn} \end{Bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{Bmatrix} \qquad ...8.26$$

Let us start iterative process by estimating a set of deflections for the right column of the above matrix equation 8.26 and then expanding the right-hand side which gives results in a column of numbers and it can be normalised. After the procedure is repeated with the normalised column itself as the new estimate and the procedure is continued till the first mode repeats itself.

The iteration process with the use of matrix equation 8.26, as explained above, converges to the **lowest value** of natural frequency 'ω^2' such that the fundamental mode of vibration is required. Further, the orthogonality principle is applied to get a modified matrix equation for the next higher modes and the natural frequency, that does not contain the **lower modes**. The same iteration process is repeated as before.

In case the matrix equation is written in terms of the **stiffness matrix** than the **flexibility matrix,** we know the equation

$$[M] \{\ddot{x}\} + [K]\{x\} = \{0\} \qquad ...8.27$$

Then multiplying by $[M]^{-1}$, as done earlier, results into

$$[I] \{\ddot{x}\} = [C] \{x\} = 0 \qquad ...8.28$$

where $[D] = [M]^{-1}[K]$

Using principal mode of vibration $\{x\} = \{X\} \sin \omega t$, differentiating twice and substituting in Eq. 8.28, we get

$$-\omega2\{X\} + [C]\{x\} = 0 \quad \text{or} \quad \{x\} = \left(\frac{1}{\omega^2}\right)[C] \{X\} \qquad ...8.29$$

The above equation is in the form of

$$\begin{Bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{Bmatrix} = \frac{1}{\omega^2} \begin{Bmatrix} c_{11} & c_{12} & & \cdots & c_{in} \\ c_{21} & c_{22} & & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n_2} & . & \cdots & c_{nn} \end{Bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{Bmatrix} \qquad ...8.30$$

The iterative process as explained earlier, is started and this converges to the lowest value of $(1/\omega^2)$ such that the highest mode of vibration is obtained [Comparing equations 8.26 and 8.30].

For next lower modes and the natural frequency, the orthogonality principle is applied to obtain a modified matrix equation that will not contain the **higher modes,** and iterative process is repeated as usual.

EXAMPLE 8.1

A single-degree-of-freedom spring-mass system has a natural frequency of 10 cyc/s. Another single degree-of-freedom spring-mass system is attached to it. The latter had a natural frequency of 20 cyc/s. What is the approximate fundamental frequency of the composite system? Use Dunkerley's method.

Solution Given: $f_1 = 10$ cps, $f_2 = 20$ cps

By Dunkerley's equation, the approximate fundamental natural frequency is given by

$$\frac{1}{\omega_{1n}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}, \quad \text{or} \quad \frac{1}{f_{1n}^2} = \frac{1}{f_1^2} + \frac{1}{f_2^2}, = \frac{1}{10^2} + \frac{1}{20^2}$$

$$f_{1n}^2 = 80 \quad \text{or} \quad f_{1n} = 8.94 \text{ cps or Hz}$$

$\therefore$ the fundamental natural frequency is $f_{1n} = 8.94$ cps or Hz.

EXAMPLE 8.2

Determine the fundamental natural frequency for a three-rotor torsional system as shown in Fig. p-8.2. using Dunkerley's method.

Solution $r_1 = 10$ cm, $k_{t1} = 500$ N-m/rad, $W_1 = 10$ kg

$r_2 = 5$ cm, $k_{t2} = 1000$ N-m/rad, $W_2 = 40$ kg

$r_3 = 10$ cm, $k_{t3} = 800$ N-m/rad, $W_3 = 20$ kg

By Dunkerley's equation, the approximate fundamental natural frequency is given by

$$\frac{1}{\omega_{1n}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \cdots \frac{1}{\omega_n^2}$$

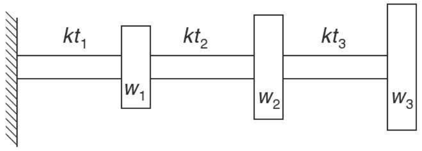


Fig. p-8.2 *Three-rotor torsional system*

For a torsional system, $\omega = \sqrt{\dfrac{k_t}{I}}$ $\quad \therefore \dfrac{1}{\omega_{1n}^2} = \dfrac{I_1}{k_{t1}} + \dfrac{I_2}{k_{t2}} + \dfrac{I_3}{k_{t3}}$

$$I_0 = \frac{mr^2}{2}$$

$$\therefore \quad I_1 = \frac{W_1 r_1^2}{2} = \frac{10 \times (0.1)^2}{2} = 0.05 \text{ kg-m}^2$$

$$I_2 = \frac{W_2 r_2^2}{2} = \frac{40 \times (0.05)^2}{2} = 0.05 \text{ kg-m}^2, \; I_3 = \frac{W_3 r_3^2}{2} = \frac{20 \times (0.1)^2}{2} = 0.1 \text{ kg-m}^2$$

$$\therefore \quad \frac{1}{\omega_{1n}^2} = \frac{0.05}{500} + \frac{0.05}{1000} + \frac{0.1}{800}$$

$$\therefore \quad \omega_{1n}^2 = 3636.36, \; \omega_{1n} = 60.30 \text{ rad/s.}$$

EXAMPLE 8.3

Find the natural frequency of the spring-mass system as shown in Fig. p-8.3 using Dunkerley's method. Take, $k_1 = k_2 = k_3 = 1$ kgf/cm and $m_1 = m_2 = m_3 = 1$ kgf/s^2/cm.

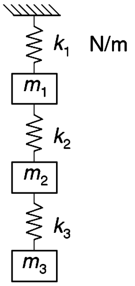


Fig. p-8.3 *Spring-mass system*

Solution $k_1 = k_2 = k_3 = 1 \text{ kgf/cm} = 1 \times 9.81 \times 100 \text{ N/m}$

$$m_1 = m_2 = m_3 = 1 \text{ kgf/s}^2\text{/cm} = 100 \text{ kgf/s}^2\text{/m}$$

$$= 100 \times 9.81 \text{ kg}, = 981 \text{ kg}$$

The Dunkerley's equation is given by

$$\frac{1}{\omega_{1n}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \cdots \frac{1}{\omega_n^2}$$

where $\omega_1^2 = \frac{k_1}{m_1}$, $\omega_2^2 = \frac{k_2}{m_2}$ and $\omega_3^2 = \frac{k_3}{m_3}$

$$\therefore \quad \frac{1}{\omega_{1n}^2} = \frac{m_1}{k_1} + \frac{m_2}{k_2} + \frac{m_3}{k_3}$$

$$\frac{1}{\omega_{1n}^2} = \frac{981}{981} + \frac{981}{981} + \frac{981}{981},$$

$$\frac{1}{\omega_{1n}^2} = 3 \quad \omega_{1n} = 0.58 \text{ rad/s},$$

$$f_{1n} = 0.092 \text{ Hz or cps.}$$

EXAMPLE 8.4

Find the natural frequency of vibration for the system as shown in Fig. p-8.4 by using Dunkerley's method. Take $E = 1.96 \times 10^{11}$ N/m^2 and $I = 4 \times 10^{-7}$ m^4.

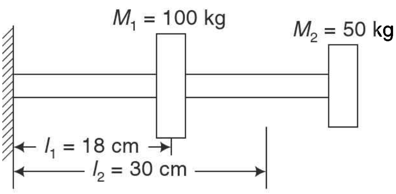


Fig. p-8.4 *Torsional system*

Solution $$y_1 = \frac{\omega_1 x^2}{6\,EI}(3a - x) = \frac{100 \times 9.81 \times 0.18^2}{6 \times 1.96 \times 10^{11} \times 4 \times 10^{-7}}(54 - 18) = 0.000024 \text{ m}$$

$$y_2 = \frac{\omega_2 x^2}{6\,EI}(3L - x) = \frac{50 \times 9.81 \times 0.30^2}{6 \times 1.96 \times 10^{11} \times 4 \times 10^{-7}} = 0.0000563 \text{ m}$$

$$\omega_n = \sqrt{\frac{g}{\delta}}, \text{ or } \omega_1 = \sqrt{\frac{g}{y_1}} = 639.3 \text{ rad/s}, \quad \omega_2 = \sqrt{\frac{g}{y_2}} = 417.43 \text{ rad/s}$$

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \frac{1}{639.3} + \frac{1}{417.43^2} \therefore \omega_n = 349.5 \text{ rad/s.}$$

EXAMPLE 8.5

A shaft of negligible weight, 6 cm diameter and 5 m length, is simply supported at the ends and carries four weights of 50 kg each at equal distance over the length of the shaft. Find the frequency of vibration by using Dunkerley's method. Take $E = 2 \times 10^6$ kg/s^2.

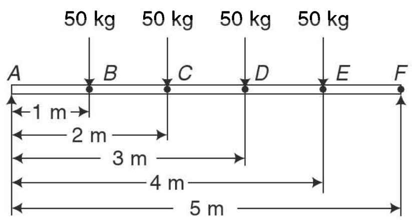


Fig. p-8.5 *Shaft*

Solution The statement of problem is as shown in Fig. p-8.5.

w = 50 kg each point, l_B = 1.0 m, l_C = 2.0 m, l_D = 3.0 m,

l_E = 4.0 m from left support 'A'

Total length 'l' = 5 m, MI of shaft $I = \dfrac{\pi}{64} d^4 = \dfrac{\pi}{64} \times 6^4 = 63.585 \text{ cm}^4$

The general expression for static deflection by Dunkerley's method because of point load 'W' is given by $y = \dfrac{W l_1^2 l_2^2}{3EIl}$

So static deflection at the point B

$$y_B = \frac{50 \times 100^2 \times 400^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.42 \text{ cm} = 4.2 \text{ mm}$$

$$y_c = \frac{50 \times 200^2 \times 300^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.95 \text{ cm} = 9.5 \text{ mm}$$

$$y_D = \frac{50 \times 300^2 \times 200^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.95 \text{ cm} = 9.5 \text{ mm}$$

$$y_E = \frac{50 \times 400^2 \times 100^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.42 \text{ cm} = 4.2 \text{ mm}$$

General expression for natural frequency is given by $\omega = \sqrt{\dfrac{g}{y}}$ rad/s

$$f = \omega/2\pi \text{ Hz}, f_B = \frac{1}{2\pi}\sqrt{\frac{9.81}{.042 \times 10^{-2}}} = 7.7 \text{ Hz}$$

$$f_c = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.95 \times 10^{-2}}} = 5.13 \text{ Hz}, \; f_D = f_C = 5.13 \text{ Hz}, f_E = 7.7 \text{ Hz}$$

According to Dunkerley's relation, we have $\dfrac{1}{f_2} = \dfrac{1}{f_1^2} + \dfrac{1}{f_2^2} + \dfrac{1}{f_3^2} + \ldots \dfrac{1}{f^2} = \dfrac{1}{(7.7)^2} + \dfrac{1}{(5.13)^2}$ $+ \dfrac{1}{(5.13)^2} + \dfrac{1}{(7.7)^2} = 0.03373 + 0.07599, f^2 = 9.113, f = 3.01$ Hz.

EXAMPLE 8.6

Determine the natural frequency of the system shown in Fig. p-8.6 by using Rayleigh's energy method.

Given $m_1 = 2$ kg, $k_1 = 6$ N/m, $m_2 = 4$ kg, $k_2 = 5$ N/m, $m_3 = 6$ kg, $k_3 = 1$ N/m.

Solution Applying unit load at the position 1, $a_{11} = \frac{1}{k_1} = \frac{1}{6} = 0.167$

$\therefore \quad a_{21} = a_{12} = 0.167, a_{31} = a_{13} = 0.167$

Applying unit load at the position 2, $a_{22} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{6} + \frac{1}{5}, a_{22} = 0.367 = a_{32} = a_{23}$

Applying unit load at the position (3), $a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}, a_{33} = \frac{1}{6} + \frac{1}{5} + \frac{1}{1}, a_{33}$ $= 1.367$

Using these influence coefficients,

$y_1 = (m_1 a_{11} + m_2 a_{12} + m_3 a_{13})\, 9.81 = (2 \times 0.167 + 4 \times 0.167 + 6 \times 0.167)\, 9.81, y_1 = 2 \times 9.81\ y_1 = 19.6$ m

$y_2 = (m_1 a_{21} + m_2 a_{22} + m_3 a_{23})\, 9.81 = (2 \times 0.167 + 4 \times 0.367 + 6 \times 0.367)\, 9.81\ y_2 = 4 \times 9.81\ y_2 = 39.24$ m

$y_3 = (m_1 a_{31} + m_2 a_{32} + m_3 a_{33})\, 9.81 = (2 \times 0.167 + 4 \times 0.367 + 6 \times 1.367)\, 9.81$

$y_3 = 10 \times 9.81 \quad y_3 = 98.10$ m

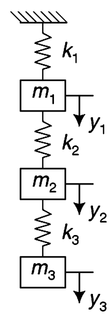


Fig. p-8.6 Spring press system

By Rayleigh's method,

$$\omega_n^2 = \frac{g \sum_{i-1}^{3} w_i y_i}{\sum_{i-1}^{3} w_i y_i^2}, \quad \omega_n^2 = \frac{g(m_1 y_1 + m_2 y_2 + m_3 y_3) \times 9.81}{\left[m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2\right](9.81)^2},$$

$$\omega_n^2 = \frac{9.81(2 \times 2 + 4 \times 4 + 60 \times 10)}{9.81(2 \times (2)^2 + 4 \times (4)^2 + 6 \times (10)^2)},$$

$$\omega_n^2 = 0.119, \text{ or } \omega_n = 0.345 \text{ rad/s}, \ f_n = 0.055 \text{ Hz}$$

Note: The above natural frequency has been found by assuming the deflection is due to static loading. For all practical purposes this value of natural frequency is quite accurate. However, if still greater accuracy is required then the deflections at various points should be obtained by considering inertia forces instead of static loads.

EXAMPLE 8.7

Find the natural frequency of the following shaft and hence determine the critical speed of the shaft by using Rayleigh's method as shown in Fig. p-8.7(a).

Given: Young's modulus of the shaft material $E = 2 \times 10^{11}$ N/m^2 and diameter of the shaft $d = 30$ mm.

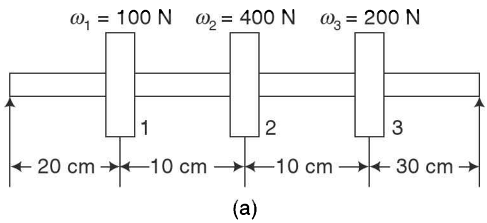


Fig. p-8.7 *Shaft*

Solution Young's modulus of the shaft material $E = 2 \times 10^{11}$ N/m^2

Diameter of the shaft $d = 30$ mm

To find the deflections at points (1), (2) and (3), the following formula is used:

$$y_x = \frac{\omega\, ax(l^2 - a^2 - x^2)}{6EIl} \text{ for } x \le (1 - a)$$

Case (1)

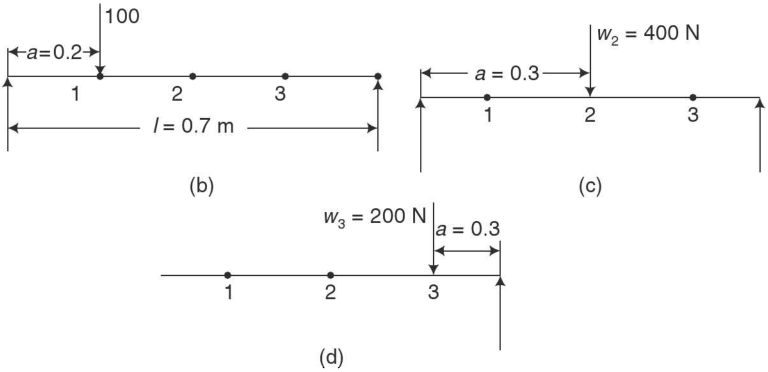


Fig.p-8.7 *To find y_{31}*

To find y_{11}, $a = 0.2$, $x = 0.5$, $l = 0.7$ m

$\therefore$ the condition $x \le (1 - a)$

$\therefore$ $0.5 \le (0.7 - 0.2)\ 0.5 = 0.5$ (True)

$$y_{11} = \frac{100 \times 0.2 \times 0.5(0.7^2 - 0.2^2 - 0.5^2)}{6 \times E \times I \times 0.7} = \frac{0.476}{E \times I}$$

(deflection at 1 due to load at 1)

To find y_{21}, $a = 0.2$, $x = 0.4$, $l = 0.73$

$\therefore$ the condition $x \le (l - a)$

$\therefore$ $0.4 \le (0.7 - 0.2)\ 0.4 < 0.5$ (True)

$$y_{21} = \frac{100 \times 0.2 \times 0.4(0.7^2 - 0.2^2 - 0.4^2)}{6 \times E \times I \times 0.7} = \frac{0.553}{E \times I}$$

(deflection at 2 due to load at 1)

To find y_{31}, $a = 0.2$ $x = 0.3$, the condition $x \leq (1 - a)$, i.e. $0.3 \leq 0.5$ is true.

$$y_{31} = \frac{100 \times 0.2 \times 0.3(0.7^2 - 0.2^2 - 0.3^2)}{6 \times E \times I \times 0.7}, y_{31} = \frac{0.514}{E \times I}$$

(deflection at 3 due to load at 1)

Case (2) $a = 0.3$, $w_2 = 400$ N

To find $y_{12,}$ $a = 0.3$, $x = 0.5$,

The condition $x \leq (1 - a)$, i.e. $0.5 \leq (0.7 - 0.3)$, $0.5 \leq 0.4$ (False)

∴ changing $a = 0.4$, $x = 0.2$

Condition $x \leq (l - a)$, i.e. $0.2 \leq (0.7 - 0.4)$, $0.2 \leq 0.3$ (True)

$$y_{12} = \frac{400 \times 0.4 \times 0.2(0.7^2 - 0.4^2 - 0.2^2)}{6 \times E \times I \times 0.7} = \frac{2.21}{E \times I}$$

(deflection at 1 due to load at 2)

To find y_{22}, $a = 0.3$, $x = 0.4$, the condition $x \leq (l - a)$, i.e. $0.4 \leq (0.7 - 0.3)$, $0.4 \leq 0.4$ (True)

$$y_{22} = \frac{400 \times 0.4 \times 0.3\,(0.7^2 - 0.3^2 - 0.4^2)}{6 \times E \times I \times 0.7} = \frac{2.743}{E \times I}$$

(deflection at 2 due to load at 2)

To find y_{32}, $a = 0.3$, $x = 0.3$,

the condition $x \leq (1 - a)$, $0.3 \leq (0.7 - 0.3)$, $0.3 \leq 0.4$ (True)

$$y_{32} = \frac{400 \times 0.3 \times 0.3\,(0.7^2 - 0.3^2 - 0.3^2)}{6 \times E \times I \times 0.7} = \frac{2.657}{E \times I}$$

(deflection at 3 due to load at 2)

Case (3) To find y_{13} $a = 0.3$, $x = 0.2$

The condition $x \leq (l - a)$, $0.2 \leq (0.7 - 0.3)$, $0.2 \leq 0.4$ (True)

$$y_{13} = \frac{200 \times 0.3 \times 0.2\,(0.7^2 - 0.2^2 - 0.3^2)}{6 \times E \times I \times 0.7} = \frac{1.029}{E \times I}$$

(deflection at 1 due to load at 3)

To find $y_{23,}$ $a = 0.3$, $x = 0.3$, the condition $x \leq (l - a)$, $0.3 \leq (0.7 - 0.3)$, $0.3 \leq 0.4$ (True)

$$y_{23} = \frac{200 \times 0.3 \times 0.3\,(0.7^2 - 0.3^2 - 0.3^2)}{6 \times E \times I \times 0.7} = \frac{1.329}{E \times I}$$

(deflection at 2 due to load at 3)

To find y_{33}, $a = 0.3$, $x = 0.4$, the condition $0.4 \le 0.4$ (True)

$$y_{33} = \frac{200 \times 0.3 \times 0.4\,(0.7^2 - 0.3^2 - 0.4^2)}{6 \times E \times I \times 0.7} = \frac{1.371}{E \times I}$$

$\therefore$ the total deflections are given by

$$y_1 = y_{11} + y_{12} + y_{13} = \frac{0.476}{EI} + \frac{2.21}{EI} + \frac{1.029}{EI} \qquad \therefore y_1 = \frac{3.715}{EI}$$

$$y_2 = y_{21} + y_{22} + y_{23} = \frac{0.553}{EI} + \frac{2.743}{EI} + \frac{1.329}{EI} \qquad \therefore y_2 = \frac{4.625}{EI}$$

$$y_3 = y_{31} + y_{32} + y_{33} = \frac{0.514}{EI} + \frac{2.675}{EI} + \frac{1.371}{EI} \qquad \therefore y_3 = \frac{4.542}{EI}$$

By Rayleigh's method,
$$\omega_n^2 = \frac{g\sum_{i-1}^{3} w_i y_i}{\sum_{i-1}^{3} w_i y_i^2} = \frac{g(w_1 y_1 + w_2 y_2 + w_3 y_3)}{w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2}$$

$$\omega_n^2 = \frac{9.81\left(100 \times \frac{3.715}{EI} + 400 \times \frac{4.625}{EI} + 200 \times \frac{4.542}{EI}\right)}{100 \times \left(\frac{3.715}{EI}\right)^2 + 400 \times \left(\frac{4.625}{EI}\right)^2 + 200 \times \left(\frac{4.542}{EI}\right)^2}$$

$$\omega_n^2 = \frac{9.8E.I(100 \times 3.7157 + 400 \times 4.625 + 200 \times 4.542)}{100 \times (3.7157)^2 + 400 \times (4.625)^2 + 200 \times (4.542)^2} = 2.183EI$$

$$I = \frac{\pi d^4}{64} = \frac{\pi \times (0.03)^4}{64} = 3.976 \times 10^{-8} m^4$$

$$\omega_n^2 = 17362.67,\ \omega_n = 131.768 \text{ rad/s}$$

Natural frequency $f_n = 20.97$ Hz

Speed can be found out by, $\omega_n = \dfrac{2\pi N}{60}$

$$\therefore \qquad N = \frac{60 \times 131.768}{2\pi},\ N = 1258.3 \text{ rev/min}$$

EXAMPLE 8.8

Use Holzer's method to determine the natural frequencies of the spring-mass system as shown in Fig. p-8.8(a) (Semidefinite system).

Given: $m_1 = m_2 = m_3 = 1$ kg and $k_1 = k_2 = 1$ N/m.

Solution

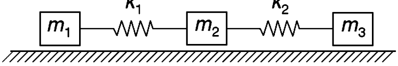


Fig. p-8.8(a) *Spring-mass system*

ω^2	Station Point	Mass 'm'	Amplitude X*	$m\omega^2 x$	$\Sigma m\omega^2 X$	Spring stiffness 'k'	$\frac{\Sigma m\omega^2 X}{k}$
0	1	1	1.00	0	0	1	0
ω_{1n}^2	2	1	1.00	0	0	1	0
	3	1	1.00	0	0	–	–
1	1	1	1.00	1	1	1	1
ω_{2n}^2	2	1	0	0	1	1	1
	3	1	−1	−1	0	–	–
2	1	1	1.00	2	2	1	2
	2	1	−1	−2	0	1	0
	3	1	−1	−2	-2_{-ve}	–	–
4	1	1	1.00	4	4	1	4
	2	1	−3	−12	−8	1	−8
	3	1	5	20	12_{+ve}	–	–
3	1	1	1.00	3	3	1	3
ω_{3n}^2	2	1	−2	−6	−3	1	−3
	3	1	1	3	0	–	–

Since the first natural frequency is equal to zero, the system is a semi definite system.

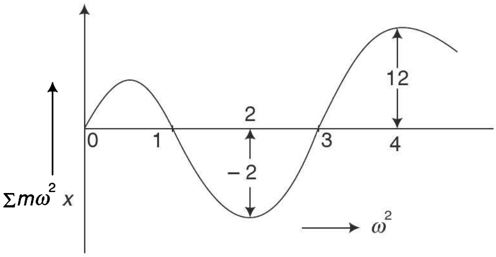


Fig. p-8.8 (b) *Graph of frequencies (ω) versus displacement (x_i)*

From the graph, Fig. p-8.8(b).

$\therefore$ $\omega_{1n} = 0$. First natural frequency. **As we know in Section 6.8, Chapter 6, in case of a semidefinite system, one of their natural frequencies is equal to zero.**

Second natural frequency $\omega_{2n} = 1$ rad/s

Third natural frequency $\omega_{3n} = 1.73$ rad/s

To draw the mode shapes

For first natural frequency ω_{1n} amplitudes are as shown in Fig. p-8.8(c).

$A_3 = 1$ $B_1 = 1$ $C_1 = 1$

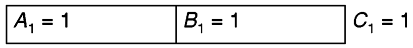


Fig. p-8.8 (c) *First mode shape*

For second natural frequency ω_{2n} amplitudes are as shown in Fig. p-8.8(d).

$A_2 = 1$ $B_2 = 0$ $C_2 = -1$

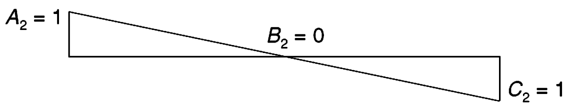


Fig. p-8.8(d) *Second mode shape*

For third natural frequency ω_{3n} amplitudes are as shown in Fig. p-8.8(e).

$A_3 = 1$ $B_3 = -2$ $C_3 = 1$

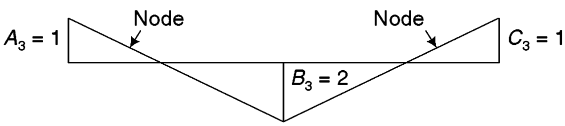


Fig. p-8.8(e) *Third mode shape*

EXAMPLE 8.9

Determine the natural frequency of the torsional system as shown in the Fig. p-8.9(a). Use Holzer's method (Semidefinite system).

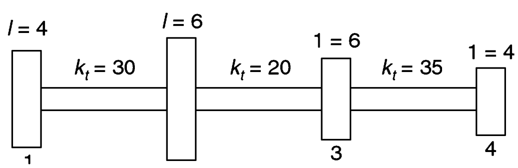


Fig. p-8.9(a) *Torsional system*

ω^2	Station Point	Inertia 'I'	Amplitude θ	$I\omega^2\theta$	$\Sigma I\omega^2\theta$	k_t	$\frac{\Sigma I\omega^2\theta}{k_t}$
	1	4	1.00	0	0	30	0
0	2	6	1.00	0	0	20	
ω_{1n}^2	3	6	1.00	0	0	35	
	4	4	1.00	0	0	–	–

1	1	4	1.00	4	4	30	0.13
	2	6	0.87	5.22	9.22	20	0.46
	3	6	0.41	2.45	11.67	35	0.33
	4	4	0.08	0.31	11.98	–	–
2	1	4	1.00	8	8	30	0.27
	2	6	0.73	8.80	16.80	20	0.84
	3	6	−0.11	−1.32	15.48	35	0.44
	4	4	−0.55	−4.42	11.06	–	–
1.5	1	4	1.00	6.00	6.00	30	0.20
	2	6	0.80	7.20	13.20	20	0.66
	3	6	0.14	1.26	14.46	35	0.41
	4	4	−0.27	−1.67	12.82	–	–
1.7	1	4	1.00	6.80	6.80	30	0.23
	2	6	0.77	7.89	14.69	20	0.73
	3	6	0.04	0.36	15.05	35	0.43
	4	4	−0.39	−2.65	12.40	–	–
3	1	4	1.00	12.00	12.00	30	0.40
	2	6	0.60	10.80	22.80	20	1.14
	3	6	−0.54	−9.72	13.08	35	0.37
		4	−0.91	−091	−10.96	2.12	– –
4	1	4	1.00	16.00	16.00	30	0.53
	2	6	0.47	11.20	27.20	20	1.36
	3	6	−0.89	−21.36	5.84	35	0.17
	4	4	−1.06	−16.91	−11.07	–	–
3.2	1	4	1.00	12.80	12.80	30	0.43
	2	6	0.57	11.01	23.81	20	1.19
	3	6	−0.62	−11.91	11.90	35	0.34
	4	4	−0.96	−12.29	−0.39	–	–
3.18 ω_{2n}^2	1	4	1.00	12.72	12.72	30	0.42
	2	6	0.58	10.99	23.71	20	1.19
	3	6	−0.61	−11.55	12.16	35	0.35
	4	4	−0.96	−12.18	−0.02	–	–
*6	1	4	1.00	24	24	30	0.80
	2	6	0.20	7.20	31.20	20	1.56
	3	6	−1.36	−48.96	−17.76	35	−0.51
	4	4	−0.85	−20.46	−38.22	–	–
10	1	4	1.00	40	40	30	1.33
	2	6	−0.33	−20	20	20	1.00
	3	6	−1.33	−79.80	−59	35	−1.71
	4	4	0.38	15.14	−44.66	–	–
13	1	4	1.00	52.00	52.00	30	1.73
	2	6	−0.73	−57.20	−5.20	20	−0.26
	3	6	−0.47	−36.66	−41.86	35	−1.20
	4	4	0.73	37.75	−4.11	–	–
14	1	4	1.00	56.00	56.00	30	1.87
	2	6	−0.87	−72.80	−16.80	20	−0.84
	3	6	−0.03	−2.52	−19.32	35	−0.55
	4	4	0.52	29.23	9.91	–	–
13.3 ω_{3n}^2	1	4	1.00	53.20	53.20	30	1.77
	2	6	−0.77	−61.71	−8.51	20	−0.43
	3	6	−0.34	−27.48	−35.99	35	−1.03
	4	4	0.69	36.62	0.63	–	–

20	1	4	1.00	80.00	80.00	30	2.67
	2	6	−1.67	−200.00	−120.00	20	−6.00
	3	6	4.33	519.6	399.60	35	11.42
	4	4	−7.09	−566.97	−167.37	–	–
16	1	4	1.00	64.00	64.00	30	2.13
	2	6	−1.13	−108.8	−44.80	20	−2.20
	3	6	1.11	106.56	61.76	35	1.76
	4	4	−0.65	−41.89	19.87	–	–
17	1	4	1.00	68.00	68.00	30	2.27
	2	6	−1.27	−129.20	−61.20	20	−3.06
	3	6	1.79	182.58	121.38	35	3.47
	4	4	−1.68	−114.10	7.28	–	–
18	1	4	1.00	72.00	72.00	30	2.40
	2	6	−1.40	−151.20	−79.20	20	−3.96
	3	6	2.56	276.48	197.28	35	5.64
	4	4	−3.08	−221.51	−24.23	–	–
17.3 ω_{4n}^2	1	4	1.00	69.20	69.20	30	2.31
	2	6	−1.31	−135.63	−66.43	20	−3.32
	3	6	2.01	208.8	142.37	35	4.07
	4	4	−2.06	−142.40	−0.03	–	–
13.27 ω_{3n}^2	1	4	1.00	53.08	53.08	30	1.77
	2	6	−0.77	−61.25	−8.17	20	−0.41
	3	6	−0.36	−28.77	−36.94	35	−1.06
	4	4	0.70	36.91	− 0.03	–	–
40	1	4	1.00	160	160	30	5.33
	2	6	−4.33	−1040	−880	20	−44
	3	6	39.67	9520.8	8640.8	35	246.88
	4	4	−207.21	−33153.60	−24512.8	–	

From the graph, the natural frequencies are given by

$$\omega_{1n} = 0,\ \omega_{2n} = 1.78 \text{ rad/s},\ \omega_{3n} = 3.64 \text{ rad/s and } \omega_{4n} = 4.16 \text{ rad/s}$$

***Graph for free-free system*, $\Sigma I\,\omega^2\,\theta$ versus ω^2**

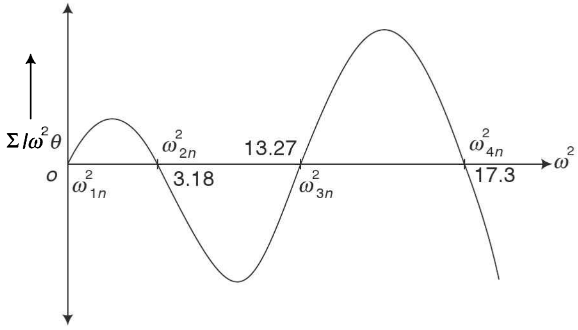


Fig. p-8.9(b) *Graph of frequencies (ω) versus displacement (θ_i)*

To draw the mode shapes

For first natural frequency 'ω_{1n}' amplitudes are as shown in Fig. p-8.9(c).

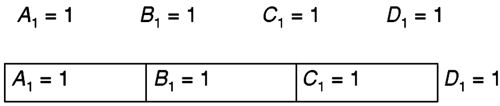


Fig. p-8.9(c) *First mode shape*

For second natural frequency ω_{2n} amplitudes are as shown in Fig. p-8.9(d).

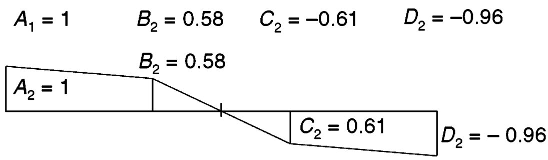


Fig. p-8.9(d) *Second mode shape*

For third natural frequency ω_{3n}, amplitudes are as shown in Fig. p-8.9(e).

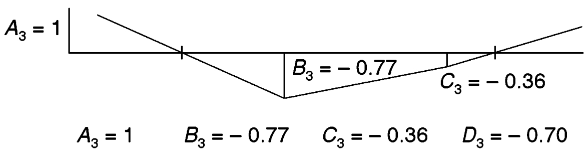


Fig. p-8.9(e) *Third mode shape*

For Fourth natural frequency ω_{4n}, amplitudes are as shown in Fig. p-8.9(f).

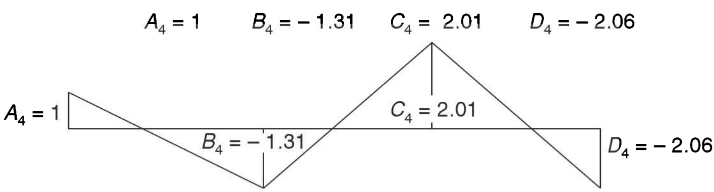


Fig. p-8.9(f) *Fourth mode shape*

EXAMPLE 8.10

Find the natural frequencies for the following system as shown in Fig.p-8.10(a) by using Holzer's method.

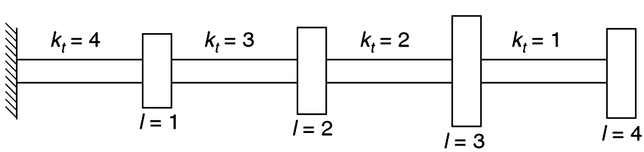


Fig. p-8.10(a) *Torsional system*

ω^2	Station Point	Inertia 'I'	Amplitude θ	$I\,\omega^2\,\theta$	$\Sigma\,I\,\omega^2\,\theta$	k_t	$\frac{\Sigma I\omega^2\theta}{k_t}$
0	1	4	1.00	0	0	1	0
	2	3	1.00	0	0	2	0
	3	2	1.00	0	0	3	0
ω_{1n}^2	4	1	1.00	0	0	4	0
	5	∞	1.00	–	–	–	–
	1	4	1.00	4	4	1	4.00
	2	3	−3.00	−9.00	−5	2	−2.5
1	3	2	−0.5	−1.00	−6	3	−2.0
	4	1	1.5	1.5	−4.5	4	−1.13
	5	∞	2.63	–	–	–	–
0.1	1	4	1.00	0.40	0.40	1	0.40
	2	3	0.6	0.18	0.58	2	0.29
	3	2	0.31	0.06	0.64	3	0.21
	4	1	0.1	0.01	0.65	4	0.16
	5	∞	−0.06	–	–	–	–
	1	4	1.00	0.36	0.36	1	0.36
0.09	2	3	0.64	0.173	0.533	2	0.266
	3	2	0.374	0.067	0.6	3	0.2
	4	1	0.174	0.016	0.616	4	0.154
	5	∞	0.02	–	–	–	–
0.092	1	4	1.00	0.368	0.368	1	0.368
	2	3	0.632	0.174	0.542	2	0.271
ω_{1n}^2	3	2	0.361	0.066	0.608	3	0.203
	4	1	0.158	0.015	0.623	4	0.156
	5	∞	0.002	–	–	–	–
0.5	1	4	1.00	2.0	2.0	1	2.0
	2	3	−1.0	−1.5	0.5	2	0.25
	3	2	−1.25	−1.25	−0.75	3	−0.25
	4	1	−1.00	−0.5	−1.25	4	−0.313
	5	∞	−0.688	–	–	–	–
	1	4	1.00	2.56	2.56	1	2.56
	2	3	−1.56	−3.0	−0.44	2	−0.22
0.64	3	2	−1.34	−1.72	−2.16	3	−0.72
	4	1	−0.62	−0.4	−2.56	4	−0.64
	5	∞	0.02	–	–	–	–
	1	4	1.00	2.552	2.552	1	2.552
0.638	2	3	−1.552	−2.971	−0.419	2	−0.209
	3	2	−1.343	−1.713	−2.132	3	−0.711
ω_{2n}^2	4	1	−0.632	−0.403	−2.535	4	−0.634
	5	∞	0.002	–	–	–	–
	1	4	1.00	8.00	8.00	1	8.00
	2	3	−7.00	−42.00	−34.00	2	−17.00
2	3	2	10.00	40.00	6.00	3	2.00
	4	1	+8.00	16.00	22.00	4	5.5
	5	∞	2.50	–	–	–	–

2.165 ω_{3n}^2	1	4	1.00	9.368	8.66	1	8.66
	2	3	−7.66	−49.75	−41.09	2	−20.55
	3	2	12.89	55.80	14.71	3	4.90
	4	1	7.99	17.29	32	4	8.00
	5	∞	−0.01	–	–	–	–
8	1	4	1.00	32	32	1	32
	2	3	−31	−744	−712	2	−356
	3	2	325	5200	4488	3	1496
	4	1	−1171	−9368	−4480	4	−1220
	5	∞	49	–	–	–	–
7.9	1	4	1.00	31.60	31.60	1	31.60
	2	3	−30.60	−725.22	−693.62	2	−346.81
	3	2	316.21	4996.12	4302.50	3	1434.17
	4	1	−1117.96	−8831.85	−4529.35	4	−1132.34
	5	∞	14.38	–	–	–	–
7.856 ω_{4n}^2	1	4	1.00	31.428	31.424	1	31.424
	2	3	−30.424	−717.033	−685.609	2	−342.804
	3	2	312.380	4908.121	4222.512	3	1407.50
	4	1	−1095.124	−8603.29	−4380.78	4	−1095.196
	5	∞	0.072	–	–	–	–

The natural frequencies are found out by plotting the graph and looking at the intercepts.

Graph for fixed-free system

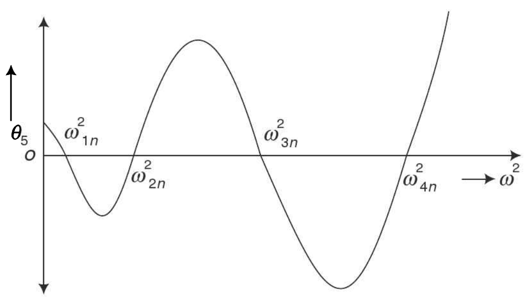


Fig. p-8.10(b) *Graph of frequencies (ω) versus displacement (θ_i)*

∴ from the graph, the natural frequencies are

$\omega_{1n}^2 = 0.092$ $\qquad \omega_{1n} = 0.303$ rad/s

$\omega_{2n}^2 = 0.638$ $\qquad \omega_{2n} = 0.799$ rad/s

$\omega_{3n}^2 = 2.165$ $\qquad \omega_{3n} = 1.471$ rad/s

$\omega_{4n}^2 = 7.856$ $\qquad \omega_{4n} = 2.803$ rad/s

EXAMPLE 8.11

Find the natural frequencies for the following system as shown in Fig. p-8.11(a), by using Holzer's method. Take $m = k = 1$.

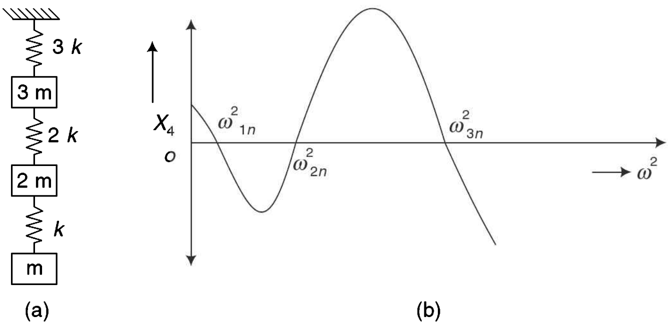


Fig. p-8.11 *Graph of frequencies (ω) against the displacement (x_i)*

ω^2	Station Point	Mass 'm'	Amplitude X	$m\omega^2 x$	$\Sigma m\omega^2 X$	Spring stiffness k	$\frac{\Sigma m\omega^2 X}{k_t}$
0	1	1	1.00	0	0	1	0
	2	2	1.00	0	0	2	0
	3	3	1.00	0	0	3	0
	4	–	1.00	–	–	–	–
1	1	1	1.00	1.00	1.00	1	1.00
	2	2	0.00	0.00	1.00	2	0.50
	3	3	−0.5	−1.5	−0.5	3	0.17
	4	–	−0.67	–	–	–	–
0.3 ω^2_{1n}	1	1	1.00	0.3	0.3	1	0.3
	2	2	0.7	0.42	0.72	2	0.36
	3	3	0.34	0.31	1.03	3	0.34
	4	–	0.00	–	–	–	–
1.3 ω^2_{2n}	1	1	1.00	1.3	1.3	1	1.3
	2	2	−0.3	−0.78	0.52	2	0.26
	3	3	−0.56	−2.18	−1.66	3	−0.55
	4	–	−0.01	–	–	–	–
2.6	1	1	1.00	2.6	2.6	1	2.6
	2	2	−1.6	−8.32	−5.72	2	−2.86
	3	3	1.26	9.83	4.11	3	1.37
	4	–	−0.11	–	–	–	–
2.56 ω^2_{3n}	1	1	1.00	2.56	2.56	1	2.56
	2	2	−1.56	− 7.99	−5.43	2	−2.72
	3	3	1.15	8.86	3.43	3	1.14
	4	–	0.01	–	–	–	–

∴ from the graph, the natural frequencies are

$$\omega^2_{1n} = 0.3\,\frac{k}{m} \quad \therefore \quad \omega_{1n} = 0.55\sqrt{\frac{k}{m}}\ \text{rad/s}\,,\ \omega^2_{2n} = 1.3\frac{k}{m}\ \therefore\ \omega_{2n} = 1.14\sqrt{\frac{k}{m}}\ \text{rad/s}$$

$$\omega^2_{3n} = 2.56\,\frac{k}{m}\ \therefore\ \omega_{3n} = 1.60\sqrt{\frac{k}{m}}\ \text{rad/s}$$

EXAMPLE 8.12

Find the natural frequencies for the following system as shown in Fig. p-8.12(a) by using Holzer's method.

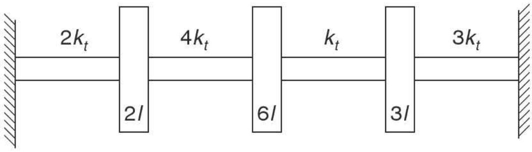


Fig. p-8.12(a) *Torsional system*

ω^2	Station Point	Inertia 'I'	Amplitude θ	$I\omega^2\theta$	$\Sigma I\omega^2\theta$	k_{ti}	$k_{ti}\theta_i - \Sigma I\omega^2\theta$	$k_{ti}\theta_i - \frac{\Sigma I\omega^2\theta}{k_{tn}}$
	1	2	1.00	0	0	2	2 − 0	0.5 = (2/5)
0	2	6	1.5	0	0	4	2 − 0	2 = 2/1
ω^2_{1n}	3	3	3.5	0	0	1	2 − 0	0.67 = 2/3
	4	–	4.17	0	0	3	–	–
	1	2	1.00	2	2	2	0	0
1	2	6	1.00	6	8	4	−6	−6
	3	3	−5	−15	−7	1	9	3
	4	–	−2.0	–	–	3	–	–
0.5	1	2	1.00	1.00	1.00	2	1.00	0.25
	2	6	1.25	3.75	4.75	4	−2.75	−2.75
	3	3	−1.5	−2.25	2.5	1	−0.5	−0.17
	4	–	−1.67	–	–	3	–	–
0.4	1	2	1.00	0.80	0.80	2	1.20	0.30
	2	6	1.30	3.12	3.92	4	−1.92	−1.92
	3	3	−0.62	−0.74	3.18	1	−1.18	−0.39
	4	–	−1.01	–	–	3	–	–
0.3	1	2	1.00	0.6	0.6	2	1.4	0.35
	2	6	1.35	2.43	3.03	4	−1.03	−1.03
	3	3	0.32	0.29	3.32	1	−1.32	−0.44
	4	–	−0.12	–	–	3	–	–
0.29	1	2	1.00	0.58	0.58	2	1.42	0.36
ω^2_{1n}	2	6	1.36	2.36	2.94	4	−0.94	−0.94
	3	3	0.42	0.37	3.31	1	−1.31	−0.44
	4	–	−0.02	–	–	3	–	–
1.5	1	2	1.00	3	3	2	−1	−0.25
	2	6	0.75	6.75	9.75	4	−7.75	−7.75
	3	3	−7	−31.5	−21.75	1	23.75	7.92
	4	–	0.92	–	–	3	–	–
1.4	1	2	1.00	2.8	2.8	2	−0.8	−0.2
	2	6	0.8	6.72	9.52	4	−7.52	−7.52
	3	3	−6.72	−28.22	−18.70	1	20.70	6.90
	4	–	0.18	–	–	3	–	–

1.375	1	2	1.00	2.75	2.75	2	−0.75	−0.188
ω^2_{2n}	2	6	0.813	6.703	9.453	4	−7.453	−7.453
	3	3	−6.64	−27.4	−17.94	1	19.938	6.646
	4	–	0.006	–	–	3	–	–
3	1	2	1.00	6	6	2	−4	−1
	2	6	0.0	0	6	4	−4	−4
	3	3	−4	−36	−30	1	32	10.67
	4	–	6.67	–	–	3	–	–
3.5	1	2	1.00	7	7	2	−5	−1.25
ω^2_{3n}	2	6	−0.25	−5.25	1.75	4	0.25	0.25
	3	3	0.00	0.0	1.75	1	0.25	0.08
	4	–	0.08	–	–	3	–	–
3.6	1	2	1.00	7.2	7.2	2	−5.2	−1.3
	2	6	−0.3	−6.48	0.72	4	1.28	1.28
	3	3	0.98	10.58	11.30	1	−9.30	− 3.10
	4	–	− 2.12	–	–	3	–	–
3.51	1	2	1.00	7.02	7.02	2	−5.02	−1.26
	2	6	−0.26	−5.37	1.65	4	0.35	0.35
	3	3	0.09	0.95	2.60	1	−0.60	−0.20
	4	–	−0.11	–	–	3	–	–

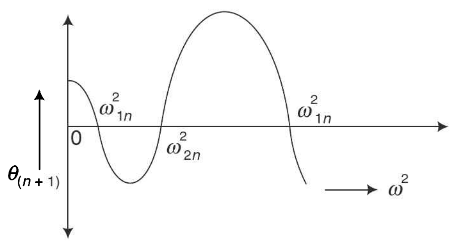


Fig. p-8.12(b) *Graph of frequencies (ω) versus displacement (θ_i)*

∴ from the graph, the natural frequencies are

$$\omega^2_{1n} = 0.29\frac{k_t}{I} \therefore \omega_{1n} = 0.54\sqrt{\frac{k_t}{I}} \text{ rad/s}, \quad \omega^2_{2n} = 1.375\frac{k_t}{I}, \therefore \quad \omega_{2n} = 1.173\sqrt{\frac{k_t}{I}} \text{ rad/s},$$

$$\omega^2_{3n} = 3.5\frac{k_t}{I} \therefore \omega_{3n} = 1.87\sqrt{\frac{k_t}{I}} \text{ rad/s}$$

EXAMPLE 8.13

Determine the lowest natural frequency of the branched system as shown in Fig. p-8.13(a) by using Holzer's method.

Solution Given: $I_1 = 10, I_2 = 15, I_3 = 20, I_4 = 10, I_5 = 10$ and $I_6 = 20$

$k_{t1} = 100, kt_2 = 200, k_{t3} = 200, k_{t4} = 100$ and $k_{t5} = 150$

Note: Unit angular displacement is assumed for the discs at the ends of the branches. Proceeding to the junction, the resulting angular displacements of the branches should be the same at the junction. If this is not true, proper adjustment must be made of the assumed values until the resulting angular displacements at the junction are same. The sum of the inertia torques of the branches is then equal to the torque acting on the main shaft.

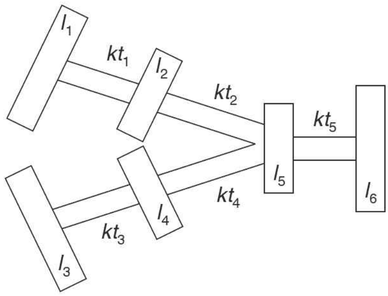


Fig. p-8.13(a) *Branched system*

ω^2	Station Point	Inertia 'I'	Amplitude θ	$I\omega^2\theta$	$\Sigma I\omega^2\theta$	k_t	$\frac{\Sigma I\omega^2\theta}{kt}$
0	1	10	1.00	0	0	100	0
	2	15	1.00	0	0	200	0
	5	10	1.00	0	0	150	0
0	3	20	1.00	0	0	200	0
ω_n^2	4	10	1.00	0	0	100	0
	5	10	1.00	0	0	150	0
	6	20	1.00	0	0	–	–
1	1	10	1.00	10	10	100	0.1
	2	15	0.9	13.5	23.5	200	0.12
	5	10	0.78	7.83	31.33	150	0.21
	3	20	1.00	20	20	200	0.1
	4	10	0.9	9	29	100	0.29
	5	10	0.61	–	–	150	–

As the station point 5 cannot have two different amplitudes at the same time, the amplitude 'θ_3' for the second branch has to be modified suitably such that $\theta_3 = \frac{0.78}{0.61}$ = 1.28 and the torque acting on kt_5 equals the sum of inertia torques developed by discs I_1, I_2, I_3 and I_4, i.e. 10 + 13.5 + 25.60 + 11.52 = 60.62.

ω^2	Station Point	Inertia 'I'	Amplitude θ	$I\,\omega^2\,\theta$	$\Sigma\, I\,\omega^2\,\theta$	k_t	$\frac{\Sigma I\omega 2\theta}{kt}$
1	3	20	1.28	25.60	25.60	200	0.13
	4	10	1.15	11.52	37.12	100	0.37
	5	10	0.78	7.79	60.62	150	0.30
	6	20	0.48	9.61	70.23	–	–
3	1	10	1.00	30	30	100	0.3
	2	15	0.7	31.50	61.50	200	0.31
	5	10	0.39	–	–	150	–
	3	20	1.00	60	60	200	0.3
	4	10	0.7	21	81	100	0.81
	5	10	−0.11		–	150	–

As the junction '5' cannot have two different amplitudes at the same time, the amplitude, 'θ_3' may be obtained for the second branch by $\theta_3 = \dfrac{0.39}{-0.11} = -3.55$ and the torque acting on $kt_5 = 61.50 - 287.55 = -226.05$.

ω^2	Station Point	Inertia 'I'	Amplitude θ	$I\,\omega^2\,\theta$	$\Sigma I\omega^2\theta$	kt	$\dfrac{\Sigma I\omega^2\theta}{kt}$
3	3	20	−3.55	−213	−213	200	−1.07
	4	10	−2.49	−74.55	−287.55	100	−2.88
	5	10	0.39	11.57	−226.05	150	−1.51
	6	20	1.90	113.82	−112.23	–	–
2.5	1	10	1.00	25	25	100	0.25
	2	15	0.75	28.125	53.125	200	0.266
	5	10	0.484	–	–	150	–
	3	20	1.00	50	50	200	0.25
	4	10	0.75	18.75	68.75	100	0.688
	5	10	0.063	–	–	150	–

As the station point 5 cannot have two different amplitudes at the same time, the amplitude 'θ_3' may be obtained for the second branch by $\theta = \dfrac{0.484}{0.063} = 7.683$ and the torque acting on k_{t5} is = 53.125 + 528.21 = 581.335.

2.5	3	20	7.683	384.15	384.15	200	1.921
	4	10	5.762	144.01	528.21	100	5.282
	5	10	0.48	–	581.34	150	3.876
	6	20	− 3.396	− 169.778	411.56	–	–
2.7	1	10	1.00	27	27	100	0.27
	2	15	0.73	20.565	56.565	200	0.283
	5	10	0.447	–	–	150	–
	3	20	1.00	54	54	200	0.27
	4	10	0.73	19.71	73.71	100	0.737
	5	10	− 0.0071	–	–	150	–

As the station point 5 cannot have two different amplitudes at the same time, the amplitude $\theta_3 = \dfrac{0.484}{0.063} = 7.683$ and the torque acting on kt_5 is = 56.565 – 4707.12.

2.7	3	20	−63.86	−3448.44	−3448.44	200	−17.24
	4	10	−46.62	−1258.68	−4707.12	100	−47.07
	5	10	0.451	–	− 4650.56	150	−31
	6	20	31.45	1698.56	− 2952	–	–

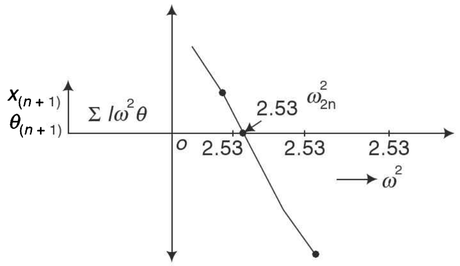


Fig. p-8.13(b) *Graph of frequencies (ω) versus displacement (θ_i)*

The fundamental natural frequency is $\omega_{2n}^2 = 2.53$

$\therefore \qquad \omega_{2n} = 1.59$ rad/s.

EXAMPLE 8.14

Using Stodola method, determine the lowest natural frequency of the four-degree-of-freedom spring-mass system as shown in Fig. p-8.14

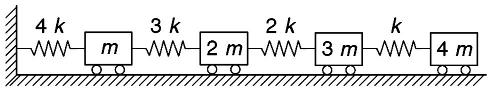


Fig. p-8.14 *Spring-mass system*

Sl. No.	Details	$k1 = 4k$	$m_1 = m$	$k_2 = 3k$	$m_2 = 2m$	$k_3 = 2k$	$m_3 = 3m$	$k_4 = k$	$m_4 = 4m$
1	Assumed deflection 'x'		4.00		3.00		2.00		1.00
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$6\omega^2$		$6\omega^2$		$4\omega^2$
3	Spring force 'S'	$20\omega^2$		$16\omega^2$		$10\omega^2$		$4\omega^2$	
4	Spring deflection δ	$5\omega^2$		$5.3\omega^2$		$5\omega^2$		$4\omega^2$	
5	Calculated deflection		$5\omega^2$		$10.3\omega^2$		$15.3\omega^2$		$19.3\omega^2$
			1		2.06		3.06		3.86
1	Assumed deflection 'x'		1		2.06		3.06		3.86
2	Inertia force '$m\omega^2 x$'		ω^2		$4.12\omega^2$		$9.18\omega^2$		$15.44\omega^2$
3	Spring force 'S'	$29.74\omega^2$		$28.74\omega^2$		$24.62\omega^2$		$15.44\omega^2$	
4	Spring deflection δ	$7.5\omega^2$		$9.4\omega^2$		$12.31\omega^2$		$15.44\omega^2$	
5	Calculated deflection		$7.5\omega^2$		$16.9\omega^2$		$29.21\omega^2$		$44.65\omega^2$
			1		2.25		3.89		5.95
1	Assumed deflection 'x'		1		2.25		3.89		5.95
2	Inertia force '$m\omega^2 x$'		ω^2		$4.5\omega^2$		$11.67\omega^2$		$23.8\omega^2$
3	Spring force 'S'	$40.97\omega^2$		$39.97\omega^2$		$35.5\omega^2$		$23.8\omega^2$	
4	Spring deflection δ	$10.25\omega^2$		$13.32\omega^2$		$17.75\omega^2$		$23.8\omega^2$	

5	Calculated deflection		$10.25\omega^2$		$23.51\omega^2$		$41.32\omega^2$		$65.12\omega^2$
			1		2.3		4.03		6.04
1	Assumed deflection 'x'		1		2.3		4.03		6.04
2	Inertia force '$m\omega^2 x$'		ω^2		$4.6\omega^2$		$15.57\omega^2$		$24.16\omega^2$
3	Spring force 'S'	$45.33\omega^2$		$44.33\omega^2$		$39.73\omega^2$		$24.16\omega^2$	
4	Spring deflection δ	$11.1\omega^2$		$14.78\omega^2$		$19.9\omega^2$		$24.16\omega^2$	
5	Calculated deflection		$11.1\omega^2$		$25.88\omega^2$		$45.78\omega^2$		$69.94\omega^2$
			1		2.35		4.12		6.3
1	Assumed deflection 'x'		1		2.35		4.12		6.3
2	Inertia force '$m\omega^2 x$'		ω^2		$4.7\omega^2$		$12.36\omega^2$		$25.2\omega^2$
3	Spring force 'S'	$43.26\omega^2$		$42.26\omega^2$		$37.56\omega^2$		$25.2\omega^2$	
4	Spring deflection δ	$10.82\omega^2$		$14.1\omega^2$		$18.78\omega^2$		$25.2\omega^2$	
5	Calculated deflection		$10.82\omega^2$		$24.92\omega^2$		$43.7\omega^2$		$68.9\omega^2$
			1	2.3		4.04		6.40	

Therefore, the first principal mode is given by [1.00 + 2.30 + 4.03 + 6.04] and the lower natural frequency is obtained from [1.00 + 2.30 + 4.03 + 6.04] = [10.25 + 23.51 + 41.32 + 65.12] ω^2

$$13.37 = 140.2\ \omega^2$$

$$\therefore \qquad \omega^2 = \frac{13.37}{140.2} = 0.095, \qquad \omega = 0.308\sqrt{\frac{k}{m}}\ \text{rad/s.}$$

EXAMPLE 8.15

Use the Stodola's method to determine the lowest natural frequency of the branched system as shown in Fig. p-8.15.

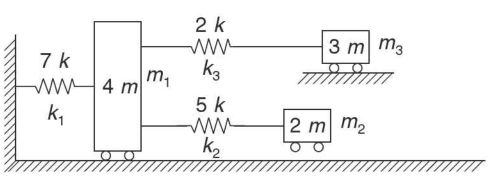


Fig.p-8.15 *Branched system*

Sl. No.	Details	$k_1 = 7$	$m_1 = 4$	$k_2 = 5$	$m_2 = 2$	$k_3 = 5$	$m_3 = 3$
1	Assumed deflection 'x'		1		1		1
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$2\omega^2$		$3\omega^2$
3	Spring force 'S'	$9\omega^2$		$5\omega^2$		$3\omega^2$	
4	Spring deflection 'δ'	$1.29\ \omega^2$		$0.4\omega^2$		$0.6\omega^2$	
5	Calculated deflection		$1.29\ \omega^2$		$1.69\ \omega^2$		$1.89\ \omega^2$
			1		1.31		1.47
1	Assumed deflection 'x'		1		1.31		1.47
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$2.62\omega^2$		$4.41\omega^2$
3	Spring force 'S'	$11.03\omega^2$		$2.62\omega^2$		$4.41\omega^2$	
4	Spring deflection 'δ'	$1.58\ \omega^2$		$0.52\omega^2$		$0.88\omega^2$	
5	Calculated deflection		$1.58\ \omega^2$		$2.1\ \omega^2$		$2.46\ \omega^2$
			1		1.33		1.56
1	Assumed deflection 'x'		1		1.33		1.56
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$2.66\omega^2$		$4.68\omega^2$
3	Spring force 'S'	$9.34\omega^2$		$2.66\omega^2$		$4.68\omega^2$	
4	Spring deflection 'δ'	$1.33\ \omega^2$		$0.53\omega^2$		$0.94\omega^2$	
5	Calculated deflection		$1.33\ \omega^2$		$1.86\ \omega^2$		$2.27\ \omega^2$
			1		1.40		1.71
1	Assumed deflection 'x'		1		1.40		1.71
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$2.8\omega^2$		$5.13\omega^2$
3	Spring force 'S'	$11.93\omega^2$		$2.8\omega^2$		$5.13\omega^2$	
4	Spring deflection 'δ'	$1.70\ \omega^2$		$0.56\omega^2$		$1.03\omega^2$	
5	Calculated deflection		$1.7\ \omega^2$		$2.26\ \omega^2$		$2.73\ \omega^2$
			1		1.33		1.61
1	Assumed deflection 'x'		1		1.33		1.61
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$2.66\omega^2$		$4.83\omega^2$
3	Spring force 'S'	$11.49\omega^2$		$2.66\omega^2$		$4.83\omega^2$	
4	Spring deflection 'δ'	$1.69\ \omega^2$		$0.53\omega^2$		$0.97\omega^2$	
5	Calculated deflection		$1.69\ \omega^2$		$2.22\ \omega^2$		$2.66\ \omega^2$
			1		1.31		1.57
1	Assumed deflection 'x'		1		1.31		1.57
2	Inertia force '$m\omega^2 x$'		$4\omega^2$		$2.62\omega^2$		$4.71\omega^2$
3	Spring force 'S'	$11.33\omega^2$		$2.62\omega^2$		$4.71\omega^2$	
4	Spring deflection 'δ'	$1.62\ \omega^2$		$0.52\omega^2$		$0.94\omega^2$	
5	Calculated deflection		$1.62\ \omega^2$		$2.14\ \omega^2$		$2.56\ \omega^2$
			1		1.32		1.58

The assumed deflections are approximately equal to the calculated deflections, the fundamental natural frequency is given

by $(1 + 1.31 + 1.57) = 1.62\ \omega^2 + 2.14\ \omega^2 + 2.56\ \omega^2$

$$\omega_{1_n} = 0.78\sqrt{\frac{k}{m}}\ \text{rad/s or } \omega_{1_n} = 0.79\sqrt{\frac{k}{m}}\ \text{rad/s.}$$

EXAMPLE 8.16

For the system shown in Fig. p-8.16, determine the lowest natural frequency by using Stodola's method carrying out two iterations.

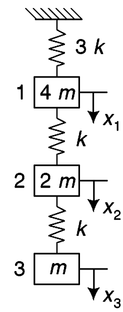


Fig. p-8.16 *Spring-mass system*

Solution Use the influence coefficients method as

$$a_{11} = a_{12} = a_{21} = a_{13} = a_{31} = \frac{1}{3k}$$

$$a_{22} = a_{32} = a_{23} = \frac{4}{3k},\ a_{33} = \frac{7}{3k}$$

I—Iteration: Assume $x_1 = 1, x_2 = 1, x_3 = 1$

The inertia forces are

$$F_1 = m\omega^2 x_1 = 4\ m\omega^2,\ F_2 = m\omega^2 x_2 = 2m\omega^2,\ F_3 = m\omega^2 x_3 = m\omega^2$$

The deflections are calculated as follows:

$$x_1^1 = F_1\alpha_{11} + F_2\alpha_{12} + F_3\alpha_{13} = 4m\omega^2 \times \frac{1}{3k} + 2m\omega^2 \times \frac{1}{3k} + m\omega^2 \times \frac{1}{3k} = \frac{7}{3}\frac{m\omega^2}{k}$$

$$x_2^1 = F_1\alpha_{21} + F_2\alpha_{22} + F_3\alpha_{23} = 4m\omega^2 \times \frac{1}{3k} + 2m\omega^2 \times \frac{4}{3k} + m\omega^2 \times \frac{4}{3k} = \frac{16}{3}\frac{m\omega^2}{k}$$

$$x_3^1 = F_1\alpha_{31} + F_2\alpha_{32} + F_3\alpha_{33} = 4m\omega^2 \times \frac{1}{3k} + 2m\omega^2 \times \frac{4}{3k} + m\omega^2 \times \frac{7}{3k} = \frac{19}{3}\frac{m\omega^2}{k}$$

$$x_1^1 : x_2^1 : x_3^1,\ \frac{7}{3}\frac{m\omega^2}{k} : \frac{16}{3}\frac{m\omega^2}{k} : \frac{19}{3}\frac{m\omega^2}{k},\ 1: 2.3: 2.7.$$

II—Iteration: Take the values of $x_1 = 1, x_2 = 2.3, x_3 = 2.7$ for second iteration.

Inertia forces are

$$F_1^1 = 4\ m\omega^2 x_1 = 4\ m\ \omega^2\ F_2^1 = m\omega^2 x_2 = 2m\omega^2 \times 2.3 = 4.6m\omega^2$$

$$F_3^1 = m\omega^2 x_3 = m\omega^2 \times 2.7 = 2.7m\omega^2$$

$$x_1^{11} = F_1^1\ \alpha_{11} + F_2^1\ \alpha_{12} + F_3^1\ a_{13} = 4m\omega^2 \times \frac{1}{3k} + 4.6m\omega^2 \times \frac{1}{3k} + 2.7m\omega^2 \times \frac{1}{3k} = \frac{11.3}{3}\frac{m\omega^2}{k}$$

$$x_2^{11} = F_1^1\ \alpha_{21} + F_2^1\ \alpha_{22} + F_3^1\ \alpha_{23} = 4m\omega^2 \times \frac{1}{3k} + 4.6m\omega^2 \times \frac{4}{3k} + 2.7m\omega^2 \times \frac{4}{3k},\ x_2^{11} = \frac{33.2}{3}\frac{m\omega^2}{k}$$

$$x_3^{11} = F_1^1 \alpha_{31} + F_2^1 \alpha_{32} + F_3^1 \alpha_{33} = 4\, m\omega^2 \times \frac{1}{3k}$$
$$+ 4.6\, m\omega^2 \times \frac{4}{3k} + 2.7 m\omega^2 \times \frac{7}{3k},\ x_3^{11} = \frac{41.3}{3}\frac{m\omega^2}{k}$$

$$x_1^{11} : x_2^{11} : x_3^{11} = \frac{11.3}{3}\frac{m\omega^2}{k} : \frac{33.2}{3}\frac{m\omega^2}{k} : \frac{41.3}{3}\frac{m\omega^2}{k}, 1{:}2.94{:}3.76 \text{ or } \frac{11.3}{3}\frac{m\omega^2}{k} = 1$$

or $$\omega = 0.46\sqrt{\frac{k}{m}} \text{ rad/s.}$$

EXAMPLE 8.17

Using Stodola's method determine the fundamental frequency and mode shape as shown in Fig. p-8.17. Take $k_{t1} = k_{t2} = k_{t3} = 0.10$ N-m/rad and $J_1 = J_2 = J_3 = 0.1$ N-m-s^2.

Solution Given $k_{t1} = k_{t2} = k_{t3} = 0.10$ N-m/rad, $J_1 = J_2 = J_3 = 0.1$ N-m-s^2

Use the influence coefficient method as

$$\alpha_{11} = \frac{1}{k_t},\ \alpha_{12} = \frac{1}{k_t},\ \alpha_{31} = \frac{1}{k_t},\ \alpha_{21} = \frac{1}{k_t},\ \alpha_{22} = \frac{2}{k_t},\ \alpha_{33} = \frac{2}{k_t},\ \alpha_{31} = \frac{1}{k_t},\ \alpha_{32} = \frac{2}{k_t},\ \alpha_{33} = \frac{3}{k_t}$$

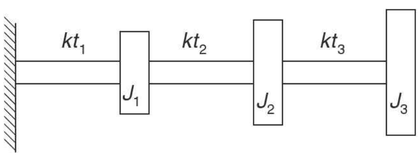


Fig. p-8.17 *Torsional system*

I—Iteration: Assume $\theta_1 = 1$, $\theta_2 = 1$, $\theta_3 = 1$ so that $\frac{\theta_1}{\theta_2} = 1$ and $\frac{\theta_1}{\theta_3} = 1$

The inertia torque can be calculated are as follows:

$$T_1 = J\omega^2\theta_1,\ T_2 = J\omega^2\theta_2,\ T_3 = J\omega^2\theta_3$$

By substituting the values of $\theta_1 = 1$, $\theta_2 = 1$, $\theta_3 = 1$, we have

$$T_1 = J\omega^2,\ T_2 = J\omega^2,\ T_3 = J\omega^2$$

The angular displacement can be calculated as follows: $\theta_1^1 = T_1\alpha_{11} + T_2\alpha_{12} + T_3\alpha_{13}$.

$$\theta_2^1 = T_1\alpha_{21} + T_2\alpha_{22} + T_3\alpha_{23}\ \theta_3^1 = T_1\,\alpha_{31} + T_2\,\alpha_{32} + T_3\,\alpha_{33}$$

By substituting the values of $T_1\alpha_{11}$, $T_2\,\alpha_{12}$ and $T_3\,\alpha_{13}$, we have

$$\theta_1^1 = T_1\,\alpha_{11} + T_2\,\alpha_{12} + T_3\,\alpha_{13},\ \theta_1^1 = \frac{J\omega^2}{k_t} + \frac{J\omega^2}{k_t} + \frac{J\omega^2}{k_t} = \frac{3J\omega^2}{k_t}$$

$$\theta_2^1 = T_1\,\alpha_{21} + T_2\,\alpha_{22} + T_3\,\alpha_{23},\ \theta_2^1 = \frac{J\omega^2}{k_t} + \frac{2J\omega^2}{k_t} + \frac{J\omega^2}{k_t} = \frac{5J\omega^2}{k_t}$$

$$\theta_3^1 = T_1\,\alpha_{31} + T_2\,\alpha_{32} + T_3\,\alpha_{33},\ \theta_3^1 = \frac{J\omega^2}{k_t} + \frac{2\,J\omega^2}{k_t} + \frac{3\,J\omega^2}{k_t} = \frac{6\,J\omega^2}{k_t}$$

$$\therefore \quad \frac{\theta_1^1}{\theta_2^1} = \frac{3J\omega^2}{k_t} \times \frac{k_t}{5J\omega^2} = 0.6 \quad \therefore \ \theta_1^1/\theta_3^1 = \frac{3J\omega^2}{k_t} \times \frac{k_t}{6J\omega^2} = 0.5.$$

II—Iteration: Assume $\theta_1 = 0.5$, $\theta_2 = 0.83$, $\theta_3 = 1$ so that $\dfrac{\theta_1}{\theta_2} = 0.6$ and $\dfrac{\theta_1}{\theta_3} = 0.5$.

$$\left(\frac{\theta_1}{\theta_2} = \frac{0.5}{0.6} = 0.883\right)$$

The inertia torque can be calculated are as follows:

$$T_1 = J\omega^2\theta_1,\ T_2 = J\omega^2\theta_2,\ T_3 = J\omega^2\theta_3$$

By substituting the values of $\theta_1 = 0.5$, $\theta_2 = 0.83$, $\theta_3 = 1$

In above equation, we have

$$T_1 = 0.5J\omega^2,\ T_2 = 0.83\ J\omega^2,\ T_3 = J\omega^2$$

$$\theta_1^1 = T_1\,\alpha_{11} + T_2\,\alpha_{12} + T_3\,\alpha_{13},\ \theta_1^1 = \frac{J\omega^2}{2k_t} + \frac{0.83\ J\omega^2}{k_t}$$

$$+ \frac{J\omega^2}{k_t} = \frac{J\omega^2}{k_t}(0.5 + 0.83 + 1.0) = 2.33\,\frac{J\omega^2}{k_t}$$

$$\theta_2^1 = T_1\,\alpha_{21} + T_2\,\alpha_{22} + T_3\,\alpha_{23},\ \theta_2^1 = \frac{0.5\ J\omega^2}{k_t} + \frac{2(0.83)J\omega^2}{k_t}$$

$$+ \frac{2\ J\omega^2}{k_t} = \frac{J\omega^2(0.5 + 0.83 + 2)}{k_t} = 4.16\,\frac{J\omega^2}{k_t}$$

$$\theta_3^1 = T_1\,\alpha_{31} + T_2\,\alpha_{32} + T_3\,\alpha_{33},\ \ \theta_3^1 = \frac{0.5\ J\omega^2}{k_t} + \frac{2(0.83)J\omega^2}{k_t} + \frac{3\ J\omega^2}{k_t}$$

$$= \frac{J\omega^2}{k_t}(0.5 + 1.66 + 3.0) = 5.16\,\frac{J\omega^2}{k_t}$$

$$\therefore \quad \frac{\theta_1^1}{\theta_2^1} = \frac{2.33\ J\omega^2}{k_t} \times \frac{k_t}{4.16\ J\omega^2} = \frac{2.33}{4.16} = 0.55$$

$$\therefore \quad \frac{\theta_1^1}{\theta_3^1} = \frac{2.33\ J\omega^2}{k_t} \times \frac{k_t}{5.16\ J\omega^2} = \frac{2.33}{5.16} = 0.45.$$

III—Iteration: Assume $\theta_1 = 0.45$, $\theta_2 = 0.82$, $\theta_3 = 1$ so that $\dfrac{\theta_1}{\theta_2} = 0.55$ and $\dfrac{\theta_1}{\theta_3} = 0.45$

$$\left(\frac{\theta_1}{\theta_2} = \frac{0.45}{0.55} = 0.82\right)$$

The inertia torque can be calculated are as follows:

$$T_1 = J\omega^2\theta_1,\ T_2 = J\omega^2\theta_2,\ T_3 = J\omega^2\theta_3$$

By substituting the values of $\theta_1 = 0.45$, $\theta_2 = 0.82$, $\theta_3 = 1$, we have

$$T_1 = 0.45\ J\omega^2,\ T_2 = 0.82\ J\omega^2,\ T_3 = J\omega^2$$

$$\theta_1^1 = T_1\alpha_{11} + T_2\alpha_{12} + T_3\alpha_{13} = \frac{0.45\ J\omega^2}{k_t} + \frac{0.82\ J\omega^2}{k_t}$$

$$+\frac{J\omega^2}{k_t} = \frac{J\omega^2}{k_t}(0.45 + 0.82 + 1.0) = 2.27\,\frac{J\omega^2}{k_t}$$

$$\theta_2^1 = T_1\,\alpha_{21} + T_2\,\alpha_{22} + T_3\,\alpha_{23} = \frac{0.45\,J\omega^2}{k_t} + \frac{2(0.82)\,J\omega^2}{k_t}$$

$$+\frac{2\,J\omega^2}{k_t} = \frac{J\omega^2(0.45 + 1.64 + 2)}{k_t} = 3.09\,\frac{J\omega^2}{k_t}$$

$$\theta_3^1 = T_1\,\alpha_{31} + T_2\,\alpha_{32} + T_3\,\alpha_{33},\ \theta_3^1 = \frac{0.45\,J\omega^2}{k_t} + 2(0.82)\frac{J\omega^2}{k_t}$$

$$+\frac{3\,J\omega^2}{k_t} = \frac{J\omega^2}{k_t}(0.45 + 1.64 + 3.0)$$

$$= 5.09\,\frac{J\omega^2}{k_t}$$

$$\therefore \qquad \frac{\theta_1^1}{\theta_2^1} = \frac{2.27}{3.09} = 0.73$$

$$\therefore \qquad \frac{\theta_1^1}{\theta_3^1} = \frac{2.27}{5.09} = 0.44.$$

IV—Iteration: Assume $\theta_1 = 0.44$, $\theta_2 = 0.6$, $\theta_3 = 1$

so that $\dfrac{\theta_1}{\theta_2} = 0.73$ and $\dfrac{\theta_1}{\theta_3} = 0.44 \left(\dfrac{\theta_1}{\theta_2} = \dfrac{0.44}{0.6} = 0.6\right)$

The inertia torque can be calculated as follows: $T_1 = J\omega^2\theta_1$, $T_2 = J\omega^2\theta_2$, $T_3 = J\omega^2\theta_3$

By substituting the values of $\theta_1 = 0.44$, $\theta_2 = 0.6$, $\theta_3 = 1$, we have

$$T_1 = 0.44\,J\omega^2,\ T_2 = 0.6\,J\omega^2,\ T_3 = J\omega^2$$

$$\theta_1^1 = T_1\,\alpha_{11} + T_2\,\alpha_{12} + T_3\,\alpha_{13} = \frac{0.44\,J\omega^2}{k_t} + \frac{0.6\,J\omega^2}{k_t}$$

$$+\frac{J\omega^2}{k_t} = \frac{J\omega^2}{k_t}(0.44 + 0.6 + 1.0) = 2.04\,\frac{J\omega^2}{k_t}$$

$$\theta_2^1 = T_1\,\alpha_{21} + T_2\,\alpha_{22} + T_3\,\alpha_{23}\cdot = \frac{0.44\,J\omega^2}{k_t} + \frac{2(0.6)\,J\omega^2}{k_t}$$

$$+\frac{2\,J\omega^2}{k_t} = \frac{J\omega^2(0.44 + 1.20 + 2)}{k_t} = 3.64\,\frac{J\omega^2}{k_t}$$

$$\theta_3^1 = T_1\,\alpha_{31} + T_2\,\alpha_{32} + T_3\,\alpha_{33}\ \theta_3^1 = \frac{0.44\,J\omega^2}{k_t} + \frac{2(0.6)\,J\omega^2}{k_t}$$

$$+\frac{3\,J\omega^2}{k_t} = \frac{J\omega^2}{k_t}(0.44 + 1.20 + 3.0)$$

$$= 4.64\,\frac{J\omega^2}{k_t}$$

$$\therefore \qquad \frac{\theta_1^1}{\theta_2^1} = \frac{2.04}{3.64} = 0.58. \quad \therefore \frac{\theta_1^1}{\theta_3^1} = \frac{2.04}{4.64} = 0.43.$$

V—Iteration: Assume $\theta_1 = 0.43$, $\theta_2 = 0.74$, $\theta_3 = 1$

so that $\quad \dfrac{\theta_1}{\theta_2} = 0.58$ and $\dfrac{\theta_1}{\theta_3} = 0.43\left(\dfrac{\theta_1}{\theta_2} = \dfrac{0.43}{0.58} = 0.6\right)$.

The inertia torque can be calculated are as follows:

$$T_1 = J\omega^2\theta_1,\ T_2 = J\omega^2\theta_2,\ T_3 = J\omega^2\theta_3$$

By substituting the values of $\theta_1 = 0.43$, $\theta_2 = 0.74$, $\theta_3 = 1$, we have

$$\theta_1^1 = T_1\,\alpha_{11} + T_2\,\alpha_{12} + T_3\,\alpha_{13},\ \theta_1^1 = \frac{0.43\,J\omega^2}{k_t} + \frac{0.74\,J\omega^2}{k_t}$$

$$+ \frac{J\omega^2}{k_t} = \frac{J\omega^2}{k_t}(0.43 + 0.74 + 1.0) = 2.17\,\frac{J\omega^2}{k_t}$$

$$\theta_2^1 = T_1\,\alpha_{21} + T_2\,\alpha_{22} + T_3\,\alpha_{23},\ \theta_2^1 = \frac{0.43\,J\omega^2}{k_t} + \frac{1.48\,J\omega^2}{k_t}$$

$$+ \frac{2\,J\omega^2}{k_t} = \frac{J\omega^2(0.43 + 1.48 + 2)}{k_t} = 3.91\,\frac{J\omega^2}{k_t}$$

$$\theta_3^1 = T_1\,\alpha_{31} + T_2\,\alpha_{32} + T_3\,\alpha_{33},\ \theta_3^1 = \frac{0.43\,J\omega^2}{k_t} + \frac{1.48\,J\omega^2}{k_t} + \frac{3\,J\omega^2}{k_t}$$

$$= \frac{J\omega^2}{k_t}(0.43 + 1.48 + 3.0)$$

$$= 4.91\,\frac{J\omega^2}{k_t}$$

$$\therefore \qquad \frac{\theta_1^1}{\theta_2^1} = \frac{2.17}{3.91} = 0.55$$

$$\therefore \qquad \frac{\theta_1^1}{\theta_3^1} = \frac{2.17}{4.91} = 0.44$$

$$\therefore \qquad 0.43 = 2.17\,\frac{J\omega^2}{k_t}$$

$$\therefore \qquad \omega^2 = \frac{0.43\,k_t}{2.17J}$$

$$\therefore \qquad \omega^2 = \frac{0.43 \times 0.1}{2.17 \times 0.1} = 0.2$$

$\therefore\ \omega_n = 0.447$ rad/s. is the fundamental frequency of the system.

EXAMPLE 8.18

Find the first two natural frequencies of vibration for the system shown in Fig. p-8.18 by using matrix iteration method.

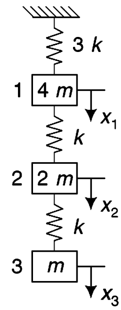


Fig.p-8.18 *Spring-mass system*

Solution By influence coefficient,

$$a_{11} = a_{21} = a_{31} = \frac{1}{3k} = a_{12} = a_{13},\ a_{22} = a_{32} = \frac{4}{3k},\ a_{33} = \frac{7}{3k}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\quad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{m\omega^2}{3k}12\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{m\omega^2}{3k}14\begin{bmatrix} 1 \\ 3.14 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3.14 \\ 4 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 1 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3.14 \\ 4 \end{bmatrix} = \frac{m\omega^2}{3k}14.28\begin{bmatrix} 1 \\ 3.16 \\ 4 \end{bmatrix}$$

$$= \frac{m\omega^2}{3k}12.28 = 1,\ \omega_1 = 0.458\sqrt{\frac{k}{m}}\ \text{rad/s},\ A_1 = 1,\ B_1 = 3.16,\ C_1 = 4$$

$$4mA_2 + 2m(3.16)B_2 + 4mC_2 = 0,\ A_2 = -1.6B_2 - C_2,\ B_2 = B_2,\ C_2 = C_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 0 & -1.6 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.4 & -3 \\ 0 & 1.6 & 0 \\ 0 & 1.6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 0 & -4.4 & -3 \\ 0 & 1.6 & 0 \\ 0 & 1.6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{m\omega^2}{3k}3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \omega_1 = \sqrt{\frac{k}{m}}\ \text{rad/s.}$$

REVIEW QUESTIONS

(1) A rotating shaft carries a number of weights w_1, w_2..., etc., at points 1,2..., etc., along its length. If y_1, y_2, ..., etc., are the static deflections of the shaft at points 1, 2, ..., etc., show that the fundamental natural frequency according to Rayleigh's method is given by expression $\omega^2 = \sqrt{\dfrac{g(w_1y_1 + w_2y_2 + w_3y_3)}{w_1y_1^2 + w_2y_2^2 + w_3y_3^2}}$.

(2) With suitable assumptions derive the Rayleigh's equation for determining the fundamental natural frequency of a multimass system.

(3) Explain Dunkerley's method for estimating the lowest working speed of a shaft carrying a number of discs.

(4) Derive an expression for natural frequency of a shaft carrying several loads using Dunkerley's method.

(5) Explain Holzer's method of analyzing multi-degree freedom systems.

(6) Explain 'Stodola's' method to estimate the natural frequency and mode shapes of multi-degree freedom systems.

(7) Distinguish between Stodola's method and Holzer's method.

(8) Differentiate between matrix iteration and matrix expansion method of solving with several degrees of freedom.

(9) Explain: (a) Dunkerley's method (b), Rayleigh's method (c) Method of matrix iteration.

(10) Write short notes on (i) Stodola's method, (ii) Rayleigh–Ritz method, and (iii) Holzer's method. What are node points and mode shapes as applied to several degrees of freedom systems?

PROBLEMS FOR PRACTICE

(1) Find the fundamental natural frequency of the spring-mass system as shown in Fig. p.p-8.1 using Dunkerley's method.

Ans. $0.279\sqrt{\dfrac{k}{4}}$ rad/s.

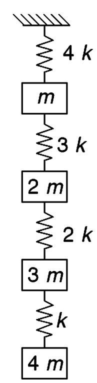


Fig. p.p-8.1

8.2 A solid steel shaft of uniform diameter with two discs is represented by a simply supported beam carrying two concentrated weights as shown in Fig. p.p-8.2.

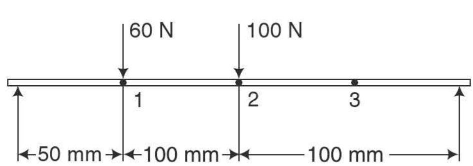


Fig. p.p-8.2

Determine the natural frequency of the system by using Rayleigh's method. Compare the result by using Dunkerley's method. $E = 1.96 \times 10^{11}$ Pa, $I = 40 \times 10^{-8} \text{m}^4$.

(3) A 40 mm diameter shaft that is 2.5 m long has a mass of 15 kg per metre length. It is simply supported at the ends and carries three masses of 90 kg, 140 kg and 60 kg at 0.8 m, 1.5 m and 2 m respectively from the left supports. Take E = 200 GPa.

Determine the natural frequency of the transverse vibration of the system by using Dunkerley's method.

Ans f_n = 2.9 cps.

(4) Determine the lowest natural frequency of transverse vibration of the system as shown in Fig. p.p-8.4. By using Rayleigh's method. Take E = 196 GPa, $I = 1 \times 10^{-6}\ m^4$, m_1 = 40 kg, m_2 = 20 kg.

Ans 1541 rad/s.

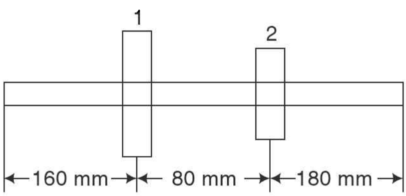


Fig. p.p-8.4

(5) Determine the natural frequencies of the spring-mass system as shown in Fig. p.p-8.5. By Holzer's method. Take m_1 = 1 kg, m_2 = 3 kg and m_3 = 2 kg; $k_1 = k_2$ = 50 N/m.

Ans ω_{n1} rad/s, ω_{n2} = 6.8 rad/s, ω_{n3} = 8.7 rad/s.

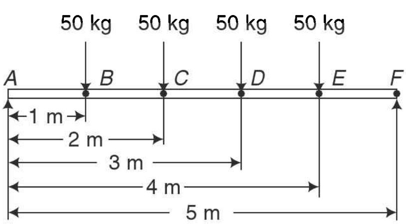


Fig. p.p-8.5

(6) Determine the fundamental natural frequency of the spring-mass system as shown in Fig. p.p-8.6 by Holzer's method.

Ans $\omega_{n3} = 1.2\sqrt{\frac{k}{m}}$ rad/s.

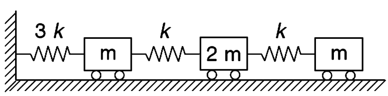


Fig. p.p-8.6

(7) Determine the natural frequencies of the spring-mass system as shown in Fig. p.p-8.7 by Holzer's method. Take m = 2.5 kg, k = 3.5 kN/m.

Ans ω_{n1} = 11 rad/s, ω_{n2} = 30 rad/s, ω_{n3} = 55 rad/s, ω_{n4} = 105 rad/s.

(8) Determine the natural frequencies of the system as shown in Fig. p.p-8.8 by Holzer's method. Take $J = 100$ kg-m^2, $k = 10^6$N-m/rad .

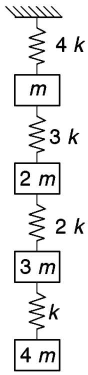


Fig. p.p-8.7

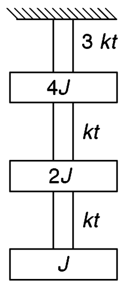


Fig. p.p-8.8

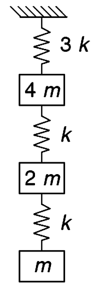


Fig. p.p-8.9

(9) Determine the natural frequencies and principal modes of vibration for the spring-mass system as shown in Fig. p.p-8.9 by matrix iteration method.

Ans $\omega_{1n} = 0.458\sqrt{\frac{k}{m}}$ rad/s, $\omega_{2n} = \sqrt{\frac{k}{m}}$ rad/s, $\omega_{3n} = 1.336\sqrt{\frac{k}{m}}$ rad/s.

Principal modes $\begin{bmatrix} 1 \\ 3.158 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1.0 \\ -3.143 \\ 4 \end{bmatrix}$.

(10) Determine the natural frequencies and principal modes of vibration for the spring-mass system as shown in Fig.p.p-8.10 by matrix iteration method.

$m_1 = 100$ kg, $m_2 = 200$ kg, and $m_3 = 300$ kg, $k_1 = 2$ kN/m, $k_2 = 1.5$ kN/m and $k_3 = 2$ kN/m.

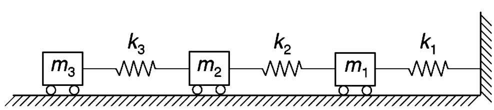


Fig. p.p-8.10

OBJECTIVE-TYPE QUESTIONS

(1) Dunkerley's equation is capable of giving
  (a) only the fundamental mode shapes
  (b) only the fundamental natural frequency
  (c) only the fundamental mode and node shapes
  (d) all of the above cases

(2) Dunkerley's method as an approximate equation can be derived
  (a) from the algebraic rules
  (b) by applying Newton's law of motion
  (c) by writing the differential equations

(d) by iterative process

(3) Rayleigh's method has been used in analysing
(a) multi-degree-freedom systems where the distributed mass was lumped up at places of known stiffness
(b) an iterative procedure to determine the principal modes of the system and its natural frequencies
(c) single-degree-of-freedom systems where the distributed mass was lumped up at places of known stiffness
(d) all of the above cases

(4) The natural frequencies is given by the equation in case of Rayleigh's method

(a) $f_n = \frac{1}{2\pi}\sqrt{\frac{\sum_{i=1}^{n} W_i Y_i}{g \sum_{i=1}^{n} W_i Y_i^2}}$

(b) $f_n = \frac{1}{2\pi}\sqrt{\frac{g\sum_{i=1}^{n} W_i Y_i}{\sum_{i=1}^{n} W_i Y_i^2}}$ Hz

(c) $f_n = \frac{1}{2}\omega_2^2 a_{22} + \frac{1}{2}\omega_1^2 a_{11} + w_2 w_1 a_{21}$ Hz

(d) $f_n = \frac{1}{\omega_{1n}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \cdots \frac{1}{\omega_n^2}$ Hz

(5) The Holzer's method is particularly useful for calculating the frequencies of
(a) the fundamental natural frequency of free vibration
(b) to derive the differential equation of motion
(c) analysing single-degree-of-freedom systems
(d) torsional vibrations of shaft

(6) The Stodola method is used to determine
(a) all the natural frequencies of a system
(b) equation of motion in terms of influence coefficient
(c) fundamental natural frequency of free undamped vibrating systems
(d) all natural frequency of free undamped vibrating systems

(7) The Holzer's method is particularly useful for calculating the frequencies of
(a) fixed-free system
(b) free-free system
(c) fixed-fixed system
(d) all of the above cases

(8) Stodola's method is usually applicable for
(a) fixed-free system
(b) free-free system
(c) fixed-fixed system
(d) none of the above cases

(9) The Rayleigh–Ritz method is usually applicable for
(a) calculating the frequencies of torsional vibrations
(b) solving for the beam problems
(c) calculating the frequencies of both linear and torsional vibrations
(d) calculating the frequencies of both linear and non-linear vibrations

(10) Matrix iteration method is
(a) An iterative procedure to determine the principal modes of the system and its natural frequencies
(b) Iterative numerical procedures are employed to eliminate the tedious mathematical work
(c) is a trial and error or tabular method used for the determination of natural frequency for free or forced vibration
(d) all of the above cases

Answers

(1) b	(2) a	(3) c	(4) b	(5) d	(6) c
(7) d	(8) a	(9) b	(10) a		

VIBRATIONS OF A CONTINUOUS SYSTEM

9

9.1 INTRODUCTION

Until now we have studied that discrete systems such as mass, elasticity and damping were assumed to be present only at certain discrete points in the system. There are many cases known as continuous or distributed systems such as beams, rods, cables, plates, etc. A continuous system will have continuously distributed mass and stiffness. Such a system is equivalent to an infinite element of masses concentrated at different points and hence it is an infinite-degrees-of-freedom system. This system consists of an infinitely large number of particles, and hence requires an infinitely large number of coordinates to specify their configuration. That is why it is also called infinite degrees of freedom.

Equations for these systems are derived by assuming that all materials or elements are homogeneous, isotropic; obey Hooke's law within the elastic limit and are of uniform cross section.

These systems have infinite principal modes of vibrations corresponding to infinite natural frequencies of the system. In general, for analysis of the vibration of continuous system knowledge of partial differential equations is very much essential. They consist of many constant boundary and initial conditions.

In this chapter, we shall consider the vibrations of simple continuous systems like vibration of strings, longitudinal vibration of rods, and torsional vibration of rods. Euler's equations of beams are considered for the analysis using exact and approximate energy method.

1. Boundary and initial conditions In case of partial differential equations, the unknown value of constants can be determined by applying either geometric or natural or both boundary conditions.

2. Geometric boundary conditions These are due to geometric compatibility.

For example, if the bar is fixed at both the ends, the displacement and slope will be zero.

3. Natural boundary conditions These are due to force and moments.

For example, if the bar is hinged at one end, the bending moment at the hinged end will be zero and so on, whereas the initial conditions are related to time.

9.1 LATERAL VIBRATION OF A STRING

Let us consider a string subjected to a transverse (lateral) vibration under tension 'T' of length 'L' as shown in Fig. 9.1(a) and let 'ρ' be the mass per unit length.

Assume that tension 'T' is large and is constant throughout its length 'L' also the amplitude of transverse vibration of the string is very small. For very small displacements of the string, $\sin\theta_1 = \tan\theta_1 = \theta_1$.

Let us consider a small element of length 'dx' at a distance 'x' from the y-axis as shown in Fig. 9.1(a). Let this element be displaced through a distance 'y' from the equilibrium position; then the FBD as shown in Fig. 9.1(b).

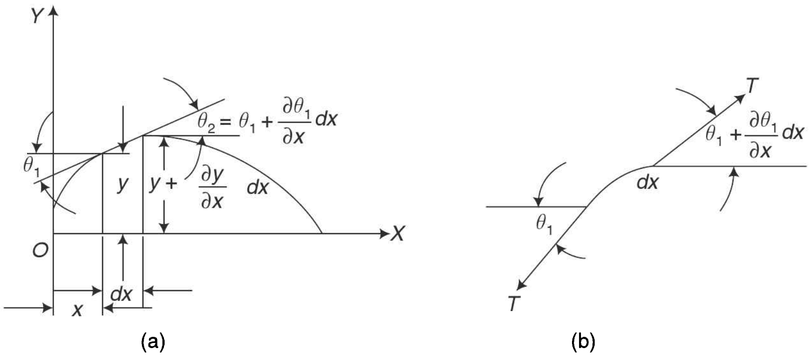


Fig. 9.1 *String in lateral vibration*

From FBD, let 'θ_1' be the angle subtended between the tangent to the small elemental string and normal to the elemental string at the left side of the string. Similarly, 'θ_2' be the angle subtended between the tangent to the elemental string and normal to the string, at the right side of the elemental string.

From geometry of the Fig. 9.1(b) in FBD, $\tan\theta_1 = \frac{\delta y}{\delta x}$.

If 'θ_1' is very small, $\tan\theta_1 = \theta_1$

$$\therefore \qquad \theta_1 = \frac{\delta y}{\delta x}, \text{ and } \tan\theta_2 = \frac{\delta y}{\delta x} + \frac{\delta}{\delta x}\left(\frac{\delta y}{\delta x}\right)dx,\ \theta_2 = \theta_1 + \frac{\delta\theta_1}{\delta x}dx \qquad \text{...9.1}$$

$$\text{since} \qquad \theta_1 = \frac{\delta y}{\delta x}.$$

Resolving the forces along y-axis,

$$T\sin\theta_1 + \left(\frac{\partial\theta_1}{\partial x}dx\right) - T\sin\theta_1 = \text{Mass} \times \text{Acceleration}$$

$$T\left(\theta_1 + \frac{\partial\theta_1}{\partial x}dx\right) - T\theta_1 = \rho \cdot dx\,\frac{\partial^2 y}{\partial t^2},$$

where $\rho.dx$ = Mass of the small element of length 'dx'

$$T\left(\theta_1 + \frac{\partial \theta_1}{\partial x}dx\right) - T\theta_1 = \rho.dx\frac{\partial^2 y}{\partial t^2},\ T\frac{\partial \theta_1}{\partial x}dx = \rho.dx\frac{\partial^2 y}{\partial t^2},\ T\frac{\partial \theta_1}{\partial x} = \rho\frac{\partial^2 y}{\partial t^2}.$$

$$\frac{T}{\rho}\left(\frac{\partial \theta_1}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2},\quad \frac{T}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2},\left(\text{since } \theta_1 = \frac{\partial y}{\partial x}\right).$$

$$\frac{T}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2} \qquad ...9.2$$

Let $a^2 = \frac{T}{\rho}$

The equation can be written as,

$$a^2\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2},\ \frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad ...9.3$$

This is a one-dimensional wave equation for lateral vibration of string and the constant 'a' as the wave propagation velocity.

This equation has four arbitrary constants and can be solved by boundary and initial conditions.

Solution of wave equation The lateral deflection 'y' along the string is a function of the variables 'x' and 't'. So it can be written as $y = y(x, t)$. ...9.4

Let us assume the harmonic mode of vibration as the system is undamped.

Thus, solution of Eq. 9.3 can be written as $y(x, t) = X(x)\ T(t)$. ...9.5

Substituting the above solution in Eq. 9.3, we get

$$\frac{a^2}{X}\cdot\frac{d^2X}{dx^2} = \frac{1}{T}\cdot\frac{d^2T}{dt^2} \qquad ...9.6$$

In this equation, LHS is a function of 'x' alone and RHS is a function of 't' alone. The above two can only be equal if each of the above equations is a constant. These constants will be 'zero', 'negative' or it may be 'positive'. If we consider 'zero' or 'positive constant' then there is no vibratory motion which is contrary to our observations for the practical systems. So we put it equal to some constant Z^2.

$$\frac{d^2X}{dx^2} + \left(\frac{Z}{a}\right)^2 X = 0 \text{ and } \frac{d^2T}{dt^2} + T^2 = 0 \qquad ...9.7$$

The solutions of the above two equations are

$$X(x) = A\cos\left(\frac{Z}{a}\right)x + B\sin\left(\frac{Z}{a}\right)x,\ T(t) = C\cos Zt + D\sin Zt.$$

The general solution can be written as

$$y(x, t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{Z}{a}\right)x + B_n\sin\left(\frac{Z}{a}\right)x\right)[C_n\cos Zt + D_n\sin Zt] \qquad ...9.8$$

In this equation, 'Z' is the frequency of vibration. The values of arbitrary parameters A_n, B_n, C_n and D_n in the above equation, can be determined by assuming boundary and initial conditions.

1. Boundary conditions Let us assume the string is fixed at both ends, i.e.

$y(0, t) = 0$ and $y(L, t) = 0$ ...9.9

2. Initial conditions Assuming the initial displacement and velocity as,

at $t = 0, \quad y(x, 0) = S(x)$

at $t = 0, \quad y(x, 0) = V(x)$ ...9.10.

Using these boundary conditions of equations 9.9 and 9.10 in Eq. 9.8, we have

$$y(0, t) = A_n(C_n \cos Zt + D_n \sin Zt) \text{ gives } A_n = 0$$

$$y(\mathrm{L}, t) = B_n \sin\left(\frac{Z}{a}\right) L(C_n \cos Zt + D_n \sin Zt), \quad \text{if } B_n \neq 0$$

which gives $\sin\left(\frac{Z}{a}\right) L = \sin n\,\pi = 0$ ...9.11.

This equation is called frequency equation.

$$\frac{Z_n}{a} L = n\pi,\ Z_n = \frac{n\pi a}{L}\left(\because a^2 = \frac{T}{\rho}\right), \text{ so frequency } Z_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} \text{ rad/s} \qquad ...9.12$$

Normal mode shape can be written as $X(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, 3$ ...9.13

Each 'n' represents a mode of vibration example for $n = 1$ (first mode) $n = 2$ (second mode) and so on. Equation 9.8 can be written as

$$y(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} C_n \cos Z_n t + D_n \sin Z_n t \qquad ...9.14$$

The values of constants 'C_n' and 'D_n' can be determined from initial conditions, i.e. displacement is $s(x)$ at $t = 0$ and velocity is $v(x)$ at $t = 0$.

Applying initial conditions for above equation, $s(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$ ...9.15

$$v(\dot{x}) = \sum_{n=1}^{\infty} Z_n D_n \sin\frac{n\pi x}{L} \qquad ...9.16$$

Multiply the equations 9.15 and 9.16 each by $\sin\frac{m\pi x}{L}$, where $m = 1, 2, 3...$

and integrate from $x = 0$ to L.

Thus, $$\int_0^L s(x) \sin\frac{m\pi x}{L}\,dx = \int_0^L C_n\left(\sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\right) dx$$

$\sin\frac{n\pi x}{L}$ and $\sin\frac{m\pi x}{L}$ are orthogonal functions and the value of the above integral will be zero except when $m = n$.

Replacing $m = n$ for nonzero value of 'C_n', we get

$$\int_0^L s(x) \sin\frac{n\pi x}{L}\,dx = \int_0^L C_n \sin^2\frac{n\pi x}{L}\,dx$$

$$\int_0^L s(x) \sin\frac{n\pi x}{L}\,dx = C_n \int_0^L \frac{1}{2}\left(1 - \cos^2\frac{n\pi x}{L}\right) dx$$

So $$C_n = \frac{2}{L}\int_0^L s(x)\sin\frac{n\pi x}{L}\,dx \qquad ...9.17$$

Similarly considering Eq. 9.16,

$$\int_0^L v(x)\sin\frac{m\pi x}{L}\,dx = Z_n D_n \int_0^L \sin\frac{n\pi x}{L}\cdot\sin\frac{m\pi x}{L}\,dx.$$

For nonzero value of D_n replace $m = n$

$$\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx = Z_n D_n \int_0^L \sin^2\frac{n\pi x}{L}$$

$$D_n = \frac{2}{Z_n L}\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx. \qquad ...9.18$$

9.3 LONGITUDINAL VIBRATION OF RODS OR BARS

Let us consider a prismatic bar of length 'L' subjected to longitudinal vibration as shown in Fig. 9.2(a). Let 'A' be the cross-sectional area of the bar, 'E' be the Young's modulus of the materials, 'ρ' be the density of the material, and 'm' be the mass per unit length.

Let us assume that the bar should be thin and of uniform cross-section throughout of its length and subjected to axial force 'F' and there will be displacements 'u' along the rod that will be a function of both positions 'x' and time 't', because the rod has an infinite number of natural modes of vibration. The distribution of the displacements will differ with each mode as shown in Fig. 9.2(a).

Let us consider a small elemental length 'dx' at a distance 'x' from the left end and 'F' be the axial force on a small elemental length. The force on the other side, i.e. right side of small elemental length is equal to $\left(F + \frac{\partial F}{\partial x}dx\right)$.

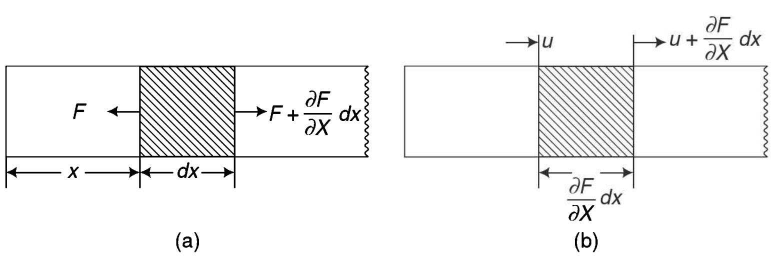


Fig. 9.2 *Longitudinal vibration of rods*

If 'u' is the displacement at a distance 'x' from the left side and $\left(u + \frac{\partial u}{\partial x}dx\right)$ displacement at a distance $x + dx$ at the right side of small elemental length. Now it is clear that from FBD as shown in Fig. 9.2(b), due to these axial forces on the small elemental length 'dx' there is a changed length by an amount equal to $\left(u + \frac{\partial u}{\partial x} dx - u\right)$ $= \left(\frac{\partial u}{\partial x} dx\right)$.

We know from mechanics of materials, when an element or a body is subjected either to the tension or compression, it undergoes stress, strain and deformation.

By definition of strain ($\in$) = Change in length /original length

$$\therefore \qquad \in = \frac{\frac{\partial u}{\partial x} dx}{dx} = \frac{\partial u}{\partial x} \qquad \text{...9.19}$$

Net force acting on the small element,

$$\left(F + \frac{\partial F}{\partial x} \partial x\right) - F = (\text{Mass}) \times (\text{Acceleration of the element})$$

$$= dm \times \frac{\partial^2 u}{\partial t^2}, \quad \text{where } dm = \text{Mass of the small elemental length}$$

$$\frac{\partial F}{\partial x} dx = (\rho dxA) \left(\frac{\partial^2 u}{\partial t^2}\right) \qquad \text{...9.20}$$

ρ = Density and $dx\, A$ = Volume of the small elemental length

We know that definition of stress (σ) equal to load /area or $\sigma = \frac{F}{A}$ or $F = \sigma A$.

$$\frac{\partial F}{\partial x} = \frac{\partial \sigma}{\partial x} A, \left(\frac{\partial F}{\partial x}\right) dx = \left(\frac{\partial \sigma}{\partial x}\right) dx\, A \qquad \text{...9.21}$$

Equation 9.20 can be written with the help of above equation as

$$\left(\frac{\partial \sigma}{\partial x}\right) dxA = (\rho dxA) \left(\frac{\partial^2 u}{\partial t^2}\right) \qquad \text{...9.22}$$

According to Hooke's law, stress $\propto$ strain 'within' elastic limit, i.e. $\sigma \propto \in$, $\sigma = E\in$ or

$$E = \frac{\sigma}{\in}, \quad \frac{\text{Stress}}{\text{Strain}} = E, \text{ where } E = \text{Young's modulus}, \sigma = \in E, \left(\frac{\partial \sigma}{\partial x}\right) dxA = \left(\frac{\partial^2 u}{\partial t^2}\right) dxAE \qquad \text{...9.23}$$

With the help of Eq. 9.22 and Eq. 9.23,

we get, $\left(\frac{\partial \in}{\partial x}\right) dxAE = (\rho dxA) \left(\frac{\partial^2 u}{\partial t^2}\right)$

But $\left(\in = \frac{\partial u}{dx}\right)$ [Eq. 9.19]

$$\left(\frac{E}{\rho}\right) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial t^2}\right), \frac{E}{\rho} \left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 u}{\partial t^2}, \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \text{ where } a^2 = E/\rho$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...9.24}$$

This is the wave equation which is identical to Eq. 9.3 or $\left(\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2}\right)$.

The general solution will be same as in the previous case of lateral vibrations.

A solution of the form is as in $u(x, t) = X(x)\, T(t)$

So $\quad X(x) = A \sin \frac{Z_n x}{a} + B \cos \frac{Z_n x}{a},\; T(t) = C \sin Z_n t + D \cos Z_n t$

will result into the general solution as

$$u(x, t) = \sum_{n=1}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t) \qquad \text{...9.25}$$

EXAMPLE 9.1

Derive the frequency equation of longitudinal vibration for a free-free beam with zero initial displacement.

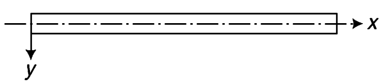


Fig. p-9.1 *Longitudinal vibration of a beam*

Solution The system is as shown in Fig. p-9.1.

We know that the general solution of longitudinal vibration of a uniform bar is given by Eq. 9.25

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t).$$

where $a = \sqrt{\frac{E}{\rho}}$ and $Z_n = 2\pi f_n$; Z_n is the natural frequency.

The boundary conditions for the above particular system (free-free beam with zero initial displacement) are $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$ and $\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0$ (for free end on both ends, strain is zero).

Differentiating the above equation (9.25) w.r.t. 'x' partially and applying these boundary conditions to the general solution, we get

$$\left(\frac{\partial u}{\partial x}\right) = \left(A \frac{Z_n}{a} \cos \frac{Z_n}{a} x - B \frac{Z_n}{a} \sin \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t) \qquad \text{...9.26}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = A \frac{Z_n}{a}(C \sin Z_n t + D \cos Z_n t) \qquad \therefore A = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \left(-B \frac{Z_n}{a} \sin \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t),$$

$$\left(-B \frac{Z_n}{a} \sin \frac{Z_n}{a} L\right)(C \sin Z_n t + D \cos Z_n t)$$

By using solution of wave equations 9.17 and 9.18, we can determine the values of constants 'C' and 'D' from initial conditions.

So $\sin \frac{Z_n}{a} L = 0$, $\sin n\pi$, $Z_n = \frac{n\pi a}{L}$, $n = 1, 2, 3...$

We know that $Z_n = 2\pi f_n$, $2\pi f_n = \frac{n\pi a}{L}$

Therefore, the natural frequency $f_n = \frac{n}{2L} a$, but $a = \sqrt{\frac{E}{\rho}}$

$\therefore \quad f_n = \frac{n}{2L}\sqrt{\frac{E}{\rho}}$, '$n$' represent the order of the mode.

EXAMPLE 9.2

Derive an expression for the free longitudinal vibration of a uniform bar of length 'L', one end of which is fixed and the other end is free as shown in Fig. p-9.2.

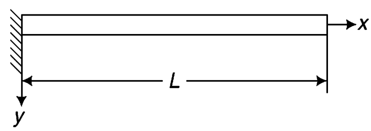


Fig. p-9.2 *Uniform bar*

Solution We know that the general solution of longitudinal vibration of a uniform bar is given by Eq. 9.25.

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t)$$

The boundary conditions for above particular system (one end of which is fixed and other end is free) are

$(u)_{x=0} = 0$, (displacement is zero at fixed end) and

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0, \text{ (strain is zero at free end).}$$

Differentiating Eq. 9.26 w.r.t. 'x' partially, we get

$$\left(\frac{\partial u}{\partial x}\right) = \left(A \frac{Z_n}{a} \cos \frac{Z_n}{a} x - B \frac{Z_n}{a} \sin \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t) \quad ...9.27$$

Applying the boundary conditions to the general solution of Eq. 9.1, we have $B = 0$

$$0 = A \frac{Z_n}{a} \cos \frac{Z_n}{a} L (C \sin Z_n t + D \cos Z_n t) \text{ or } \cos \frac{Z_n}{a} L = 0 = \cos \frac{n\pi}{2},$$

where $n = 1, 3, 5...$

And $\quad A \neq 0.\ \frac{Z_n}{a} L = \frac{n\pi}{2}, \quad Z_n = \frac{n\pi a}{2L} \quad$ But $Z_n = 2\pi f_n$, $2\pi f_n = \frac{na\pi}{2L}$

$\therefore \quad f_n = \frac{n}{4L}\sqrt{\frac{E}{\rho}} \quad \therefore a = \sqrt{\frac{E}{\rho}}$

The general solution of longitudinal vibration of a uniform bar can be written as

$$u(x, t) = \sum_{n=1,3,5.}^{\infty} \sin \frac{nx\pi}{2L}\left(C \sin \frac{na\pi}{2L} t + D \cos \frac{na\pi}{2L} t\right).$$

EXAMPLE 9.3

A bar of uniform cross-section having length '*L*' is fixed at both ends as shown in Fig. p-9.3. A bar is subjected to longitudinal vibrations having a constant velocity 'v_0' at all points. Derive suitable mathematical expression of longitudinal vibrations in the bar.

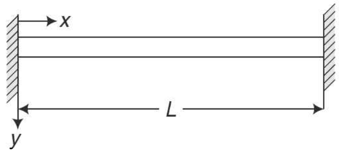


Fig. p-9.3 *Uniform bar*

Solution As we know that the general solution of longitudinal vibration of a uniform bar (9.25) can be written as

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \text{ Cos } \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t)$$

***The boundary conditions, for the above particular system (fixed at both ends) are x = 0, displacement* = 0,**

i.e. $u(0, t) = 0, \quad u(L, t) = 0$

By using the first boundary condition, in the above general solution of longitudinal vibration of a uniform bar (9.25), we get

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \sin \frac{n\pi x}{L}(C \sin Z_n t + D \cos Z_n t)$$

$$B = 0$$

And by using second boundary conditions, we have $\frac{Z_n L}{a} = 0 = \sin n\pi$

$$n = 1,2,3..., \text{ but } Z_n = \frac{n\pi a}{L}, \; a = \sqrt{\frac{E}{\rho}}$$

Substituting these value of Z_n in Eq. 9.20, we get

Again the initial conditions are $u(x, 0) = 0, \quad \dot{u}(x, 0) = V_0$

By using the first initial condition in the above general solution, we get

$$0 = \sum_{n=1,2,3,...}^{\infty} \sin \frac{n\pi x}{L} \cdot D, \; D = 0$$

Then the equation is $u(x, t) = \sum_{n=1,2,3,...}^{\infty} \sin \frac{n\pi x}{L} \cdot C \sin Z_n t$

By using the second initial condition in the above general solution, we get

Then the equation is $\dot{u}(x, t) = \sum_{n=1,2,3,...}^{\infty} \sin \frac{n\pi x}{L} \cdot C Z_n \cos Z_n t$

$$\dot{u}(x, 0) = \sum_{n=1,2,3,...}^{\infty} C Z_n \sin \frac{n\pi x}{L} = V_0$$

or $$C = \frac{2}{n\pi a}\int_0^L V_0 \sin \frac{n\pi x}{L} dx \text{ (Eq. 9.18)} \quad C = \frac{2V_0 L}{n^2\pi^2 a}(1 - \cos n\pi)$$

So $$C = \frac{4V_0 L}{n^2\pi^2 a} \text{ when } n = 1, 3, 5... \quad \text{and } C = 0 \text{ when } n = 2, 4, 6...$$

Finally, the required expression can be written as

$$u(x, t) = \frac{4V_0L}{\pi^2 a} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2} \sin\frac{n\pi x}{L} \sin\frac{n\pi a}{L} t$$

EXAMPLE 9.4

Determine the normal function for free longitudinal vibration of a uniform bar of length 'L' and uniform cross-section. Both ends of the bar are fixed as shown in Fig. p-9.4.

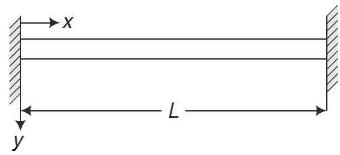


Fig. p-9.4 *Longitudinal vibration of a uniform bar*

Solution As we know, the general solution of a longitudinal vibration of a uniform bar (9.25) can be written as

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin\frac{Z_n}{a} x + B \cos\frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t)$$

The boundary conditions for the above particular system (fixed at both ends) are

$$(u)_{x=0} = (u)_{x=1} = 0$$

The displacements of this bar at its ends are equal to zero.

Substituting these boundary conditions into the general solution, we have

$$(u)_{x=0} = \sum_{n=1,2,3,...}^{\infty} T_n \left[C \cos\left(\frac{Z_n}{a}\right) x + D \sin\left(\frac{Z_n}{a}\right) x\right] = 0 \text{ or } C = 0$$

$$(u)_{x=L} = \sum_{n=1,2,3,...}^{\infty} T_n \left[D \sin\left(\frac{Z_n}{a}\right) x\right] = 0 \text{ or } \sin\left(\frac{Z_n}{a} L\right) = 0 \text{ and } Z_n = \frac{n\pi a}{L}, \text{ where}$$

$n = 1,2,3, ...$

Hence, the normal function is $X_n(x) = D \sin\frac{n\pi x}{L}$, $n = 1, 2, 3...$

EXAMPLE. 9.5

A bar of length 'L' fixed at one end and connected at the other end by a spring of stiffness 'k' is as shown in Fig. p-9.5. Derive a suitable expression of motion for longitudinal vibrations.

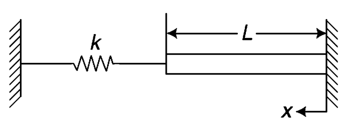


Fig. p-9.5 *Bar fixed at one end and connected at the other end by a spring*

Solution As we know, the general solution of longitudinal vibration of the bar (9.25) can be written as

$$u(x, t) = \sum_{n=1,2,3.}^{\infty} \left(A \sin\frac{Z_n}{a} x + B \cos\frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t) \quad ...9.25$$

The general solution of longitudinal vibration of a uniform bar whose one end is fixed and other end is free can be written as (similar to Example 9.2).

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{nx\pi}{2L}\left(C \sin \frac{nx\pi}{2L} t + D \cos \frac{nx\pi}{2L} t\right)$$

The boundary conditions for the above particular system are

$$(u)_{x=0} = 0 \quad AE = \frac{\partial u}{\partial x}(L, t) = ku(L, t) \text{ (Tensile force = Spring force)}$$

Applying the second boundary conditions, we get

$$AE \frac{Z_n}{a} \cos \frac{Z_n}{a} L\left(C \sin \frac{n\pi a}{2L} t + D \cos \frac{n\pi a}{2L} t\right) = k \sin \frac{Z_n}{a} L\left(C \sin \frac{n\pi a}{2L} t + D \cos \frac{n\pi a}{2L} t\right)$$

$\tan \frac{Z_n L}{a} = \frac{AE}{k} \frac{Z_n}{a}$ is the required equation.

9.4 TORSIONAL VIBRATION OF UNIFORM SHAFT OR RODS

The equation of motion for the torsional vibration of the circular uniform shafts are same as the longitudinal vibration of the uniform bars discussed in one-dimensional wave equation for lateral vibration of string Eq. 9.3.

Also the method of derivation of these equations is same as that of longitudinal vibration of bars in Eq. 9.24.

Let us consider a prismatic shaft of length 'L' subjected to torsional vibration as shown in Fig. 9.3(a).

Let us consider a small elemental length of rod 'dx' at a distance 'x' from left end, let 'T' be the applied torque and 'θ' be the angle of twist at left side of small elemental length of rod, $\left(\theta + \frac{\partial \theta}{\partial x} dx\right)$. Twist at a distance $x + dx$ from right side due to a applied torque $\left(T + \frac{\partial T}{\partial x} dx\right)$, as in Fig. 9.3(b), similar to the longitudinal vibration of rods.

J = Polar moment of inertia of shaft per unit length

I = Mass moment of inertia

G = Modulus of rigidity of the shaft material

d = Diameter of the shaft

ρ = Mass density of the material = (Mass × Volume)

From Newton's second law of motion,

Applied torque (T) = Inertia force × Angular acceleration or $T = I \times \omega$

$$\text{Net torque} = \left(T + \frac{\partial T}{\partial x} dx\right) - T = I \times \frac{d^2\theta}{dt^2} \text{ or } \left(\frac{\partial T}{\partial x} dx\right) = I \times \frac{d^2\theta}{dt^2} \qquad ...9.28$$

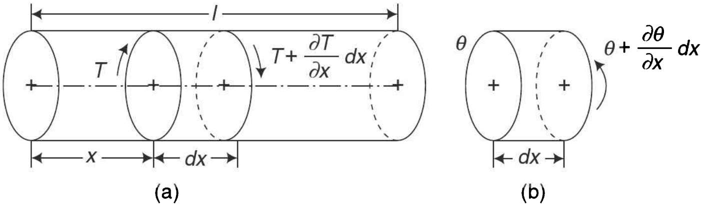


Fig. 9.3 *Torsional vibration of uniform shaft*

From mechanics of materials, the elementary torsion theory equation $\frac{T}{J}=\frac{G\theta}{l}$.

But $\frac{\theta}{l}=\frac{d\theta}{dx}$ = Twist per unit length or the rate of twist of small elemental length of rod dx.

$$\frac{T}{J}=G\frac{d\theta}{dx},\quad T=GJ\frac{d\theta}{dx},\left(\frac{\partial T}{\partial x}dx\right)=GJ\frac{\partial}{\partial x}\left(\frac{d\theta}{dx}\right)dx \qquad ...9.29$$

Comparing Eq. 9.28 and Eq. 9.29, we have

$$GJ\frac{\partial}{\partial x}\left(\frac{d\theta}{dx}\right)dx=I.\frac{d^2\theta}{dt^2} \qquad ...9.30$$

For a shaft of constant cross-section, 'GJ' is constant

$$J=\frac{\pi}{32}d^4,\ I=\frac{\pi}{32}d^4\,p.dx \text{ (mass moment of inertia)}$$

Substitute the values of 'I' and 'J' in Eq. 9.30. we get,

$$\frac{G}{\rho}\frac{\partial^2\theta}{\partial x^2}=\frac{\partial^2\theta}{\partial t^2},\ \frac{\partial^2\theta}{\partial x^2}(x,t)=\frac{1}{a^2}\frac{\partial^2\theta}{\partial t^2}(x,t) \quad \text{where } a^2=G/\rho$$

This is wave equation identical to Eq. 9.3.

$$\frac{\partial^2 y}{\partial t^2}=\frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad ...9.31$$

The general solution of the above equation can be written as

$$\theta(x,t)=\sum_{n=1}^{\infty}\left(A\sin\frac{Z_n x}{a}+B\cos\frac{Z_n x}{a}\right)(C\sin Z_n t+D\cos Z_n t). \qquad ...9.32$$

EXAMPLE 9.6

Derive the frequency equation of torsional vibrations for a free–free shaft of length 'l'.

Solution The general solution for equation of torsional vibrations for shaft can be written as Eq. 9.32:

$$\theta(x,t)=\sum_{n=1}^{\infty}\left(A\sin\frac{Z_n x}{a}+B\cos\frac{Z_n x}{a}\right)(C\sin Z_n t+D\cos Z_n t).$$

The boundary conditions for above particular system are

$\frac{\partial \theta}{\partial x}(0 \cdot t) = 0$ (strain in zero at both ends), $\frac{\partial \theta}{\partial x}(L \cdot t) = 0$

Applying the above two boundary conditions to the general solution of torsional vibrations, we get

$$\frac{\partial \theta}{\partial x} = \left(A \cos \frac{Z_n}{a} x \frac{Z_n}{a} - B \sin \frac{Z_n}{a} x \frac{Z_n}{a}\right)(C \sin Z_n t + D \cos Z_n t) \text{ or}$$

$$\frac{\partial \theta}{\partial x} = \frac{Z_n}{a}\left(A \cos \frac{Z_n}{a} x - B \sin \frac{Z_n}{a} x\right)(C \sin Z_n t + D \cos Z_n t)$$

$$\frac{\partial \theta}{\partial x} = 0 \text{ at } x = 0$$

$$\therefore \qquad A = 0, \text{ and } \frac{\partial \theta}{\partial x} = 0 \text{ at } x = l$$

$$0 = B\frac{Z_n}{a} \sin \frac{Z_n}{a} L\,(C \sin Z_n t + D \cos Z_n t)$$

$$\sin \frac{Z_n}{a} l = \sin n\pi, \quad Z_n = \frac{n\pi a}{L} \text{ or } 2\pi, f_n = \frac{n\pi a}{L}, f_n = \frac{n}{2L}\sqrt{\frac{G}{\rho}}$$

where $a = \sqrt{\frac{G}{\rho}}$ and $n = 1, 2, 3, \ldots$

The general solution can be expressed as

$$\theta(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} \cos \frac{n\pi x}{aL}\left(C \sin \frac{n\pi at}{L} + D \cos \frac{n\pi at}{L}\right)$$

EXAMPLE 9.7

A uniform shaft of length '*L*' fixed at one end and free at the other end is as shown in Fig. p-9.7. Determine the free torsional vibration of the shaft.

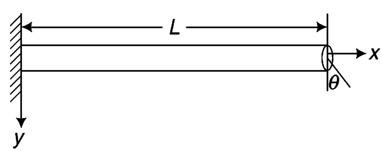


Fig. p-9.7 *Uniform shaft fixed at one end and free at the other end*

Solution The differential equation of motion for free torsional vibration of a shaft is given by $\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial \theta}{\partial x^2}$

where θ = Angular displacement, $a^2 = G/\rho$ and Z_n = Natural frequencies of the shaft ($Z_n = 2\pi f_n$)

The general solution for equation of torsional vibrations for a shaft can be written as

$$\theta(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} (A_n \cos Z_n t + B_n \sin Z_n t)\left(C_n \cos \frac{Z_n}{a} x + D_n \sin \frac{Z_n}{a} x\right)$$

The boundary conditions for the above particular system are

At $\qquad x = 0, \quad \theta(0, t) = 0$

At $x = L, \quad GI_p(\partial\theta/\partial x) = 0$

where I_p is the polar moment of inertia of the shaft

Using the first boundary conditions, we get

$$\theta(0, t) = \sum_{t=1,2,3,...}^{\infty} C_n(A_n \cos Z_n t + B_n \sin Z_n t) = 0 \text{ or } C_n = 0.$$

And from second boundary condition, we get

$$\theta(x, t) = \sum_{n=1,2,3,...}^{\infty} \left(\sin \frac{Z_n}{a} x\right)(A_n \cos Z_n t + B_n \sin Z_n t)$$

$$(\partial\theta/\partial x)_{x=L} = \sum_{n=1,2,3,...}^{\infty} \frac{Zn}{a}\left(\cos \frac{Z_n}{a} L\right)(A_n \cos Z_n t + B_n \sin Z_n t) = 0$$

$$\cos \frac{Z_n}{a} L = 0, Z_n = \frac{n\pi a}{2L}, \quad \text{where } n = 1, 3, 5, ...$$

Hence, the torsional vibration of the shaft is

$$\theta(x, t) = \sum_{n=1,3,...}^{\infty} \sin \frac{n\pi x}{2L}\left(A_n \cos \frac{n\pi at}{2L} + B_n \sin \frac{n\pi at}{2L}\right)$$

where 'A_n' and 'B_n' are constants determined by initial conditions of the problem.

EXAMPLE 9.8

Derive the frequency equation for the torsional vibration of a uniform circular shaft with rotors attached rigidly at the ends as shown in Fig. p-9.8.

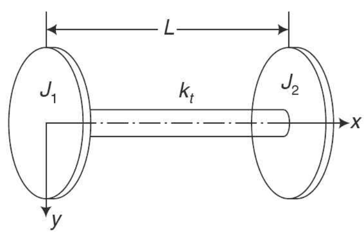


Fig. p-9.8 *Two-rotor system*

Solution The general solution for the torsional vibration of circular shafts can be expressed as

$$\theta(x, t) = \sum_{n=1,2,3,...}^{\infty} (A_n \cos Z_n t + B_n \sin Z_n t)\left(C_n \cos \frac{Z_n}{a} x + D_n \sin \frac{Z_n}{a} x\right)$$

where $a^2 = G/\rho$ and Z_n = Natural frequencies

($\because$ the equations of motion for torsional vibration of a circular shaft and for longitudinal vibration of uniform bars are identical)

The twisting of the shaft at both ends is produced by the inertia forces of the rotors.

The boundary conditions for the above particular system are

At $x = 0, \quad J_1(\partial^2\theta/\partial t^2) = GI_p(\partial\theta/\partial x)$

At $x = L, \quad J_2(\partial^2\theta/\partial t^2) = -GI_p(\partial\theta/\partial x)$

where G = Shear modulus of elasticity, I_p = Polar moment of inertia.

From first boundary condition, we get

$$Z_n^2 J_1 C_n + (Z_n GI_p/a) D_n = 0$$

And from second boundary condition we get

$$\left(Z_n^2 J_2 \cos Z_n L/a + \frac{Z_n GI_p}{a} \sin Z_n L/a\right) C_n + \left(Z_n^2 J_2 \sin Z_n L/a - \frac{Z_n GI_p}{a} \cos Z_n L/a\right) D_n = 0$$

The frequency equation obtained by equating to zero the determinant of the coefficients of 'C_n' and 'D_n' is

$$Z_n^2 \left(\cos Z_n L/a - \frac{Z_n a J_1}{GI_p} \sin Z_n L/a\right) J_2 + \frac{Z_n GI_p}{a}\left(\sin Z_n L/a + \frac{Z_n a J_1}{GI_p} \cos Z_n L/a\right) = 0$$

EXAMPLE 9.9

A pulley of moment of inertia 'J' is rigidly attached to the free end of a uniform shaft of length 'L' as shown in Fig. p-9.9. Determine the frequency equation for torsional vibration.

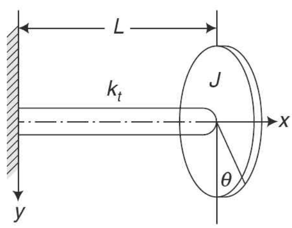


Fig. p-9.9 *Uniform shaft and pulley*

Solution The differential equation of motion for torsional vibration of the shaft are given by

$$\frac{\partial^2\theta}{\partial t^2} = a^2 \frac{\partial^2\theta}{\partial x^2} \qquad \text{... 9.33(a)}$$

The general solution for the torsional vibration of circular shafts can be expressed as

$$\theta(x, t) = \sum_{n=1,2,3,\ldots}^{\infty} (A_n \cos Z_n t + B_n \sin Z_n t)\left(C_n \cos \frac{Z_n}{a} x + D_n \sin \frac{Z_n}{a} x\right) \qquad \text{... 9.33(b)}$$

where $a^2 = G/\rho$ and Z_n = Natural frequencies

The boundary conditions for the above particular system are

$$\theta(0, t) = 0, \; -GI_p(\partial\theta/\partial x)_{x=L} = J(\partial^2\theta/\partial t^2)$$

i.e. the angular displacement of the shaft at the fixed end is equal to zero and the restoring torque of the shaft at the free end is equal to the inertia moment of the pulley.

From first boundary condition $C_n = 0$; and from secondary boundary condition,

$-\dfrac{GI_p Z_n}{a} \cos \dfrac{Z_n L}{a} = -JZ_n^2 \sin \dfrac{Z_n L}{a}$ or $-\tan \dfrac{Z_n L}{a} = \dfrac{GI_p}{aJZ_n}$, which is the frequency equation.

9.4 TRANSVERSE VIBRATION OF BEAMS OR EULER'S EQUATION OF BEAMS

Let us consider a simply supported beam of uniform cross section subjected to transverse vibration as shown in Fig. 9.4.

Assumption made while deriving the expression for transverse vibration of beams are the following:

1. The deformation of the beam is assumed due to moment and shear force.
2. There are no axial forces acting on the beam and effects of shear deflection are neglected.

We know from mechanics of materials, the differential equation of motion for the transverse vibration of beam, the deflection curve of a beam is given by

$$EI\frac{d^2y}{dx^2} = -M \quad \text{... 9.34}$$

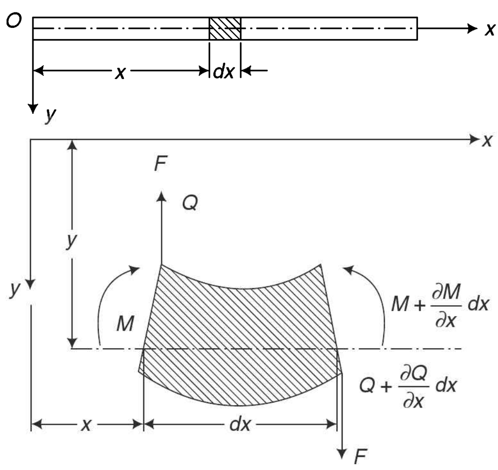


Fig. 9.4 *Transverse vibration of beams*

where y = Deflection of the beam, M = Bending moment at any cross-section

EI = known as the Flexural rigidity of the beam and is assumed as a constant.

Differentiating Eq. 9.34 twice, we get

$$EI\frac{d^3y}{dx^3} = -F \quad \text{...9.35}$$

$$EI\frac{d^4y}{dx^4} = W \quad \text{...9.36}$$

where F = Shear force, W = Intensity of loading.

(As we know the relationship between the shear force 'F', the intensity of loading 'W' and bending moment 'M').

In case of free transverse vibration of beams without application of external loading, it is very important to consider the inertia forces $\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2}$ as the loading intensity along the entire length of beam. Then Eq. 9.36 becomes

$$EI\frac{\partial^4 y}{\partial x^4} = -\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2} \qquad ...9.37$$

Here, partial derivatives are used because of the deflection of the beam 'y' is a function of 'x' and 't'.

$$\frac{EIg}{\rho A}\frac{\partial^4 y}{\partial x^4} = -\frac{\partial^2 y}{\partial t^2}$$

Let $$a^2 = \frac{EIg}{\rho A}$$

$\therefore$ $$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \qquad ...9.38$$

is the differential equation of motion for the transverse vibration for a simply supported beam of uniform cross-section including transverse inertia and stiffness of the beam.

The general solution for transverse vibration of beams is given by the expression

$$y(x, t) = A\cos h\,cx + B\sin h\,cx + C\cos cx + D\sin cx. \qquad ...9.39$$

EXAMPLE 9.10

A uniform beam fixed at one end and simply supported at the other end is having transverse vibrations. Derive a suitable expression for frequency.

Solution The general solution for transverse vibration is given by the expression 9.39.

$$y(x, t) = A\cos h\,cx + B\sin h\,cx + C\cos cx + D\sin cx$$

The boundary conditions for the above particular system are

$$\left.\begin{array}{l} y(0, t) = 0 \\ \frac{dy}{dx}(0, t) = 0 \end{array}\right\} \text{for fixed end,} \quad \left.\begin{array}{l} y(L, t) = 0 \\ \frac{d^2y}{dx^2}(L, t) = 0 \end{array}\right\} \text{for simply supported end}$$

Applying the above boundary conditions for the general solution of transverse vibration is given by expression,

$$y(x, t) = A\cos h\,cx + B\sin h\,cx + C\cos cx + D\sin cx$$

We get $y(0, t) = A + C = 0$

Differentiating the above equation w.r.t. x,

$$\frac{dy}{dx}(x, t) = c\,[A \sin h\, cx + B \cos h\, cx - C \sin cx + D \cos cx\,]$$

Differentiating again the above equation w.r.t. x,

$$\frac{d^2y}{dx^2}(x, t) = c^2[A \cos h\, cx + B \sin h\, cx - C \cos cx - D \sin cx]$$

$$\therefore \quad \frac{dy}{dx}(0, t) = B + D = 0$$

$$y(L, t) = A\,(\cos h\, cL - \cos cL) + B\,(\sin h\, cL - \sin cL) = 0 \text{ and}$$

$$\frac{d^2y}{dx^2}(L, t) = c^2\,[A \cos h\, cL + B \sin h\, cL - C \cos cL - D \sin cL] = 0$$

$$A\,(\cos h\, cl + \cos cl\,) + B\,(\sin h\, cl + \sin cl) = 0$$

$$A\,(\cos h\, cl - \cos cl\,) + B\,(\sin h\, cl - \sin cl) = 0$$

$$A\,(\cos h\, cl + \cos cl) + B\,(\sin h\, cl + \sin cl) = 0$$

Eliminating 'A' and 'B' from the above two equations, we get

$$(\cos h\, cl - \cos cl\,)\,(\sin h\, cl + \sin cl) - (\sin h\, cl - \sin cl)\,(\cos h\, cl + \cos cl) = 0$$

Solving it, we get frequency equations as

$$\cos cl \sin h\, cl - \sin cl \cos h\, cl = 0$$

$$\tan cl = \tan h\, cl$$

EXAMPLE 9.11

Find frequency equation of a uniform beam fixed at one end and free at the other end for transverse vibrations.

Solution The general solution for transverse vibration is given by the expression 9.39

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx.$$

The boundary conditions for the above particular system are

$$y(0, t) = 0 \text{ (zero deflection at fixed end)}, \quad \frac{dy}{dx}(0, t) = 0 \text{ (zero slope)}$$

$$\frac{d^2y}{dx^2}(L, t) = 0 \text{ (zero bending moment)}, \quad \frac{d^3y}{dx^3}(L, t) = 0 \text{ (zero shear force)}$$

Applying boundary conditions, we get $0 = A + C$, $A = -C$

$$\frac{dy}{dx}(x, t) = c\,(A \sin h\, cx + B \cos h\, cx - C \sin cx + D \cos cx) = 0$$

$$\frac{dy}{dx}(0, t) = 0 = B + D \qquad \therefore\ B = -D$$

$$\frac{d^2y}{dx^2}(L, t) = c^2 [A (\cos h\, cL + \cos cL) + B (\sin h\, cL + \sin cL)] = 0$$

$$\frac{d^3y}{dx^3}(L, t) = c^3 [A (\sin h\, cL - \sin cL) + B (\cos h\, cL + \cos cL)] = 0$$

$$[\cos h\, cL + \cos cL]^2 - (\sin h^2\, cL - \sin^2 cL) = 0$$

$$\cos h^2\, cL + \cos^2 cL + 2 \cos h\, cL \cos cL - \sin h^2\, cL + \sin^2 cL = 0$$

Solving, we get

$$\cos h\, cL \cos cL + 1 = 0$$

The above equation can be solved for cL to find natural frequency of the system.

EXAMPLE 9.12

Derive frequency equation for a beam with both ends free and having transverse vibrations.

Solution The general solution for transverse vibration is given by the expression 9.39

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx, \text{ where } c^2 = Z_n \sqrt{\frac{\rho A}{EI}}.$$

The boundary conditions for the above particular system are

$$\frac{d^2y}{dx^2}(0, t) = 0 \quad \text{(Because bending moment should be zero)}$$

$$\frac{d^2y}{dx^2}(L, t) = 0 \quad \text{(Because bending moment should be zero)}$$

$$\frac{d^3y}{dx^3}(0, t) = 0 \quad \text{(Because shear force should be zero)}$$

$$\frac{d^3y}{dx^3}(L, t) = 0 \quad \text{(Because shear force should be zero)}$$

Now applying the boundary conditions, for general solution of transverse vibration, we get

$$\frac{d^2y}{dx^2}(x, t) = c^2 [A \cos h\, cx + B \sin h\, cx - C \cos cx - D \sin cx]$$

$$\frac{d^2y}{dx^2}(0, t) = c^2 (A - C) = 0$$

$$\therefore \quad A = C$$

$$\frac{d^3y}{dx^3}(x, t) = c^3 [A \sin h\, cx + B \cos h\, cx + C \sin cx - D \cos cx]$$

$$\frac{d^3y}{dx^3}(0, t) = c^3 [B - D] = 0$$

$$\therefore \quad B = D$$

$$\frac{d^2y}{dx^2}(L, t) = c^2 [A (\cos h\, cL - \cos cL) + B (\sin h\, cL - \sin cL)] = 0$$

$$\frac{d^3y}{dx^3}(L, t) = c^3 [A (\sin h\, cL + \sin cL) + B (\cos h\, cL - \cos cL)] = 0$$

$$A (\cos h\, cL - \cos cL) + B (\sin h\, cL - \sin cL) = 0$$

$$A (\sin h\, cL + \sin cL) + B (\cos h\, cL - \cos cL) = 0 \text{ or}$$

$$(\cos h\, cL - \cos cL)^2 - (\sin h^2\, cL - \sin^2 cL) = 0$$

$$\cos h^2\, cL + \cos^2 cL - 2\cos h\, cL \cos cL - \sin h^2\, cL + \sin^2 cL = 0$$

$$\cos h^2\, cL - \sin h^2\, cL = 1 \text{ and } \cos^2 cL + \sin^2 cL = 1$$

$$\cos h\, cL + \cos cL = 1$$

9.6 IMPORTANT EQUATIONS IN VIBRATIONS OF A CONTINUOUS SYSTEM

1. Lateral vibration of a string One-dimensional wave equation for lateral vibration of a string is given by the expression

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{...9.40}$$

The general solution for lateral vibration of a string is given by the expression

$$y(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{Z}{a}\right)x + B_n \sin\left(\frac{Z}{a}\right)x\right) [C_n \cos Zt + D_n \sin Zt] \qquad \text{...9.41}$$

2. Longitudinal vibration of bars The differential equation of motion for longitudinal vibration of bars is given by the expression

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...9.42}$$

The general solution for longitudinal vibration of bars is given by the expression

$$u(x, t) = \sum_{n=1}^{\infty} \left(A \sin \frac{Z_n}{a} x + B \cos \frac{Z_n}{a} x\right) [C \sin Z_n t + D \cos Z_n t] \qquad \text{...9.43}$$

3. Torsional vibration of circular rods or shafts The differential equation of motion for torsional vibration of circular rods or shafts is given by the expression

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2} \qquad ...9.44$$

The general solution for torsional vibration of circular rods or shafts is given by the expression

$$\theta(x, t) = \sum_{n=1}^{\infty} \left(A \sin \frac{Z_n x}{a} + B \cos \frac{Z_n x}{a} \right) (C \sin Z_n t + D \sin Z_n t) \qquad ...9.45$$

4. Transverse vibration of beams The differential equation of motion for transverse vibration of beams is given by the expression

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0 \qquad ...9.46$$

The general solution for transverse vibration of beams is given by the expression

$$y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx \qquad ...9.47$$

REVIEW QUESTIONS

1. What is a continuous system? How does a continuous system differ from a discrete system in the nature of its equations of motion?
2. How many natural frequencies does a continuous system have?
3. Derive the one-dimensional wave equations for lateral vibrations of a string.
4. Derive the wave equations for lateral vibrations of a string. Obtain general expression for the lateral vibrations of string.
5. Write notes on (i) longitudinal vibration of rods or bars, (ii) torsional vibration of circular shaft, and (iii) Euler's equation of beams.
6. Derive an expression for the longitudinal vibration of a uniform bar of length 'L', one end of which is fixed and the other end is free.
7. Given a bar of cross-sectional area 'A', length 'L', Young's modulus E, mass/unit volume ρ. (i) Derive the equation governing the longitudinal vibration of the bar. (ii) Obtain the general solution of the differential equation derived above.
8. Show that the differential equation of motion for the transverse vibration for a simply supported beam of uniform cross-section is given by $\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0$ where $a = \sqrt{\frac{EI}{\rho A}}$ and obtain a general solution for the governing differential.

PROBLEMS FOR PRACTICE

(1) Derive the frequency equation for the Longitudinal vibration of a rod of two different cross-sectional areas 'A_1' and 'A_2' as shown in Fig. p.p-9.1.

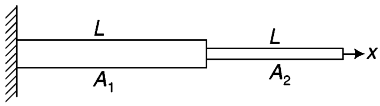


Fig. p.p-9.1

(2) Derive the orthogonality principle of normal modes for longitudinal vibration of uniform bars.

(3) Derive the frequency for the transverse vibration of a uniform beam of length 'L' if one end is fixed and the other end is free. Draw the principal modes of vibration.

Ans. $\cos cL . \cos h\, cL \infty - 1.$

(4) A uniform bar of length 'L' is acted upon by a forcing function $F_0 \sin \omega t$ at the end $x = 0$ as shown in Fig. p.p-9.4. If both ends are free, find the steady-state response of the bar.

Ans. $y(y, t) = \frac{F_0 L}{AE}\left(\frac{a}{\omega L}\right) \operatorname{cosec} \frac{\omega L}{a} \cos \left[\frac{\omega}{a}(1 - x)\right] \sin \omega t.$

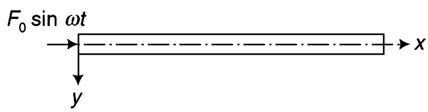


Fig. p.p-9.4

(5) An external torque '$T_0 \sin \omega t$' is applied to the free end of a uniform shaft of length 'L' as shown in Fig. p.p-9.5. Find the steady-state vibration of the shaft.

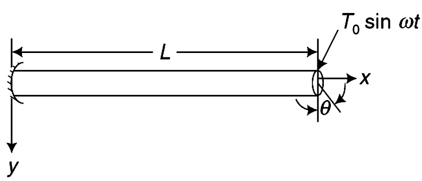


Fig. p.p-9.5

Ans. $\theta(x, t) = \frac{T_0 a}{GI_p \omega}$ s, $\frac{\omega L}{a} \sin \frac{\omega x}{a} \sin \omega t.$

(6) Compare the fundamental natural frequency of a round bar of steel of 200 metre length having 40 mm diameter. Assume the bar to be free at both ends.

OBJECTIVE-TYPE QUESTIONS

(1) Infinite number of degrees-of-freedom system means
- (a) maximum number of coordinates to specify their configuration.
- (b) infinitely large number of coordinates to specify their configuration.
- (c) minimum number of coordinates to specify their configuration
- (d) infinite number of natural frequencies of the system.

(2) In the vibration of continuous system for analysis of problems,
- (a) the knowledge of partial differential equation is very much essential.
- (b) constant boundary and initial conditions
- (c) the knowledge of partial differential equation is very much essential and constant boundary conditions and initial conditions
- (d) all of the above cases

(3) The one-dimensional wave for lateral vibration of a string is given by
- (a) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{\rho a^2}\frac{\partial^2 y}{\partial t^2}$
- (b) $\frac{\partial^2 y}{\partial x^2} = \rho a^2 \frac{\partial^2 y}{\partial t^2}$
- (c) $\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2}$
- (d) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2}$

(4) Examples of continuous systems are
- (a) spring-mass system
- (b) spring-mass-damper system
- (c) beams, rods, cables, plates
- (d) all of the above cases

(5) in case of continuous systems
- (a) finite number of coordinates specify their configuration
- (b) infinitely large number of coordinates specify their configuration
- (c) finite number of natural frequencies specify their configuration
- (d) none of the above cases

(6) longitudinal vibration of rods or bars is given by
- (a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$
- (b) $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$
- (c) $\frac{\partial^2 u}{\partial x^2}\frac{1}{a^2} = \frac{\partial^2 u}{\partial t^2}$
- (d) $\frac{G}{\rho}\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$

(7) Torsional vibration of uniform shaft or rods is given by
- (a) $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 \theta}{\partial t^2}(x, t)$
- (b) $\frac{\partial^2 \theta}{\partial x^2}(x, t) = \frac{\partial^2 \theta}{\partial t^2}$
- (c) $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$
- (d) $\frac{\partial^2 \theta}{\partial t^2}(x, t) = \frac{1}{a^2}\frac{\partial^2 \theta}{\partial t^2}(x, t)$

(8) The general solution for transverse vibration of beams is given by
- (a) $y(x, t) = C \cos cx + D \sin cx$
- (b) $y(x, t) = A \cos h\, cx + B \sin h\, cx$
- (c) $A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx$
- (d) $y(x, t) = A \cos h\, cx + B \sin h\, cx + C \cos cx + D \sin cx$

Answers

(1) b	(2) c	(3) d	(4) c	(5) b	(6) a
(7) d	(8) d				

TRANSIENT VIBRATION

10

10.1 INTRODUCTION

The response of a vibratory system to sudden blows or impacts is known as transient vibration. It is now gaining lots of importance in the field of impact engineering and study of falling bodies. Behaviour of vibratory systems under sudden release of the displaced mass of a vibratory system is not new and is not fundamentally different from the sudden application of load. In short, impact and transient loading will be synonymous.

In transient vibration, the concept of impact gives rise to analytical methods for predicting the response of a system to a wide variety of forcing functions. The general solution for periodic forcing can be expressed in terms of the solution for harmonic forcing. The link is used to tie two solutions together in Fourier series. Similarly, in this chapter we find a method for expressing the solution of an arbitrary forcing function and shall build this method upon the response to a simple impact. The methods developed in this chapter will be found most useful for transient loading (that is, periodic forcing functions).

In this chapter, we shall discuss two cases. If the impacting mass is large compared to the mass of the system, the two may cling together and move as one. Adherence may result, regardless of size, if the surfaces are sufficiently inelastic. Conversely, If the dropped mass is small compared with the mass of the system, and if both are elastic, the impacting mass may rebound immediately after it strikes. These two extreme cases are quite simple to handle. But between these interactions lie a broad range of interactions where the elasticity or inelasticity of the impacting bodies must be taken into account along with other phenomena.

Case (i) Impact of bodies that cling together As stated earlier, If the impacting mass is large compared to the mass of the system, the two may cling together and move as one.

Let us suppose that a rigid body is dropped from a height on the mass element of a simple spring-mass system. At the very instant of contact, the system mass gains momentum. At the same time, the falling body loses momentum and the two bodies begin to move as one. Now that they are joined, they can be dealt with as a single, spring-suspended body, and the complete system may be thought of as a simple spring-mass system in a state of free vibration.

The analysis of the problem can be divided into two parts:

(i) The conservation of momentum at the instant of impact, and

(ii) The conservation of energy in the ensuing vibration of the combined system.

If the bodies are sufficiently elastic, the duration of impact may be extended so that the condition (i) becomes complicated by an alteration of spring force during the period of momentum exchange. We shall assume that this does not take place, and that the transfer of momentum is instantaneous. Equating the momentum of a falling body just before the impact to the combined momentum of the two bodies just after impact, we have

$$m_1 v_1 = (m_1 + m_2) v_2 \quad ...10.1$$

where m_1 is the falling mass, m_2 is the system mass, v_1 is the velocity of the falling body just before impact, and v_2 is the velocity of combination just after impact as shown in Fig. 10.1.

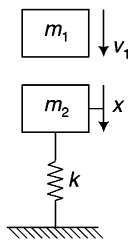


Fig. 10.1 *Impact when falling body encounters a spring-mass system*

The kinetic energy of the complete system immediately after impact is found to be

$$E_k = \frac{(m_1 + m_2)v_2^2}{2} = \frac{m_1 v_1^2}{2(m_1 + m_2)} \quad ...10.2$$

Noting that when the system comes to rest at its lowest position the decrease of kinetic energy will equal to the increase of potential energy, we can write an equation for the system in that position:

$$\frac{m_1 v_1^2}{2(m_1 + m_2)} = -(m_1 + m_2)g\,\Delta x + \frac{k(\Delta x)^2}{2} - f_0 \Delta x \quad ...10.3$$

Note that the initial spring force f_0 is equal to $m_2 g$. We may now solve this equation for the maximum deflection of the spring resulting from impact:

$$\Delta x = [m_1 g \pm \sqrt{(m_1 g) + \frac{k(m_1 v_1)^2}{m_1 + m_2}}\, \frac{1}{k} \quad ...10.4$$

The spring will be subject to maximum stress at the value of the solution given by the plus sign; this is the downward displacement of the system. The solution using the minus sign represents magnitude of the displacement at which the system stops at the top of its upswing.

Case (ii) Due to sudden impulse If the dropped mass is small compared with the mass of the system, and if both are elastic, the impacting mass may rebound immediately after it strikes.

Let us revisit Newton's second law.

The equation for Newton's second law can be integrated to read as follows:

$$\Delta(mv) = \int f dt \quad ...10.5$$

The right-hand term is given the name impulse. Whenever two bodies contact and rebound, the change of momentum of either is the value of impulse on the other, whether or not the impact is elastic. The concept of impulse will be shown to permit the extension of the equation derived for sudden impacts to problems of force application. Instead of conservation of momentum, the first condition that the system meets is the impulse equation.

For the case where the impacting body rebounds, the velocity of the mass m of the system will be given by

$$v = \frac{\text{impulse}}{m} \qquad \text{...10.6}$$

Following the impact, the system will be in harmonic-free vibration. The amplitude of this vibration is not difficult to find when we observe that Eq. 10.6 gives the velocity of the system at zero displacement. In terms of the parameters of the harmonic motion, the velocity at zero displacement is already known to be

$$(v)_{x=0} = v = v\omega \qquad \text{...10.7}$$

Combining Eq. 10.7 with 10.6, we find a relationship in between the strength of the impulse and the amplitude of vibration:

$$X = \frac{\text{impulse}}{m\omega} \qquad \text{...10.8}$$

Letting the instant of impact be the time of origin, and taking the positive direction of displacement as that of the impulse, the equation of motion becomes

$$X = \frac{\text{impulse}}{m\omega} \sin \omega t \qquad \text{...10.9}$$

Let us illustrate the above concepts by the following examples.

EXAMPLE 10.1

A pendulum is sometimes used to determine the speed of rifle bullets in Fig. p-10.1. A bullet of mass m_1 and velocity v_1 strikes a pendulum of mass m_2 and arm length l. Assuming that the two masses cling together, write an equation of the pendulum swing is known.

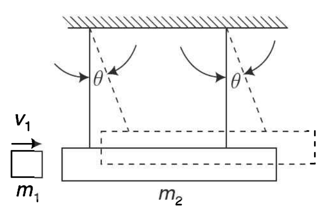


Fig. p-10.1 *Ballistic pendulum*

Solution The kinetic energy immediately after impact is given by Eq. 10.2.

$$E_k = \frac{(m_1 v_1)^2}{2(m_1 + m_2)}$$

The potential energy at the end of the pendulum swing is

$$E_p = (m_1 + m_2)\, gl(1 - \cos\theta)$$

Combining these two equations and solving for the velocity, we obtain

$$v_1^2 = 2gl\frac{(m_1 + m_2)^2}{m_1^2}(1 - \cos\theta) \qquad \text{...10.10}$$

If the weight of the bullet is small compared to that of the pendulum, we may neglect in the terms involving combined weight. Additional simplification can be made if the angle is sufficiently small so that we may write cos $\theta = 1 - (\theta^2/2)$. In these circumstances, Eq. 10.10 reduces to the following approximate equation:

$$v_1^2 = gl\theta^2 \left(\frac{m_2}{m_1}\right)^2 \qquad \text{...10.11}$$

EXAMPLE 10.2

Equipment packaged for shipment may be sometimes thought of as a group of vibratory systems in a rigid box. When the box is dropped, each of the vibratory systems is subjected to an impact, and stress is produced in the portion of the system which acts as the spring to obtain insight into the factors governing failure of such system. Find the equation for the spring force produced when the postman drops the system in Fig. p-10.2 from a height '*h*'.

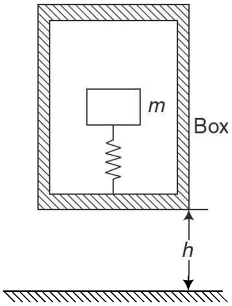


Fig. p-10.2 *Falling packed equipment*

Solution If we assume that the box falls, strikes the floor, and comes to rest, the mass will at the instant of impact, exert zero force on the spring. The box will however have a downward velocity equal to that of the system just before impact. The situation would be identical to that of falling the same distance h onto a spring; thus it can be described by Eq. 10.4 if the system mass m_2 is allowed to be zero.

$$f_s = k\Delta x = m\left(g \pm \sqrt{g^2 + \omega^2 v_1^2}\right)$$

We see that the spring force 'f_s' is increased when the natural frequency, the mass or the velocity is increased. Since $v = \sqrt{2gh}$ for falling bodies, this equation becomes

$$f_s = k\Delta x = mg\left[1 \pm \sqrt{1 + \frac{2\omega^2 h}{g}}\right] \qquad \text{...10.12}$$

Whether the system will fail depends upon how the spring is designed, since for a given spring constant, springs may vary in the maximum amount of force they will sustain.

EXAMPLE 10.3

Let us consider a small change in level in a road as an inclined step of height δy; any sort of contour can be synthesised from a sequence of such bumps. What will be the response of a one-degree-of-freedom vehicle when it goes over such a bump?

Solution Part of the vibration will result from sudden displacement of the spring, and part will result from the force transmitted to the mass by way of the damper. We may solve for each of these effects separately.

If it is assumed that the bump is of short duration, the spring will depress without appreciable movement of mass, and the increase of strain energy will be

$$E_p = k\frac{(\delta y)^2}{2}$$

Again assuming that the mass does not move appreciably, the damping force will be proportional to the rate of displacement of the wheel:

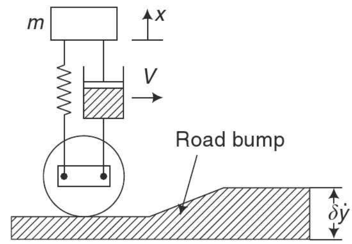


Fig. p-10.3 *Simple road bump*

$f = c\dot{y}$. Thus, the impulse transmitted to the mass by way of the damper is $fdt = c\dot{y}\,dt$

However, $\dot{y}dt = \delta y$ and the velocity of the mass immediately after the impulse is

$$v = \frac{fdt}{m} = \frac{c\delta y}{m}$$

From this, the kinetic energy is found to be $E_k = \dfrac{(c\delta y)^2}{2m}$

The total vibration energy can be expressed in terms of amplitude as $E_{P\,\max} = \dfrac{kX^2}{2}$

$$\frac{EX^2}{2} = \frac{1}{2}\left[K + \left(\frac{c^2}{m}\right)\right](\delta y)^2$$

Simplifying and solving for 'X', we obtain

$$X = \delta y\sqrt{1 + \left(\frac{c^2}{km}\right)} = \delta y\sqrt{1 + 4\left(\frac{c}{c_c}\right)^2} \qquad \text{...10.13}$$

From Eq.10.13, we note that the response to a single bump is worsened by the increase in damping.

EXAMPLE 10.4

We have already derived equations for the response of a vibratory system to an impulse of short duration. For the case of constant force applied to the mass of a spring-mass system over a period of time 't^1', determine the value of amplitude of vibration just after release of the load.

Solution When a steady force is suddenly applied to a vibratory system, it is as though the static equilibrium position of the system mass were suddenly changed. The system mass is a distance $\frac{f}{k}$ from this new rest position and is stationary. The free vibration that follows this initial condition is given by

$$x = \frac{f}{k}(1 - \cos \omega t) \qquad \text{...10.14}$$

where 'x' is measured from the original neutral position of the system and not from the new one. So long as the steady force persists, this equation will describe the motion of the system, but when it ceases the system finds itself in a state of vibration relative to the original neutral position. The amplitude of this vibration can be evaluated in terms of the velocity and displacement at the instant of cessation of the force.

Using the principle of conservation of energy, we note that the sum of the kinetic and potential energies at the instant of load removal will equal the maximum strain energy of the ensuing vibration, $\frac{kX^2}{2}$

$$\frac{kX^2}{2} = \frac{kx^2}{2} + \frac{mx^2}{2} \qquad ..10.15$$

Note that 'X' may be a variable in this particular problem, depending upon the time of release of load. Eq. 10.15 simplifies to $X = \sqrt{x^2 + \left(\frac{\dot{x}}{\omega}\right)^2}$

Substituting into this from Eq. (10.14), we obtain the desired expression for the amplitude:

$$X = \left(\frac{f}{m\omega^2}\right)\sqrt{(1 - \cos\omega + 1)^2 + (\sin \omega t^1)^2} \qquad ...10.16$$

when 't^1' is the duration of the loading.

The following table of amplitudes for applications of varying duration was compiled from the information given by Eq. 10.16.

Table 10.1 *Amplitudes of varying durations*

$\omega t'$	X
0 (sudden impact)	1.00 $ft^1/m\omega$
$\frac{\pi}{4}$	0.98 $ft^1/m\omega$
$\frac{\pi}{2}$	0.90 $ft^1/m\omega$
$\frac{3\pi}{4}$	0.78 $ft^1/m\omega$
π	0.60 $ft^1/m\omega$
$\frac{3\pi}{2}$	0.30 $ft^1/m\omega$
2π	0

From Table 10.1, it can be seen that impacts as long as $\frac{1}{4}$ cycle can be treated as sudden impacts with an error of only 10%. If the duration of an impulse of this simple type is for one whole cycle of the system, there is no vibration upon its release.

10.1 RESPONSE TO IMPULSE INPUT

A force of large magnitude which acts over a very short time interval is known as impulsive force. Impulse is a time integral of the force which is finite. The zero initial state response to and short, sharp blow can be obtained by idealising the blow as a Dirac-delta function: an infinitesimally brief force of infinitely large amplitude. The product of force and time, obtained from $\hat{I} = \int F(t)\,dt$ across this singularity has a non zero non-infinite value. For the Dirac–delta function, this value is unity, but we can multiply the function by the factor $\hat{I}$ to get the size of blow we need.

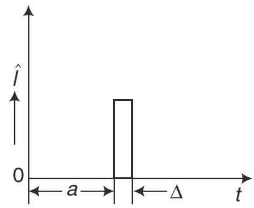


Fig 10.2 *Response to impulse input*

From Newton's law, this blow on a mass causes a change in velocity $mdv/g_c = Fdt$. If the velocity $v_{(0-)}$ before the impulse is zero then the velocity immediately after the impulse is $v_{(0+)} = \hat{I}g_c/m$. After the excitation is over the motion will proceed like the homogeneous equation, solution of which is given as

$$x = \frac{\hat{I}g_c}{mw_d} e^{-\xi w_n \tau} \sin(w_d t) \qquad ...10.17$$

which we can express as the product of an excitation factor $\hat{I}$ with a system response function $h_{imp}(t)$:

$$x = \hat{I} \cdot h_{imp}(t) \qquad ...10.18$$

where $$h_{imp}(t) = \frac{w_n}{k\sqrt{1-\xi^2}} e^{-\xi w_n t} \sin\left(\sqrt{1-\xi^2}\, w_n t\right) = \frac{w_n}{k} \sin(w_n t) \qquad ...10.19$$

We should ask two questions: How short the pulse must be in order for the analysis to be appropriate? The answer is that the mass should have no significant motion while the impulse takes place, compared to the later motion described by the equation. Therefore, the time duration Δt of the blow must be smaller than the quarter-period of the system, or $w_n\, \Delta t << \pi/2$.

Secondly, we can use the simple undamped solution at the right rather than the full damped solution. To use the simple expression, we must limit ourself both to moderate damping $\zeta << 1$ and also to a limited time frame $w_n\, \Delta t < 2\pi$. The logarithmic decrement δn gives us an indication of the difference between damped and undamped solutions during one period of free oscillation: at 1.5% of critical damping, the damped solution delays about 10%. Since many structural systems are lightly damped, and since we are interested mostly in the maximum amplitudes, which occur soon (about a quarter-cycle) after the impulse, the undamped solution is often adequate.

We can generalise this solution to impulses occurring at some time t_1 other than zero:

$$h_{imp}(t - t_1) = \frac{w_n}{k\sqrt{1-\zeta^2}} e^{-\zeta w_n (t - t_1)} \sin\left(\sqrt{1-\zeta^2}\, w_n (t - t_1)\right) \qquad ...10.20$$

$$\approx \frac{w_n}{k} \sin(w_n (t - t_1)) \qquad ...10.21$$

And we can superpose many different impulses

$$x = \sum_n \hat{I}_n h_{imp}(t - t_n) \qquad ...10.22$$

Therefore, the solution to a series of impulses is always the summation of the responses of all previous impulses—in the other words, the form of the solution and the additional term changes each time an impulse passes by.

10.1 RESPONSE TO STEP INPUT

The response to a step of height F_0 imposed at $t = 0$ can be described as an initial condition response due to a steady force which causes a shift in equilibrium:

$$\frac{m}{g_c}\ddot{x} + c\dot{x} + kx = F_o \qquad \text{...10.23}$$

Or, in standard form,

$$\ddot{x} + 2\xi\,\omega n\dot{x} + \left(\omega_n^2\right)x = \left(\frac{F_o}{k}\right)\left(\omega_n^2\right) \qquad \text{...10.24}$$

where $x(0) = 0$ and $\dot{x} = 0$; this results in this solution:

$$x = \frac{F_o}{k}e^{-\xi\,\omega_n t}\left(1 - \cos\left(\omega_d t\right) - \frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d t\right)\right.$$

For an undamped case, the response equation can be written using $\zeta = 0$

$$\frac{F_o}{k}\left(1 - \cos\left(\omega_n t\right)\right) \qquad \text{...10.25}$$

10.4 RESPONSE TO RAMP INPUT

Although we can simulate arbitrary shaped pulses as a series of small steps, we can do it more efficiently if we obtain the response of a ramp function of slope $\overset{o}{F}$, by solving the differential equation

$$\frac{m}{g_c}\ddot{x} + c\dot{x} + kx = \overset{o}{F}t \qquad \text{...10.26}$$

Or, in standard form,

$$\ddot{x} + \left(2\xi\,\omega_n\right)\dot{x} + \left(\omega_n^2\right)x = \left(\frac{\overset{o}{F}}{k}\right)\left(\omega_n^2\right)t$$

which has the approximate solution

$$x = \frac{\overset{o}{F}}{k}\left(t - \frac{e^{-\zeta\omega_n\tau}}{\omega_n}\left(2\xi(1 - \cos\omega_d t) + \frac{1-2\xi^2}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right)$$

$$\approx \frac{\overset{o}{F}}{k}\left(t - \frac{\sin\omega_n t}{\omega_n}\right) \qquad \text{...10.27}$$

9.6 PHASE PLANE METHOD TO SOLVE TRANSIENT PROBLEMS

A spring-mass system with initial conditions X_0 and Y_0 has its differential equation written as

$$\ddot{x} + \omega_n^2 x = 0$$

Its solution can be written as

$$x = A \sin(\omega_n t + \phi) \qquad ...10.28$$

where $$A = \sqrt{x_o^2 + \frac{V_o^2}{\omega_n^2}}$$

and $$\phi = \tan^{-1}\left(\frac{\omega_n x_o}{v_o}\right)$$

Differentiating Eq. 10.28 for velocity, we have

$$\dot{x} = A\,\omega_n \cos(\omega_n t + \phi)$$

$$\frac{\dot{x}}{\omega_n} = A \cos(\omega_n t + \phi) \qquad ...10.29$$

Squaring and adding equations 10.28 and 10.29, we have

$$x^2 + \left(\frac{\dot{x}}{\omega_n}\right)^2 = A^2 \qquad ...10.30$$

The above is an equation of a circle with coordinates x and $\left(\frac{\dot{x}}{\omega_n}\right)$. Its radius is A and centre is at the origin. This is shown in Fig. 10.3(a). The starting point on this displacement velocity is marked as P. At t_1 seconds later, the displacement and velocity of the system are represented by the point Q, where angle $POQ = \omega_n t_1$ radians. From this diagram, the displacement and velocity phase of the motion are available from the single point which corresponds to a particular time. This is the **phase plane plot**. The horizontal projection of the phase trajectory on a time base gives the displacement time plot of the motion and is shown in Fig. 10.3. Similarly, the vertical projection on the time base gives the velocity time plot of the motion.

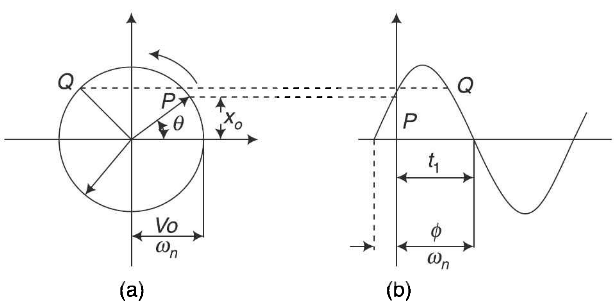


Fig. 10.3 *(a) Phase plane plot, and (b) Displacement–time plot*

It may be noted that the centre of phase trajectory always lies on the x-axis at a distance equal to the static equilibrium displacement of the system. In this case, the static equilibrium displacement was zero and therefore the centre of the circle was located at the origin. In case of a step force input F_o, the static equilibrium position suddenly changes through a distance of F_o/k. Thus, the phase plane plot for such a motion will be a circle whose centre lies F_o/k above the centre. The radius of this circle will be F_o/k so that the trajectory starts from the origin corresponding to zero initial conditions.

9.6 LAPLACE TRANSFORMATION METHOD TO SOLVE TRANSIENT PROBLEMS

Laplace transform or transformation is a very useful tool for the solution of differential equation and especially so, where transients are involved. It is that branch of operational calculus wherein a function is transformed from time (t) domain to new (f) domain. The original differential equation in the (t) domain, by the use of Laplace transform changes itself into an algebraic equation is very easy as compared to that of differential equation. Once the solution in (f) domain is obtained, the process of inverse transformation gives the solution back in the t domain. Manipulation with transformation and inverse transformation is facilitated by the use of a table of transforms pairs.

EXAMPLE 10.5

Determine the response of the overdamped, single-degree-of-freedom spring-mass system (ξ > 1) to a unit impulse force $\delta(t)$ using the Laplace transform method.

Solution The equation of motion is given as

$$m\ddot{x} + c\dot{x} + kx = \delta(t)$$

Or

$$\ddot{x} + (2\xi\ \omega_n)\dot{x} + (\omega^2)x = (m)\ \delta(t)\ \xi > 1$$

Taking Laplace transform on both sides, with zero initial conditions:

$$s^2X(S) + (2\xi\ \omega)\ X(S) + (\omega^2)\ X(S) = \frac{1}{m}$$

If r_1 and r_2 are the roots of the LHS equation then

$$X(S) = \frac{\frac{1}{m}}{s^2 + (2\xi\ \omega)s + (\omega^2)} = \frac{1}{m(r_1 - r_2)}\left[\frac{1}{(s - r_1)} - \frac{1}{(s - r_2)}\right]$$

Using inverse Laplace transform, we get

$$x(t) = \frac{\frac{1}{m}}{s^2 + (2\xi\ \omega)s + (\omega^2)}$$

$$x(t) = \frac{e^{-\xi\omega t}}{2m\omega\sqrt{\xi^2 - 1}}\left(e^{\omega t\sqrt{\xi^2 - 1}} - e^{-\omega t\sqrt{\xi^2 - 1}}\right).$$

10.1 IMPULSE SEQUENCES

When one impulse of short duration follows another, it will initiate harmonic vibration and be completely independent of the vibration produced by the first impulse.

This is known as an impulse sequence. Equation 10.9 is the key to the calculation of the response of a spring-mass system to the impulse sequences. This independence stems from the fact that the equation of vibration for such a system is linear, and that the principle of superposition will apply. That is, the solution for two forcing functions taken separately may be superimposed to give the solution for the combined forcing functions.

Let us rewrite Eq. 10.9 for the time t, the displacement due to the impulse in a sequence will be $x_i = X_i \sin \omega(t - t_i)$

where t_i is the time at which the impulse was applied, and $(t - t_i)$ the elapsed time since the application of the impulse. For a sequence of impulses, the displacement due to each impulse will be

$$x_a = X_a \sin \omega(t - t_a), \quad x_b = X_b \sin \omega(t - t_b), x_c = X_c \sin \omega(t - t_c).$$

The total displacement of the system at time 't' will be

$$x = x_a + x_b + x_c \qquad \text{...10.31}$$

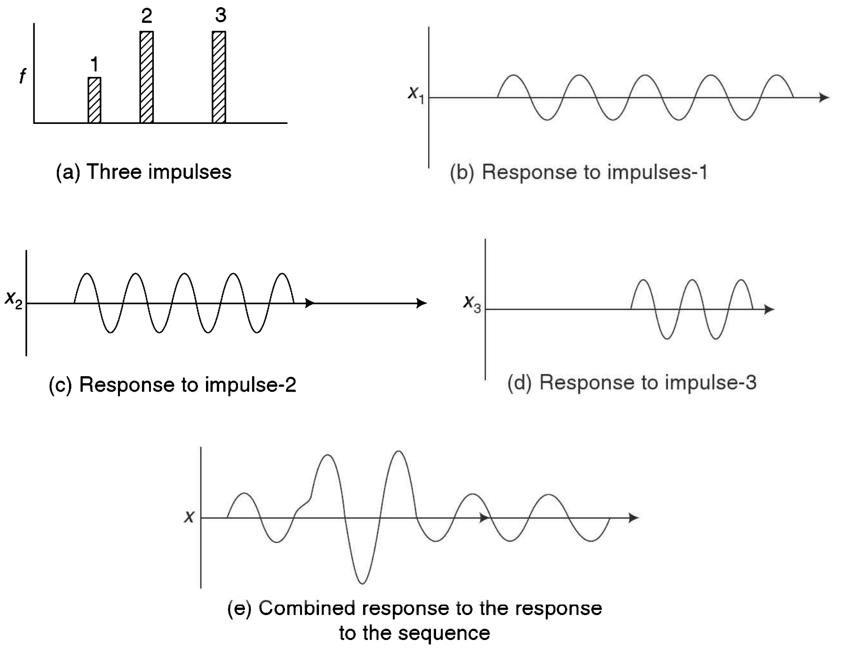


Fig. 10.4 *Synthesis of the response to an impact sequence*

This is shown graphically in Fig. 10.4. From this diagram and the above equations, it is clear that the effect of a sequence of impacts may not be that of increasing the amplitude of vibration with each impact. It is possible for an impact to bring a vibrating system to a halt. In a sense repeating what has just been said, we may write the complete equation of motion for an impact sequence:

$$x = X_a \sin \omega(t - t_a) + X_b \sin \omega(t - t_b) + X_c \sin \omega(t - t_c) + \cdots \qquad \text{...10.32}$$

Applying Eq. 10.31, so that the above equation may be written in terms of the impulses acting on the system, we obtain for the impact sequence the equivalent of Eq. 10.32,

$$x = \frac{(\text{Impulse})_a}{m\omega}\sin\omega(t - t_a) + \frac{(\text{Impulse})_b}{m\omega}\sin\omega(t - t_b)$$
$$+ \frac{(\text{Impulse})_c}{m\omega}\sin\omega(t - t_c) \quad \text{... 10.33}$$

Let us illustrate the above concept by the following example.

EXAMPLE 10.6

A machine member is so designed that in operation it is suddenly thrust forward and after a dwell of 0.5 second, is suddenly drawn back. Part of this member acts as a vibratory system, with a mass of 10 kg and a natural frequency of 8.75 cycles/second. The forward thrust acts on this system as an impulse of 3090 Nm-s and the backward thrust, acts as an impulse of –3090 Nm-s. Find the equation for vibration of the system after the second impulse has been applied.

Solution Using Eq. 10.32, we find that the motion due to first impulse is

$$x_a = \frac{\text{Impulse}}{m\omega}\sin\omega t,\ x_a = \frac{80}{10 \times 55}\sin\omega t = 5.62\sin\omega t$$

The motion due to second impulse alone is $x_b = -5.62\sin\omega(t - 0.5)$

The complete solution after the second impulse is

$$x = 5.62[\sin\omega t - \sin\omega(t - 0.5)]$$

The maximum value of 'x' can take in this equation is $X = (5.62)(0.765)$

The maximum value of 'x' for the first impulse alone is $X_a = 5.62$. In this case, the second impulse diminished the amplitude of vibration by about 23%.

10.8 CONTINUOUS FORCING FUNCTIONS

A continuous forcing function can be analysed by the same methods as a sequence of impacts if it is first converted into such a sequence, as shown in Fig. 10.5. The continuous force time curve is broken into increments with respect to the time scale.

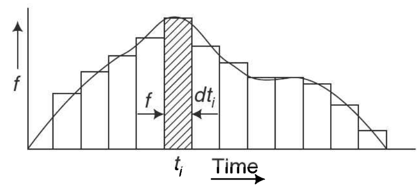


Fig. 10.5 *Reduction of a continuous forcing function to a sequence of impulses*

Each increment is an impulse of the form $f dt$. In order to apply Eq. 10.33 to this problem, we shall write it in the form

$$x = \Sigma\left(\frac{(\text{impulse})_i}{m\omega}\sin\omega(t - t_i)\right)$$

For a continuous sequence of impulses, the summation sign may be replaced by an integral sign and for the impulse we substitute $f\,dt_i$ so that

$$x = \int_0^t \frac{f dt_i}{m\omega} \sin \omega(t - t_i) \qquad ...10.34$$

Remember that t_i is the time at which the impulse $f\,dt_i$ was applied and is a variable in the equation. The time 't' is a constant so far as the integration is concerned, since it is the time at which we observe the system; consequently 't' is the limit of the integration.

By means of a trigonometric identity, the integrand may be expanded:

$$x = \int_0^t \frac{1}{m\omega} (\sin \omega t \cos \omega t_i - \cos \omega t \sin \omega t_i) dt_i \qquad ...\ 10.35$$

When we hold the natural frequency of the system constant, we get

$$x = \frac{\sin \omega t}{m\omega} \int_0^t (f \cos \omega t_i) dt_i - \frac{\cos \omega t}{m\omega} \int_0^t (f \sin \omega t_i) dt_i \qquad ...\ 10.36$$

This powerful equation is not particularly difficult to use, provided 'f' is a sufficiently simple function of time. Examples 10.6 and 10.7 illustrate its use. The first involves transient behaviour of a system under the influence of a harmonic forcing function and the second is the response of a system to a single cycle of sawtooth function.

EXAMPLE 10.7

Determine how the amplitude grows for an undamped spring-mass system subject to harmonic forcing at the natural frequency of the system. This means that the applied force will be $f = F \sin \omega t_i$, where ω is the natural frequency of the system.

Solution We shall use the above equation, substituting into it $f = F \sin \omega t_i$. The first integral becomes

$$\int_0^t F \sin \omega t_i \cos \omega t_i dt_i = -\frac{F}{2} \frac{\cos^2 \omega t}{\omega} + \frac{F}{2\omega}$$

The second integral becomes,

$$\int_0^t F \sin^2 \omega t_i \, dt_i = -\frac{F \sin \omega t \cos \omega t}{2\omega} + \frac{F}{2} t_i$$

And the complete equation is

$$x = \frac{F}{2m\omega} \left[\frac{-\sin \omega t \cos^2 \omega t}{\omega} + \frac{\cos^2 \omega t \sin \omega t}{\omega} - t \cos \omega t \right] + \frac{\sin \omega t}{\omega}$$

Simplifying, we obtain $x = \dfrac{Ft \cos \omega t}{2m\omega} + \dfrac{F \sin \omega t}{m\omega^2}$

We have now shown that the variation is the result of a harmonic component and a component which grows linearly with respect to time. If the system is initially at rest and is then subjected to harmonic forcing at its natural frequency, the amplitude does not become infinite at once, but grows linearly with time. This analysis shows why it is possible to pass a system hurriedly through the region of resonance without extreme response.

EXAMPLE 10.8

The effect of an automobile moving across a cable supported drawbridge in Fig. p-10.8 may be approximated as the steady movement of a concentrated load along the bridge. The moment about the bridge pivot increases steadily until the car leaves the bridge. Assume that the moment of inertia and the natural frequency of the bridge do not change appreciably, and find the equation for the deflection of the bridge under the weight of the car.

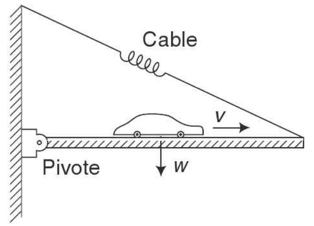


Fig. p-10.8 *Effect of an automobile moving across a cable-supported drawbridge*

Solution From the continuing forcing function we have in Eq. 10.36,

$$x = \frac{\sin \omega t}{m\omega}\int_0^t (f \cos \omega t_i) dt_i - \frac{\cos \omega t}{m\omega}\int_0^t (f \sin \omega t_i) dt_i$$

into torsion terminology, taking note of the fact that the applied torque is given by $\tau = w\ v\ t$, where, 'v' is the velocity of car and 'w' is the weight. In rewriting the equation, we shall replace x by θ, m by I and f by τ.

$$\theta = \frac{\sin \omega t}{I\omega}\int_0^t \tau \cos \omega t_i dt_i - \frac{\cos \omega t}{I\omega}\int_0^t \tau \sin \omega t_i$$

When $\tau = w.v.t$ is substituted for the torque, the first integral becomes

$$\int_0^t wvt_i \cos \omega t_i dt_i = w\ v\left[\frac{\cos \omega t}{\omega^2} + \frac{t \sin \omega t}{\omega} - \frac{1}{\omega^2}\right]$$

The second integral becomes

$$\int_0^t wvt_i \sin \omega t_i\, dt_i = wv\left(\frac{\sin \omega t}{\omega^2} - \frac{t \cos \omega t}{\omega}\right)$$

And the complete equation takes the form

$$\theta = \frac{wv}{I\omega}\left(\frac{\sin \omega t \cos \omega t}{\omega^2} + \frac{t \sin^2 \omega t}{\omega} - \frac{\cos \omega t \sin \omega t}{\omega^2} + \frac{t \cos^2 \omega t}{\omega} - \frac{\sin \omega t}{\omega^2}\right)$$

Upon simplifying, we obtain

$$\theta = \frac{v\omega t}{I\omega^2}\left(t - \frac{\sin \omega t}{\omega}\right)$$

This expression holds so long as the car is on the bridge.

REVIEW QUESTIONS

1. What is the difference between transient vibration and random vibration?
2. Discuss the response of single-degree-spring-mass system subjected to unit impulsive force.

3. For a vibratory system subjected to step unit, plot the system response to step input for different amount of damping.
4. Draw the phase plane plot and the displacement time plot for a spring-mass system subjected to a rectangular pulse of duration t.
5. Explain in detail the phase plane method to solve transient problems.

PROBLEMS FOR PRACTICE

1. A single-degree-of-freedom system is subjected to impulsive excitation of 0.5 Ns. Obtain the expression for the response of the system. Use mass of body = 1.5 kg; c = 0.5 kg/s and stiffness = 5 N/m.
2. Determine the response of an undamped, single-degree-of-freedom spring-mass system subjected to the triangular impulse as shown Fig. p.p-10.1.
3. Determine the response of an underdamped single-degree-of-freedom system using Laplace transform method. Equation of motion is given by $mx + c\ddot{x} + k\dot{x} = F_1 u(t)$. Where $u(t)$ is the unit step function and initial condition are $x(o) = 0$ and $x(o)^* = 0$.

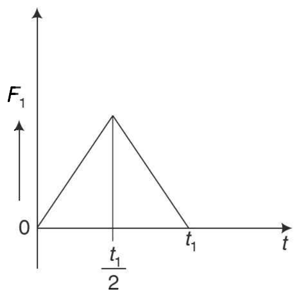


Fig. p.p-10.1

OBJECTIVE-TYPE QUESTIONS

1. The response of the vibratory system is purely __________ if excitation is of the periodic nature like shock pulse or a transient excitation:
   (a) transient (b) random
   (c) linear (d) nonlinear
2. An explosion occurring on a system with a comparatively __________ natural period will be an impulse while the same explosion occurring on a system with __________ natural period will be a pulse.
   (a) smaller/larger
   (b) larger/smaller
   (c) zero/smaller
   (d) larger/small
3. Phase plane method can be used to analyse the systems subjected to __________ steps.
   (a) single
   (b) multiple
   (c) irregular
   (d) multiple/irregular.

Answers

(1) a (2) b (3) d

RANDOM VIBRATIONS

11

11.1 INTRODUCTION

In our study of vibration, we talk about three types of excitation functions: (i) harmonic, (ii) periodic, and (iii) nonperiodic. Values of the excitation are known at any given time and hence such functions are also called **deterministic functions**. Figure 11.1 shows typical examples. It is to be noted that response of system to deterministic excitation is also deterministic.

The dynamic loads which have been considered in deterministic problems, have fixed or definite values of amplitude, frequency, period and phase. But for many cases, such as wave forces on offshore structures, jet engine noise, earthquake effects on buildings, bridges and dams, air pressure on aeroplanes, rumble of ball bearings, hiss between stations of a radio and vibrations of ships in rough seas, there is an uncertainty involved with the exactness of the loading parameters.

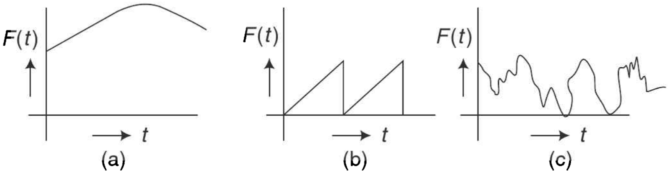


Fig. 11.1 *Deterministic excitation functions*

These uncertainties are related to the random time functions. Their values are never known at any given time and vary from time to time, place to place. For example, the intensity of an earthquake tremor will be different in Japan compared to an earthquake tremor at India. The reasons for variations are many and there are too many factors that affect the outcome. These vibratory phenomenon whose outcome at future instants of time cannot be predicted are classified as **nondeterministic** and referred to as **random vibrations.**

Figure 11.1(c) shows typical random force function.

The degree of randomness in a vibratory process depends on

1. Parameters that we vary in the experiments, and
2. The ability to control these parameters.

The more the parameters associated, more is the randomness and more precise control of the parameters results in less randomness. The total number of records together is referred to as an ensemble. To form a rational basis of the understanding of the random process, concepts of probability are used.

11.2 IMPORTANT TERMINOLOGY USED IN RANDOM VIBRATIONS

The following are the important terminologies used in random vibrations.

11.2.1 Random Phenomenon

If the time histories cannot be expressed in terms of known functions of time then such phenomenon is called random phenomenon**.** For example, suppose we are interested to measure the displacement of landing gear of aircraft while landing and denote time history by $X_1(t)$. At some other time the same aircraft lands in the same runway and if we measure the displacement and denote the time history by $X_2(t)$. $X_1(t)$ and $X_2(t)$ will never be identical because there is always randomness involved due to the complex set of variables and hence we call displacement under our consideration as a random phenomenon.

11.2.2 Sample Function

An individual time history describing the random phenomenon is called a sample function. In the above example, $X_1(t)$ and $X_2(t)$ are called sample displacement functions.

11.2.3 Random Variable

When we consider infinitely many sample functions such as $X_1(t)$, $X_2(t)$..., etc, at a particular value of time then we obtain the value of the random variable $X(t)$ at a given time.

11.2.4 Random Process, or Stochastic Process

The entire collection or ensemble of all possible time histories that might result from the experiment is called a stochastic process or random process. It is denoted by $[x_k(t)]$.

11.2.5 Statistical Regularity

When the number of sample functions become large and averages tends to recognizable limits; then the random process is said to exhibit statistical regularity.

Excitation is random and hence the response is also random, Usually, hundreds of sampling functions are involved in random process and it becomes extremely difficult to analyse if we consider each sample function. Instead, averages of the sample functions are considered. These averages are sometime referred to as statistics.

11.2.6 Ensemble Averages, Mean and Autocorrelation

Ensemble is the entire collection of all possible time histories that might result from the experiments.

Let $f(x)$ be a function of 'x', whose first-order probability density function is $\rho(x)$. The expected value of $f(x)$, which is considered as a continuous variable is

$$E[f(x)] = \int_{-\infty}^{\infty} f(x)\rho(x)dx \qquad \text{...11.1}$$

This ensemble average is called the mathematical expectation of $f(x)$ and the operator 'E' is used to denote this ensemble average.

When $$f(x) = X$$

then $E(x) = \int_{-\infty}^{\infty} x\rho(x)dx$ = Mean of ensemble average = Expected value of 'X'.

When $f(x) = x^2$ then Eq. 11.1 becomes

$$E\lfloor X^2 \rfloor = \int_{-\infty}^{\infty} X^2 \rho(x)dx = \text{Mean of the square of random variable '}x\text{'.} \qquad \text{...11.2}$$

The variance σ^2 is defined as the ensemble average of the square of the deviation from the mean.

$$\sigma^2 = E\lfloor X - (E\{X\})^2 \rfloor = \int_{-\infty}^{\infty} (x - E\{X\}^2 \rho(x)dx \qquad \text{...11.3}$$

Expanding right-hand side of Eq. 11.3, we get

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 P(x)dx - \int_{-\infty}^{\infty} 2xE[X]p(x)dx + \int_{-\infty}^{\infty} [E(X)]^2 P(x)dx$$

$$= E[X^2] - 2[E(X)]^2 + [E(X)]^2 \quad \text{or} \quad \sigma^2 = E\lfloor X^2 \rfloor - E[(X)]^2$$

Standard deviation of the random variable 'X' is given by

$$\sigma = \sqrt{\sigma^2} = \sqrt{\text{Variance of } x}$$

Standard deviation indicates the spread of the random variable about the mean.

Let us consider two random variables. If $f(x_1)$ and $g(x_2)$ are functions of x_1 and x_2 respectively then the mathematical expectation of their product can be written as

$$E[f(x_1)\, g(x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)\, g(x_2)\, p(x_1 x_2)\, dx_1 dx_2$$

When $$f(x_1) = X_1 \text{ and } g(x_2) = X_2$$

then $$E[X_1 X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p(x_1 x_2) dx_1 dx_2$$

= Average across the ensemble of all products $X_1 X_2$.

$E[X_1X_2]$ is called the autocorrelation function. Thus autocorrelation is a different type of ensemble average obtained by summing the product of instantaneous values of sample functions at two different times and dividing by the number of sample function.

A covariance is obtained by averaging the deviations from the means at two time instants.

$$\text{Cov}(X_1, X_2) = E[\{X_1 - E(X_1)\}\ \{X_2 - E(X_2)\}]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [X_1 - E(X_1)][X_2 - E(X_2)]p(x_1x_2)\,dx_1dx_2$$

We can show that

$$\text{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]\,E[X_2]$$

If X_1 and X_2 have zero means, the covariance is identical to the autocorrelation.

Mean and variance provide gross or limited information about a process. In order to make a more refined study of the random processes, a detailed information about probability distributions is required.

11.2.7 Stationary Process

In general, mean value and autocorrelation functions depend on initial time. When mean and autocorrelation functions depend on initial time then the random process is called nonstationary. In the special case in which mean value and auto correlation do not depend on initial time, the random process is said to be weakly stationary. For a weakly stationary random process, the mean value is constant and autocorrelation depends on time shift only. When all the possible averages are independent of initial time then the random process is said to be strongly stationary.

A random process is said to be stationary if its probability distributions do not vary if we shift time scale, or in other words if we shift the origin. The probability density at any time instant is valid for different segments of time. For a process to be strictly stationary, it can have no beginning and no end and each sample should extend from $-\infty$ to $+\infty$.

For example, wave and wind generation can be regarded as stationary random processes. Earthquake motions cannot be treated as stationary processes because probability distributions change with respect to time.

For stationary processes, the first-order probability distribution $p(x)$ is independent of time. Therefore, $E[X]$ and σ^2 are also independent of time.

11.2.8 Temporal Statistics and Ergodic Hypothesis

Generally speaking to get required statistics (such as ensemble, mean value and auto correlation) and satisfy statistical regularity, a large number of sample functions are needed. Under certain circumstances however, for certain stationary processes, it is possible to get the same mean value and autocorrelation function for a random process by using a single representative but a long sample function of time can give the required information, when averaging is done with respect to time along the sample. Such an average is called the temporal average or time average.

Temporal mean and temporal mean square are defined as

$$<X(t)> = \lim \tau \to \infty \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt$$

and $$<X(t)^2> = \lim \tau \to \infty \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt$$

Temporal autocorrelation is defined as

$$\Phi(\tau) = <X(t)X(t+\tau)> = \lim_{\tau\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt$$

If the random process is stationary and if the temporal mean value and temporal autocorrelation functions are same, irrespective of time history over which these averages are calculated, then the random process is called ergodic random process. Hence, for ergodic processes the temporal mean value and autocorrelation functions calculated over a representative sample function must be equal to ensemble mean value and autocorrelation function. Thus, when $\tau = 0$, $\Phi(0) = < X^2(t) >$ = Temporal mean square, the ergodic process is one for which ensemble averages are equal to temporal averages.

$$< X(t) > = E[X(t)]$$

$$\Phi(\tau) = R(\tau)$$

The symbol < >is used to indicate temporal averages.

When samples obtained at a particular location are spatially valid at other locations, the process is said to be a homogeneous process. A random process may be stationary without being ergodic. If a given process is not ergodic but stationary then we must calculate ensemble averages instead of time averages.

EXAMPLE. 11.1

Determine the ensemble mean, ensemble mean square, temporal mean, temporal mean square, the temporal autocorrelation and ensemble autocorrelation of the following function,

$$X(t) = A \sin(\omega t + \alpha)$$

where 'A' and 'ω' have fixed values, but the phase angle 'α' is a random variable.

The probability density function of $p(\alpha)$ is shown in Fig. p-11.1.

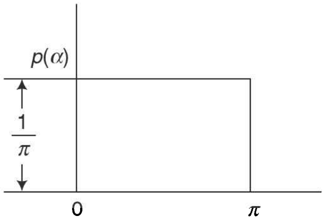


Fig. p-11.1 *Probability density function*

Solution The ensemble mean is

$$E[X(t)] = A\int_0^{\pi} \sin(\omega t + \alpha)p(\alpha)d\alpha$$

$$= \frac{A}{\pi}\int_0^{\pi} \sin(\omega t + \alpha)d\alpha = \frac{2A}{\pi}\cos \omega t$$

The ensemble mean square is

$$E\lfloor X^2(t)\rfloor = A^2\int_0^{\pi} \sin^2(\omega t + \alpha)p(\alpha) = \frac{A^2}{\pi}\int_0^{\pi} \frac{[(1-\cos^2)(\omega t + \alpha)]d\alpha}{2} = \frac{A^2}{2}$$

The temporal mean is $< X(t) > = \lim_{T\to\infty}\frac{1}{T}\int_0^{T} A\sin(\omega t + \alpha)dt = 0$

The temporal mean square is $<X^2(t)> = \lim_{T\to\infty}\frac{1}{T}\int_0^{T} A^2\sin^2(\omega t + \alpha)dt = \frac{A^2}{2}$

The temporal autocorrelation is

$$\Theta(\tau) = \lim_{\tau\to\infty}\frac{1}{T}\int_0^{T} A\sin(\omega t + \alpha)\, A\sin(\omega t + \alpha + \omega t)dt = \frac{A^2}{2}\cos \omega t$$

The ensemble autocorrelation is

$$R(\tau) = \int_0^{\pi} A\sin(\omega t = \alpha)\, A\sin(\omega t + \alpha + \omega t)\frac{1}{\pi}d\alpha$$

$$R(\tau) = \frac{2A}{\pi}\cos(\omega\tau)\cos(\omega\tau)$$

It is clear that the random process described is not stationary.

11.1 POWER SPECTRAL DENSITY

For a stationary ergodic process, the temporal mean square is

$$<F^2(t)> = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(t)F(t)dt \qquad \ldots 11.4$$

We can reformulate the Fourier series expression to take in to account the infinite time range and can be written as

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega \quad \text{(in time domain)}$$

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}\, d\omega \quad \text{(in frequency domain)}$$

$F(\omega)$ is called the Fourier transform of $F(t)$. In general, $F(\omega)$ is complex, $F(t)$ is real. $F(-\omega)$ is the complex conjugate of $F(\omega)$. $F(t)$ and $F(\omega)$ are known as Fourier transfer pairs.

Using $\omega = 2\pi f$ in the above Fourier transfer pairs, we get

$$F(t) = \int_{-\infty}^{\infty} F(t)e^{i\pi 2tf}\, dt \qquad ...11.5$$

$$F(f) = \int_{-\infty}^{\infty} F(t)e^{-i\pi 2tf}\, dt \qquad ...11.6$$

Substituting one $F(t)$ of the integral of Eq. 11.4 by Eq. 11.5,we get

$$<F^2(t)> = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(t)\int_{-\infty}^{\infty} F(t)e^{i\pi 2tf}\, dfdt$$

$$= \lim_{T\to\infty}\int_{-\infty}^{\infty} F(f)\frac{1}{T}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} F(f)e^{i\pi 2tf}\, dt\right]df \qquad ...11.7$$

A comparison of the integral within the parenthesis of Equations 11.7 and 11.6 reveals that the integral is the complex conjugate of $F(f)$. We shall indicate it as $F^*(f)$.

Therefore, Eq.11.7 becomes

$$<F^2(t)> = \lim_{T\to\infty}\int_{-\infty}^{\infty} F(f)\,\frac{1}{T}F(f)\,F^*(f)df$$

$$= \lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{1}{T}[F(f)]^2 df \quad \text{or} \quad <F^2(t)> = \int_{-\infty}^{\infty} S(f)df \qquad ...11.8$$

where $$S(f) = \frac{1}{T}[F(f)]^2 \qquad ...11.9$$

$S(f)$ is called the power spectral density and can be obtained from the square of the Fourier transform of the random process. If the spectral density can be obtained for the whole range of frequencies, the mean square of the process can be determined from Eq. 11.9.

This implies that the autocorrelation function $\phi(\tau)$ is the Fourier transform of the spectral density $S(f)$ of the process.

Therefore, $$S(f) = \int_{-\infty}^{\infty} \Phi(\tau)e^{-2\pi f\tau}d\tau.$$

11.2 RANDOM RESPONSE OF SINGLE DEGREE OF FREEDOM SYSTEMS

There are two approaches of studying the response of a system:

1. Time-domain analysis
2. Frequency-domain analysis.

These concepts are extended here for single degree of freedom systems.

11.4.1 Time-domain Analysis

Indicating $x(t)$ as the response function of the structure, the Duhamel's integral can be written in terms of unit impulse response function as

$$x(t) = \int_{-\infty}^{\infty} h(\tau)F(t-\tau)d\tau \quad ...11.10$$

where $h(\tau)$ is the unit impulse response function. If $x(t)$ and $F(t)$ are stationary random processes then the expectation of both sides of the integral is

$$E[x(t)] = \int_{-\infty}^{-\infty} h(\tau)E[F(t-\tau)]d\tau \quad ...11.11$$

If m_x and m_f denote the mean values of $x(t)$ and $F(t)$ then Eq. 11.11 can be written as

$$m_x = m_f \int_{-\infty}^{\infty} h(\tau)d\tau \quad ...11.12$$

Eq. 11.12 gives the relationship between the mean values of the input and output of the process.

If we consider the autocorrelation $R_{FF}(\tau)$ for the excitation and $R_{xx}(\tau)$ for the response then

$$R_{xx}(\tau) = E[x(t)\,x(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\theta_1)\,h(\theta_2)\,E[F(t-\theta_1)\,F(\tau+\tau-\theta_2)]\,d\theta_1\,d\theta_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\theta_1)\,h(\theta_2)R_{FF}(\tau+\theta_1-\theta_2)d\theta_1\,d\theta_2 \quad ...11.13$$

11.4.2 Frequency Domain Analysis

If $x(t)$ is the response of the system due to an excitation $F(t)$ then we know,

$$X(\omega) = H(\omega)\,F(\omega) \quad ...11.14$$

where $X(\omega)$ and $F(\omega)$ are Fourier transforms of $x(t)$ and $F(t)$ and $H(\omega)$ is the complex frequency response function. For a single-degree-freedom-system having damping and stiffness, we know that

$$H(\omega) = \frac{1}{-m\omega^2 + ic\omega + k} = \frac{\frac{1}{k}}{\left[1-\left(\frac{\omega}{p}\right)^2\right] + i2\xi\left(\frac{\omega}{p}\right)}$$

Taking complex conjugate of all the quantities in Eq. 11.14, we get

$$X^*(\omega) = H^*(\omega)\, F^*(\omega) \qquad ...11.15$$

Multiplying equations 11.14 and 11.15, we obtain

$$X(\omega)\, X^*(\omega) = H(\omega)\, H^*(\omega)\, F(\omega)\, F^*(\omega) \qquad ...11.16$$

or

$$X(\omega)\, X^*(\omega) = [H(\omega)]^2\, F(\omega)\, F^*(\omega) \qquad ...11.17$$

Dividing both sides of the Eq. 11.17 by T, and taking limits as $T \to \infty$, we get

$$\lim_{T\to\infty} \frac{1}{T} [X(\omega)]^2 = [H(\omega)]^2 \lim_{T\to\infty} \frac{1}{T} [F(\omega)]^2 \qquad ...11.18$$

From the definition of power spectral density,

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{T} [X(\omega)]^2 \qquad ...11.19$$

$$S_{FF}(\omega) = \lim_{T\to\infty} \frac{1}{T} [F(\omega)]^2 \qquad ...11.20$$

$S_{XX}(\omega)$ and $S_{FF}(\omega)$ are the spectral densities of the response and the excitation. Combining equations 11.18 to 11.20,we get

$$S_{XX}(\omega) = [H(\omega)]^2\, S_{FF}(\omega) \qquad ...11.21$$

Equation 11.21 gives the relationship between spectral densities of excitation of the response.

REVIEW QUESTIONS

(1) Differentiate among deterministic and nondeterministic forcing functions.

(2) Define mean value between nonstationary and stationary random processes.

(4) Differentiate between mean value and time average.

(5) What are the characteristics of ergodic random processes?

(6) Define mean square value, variance and standard deviation of a random process.

(7) Define probability density functions. What are their uses?

(8) Define power spectral density.

(9) List a few examples in which forcing functions are nondeterministic.

(10) Discuss time and frequency domain analyses of random response of single DOF system.

PROBLEMS FOR PRACTICE

(1) Calculate the temporal mean valve and autocorrelation function of the function given in Fig. p.p-11.1 and plot the autocorrelation function.

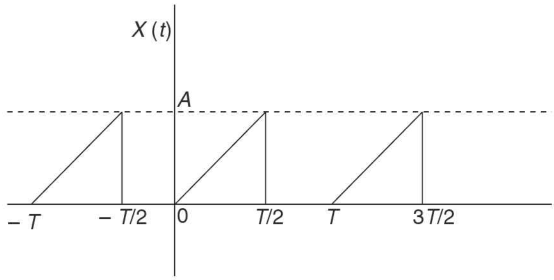


Fig p.p. 11.1

(2) Calculate the mean square value and the variance and standard deviation of the function of Fig. p.p-11.1.

OBJECTIVE-TYPE QUESTIONS

1. When dealing with random vibration problems, both excitation and response process are modelled as _________ process.
   (a) nonlinear (b) linear
   (c) stochastic (d) variable
2. For ergodic processes the temporal mean value and autocorrelation functions calculated over a representative sample function must be equal to _________
   (a) ensemble mean value
   (b) autocorrelation function
   (c) both (a) and (b)
   (d) zero
3. When mean value and autocorrelation do not depend on initial time, the random process is said to be _________
   (a) weakly stationary
   (b) strong stationary
   (c) stationary
   (d) nonstationary

Answers

(1) c (2) c (3) a

APPENDIX

Some Important Symbols

1. ω = omega
2. α = alpha
3. β = beta
4. ϕ = phi
5. θ = theta
6. ν = nu
7. μ = mu
8. τ = tau
9. $\in$ = epsilon
10. λ = lambda
11. η = eta
12. ρ = rho
13. δ = delta
14. ξ = zeta
15. σ = sigma
16. ψ = psi
17. Δ = cap. delta
18. Σ = cap. sigma

Some Important Formulas

1. $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$
2. $\cos (A + B) = \cos A \cos B - \sin A \sin B$
3. $\cos (A - B) = \cos A \cos B + \sin A \sin B$
4. $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
5. $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
6. $\sin A . \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)]$,

   $\cos A . \cos B = \frac{1}{2} [\cos (A - B) + \cos (A + B)]$
7. $\sin A + \sin B = 2 \sin \left(\frac{A + B}{2}\right) \cos \left(\frac{A - B}{2}\right)$

8. $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

9. $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

10. $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

11. $\sin^2 A + \cos^2 A = 1$

12. $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A$

13. $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$
    $= 1 - 2\sin^2\theta$

14. $\sin 3\theta = 3\sin\theta - 4\sin^2\theta$, $\cos 3\theta = 4\cos^3\theta - 3\sin\theta$

15. $\pi = 3.141592$ rad $= 57.3°$, $1° = 0.01745$ rad $e = 2.7183$

16. $e^{ix} = \cos x + i\sin x$, $\sin ix = \frac{e^{ix} - e^{-ix}}{2i}$, $\cos ix = \frac{e^{ix} + e^{-ix}}{2i}$

17. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

18. $\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

19. $Z = x + iy = A\,e^{i\theta}$ with $A = \sqrt{x^2 + y^2}$ and $\theta + \tan^{-1}\left(\frac{y}{x}\right)$

20. $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $z_1 \pm z_2 = x_1 \pm x_2 + i(y_1 \pm y_2)$

SI Units

	Parameter	***In SI Units***
(1)	Mass	kg
(2)	Force or weight	N
(3)	Linear dimensions	m' or mm
(4)	Spring stiffness	N/m or N/mm
(5)	Moment of Inertia	m^4 or mm^4
(6)	Mass moment of inertia	kg-m^2 or kg-mm^2
(7)	Torsional spring stiffness	N-m/rad or N-mm/rad
(8)	Modulus of elasticity and modulus of rigidity	N/m^2 or N/mm^2

The General Bending Moment Equation is $\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$

where M = Bending moment at a section in N-m or N-mm

I = Moment of inertia of a particular cross-section in m^4 or mm^4

E = Young's modulus of the material of the beam in N/mm^2

σ = Bending stress in N/m^2 or N/mm^2

R = Radius of curvature of bent beam in m or mm

y = Distance of the fiber under consideration from neutral axis in m or mm

The General Torsion Equation is $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$

where T = Applied torque in N-m or N-mm

J = Polar moment of inertia of circular section in mm^2

r = Radius of the circular section in mm

τ = Shear stress at the point distance 'r' from longitudinal axis in N/mm^2

G = Rigidity modulus of material of the shaft in N/mm^2

θ = Angle of twist in radians

l = Length of shaft in mm

Also $\frac{T}{J} = \frac{G\theta}{l} = \frac{T}{\theta} = k_t = \frac{GJ}{l}$, k_t = Stiffness of the Spring in N/mm

Moment of Inertia of Some Important Sections

Moment of inertia of rectangular section $I = \frac{bh^3}{12}$ shown in Fig. A.1(a).

Moment of inertia of a hollow rectangular section is shown in Fig. A.1(b). $I = \frac{bh^3}{12} - \frac{b_1 h_1^3}{12}$

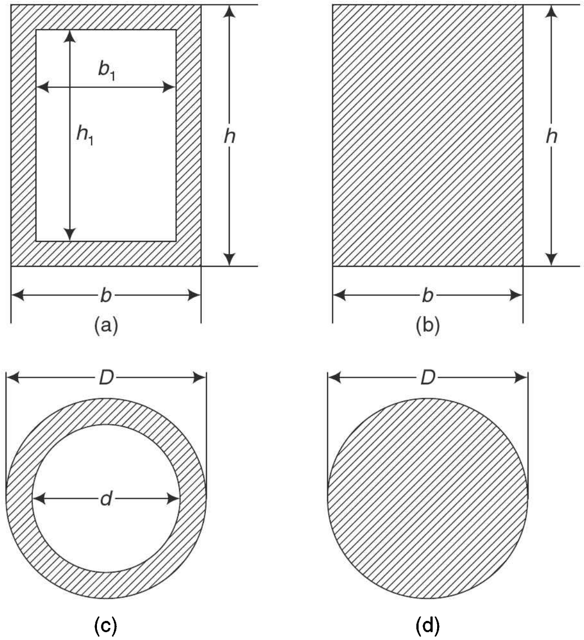


Fig. A.1

Moment of inertia of circular cross section shown in Fig. A.1(c) is $I = \dfrac{\pi D^4}{64}$

Moment of inertia of hollow circular cross section shown in Fig. A.1 (d) is

$$I = \frac{\pi D^4}{64} - \frac{\pi D^4}{64}$$

Moment of inertia of the triangular section about an axis through its centre of gravity and parallel to x–x axis shown in Fig. A.1(e) is $I = \dfrac{bh^3}{36}$

Moment of inertia of uniform thin rod shown in Fig. A.1(f) is $I = \dfrac{ml^2}{3}$

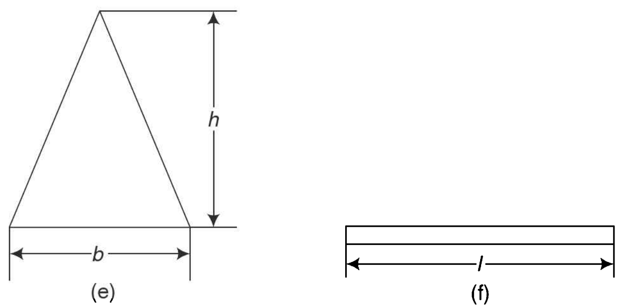


Fig. A.1

Moment of inertia of square plate shown in Fig. A.1(g) is $i = \dfrac{5ml^2}{12}$

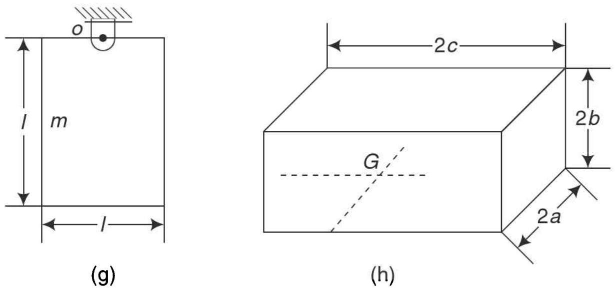


Fig. A.1

Mass moment of inertia of a rectangular block of dimension $2a$, $2b$, $2c$, about an axis passing through the centre and parallel to edge $2a$ shown in Fig. A.1(h) is

$$IG = m\left[\frac{b^2 + c^2}{3}\right]$$

Moment of inertia of circular disc, $i = 1/2\ mr^2$

Moment of inertia of slender bar, $i = \frac{ml^2}{6}$

Polar moment of inertia of circular section, $I_p = \frac{\pi d^4}{32}$

Mass moment of inertia of a semicircular shell of mass 'm' and radius 'r' about its centre 'o' is $i_o = mr^2$

Similarly, mass moment of inertia of a thin cylindrical tube having mass 'm' and radius 'r' about its axis is $i_o = mr^2$

Compound Pendulum

(a) Parallel axis theorem: Fig. A.2(a), (b)

Statement: If the MI of a plane area about an axis through its CG is denoted by i_{xx} then MI of the area about any reference (1)-(1) parallel to x-axis and at a distance of $\bar{y}$ from CG is given by

$$I_{1-1} = I_{xx} + A\bar{y}^2 \quad ...(1)$$

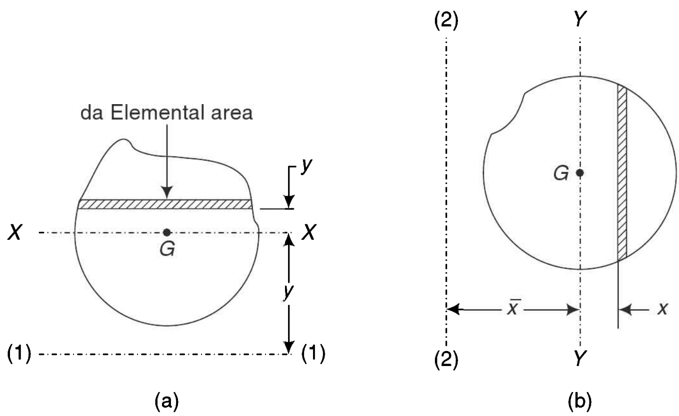


Fig. A.2 *Parallel axis theorem*

Proof: Consider an elemental area 'da' parallel to x-x axis

Let i_x = MI about x-x axis

i_{1-1} = MI about reference axis (1)-(1)

a = Area of body

$\bar{y}$ = Distance of CG of body from (1)-(1)

In Fig. A.2(a), i_{xx}, $i_{1\text{-}1}$ and elemental area are parallel to each other. Hence, it is called parallel axis theorem.

MI of element about (1)-(1) = $da(\bar{y} + y)^2$

MI of whole body about (1)-(1), $i_{1\text{-}1} = \Sigma da(\bar{y} + y)^2 = \Sigma da(\bar{y}^2 + y^2 + 2y\bar{y}) = \Sigma da\bar{y}^2 + \Sigma day^2 + 2\Sigma day\bar{y}$

or $\qquad I_{1-1} = \bar{y}^2\,\Sigma da + \Sigma day^2 + 2\bar{y}\,\Sigma day$

Now, $\quad \Sigma day^2 = \text{MI about } xx = i_{xx}$

$\Sigma day = 0$ (i.e. moment of areas about centroid = 0)

$\therefore \quad I_{1-1} = Ay^{-2} + I_{xx} + 0$ or $I_{1-1} = I_{xx} + Ay^{-2}$

Similarly, *if* i_{2-2} axis is parallel to y–y axis at a distance $\bar{x}$ from centroid then

$$I_{2-2} = I_{yy} + Ax^{-2}$$

(b) Perpendicular axis theorem

From the geometry of Fig. A.2(c), we know that $r^2 = x^2 + y^2$

We know that the moment of inertia of the lamina p about x–x axis

$$i_{xx} = da\,.y^2$$

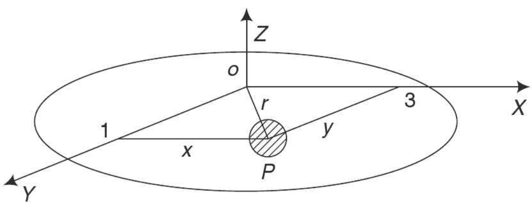


Fig. A.2 *Perpendicular axis theorem*

Similarly, $\quad i_{yy} = da.\ x^2$

and $\quad i_{zz} = da.\ r^2 = da.\ (x^2 + y^2)$

$\quad = da.\ x^2 + da.\ y^2 = i_{yy} + i_{xx}$

Equivalent Masses

Mass (m) attached at the end of spring of mass 'm' shown in Fig. A.3(a) is given by

$$m_{eq} = M + \frac{m}{3}$$

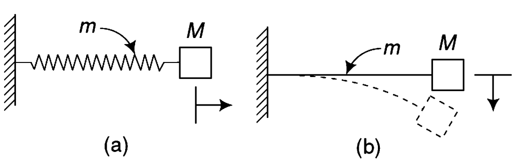
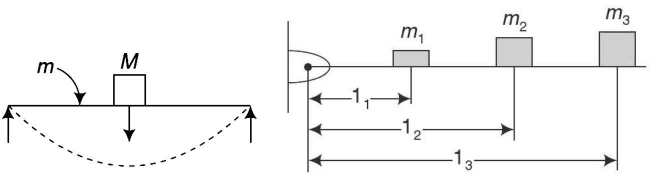


Fig. A.3

Cantilever beam of mass 'm' carrying an end mass shown in Fig. A.3(b) is given by

$$m_{\text{eq}} = m + 0.23\ m$$

Simply supported beam of mass 'm' carrying a mass 'm' at the middle shown in Fig. A.3(c) is given by $m_{eq} = m + 0.5\ m$



(c) (d)

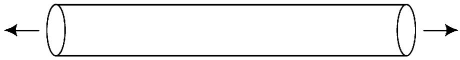
Fig. A.3

Mass on a hinged bar shown in Fig. A.3(d) is given by

$$m_{eq1} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3.$$

Equivalent Spring Stiffness

Rod under axial load shown in Fig. A.4(a) is given by $K_{eq} = \frac{EA}{I}$ N/mm.



(a)

Fig. A.4

(l = Length in mm, a = Cross-sectional area in mm^2)

Tapered rod under axial load shown in Fig. A.4(b) is given by $k_{eq} = \frac{\pi EDd}{4l}$ N/mm

(b)

Fig. A.4

d, D = End diameters, l = Length of the tapered rod in mm

E = Young's modulus of the material in N/mm^2.

(c)

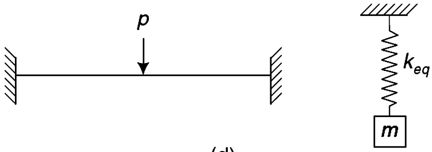
Fig. A.4

Helical spring under axial load shown in Fig. A.4(c) is given by $k_{eq} = \frac{Gd^4}{8nD^3}$ N/mm.

d = Wire diameter, D = Mean coil diameter in mm, n = Number of active turns.

(i) When we have a both-ends-fixed beam with load acting at the midpoint of the beam, the spring stiffness shown in Fig. A.4(d) is given by

$$k_{eq} = \frac{p}{\delta}, \qquad \delta = \frac{PL^3}{192EI}, \qquad k_{eq} = \frac{192EI}{l^3} \text{ N/mm}$$



(d)

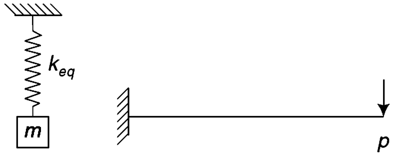
Fig. A.4

(ii) For a cantilever beam with load acting at end point, the spring stiffness shown in Fig. A.4(e) is given by $k_{eq} = \frac{3EI}{l^3}$ N/mm.

i = Moment of inertia of a particular cross-section in mm^4

e = Young's modulus of the material in N/mm

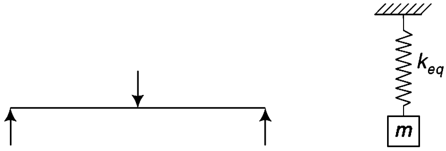
$$k_{eq} = \frac{p}{\delta}, \qquad \delta = \frac{PL^3}{3EI}, \qquad k_{eq} = \frac{3EI}{l^3} \text{ N/mm}$$



(e)

Fig. A.4

(iii) For a simply supported beam with load acting at the midpoint, the spring stiffness shown in Fig. A.4(f) is given by $k_{eq} = \frac{48EI}{l^3}$ N/mm.



(f)

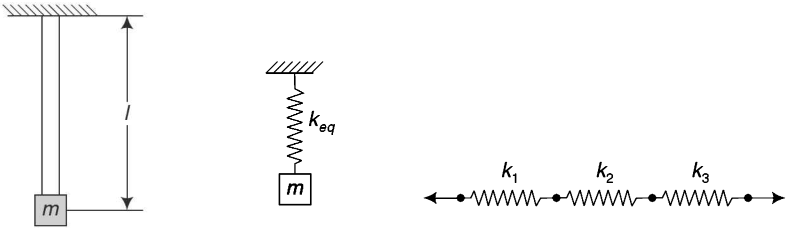
Fig. A.4

i = Moment of inertia of a particular cross-section in mm^4

E = Young's modulus of the material in N/mm^2

$$k_{eq} = \frac{p}{\delta}, \qquad \delta = \frac{Pl^3}{48EI}, \qquad k_{eq} = \frac{48EI}{l^3} \text{ N/mm}$$

(iv) Stiffness of the slender bar under longitudinal vibration shown in Fig. A.4(g) is given by $k_{eq} = \frac{AE}{l}$ N/mm.



(g) (h)

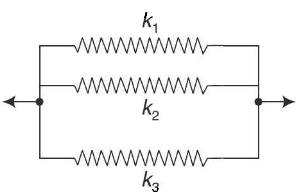
Fig. A.4

When springs are in series, the equivalent spring stiffness shown in Fig. A.4(h) is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots\ldots + \frac{1}{k_n} \text{ N/mm}$$

When springs are in parallel, the equivalent spring stiffness shown in Fig. A.5 (i) is given by

$$k_{eq} = k_1 + k_2 + k_3 + \ldots\ldots + k_n \text{ N/mm}$$

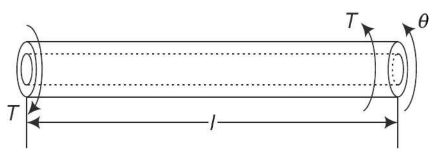


(i)

Fig. A.4

When a hollow shaft is under torsion (t), the equivalent spring stiffness shown in Fig. A.4(j) is given by

$$k_{eq} = \frac{\pi G}{32l}(D^4 - d^4) \text{ N/mm}$$



(j)

Fig. A.4

l = Length, D = Outer diameter, d = Inner diameter in mm, g = Modulus of rigidity in N/mm^2.

Static Deflection of the Beam with Different End Conditions

(a) Cantilever Beam The static deflection of the cantilever beam when the mass acting at one end shown in Fig. A.5(a) is given by the equation $\delta = \frac{PL^3}{3EI}$.

Angular frequency $\omega_n = \sqrt{\frac{3EI}{ml^3}}$ rad/s

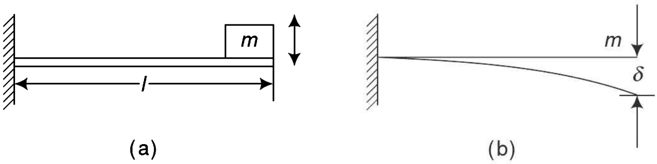


Fig. A.5

(b) Fixed-fixed beam For a beam with both ends fixed and load acting at the midpoint of beam, the static deflection of the beam shown in Fig. A.5(c) is given by the equation $\delta = \dfrac{Pl^3}{192EI}$

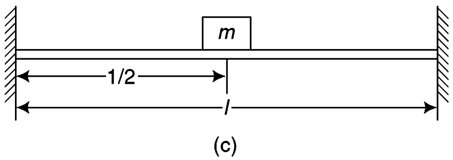


Fig. A.5

Angular frequency $\omega_n = \sqrt{\dfrac{192EI}{ml^3}}$ rad/s.

(c) Simply supported beam Maximum deflection of simply supported beam load acting at midpoint shown in Fig. A.5(d) is given by the equation $\delta_{\max} = \dfrac{Fl^3}{48EI}$

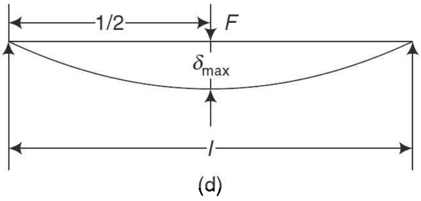


Fig. A.5

BIBLIOGRAPHY

(1) *Theory of Vibration with Applications:* W T Thomsom and Marie Dillon Dahleh, Pearson Education, 5th edition, 2007

(2) *Theory and Problems of Mechanical Vibrations:* William W Seto, McGraw-Hill Education, 1964

(3) *Theory and Practice of Mechanical Vibrations:* J S Rao and K Gupta, New Age International Publishers, 2nd edition, 2001

(4) *Mechanical Vibrations:* G K Grover, Nem Chand and Bros, Roorkee, 6th edition, 1998

(5) *Mechanical Vibration:* V P Singh, Dhanat Rai and Co. (P) Ltd. 3rd edition, 2006

(6) *Elements of Vibrations Analysis:* Leonanrd Meirovitch, Tata McGraw Hill, Special Indian Edition, 2007

(7) *Mechanical Vibrations:* Graham Kelly, Schaum's Outline Series, Tata McGraw-Hill, Special Indian Edition, 2007

(8) *Mechanical Vibration:* S S Rao, Pearson Education, 4th edition, 2003

(9) *Mechanical Vibration Analysis:* P Srinivasan, Tata McGraw-Hill, 1982

(10) *Vibrations Problems in Engineering:* S Timoshenko, D H Young, 'D Van Nostrand Co. Inc., 3rd edition, 1956

(11) *Vibrations:* Balakumar Balachandran and Edward B Magrab, Thomson Asia, 2003

INDEX